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MOTIVIC MEASURES AND $\mathbb{F}_1$-GEOMETRIES

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Abstract. Right adjoints for the forgetful functors on λ-rings and bi-rings are applied to motivic measures and their zeta functions on the Grothendieck ring of $\mathbb{F}_1$-varieties in the sense of Lorscheid and Lopez-Pena (torified schemes). This leads us to a specific subring of $\mathbb{W}(\mathbb{Z})$, properly containing Almkvist's ring $\mathbb{W}_0(\mathbb{Z})$, which might be a natural receptacle for all local factors of completed zeta functions.

1. Introduction

In [2] Jim Borger proposes to consider integral λ-rings as $\mathbb{F}_1$-algebras, with the λ-structure viewed as the descent data from $\mathbb{Z}$ to $\mathbb{F}_1$. Crucial is the fact that the functor of forgetting the λ-structure has the Witt-ring functor $\mathbb{W}(-)$ as its right adjoint.

Recall that the λ-ring $\mathbb{W}(\mathbb{Z}) = 1 + t\mathbb{Z}[[t]]$ has addition ordinary multiplication of power series, and a new multiplication induced functorially by demanding that $(1-mt)^{-1} * (1-nt)^{-1} = (1-mnt)^{-1}$. We will view $\mathbb{W}(\mathbb{Z})$ as a receptacle for motivic data, such as zeta-functions.

A counting measure is a ringmorphism $\mu : K_0(\mathbf{Var}_{\mathbb{Z}}) \longrightarrow \mathbb{Z}$, with $K_0(\mathbf{Var}_{\mathbb{Z}})$ the Grothendieck ring of schemes of finite type over $\mathbb{Z}$. A classic example being $\mu_{\mathbb{F}_p}([X]) = \#\overline{X}_p(\mathbb{F}_p)$ where $\overline{X}_p$ is the reduction of X modulo p. The $\mathbb{F}_p$-counting measure $\mu_{\mathbb{F}_p}$ is *exponentiable* meaning that it defines a ringmorphism

$$\zeta_{\mathbb{F}_p} : K_0(\mathbf{Var}_{\mathbb{Z}}) \longrightarrow \mathbb{W}(\mathbb{Z}) \qquad [X] \mapsto \zeta_{\mathbb{F}_p}(\overline{X}_p, t) = exp(\sum_{r \geq 1} \#\overline{X}_p(\mathbb{F}_{p^r})\frac{t^r}{r})$$

and is *rational*, meaning that $\zeta_{\mathbb{F}_q}$ factors through the Almkvist subring $\mathbb{W}_0(\mathbb{Z})$ of $\mathbb{W}(\mathbb{Z})$, consisting of all rational functions.

For a scheme X of finite type over $\mathbb{Z}$, let $N(x)$ for every closed point $x \in |X|$ be the cardinality of the finite residue field at x, then the *Hasse-Weil zeta function* of X decomposes as a product

$$\zeta_X(s) = \prod_{x \in |X|} \frac{1}{(1-N(x)^{-s})} = \prod_p \zeta_{\mathbb{F}_p}(\overline{X}_p, p^{-s})$$

over the non-archimedean local factors. If we take the product with the archimedean factors (Γ-factors) we obtain the completed zeta function $\hat{\zeta}_X(s)$.

One of the original motivations for constructing $\mathbb{F}_1$-geometries was to understand these Γ-factors, see the lecture notes [20] by Yuri I. Manin. For example, Manin conjectured that Deninger's Γ-factor $\prod_{n \geq 0} \frac{s-n}{2\pi}$ of $\overline{\mathbf{Spec}(\mathbb{Z})}$ at complex infinity should be the zeta function of (the dual of) infinite dimensional projective space $\mathbb{P}^{\infty}_{\mathbb{F}_1}$, see [19, 4.3] and [21, Intro].

As a step towards this conjecture, we proposed in [14] to consider integral bi-rings as $\mathbb{F}_1$-algebras, this time with the co-ring structure as the descent data from $\mathbb{Z}$ to $\mathbb{F}_1$. Here again, the forgetful functor has a right adjoint with assigns to $\mathbb{Z}$ the bi-ring $\mathbb{L}(\mathbb{Z})$ of all integral recursive sequences equipped with the Hadamard product. These two approaches to $\mathbb{F}_1$-geometry are related, that is, we have a commuting diagram of (solid) ringmorphisms (dashed morphisms are explained below)

$$\begin{tikzcd}
& \mathbb{W}_0(\mathbb{Z}) \arrow[dr] & & \\
& & \mathbb{M}(\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{W}(\mathbb{Z}) \arrow[d, "\text{👻}"] \\
\mathbb{Z}[\mathbb{L}] \arrow[uur, dotted, "\zeta_{\mathbb{F}_p}"] \arrow[urr, dotted, "\zeta_{\mathbb{F}_1}"'] & & \mathbb{L}(\mathbb{Z}) \arrow[r, "i"'] & \mathbb{Z}^{\infty}
\end{tikzcd}$$

with the ghost-map $\text{👻} = t\frac{d}{dt}log(-)$ and $\mathbb{M}(\mathbb{Z})$ the pull-back of 👻 and the natural inclusion map i. One might speculate that the relevant counting measures $\mu : K_0(\mathbf{Var}_{\mathbb{Z}}) \longrightarrow \mathbb{Z}$ are those which determine a ring-morphism $\zeta_\mu : K_0(\mathbf{Var}_{\mathbb{Z}}) \longrightarrow \mathbb{M}(\mathbb{Z})$, with those factoring over $\mathbb{W}_0(\mathbb{Z})$ corresponding to the non-archimedean factors, and the remaining ones related to the Γ-factors.

This is motivated by our description of the $\mathbb{F}_1$-zeta function of Lieber, Manin and Marcolli in [15]. Here, one considers integral schemes with a decomposition into tori $\mathbb{G}_m^n$ as $\mathbb{F}_1$-varieties and with morphisms respecting the decomposition and with all restrictions to tori being morphisms of group schemes. The corresponding Grothendieck ring $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1})$ can then be identified with the subring $\mathbb{Z}[\mathbb{L}]$ of $K_0(\mathbf{Var}_{\mathbb{C}})$. Kapranov's motivic zeta function induces a natural λ-ring structure on $\mathbb{Z}[\mathbb{L}]$ and we can also define a bi-ring structure on it by taking $\mathbb{D} = \mathbb{L} - 2$ to be a primitive generator. By right adjointness we then have natural one-to-one correspondences

$$\mathbf{comm}^+_{bi}(\mathbb{Z}[\mathbb{L}], \mathbb{L}(\mathbb{Z})) \leftrightarrow \mathbf{comm}(\mathbb{Z}[\mathbb{L}], \mathbb{Z}) \leftrightarrow \mathbf{comm}^+_{\lambda}(\mathbb{Z}[\mathbb{L}], \mathbb{W}(\mathbb{Z}))$$

To a counting measure $\mathbb{L} \mapsto m$ corresponds a λ-ring morphism $\zeta_m : \mathbb{Z}[\mathbb{L}] \longrightarrow \mathbb{W}(\mathbb{Z})$ which factors through $\mathbb{W}_0(\mathbb{Z})$ and coincides with $\zeta_{\mathbb{F}_p}$ when $m = p$. If X is an integral scheme with toric decomposition, its $\mathbb{F}_1$-zeta function is defined to be the ringmorphism

$$\zeta_{\mathbb{F}_1} : \mathbb{Z}[\mathbb{L}] \longrightarrow \mathbb{W}(\mathbb{Z}) \qquad \zeta_{\mathbb{F}_1}(X, t) = exp(\sum_{r \geq 1} \#X(\mathbb{F}_{1^m})\frac{t^r}{r})$$

with $\#X(\mathbb{F}_{1^m})$ being the total number of m-th roots of unity in the tori making up X, see [15]. This $\zeta_{\mathbb{F}_1}$ is not a λ-ring morphism and does not factor through $\mathbb{W}_0(\mathbb{Z})$. However, the counting measure $\mathbb{L} \mapsto 3$ corresponds to a bi-ring morphism $c_3 : \mathbb{Z}[\mathbb{L}] \longrightarrow \mathbb{L}(\mathbb{Z})$ which factors through $\mathbb{M}(\mathbb{Z})$ and such that the composition with $\mathbb{M}(\mathbb{Z}) \longrightarrow \mathbb{W}(\mathbb{Z})$ is the zeta-morphism $\zeta_{\mathbb{F}_1}$.

1.1. **Structure of this paper.** In section 2 we use right adjointness of the functor $\mathbb{W}(-)$ to give quick proofs of the facts that the pre λ-structure on $K_0(\mathbf{Var}_{\mathbb{C}})$ given by Kapranov's motivic zeta function does not define a λ-ring structure, and that its universal motivic measure is not exponentiable.

In section 3 we relate the versions of $\mathbb{F}_1$-geometry determined by λ-rings resp. bi-rings to the concrete resp. abstract Bost-Connes systems associated to cyclotomic Bost-Connes data as in [24]. This allows to have relative versions of $\mathbb{W}_0(\mathbb{Z})$ and $\mathbb{L}(\mathbb{Z})$ by imposing conditions on the eigenvalues of actions of Frobenii on (co)homology or on the roots and poles of zeta-polynomials.

In section 4 we study counting measures on the Grothendieck ring of torified integral schemes, proving the results mentioned above. It turns out that the pull-back $\mathbb{M}(\mathbb{Z})$ of $\mathbb{W}(\mathbb{Z})$ and $\mathbb{L}(\mathbb{Z})$ might be the appropriate receptacle for local factors of zeta functions of integral schemes. These results can be extended to other subrings of $K_0(\mathbf{Var}_{\mathbb{Z}})$ which are λ-rings and admit a bi-ring structure.

In section 5 we introduce the category of all linear dynamical systems which plays the same role for $\mathbb{L}(\mathbb{Z})$ as does the endomorphism category for $\mathbb{W}_0(\mathbb{Z})$. To completely reachable systems we associate their transfer functions which are strictly proper rational functions. As such, these systems may be relevant in the study of zeta-polynomials, as introduced by Manin in [21].

Acknowledgements This paper owes much to recent work of Yuri I. Manin, Matilde Marcolli and co-authors, [23],[15] and [24]. Unconventional symbols are taken from the LaTeX-package `halloweenmath` [25], befitting the current topic.

2. Motivic measures on $K_0(\mathbf{Var}_k)$

Let $\mathbf{Var}_k$ be the category of varieties over a field k. The Grothendieck ring $K_0(\mathbf{Var}_k)$ is the quotient of the free abelian group on isomorphism classes $[X]$ of varieties by the relations $[X] = [Y] + [X - Y]$ whenever Y is a closed subvariety of X, and multiplication is induced by products of varieties, that is, $[X].[Y] = [X \times Y]$. As the structure of $K_0(\mathbf{Var}_k)$ is fairly mysterious, we try to probe its properties via motivic measures.

Definition 1. *A* motivic measure on $K_0(\mathbf{Var}_k)$ *with values in a commutative ring R is a ringmorphism*

$$\mu : K_0(\mathbf{Var}_k) \longrightarrow R$$

The archetypical example of a motivic measure on the Grothendieck ring of varieties over a finite field $\mathbb{F}_q$ is the *counting measure* with values in $\mathbb{Z}$

$$\mu_{\mathbb{F}_q} : K_0(\mathbf{Var}_{\mathbb{F}_q}) \longrightarrow \mathbb{Z} \qquad [X] \mapsto \#X(\mathbb{F}_q)$$

An example of a motivic measure on the Grothendieck ring of complex varieties $K_0(\mathbf{Var}_{\mathbb{C}})$ with values in $\mathbb{Z}$ is the *Euler characteristic measure*

$$\chi_c : K_0(\mathbf{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z} \qquad [X] \mapsto \chi_c(X) = \sum_i (-1)^i dim_{\mathbb{Q}}\ H^i_c(X^{an}, \mathbb{Q})$$

There are plenty of motivic measures with values in other rings such as the *Hodge characteristic measure* μ_H with values in $\mathbb{Z}[u,v]$, see [16, §4.1], the *Poincaré characteristic measure* P_X with values in $\mathbb{Z}[u]$, see [16, §4.1], the *Gillet-Soulé measure* μ_{GS} with values in the Grothendieck ring if Chow motives, see [6].

Of particular importance to us are the 'exotic' Larsen-Luntz measure μ_{LL} on $K_0(\mathbf{Var}_{\mathbb{C}})$ with values in the quotient field of the monoid ring $\mathbb{Z}[C]$ with C the multiplicative monoid of polynomials in $\mathbb{Z}[t]$ with positive leading coefficient, see [12], and the *universal motivic measure*, which is the identity morphism $id : K_0(\mathbf{Var}_k) \longrightarrow K_0(\mathbf{Var}_k)$.

For a commutative ring R let $\mathbb{W}(R)$ be the set $1 + tR[[t]]$ of all formal power series over R with constant term equal to one, and let multiplication of formal power series be the *addition* on $\mathbb{W}(R)$. We say that R admits a *pre λ-structure* if there exists a morphism of additive groups

$$\lambda_t : R \longrightarrow \mathbb{W}(R) = 1 + tR[[t]] \qquad a \mapsto \lambda_t(a) = 1 + at + \ldots = \sum_{m \geq 0} \lambda^m(a) t^m$$

that is, it satisfies $\lambda_0(a) = 1, \lambda_1(a) = a$, and

$$\lambda_t(a+b) = \lambda_t(a).\lambda_t(b) \quad \text{that is} \quad \lambda^m(a+b) = \sum_{i+j=m} \lambda^i(a)\lambda^j(b)$$

Given a pre λ-structure λ_t on R we can define the *Adams operations* Ψ_m on R via

$$t\frac{d}{dt}\, log(\lambda_t(a)) = t\frac{1}{\lambda_t(a)}\frac{d\lambda_t(a)}{dt} = \sum_{m \geq 1} \Psi_m(a) t^m$$

and note that for all $m \in \mathbb{N}$ and all $a, b \in R$ we have $\Psi_m(a+b) = \Psi_m(a) + \Psi_m(b)$. We say that a pre λ-ring R is a *λ-ring* if for all $m, n \in \mathbb{N}$ we have these conditions on the Adams operations

$$\Psi_m(a.b) = \Psi_m(a).\Psi_m(b) \quad \text{and} \quad \Psi_m \circ \Psi_n = \Psi_n \circ \Psi_m$$

Equivalently, if we define a multiplication $*$ on $\mathbb{W}(R)$ induced by the functorial requirement that $(1-at)^{-1} * (1-bt)^{-1} = (1-abt)^{-1}$ for all $a, b \in R$, then the map λ_t is a morphism of rings. For more on λ-rings, see [9], [11] and [33].

A morphism $\phi : (R, \lambda_t) \longrightarrow (R', \lambda'_t)$ between two λ-rings is a ringmorphism such that for all $a \in R$ we have that $\lambda'_t(\phi(a)) = \mathbb{W}(\phi)(\lambda_t(a))$ where $\mathbb{W}(\phi)$ is the map on $\mathbb{W}(R) = 1 + tR[[t]]$ induced by ϕ. With $\mathbf{comm}^+_\lambda$ we will denote the category of all (commutative) λ-rings. If $\mathbf{comm}$ is the category of all commutative rings, then

$$\mathbb{W} \ : \ \mathbf{comm} \longrightarrow \mathbf{comm}^+_\lambda \qquad A \mapsto \mathbb{W}(A)$$

is a functor, which is right adjoint to the forgetful functor $F : \mathbf{comm}^+_\lambda \longrightarrow \mathbf{comm}$. That is, for every λ-ring (R, λ_t) and every commutative ring A we have a natural one-to-one corespondence

$$\mathbf{comm}^+_\lambda(R, \mathbb{W}(A)) \leftrightarrow \mathbf{comm}(R, A) \qquad \phi \leftrightarrow \ghost_1 \circ \phi$$

with the *ghost components* $\ghost_m : \mathbb{W}(A) \longrightarrow A$ defined by

$$t\frac{1}{P}\frac{dP}{dt} = \sum_{m=1}^{\infty} \ghost_m(P) t^m \qquad \text{for all } P \in \mathbb{W}(A) = 1 + tA[[t]]$$

Kapranov's motivic zeta function ζ defines a natural pre λ-structure on $K_0(\mathbf{Var}_k)$

$$\zeta : K_0(\mathbf{Var}_k) \longrightarrow \mathbb{W}(K_0(\mathbf{Var}_k)) \quad [X] \mapsto \zeta_X(t) = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots$$

where $S^nX = X^n/S_n$ is the n-th symmetric product of X.

Definition 2. *A motivic measure $\mu : K_0(\mathbf{Var}_k) \longrightarrow R$ with values in R is said to be* exponentiable *if the uniquely determined map $\zeta_\mu : K_0(\mathbf{Var}_k) \longrightarrow \mathbb{W}(R)$ by*

$$\zeta_\mu([X]) = 1 + \mu([X])t + \mu([S^2X])t^2 + \mu([S^3X])t^3 + \dots$$

is a ringmorphism.

Again, the archetypical example being the counting measure $\mu_{\mathbb{F}_q}$ on $K_0(\mathbf{Var}_{\mathbb{F}_q})$ which is exponentiable, with corresponding zeta-function

$$\zeta_{\mu_{\mathbb{F}_q}} : K_0(\mathbf{Var}_{\mathbb{F}_q}) \longrightarrow \mathbb{W}(\mathbb{Z}) \qquad \zeta_{\mu_{\mathbb{F}_q}}([X]) = \sum_{m=1}^{\infty} \#X(\mathbb{F}_{q^m})t^m = Z_{\mathbb{F}_q}(X,t)$$

the classical Hasse-Weil zeta function, see [26, Prop. 8] or [29, Thm. 2.1]. Also the Euler characteristic measure on $K_0(\mathbf{Var}_{\mathbb{C}})$ is exponentiable with corresponding zeta function

$$\zeta_{\mu_c} : K_0(\mathbf{Var}_{\mathbb{C}}) \longrightarrow \mathbb{W}(\mathbb{Z}) \qquad \zeta_{\mu_c}([X]) = \frac{1}{(1-t)^{\chi_c(X)}}$$

However, as shown in [30, §4] the Larsen-Luntz motivic measure μ_{LL} on $K_0(\mathbf{Var}_{\mathbb{C}})$ is *not* exponentiable. For this would imply that

$$\zeta_{\mu_{LL}}(C_1 \times C_2) = \zeta_{\mu_{LL}}(C_1) * \zeta_{\mu_{LL}}(C_2)$$

for any pair of projective curves C_1 and C_2. Kapranov proved that $\zeta_\mu(C)$ is a rational function for every curve and every motivic measure, which would imply that $\mu_{LL}(C_1 \times C_2)$ would be rational too, by [30, Prop. 4.3], which contradicts [12, Thm 7.6] in case C_1 and C_2 have genus ≥ 1.

It is a natural to ask whether the pre λ-structure on $K_0(\mathbf{Var}_k)$ defined by Kapranov's motivic zeta function defines a λ-ring structure on $K_0(\mathbf{Var}_k)$, see [29, §3 Questions] or [7, §2.2]. The following is well-known to the experts, but we cannot resist including the short proof.

Proposition 1. *If Kapranov's motivic zeta function makes $K_0(\mathbf{Var}_k)$ into a λ-ring, then every motivic measure*

$$\mu : K_0(\mathbf{Var}_k) \longrightarrow R$$

is exponentiable.

As a consequence, Kapranov's zeta function does not define a λ-ring structure on $K_0(\mathbf{Var}_{\mathbb{C}})$.

Proof. If $K_0(\mathbf{Var}_k)$ is a λ-ring, then by right adjunction of $\mathbb{W}(-)$ with respect to the forgetful functor, we have a natural one-to-one correspondence

$$\mathbf{comm}(K_0(\mathbf{Var}_k), R) \leftrightarrow \mathbf{comm}^+_\lambda(K_0(\mathbf{Var}_k), \mathbb{W}(R))$$

and under this correspondence the motivic measure μ maps to a unique λ-ring morphism $\zeta_\mu : K_0(\mathbf{Var}_k) \longrightarrow \mathbb{W}(R)$.

Because the Larsen-Luntz motivic measure μ_{LL} on $K_0(\mathbf{Var}_{\mathbb{C}})$ is not exponentiable, it follows that $K_0(\mathbf{Var}_{\mathbb{C}})$ cannot be a λ-ring. $\square$

Another immediate consequence is this negative answer to [29, §3 Questions].

Proposition 2. *The universal motivic measure on $K_0(\mathbf{Var}_{\mathbb{C}})$ is not exponentiable.*

Proof. By functoriality, any motivic measure $\mu : K_0(\mathbf{Var}_{\mathbb{C}}) \longrightarrow R$ gives rise to a morphism of λ-rings $\mathbb{W}(\mu) : \mathbb{W}(K_0(\mathbf{Var}_{\mathbb{C}})) \longrightarrow \mathbb{W}(R)$.

If the universal measure would be exponentiable, this would give a ringmorphism $\zeta : K_0(\mathbf{Var}_{\mathbb{C}}) \longrightarrow \mathbb{W}(K_0(\mathbf{Var}_{\mathbb{C}}))$ and composition

$$\mathbb{W}(\mu) \circ \zeta : K_0(\mathbf{Var}_{\mathbb{C}}) \longrightarrow \mathbb{W}(R)$$

would then imply that μ is exponentiable, which cannot happen for μ_{LL}. $\square$

An important condition on a motivic measure $\mu : K_0(\mathbf{Var}_k) \longrightarrow R$ is its *rationality*. In order to define this, we need to recall the *endomorphism category* and its Grothendieck ring, see [1] and [8].

For a commutative ring R consider the category $\mathcal{E}_R$ consisting of pairs (E, f) where E is a projective R-module of finite rank and f is an endomorphism of E. Morphisms in $\mathcal{E}_R$ are module morphisms compatible with the endomorphisms. There is a duality $(E, f) \leftrightarrow (E^*, f^*)$ on $\mathcal{E}_R$ and we have $\oplus$ and $\otimes$ operations

$$(E_1, f_1) \oplus (E_2, f_2) = (E_1 \oplus E_2, f_1 \oplus f_2) \quad (E_1, f_1) \otimes (E_2, f_2) = (E_1 \otimes E_2, f_1 \otimes f_2)$$

with a zero object $\mathbf{0} = (0, 0)$ and a unit object $\mathbf{1} = (R, 1)$. These operations turn the Grothendieck ring $K_0(\mathcal{E}_R)$ into a commutative ring, having an ideal consisting of the pairs $(E, 0)$, with quotient ring $\mathbb{W}_0(R)$.

The ring $\mathbb{W}_0(R)$ comes equipped with Frobenius ring endomorphisms $Fr_n(E, f) = (E, f^n)$, Verschiebung additive maps

$$V_n(E, f) = (E^{\oplus n}, \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & f \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix})$$

and ghost ringmorphisms $\ghost_n(E, f) = Tr(f^n) : \mathbb{W}_0(R) \longrightarrow R$. For various relations among the maps Fr_n, V_n and $\ghost_n$ see for example [4, Prop. 2.2].

The connection between Almkvist's functor $\mathbb{W}_0(-)$ and $\mathbb{W}(-)$ is given by the ringmorphisms

$$L_R \; : \; \mathbb{W}_0(R) \longrightarrow \mathbb{W}(R) \qquad L_R(E, f) = \frac{1}{det(1 - tM_f)}$$

where M_f is the matrix associated to f (that is, if $f = \sum_i x_i^* \otimes x_i \in End_R(E) = E^* \otimes E$, then $M_f = (a_{ij})_{i,j}$ with $a_{ij} = x_i^*(x_j)$. By [1, Thm 6.4] we know that L_R is injective with image all *rational* formal power series of the form

$$\frac{1 + a_1 t + \dots + a_n t^n}{1 + b_1 t + \dots + b_m t^m} \qquad a_i, b_i \in R, m, n \in \mathbb{N}_+$$

Definition 3. *We say that a motivic measure $\mu : K_0(\mathbf{Var}_k) \longrightarrow R$ is* rational *if it is exponentiable and if the corresponding zeta-function ζ_μ factors through $\mathbb{W}_0(RT)$. That is, there is a unique ringmorphism*

$$r_\mu : K_0(\mathbf{Var}_k) \longrightarrow \mathbb{W}_0(R)$$

such that $\zeta_\mu = L_R \circ r_\mu$.

By a classic result of Dwork we know that the counting measure $\mu_{\mathbb{F}_q}$ is rational, as is the Euler characteristic measure μ_c.

3. Cyclotomic Bost-Connes data

Let R be an integral domain with field of fractions K of characteristic zero and with algebraic closure $\overline{K}$. Let $\overline{K}^*_\times$ be the multiplicative group of all non-zero elements and $\boldsymbol{\mu}_\infty$ the subgroup consisting of all roots of unity. The power maps $\sigma_n : x \mapsto x^n$ for $n \in \mathbb{N}_+$ form a commuting family of endomorphisms of $\overline{K}^*_\times$ and its subgroups. Following M. Marcolli en G. Tabuada in [24] we define:

Definition 4. *A* cyclotomic Bost-Connes datum *is a divisible subgroup* Σ

$$\boldsymbol{\mu}_\infty \subseteq \Sigma \subseteq \overline{K}^*_\times$$

stable under the action of the Galois group $G = Gal(\overline{K}/K)$.

The subgroup Σ should be considered as 'generalised' Weil numbers (recall that for each prime power $q = p^r$ the Weil q-numbers are an instance, see [24, Example 4]).

Observe that cyclotomic Bost-Connes data are special cases of *concrete* Bost-Connes data as in [24, Def. 2.3] with the endomorphisms σ_n the n-th power maps $\sigma_n(x) = x^n$ and $\rho_n(x) = \boldsymbol{\mu}_n \sqrt[n]{x} \subset \Sigma$. In [24, §4] Marcolli and Tabuada associate to a cyclotomic Bost-Connes system with $\overline{K} = \overline{\mathbb{Q}}$ a quantum statistical mechanical system. Further, in [24, §2] both *concrete* and *abstract* Bost-Connes systems are associated to a cyclotomic Bost-Connes datum Σ. We will relate these to $\mathbb{F}_1$-geometries.

A powerful idea, due to Jim Borger [2] and [3], to construct 'geometries' under $\mathbf{Spec}(\mathbb{Z})$ is to consider a subcategory $\mathbf{comm}^+_{\mathbf{X}}$ of commutative rings $\mathbf{comm}$ which allows a right adjoint R to the forgetful functor $F : \mathbf{comm}^+_{\mathbf{X}} \longrightarrow \mathbf{comm}$.

The additional structure $\mathbf{X}$ should be thought of as descent data from $\mathbb{Z}$ to $\mathbb{F}_1$, the elusive field with one element. As a consequence, the commutative ring $F(R(\mathbb{Z}))$ can then be considered to be the coordinate ring of the arithmetic square $\mathbf{Spec}(\mathbb{Z}) \times_{\mathbf{Spec}(\mathbb{F}_1)} \mathbf{Spec}(\mathbb{Z})$.

We propose to view the object $R(\mathbb{Z}) \in \mathbf{comm}^+_{\mathbf{X}}$ as a receptacle for motivic data. That is, (co)homology groups with actions of Frobenii and zeta-functions determine elements in $R(\mathbb{Z})$ and the subobject in $\mathbf{comm}^+_{\mathbf{X}}$ they generate can then be seen as its representative in the corresponding version of $\mathbb{F}_1$-geometry.

3.1. **Concrete Bost-Connes systems and $\mathbf{comm}^+_\lambda$.** Following [24, Def. 2.6] one associates to Σ the *concrete Bost-Connes system* which consists of the integral group ring $\mathbb{Z}[\Sigma]$ equipped with

(1) the induced $G = Gal(\overline{K}/K)$-action,
(2) G-equivariant ring endomorphisms $\tilde{\sigma}_n$ induced by $\tilde{\sigma}_n(x) = x^n$ for all $x \in \Sigma$,
(3) G-equivariant $\mathbb{Z}$-module maps $\tilde{\rho}_n$ induced by $\tilde{\rho}_n(x) = \sum_{x' \in \rho_n(x)} x'$ for all $x \in \Sigma$.

Proposition 3. *For a cyclotomic Bost-Connes datum* Σ*, the concrete Bost-Connes system* $(\mathbb{Z}[\Sigma], \tilde{\sigma}_n, \tilde{\rho}_n)$ *is a sub-system of* $(\mathbb{W}_0(\overline{K}), Fr_n, V_n)$.

Proof. From [4, Prop. 2.3] we recall that $\mathbb{W}_0(\overline{K})$ is isomorphic to the integral group ring $\mathbb{Z}[\overline{K}^*_\times]$ via the map which assigns to (E,f) the divisor of non-zero eigenvalues of f (with multiplicities).

Under this isomorphism the Frobenius maps Fr_n becomes $\tilde{\sigma}_n$ and the Verschiebung V_n the map $\tilde{\rho}_n$ for the cyclotomic Bost-Connes datum $\overline{K}^*_\times$. □

Definition 5. *For a cyclotomic Bost-Connes datum Σ, let $\mathcal{E}_{\Sigma,R}$ be the full subcategory of $\mathcal{E}_R$ consisting of pairs (E,f) with E a projective R-module and M_f a $\overline{K}$-diagonalisable matrix having all its eigenvalues in Σ. With $\mathbb{W}_0(\Sigma,R)$ we denote the subring of $\mathbb{W}_0(R)$ generated by $\mathcal{E}_{\Sigma,R}$.*

Example 1. *Consider Yuri I. Manin's idea to replace the action of the Frobenius map on étale cohomology of an $\mathbb{F}_q$-variety at $q=1$ by pairs $(H_k(M,\mathbb{Z}),f_{*k})$ where f_{*k} is the action of a Morse-Smale diffeomorphism f on a compact manifold M upon its homology $H_k(M,\mathbb{Z})$,* [18, §0.2]. *This implies that each f_{*k} is quasi-unipotent, that is all its eigenvalues are roots of unity. This fits in with Manin's view that 1-Frobenius morphisms acting upon their (co)homology have eigenvalues which are roots of unity.*

In [23, §2]*, Manin and Matilde Marcolli assign an object in* $\mathbf{comm}^+_\lambda$ *to the Morse-Smale setting (M,f) as follows. Each $H_k(M,\mathbb{Z})$ is viewed as a $\mathbb{Z}[t,t^{-1}]$-module by letting t act as f_{*k}. Next, they consider the minimal category $\mathcal{C}_M$ of $\mathbb{Z}[t,t^{-1}]$-modules, containing all $H_k(M,\mathbb{Z})$, and closed with respect to direct sums, tensor products and exterior products. Then, its Grothendieck ring $K_0(\mathcal{C}_M)$ comes equipped with a λ-ring structure coming from the exterior products, which is then said to be the representative of $\{(H_k(M,\mathbb{Z}),f_{*k});k\}$ in $\mathbb{F}_1$-geometry, see* [23, Def. 2.4.2].

Alternatively, one can assign to each $(H_k(M,\mathbb{Z}),f_{*k})$ the element

$$det(1-t(f_{*k}|H_k(M,\mathbb{Z})))^{-1} \in 1+t\mathbb{Z}[[t]] = \mathbb{W}(\mathbb{Z})$$

and consider the λ-subring of $\mathbb{W}(\mathbb{Z})$ generated by these elements. Clearly, all $(H_k(M,\mathbb{Z}),f_{*k})$ lie in $\mathcal{E}_{\boldsymbol{\mu}_\infty,\mathbb{Z}}$.

3.2. **Abstract Bost-Connes systems and $\mathbf{comm}^+_{\mathbf{bi}}$.** Following [24, Def. 2.5] one can associate to a cyclotomic Bost-Connes datum Σ the *abstract Bost-Connes system* which consists of the Galois-invariants of the group ring of Σ over $\overline{K}$, that is,

(1) the K-algebra $\overline{K}[\Sigma]^{Gal(\overline{K}/K)}$, equipped with
(2) K-algebra morphisms $\tilde{\sigma}_n$ induced by $x \mapsto x^n$ for all $x \in \Sigma$, and
(3) K-linear maps $\tilde{\rho}_n$ induced by $x \mapsto_{x' \in \rho_n(x)} x'$ for all $x \in \Sigma$.

Clearly, $\overline{K}[\Sigma]^G$ is a Hopf-algebra and from [24, Thm. 1.5.(iv)] we recall that the affine group K-scheme $\mathbf{Spec}(\overline{K}[\Sigma]^G)$ agrees with the Galois group of the neutral Tannakian category $\mathbf{Aut}^{\overline{K}}_\Sigma(\mathbb{Q})$ consisting of pairs (V,Φ) with V a finite dimensional K-vectorspace and

$$\Phi : V \otimes \overline{K} \longrightarrow V \otimes \overline{K}$$

a G-equivariant diagonalisable automorpism all of whose eigenvalues belong to Σ, that is, the category $\mathcal{E}_{\Sigma,K}$ introduced above.

In [14] we proposed to consider the category $\mathbf{comm}^+_{\mathbf{bi}}$ of all (torsion free) commutative and co-commutative $\mathbb{Z}$-birings. This time, the forgetful functor

$F : \mathbf{comm}^+_{\mathbf{bi}} \longrightarrow \mathbf{comm}$ has as right adjoint $C(-)$ where $C(A)$ is the free co-commutative co-ring on A. In particular, $C(\mathbb{Z}) = \mathbb{L}(\mathbb{Z})$, the coring of all integral linear recursive sequences, equipped with the Hadamard product, see [14, Thm. 2].

For a commutative domain R, consider the polynomial ring $R[t]$ with coring structure defined by letting t be a group-like element, that is, $\Delta(t) = t \otimes t$ and $\epsilon(t) = 1$.

The full linear dual $R[t]^*$ can be identified with the module of all infinite sequences $f = (f_n)_{n=0}^{\infty} \in R^{\infty}$ with $f(t^n) = f_n$. $\mathbb{L}(R)$ will be $R[t]^o$, that is, the submodule of all sequences f such that $Ker(f) = (m(t))$ with $m(t) = t^r - a_1 t^{r-1} - \ldots - a_r$ is a monic polynomial. As $f(t^n m(t)) = 0$ it follows that f is a linear recursive sequence, that is, for all $n \geq r$ we have $f_n = a_1 f_{n-1} + a_2 f_{n-2} + \ldots + a_r f_{n-r}$. Therefore,

$$\mathbb{L}(R) = R[t]^o = \lim_{\rightarrow} \left(\frac{R[t]}{(m(t))}\right)^*$$

where the limit is taken over the multiplicative system of monic polynomials with coefficients in R.

We define a coring structure on $\mathbb{L}(R)$ dual to the ring structure on $R[t]/(m(t))$. With this coring structure, $\mathbb{L}(R)$ becomes an integral biring if we equip $\mathbb{L}(R)$ with the Hadamard product of sequences, that is, componentwise multiplication $(f.g)_n = f_n.g_n$ and unit $1 = (1, 1, 1, \ldots)$.

If K is a field of characteric zero, one can describe the co-algebra structure on $\mathbb{L}(K)$ explicitly, see [28] for more details.

On the linear recursive sequence $f = (f_i)_{i=0}^{\infty} \in K^{\infty}$ the counit acts as $\epsilon(f) = f_0$, projection on the first component. To define the co-multiplication recall that the *Hankel matrix* $M(f)$ of the sequence f is the symmetric $k \times k$ matrix

$$H(f) = \begin{bmatrix} f_0 & f_1 & f_2 & \ldots & f_{k-1} \\ f_1 & f_2 & f_3 & \ldots & f_k \\ f_2 & f_3 & f_4 & \ldots & f_{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ f_{k-1} & f_k & f_{k+1} & \ldots & f_{2k-2} \end{bmatrix}$$

with k maximal such that $H(f)$ is invertible. If $H(f)^{-1} = (s_{ij})_{i,j} \in M_n(K)$ then we have in $\mathbb{L}(K)$

$$\Delta(f) = \sum_{i,j=0}^{k-1} s_{ij} (D^i f) \otimes (D^j f)$$

where D is the shift operator $D(f_0, f_1, f_2, \ldots) = (f_1, f_2, \ldots)$. Clearly, if K is the fraction field of R, and if a sequence $f \in \mathbb{L}(R)$ has Hankel matrix $H(f)$ with determinant a unit in R, the same formula applies for $\Delta(f)$ as $\mathbb{L}(R)$ is a sub-biring of $\mathbb{L}(K)$. In general however, $\Delta(f)$ cannot be diagonalized in terms of $f, Df, D^2 f, \ldots$ with R-coefficients and we have no other option to describe the comultiplication than as the direct limit of linear duals of the ringstructures on $R[t]/(m(t))$.

Proposition 4. *For a cyclotomic Bost-Connes datum Σ, the Hopf-algebra $\overline{K}[\Sigma]^G$ describing the abstract Bost-Connes system is a sub-bialgebra of $\mathbb{L}(K)$.*

Proof. We can describe the bialgebra $\mathbb{L}(\overline{K})$ of linear recursive sequences over $\overline{K}$ using the structural results for commutative and co-commutative Hopf algebras over an algebraically closed field of characteristic zero, see [13].

Let T be the set of all sequences over $\overline{K}$ which are zero almost everywhere, then T is a bialgebra ideal in $\mathbb{L}(\overline{K})$ and we have a decomposition

$$\mathbb{L}(\overline{K}) = \overline{K}[t]^o \simeq \overline{K}[t,t^{-1}]^o \oplus T$$

One verifies that in the Hopf-dual $\overline{K}[t,t^{-1}]^o$ the group of group-like elements is isomorphic to the multiplicative group $\overline{K}^*_\times$, with $s \in \overline{K}^*_\times$ corresponding to the geometric sequence $(1,s,s^2,s^3,\ldots)$. Further, there is a unique primitive element corresponding to the sequence $d=(0,1,2,3,\ldots)$. Then, the structural result implies that, as bialgebras, we have an isomorphism

$$\mathbb{L}(\overline{K}) \simeq (\overline{K}[\overline{K}^*_\times] \otimes \overline{K}[d]) \oplus T$$

As the Galois group $G = Gal(\overline{K}/K)$ acts on this bialgebra and as $\mathbb{L}(K) = \mathbb{L}(\overline{K})^G$, the claim follows. □

Example 2. *Continuing Example 1 on Morse-Smale diffeomorphism, as anticipated in* [23, remark 2.4.3]*, in the* $\mathbf{comm}^+_{\mathbf{bi}}$*-proposal, one can associate to each* $(H_k(M,\mathbb{Z}), f_{*k})$ *the element*

$$(Tr(f_{*k}|H_k(M,\mathbb{Z})), Tr(f^2_{*k}|H_k(M,\mathbb{Z})), Tr(f^3_{*k}|H_k(M,\mathbb{Z})), \ldots) \in \mathbb{L}(\mathbb{Z})$$

and considers the sub-biring of $\mathbb{L}(\mathbb{Z})$ *generated by these elements.*

3.3. **Motivic measures and $\mathbb{L}(R)$.** By taking the trace of the Cayley-Hamilton polynomial we have a ghost ringmorphism $\ghost : \mathbb{W}_0(R) \longrightarrow \mathbb{L}(R)$

$$(E,f) \mapsto (\ghost_1(E,f), \ghost_2(E,f), \ldots) = (Tr(M_f), Tr(M_f^2), \ldots)$$

Further, we have a traditional ghost morphism $\ghost : \mathbb{W}(R) \longrightarrow R^\infty$ determined by $t\frac{d}{dt}log(-)$ on $\mathbb{W}(R) = 1 + tR[[t]]$

$$\ghost(f(t)) = (a_1, a_2, \ldots) \quad \text{where} \quad t\frac{d}{dt}log(f(t)) = \sum_{m=1}^{\infty} a_m t^m$$

Proposition 5. *Let R be a commutative ring and $\mu : K_0(\mathbf{Var}_k) \longrightarrow R$ a motivic measure. The measure μ is exponentiable if there exists a ringmorphism ζ_μ, and is rational if there is a ringmorphism r_μ, making the diagram below commute*

$$\begin{tikzcd}
 & K_0(\mathbf{Var}_k) \arrow[r, "\mu"] \arrow[d, dotted, "r_\mu"'] \arrow[dr, dotted, "\zeta_\mu"] & R \\
\mathcal{E}_R \arrow[r, dotted] & \mathbb{W}_0(R) \arrow[r, "L_R"] \arrow[d, "\ghost"] & \mathbb{W}(R) \arrow[d, "\ghost"] \\
\mathcal{S}_R^{cr} \arrow[r, dotted, two heads] & \mathbb{L}(R) \arrow[r, "i"] & R^\infty
\end{tikzcd}$$

The left-most maps are additive and multiplicative from the endomorphism category, resp. the category of completely reachable systems, to be defined in § 5.

Proof. This follows from the definitions above and the fact that $log(L_R(E,f)) = \sum_{m\geq 1} Tr(M_f^m)\frac{t^m}{m}$. □

Example 3. *As a consequence, an exponentiable motivic measure μ assigns to a k-variety X the element $\zeta_\mu([X]) \in \mathbb{W}(R)$, and a rational motivic measure μ assigns to X elements 👻$(r_\mu([X])) \in \mathbb{L}(R)$ and $L_R(r_\mu([X])) \in \mathbb{W}(R)$.*

4. Motivic measures on $K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor})$

In this section we consider yet another approach to $\mathbb{F}_1$-geometry based on the notion of *torifications* as introduced by Lorscheid and Lopez Pena in [17] and generalized by Manin and Marcolli in [22].

A *torification* of a complex algebraic variety, defined over $\mathbb{Z}$, is a decomposition into algebraic tori

$$X = \sqcup_{i \in I} T_i \quad \text{with} \quad T_i \simeq \mathbb{G}_m^{d_i}$$

We consider here *strong morphisms* between torified varieties (see [15, §5.1] for weaker notions), that is a morphism of varieties, defined over $\mathbb{Z}$,

$$f : X = \sqcup_{i \in I} T_i \longrightarrow Y = \sqcup_{j \in J} T'_j$$

together with a map $h : I \longrightarrow J$ of the indexing sets such that the restriction of f to any torus

$$f_i = f|_{T_i} : T_i \longrightarrow T'_{h(i)}$$

is a morphism of algebraic groups. With $K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor})$ we denote the Grothendieck ring generated by the strong isomorphism classes $[X = \sqcup_i T_i]$ of torified varieties, modulo the *scissor relations*

$$[X = \sqcup_i T_i] = [Y = \sqcup_j T'_j] + [X \backslash Y = \sqcup_k T"_k]$$

whenever the decomposition in tori in the torifications of Y and $X \backslash Y$ is a union of tori of the torification of X. This condition is very strong and implies that the class of any torified variety in $K_o(\mathbf{Var}_{\mathbb{F}_1}^{tor})$ is of the form

$$[X = \sqcup_i T_i] = \sum_{n \geq 0} a_n \mathbb{T}^n \quad \text{with } a_n \in \mathbb{N}_+ \text{ and } \mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1 \in K_0(\mathbf{Var}_{\mathbb{C}})$$

That is,

$$K_0(\mathbf{Var}_{\mathbb{F}_q}^{tor}) = \mathbb{Z}[\mathbb{T}] = \mathbb{Z}[\mathbb{L}] \subset K_0(\mathbf{Var}_{\mathbb{C}})$$

with $\mathbb{L} = [\mathbb{A}^1]$ the Lefschetz motive. Whereas Kapranov's motivic zeta function does not make $K_0(\mathbf{Var}_{\mathbb{C}})$ into a λ-ring, it does define a λ-structure on certain subrings, including $\mathbb{Z}[\mathbb{L}]$, see [7, §2.2 Example], with $S^n(\mathbb{L}) = \mathbb{L}^n$

Proposition 6. *Any motivic measure $\mu : K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor}) \longrightarrow R$ with values in a commutative ring R is exponentiable and rational.*

Proof. Because $K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor}) = \mathbb{Z}[\mathbb{L}]$ is a λ-ring, we have by right adjointness of $\mathbb{W}(-)$ a natural one-to-one correspondence

$$\mathbf{comm}(K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor}), R) \leftrightarrow \mathbf{comm}_\lambda^+(K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor}), \mathbb{W}(R))$$

with μ corresponding to a unique λ-ring morphism

$$\zeta_\mu : K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor}) \longrightarrow \mathbb{W}(R) = 1 + tR[[t]] \qquad \mathbb{L} \mapsto 1 + rt + r^2t^2 + \ldots = \frac{1}{1 - rt}$$

with $r = \mu(\mathbb{L})$. That is, μ is exponentiable and rational as it factors through the ringmorphism $r_\mu : K_0(\mathbf{Var}_{\mathbb{F}_1}^{tor}) \longrightarrow \mathbb{W}_0(R)$ defined by $\mathbb{L} \mapsto [R, r]$. $\square$

If we equip $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}) = \mathbb{Z}[\mathbb{L}]$ with the bi-ring structure induced by letting $\mathbb{L}$ be a group-like generator, that is $\Delta(\mathbb{L}) = \mathbb{L} \otimes \mathbb{L}$ and $\epsilon(\mathbb{L}) = 1$, we have a bi-ring morphism $c_\mu : K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}) \longrightarrow \mathbb{L}(R)$ defined by $\mathbb{L} \mapsto (1, r, r^2, \dots)$ making the diagram below commutative.

$$\begin{tikzcd}
K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}) \arrow[dr, "r_\mu"] \arrow[drr, bend left, "\zeta_\mu"] \arrow[ddr, bend right, "c_\mu"'] & & \\
& \mathbb{W}_0(R) \arrow[r, "L_{\mathbb{Z}}"] \arrow[d] & \mathbb{W}(R) \arrow[d] \\
& \mathbb{L}(R) \arrow[r] & R^\infty
\end{tikzcd}$$

For example, any motivic measure with values in $\mathbb{Z}$ is of the form

$$\mu_m : K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}) = \mathbb{Z}[\mathbb{L}] \longrightarrow \mathbb{Z} \qquad \mathbb{L} \mapsto m+1$$

and if $m+1 = p$ with p a prime number, the corresponding zeta function $\zeta_{\mu_m}(X,t)$ coincides with the Hasse-Weil zeta function of the reduction mod p of the torified variety X. The reason for choosing $m+1$ rather than m will be explained in 4.1 below.

Similarly, we can define $\mathbb{F}_{1^m}$*-varieties* to be torified varieties $X = \sqcup T_i$ with the natural action of the group of m-th roots of unity $\boldsymbol{\mu}_m$ on each torus T_i. As a consequence we have

$$K_0(\mathbf{Var}^{tor}_{F_{1^m}}) = \mathbb{Z}[\mathbb{T}] = \mathbb{Z}[\mathbb{L}]$$

and the previous result holds also for $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}})$.

4.1. **Counting $\mathbb{F}_{1^m}$-points.** The motivic measure μ_{2m} can be interpreted as a 'counting measure' associated to the $\mathbb{F}_1$-extension $\mathbb{F}_{1^m}$.

Indeed, in [15, Lemma 5.6] Joshua Lieber, Yuri I. Manin and Matilde Marcolli define for a torified variety X with Grothendieck class $[X] = \sum_{i=0}^N a_i \mathbb{T}^i \in K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1})$ that

$$\#X(\mathbb{F}_{1^m}) = \sum_{i=0}^N a_i m^i$$

That is, $\#X(\mathbb{F}_1)$ counts the number of tori in the torified variety X, and $\#X(\mathbb{F}_{1^m})$ counts the number of m-th roots of unity in the tori-decomposition of X. Therefore, $\mu_m = \mu_{\mathbb{F}_{1^m}}$.

In analogy with this Hasse-Weil zeta function of varieties over $\mathbb{F}_q$, Lieber, Manin and Marcolli then define the $\mathbb{F}_1$- *zeta function* to be the ring morphism, by [15, Prop. 6.2]

$$\zeta_{\mathbb{F}_1} \; : \; K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}) \longrightarrow \mathbb{W}(\mathbb{Z}) \qquad [X] = \sum_{k=0}^N a_k \mathbb{T}^k \mapsto exp(\sum_{k=0}^N a_k Li_{1-k}(t))$$

where $Li_s(t)$ is the polylogarithm function, that is, $Li_{1-k}(t) = \sum_{l \geq 1} l^{k-1} t^l$. This gives us a motivic measure on $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1})$ with values in $\mathbb{W}(\mathbb{Z})$, but it does not correspond to any of the zeta-functions ζ_{μ_k} corresponding to the motivic measure μ_k. In particular, $\zeta_{\mathbb{F}_1}$ is *not* a morphism of λ-rings.

Mutatis mutandis we can define similarly the $\mathbb{F}_{1^m}$-zeta function, for the field extension $\mathbb{F}_{1^m}$ of $\mathbb{F}_1$, to be the ring morphism

$$\zeta_{\mathbb{F}_{1^m}} \ : \ K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) \longrightarrow \mathbb{W}(\mathbb{Z}) \qquad [X] = \sum_{k=0}^{N} a_k \mathbb{T}^k \mapsto exp(\sum_{k=0}^{N} a_k m^k Li_{1-k}(t))$$

and again, this zeta function does not come from any of the motivic measures μ_k on $K_0(\mathbf{Var}_{\mathbb{F}_{1^m}})$.

However, we can define another bi-ring (actually, Hopf-ring) structure on $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) = \mathbb{Z}[\mathbb{T}]$ induced by taking $\mathbb{D} = \mathbb{T} - m$ (observe that $\#\mathbb{D}(\mathbb{F}_{1^m}) = 0$) to be the primitive generator, that is,

$$\Delta(\mathbb{D}) = \mathbb{D} \otimes 1 + 1 \otimes \mathbb{D} \quad \text{and} \quad \epsilon(\mathbb{D}) = 0$$

We will call this the *Lie algebra structure* on $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}})$.

Proposition 7. *If we equip $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) = \mathbb{Z}[\mathbb{T}]$ with the Lie-algebra structure, then under the natural one-to-one correspondence*

$$\mathbf{comm}(K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}), \mathbb{Z}) \leftrightarrow \mathbf{comm}^+_{bi}(K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}), \mathbb{L}(\mathbb{Z}))$$

the motivic measure $\mu_{2m} : K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) \longrightarrow \mathbb{Z}$ corresponds to a unique bi-ring morphism $c_{\mu_{2m}} : K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) \longrightarrow \mathbb{L}(\mathbb{Z})$, making the diagram below commutative

$$\begin{tikzcd}
K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) \arrow[drr, bend left, "\zeta_{\mathbb{F}_{1^m}}"] \arrow[dr, dotted, "\nexists"] \arrow[ddr, bend right, "c_{\mu_{2m}}"'] & & \\
& \mathbb{W}_0(\mathbb{Z}) \arrow[r, "L_{\mathbb{Z}}"] \arrow[d, "\mathsf{gh}"'] & \mathbb{W}(\mathbb{Z}) \arrow[d, "\mathsf{gh}"] \\
& \mathbb{L}(\mathbb{Z}) \arrow[r] & \mathbb{Z}^{\infty}
\end{tikzcd}$$

Proof. By definition we have that $\zeta_{\mathbb{F}_{1^m}}(\mathbb{T}^i) = exp(\sum_{k \geq 1} m^i k^{i-1} t^k)$, and therefore, because gh corresponds to $t\frac{d}{dt}log(-)$, we have that

$$\mathsf{gh}(\zeta_{\mathbb{F}_{1^m}}(\mathbb{T}^i)) = (m^i, m^i 2^i, m^i 3^i, \ldots) = \mathsf{gh}(\zeta_{\mathbb{F}_{1^m}}(\mathbb{T}))^i$$

To enforce commutativity with a ringmorphism c_μ we must have that

$$c_\mu(\mathbb{T}) = (m, 2m, 3m, \ldots) = m.d + m.1$$

for the primitive element $d = (0, 1, 2, \ldots) \in \mathbb{L}(\mathbb{Z})$, that is, $\Delta(d) = d \otimes 1 + 1 \otimes d$ and $\epsilon(d) = 0$ and with $1 = (1, 1, 1, \ldots) \in \mathbb{L}(\mathbb{Z})$.

But then, for the Lie algebra structure on $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}})$ we have that $c_\mu(\mathbb{D})$ is the primitive element $m.d \in \mathbb{L}(\mathbb{Z})$, and therefore c_μ is the unique bi-ring morphism $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) \longrightarrow \mathbb{L}(\mathbb{Z})$ corresponding to the motivic measure μ_{2m} : $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) \longrightarrow \mathbb{Z}$ because the second component of $c_\mu(\mathbb{T}) = 2m$.

Suppose there would be a ringmorphism $r : K_0(\mathbf{Var}^{tor}_{\mathbb{F}_{1^m}}) \longrightarrow \mathbb{W}_0(\mathbb{Z})$, then we must have that $\mathsf{gh}(r(\mathbb{T} - m)) = m.d \in \mathbb{L}(\mathbb{Z})$. By functoriality we have a commuting square

$$\begin{tikzcd} \mathbb{W}_0(\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{W}_0(\overline{\mathbb{Q}}) = \mathbb{Z}[\overline{\mathbb{Q}}^*_\times] \arrow[d] \\ \mathbb{L}(\mathbb{Z}) \arrow[r] & \mathbb{L}(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}[\overline{\mathbb{Q}}^*_\times] \otimes \overline{\mathbb{Q}}[d]) \oplus K \end{tikzcd}$$

and d does not lie in the image of the rightmost map. $\square$

Because $K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1})$ is both a λ-ring (with $\Psi_k(\mathbb{L}^i) = \mathbb{L}^{ki}$) and a bi-ring (with the Lie algebra structure with primitive element $\mathbb{D} = \mathbb{T} - 1$) we have natural one-to-one correspondences

$$\mathbf{comm}^+_{bi}(K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}), \mathbb{L}(\mathbb{Z})) \leftrightarrow \mathbf{comm}(K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}), \mathbb{Z}) \leftrightarrow \mathbf{comm}^+_\lambda(K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}), \mathbb{W}(\mathbb{Z}))$$

Under the left correspondence, the motivic measure μ_m defined by $\mu_m(\mathbb{T}) = m$ corresponds to the bi-ring morphism

$$b_m : K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}) = \mathbb{Z}[\mathbb{D}] \longrightarrow \mathbb{L}(\mathbb{Z}) \qquad \mathbb{D} \mapsto (m-1).d = (0, m-1, 2(m-1), \dots)$$

as $b_m(\mathbb{T}) = (1, m, 2m-1, \dots)$ and the coresponding ring-morphism to $\mathbb{Z}$ is composing with projection on the second factor.

Under the right correspondence, the motivic measure μ_m corresponds to the λ-ring morphism $l_m : K_0(\mathbf{Var}^{tor}_{\mathbb{F}_1}) = \mathbb{Z}[\mathbb{L}] \longrightarrow \mathbb{W}(\mathbb{Z})$

$$\mathbb{L} \mapsto \frac{1}{1-(m+1)t} = 1 + (m+1)t + (m+1)^2t^2 + \dots$$

as $l_m(\mathbb{T}) = (1-t).b_l(\mathbb{L}) = 1 + mt + m(m+1)t^2 + \dots$ and the corresponding ring morphism to $\mathbb{Z}$ is $\ghost_1(l_m(\mathbb{T})) = m$.

It follows from propositions 6 and 7 that these morphisms factor through the pull-back $\mathbb{M}(\mathbb{Z})$.

$$\begin{tikzcd} \mathbb{W}_0(\mathbb{Z}) \arrow[dr] \arrow[drr, bend left, "L_{\mathbb{Z}}"] \arrow[ddr, bend right, "\ghost"'] & & \\ & \mathbb{M}(\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{W}(\mathbb{Z}) \arrow[d, "\ghost"] \\ & \mathbb{L}(\mathbb{Z}) \arrow[r] & \mathbb{Z}^\infty \end{tikzcd}$$

Motivated by this, one might view $\mathbb{M}(\mathbb{Z})$ as the correct receptacle for ringmorphisms $K_0(\mathbf{Var}_{\mathbb{Z}}) \longrightarrow \mathbb{W}(\mathbb{Z})$ determined by a counting measure $K_0(\mathbf{Var}_{\mathbb{Z}}) \longrightarrow \mathbb{Z}$. Here, local factors corresponding to non-archimedean places can be distinguished from the Γ-factors by the fact that they factor through $\mathbb{W}_0(\mathbb{Z})$.

5. Linear systems and zeta-polynomials

The original motivation for proposing bi-rings as $\mathbb{F}_1$-algebras was to give a potential explanation of Manin's interpretation of Deninger's Γ-factor $\prod_{n \geq 0} \frac{s-n}{2\pi}$ at complex infinity as the zeta function of (the dual of) infinite dimensional projective space $\mathbb{P}^\infty_{\mathbb{F}_1}$, see [19, 4.3] and [21, Intro]. In [14] a noncommutative moduli space was constructed using linear dynamical systems having the required motive. This suggests the introduction of the category $\mathcal{S}_R$ of discrete R-linear dynamical systems, which plays a similar role for $\mathbb{L}(R)$ as does the endomorphism category $\mathcal{E}_R$ for $\mathbb{W}_0(R)$ and $\mathbb{W}(R)$.

For R a commutative ring consider the category $\mathcal{S}_R$ with objects quadruples (E, f, v, c) with E a projective R-module of finite rank, $f \in End_R(E)$, $v \in E$ and $c \in E^*$ and with morphisms R-module morphisms $\phi : E \longrightarrow E'$ such that $\phi \circ f = f' \circ \phi$, $\phi(v) = v'$ and $c = c' \circ \phi$. A quadruple (E, f, v, c) can be seen as an R-representation of the quiver

$$\begin{tikzcd} \textcircled{R} \arrow[r, bend left, "v"] & \textcircled{E} \arrow[l, bend left, "c"] \arrow[loop right, "f"] \end{tikzcd}$$

and morphisms correspond to quiver-morphisms.

Again, there is a duality $S = (E, f, v, c) \leftrightarrow S^* = (E^*, f^*, c^*, v^*)$ on $\mathcal{S}_R$ and we have $\oplus$ and $\otimes$ operations

$$\begin{cases} (E_1, f_1, v_1, c_1) \oplus (E_2, f_2, v_2, c_2) = (E_1 \oplus E_2, f_1 \oplus f_2, v_1 \oplus v_2, c_1 \oplus c_2) \\ (E_1, f_1, v_1, c_1) \otimes (E_2, f_2, v_2, c_2) = (E_1 \otimes E_2, f_1 \otimes f_2, v_1 \otimes v_2, c_1 \otimes c_2) \end{cases}$$

with a zero object $\mathbf{0} = (0, 0, 0, 0)$ and a unit object $\mathbf{1} = (R, 1, 1, 1)$.

We will call a quadruple $S = (E, f, v, c)$ a *discrete R-linear dynamical system.* Borrowing terminology from system theory, see for example [32, [VI.§5], we define:

Definition 6. *For $S = (E, f, v, c) \in \mathcal{S}_R$ with E of rank n, we say that*

(1) S is completely reachable *if E is generated as R-module by the elements $\{v, f(v), f^2(v), \dots\}$.*
(2) S is completely observable *if the R-module morphism $\phi : E \longrightarrow R^n$ given by $\phi(x) = (c(x), c(f(x)), \dots, c(f^{n-1}(x)))$ is injective.*
(3) S is a canonical system *if S is both completely reachable and completely observable.*
(4) S is a split system *if both S and S^* are completely reachable.*

Definition 7. *There is an additive and multiplicative bat-map*

$$🦇_R \ : \ \mathcal{S}_R \longrightarrow \mathbb{L}(R) \qquad (E, f, v, c) \mapsto (c(v), c(f(v)), c(f^2(v)), c(f^3(v)), \dots)$$

sending a linear dynamical system to its input-output or transfer sequence. We say that a linear recursive sequence $s = (s_0, s_1, s_2, \dots) \in \mathbb{L}(R)$ is realisable *by the system $(E, f, v, c) \in \mathcal{S}_R$ if* 🦇$_R(E, f, v, c) = s$.

Remark 1. *In system theory, see for example* [32, VI.§5], *one relaxes the condition on the state-space E which is merely an R-module and replaces the $rk(E) = n$ condition by the requirement that E is generated by n elements.*

We will now prove that every element $s \in \mathbb{L}(R)$ is realisable by a completely reachable system and verify when this system is in addition canonical, respectively split.

For $s = (s_0, s_1, s_2, \dots) \in \mathbb{L}(R)$ satisfying the recurrence relation $s_n = a_1 s_{n-1} + a_2 s_{n-2} + \dots + a_r s_{n-r}$ of depth r, valid for all $n \in \mathbb{N}$ with the $a_i \in R$. Consider the system $S_s = (E_s, f_s, v_s, c_s) \in \mathcal{S}_R$ with

$$E_s = \frac{R[x]}{(x^r - a_1x^{r-1} - \ldots - a_r)}, \quad f_s = x.|E_s, \quad v_s = 1 \in E_s, \quad c_s(x^i) = s_i$$

and consider the $r \times r$ matrix, with r the depth of the recurrence relation

$$H_r(s) = \begin{bmatrix} s_0 & s_1 & s_2 & \ldots & s_{r-1} \\ s_1 & s_2 & s_3 & \ldots & s_r \\ s_2 & s_3 & s_4 & \ldots & s_{r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ s_{r-1} & s_r & s_{r+1} & \ldots & s_{2r-2} \end{bmatrix}$$

Proposition 8. *With notations as above, $s \in \mathbb{L}(R)$ is realisable by the system $S_s = (E_s, f_s, v_s, c_s) \in \mathcal{S}_R$, and*

1. *S_s is completely reachable,*
2. *S_s is canonical if and only if $det(H_r(s)) \neq 0$,*
3. *S_s is split if and only if $det(H_r(s)) \in R^*$.*

Proof. Clearly, E_s is a free R-module of rank r and one verifies that $\leftrightsquigarrow_R(S_s) = s$. Further, $\{v_s, f_s(v_s), f_s^2(v_s), \ldots, f_s^{r-1}(v_s)\} = \{1, x, x^2, \ldots, x^{r-1}\}$ and these elements generate E_s whence S_s is completely reachable. The R-module morphism ϕ : $E_s \longrightarrow R^r$ defined by $\phi(e) = (c_s(e), c_s(f_s(e)), \ldots, c(f^{r-1}(e)))$ is determined by the images

$$\phi(x^i) = (s_i, s_{i+1}, \ldots, s_{i+r-1})$$

for $0 \leq i < r$ and as these x^i form an R-basis for E_s, the map ϕ is injective, or equivalently that S_s is completely observable if and only if $det(H_r(s)) \neq 0$.

The dual module, $E_s^* = R\epsilon_0 \oplus \ldots \oplus R\epsilon_{r-1}$ where $\epsilon_i(x^j) = \delta_{ij}$. With respect to this basis we have $f_s^*(\epsilon_i) = \epsilon_{i-1} + a_{r-i}\epsilon_{r-1}$ for $i \geq 1$ and $f_s^*(\epsilon_0) = a_r\epsilon_{r-1}$, that is

$$M_{f_s^*} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ a_r & a_{r-1} & \ldots & a_1 \end{bmatrix} \qquad c_s^* = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{r-1} \end{bmatrix}$$

and $v_s^* = (1, 0, \ldots, 0)$. It follows that $\{c_s^*, f_s^*(c_s^*), f_s^{*2}(c_s^*), \ldots, f_s^{*n}(c_s^*)\}$ generate E_s^* if and only if $H_r(s) \in GL_r(R)$. $\square$

Example 4. *Consider the sequence $s = (1, 2, 3, \ldots)$ which we encountered in our study of the $\mathbb{F}_1$-zeta function. We have*

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \in GL_2(\mathbb{Z}) \quad \textit{and} \quad \textit{det} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = 0$$

leading to the (minimal) recurrence relation $x^2 - 2x + 1 = (x-1)^2$. The corresponding system $S_s = (E_s, f_s, v_s, c_s)$ is split and determined by

$$E_s = \frac{\mathbb{Z}[x]}{(x-1)^2}, \quad f_s = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad v_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \textit{and} \quad c_s = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Clearly, if $S = (E, f, v, c)$ is split, it is a canonical system. Over a field K the converse is also true. Note that the difference between canonical and split systems over R is also important for the co-multiplication on $\mathbb{L}(R)$.

Over a field K every recursive sequence $s = (s_0, s_1, \dots) \in \mathbb{L}(K)$ has a *minimal* canonical realisation, that is, one with the dimension of the state-space E minimal. To find it, start with a recursive relation $s_n = a_1 s_{n-1} + a_2 s_{n-2} + \dots + a_r s_{n-r}$ of depth r and form as above the matrix $H_r(s)$ with columns $H_0, H_1, \dots, H_{r-1}$. Let t be the largest integer such that the columns $H_0, H_1, \dots, H_{t-1}$ are linearly independent. If $t = r$ then the previous lemma gives a minimal canonical realisation. If $t < r$ then we have unique coefficients $\alpha_i \in K$ such that $H_t = \alpha_1 H_{t-1} + \alpha_2 H_{t-2} + \dots + \alpha_t H_0$. But then, it follows that

$$s_n = \alpha_1 s_{n-1} + \alpha_2 s_{n-2} + \dots + \alpha_t s_{n-t}$$

is a recursive relation for s of minimal depth t. Using this recursive relation we can then construct a canonical realisation as in the previous lemma, with. this time a state-space of minimal dimension. Over a Noetherian domain R one always has a canonical realisation (in the weak sense that the state module E does not have to be projective) see [32, Theorem IV.5.5] and if R is a principal ideal every linear recursive sequence has a minimal canonical realisation, with free state module, see [32, VI.5.8.iii].

Over a field K we know that canonical systems $S_K = (E_K, f_K, v_K, c_K)$, with $dim(E_K) = n$ are also classified up to isomorphism by their *transfer function*

$$T_{S_K}(z) = c_K(zI - M_{f_K})^{-1} v_K = \frac{Y(z)}{X(z)} = \frac{c_{n-1}z^{n-1} + \dots + c_1 z + c_0}{z^n + d_{n-1}z^{n-1} + \dots + d_1 z + d_0}$$

which are strictly proper rational functions of McMillan degree n, that is, $deg(Y(z)) < deg(X(z)) = n$ (this is immediate from Cramer's rule) and $(Y(z), X(z)) = 1$, see for example [32, II.§5].

Proposition 9. *Let $T(z) = \frac{Y(z)}{X(z)}$ be a strictly proper rational K-function with $Y(z), X(z) \in R[z]$, then there is a completely reachable R-linear system $S = (E, f, v, c)$ such that $T(z) = c(zI - M_f)^{-1}v$. If R is a principal ideal domain, this can be achieved by a minimal canonical system.*

Proof. We can always find an R-system $S' = (E', f', v', c')$ with transfer function $T(z) = c'.(zI - M_{f'})^{-1}.v'$, with $E' = R^n$

$$f' = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ -d_0 & -d_1 & -d_2 & \dots & -d_{n-1} \end{bmatrix} \quad v' = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad c' = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \end{bmatrix}$$

and this system is completely reachable as $\{v', f'(v'), f'^2(v'), \dots\}$ generate R^n. However, it need not be canonical in general. Still, we can consider its input-output sequence

$$\leftrightsquigarrow_R(S') = (c'.v', c'.M_{f'}.v', c'.M^2_{f'}.v', \dots) \in \mathbb{L}(R)$$

By surjectivity on canonical systems in case R is a principal ideal domain, there is a canonical R-system $S = (E, f, v, c)$ with $\mathbat_R(S) = \mathbat_R(S')$, that is,

$$c'.v' = c.v,\ c'.M_{f'}.v' = c.M_f.v,\ c'.M_{f'}^2.v' = c.M_f^2.v,\ \dots$$

But, as $T(z) = c'.(zI - M_{f'})^{-1}.v' = c'.v'z^{-1} + c'.M_{f'}.v'z^{-2} + c'.M_{f'}^2.v'.z^{-3} + \dots$ we see that $T(z)$ is also the transfer function of the canonical R-system S, proving the claim. □

Definition 8. *For a cyclotomic Bost-Connes datum Σ, let $\mathcal{S}_{\Sigma,R}^{cr}$ be the full subcategory of $\mathcal{S}_R$ consisting of all completely reachable systems $S = (E, f, v, c)$ such that all zeroes and poles of the transfer function*

$$T_S(z) = c.(zI - M_f)^{-1}.v$$

are in Σ.

Example 5. *Continuing example 4, we have for T_{S_s}*

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} z & 1 \\ -1 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{z}{(z-1)^2} = Li_{-1}$$

5.1. **Zeta polynomials.** An interesting class of strictly proper rational functions is associated to Manin's 'zeta polynomials' introduced in [21, §1] and generalized in [10] and [27], see also [23, §2.5]. The terminology comes from a result of F. Rodriguez-Villegas [31]. Let $U(z)$ be a polynomial of degree e with $U(1) \neq 0$ and consider the strictly proper rational function

$$P(z) = \frac{U(z)}{(1-z)^{e+1}}$$

There is a polynomial $H(z)$ of degree e such that the power series expansion of $P(z)$ is

$$P(z) = \sum_{n=0}^{\infty} H(n) z^n$$

If all roots of $U(z)$ lie on the unit circle, Rodriguez-Villegas proved that the polynomial $Z(z) = H(-z)$ has zeta-like properties: all roots of $Z(z)$ lie on the vertical line $Re(z) = \frac{1}{2}$ and if all coefficients of $U(z)$ are real then $Z(z)$ satisfies the functional equation

$$Z(1-z) = (-1)^e Z(s)$$

In [21, §1] Yuri I. Manin associates such a zeta-polynomial to each cusp f form of $\Gamma = PSL_2(\mathbb{Z})$ which is an eigenform for all Hecke operators, and views this polynomial as 'the local zeta factor in characteristic one'. The corresponding numerator $U_f(z)$ of the strictly proper rational function comes from the period polynomial divided by the real zeroes and by [5] the remaining zeros all lie on the unit circle.

In [10] this construction was generalised to the case of cusp newforms of even weight for the congruence subgroups $\Gamma_0(N)$, where this time the zeroes of period polynomials all lie on the circle with radius $\frac{1}{\sqrt{N}}$.

Let $Z_i(z)$ be a suitable collection of zeta-polynomials determined by strictly proper rational functions $P_i(z) = \frac{U_i(z)}{(1-z)^{d_i}}$ with $U_i(z) \in R[z]$ then we can view the sub bi-ring of $\mathbb{L}(\mathbb{Z})$ generated by the elements $\mathbat_R(S_i) \in \mathbb{L}(R)$, where S_i is a completely reachable or minimal canonical system realizing $P_i(z)$, as a representative for the collection of zeta-polynomials in the $\mathbf{comm}^+_{\mathbf{bi}}$-version of $\mathbb{F}_1$- geometry. Again, we can define similarly versions relative to a cyclotomic Bost-Connes datum Σ by imposing that the zeroes of the zeta-polynomials must lie in Σ.

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