WHAT IS A NONCOMMUTATIVE TOPOS?

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Abstract. In [1] noncommutative frames were introduced, generalizing the usual notion of frames of open sets of a topological space. In this paper we extend this notion to noncommutative versions of Grothendieck topologies and their associated noncommutative toposes of sheaves of sets.

For Fred Van Oystaeyen on his 70th birthday.

1. Introduction

The set Ω of all open sets of a topological space X is a complete Heyting algebra: it is partially ordered under inclusion, the join $\vee$ and meet $\wedge$ operations are resp. union and intersection of opens, the implication operator $U \to V$ is defined to be the largest open set W satisfying $W \cap U \subseteq V$, and it has a unique bottom element $0 = \emptyset$ and top element $1 = X$, see for example [3, §I.8].

Let $\mathcal{F}$ be a sheaf of sets over the constructible topology on X, that is the topology generated by all open *and* all closed subsets of X. For every open set U in X we consider $\{(U,s) \mid s \in \Gamma(U,\mathcal{F})\}$. The set H of all such possible (U,s) is partially ordered under $(U,s) \leq (V,t)$ if and only if $U \subseteq V$ and $t|U = s$. Fix a distinguished global section $g \in \Gamma(X,\mathcal{F})$. We now define noncommutative operations of H as follows

- $(U,s) \wedge (V,t) = (U \cap V, s|U \cap V)$,
- $(U,s) \vee (V,t) = (U \cup V, t \cup s|U - V)$,
- $(U,s) \to (V,t) = (U \to V, t \cup g|(U \to V) - V)$

H still has a unique bottom element corresponding to $0 = \emptyset$, but now has a family $\{(X,t) \mid t \in \Gamma(X,\mathcal{F})\}$ of top elements, and observe that the downset of each of them $(X,t)_{\downarrow}$ is isomorphic to the Heyting algebra Ω, and if we consider Green's equivalence relation $\mathcal{D}$

$$(U,s)\ \mathcal{D}\ (V,t) \quad \text{if and only if} \quad \begin{cases} (U,s) \wedge (V,t) \wedge (U,s) = (U,s) \\ (V,t) \wedge (U,s) \wedge (V,t) = (V,t) \end{cases}$$

then the equivalence classes $H/\mathcal{D}$ with the induced structures are isomorphic to Ω as Heyting algebras. H is an example of a noncommutative complete Heyting algebra as introduced and studied in [1]. We can view H as the set of opens of a noncommutative topological space with commutative shadow X.

In this paper we aim to define, in a similar way, noncommutative counterparts of toposes $\mathbf{Sh}(\mathbf{C},J)$ of sheaves of sets with respect to a Grothendieck topology J on

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a small category $\mathbf{C}$. Fred Van Oystaeyen suggested in his book 'Virtual topology and functor geometry' a possible approach:

> "One easily finds that the first main problem is to circumvent the notion of subobject classifier. An approach may be to allow a *family* of 'subobject classifiers' defined in a suitable way." [4, p. 44]

Let $\widehat{\mathbf{C}}$ be the topos of presheaves on $\mathbf{C}$, that is, with objects all contravariant functors $\mathbf{P} : \mathbf{C} \longrightarrow \mathbf{Sets}$ and with morphisms all natural transformations. Recall from [3, §III.7] that the natural transformation $true : \mathbf{1} \longrightarrow \mathbf{\Omega}$ is the subobject classifier of $\widehat{\mathbf{C}}$, where for every object C of $\mathbf{C}$ we take $\mathbf{\Omega}(C)$ to be the set of all sieves on C and where the global section $true$ picks out the unique maximal sieve $\mathbf{y}(C)$ of all morphisms with codomain C. Each $\mathbf{\Omega}(C)$ is a complete Heyting algebra, that is, $\mathbf{\Omega}$ is a presheaf of complete Heyting algebras on $\mathbf{C}$. We will define a *noncommutative subobject classifier* $\mathbf{H}$ to be a presheaf of noncommutative complete Heyting algebras making the diagram below commute

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "\mathbf{\Omega}"] \arrow[dr, "\mathbf{H}"'] & & \mathbf{cHA} \\ & \mathbf{ncHA} \arrow[ur, "./\mathcal{D}"'] & \end{tikzcd}$$

where $./\mathcal{D} : \mathbf{ncHA} \longrightarrow \mathbf{cHA}$ is the covariant functor sending a noncommutative complete Heyting algebra H to its commutative shadow $H/\mathcal{D}$. Note that $\mathbf{H}$ has a subobject $t_{\mathbf{H}} : \mathbf{T} \longrightarrow \mathbf{H}$ where $\mathbf{T}$ is the presheaf of top elements of $\mathbf{H}$. We will often recite these two mantras:

(1) : Occurrences of the terminal object $\mathbf{1}$ and $\mathbf{\Omega}$ in classical definitions should be replaced by the presheaves $\mathbf{T}$ and $\mathbf{H}$.

(2) : All noncommutative structures will determine *families* of classical structures, parametrized by the global sections of $\mathbf{T}$.

Let us illustrate this in the definition of the noncommutative Heyting algebra $\mathbf{Sub_H}(\mathbf{P})$ generalizing the classical Heyting algebra of subobjects $\mathbf{Sub}(\mathbf{P})$ of $\mathbf{P} \in \widehat{\mathbf{C}}$. Subobjects of $\mathbf{P}$ are in one-to-one correspondence with natural transformations $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ via the pullback diagram on the left below

$$\begin{tikzcd} \mathbf{Q} \arrow[r, "N"] \arrow[d] & \mathbf{1} \arrow[d, "true"] \\ \mathbf{P} \arrow[r, "N"'] & \mathbf{\Omega} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Q} \arrow[r, "N"] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\ \mathbf{P} \arrow[r, "N"'] & \mathbf{H} \end{tikzcd}$$

Similarly, elements of $\mathbf{Sub_H}(\mathbf{P})$ will be pairs $(\mathbf{Q}, N)$ where $N : \mathbf{P} \longrightarrow \mathbf{H}$ is a natural transformation and $\mathbf{Q}$ is the pullback subobject of the diagram on the right above. Because $\mathbf{H}$ is a presheaf of noncommutative Heyting algebras we have that if N and N' are natural transformations from $\mathbf{P}$ to $\mathbf{H}$ then so are $N \wedge N'$, $N \vee N'$ and $N \to N'$ as defined in lemma 3. This then allows us to define operations on $\mathbf{Sub_H}(\mathbf{P})$

$$\begin{cases} (\mathbf{Q}, N) \wedge (\mathbf{Q}', N') = (\mathbf{Q} \wedge \mathbf{Q}', N \wedge N') \\ (\mathbf{Q}, N) \vee (\mathbf{Q}', N') = (\mathbf{Q} \vee \mathbf{Q}', N \vee N') \\ (\mathbf{Q}, N) \to (\mathbf{Q}', N') = (\mathbf{Q} \to \mathbf{Q}', N \to N') \end{cases}$$

where we have the pull-back diagrams

$$\begin{tikzcd} \mathbf{Q} \wedge \mathbf{Q}' \arrow[r] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\ \mathbf{P} \arrow[r, "N \wedge N'"'] & \mathbf{H} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Q} \vee \mathbf{Q}' \arrow[r] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\ \mathbf{P} \arrow[r, "N \vee N'"'] & \mathbf{H} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Q} \to \mathbf{Q}' \arrow[r] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\ \mathbf{P} \arrow[r, "N \to N'"'] & \mathbf{H} \end{tikzcd}$$

defining a noncommutative Heyting algebra structure. Let $\Gamma(\mathbf{T})$ be the set of global sections $g : \mathbf{1} \longrightarrow \mathbf{T}$ of the presheaf of top elements $\mathbf{T}$, then there is a morphism

$$\mathbf{sub}_{\mathbf{H}}(\mathbf{P}) \longrightarrow \prod_{g \in \Gamma(\mathbf{T})} \mathbf{Sub}(\mathbf{P}) \qquad (\mathbf{Q}, N) \mapsto (\mathbf{Q}_g)_{g \in \Gamma(\mathbf{T})}$$

with $\mathbf{Q}_g$ determined by the diagram below

$$\begin{tikzcd} \mathbf{Q}_g \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "g"] \arrow[dr, "id"] & \\ \mathbf{Q} \arrow[r, "N"] \arrow[d] & \mathbf{T} \arrow[r] \arrow[d, "t_{\mathbf{H}}"] & \mathbf{1} \arrow[d, "true"] \\ \mathbf{P} \arrow[r, "N"'] & \mathbf{H} \arrow[r, "./\mathcal{D}"'] & \mathbf{\Omega} \end{tikzcd}$$

Having defined noncommutative subobject classifiers $\mathbf{H}$, we approach defining noncommutative Grothendieck topologies via generalizing Lawvere-Tierney topologies on $\widehat{\mathbf{C}}$, see for example [3, §V.1]. A *noncommutative Lawvere topology* will then be a natural transformation $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ satisfying

(NLT1) : $j_{\mathbf{H}} \circ t_{\mathbf{H}} = t_{\mathbf{H}}$,

(NLT2) : $j_{\mathbf{H}} \circ j_{\mathbf{H}} = j_{\mathbf{H}}$,

$$\begin{tikzcd} \mathbf{T} \arrow[r, "t_{\mathbf{H}}"] \arrow[dr, "t_{\mathbf{H}}"'] & \mathbf{H} \arrow[d, "j_{\mathbf{H}}"] \\ & \mathbf{H} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, "j_{\mathbf{H}}"] \arrow[dr, "j_{\mathbf{H}}"'] & \mathbf{H} \arrow[d, "j_{\mathbf{H}}"] \\ & \mathbf{H} \end{tikzcd}$$

(NLT3) : For every object C in $\mathbf{C}$, every top-element $t \in \mathbf{T}(C)$ and all $x, y \in t_{\downarrow} \subset \mathbf{T}(C)$ we have the condition

$$j_{\mathbf{H}}(C)(x \wedge y) = j_{\mathbf{H}}(C)(x) \wedge j_{\mathbf{H}}(C)(y)$$

Again, every global section $g : \mathbf{1} \longrightarrow \mathbf{T}$ determines a Lawvere-Tierney topology on $\widehat{\mathbf{C}}$ via the restriction of $j_{\mathbf{H}}$ on $g_{\downarrow} \simeq \mathbf{\Omega}$.

As $\mathbf{C}$ is a small category there is a one-to-one correspondence between Lawvere-Tierney topologies on $\widehat{\mathbf{C}}$ and Grothendieck topologies on $\mathbf{C}$. Extending this, we have that a noncommutative Lawvere topology determines a *noncommutative Grothendieck topology* by associating to every object C the following collection of elements from $\mathbf{Sub}_{\mathbf{H}}(\mathbf{y}C)$

$$J_{\mathbf{H}}(C) = \{(S, x) \in \mathbf{\Omega}(C) \times \mathbf{H}(C) \mid (S, x) \in \mathbf{Sub}_{\mathbf{H}}(\mathbf{y}C) \text{ and } j_{\mathbf{H}}(C)(x) \in \mathbf{T}(C)\}$$

This then allows us to define a presheaf $\mathbf{F}$ in the slice category $\widehat{\mathbf{C}}/\mathbf{T}$ to be a *sheaf* for the noncommutative Grothendieck topology $J_{\mathbf{H}}$ if and only if for every object

C of $\mathbf{C}$, every element $(S,x) \in J_{\mathbf{H}}(C)$, and every morphism g in $\widehat{\mathbf{C}}/\mathbf{T}$

$$\begin{tikzcd}
& \mathbf{y}C \arrow[dr, dotted, "\exists!"] & \\
S \arrow[ur] \arrow[rr, "g"] \arrow[dr, "x"'] & & \mathbf{F} \arrow[dl, "\pi_{\mathbf{F}}"] \\
& \mathbf{T} &
\end{tikzcd}$$

there is a unique morphism $\mathbf{y}C \longrightarrow \mathbf{F}$ in $\widehat{\mathbf{C}}$. Here $S \xrightarrow{x} \mathbf{T}$ is the pull-back map induced by the natural transformation $x : \mathbf{y}C \longrightarrow \mathbf{H}$. The category of all such sheaves $\mathbf{Sh}(\mathbf{C}, J_{\mathbf{H}})$ is then called a *noncommutative topos*.

In the last section we present a large class of examples of noncommutative subobject classifiers and give an explicit example of a noncommutative topos which is *not* a Grothendieck topos, nor even an elementary topos.

2. Noncommutative Heyting algebras

In this section we will recall the main structural results on noncommutative (complete) Heyting algebras obtained in [1].

Recall that a bounded lattice L is a set with two distinguished elements 0 and 1 and two binary operations $\vee$ and $\wedge$ which are both idempotent, associative and commutative and satisfy the identities

$$1 \wedge x = x, \qquad 0 \vee x = x$$

$$x \wedge (y \vee x) = x = (x \wedge y) \vee x$$

L is said to be distributive if we have the added identity

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

A *Heyting algebra* H is a bounded distributive lattice $(H, 0, 1, \vee, \wedge)$ which is also a partially ordered set under $\leq$ and has a binary operation $\rightarrow$ satisfying the following set of axioms

(H1): $(x \rightarrow x) = 1$,
(H2): $x \wedge (x \rightarrow y) = x \wedge y$,
(H3): $y \wedge (x \rightarrow y) = y$,
(H4): $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

Equivalently, these axioms can be replaced by the following single axiom

(HA): $x \wedge y \leq z$ iff $x \leq y \rightarrow z$.

A Heyting algebra H is said to be complete if every subset $\{x_i : i \in I\}$ of H has a supremum $\bigvee_i x_i$ and an infimum $\bigwedge_i x_i$, satisfying the infinite distributive law $\bigvee_i (y \wedge x_i) = y \wedge \bigvee_i x_i$. With $\mathbf{cHA}$ we denote the category of all join-complete Heyting algebras with morphisms the lattice, order preserving maps, preserving 0 and 1.

In [1] noncommutative Heyting algebras were introduced and studied. A *skew lattice* is an algebra $(L, \wedge, \vee)$ where $\wedge$ and $\vee$ are idempotent and associative binary operations satisfying the identities

$$x \wedge (x \vee y) = x = x \vee (x \wedge y) \quad \text{and} \quad (x \wedge y) \vee y = y = (x \vee y) \wedge y$$

A skew lattice is *strongly distributive* if it satisfies the additional identities

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \quad \text{and} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Green's equivalence relation $\mathcal{D}$ on a skew lattice is defined via $x \, \mathcal{D} \, y$ iff $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$. We will denote the $\mathcal{D}$-equivalence class of $x \in L$ by $\mathcal{D}_x$. The set of equivalence classes $L/\mathcal{D}$ with the induced operations is a distributive lattice and if $L/\mathcal{D}$ has a maximal element 1 we call the corresponding $\mathcal{D}$-class in L the set of *top elements* and denote it with T.

A skew lattice has a natural partial order defined by $x \leq y$ iff $x \wedge y = x = y \wedge x$. With $x_\downarrow$ we will denote the subset consisting of all $y \in L$ such that $y \leq x$. By a result of Leech [2], $x_\downarrow$ is a distributive lattice for any x in a strongly distributive skew lattice$S$. If S has a maximal element 1 then $S = 1_\downarrow$, which implies that S is necessarily commutative. That is, we have to sacrifice a unique top element when passing to the noncommutative setting.

From [1, §3] we recall that a *noncommutative Heyting algebra* is an algebra $(H, \wedge, \vee, 0, t)$ where $(H, \wedge, \vee, 0)$ is a strongly distributive lattice with bottom 0 and a top $\mathcal{D}$-class T, t is a distinguished element of T and $\to$ is a binary operation satisfying the following conditions

(NH1) $x \to y = (y \vee (t \wedge x \wedge t) \vee y) \to y$,
(NH2) $x \to x = x \vee t \vee x$,
(NH3) $x \wedge (x \to y) \wedge x = x \wedge y \wedge x$,
(NH4) $y \wedge (x \to y) = y$ and $(x \to y) \wedge y = y$,
(NH5) $x \to (t \wedge (y \wedge z) \wedge t) = (x \to (t \wedge y \wedge t)) \wedge (x \to (t \wedge z \wedge t))$.

The main structural result on noncommutative Heyting algebras, [1, Thm. 3.5] asserts that if $(H, \wedge, \vee, \to, 0, t)$ is a noncommutative Heyting algebra, then

(1) $(t_\downarrow, \wedge, \vee, \to, 0, t)$ is a Heyting algebra with a unique top element t, isomorphic to $H/\mathcal{D}$.
(2) For any $t' \in T$ also $(t'_\downarrow, \wedge, \vee, \to, 0, t')$ is a Heyting algebra and the map

$$\phi \; : \; t_\downarrow \longrightarrow t'_\downarrow \qquad x \mapsto t' \wedge x \wedge t'$$

is an isomorphism of Heyting algebras and for all $x \in t_\downarrow$ we have $x \, \mathcal{D} \, \phi(x)$.

From now on we will assume that the noncommutative Heyting algebra is *complete*, that is if all *commuting* subsets have suprema and infima in their partial ordering, and they satisfy the infinite distributive laws

$$(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y) \quad \text{and} \quad x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i)$$

for all $x, y \in H$ and all commuting subsets $(x_i)_i$ and $(y_i)_i$.

With **ncHA** we denote the category with objects all complete noncommutative Heyting algebras and maps preserving $\leq$, $\wedge$, $\vee$, $\to$, 0 and the distinguished top element t.

From [1, Thm. 3.5.(iii)] we recall that Green's relation $\mathcal{D}$ is a congruence on a noncommutative Heyting algebra H and that the Heyting algebra $H/\mathcal{D}$ is its maximal lattice image, that is, every noncommutative Heyting algebra morphism $H \longrightarrow H_c$ to a (commutative) Heyting algebra H_c factors through the quotient $\pi_{\mathcal{D}} : H \twoheadrightarrow H/\mathcal{D}$. We can rephrase this as

Lemma 1. *Green's relation $\mathcal{D}$ induces a covariant functor*

$$/\mathcal{D} \;:\; \mathbf{ncHA} \longrightarrow \mathbf{cHA} \qquad H \mapsto H/\mathcal{D}$$

3. Noncommutative subobject classifiers

Let $\mathbf{C}$ be a small category and $\mathbf{P}$ a presheaf on $\mathbf{C}$, that is, a contravariant functor $\mathbf{P} : \mathbf{C} \longrightarrow \mathbf{Sets}$. We recall that subobjects of $\mathbf{P}$ correspond to natural transformations $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ to the subobject classifier $\mathbf{\Omega}$, which is a presheaf of complete Heyting algebras on $\mathbf{C}$.

Motivated by this, we will consider the set $(\mathbf{P}, \mathbf{H})$ of all natural transformations $N : \mathbf{P} \longrightarrow \mathbf{H}$ to a presheaf $\mathbf{H}$ of noncommutative complete Heyting algebras on $\mathbf{C}$ and equip this set with a noncommutative Heyting algebra structure.

Let C be an object of $\mathbf{C}$. A sieve S on C is a set of morphisms in $\mathbf{C}$, all with codomain C, such that if $g \in S$ then $h \circ g \in S$ whenever this composition makes sense. With $\mathbf{\Omega}(C)$ we will denote the set of all sieves on C. If S is a sieve on C and $h : D \longrightarrow C$ a morphism in $\mathbf{C}$, then

$$h^*(S) = \{ g \mid \mathit{codom}(g) = D, h \circ g \in S \}$$

is a sieve on D. Hence, $\mathbf{\Omega}$ is a contravariant functor $\mathbf{\Omega} : \mathbf{C} \longrightarrow \mathbf{Sets}$, that is, a presheaf on $\mathbf{C}$. In fact, as unions and intersections of sieves on C are again sieves on C, each $\mathbf{\Omega}(C)$ is a complete Heyting algebra with bottom element $0 = \emptyset$ and unique maximal element $1 = \mathbf{y}(C)$ the set of all morphisms with codomain C. Moreover, for any $h : D \longrightarrow C$ we have that $h^* : \mathbf{\Omega}(C) \longrightarrow \mathbf{\Omega}(D)$ is a morphism of Heyting algebras. That is, we have a contravariant functor

$$\mathbf{\Omega} : \mathbf{C} \longrightarrow \mathbf{cHA}$$

to the category $\mathbf{cHA}$ of complete Heyting algebras. Assigning to each C the maximal element $1 = \mathbf{y}(C)$ defines a global section of $\mathbf{\Omega}$

$$\mathit{true} \;:\; \mathbf{1} \longrightarrow \mathbf{\Omega}$$

which is the subobject classifier in $\widehat{\mathbf{C}}$, the topos of all presheaves of sets on $\mathbf{C}$., see [3, p. 37-39]. That is, for every presheaf $\mathbf{P} \in \widehat{\mathbf{C}}$ there is a natural one-to-one correspondence between natural transformations $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ and subobjects $\mathbf{Q}$ of $\mathbf{P}$ in $\widehat{\mathbf{C}}$, given by the pullback diagram

$$\begin{tikzcd}
\mathbf{Q} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "true"] \\
\mathbf{P} \arrow[r, "N"'] & \mathbf{\Omega}
\end{tikzcd}$$

With this in mind, let us start with a presheaf $\mathbf{H}$ of noncommutative complete Heyting algebras on $\mathbf{C}$, that is, a contravariant functor

$$\mathbf{H} \;:\; \mathbf{C} \longrightarrow \mathbf{ncHA}$$

Every morphism $D \xrightarrow{f} C$ in $\mathbf{C}$ induces a morphism of noncommutative complete Heyting algebras

$$H(f) : \mathbf{H}(C) \longrightarrow \mathbf{H}(D)$$

and, in particular, it induces a map on the sets of top elements of these noncommutative Heyting algebras

$$\mathbf{T}(f) \ : \ \mathbf{T}(C) = T(\mathbf{H}(C)) \xrightarrow{\mathbf{H}(f)} T(\mathbf{H}(D)) = \mathbf{T}(D)$$

That is, taking for every object C in $\mathbf{C}$ the set of top elements $\mathbf{T}(C)$ of the noncommutative complete Heyting algebra $\mathbf{H}(C)$ is a presheaf of sets on $\mathbf{C}$, and the inclusions $\mathbf{T}(C) \subseteq \mathbf{H}(C)$ define a natural transformation

$$t_{\mathbf{H}} \ : \ \mathbf{T} \longrightarrow \mathbf{H}$$

Lemma 2. *Let $\mathbf{P} \in \widehat{\mathbf{C}}$ and let $N, N' : \mathbf{P} \longrightarrow \mathbf{H}$ be natural transformations, then the maps*

$$\begin{cases} (N \wedge N')(C) : \mathbf{P}(C) \longrightarrow \mathbf{H}(C) & x \mapsto N(C)(x) \wedge N'(C)(x) \\ (N \vee N')(C) : \mathbf{P}(C) \longrightarrow \mathbf{H}(C) & x \mapsto N(C)(x) \vee N'(C)(x) \\ (N \rightarrow N')(C) : \mathbf{P}(C) \longrightarrow \mathbf{H}(C) & x \mapsto N(C)(x) \rightarrow N'(C)(x) \end{cases}$$

define natural transformation $N \wedge N', N \vee N', N \rightarrow N' : \mathbf{P} \longrightarrow \mathbf{H}$.

Proof. For every morphism $D \xrightarrow{f} C$ in $\mathbf{C}$ we have to verify that the diagram below is commutative

$$\begin{tikzcd} \mathbf{P}(C) \arrow[r, "(N\wedge N')(C)"] \arrow[d, "\mathbf{P}(f)"'] & \mathbf{H}(C) \arrow[d, "\mathbf{H}(f)"] \\ \mathbf{P}(D) \arrow[r, "(N\wedge N')(D)"'] & \mathbf{H}(D) \end{tikzcd}$$

For every $x \in \mathbf{P}(C)$ we have that $\mathbf{H}(f)((N \wedge N')(C)(x)) =$

$$\mathbf{H}(f)(N(C)(x) \wedge N'(C)(x)) = \mathbf{H}(f)(N(C)(x)) \wedge \mathbf{H}(f)(N'(C)(x))$$

where the last equality follows from $\mathbf{H}(f)$ being a morphism of noncommutative complete Heyting algebras. Because N and N' are natural transformations, we have the equalities

$$\mathbf{H}(f)(N(C)(x)) = N(D)(\mathbf{P}(f)(x)) \quad \text{and} \quad \mathbf{H}(f)(N'(C)(x)) = N'(D)(\mathbf{P}(f)(x))$$

and so the term above is equal to

$$N(D)(\mathbf{P}(f)(x)) \wedge N'(D)(\mathbf{P}(f)(x)) = (N \wedge N')(D)(\mathbf{P}(f)(x))$$

The proofs for $N \vee N'$ and $N \rightarrow N'$ proceed similarly. □

Every natural transformation $N : \mathbf{P} \longrightarrow \mathbf{H}$ determines a pair $(\mathbf{Q}, N)$ where $\mathbf{Q}$ is a subobject of $\mathbf{P}$ via the pullback diagram

$$\begin{tikzcd} \mathbf{Q} \arrow[r, "N"] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{T}}"] \\ \mathbf{P} \arrow[r, "N"'] & \mathbf{H} \end{tikzcd}$$

With $\mathbf{Sub_H}(\mathbf{P})$ we denote the set of all such pairs $(\mathbf{Q}, N)$ determined by a natural transformation $N : \mathbf{P} \longrightarrow \mathbf{H}$.

Lemma 3. *On the poset* $\mathbf{Sub_H}(\mathbf{P})$ *we can define operations*

$$\begin{cases} (\mathbf{Q}, N) \wedge (\mathbf{Q}', N') = (\mathbf{Q} \wedge \mathbf{Q}', N \wedge N') \\ (\mathbf{Q}, N) \vee (\mathbf{Q}', N') = (\mathbf{Q} \vee \mathbf{Q}', N \vee N') \\ (\mathbf{Q}, N) \to (\mathbf{Q}', N') = (\mathbf{Q} \to \mathbf{Q}', N \to N') \end{cases}$$

where we have the pull-back diagrams

$$\begin{tikzcd} \mathbf{Q} \wedge \mathbf{Q}' \arrow[r] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\ \mathbf{P} \arrow[r, "N \wedge N'"'] & \mathbf{H} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Q} \vee \mathbf{Q}' \arrow[r] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\ \mathbf{P} \arrow[r, "N \vee N'"'] & \mathbf{H} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Q} \to \mathbf{Q}' \arrow[r] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\ \mathbf{P} \arrow[r, "N \to N'"'] & \mathbf{H} \end{tikzcd}$$

These operations turn the set $\mathbf{Sub_H}(\mathbf{P})$ *into a noncommutative complete Heyting algebra with minimal element* $(\emptyset, N_0)$ *and distinguished top element* $(\mathbf{P}, N_d)$*, where the natural transformations* $N_0, N_d : \mathbf{P} \longrightarrow \mathbf{H}$ *are the compositions*

$$N_0 : \mathbf{P} \longrightarrow \mathbf{1} \xrightarrow{0} \mathbf{H} \quad \textit{and} \quad N_d : \mathbf{P} \longrightarrow \mathbf{1} \xrightarrow{d} \mathbf{H}$$

with the left-most morphism the unique map to the terminal object $\mathbf{1}$ *and* d *the global section of* $\mathbf{H}$ *determined by the distinguished elements. The top-elements are exactly the pairs* $(\mathbf{P}, N)$ *where* $N : \mathbf{P} \longrightarrow \mathbf{T}$ *is a natural transformation.*

Proof. Follows from the previous lemma and uniqueness of pull-backs. $\square$

Definition 1. *A presheaf* $\mathbf{H}$ *of noncommutative complete Heyting algebras on* $\mathbf{C}$ *is said to be a* noncommutative subobject classifier *if* $\mathbf{H}/\mathcal{D} \simeq \mathbf{\Omega}$.

Lemma 4. *If* $\mathbf{H}$ *is a noncommutative subobject classifier, then for every presheaf* $\mathbf{P}$ *on* $\mathbf{C}$*, we have a surjective morphism of (noncommutative) complete Heyting algebras*

$$\mathbf{Sub_H}(\mathbf{P}) \twoheadrightarrow \mathbf{Sub}(\mathbf{P})$$

Proof. The map is determined by sending a pair $(\mathbf{Q}, N)$ to $\mathbf{Q}$. Or, equivalently, by composing with the quotient map of noncommutative complete Heyting algebras dividing out Green's relation

$$\begin{tikzcd} \mathbf{Q} \arrow[r, "N"] \arrow[d] & \mathbf{T} \arrow[r] \arrow[d, "t_{\mathbf{H}}"] & \mathbf{1} \arrow[d] \\ \mathbf{P} \arrow[r, "N"'] & \mathbf{H} \arrow[r, "./\mathcal{D}"'] & \mathbf{\Omega} \end{tikzcd}$$

Let $d : \mathbf{1} \longrightarrow \mathbf{H}$ be the global section corresponding to the distinguished top element, then the maps (of noncommutative complete Heyting algebras)

$$\mathbf{\Omega}(C) \xrightarrow{\simeq} d(C)(1)_{\downarrow} \hookrightarrow \mathbf{H}(C)$$

determine a natural transformation $\mathbf{\Omega} \xrightarrow{i} \mathbf{H}$. If $\mathbf{Q}$ is the subobject of $\mathbf{P}$ corresponding to the natural transformation $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ then the composition $i \circ N$ is an element of $(\mathbf{P}, \mathbf{H})$ mapping to $\mathbf{Q}$. $\square$

4. Noncommutative Grothendieck topologies

In this section we will introduce noncommutative Grothendieck topologies and their corresponding toposes of sheaves. We will first extend the notion of Lawvere-Tierney topologies, which are certain closure operations on $\mathbf{\Omega}$, to noncommutative subobject classifiers. As Lawvere-Tierney topologies coincide with Grothendieck topologies when the category $\mathbf{C}$ is small, we will then determine the corresponding noncommutative Grothendieck topologies and define sheaves over them.

A *Lawvere-Tierney topology* on $\widehat{\mathbf{C}}$, see for example [3, V.§1], is a natural transformation $j : \mathbf{\Omega} \longrightarrow \mathbf{\Omega}$ satisfying the following three properties
(LT1): $j \circ \text{true} = \text{true}$;
(LT2): $j \circ j = j$;
(LT3): $j \circ \wedge = \wedge \circ (j \times j)$.

$$\begin{tikzcd} \mathbf{1} \arrow[r, "true"] \arrow[dr, "true"'] & \mathbf{\Omega} \arrow[d, "j"] \\ & \mathbf{\Omega} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{\Omega} \arrow[r, "j"] \arrow[dr, "j"'] & \mathbf{\Omega} \arrow[d, "j"] \\ & \mathbf{\Omega} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{\Omega} \times \mathbf{\Omega} \arrow[r, "\wedge"] \arrow[d, "j \times j"'] & \mathbf{\Omega} \arrow[d, "j"] \\ \mathbf{\Omega} \times \mathbf{\Omega} \arrow[r, "\wedge"'] & \mathbf{\Omega} \end{tikzcd}$$

Motivated by this we define, for a noncommutative subobject classifier $\mathbf{H}$ with presheaf of top-elements $t_{\mathbf{T}} : \mathbf{T} \longrightarrow \mathbf{H}$, a *noncommutative Lawvere topology* to be a natural transformation (of presheaves of sets)

$$j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$$

satisfying the properties
(NLT1) : $j_{\mathbf{H}} \circ t_{\mathbf{H}} = t_{\mathbf{H}}$,
(NLT2) : $j_{\mathbf{H}} \circ j_{\mathbf{H}} = j_{\mathbf{H}}$,

$$\begin{tikzcd} \mathbf{T} \arrow[r, "t_{\mathbf{H}}"] \arrow[dr, "t_{\mathbf{H}}"'] & \mathbf{H} \arrow[d, "j_{\mathbf{H}}"] \\ & \mathbf{H} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, "j_{\mathbf{H}}"] \arrow[dr, "j_{\mathbf{H}}"'] & \mathbf{H} \arrow[d, "j_{\mathbf{H}}"] \\ & \mathbf{H} \end{tikzcd}$$

and where we replace the third commuting diagram by
(NLT3) : For every object C in $\mathbf{C}$, every top-element $t \in \mathbf{T}(C)$ and all $x, y \in t_{\downarrow} \subset \mathbf{T}(C)$ we have the condition

$$j_{\mathbf{H}}(C)(x \wedge y) = j_{\mathbf{H}}(C)(x) \wedge j_{\mathbf{H}}(C)(y)$$

Lemma 5. *A noncommutative Lawvere topology $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ induces for every presheaf $\mathbf{P}$ a closure operator on the noncommutative complete Heyting algebra $\mathbf{Sub}_{\mathbf{H}}(\mathbf{P})$.*

Proof. Let $N : \mathbf{P} \longrightarrow \mathbf{H}$ be a natural transformation and consider the inner pullback square

$$\begin{tikzcd}
\overline{\mathbf{Q}} \arrow[rrr] \arrow[ddd] & & & \mathbf{T} \arrow[ddd] \\
& \mathbf{Q} \arrow[r] \arrow[d] \arrow[ul, dotted] & \mathbf{T} \arrow[d] \arrow[ur, "id"] & \\
& \mathbf{P} \arrow[r, "N"] \arrow[dl, "id"'] & \mathbf{H} \arrow[dr, "j_{\mathbf{H}}"] & \\
\mathbf{P} \arrow[rrr, "j_{\mathbf{H}} \circ N"'] & & & \mathbf{H}
\end{tikzcd}$$

then the composed morphism $j_{\mathbf{H}} \circ N$ gives the outer square, and hence determines an element in $\mathbf{Sub_H}(\mathbf{P})$

$$\overline{(\mathbf{Q}, N)} = (\overline{\mathbf{Q}}, j_{\mathbf{H}} \circ N)$$

The dashed morphism exists because the outer square is a pullback diagram, and hence we have $\mathbf{Q} \subseteq \overline{\mathbf{Q}}$ and therefore

$$(\mathbf{Q}, N) \leq \overline{(\mathbf{Q}, N)} \quad \text{and} \quad \overline{\overline{(\mathbf{Q}, N)}} = \overline{(\mathbf{Q}, N)}$$

where the latter follows from $j_{\mathbf{H}} \circ j_{\mathbf{H}} = j_{\mathbf{H}}$. □

Recall that a *Grothendieck topology* on $\mathbf{C}$, see for example [3, III.§2], is a function J which assigns to each object C a collection $J(C)$ of sieves on C, satisfying the following requirements

(GT1): the maximal sieve $\mathbf{y}(C) = \{ f \mid codom(f) = C \} \in J(C)$;
(GT2): if $S \in J(C)$, then $h^*(C) \in J(D)$ for all arrows $h : D \longrightarrow C$,
(GT3): if R is a sieve on C such that $h^*(R) \in J(D)$ for all $h : D \longrightarrow C \in S \in J(C)$, then $R \in J(C)$.

If $\mathbf{C}$ is a small category, Lawvere-Tierney topologies on $\widehat{\mathbf{C}}$ are in one-to-one correspondence with Grothendieck topologies on $\mathbf{C}$, see for example [3, Thm. V.4.1]. One recovers the collection $J(C)$ from a Lawvere-Tierney topology j as the set of all sieves S on C such that $j(S) = \mathbf{y}(C)$ in $\mathbf{\Omega}(C)$.

Let us specify the construction of $\mathbf{Sub_H}(\mathbf{P})$ for the presheaf $\mathbf{P} = \mathbf{y}C$ determined by

$$\mathbf{y}C \; : \; \mathbf{C} \longrightarrow \mathbf{Sets} \qquad D \mapsto Hom_{\mathbf{C}}(D, C)$$

Note that the subobjects of $\mathbf{y}C$ are exactly the sieves S on C and that by Yoneda's lemma every natural transformation $N : \mathbf{y}C \longrightarrow \mathbf{H}$ determines (and is determined by) $x = N(C)(id_C) \in \mathbf{H}(C)$. Conversely, every element $x \in \mathbf{H}(C)$ determines the pull-back diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \mathbf{T} \arrow[d, "t_{\mathbf{H}}"] \\
\mathbf{y}C \arrow[r, "x"'] & \mathbf{H}
\end{tikzcd}$$

where S is the sieve on C specified by

$$S = \{D \xrightarrow{f} C \ : \ \mathbf{H}(f)(x) \in \mathbf{T}(D)\}$$

Observe that S is indeed a sieve as the maps $\mathbf{H}(g)$ for $E \xrightarrow{g} D$ induce a map on the top-elements $\mathbf{T}(D) \longrightarrow \mathbf{T}(E)$. Therefore,

$$\mathbf{Sub_H}(\mathbf{y}C) = \{(S,x) \in \mathbf{\Omega}(C) \times \mathbf{H}(C) \mid S = \{D \xrightarrow{f} C \ : \ \mathbf{H}(f)(x) \in \mathbf{T}(D)\}\}$$

We have seen that $\mathbf{Sub_H}(\mathbf{y}C)$ is a noncommutative complete Heyting algebra, having as its set of top-elements

$$T(\mathbf{Sub_H}(\mathbf{y}C)) = \{(\mathbf{y}(C), t) \mid t \in \mathbf{T}(C)\}$$

and with minimal element $(\emptyset, 0)$. If $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ is a noncommutative Lawvere topology, the corresponding closure operation on $\mathbf{Sub_H}(\mathbf{y}C)$ can be specified as

$$\overline{(S,x)} = (\overline{S}, j_{\mathbf{H}}(C)(x)) \quad \text{with} \quad \overline{S} = \{D \xrightarrow{f} C \ : \ \mathbf{T}(f)(j_{\mathbf{H}}(C)(x)) \in \mathbf{T}(D)\}$$

Motivated by the above correspondence between Lawvere-Tierney and Grothendieck topologies, we can now define:

Definition 2. *Let $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ be a noncommutative Lawvere topology, then the corresponding* noncommutative Grothendieck topology $J_{\mathbf{H}}$ *assigns to every object C of $\mathbf{C}$ the collection of elements from $\mathbf{Sub_H}(\mathbf{y}C)$*

$$J_{\mathbf{H}}(C) = \{(S,x) \in \mathbf{\Omega}(C) \times \mathbf{H}(C) \mid (S,x) \in \mathbf{Sub_H}(\mathbf{y}C) \ \textit{and} \ j_{\mathbf{H}}(C)(x) \in \mathbf{T}(C)\}$$

If J is a Grothendieck topology on $\mathbf{C}$ then a presheaf $\mathbf{P}$ of sets on $\mathbf{C}$ is called a sheaf for J if and only if for every object C of $\mathbf{C}$, every sieve $S \in J(C)$ (considered as a subobject of $\mathbf{y}C$) and every natural transformation $g : S \longrightarrow \mathbf{P}$, there is a unique natural transformation $\mathbf{y}C \longrightarrow \mathbf{P}$ making the diagram below commute

$$\begin{tikzcd}
& \mathbf{y}C \arrow[dr, dotted, "\exists!"] & \\
S \arrow[ur] \arrow[rr, "g"] \arrow[dr] & & \mathbf{P} \arrow[dl] \\
& \mathbf{1} &
\end{tikzcd}$$

Clearly, the canonical bottom maps to the terminal object $\mathbf{1}$ are superfluous in the definition, but they may help to motivate the definition below.

Let $\mathbf{H}$ be a noncommutative subobject generator with presheaf of top-elements $\mathbf{T}$ and let $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ be a noncommutative Lawvere topology, then the corresponding noncommutative Grothendieck topology $J_{\mathbf{H}}$ assigns to every object C a collection $J_{\mathbf{H}}(C)$ of couples (S,x) where S is a subobject of $\mathbf{y}C$ and $x : S \longrightarrow \mathbf{T}$ is a natural transformation which is the restriction to S of a natural transformation $x : \mathbf{y}C \longrightarrow \mathbf{H}$ determined by $x \in \mathbf{H}(C)$.

So, instead of the canonical morphism $S \longrightarrow \mathbf{1}$ we have to consider certain morphisms $x : S \longrightarrow \mathbf{T}$. Therefore it makes sense to define the category of all presheaves with respect to the noncommutative Grothendieck topology $J_{\mathbf{H}}$ to be the slice category $\widehat{\mathbf{C}}/\mathbf{T}$. That is, the objects are pairs $(\mathbf{F}, \pi_{\mathbf{F}})$ with $\mathbf{F} \in \widehat{\mathbf{C}}$

and $\pi_{\mathbf{F}}$ a natural transformation $\mathbf{F} \longrightarrow \mathbf{T}$, and morphisms compatible natural transformations g

$$\begin{tikzcd} \mathbf{F} \arrow[d, "\pi_{\mathbf{F}}"'] \\ \mathbf{T} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{F} \arrow[rr, "g"] \arrow[dr, "\pi_{\mathbf{F}}"'] & & \mathbf{G} \arrow[dl, "\pi_{\mathbf{G}}"] \\ & \mathbf{T} & \end{tikzcd}$$

Definition 3. *A presheaf* $(\mathbf{F}, \pi_{\mathbf{F}})$ *is a sheaf with respect to the noncommutative Grothendieck topology* $J_{\mathbf{H}}$ *if and only if for every object* C *of* $\mathbf{C}$*, every element* $(S, x) \in J_{\mathbf{H}}(C)$*, and every morphism* g *in* $\widehat{\mathbf{C}}/\mathbf{T}$

$$\begin{tikzcd} & \mathbf{y}C \arrow[dr, dotted, "\exists !"] & \\ S \arrow[ur] \arrow[rr, "g"] \arrow[dr, "x"'] & & \mathbf{F} \arrow[dl, "\pi_{\mathbf{F}}"] \\ & \mathbf{T} & \end{tikzcd}$$

there is a unique morphism $\mathbf{y}C \longrightarrow \mathbf{F}$ *in* $\widehat{\mathbf{C}}$*. Here* $S \xrightarrow{x} \mathbf{T}$ *is the pull-back map induced by the natural transformation* $x : \mathbf{y}C \longrightarrow \mathbf{H}$*.*

The noncommutative topos $\mathbf{Sh}(\mathbf{C}, J_{\mathbf{H}})$ *has as its objects all sheaves with respect to the noncommutative Grothendieck topology* $J_{\mathbf{H}}$ *and morphisms as in* $\widehat{\mathbf{C}}/\mathbf{T}$*.*

5. A class of examples

In this section we will construct examples of noncommutative subobject classifiers and show that a noncommutative topos does not have to be an elementary topos.

First, we will construct complete noncommutative Heyting algebras. By a result of [1] complete noncommutative Heyting algebras are exactly noncommutative frames (together with a distinguished element in the top $\mathcal{D}$-class), where a *noncommutative frame* is a strongly distributive, join complete skew lattice that satisfies the infinite distributive laws.

Let h be a (commutative) complete Heyting algebra. Since h is a distributive lattice it embeds into $\prod_{i \in I} \mathbf{2}$ for some index set I, where $\mathbf{2}$ is the two element lattice

$$\mathbf{2} = \begin{array}{c} 1 \\ | \\ 0 \end{array} \qquad \text{and define} \qquad \widehat{P} = \begin{array}{c} \cdots\cdots\; p \;\cdots\cdots \\ | \\ 0 \end{array}$$

to be the skew lattice on $\widehat{P} = \{0\} \cup P$, with a unique bottom element 0 and a set P of top elements, and operations are defined by:

$$x, y \in P : x \wedge y = x, \qquad x \vee y = y.$$

$$x \wedge 0 = 0 = 0 \wedge x, \qquad x \vee 0 = x = 0 \vee x,$$

Note that $\widehat{P}$ is a strongly distributive skew lattice and has two $\mathcal{D}$-classes: bottom class $\{0\}$ and top class P, whence $\widehat{P}/\mathcal{D} \simeq \mathbf{2}$.

Let H be the pullback (in **Sets**) of the following diagram:

$$\begin{tikzcd} H \arrow[r] \arrow[d] & \prod_{i\in I} \widehat{P} \arrow[d, "/\mathcal{D}"] \\ h \arrow[r, "i"] & \prod_{i\in I} \mathbf{2} \end{tikzcd}$$

Denoting by π_i the projection to the i-th factor we obtain a commutative diagram:

(1)
$$\begin{tikzcd} H \arrow[r, "\pi_i"] \arrow[d] & \widehat{P} \arrow[d, "/\mathcal{D}"] \\ h \arrow[r] & \mathbf{2} \end{tikzcd}$$

Lemma 6. *With notations as above, H becomes a noncommutative frame with bottom 0 and top $\mathcal{D}$-class $T(H) = \prod_{i\in I} P$ under the operations*

$$(x_i)_i \wedge (y_i)_i = (x_i \wedge y_i)_i \qquad \textit{and} \qquad (x_i)_i \vee (y_i)_i = (x_i \vee y_i)_i$$

where the bracketed operations are performed in the skew lattice $\widehat{P}$. In particular, $H/\mathcal{D} \simeq h$. If we fix a distinguished element $d \in H$ s.t. $\pi_i(d) \neq 0$ for all $i \in I$ then H is a complete noncommutative Heyting algebra.

Proof. First we observe that H is a strongly distributive skew lattice because it embeds into a power of $\widehat{P}$ and strongly distributive skew lattices form a variety. Note that elements $x, y \in H$ are $\mathcal{D}$-equivalent exactly when for all $i \in I$: $(\pi_i(x) = 0$ iff $\pi_i(y) = 0)$. A commuting subset in H is of the form $\{x_j \,|\, j \in J\}$ s.t. $\pi_i(x_j) \neq 0$ together with $\pi_i(x_k) \neq 0$ implies $\pi_i(x_j) = \pi_i(x_k)$, for all $j, k \in J$ and all $i \in I$. Skew lattice H is join complete because h is complete and the diagram (1) commutes. It remains to prove that H satisfies the infinite distributive laws. Given a commuting subset $\{x_j\} \subseteq H$, $y \in H$ and $i \in I$ we need to show that:

$$\pi_i(\bigvee x_j \wedge y) = \pi_i(\bigvee (x_j \wedge y)) \text{ and } \pi_i(y \wedge \bigvee x_j) = \pi_i(\bigvee (y \wedge x_j))$$

First we observe that $\{x_j \wedge y \,|\, j \in J\}$ and $\{y \wedge x_j \,|\, j \in J\}$ are again commuting subsets. Note that if $\pi_i(x_j) \neq 0$ for some j then $\pi_i(\bigvee x_j \wedge y) = \pi(x_j \wedge y) = \pi_i(\bigvee (x_j \wedge y))$. If $\pi_i(x_j) = 0$ for all j then $\pi_i(\bigvee x_j \wedge y) = 0 = \pi_i(\bigvee (x_j \wedge y))$. □

Lemma 7. *For every contravariant functor*

$$\mathbf{h} \;:\; \mathbf{C} \longrightarrow \mathbf{cHA}$$

and every presheaf $\mathbf{P} \in \widehat{\mathbf{C}}$ with a global section $d : \mathbf{1} \longrightarrow \mathbf{P}$ there is a contravariant functor

$$\mathbf{H} \;:\; \mathbf{C} \longrightarrow \mathbf{ncHA} \qquad C \mapsto \mathbf{H}(C)$$

where $\mathbf{H}(C)$ is the complete noncommutative Heyting algebra constructed in the previous lemma from the complete Heyting algebra $h = \mathbf{h}(C)$ and the set $P = \mathbf{P}(C)$, with presheaf of top elements $\mathbf{T}$. Moreover, $\mathbf{H}/\mathcal{D} \simeq \mathbf{h}$.

In the special case when $\mathbf{h} = \mathbf{\Omega}$ we obtain for every presheaf $\mathbf{P}$ with a global section a noncommutative subobject classifier $\mathbf{H}$ with $\mathbf{H}/\mathcal{D} \simeq \mathbf{\Omega}$.

Proof. Follows immediately from the previous lemma. □

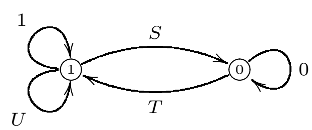
Let us work out an explicit example. Let $\mathbf{C}$ be the category having two objects V and E and two non-identity morphisms $s, t : V \longrightarrow E$, then it is easy to see that the presheaf topos

$$\widehat{\mathbf{C}} \simeq \mathbf{diGraph}$$

is the category of directed graphs. A presheaf $\mathbf{P} : \mathbf{C} \longrightarrow \mathbf{Sets}$ determines a set of vertices $\mathbf{P}(V)$ and edges $\mathbf{P}(E)$ and the two maps $\mathbf{P}(s), \mathbf{P}(t) : \mathbf{P}(E) \longrightarrow \mathbf{P}(V)$ assign to an edge its starting resp. terminating vertex. The subobject classifier $\mathbf{\Omega}$ is given by

$$\begin{cases} \mathbf{\Omega}(E) = \{1 = \{id_E, s, t\}, U = \{s, t\}, S = \{s\}, T = \{t\}, 0 = \emptyset\} \\ \mathbf{\Omega}(V) = \{1 = \{id_V\}, 0 = \emptyset\} \end{cases}$$

and corresponds to the directed graph

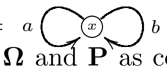


with the terminal subobject $\mathbf{1}$ corresponding to the subgraph on the loop 1. The Heyting algebras have poset structure

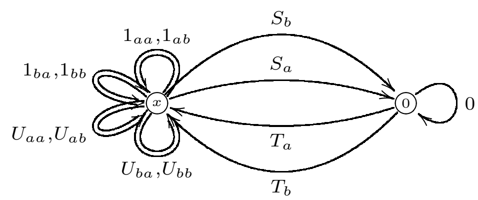
$$\mathbf{\Omega}(E) = \begin{tikzcd} & 1 \arrow[d, dash] & \\ & U \arrow[dl, dash] \arrow[dr, dash] & \\ S \arrow[dr, dash] & & T \arrow[dl, dash] \\ & 0 & \end{tikzcd} \qquad \mathbf{\Omega}(V) = \begin{tikzcd} 1 \arrow[d, dash] \\ 0 \end{tikzcd}$$

It is easy to verify that there are exactly 4 Lawvere-Tierney topologies on $\widehat{\mathbf{C}}$ with corresponding Grothendieck topologies on $\mathbf{C}$ and corresponding sheafifications:

(1) $J_1(V) = \{1\}$ and $J_1(E) = \{1\}$, the chaotic topology. All presheaves are J_1-sheaves and the sheafification functor is the identity.
(2) $J_2(V) = \{1\}$ and $J_2(E) = \{1, U\}$. The sheaf condition for $\mathbf{P}$ asserts that for all $v, w \in \mathbf{P}(V)$ there is a unique edge e with $s(e) = v$ and $t(e) = w$. That is, sheaves are the complete directed graphs, and the sheafification of a directed graph is the complete directed graph on the vertices.
(3) $J_3(V) = \{1, 0\}$ and $J_3(E) = \{1\}$. The only non-maximal covering sieve on V is the empty sieve. A presheaf $\mathbf{P}$ is a J_3-sheaf if and only if $\mathbf{P}(V)$ is a singleton. The sheafification sends the vertices of a directed graph all to the same vertex and each edge to a different loop.
(4) $J_4(V) = \{1, 0\}$ and $J_4(E) = \{1, U, S, T, 0\}$, the discrete topology. Here the only sheaf is the terminal object (a one loop graph) and sheafification is the unique map to the terminal object.

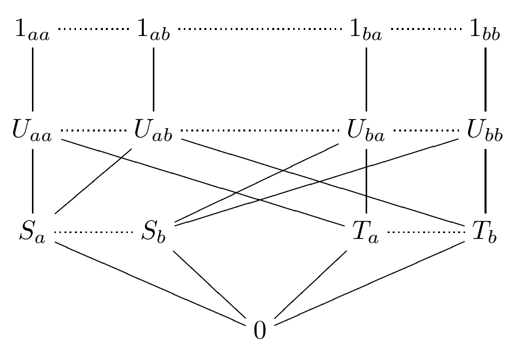
Consider the presheaf $\mathbf{P} =$ 

, then the noncommutative subobject classifier $\mathbf{H}$ corresponding to $\mathbf{\Omega}$ and $\mathbf{P}$ as constructed in lemma 7 can be slightly simplified such that $\mathbf{H}(E)$ has only 4 top elements, rather than the 8 given by the construction. The corresponding directed graph is



with the subobject $\mathbf{T} \longrightarrow \mathbf{H}$ corresponding to the subgraph on the 4 loops $1_{aa}, 1_{ab}, 1_{ba}$ and 1_{bb}. The poset structure on the noncommutative Heyting algebras is $\mathbf{H}(V) \simeq \mathbf{\Omega}(V) \simeq \mathbf{2}$ and

$\mathbf{H}(E) =$



We will next determine the noncommutative toposes determined by noncommutative Grothendieck topologies associated to $\mathbf{H}$. The category of presheaves is the slice category $\widehat{\mathbf{C}}/\mathbf{T}$. A directed graph with a morphism $\pi_{\mathbf{F}} : \mathbf{F} \longrightarrow \mathbf{T}$ is a directed graph with a 4-coloring of its edges. Morphisms in $\widehat{\mathbf{C}}/\mathbf{T}$ are directed graph morphisms preserving the coloring of edges.

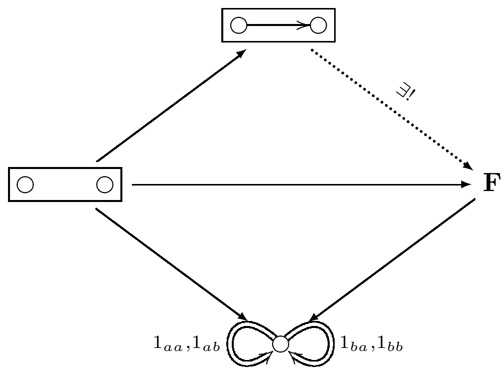
Lemma 8. *There are exactly* 16 *noncommutative Grothendieck topologies associated to* $\mathbf{H}$*:*

$$j_{\mathbf{H}}(E) = \{1_{aa}, 1_{ab}, 1_{ba}, 1_{bb}\} \cup S \quad \textit{with} \quad S \subseteq \{U_{aa}, U_{ab}, U_{ba}, U_{bb}\}$$

Any 4*-colored digraph satisfies the sheaf condition if* $S = \emptyset$*. For the noncommutative Grothendieck topologies with* $S \neq \emptyset$ *the sheaves are exactly the complete digraphs with a* 4*-coloring.*

Proof. Assume that a noncommutative Lawvere topology $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ is such that $j_{\mathbf{H}}(V)(0) = 1$, then $j_{\mathbf{H}}(E)(0) \in \{1_{aa}, 1_{ab}, 1_{ba}, 1_{bb}, U_{aa}, U_{ab}, U_{ba}, U_{bb}\}$ which is impossible because $j_{\mathbf{H}}(E)$ must be order preserving. Therefore $j_{\mathbf{H}}(V) = id_{\mathbf{H}(V)}$. As a consequence the Grothendieck topologies on $\mathbf{C}$ corresponding to the global sections $1_{aa}, 1_{ab}, 1_{ba}, 1_{bb}$ can only be either J_1 or J_2, giving the 16 cases. If $S = \emptyset$ we have no conditions to satisfy for $\mathbf{F} \longrightarrow \mathbf{T}$.

If, however $S \neq \emptyset$, each occurrence of U_{aa}, U_{ab}, U_{ba} or U_{bb} gives rise to a condition



which means that for every pair of vertices $v, w \in \mathbf{F}(V)$ there must be a unique edge $v \longrightarrow w$. Note that the color of this unique edge is not imposed by U_{aa}, U_{ab}, U_{ba} or U_{bb}. Therefore, $\mathbf{F}$ is a sheaf for the noncommutative Grothendieck topology if and only if $\mathbf{F}$ is a complete digraph with a certain 4-coloring of the edges determined by the map $\mathbf{F} \longrightarrow \mathbf{T}$. $\square$

It does follow that for any noncommutative Grothendieck topology $J_{\mathbf{H}}$ with $S \neq \emptyset$ the noncommutative topos $\mathbf{Sh}(\mathbf{C}, J_{\mathbf{H}})$ is *not* a Grothendieck topos, nor even an elementary topos, as it fails to have a terminal object (the four loop graph with one loop of each color is *not* a sheaf).

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