THE SUPERPOTENTIAL $\mathbf{XYZ + XZY + \frac{c}{3}(X^3 + Y^3 + Z^3)}$

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Abstract. The motivic Donaldson-Thomas series associated to an elliptic Sklyanin algebra corresponding to a point of order two differs from the conjectured series in [5, Conjecture 3.4].

1. Introduction

A 3-*dimensional elliptic Sklyanin algebra* $S = S_{a,b,c}$ is a quotient of the free algebra $\mathbb{C}\langle X, Y, Z\rangle$ modulo the graded ideal generated by the three quadratic relations

$$\begin{cases} aXY + bYX + cZ^2 &= 0 \\ aYZ + bZY + cX^2 &= 0 \\ aZX + bXZ + cY^2 &= 0 \end{cases}$$

If $abc \neq 0$ and $3(abc)^3 \neq (a^3 + b^3 + c^3)^3$ these algebras have excellent ringtheoretic and homological properties, as proved by M. Artin, J. Tate and M. Van den Bergh in [1],[2]. They are determined by the plane elliptic curve

$$E_{pt} \ : \ (a^3 + b^3 + c^3)XYZ - abc(X^3 + Y^3 + Z^3) = 0 \subset \mathbb{P}^2$$

and translation by the point $\tau = [a : b : c] \in E_{pt}$ on it. The tools of noncommutative projective algebraic geometry have been used to classify the finite dimensional simple representations of $S_{a,b,c}$ in case $\tau \in E_{pt}$ is a point of finite order, see [18], [7], and more recently [19]. We recall these result in section 2 and make them explicit in the case when τ has order two, using the theory of Clifford algebras.

The Sklyanin algebra $S_{a,b,c}$ can also be realized as the Jacobi algebra associated to the superpotential

$$W = aXYZ + bXZY + \frac{c}{3}(X^3 + Y^3 + Z^3)$$

That is, if ∂_V denotes the cyclic derivative with respect to the variable V, then

$$S_{a,b,c} = \frac{\mathbb{C}\langle X, Y, Z\rangle}{(\partial_X(W), \partial_Y(W), \partial_Z(W))}$$

$Tr(W)$ determines the Chern-Simons functional $M_n(\mathbb{C}) \oplus M_n(C) \oplus M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ and for every $\lambda \in \mathbb{C}$ we will denote by $\mathbb{M}_n^W(\lambda)$ the fiber $Tr(W)^{-1}(\lambda)$. Because the degeneracy locus of $Tr(W)$ coincides with the scheme of n-dimensional representations of $S_{a,b,c}$ it is conjectured in [5] that the motivic Donaldson-Thomas series

$$U_W(t) = \sum_{n=0}^{\infty} \mathbf{L}^{\frac{-2n^2}{2}} \frac{[\mathbb{M}_n^W(0)] - [\mathbb{M}_n^W(1)]}{[GL_n]} t^n$$

is determined by the virtual motives of simple representations of $S_{a,b,c}$. If τ has order n and $(n,3)=1$ it is known that apart from the trivial 1-dimensional representation all finite dimensional simple representations of $S_{a,b,c}$ have dimension n and [5, Conjecture 3.4] conjectures that in this case we have

$$U_W(t) = \mathbf{Exp}\left(-\frac{M_1}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t} - \frac{M_n}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t^n}{1-t^n}\right)$$

with $M_1 = \mathbb{L}^{-\frac{3}{2}}([X_{DT}=1,\mu_3]-[X_{DT}=0])$ where X_{DT} is the cubic in $\mathbb{A}^3$

$$X_{DT} = (a+b)xyz + \frac{c}{3}(x^3+y^3+z^3)$$

and where $M_n = \mathbb{L}^{1/2}([\mathbb{P}^2]-[E_c])$ where E_c is the plane elliptic curve $E_{pt}/\langle\tau\rangle$ isogenous to E_{pt} by dividing out the cyclic subgroup generated by τ.

In [12] we developed a method to verify such conjectures inductively by calculating the motives of certain Brauer-Severi schemes. In this paper we will compute the second term of $U_W(t)$ for the Sklyanin algebra $S_{1,1,c}$, that is when τ is a point of order two. By [5, Conjecture 3.4] one would expect this coefficient to involve the motives of at least two different elliptic curves $[E_c]$ and $[E_{DT}]$ (which have different j-invariants). However, the computed term only involves the motif $[E_{DT}]$.

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2. Simple representations of Sklyanin algebras

The elliptic curve associated to the Sklyanin algebra $S_{a,b,c}$

$$E_{pt} \;:\; (a^3+b^3+c^3)XYZ - abc(X^3+Y^3+Z^3) = 0$$

is the locus of all *point modules* of $S_{a,b,c}$, that is, graded (critical) left-modules $A/(Al_1+Al_2)$ with the l_i linear in X,Y,Z (and hence l_1,l_2 determine a point in $\mathbb{P}^2$) such that its Hilbert series is $(1-t)^{-1}$. Addition by the point $p=[a:b:c]\in E_{pt}$ describes the automorphism on point modules given by the shift-by-1 functor. A *line module* of $S_{a,b,c}$ is a graded (critical) left-module A/Al with l linear and Hilbert series $(1-t)^{-2}$. As $S_{a,b,c}$ is a domain, line modules correspond to lines in $\mathbb{P}^2$.

We are particularly interested in elliptic Sklyanin algebras which are finite modules over their centers. S. P. Smith and J. Tate [16] proved that this is the case if and only if $\tau\in E_{pt}$ is a point of finite order n. In this case $S_{a,b,c}$ is a maximal order in a division algebra of dimension n^2 over its center and the center of $S_{a,b,c}$ is isomorphic to

$$Z_{a,b,c} = \frac{\mathbb{C}[u_1,u_2,u_3,c_3]}{\Phi(u_1,u_2,u_3)-c_3^n}$$

where the u_i are central elements of degree n, c_3 is a central element of degree 3 and Φ is a homogeneous polynomial of degree 3 in the u_i describing the isogenous elliptic curve $E_c = E_{pt}/\langle\tau\rangle$. In [18] and [7] it is shown that when $(n,3)=1$ all finite dimensional simple representations of $S_{a,b,c}$ (apart from the trivial 1-dimensional simple) are of dimension n and correspond to the smooth points of the central variety, which has an isolated singularity at the top.

In principle, one can give an explicit description of the triple of $n \times n$ matrices describing the simple n-dimensional representation M_q corresponding to the maximal (non-graded) ideal $\mathfrak{m}_q$ of $Z_{a,b,c}$ using the isogeny $E_{pt} \twoheadrightarrow E_c$, see [11] or [7]. If c_3 does not vanish in q, the ruling from the top-singularity through q determines a point $\overline{q}$ in $\mathbf{Proj}(Z_{a,b,c}) = \mathbb{P}^2 = \mathbb{P}(u_1^*, u_2^*, u_3^*)$ not lying on the elliptic curve E_c. Write $\overline{q}$ as the intersection of two lines L_1 and L_2 in $\mathbb{P}^2$ and lift L_1 through the isogeny to a line L in $\mathbb{P}^2 = \mathbb{P}(X^*, Y^*, Z^*)$, then $\overline{q}$ determines the *fat point of multiplicity* n, that is, the graded (critical) left-module with Hilbert series $n/(1-t)$

$$F_{\overline{q}} = \frac{A}{Al + Al_2}$$

where l is the linear form in X, Y, Z determining L and l_2 the degree n central element which is the linear form in u_1, u_2, u_3 determining L_2. The central localization of $S_{a,b,c}$ at c_3 has a central element t of degree 1 and the simple representation M_q is then the quotient of $F_{\overline{q}}$ by $t - \lambda$ where λ is the evaluation of t in q. If c_3 is zero in q, the ruling determines a point $\overline{q} \in E_c$ which lifts through the isogeny to n point modules which form of τ-orbit. The coordinates of the corresponding n points on E_{pt} can then be used to give explicit $n \times n$ matrices of the corresponding simple representation M_q, see [7, §3.1].

Clearly, this approach is only as effective as we have explicit formulas for lifting through the isogeny $E_{pt} \twoheadrightarrow E'$, that is for small n. Next, we give explicit matrices describing the simple representations in the case when $n = 2$, that is when $a = b = 1$, not using the isogeny but the fact that in this case the Sklyanin algebras $S_c = S_{1,1,-c}$ can be viewed as Clifford algebras of ternary symmetric bilinear forms and we can apply the theory of quadratic forms to describe its simple 2-dimensional representations.

In a recent paper [14] D.J. Reich and C. Walton describe a Maple algorithm to obtain explicit representations of 3-dimensional Sklyanin algebras associated to a point of order two. Here we give a pen-and-paper approach, using classical quadratic form theory.

Let $A = (a_{ij})_{i,j} \in M_3(\mathbb{C})$ be a symmetric 3×3 matrix of rank ≥ 2. The associated *Clifford algebra* $\mathbf{Cliff}_{\mathbb{C}}(A)$ is the 8-dimensional $\mathbb{C}$-algebra generated by three elements x_1, x_2 and x_3 with defining relations

$$x_i.x_j + x_j.x_i = a_{ij} \qquad \text{for all } 1 \leq i, j \leq 3$$

The symmetric bilinear form on $V = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3$ defined by A coincides with $\langle v, w \rangle = Tr(v.w)$ for all $v, w \in V$, where the product is taken in the Clifford algebra. The structure of Clifford algebras is well-known, see for example [9].

$$\mathbf{Cliff}_{\mathbb{C}}(A) \simeq \begin{cases} M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) & \text{if } rk(A) = 3 \\ M_2(\mathbb{C}) \otimes \mathbb{C}[\epsilon] & \text{if } rk(A) = 2 \end{cases}$$

That is, $\mathbf{Cliff}_{\mathbb{C}}(A)$ has two distinct simple 2-dimensional representations $\psi_{\pm}$, which coincide when $det(A) = 0$. We want to describe these explicitly, that is determine the 2×2 matrices $\psi_{\pm}(x_i)$. There is an invertible matrix $P \in GL_3(\mathbb{C})$ such that

$$P^\tau.A.P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \langle 1, 1, 1 \rangle \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \langle 1, 1, 0 \rangle$$

The *Pauli matrices* describe the simple representations of $\mathbf{Cliff}_{\mathbb{C}}(\langle 1,1,\delta\rangle)$. If

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then we have

$$\psi_{\pm}(u_1) = \sigma_1, \quad \psi_{\pm}(u_2) = \sigma_2 \quad \text{and} \quad \psi_{\pm}(u_3) = \pm\delta\sigma_3$$

for the new basis $(u_1,u_2,u_3)^\tau = P.(x_1,x_2,x_3)^\tau$ of V. But then, if $P^{-1} = (q_{ij})_{i,j}$ we have:

Lemma 1. *The simple* 2*-dimensional representation(s) of* $\mathbf{Cliff}_{\mathbb{C}}(A)$ *are given by*

$$\psi_{\pm}(x_i) = \sum_{j=1}^{3} q_{ji}\psi_{\pm}(u_j) = q_{i1}\sigma_1 + q_{i2}\sigma_2 + \pm q_{i3}\delta\sigma_3$$

The 3*-dimensional quaternion Sklyanin algebra* $S_c = S_{1,1,-c}$ is the $\mathbb{C}$-algebra generated by three elements $X = x_1, Y = x_2, Z = x_3$ with defining quadratic relations

$$XY + YX = cZ^2, \quad YZ + ZY = cX^2 \quad \text{and} \quad ZX + XZ = Y^2$$

It follows that $u = X^2, v = Y^2$ and $Z^2 = w$ are central elements and hence that S_c is the Clifford algebra over $R = \mathbb{C}[u,v,w]$ as in [3] associated with the ternary symmetric bilinear form on the free module $V = Rx_1 \oplus Rx_2 \oplus Rx_3$ determined by the symmetric matrix in $M_3(R)$

$$Q = \begin{bmatrix} 2u & cw & cv \\ cw & 2v & cu \\ cv & cu & 2w \end{bmatrix}$$

Evaluating the entries of Q in a point $p = (\alpha,\beta,\gamma) \in \mathbb{A}^3_{\mathbb{C}} = \mathbf{max}(R)$ we obtain a symmetric matrix $A = Q(p) \in M_3(\mathbb{C})$ which is of rank at least two if and only if $p \neq (0,0,0)$. Lemma 1 gives us explicit representations of the two (or one) simple 2-dimensional representations $\psi_{\pm}(p)$ of S_c lying over the point p.

It follows from [10] or [16] that the center $Z(S_c) = R \oplus R.Tr(x_1x_2x_3)$ where $Tr(x_1x_2x_3)^2 = D = det(Q)$. As a result $\mathbf{max}(Z(S_c))$ is a two-fold cover of $\mathbb{A}^3_{\mathbb{C}} = \mathbf{max}(R)$ ramified along the surface where D vanishes. By the above, points of $\mathbf{max}(Z(S_c))$ (apart from the unique point lying over $0 = (0,0,0)$) are in one-to-one correspondence with the isomorphism classes of 2-dimensional simple representations of S_c.

We will now construct families of explicit representations as in [14]. The idea is to diagonalize Q over $\mathbb{A}^3 - \{0\}$ and to keep track of the base-change matrix $P \in M_3(\mathbb{C}[u,v,w])$. For this we apply the classical diagonalization algorithm which in this case involves the choice of just two pivots.

As $p \neq (0,0,0)$ we may assume (after permuting the variables x_i if necessary) that $2u \neq 0$ which will be our first pivot. One starts off with the 3×6 matrix $(Q|I_3)$ and uses the pivot to obtain zeroes in positions $2,3$ of the first column and positions $2,3$ in the first row by the usual trick of adding suitable multiples of rows and columns. The row-operations also have an effect on the right-hand side 3×3 matrix. After this step one obtains the matrix

$$\begin{bmatrix} 2u & 0 & 0 & 1 & 0 & 0 \\ 0 & 2u(4uv - c^2w^2) & 2u(2cu^2 - c^2vw) & -cw & 2u & 0 \\ 0 & 2u(2cu^2 - c^2vw) & 2u(4uw - c^2v^2) & -cv & 0 & 2u \end{bmatrix}$$

Case 1 : If $A = 4uv - c^2w^2 \neq 0$ (or, after permuting the variables, $4uw - c^2v^2 \neq 0$) use this as pivot. After this step one obtains the diagonal matrix Δ and the basechange matrix P

$$(\Delta|P^\tau) = \begin{bmatrix} 2u & 0 & 0 & 1 & 0 & 0 \\ 0 & 2uA & 0 & -cw & 2u & 0 \\ 0 & 0 & 4u^2AD & 2cuB & 2cuC & 2uA \end{bmatrix}$$

where $B = cuw - 2v^2$ and $C = cvw - 2u^2$. Clearly, P is invertible on the open set where $uA \neq 0$.

Case 2 : If $4uv - c^2w^2 = 0 = 4uw - c^2v^2$, we have $2cu^2 - c^2vw \neq 0$. In this case we add the third row to the second and the third column to the second, use the resulting $(2,2)$-entry as pivot in order to arrive at

$$(\Delta|P^\tau) = \begin{bmatrix} 2u & 0 & 0 & 1 & 0 & 0 \\ 0 & -2uL & 0 & -cv-cw & 2u & 2u \\ 0 & 0 & -16u^4LD & 4cu^2Q_0 & 4u^2Q_1 & -4u^2Q_2 \end{bmatrix}$$

where

$$\begin{cases} Q_0 &= (w-v)(2w+2v+cu) \\ Q_1 &= c^2vw - 4uw + c^2v^2 - 2cu^2) \\ Q_2 &= c^2w^2 + c^2vw - 4uv - 2cu^2) \end{cases}$$

and $L = Q_1 + Q_2$. The determinant of the basechange matrix is $-8u^3L$. In a point where $4uv - c^2w^2 = 0 = 4uw - c^2v^2$, L is equal to $-2(2cu^2 - c^2vw)$ so P is invertible in those points. Observe that these two cases cover all points in $\mathbf{max}(Z(S_c))$ where $u \neq 0$.

Lemma 2. *With notations as above, let $\Delta = diag(D_1, D_2, D_3)$ and $P^{-1} = (Q_{ij})_{i,j}$. Then, the maps (remember that $x_1 = X, x_2 = Y$ and $x_3 = Z$)*

$$\psi_\pm(x_i) = Q_{i1}\sqrt{D_1}\sigma_1 + Q_{i2}\sqrt{D_2}\sigma_2 \pm \sqrt{D_3}\sigma_3$$

give a family of explicit representations of S_c, with a unique representative for all simple 2-dimensional representations on the open set of $\mathbf{max}(Z(S_c))$ where $u \neq 0$. Here we take the matrices of the first case if $uA \neq 0$ and those of the second case on the locus where $4uv - c^2w^2 = 0 = 4uw - c^2v^2$. Permuting the variables covers the entire Azumaya-locus of S_c which is $\mathbf{max}(Z(S_c))$ with the unique isolated singularity lying over $(0,0,0)$ removed.

For example, on the open set where $uA \neq 0$ we have the following explicit matrix-representations:

$$\begin{cases} \psi_\pm(X) &= \begin{bmatrix} 0 & \sqrt{2u} \\ \sqrt{2u} & 0 \end{bmatrix} \\ \\ \psi_\pm(Y) &= \frac{cw}{2u}\begin{bmatrix} 0 & \sqrt{2u} \\ \sqrt{2u} & 0 \end{bmatrix} + \frac{1}{2u}\begin{bmatrix} 0 & -i\sqrt{2uA} \\ i\sqrt{2uA} & 0 \end{bmatrix} \\ \\ \psi_\pm(Z) &= \frac{cv}{2u}\begin{bmatrix} 0 & \sqrt{2u} \\ \sqrt{2u} & 0 \end{bmatrix} - \frac{cC}{2uA}\begin{bmatrix} 0 & -i\sqrt{2uA} \\ i\sqrt{2uA} & 0 \end{bmatrix} \mp \frac{1}{2uA}\begin{bmatrix} 2u\sqrt{AD} & 0 \\ 0 & -2u\sqrt{AD} \end{bmatrix} \end{cases}$$

3. Superpotentials and motives

Consider the cubic superpotential $W = aXYZ + bXZY + \frac{c}{3}(X^3 + Y^3 + Z^3)$ in the noncommutative variables X, Y and Z. For every dimension $n \geq 1$, the superpotential W determines the Chern-Simons functional

$$Tr(W) \ : \ M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

obtained by replacing X, Y and Z by the first, second resp. third component matrix and taking the trace. The representation theoretic interest of the degeneracy locus $\{ \ dTr(W) = 0 \ \}$ of this functional is that it coincides with the scheme of n-dimensional representations $\mathbf{rep}_n(R_W)$ of the associated Jacobi algebra

$$R_W = \frac{\mathbb{C}\langle X, Y, Z\rangle}{(\partial_X(W), \partial_Y(W), \partial_Z(W))}$$

where the ∂_V are the cyclic derivative with respect to the variables V, which in the case of the above superpotential W gives us the defining equations of $S_{a,b,c}$. That is, the degeneracy locus of the superpotential W

$$\{ \ dTr(W) = 0 \ \} = \mathbf{rep}_n(S_{a,b,c})$$

By the Denef-Loeser theory of motivic nearby cycles, see [8], the motive of this degeneracy locus can often be computed as the difference of the motives of the general fiber and the zero-fiber of the functional. For this reason we are interested in the (naive, equivariant) motive of the λ-fiber of the functional $Tr(W)$ which we denote by $\mathbb{M}_n^W(\lambda) = Tr(W)^{-1}(\lambda)$.

Recall that to each isomorphism class of a complex variety X (equipped with a good action of a finite group of roots of unity) we associate its naive equivariant motive $[X]$ which is an element in the ring $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1/2}]$ (see [6] or [5]) and is subject to the scissor- and product-relations

$$[X] - [Z] = [X - Z] \quad \text{and} \quad [X].[Y] = [X \times Y]$$

whenever Z is a Zariski closed subvariety of X. A special element is the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}, id]$ and we recall from [13, Lemma 4.1] that $[GL_n] = \prod_{k=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^k)$ and from [5, 2.2] that $[\mathbb{A}^n, \mu_k] = \mathbb{L}^n$ for a linear action of μ_k on $\mathbb{A}^n$. This ring is equipped with a plethystic exponential $\mathbf{Exp}$, see for example [4] and [6].

As W is homogeneous it follows from [6, Thm. 1.3] that the virtual motive of the degeneracy locus is equal to

$$[dTr(W) = 0]_{virt} = [\mathbf{rep}_n(S_{a,b,c})]_{virt} = \mathbb{L}^{-\frac{2n^2}{2}}([\mathbb{M}_n^W(0)] - [\mathbb{M}_n^W(1)])$$

where $\hat{\mu}$ acts via μ_d on $\mathbb{M}_n^W(1)$ and trivially on $\mathbb{M}_n^W(0)$. These virtual motives can be packaged together into the motivic Donaldson-Thomas series

$$U_W(t) = \sum_{n=0}^{\infty} \mathbb{L}^{-\frac{2n^2}{2}} \frac{[\mathbb{M}_n^W(0)] - [\mathbb{M}_n^W(1)]}{[GL_n]} t^n$$

By the Jordan-Hölder theorem, the sequence $\{ \ [\mathbf{rep}_n(S_{a,b,c})]_{virt} \ \}$ is expected to jump at every dimension n where $S_{a,b,c}$ has simple n-dimensional representations. For this reason A. Cazzaniga, A. Morrison, B. Pym and B. Szendrői conjecture in [5] that the generating sequence $U_W(t)$ has an exponential expression involving rational functions of virtual motives connected to the simple representations of the

Jacobi algebra $S_{a,b,c}$. Explicitly, their conjecture [5, Conjecture 3.4] asserts that in case $\tau \in E_{pt}$ has infinite order that then

$$U_W(t) = \mathbf{Exp}(-\frac{M_1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t})$$

where $M_1 = \mathbb{L}^{-3/2}([X_{DT} = 1] - [X_{DT} = 0])$ where X_{DT} is the cubic function in the three commuting variables x, y, z

$$X_{DT} = (a+b)xyz + \frac{c}{3}(x^3 + y^3 + z^3)$$

which gives $Tr(W)$ for $n = 1$. Note that X_{DT} determines an elliptic curve in $\mathbb{P}^2$, usually with a different j-invariant than E_{pt} and E_c. If however $\tau \in E_{pt}$ is a point of finite order n and $(n,3) = 1$ one expects another term in the exponential expression coming from the simples in dimension n. In [5, Conjecture 3.4] it is conjectured that in this case

$$U_W(t) = \mathbf{Exp}(-\frac{M_1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t} - \frac{M_n}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t^n}{1-t^n})$$

where $M_n = \mathbb{L}^{1/2}([\mathbb{P}^2] - [E_c])$. Observe already from section 2 that this term only encodes the simple n-dimensional representations determined by points $q \in \mathbf{Spec}(Z_{a,b,c})$ not lying on the cone over E_c.

Lemma 3. *If we denote with*

$$N_1 = (\mathbb{L} - 1)[E_{DT}] + 1 - [S_{DT}, \mu_3] \quad \text{and} \quad N_2 = [E_c] - [\mathbb{P}^2]$$

then the coefficient of t^2 in the conjectured series $U_W(t)$ is equal to

$$\frac{\mathbb{L}(\mathbb{L}^2 - 1)N_2 + \mathbb{L}^{-2}N_1^2 + \mathbb{L}^{-1}(\mathbb{L}^2 - 1)N_1 + \mathbb{L}^{-2}(\mathbb{L} - 1)\sigma_2(N_1)}{(\mathbb{L}^2 - 1)(\mathbb{L} - 1)}$$

Proof. With these notations, the conjecture [5, Conjecture 3.4] can be rewritten as

$$U_W(t) = \mathbf{Exp}(\frac{\mathbb{L}(\mathbb{L}^{-2}N_1)}{\mathbb{L} - 1}\frac{t}{1-t}).\mathbf{Exp}(\frac{\mathbb{L}N_2}{\mathbb{L} - 1}\frac{t^2}{1-t^2})$$

The second term is equal to

$$\mathbf{Exp}(\sum_{k\geq 1}\sum_{j\geq 0}\mathbb{L}^{-j}N_2t^{2k}) = \prod_{k\geq 1}\prod_{j\geq 0}\mathbf{Exp}(\mathbb{L}^{-j}N_2t^{2k}) =$$

$$\prod_{k\geq 1}\prod_{j\geq 0}(\sum_{n\geq 0}\sigma_n(\mathbb{L}^{-j}N_2t^{2k})) = \prod_{k\geq 1}\prod_{j\geq 0}(\sum_{n\geq 0}\mathbb{L}^{-nj}\sigma_n(N_2)t^{2kn})$$

As we are only interested in the coefficient of t^2 we need only consider the term in the first product where $k = 1$ and then get

$$(1 + N_2t^2 + \ldots)(1 + \mathbb{L}^{-1}N_2t^2 + \ldots)(1 + \mathbb{L}^{-2}N_2t^2 + \ldots)\ldots = 1 + \frac{N_2}{1 - \mathbb{L}^{-1}}t^2 + \ldots$$

For the first term, we get likewise

$$\mathbf{Exp}(\sum_{k\geq 1}\sum_{j\geq 2}\mathbb{L}^{-j}N_1t^k) = \prod_{k\geq 1}\prod_{j\geq 2}\mathbf{Exp}(\mathbb{L}^{-j}N_1t^k) =$$

$$\prod_{k\geq 1}\prod_{j\geq 2}(\sum_{n\geq 0}\sigma_n(\mathbb{L}^{-j}N_1t^k)) = \prod_{k\geq 1}\prod_{j\geq 2}(\sum_{n\geq 0}\mathbb{L}^{-nj}\sigma_n(N_1)t^{kn})$$

As we only want the coefficient of t^2 we have to consider three contributions:

$k=1,n=1$ in two brackets with $j_2 > j_1 \geq 2$ this gives

$$\sum_{2 \leq j_1 < j_2} N_1^2 \mathbb{L}^{-(j_1+j_2)} = \sum_{j \geq 2} \sum_{k \geq 0} \mathbb{L}^{-2j-k-1} N_1^2 =$$

$$\mathbb{L}^{-5} N_1^2 (\sum_{j \geq 0} \mathbb{L}^{-2j})(\sum_{k \geq 0} \mathbb{L}^{-k}) = \frac{\mathbb{L}^{-5} N_1^2}{(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-1})} = \frac{\mathbb{L}^{-2} N_1^2}{(\mathbb{L}^2-1)(\mathbb{L}-1)}$$

$k=2,n=1$ in one bracket and $n=0$ in all others. This gives

$$\sum_{j \geq 2} \mathbb{L}^{-j} N_1 = \frac{\mathbb{L}^{-2} N_1}{1-\mathbb{L}^{-1}} = \frac{\mathbb{L}^{-1} N_1}{\mathbb{L}-1}$$

$k=1,n=2$ in one bracket and $n=0$ in all others. Then we get

$$\sum_{j \geq 2} \mathbb{L}^{-2j} \sigma_2(N_1) = \frac{\mathbb{L}^{-4} \sigma_2(N_1)}{1-\mathbb{L}^{-2}} = \frac{\mathbb{L}^{-2} \sigma_2(N_1)}{\mathbb{L}^2 - 1}$$

Summing up all terms gives the claimed expression. □

4. Brauer-Severi motives

In [12] an inductive method was proposed to compute the coefficients of the series $U_W(t)$ inductively. For every $n \geq 1$ and every $\lambda \in \mathbb{C}$ introduce the following quotient of the trace ring $\mathbb{T}_{3,n}$ of 3 generic $n \times n$ matrices

$$\mathbb{T}_n^W(\lambda) = \frac{\mathbb{T}_{3,n}}{(Tr(W)-\lambda)}$$

The reason being that the λ-fiber $Tr(W)^{-1}(\lambda)$ is the scheme of n-dimensional trace preserving representations of $\mathbb{T}_n^W(\lambda)$

$$Tr(W)^{-1}(\lambda) = \mathbf{trep}_n(\mathbb{T}_n^W(\lambda))$$

Now, consider the associated Brauer-Severi scheme in the sense of M. Van den Bergh [17]. That is, consider the open subscheme U_n^W of $\mathbf{trep}_n(\mathbb{T}_n^W(\lambda)) \times \mathbb{C}^n$ consisting of couples

$$U_n^W(\lambda) = \{(\phi, v) \in \mathbf{trep}_n(\mathbb{T}_n^W(\lambda)) \times \mathbb{C}^n \mid \phi(\mathbb{T}_n^W(\lambda)).v = \mathbb{C}^n\}$$

on which GL_n acts freely and let the Brauer-Severi scheme be the corresponding quotient variety $\mathbf{BS}_n^W(\lambda) = U_n^W(\lambda)/GL_n$. Then it is shown in [12, Prop. 5] that one can compute the fiber-motives at n from knowledge of the Brauer-Severi-motives for all dimensions $k \leq n$ and the fiber-motives at all $k < n$. Explicitly,

$$(\mathbb{L}^n - 1)\frac{[\mathbb{M}_n^W(0)] - [\mathbb{M}_n^W(1)]}{[GL_n]}$$

is equal to

$$([\mathbf{BS}_n^W(0)] - [\mathbf{BS}_n^W(1)]) + \sum_{k=1}^{n-1} \frac{\mathbb{L}^{2k(n-k)}}{[GL_{n-k}]}(\mathbf{BS}_k^W(0)] - [\mathbf{BS}_k^W(1)])([\mathbb{M}_k^W(0)] - [\mathbb{M}_k^W(1)])$$

We will next compute the first two terms in $U_W(t)$ and for $n=2$ the previous formula reduces to

$$(\mathbb{L}^2-1)\frac{[\mathbb{M}_2^W(0)] - [\mathbb{M}_2^W(1)]}{[GL_2]} = [\mathbf{BS}_2^W(0)] - [\mathbf{BS}_2^W(1)] + \frac{\mathbb{L}^2}{(\mathbb{L}-1)}([\mathbb{M}_1^W(0)] - [\mathbb{M}_1^W(1)])^2$$

and we have already that

$$[\mathbb{M}_1^W(1)] = [X_{DT}=1] \quad \text{and} \quad [\mathbb{M}_1^W(0)] = [X_{DT}=0] = (\mathbb{L}-1)[E_{DT}]+1$$

so it remains to compute the difference of the Brauer-Severi motives $[\mathbf{BS}_2^W(0)] - [\mathbf{BS}_2^W(1)]$.

From [15] we deduce that $\mathbf{BS}_2(\mathbb{T}_{3,2})$ has a cellular decomposition as $\mathbb{A}^{10}\sqcup\mathbb{A}^8\sqcup\mathbb{A}^8$ where the three cells have representatives

$$\begin{cases} \mathbf{cell}_1 \ : \ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & p \\ 1 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ \\ \mathbf{cell}_2 \ : \ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & t \\ 1 & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ \\ \mathbf{cell}_3 \ : \ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ 0 & v \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & x \\ 1 & z \end{bmatrix} \end{cases}$$

It follows that $\mathbf{BS}_{3,2}^W(\lambda)$ decomposes as $\mathbf{S_1}(\lambda)\sqcup\mathbf{S_2}(\lambda)\sqcup\mathbf{S_3}(\lambda)$ where the subschemes $\mathbf{S_i}(\lambda)$ of $\mathbb{A}^{11-i}$ have defining equations

$$\begin{cases} \mathbf{S_1}(\lambda) \ : \ (C+Q_u.u+Q_y.y+Q_q)|_{n=0} = \lambda \\ \mathbf{S_2}(\lambda) \ : \ (C+Q_y.y+Q_u)|_{s=0} = \lambda \\ \mathbf{S_3}(\lambda) \ : \ (C+Q_y)|_{w=0} = \lambda \end{cases}$$

where

$$\begin{cases} C &= \frac{c}{3}(n^3+r^3+s^3+v^3+w^3+z^3)+(a+b)(rvz+nsw) \\ Q_q &= a(tz+sx)+b(vx+tw)+cp(r+n) \\ Q_u &= a(rx+pw)+b(pz+nx)+ct(v+s) \\ Q_y &= a(pv+nt)+b(rt+ps)+cx(z+w) \end{cases}$$

Note that in using the cellular decomposition, we set a variable equal to 1. So, in order to retain a homogeneous form we let $\mathbb{G}_m$ act on n,s,w,r,v,z with weight one, on q,u,y with weight two and on x,t,p with weight zero. Thus, we need a slight extension of [6, Thm. 1.3] as to allow $\mathbb{G}_m$ to act with weight two on certain variables.

We will restrict to the case of a Sklyanin algebra with a point of order two, that is the case when $a=b$, which we may assume to be equal to 1, and with $c\neq 0$.

Lemma 4. *With $a=b=1$ and $c\neq 0$ we have*

$$\begin{cases} [\mathbf{S_3}(0)] = \mathbb{L}^7+\mathbb{L}^5-\mathbb{L}^4 \\ [\mathbf{S_3}(1)] = \mathbb{L}^7-\mathbb{L}^4 \end{cases}$$

and therefore $[\mathbf{S_3}(0)]-[\mathbf{S_3}(1)] = \mathbb{L}^5$.

Proof. The defining equation of $\mathbf{S_3}(\lambda)$ in $\mathbb{A}^8$ is

$$\frac{c}{3}(n^3+r^3+s^3+v^3+z^3)+2rvz+(v+s)p+(n+r)t+czx = \lambda$$

$(1):$ If $v+s \neq 0$ we can eliminate p from the equation and get a contribution $\mathbb{L}^5(\mathbb{L}^2-\mathbb{L})$ as there are five free variables and $[v+s \neq 0]_{\mathbb{A}^2} = \mathbb{L}^2 - \mathbb{L}$. Note that this is independent of the value of λ.

$(2):$ If $v+s=0$ we get the equation

$$\frac{c}{3}(n^3+r^3+z^3)+2rvz+(n+r)t+czx=\lambda$$

If we assume that in addition $n+r \neq 0$ we can eliminate t, then by an argument as above we obtain a contribution $\mathbb{L}^4(\mathbb{L}^2-\mathbb{L})$, again independent of the value of λ.

$(3):$ If $v+s=0$ and $n+r=0$ we get as equation $\frac{c}{3}z^3+2rvz+czx=\lambda$. So, if $z \neq 0$ we can eliminate x and get a term $\mathbb{L}^4(\mathbb{L}-1)$, independent of λ.

$(4):$ If $v+s=0, n+r=0$ and $z=0$ we get the equation $0=\lambda$. Hence, if $\lambda=1$ this gives no contribution, but if $\lambda=0$ we get a contribution $\mathbb{L}^5$.

Summing up we get the claimed motives. $\square$

As we are only interested in the differences $[\mathbf{S_k}(0)]-[\mathbf{S_k}(1)]$ we will in the remaining computations only determine the difference of the motives in those subcases where the result can depend on the value of λ.

Lemma 5. *With $a=b=1$ and $c \neq 0$ we have*

$$[\mathbf{S_2}(0)]-[\mathbf{S_2}(1)] = \mathbb{L}^6+\mathbb{L}^3.[\mu_3].([X_0]-[X_1])$$

where X_λ is the locally closed subset in $\mathbb{A}^3$ (with variables x,y,z) defined by

$$X_\lambda = \begin{cases} x \neq 0 \\ x(3\rho c z^2 - 3\rho^2 cxz + 6yz + (c^4+2c)x^2 - 3\rho c^3 xy + 3\rho^2 c^2 y^2) = 3\lambda \end{cases}$$

and $\rho^3=1$.

Proof. The defining equation of $\mathbf{S_2}(\lambda)$ in $\mathbb{A}^9$ is

$$\frac{c}{3}(n^3+r^3+v^3+w^3+z^3)+2rvx+(vp+(n+r)t+c(z+w)x)y+$$
$$((r+n)x+(w+z)p+cvt)=\lambda$$

$(1):$ If $vp+(n+r)t+c(z+w)x \neq 0$ we can eliminate y from the equation, independent of the value of λ.

$(2):$ If $vp+(n+r)t+c(z+w)x=0$ and $v \neq 0$ we have

$$p = -\tfrac{n+r}{v}t - c\tfrac{z+w}{v}x$$

and after substitution the equation becomes

$$\frac{c}{3}(n^3+r^3+v^3+w^3+z^3)+2rvx+((r+n)-c\frac{(z+w)^2}{v})x+(cv-\frac{(n+r)(w+z)}{v})t=\lambda$$

If $v(r+n)-c(w+z)^2 \neq 0$ we can eliminate x from the equation, and the remaining motive to consider, that is,

$$[vp+(n+r)t+c(z+w)x=0, v\neq 0, v(r+n)-c(w+z)^2 \neq 0]_{\mathbb{A}^7}$$

does not depend on λ.

If $v(r+n)-c(w+z)^2=0$ but $cv^2-(n+r)(w+z) \neq 0$ we can eliminate t, and again the resulting motive independent of λ, so does not contribute.

(3) : We arrive at the first subcase which depends on λ. The defining equations of the locally closed subset of $\mathbb{A}^5$ (we have eliminated p and the variables y, x and t are free) are

$$\begin{cases} v \neq 0 \\ v(r+n) - c(w+z)^2 = 0 \\ cv^2 - (n+r)(w+z) = 0 \\ \frac{c}{3}(n^3 + r^3 + v^3 + w^3 + z^3) + 2rvz = \lambda \end{cases}$$

From the first equation we obtain $r + n = \frac{c(w+z)^2}{v}$, and substituting this in the second equation gives

$$v^3 = (w+z)^3 \qquad \text{whence} \qquad \begin{cases} w = \rho v - z \\ n = c\rho^2 v - r \end{cases}$$

for $\rho^3 = 1$, so we have three subcases to consider which are clearly isomorphic, giving a factor $[\mu_3]$.

If we substitute the obtained equations in the last equation, we obtain the locally closed subset in $\mathbb{A}^3$ (with remaining coefficients r, v, z)

$$X_\lambda = \begin{cases} v \neq 0 \\ v(3\rho c z^2 - 3\rho^2 c v z + 6rz + (c^4 + 2c)v^2 - 3\rho c^3 r v + 3\rho^2 c^2 r^2) = 3\lambda \end{cases}$$

Therefore, this subcase contributes a term equal to

$$\mathbb{L}^3.[\mu_3].([X_0] - [X_1])$$

(4) : We have exhausted the $v \neq 0$ case, so from now on $v = 0$ and we have to solve in $\mathbb{A}^7$

$$\begin{cases} (n+r)t + c(z+w)x = 0 \\ \frac{c}{3}(n^3 + r^3 + w^3 + z^3) + (r+n)x + (w+z)p = \lambda \end{cases}$$

If $w + z \neq 0$ we can eliminate x from the first equation, substitute it in the second and eliminate p from the second, all this independent of λ.

(5) : If $w + z = 0$ we have

$$\begin{cases} (n+r)t = 0 \\ \frac{c}{3}(n^3 + r^3) + (r+n)x = \lambda \end{cases}$$

So, if $r + n \neq 0$ we must have that $t = 0$ and can eliminate x from the second equation, independent of λ.

(6) : The remaining case is when y, x, t and p are free variables and we have

$$\begin{cases} v = 0 \\ w + z = 0 \\ r + n = 0 \end{cases}$$

and the remaining equation is $0 = \lambda$. So, for $\lambda = 1$ we get no contribution, whereas for $\lambda = 0$ we get a contribution $\mathbb{L}^6$. $\square$

Lemma 6. *With $a = b = 1$ and $c \neq 0$ we have*

$$[\mathbf{S_1}(0)] - [\mathbf{S_1}(1)] = \mathbb{L}^7 + \mathbb{L}^3.[\mu_3].([X_0] - [X_1])$$

where X_λ is the locally closed subset in $\mathbb{A}^3$ (with variables x, y, z) defined by

$$X_\lambda = \begin{cases} x \neq 0 \\ x(3\rho c z^2 - 3\rho^2 cxz + 6yz + (c^4 + 2c)x^2 - 3\rho c^3 xy + 3\rho^2 c^2 y^2) = 3\lambda \end{cases}$$

and $\rho^3 = 1$.

Proof. The defining equation of $\mathbf{S_1}(\lambda)$ in $\mathbb{A}^{10}$ is equal to

$$\frac{c}{3}(r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz + ((w+z)p + c(v+s)t + rx)u +$$

$$((v+s)p + rt + c(z+w)x)y + (crp + (z+w)t + (s+v)x) = \lambda$$

Again, we will split the computations is subcases and only work out those for which the difference of motives may depend on λ.

(1) : If $(w + z)p + c(v + s)t + rx \neq 0$ we can eliminate u from the equation, independent of the value of λ.

(2) : If $(w+z)p+c(v+s)t+rx = 0$, u is a free variable and the equation becomes

$$\frac{c}{3}(r^3+s^3+v^3+w^3+z^3)+2rvz+((v+s)p+rt+c(z+w)x)y+(crp+(z+w)t+(s+v)x) = \lambda$$

If $ry + (z + w) \neq 0$ we can eliminate t from the equation, independent of λ.

(3) : If $(w + z)p + c(v + s)t + rx = 0$ and $ry + (z + w) = 0$ and $r \neq 0$, then we have the equations

$$\begin{cases} y = -\frac{z+w}{r} \\ x = -\frac{w+z}{r}p - \frac{c(v+s)}{r}t \end{cases}$$

and substitution gives us the equation

$$\frac{c}{3}(r^3+s^3+v^3+w^3+z^3)+2rvz-\frac{z+w}{r}((v+s)p+c(z+w)(-\frac{w+z}{r}p-\frac{c(v+s)}{r}t)+$$

$$(crp + (s+v)(-\frac{w+z}{r}p - \frac{c(v+s)}{r}t)) = \lambda$$

The coefficient of t is equal to $-\frac{c(v+s)^2}{r} + \frac{z+w}{r}\frac{c^2(z+w)(v+s)}{r}$. Hence, if $c(z + w)^2(v + s) - r(v + s)^2 \neq 0$ we can eliminate t from the equation, independent of λ.

(4) : If $r \neq 0$, $(w + z)p + c(v + s)t + rx = 0$ and $ry + (z + w) = 0$ and $c(z + w)^2(v + s) - r(v + s)^2 = 0$, the equation becomes

$$\frac{c}{3}(r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz + (\frac{c(z+w)^3}{r^2} - 2\frac{(z+w)(s+v)}{r} + cr)p = \lambda$$

That is, if $c(z + w)^3 - 2r(z + w)(s + v) + cr^3 \neq 0$ we can eliminate p, independent of λ.

(5) : The first subcase dependent on λ is now that u, p and t are free variables and we have the following locally closed subset of $\mathbb{A}^5$ (in the remaining variables

$r, s, v, w, z)$

$$\begin{cases} r \neq 0 \\ c(z+w)^2(v+s) - r(v+s)^2 = 0 \\ c(z+w)^3 - 2r(z+w)(s+v) + cr^3 = 0 \\ \frac{c}{3}(r^3+s^3+v^3+w^3+z^3) + 2rvz = \lambda \end{cases}$$

If $v+s \neq 0$ we have $r(v+s) = c(z+w)^2$ and substituting in the third equation gives $r^3 = (z+w)^3$ whence $z+w = \rho r$ for $\rho^3 = 1$, but then also $c\rho^2 r = v+s$. If we substitute

$$\begin{cases} w = \rho r - z \\ s = c\rho^2 - v \end{cases}$$

in the last equation, we get the locally closed subset in $\mathbb{A}^3$, isomorphic to X_λ of the previous case (interchanging the variables r and v)

$$X_\lambda = \begin{cases} r \neq 0 \\ r(3\rho c z^2 + 6vz - 3\rho^2 crz + 3\rho^2 c^2 v^2 - 3\rho c^3 rv + (c^4+2c)r^2) = \lambda \end{cases}$$

Therefore, this subcase contributes a term equal to

$$\mathbb{L}^3.[\mu_3].([X_0] - [X_1])$$

(6) : From now on we may assume that $r = 0$, together with $(w+z)p + c(v+s)t + rx = 0$ and $ry + (z+w) = 0$. But then, $z + w = 0$ and the conditions are equivalent to the following system of equations in $\mathbb{A}^6$ (in the variables s, t, v, p, x, y). Observe that we have u and w as extra free variables

$$\begin{cases} c(s+v)t = 0 \\ \frac{c}{3}(s^3+v^3) + (s+v)py + (s+v)x = \lambda \end{cases}$$

If $s + v \neq 0$ we have $t = 0$ and can eliminate x from the last equation, independent of λ.

(7) : If $s + v = 0$ we have u, w, t, p, y, x, s as free variables and the remaining condition is $0 = \lambda$. That is, if $\lambda = 1$ there is no contribution and for $\lambda = 0$ we get a term $\mathbb{L}^7$. □

Summing up the three contributions, we have:

Lemma 7. *For the Brauer-Severi motives we have*

$$[\mathbf{BS}_2^W(0)] - [\mathbf{BS}_2^W(1)] = \mathbb{L}^7 + \mathbb{L}^6 + \mathbb{L}^5 + 2\mathbb{L}^3[\mu_3]([X_0] - [X_1])$$

Therefore, the coefficient of t^2 in the series $U_W(t)$ is equal to

$$\mathbb{L}^{-4}\frac{[\mathbb{M}_2^W(0)] - [\mathbb{M}_2^W(1)]}{[GL_2]} = \frac{\mathbb{L}(\mathbb{L}^3-1) + 2[\mu_3]([X_0]-[X_1])\mathbb{L}^{-1}(\mathbb{L}-1) + \mathbb{L}^{-2}N_1^2}{(\mathbb{L}^2-1)(\mathbb{L}-1)}$$

Remains to compute the motives $[X_\lambda]$ where

$$X_\lambda = \begin{cases} x \neq 0 \\ x.(\rho c z^2 - \rho^2 cxz + 2yz + \frac{c^4+2c}{3}x^2 - \rho c^3 xy + \rho^2 c^2 y^2) = \lambda \end{cases}$$

After performing the linear change of variables

$$\begin{cases} X = \sqrt{\frac{c^4+8c}{12}}x + i\frac{\sqrt{(c^3-1)\rho}}{c}z \\ Y = -\frac{c^2}{2}x + \rho c y + \frac{\rho^2}{c}z \\ Z = \sqrt{\frac{c^4+8c}{12}}x - i\frac{\sqrt{(c^3-1)\rho}}{c}z \end{cases}$$

we can express

$$X_\lambda = \begin{cases} X + Z \neq 0 \\ (X+Z)(Y^2+XZ) = \lambda \end{cases}$$

Lemma 8. *With notations as above we have*

$$[X_0] = (\mathbb{L}-1)^2 \quad \textit{and} \quad [X_1] = (\mathbb{L}-1)^2 + [\mu_3]\mathbb{L}$$

Proof. We have $[X_0] = [Y^2 + XZ = 0]_{\mathbb{A}^3} - [Y^2 + XZ = 0, X + Z = 0]_{\mathbb{A}^3}$ which equals

$$[Y^2 + XZ = 0]_{\mathbb{A}^3} - [(X+Y)(X-Y) = 0]_{\mathbb{A}^2} = \mathbb{L}^2 - (2\mathbb{L}-1)$$

As for X_1, we have for every $X + Z = a \neq 0$

$$[Y^2 - X^2 + aX = \frac{1}{a}]_{\mathbb{A}^2} = \begin{cases} \mathbb{L} - 1 \text{ if } a^3 \neq 4 \\ 2\mathbb{L} - 1 \text{ if } a^3 = 4 \end{cases}$$

as this is the affine part of a quadric $Y^2 - X^2 + aXU - \frac{1}{a}U^2 = 0$ in $\mathbb{P}^2$, having two points at infinity $U = 0$, for every $a \neq 0$. The quadric has a unique singular point $[\frac{a}{2} : 0 : 1]$ if and only if $a^3 = 4$.Therefore,

$$[X_1] = (\mathbb{L} - 1 - [\mu_3])(\mathbb{L}-1) + [\mu_3](2\mathbb{L}-1).$$

□

Theorem 1. *For the quaternionic Sklyanin algebra $S_{1,1,c}$ we have that the coefficient of the second term in the motivic Donaldson-Thomas series $U_W(t)$ is equal to*

$$\mathbb{L}^{-4}\frac{[\mathbb{M}_2^W(0)] - [\mathbb{M}_2^W(1)]}{[GL_2]} = \frac{\mathbb{L}(\mathbb{L}^3-1) - 2[\mu_3]^2(\mathbb{L}-1) + \mathbb{L}^{-2}N_1^2}{(\mathbb{L}^2-1)(\mathbb{L}-1)}$$

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