AZUMAYA REPRESENTATION SCHEMES

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ABSTRACT. We extend Grothendieck topologies on commutative algebras to the category of all Azumaya algebras and we show that the functor assigning to an Azumaya algebra A the set of all algebra maps $R \longrightarrow A$ from a fixed $\mathbb{C}$-algebra R, is a sheaf for all such Grothendieck topologies coarser than the maximal flat topology. We construct Azumaya representation schemes representing algebra maps from R to a fixed Azumaya algebra A, which is relevant in the study of the representation stack $[\mathtt{rep}_n(R)/\mathrm{PGL}_n]$. Finally, we describe the related quotient stack $[\mathtt{rep}_\alpha(R)/\mathrm{PGL}(\alpha)]$ in terms of twisted representations of quivers.

1. INTRODUCTION

Throughout, all algebras R will be associative, unital, finitely generated $\mathbb{C}$-algebras, not necessarily commutative. With $\mathtt{rep}_n(R)$ we denote the affine scheme of all n-dimensional representations of R, that is, all $\mathbb{C}$-algebra maps $R \longrightarrow M_n(\mathbb{C})$. Conjugation in $M_n(\mathbb{C})$ defines a PGL_n-action on $\mathtt{rep}_n(R)$, its orbits corresponding to isomorphism classes of n-dimensional representations. By results of M. Artin [2] and C. Procesi [15] it is known that the geometric points of the quotient scheme $\mathtt{rep}_n(R)/\mathrm{PGL}_n$ classify isomorphism classes of n-dimensional semi-simple representations of R.

In order to classify the isomorphism classes of all n-dimensional representations, one has to consider the representation stack of n-dimensional representations $[\mathtt{rep}_n(R)/\mathrm{PGL}_n]$, which by the results of [9] is the functor from the category $\mathtt{Comm}$ of all commutative $\mathbb{C}$-algebras to $\mathtt{Groupoids}$ the category of all groupoids

$$[\mathtt{rep}_n(R)/\mathrm{PGL}_n] : \mathtt{Comm} \longrightarrow \mathtt{Groupoids} \qquad C \mapsto \mathtt{Azu}_n^C(R)$$

where the objects of the groupoid $\mathtt{Azu}_n^C(R)$ are the $\mathbb{C}$-algebra maps $\phi : R \longrightarrow A$ where A is a constant degree n Azumaya algebra with center C, and morphisms $\alpha : \phi \longrightarrow \phi'$ are given by C-algebra morphisms $\alpha : A \longrightarrow A'$ making the diagram below commute.

$$
\begin{array}{ccc}
 & & A \\
 & \nearrow{\scriptstyle \phi} & \big\downarrow{\scriptstyle \alpha} \\
R & & \\
 & \searrow{\scriptstyle \phi'} & \big\downarrow \\
 & & A'
\end{array}
$$

The information contained in these representation stacks, for varying n, can also be expressed in the following way. Consider the category $\mathtt{Azu}$ with objects all Azumaya

algebras and with morphisms all $\mathbb{C}$-algebra maps preserving centers. Given an affine $\mathbb{C}$-algebra R we can then consider the covariant functor

$$\mathrm{Alg}(R,-) \; : \; \texttt{Azu} \longrightarrow \texttt{Sets} \qquad A \mapsto \mathrm{Alg}_{\mathbb{C}}(R,A)$$

and hence a contravariant functor on the (geometric) opposite category $\texttt{Azu}^{\mathrm{op}}$. A first aim of the present paper is to investigate Grothendieck topologies on $\texttt{Azu}^{\mathrm{op}}$ for which the functor $\mathrm{Alg}(R,-)$ is a (set-valued) sheaf. For this reason we study in section 2 the problem of extending Grothendieck topologies on $\texttt{Comm}^{\mathrm{op}} = \texttt{Aff}$ to the category $\texttt{Azu}^{\mathrm{op}}$. It will transpire that often a Grothendieck topology on $\texttt{Comm}^{\mathrm{op}}$ can be extended in uncountable many ways to a Grothendieck topology on $\texttt{Azu}^{\mathrm{op}}$, depending on the chosen Grothendieck topology on the category $\mathbb{N}_+^{\times}$ with objects the strictly positive integers and morphisms given by division. This gives a perhaps surprising connection between the extension problem for Grothendieck topologies and the so called 'arithmetic site' introduced and studied by A. Connes and C. Consani [5].

In section 3 we will show that the functor $\mathrm{Alg}(R,-)$ on $\texttt{Azu}$ is a sheaf for every Grothendieck topology on $\texttt{Azu}^{\mathrm{op}}$ coarser than the maximal flat topology, that is the extension of the flat topology on $\texttt{Comm}^{\mathrm{op}}$ to $\texttt{Azu}^{\mathrm{op}}$ corresponding to the discrete topology on $\mathbb{N}_+^{\times}$. If we fix an Azumaya algebra A with center C it follows that the covariant set-valued functor from the category $\texttt{Comm}_C$ of all commutative C-algebras to $\texttt{Sets}$

$$\texttt{Comm}_C \longrightarrow \texttt{Sets} \qquad D \mapsto \mathrm{Alg}_{\mathbb{C}}(R, A \otimes_C D)$$

is a sheaf with respect to any Grothendieck topology coarser than the flat topology. The main result of this section shows that this sheaf is in fact representable by a scheme over $\texttt{Spec}(C)$, which we call the Azumaya representation scheme of R associated to the Azumaya algebra A.

If R is a basic finite dimensional $\mathbb{C}$-algebra it is isomorphic to $\mathbb{C}Q/I$ where Q is a quiver on k vertices and I is an ideal of the path algebra $\mathbb{C}Q$. In this case the geometric points of the quotient scheme $\texttt{rep}_n(R)/\mathrm{PGL}_n$ correspond to the dimension vectors $\alpha = (d_1,\ldots,d_k)$ such that $|\alpha| = \sum_{i=1}^k d_i = n$. Moreover, the representation scheme itself decomposes as

$$\texttt{rep}_n(R) = \bigsqcup_{|\alpha|=n} \mathrm{GL}_n \times^{\mathrm{GL}(\alpha)} \texttt{rep}_\alpha(R)$$

where $\mathrm{GL}(\alpha) = \prod_i \mathrm{GL}_{a_i}$ and $\texttt{rep}_\alpha(R)$ is the scheme of all α-dimensional representations of Q satisfying the equations given by elements of the ideal I. Therefore, it is natural to consider for an affine $\mathbb{C}^k$-algebra the α-dimensional representation stack $[\texttt{rep}_\alpha(R)/\mathrm{PGL}(\alpha)]$ where $\mathrm{PGL}(\alpha) = \mathrm{GL}(\alpha)/\mathbb{C}^*(1_{a_1},\ldots,1_{a_k})$.

It is well-known, see for example [11], that principal $\mathrm{PGL}(\alpha)$-bundles over $\texttt{Spec}(C)$ correspond to Azumaya algebras A with center C having a distinguished embedding $\mathbb{C}^k \hookrightarrow A$. In section 4 we will give a structural result for such Azumaya algebras and determine their automorphisms. This then allows us to interpret the C-points of the stack $[\texttt{rep}_\alpha(R)/\mathrm{PGL}(\alpha)]$ for $R = \mathbb{C}Q/I$ as twisted quiver representations.

2. Grothendieck topologies on Azumaya algebras

Let C be a commutative algebra. Recall from [6] that an algebra A is said to be an *Azumaya algebra* over C if and only if

(1) The center $Z(A)$ of A equals C.

(2) There is a *separability idempotent* $e = \sum a_i \otimes b_i \in A \otimes_C A^{\mathrm{op}}$, that is, $\mu(e) = \sum_i a_i b_i = 1$ and $e^2 = e$.

If only the second condition is satisfied we say that A is *separable* over C. Equivalently, A is an Azumaya algebra over C if and only if there is an étale cover

$$\{C \to C_i\}_{i=1}^k$$

such that for each $i \in \{1, \ldots, k\}$ there is an $n_i \in \mathbb{N}_+$ for which $A \otimes_C C_i \cong \mathrm{M}_{n_i}(C_i)$, the algebra of $n_i \times n_i$-matrices with coefficients in C_i. So A is projective over C and we can often assume that A is of constant rank n^2, in which case n will be called the *degree* of A.

Definition 2.1. *With* Azu *we will denote the category having as its objects all Azumaya algebras A over commutative algebras, and an algebra morphism $f : A \longrightarrow A'$ is a morphism in* Azu *if it preserves centers, that is if $f(Z(A)) \subset Z(B)$. Note that when A and A' are Azumaya algebras of the same constant degree n this condition is always satisfied.*

We will often invoke the *(double) centralizer theorem* (see [6, Thm. II.4.3]): let A be an Azumaya algebra with center C and let $C \subseteq B \subseteq A$ be any subalgebra of A separable over C. Then the *centralizer*

$$A^B = \{a \in A \mid \forall b \in B \; : \; a.b = b.a\}$$

is also separable over C and $A^{(A^B)} = B$. If B is in addition an Azumaya algebra over C, then so is A^B and we have

$$A \simeq B \otimes_C A^B$$

It is well known that the category $\mathtt{Azu}_C$ of all Azumaya algebras with the same center C is a symmetric monoidal category under $\otimes_C$. More generally, if A and B are separable over the commutative ring C, then so is $A \otimes_C B$. An immediate consequence of the double centralizer theorem is:

Proposition 2.2. *If $f_i : A \longrightarrow A_i$ (for $i = 1, 2$) are morphisms in* Azu *then the tensor product*

$$A_1 \otimes_A A_2$$

is again an Azumaya algebra, with center $Z(A_1) \otimes_{Z(A)} Z(A_2)$.

Proof. Let C_i be the center of A_i, then as $A \otimes_C C_i$ is a C_i-Azumaya subalgebra of A_i it follows from the centralizer theorem that

$$A_i \cong (A \otimes_C C_i) \otimes_{C_i} A_i^A \cong A \otimes_C A_i^A$$

But then we have the following isomorphisms.

$$\begin{aligned} A_1 \otimes_A A_2 \cong \;\; & A_1^A \otimes_C A \otimes_A A \otimes_C A_2^A \\ \cong \;\; & A_1^A \otimes_C A \otimes_C A_2^A \\ \cong \;\; & A_1 \otimes_C A_2^A \cong A_1^A \otimes_C A_2 \end{aligned}$$

As all A_i and A_i^A are separable over C (by transitivity of separability) it follows that $A_1 \otimes_C A_2^A$ and $A_1^A \otimes_C A_2$ are separable over C and hence are Azumaya algebras over their center. $\square$

If a category C^{op} has pullbacks (or, equivalently, the category C has pushouts) then one can restrict to a basis to define a Grothendieck topology on C^{op}. As we want to describe Grothendieck topologies on the (geometric) opposite category $\mathsf{Azu}^{\mathrm{op}}$, the previous result would be useful if the tensor product would be a pushout in Azu. However, this is *not* the case. Indeed, let A be an Azumaya algebra with center C and degree $n > 1$, then $A \otimes_C A$ is Azumaya of degree n^2 so cannot satisfy the condition for the diagram

$$
\begin{array}{ccc}
C & \longrightarrow & A \\
\downarrow & & \\
A & \longrightarrow & A \otimes_C A \\
& & \\
& \xrightarrow{\mathrm{id}} & A
\end{array}
$$

with arrows labeled id and $\nexists h$.

In fact, some diagrams in Azu cannot have *any* pushout.

Example 2.3. *Consider the diagram*

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \mathrm{M}_n(\mathbb{C}) \\
\downarrow & & \\
\mathrm{M}_n(\mathbb{C}) & &
\end{array}
$$

If the pushout of above diagram exists, then it is unique. Call it A_n and write $C = Z(A_n)$. We will derive a contradiction from the existence of A_n, for $n > 1$. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \mathrm{M}_n(\mathbb{C}) \\
\downarrow & & \\
\mathrm{M}_n(\mathbb{C}) & \longrightarrow & A_n \\
& \xrightarrow{h} & \\
& & \mathrm{M}_n(\mathbb{C})
\end{array}
$$

with arrow labeled id.

By definition of the pushout, the dashed arrow h exists. It induces a morphism $C \to \mathbb{C}$ on centers, namely the unique morphism $C \to \mathbb{C}$ for which

$$
A_n \otimes_C \mathbb{C} \cong \mathrm{M}_n(\mathbb{C}).
$$

This implies however that the dashed arrow h' in

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \mathrm{M}_n(\mathbb{C}) \\
\downarrow & & \\
\mathrm{M}_n(\mathbb{C}) & \longrightarrow & A_n \\
& \xrightarrow{h'} & \\
& & \mathrm{M}_n(\mathbb{C})
\end{array}
$$

with arrows labeled α and id.

still induces the same morphism on centers $C \to \mathbb{C}$, for some nontrivial automorphism α. So $h' = \beta \circ h$ for some automorphism β of $\mathrm{M}_n(\mathbb{C})$. Commutativity of the diagram shows both $\beta = \alpha$ and $\beta = \mathrm{id}$, a contradiction.

So we will have to define Grothendieck topologies on $\mathtt{Azu}^{\mathrm{op}}$ via sieves, see for example [13, III.2]. As we like to retain an algebraic description we will work in $\mathtt{Azu}$. Therefore, a *sieve* S on an Azumaya algebra A is a collection of morphisms in $\mathtt{Azu}$

$$S = \{A \xrightarrow{f} B_f\} \quad \text{such that if } f \in S \text{ then } g \circ f : A \longrightarrow B_f \longrightarrow D \in S$$

for every morphism $g : B_f \longrightarrow D$ in $\mathtt{Azu}$. A *Grothendieck topology* on $\mathtt{Azu}$ is a function J which assigns to each Azumaya algebra A a collection $J(A)$ of sieves on A satisfying the following properties

   (1) The maximal sieve $T_A = \{f : A \longrightarrow B \in \mathtt{Azu}\}$ of all morphisms from A is an element of $J(A)$
   (2) Stability: If $S \in J(A)$, then for any morphism $h : A \longrightarrow B$ in $\mathtt{Azu}$, $h^{-1}(S) = \{g : B \longrightarrow D : g \circ h \in S\} \in J(B)$
   (3) Transitivity: If $S \in J(A)$ and R is a sieve on A such that $h^{-1}(R) \in J(B)$ for all morphisms $h : A \longrightarrow B$ in S, then $R \in J(A)$.

We say that a collection of morphisms $\{A \longrightarrow A_i\}_{i \in I}$ is a cover of A with respect to the Grothendieck topology J if these morphisms generate (by post-composition) a sieve in $J(A)$. Thus as an alternative to specifying a Grothendieck topology by an assignment of sieves we can specify it by an assignment of collections of covers. This approach is similar to using bases for a topology, but the lack of pushouts in $\mathtt{Azu}$ means that we have to be a bit careful.

We will first give a combinatorial description of sieves and Grothendieck topologies on the full subcategory $\mathtt{Mat}$ of $\mathtt{Azu}$ on the matrix algebras $M_n(\mathbb{C})$ for all $n \in \mathbb{N}_+$. Let $\mathbb{N}_+^{\times}$ be the poset category with objects $n \in \mathbb{N}_+$ and morphisms $n \longrightarrow m$ iff $n | m$. Clearly, we have a projection $\pi : \mathtt{Mat} \longrightarrow \mathbb{N}_+^{\times}$ sending a morphism $M_n(\mathbb{C}) \longrightarrow M_{nk}(\mathbb{C})$ to $n \longrightarrow nk$.

Lemma 2.4. *Sieves on $M_n(\mathbb{C})$ in $\mathtt{Mat}$ are in bijection with sieves on n in $\mathbb{N}_+^{\times}$ via $S \mapsto \pi(S)$. As a consequence, Grothendieck topologies on $\mathtt{Mat}$ are in bijection with Grothendieck topologies on $\mathbb{N}_+^{\times}$.*

Proof. The result follows if we can show that a sieve S on $M_n(\mathbb{C})$ is fully determined by the multiples of n such that there is a morphism $\alpha : M_n(\mathbb{C}) \longrightarrow M_{nk}(\mathbb{C}) \in S$ and not on the actual morphism α. So, let $\beta : M_n(\mathbb{C}) \longrightarrow M_{nk}(C)$ be another morphism, then it follows from the double centralizer theorem that there is an automorphism γ of $M_{nk}(\mathbb{C})$ such that $\gamma \circ \alpha = \beta$. But then we have

$$\alpha \in S \Leftrightarrow \beta \in S$$

from which the claims follow. $\qquad\qquad\square$

Grothendieck topologies on $\mathbb{N}_+^{\times}$ have been studied in [10] in connection with the arithmetic site of Connes and Consani [5]. Sieves S on n correspond one-to-one with submonoids $M = \cup_i n_i \mathbb{N}_+$ of the multiplicative monoid $\mathbb{N}_+^{\times}$ via $n \longrightarrow nk \in S$ iff $k \in M$. Further, if $h : n \longrightarrow n'$ is a morphism in $\mathbb{N}_+^{\times}$ and a sieve S corresponds to $M = \cup_i n_i \mathbb{N}_+$, then $h^{-1}(S)$ corresponds to $\cup_i \mathrm{lcm}(n', n_i) \mathbb{N}_+$. These observations allow to construct uncountable many different Grothendieck topologies on $\mathbb{N}_+^{\times}$:

Example 2.5. *Consider a set Σ of prime numbers. To this set we can associate a Grothendieck topology K_Σ on $\mathbb{N}_+^\times$ by taking for the collection $K_\Sigma(n)$ of the sieves on n all sieves corresponding to submonoids $M = \cup_i n_i \mathbb{N}_+$ such that at least one n_i has all its prime divisors in Σ. It is easy to check that K_Σ defines a Grothendieck topology on $\mathbb{N}_+^\times$. Moreover $K_\Sigma = K_{\Sigma'}$ implies that $\Sigma = \Sigma'$, for two sets of primes Σ and Σ'.*

As the collection of submonoids describing the sieves on elements is equal for all $n \in \mathbb{N}_+$, these Grothendieck topologies are stable under multiplication, *that is they have the property that if $\{n \longrightarrow n_r\}_{r \in R}$ is in $K_\Sigma(n)$ and if k is any positive integer, then $\{nk \longrightarrow n_r k\}_{r \in R} \in K_\Sigma(nk)$. If, in particular $\Sigma = \mathbb{P}$ is the set of all prime numbers, then the corresponding Grothendieck topology $K_+ = K_\mathbb{P}$ will be called the* maximal topology *on $\mathbb{N}_+^\times$. In contrast, the topology $K_- = K_\emptyset$ corresponding to the empty set will be called the* minimal topology *on $\mathbb{N}_+^\times$.*

Definition 2.6. *Let J be a Grothendieck topology on $\mathtt{Comm}^{\mathrm{op}}$ and K a Grothendieck topology on $\mathbb{N}_+^\times$. Let A be an Azumaya algebra with center C. We say that a family of morphisms $\{A \to A_{ir}\}_{i \in I, r \in R_i}$ in $\mathtt{Azu}$ covers A if the following two conditions are satisfied.*

- *For a fixed $i \in I$ all maps $A \to A_{ir}$ induces the same map $C \to C_i$ on the centers, and the family $\{C \to C_i\}_{i \in I}$ is a covering for J.*
- *For all $i \in I$, $A \otimes_C C_i$ is of constant degree n_i, A_{ir} is of constant degree n_{ir} for all $r \in R_i$, and $\{n_i \to n_{ir}\}_{r \in R_i}$ is a covering for K for all $r \in R_i$.*

Now we define J_K by

$$S \in J_K(A) \Leftrightarrow S \text{ contains a family } \{A \to A_{ir}\}_{i \in I, r \in R_i} \text{ covering } A$$

for each sieve S on A in the category $\mathtt{Azu}$.

We want to prove that under certain conditions the above collection of sieves J_K is a Grothendieck topology on $\mathtt{Azu}$. We can reformulate the axioms of a Grothendieck topology using Definition 2.6. Take an Azumaya algebra A and a sieve S in $J_K(A)$. Consider a family $\{A \to A_{ir}\}_{i \in I, r \in R_i} \subseteq S$ as in the definition, with corresponding coverings $\{C \to C_i\}_{i \in I}$ and $\{n_i \to n_{ir}\}_{r \in R_i}$ for all $i \in I$.

(1) There exists a covering $\{C \to C_i'\}_{i \in I}$ for J such that $A \otimes_C C_i'$ is of constant degree for each $i \in I$.
(2) For each $h : A \to B$, there is a covering $\{B \to B_{pl}\}_{p \in P, l \in L_p}$ as in the definition, such that each composition $A \to B_{pl}$ factors along some morphism $A \to A_{ir}$.
(3) Suppose that we have a covering $\{B^h \to B_{pl}^h\}_{p \in P, l \in L_p}$ for each $h : A \to B^h$ in S, such that each $A \to B_{pl}^h$ is contained in some sieve R. Then R contains a covering.

Theorem 2.7. *Let J be a Grothendieck topology on $\mathtt{Comm}^{\mathrm{op}}$ and K a Grothendieck topology on $\mathbb{N}_+^\times$. Suppose that*

(1) J is finer than the étale topology, or
(2) J is finer than the Zariski topology and K is stable under multiplication.

Then the collection of sieves J_K of Definition 2.6 defines a topology on $\mathtt{Azu}^{\mathrm{op}}$. Moreover, two such topologies J_K and $J'_{K'}$ agree only if $J = J'$ and $K = K'$.

Proof. We prove the three axioms for a Grothendieck topology, in the second form mentioned above.

(1). Take a Zariski covering $\{C \to C_i'\}_{i \in I}$ such that $A \otimes_C C_i'$ is of constant degree n_i', and consider the covering $\{A \to A \otimes_C C_i'\}_{i \in I}$. This corresponds to the covering $\{C \to C_i'\}_{i \in I}$ for J and the trivial covering $\{n_i' \to n_i'\}$ for K, for each $i \in I$.

(2). Case (1). By possibly refining the family $\{C \to C_i\}_{i \in I}$, we can assume that $A_{ir} = \mathrm{M}_{n_i r}(C_i)$. Take a covering $\{D \to D_{il}\}_{i \in I, l \in L_i}$ refining $\{D \to D \otimes_C C_i\}_{i \in I}$. We can then assume that $B \otimes_D D_{il} = \mathrm{M}_{n_i k}(D_{il})$ for some natural number k depending on $i \in I$ and $l \in L_i$. Now it suffices to consider the covering

$$\{B \to \mathrm{M}_{n_i \operatorname{lcm}(k,r)}(D_{il})\}_{i \in I, l \in L_i}.$$

Case (2). Take a covering $\{D \to D_{il}\}_{i \in I, l \in L_i}$ refining $\{D \to D \otimes_C C_i\}_{i \in I}$. We can then assume that $B_{il} = B \otimes_D D_{il}$ is of constant degree $n_i k_{il}$. In this case consider the covering

$$\{B \to B_{il} \otimes_{A_i} A_{ir}\}_{i \in I, r \in R_i, l \in L_i},$$

where we write $A_i = A \otimes_C C_i$. Note that $B_{il} \otimes_{A_i} A_{ir}$ is of constant degree $n_{ir} k_{il}$ over its center D_{il}.

(3). Consider specifically the morphisms $h : A \to A_{ir}$ for each $i \in I$ and $r \in R_i$. Then by the assumption there is a covering $\{A_{ir} \to A_{irl}\}_{l \in L_r}$ of A_{ir} and moreover the compositions $A \to A_{irl}$ are in R for all $i \in I$, $r \in R_i$, $l \in L_r$. So $R \in J_K(A)$ because it contains the covering

$$\{A \to A_{irl}\}_{i \in I, r \in R_i, l \in L_r}.$$

The last statement is immediate if we can recover both J and K from J_K. This is precisely the content of the following proposition. $\qquad\square$

Proposition 2.8. *We can recover J as the collection of sieves $\{C \to Z(A_i)\}_{i \in I}$ for each sieve $\{C \to A_i\}_{i \in I}$ in J_K. Similarly, we can recover K as the collection of sieves $\{n \to n_i\}_{i \in I}$ for each sieve $\{\mathrm{M}_n(\mathbb{C}) \to A_i\}_{i \in I}$ in J_K with A_i of constant degree n_i.*

Proof. Let S_0 be a sieve in $J(C)$. The elements of S_0 generate a sieve $S \in J_K(C)$. It is easy to see that taking centers gives back S_0. Conversely, for any sieve $\{C \to A_i\}_{i \in I}$ in $J_K(C)$ we have that $\{C \to Z(A_i)\}_{i \in I}$ is a sieve in $J(C)$, because it contains a family $\{C \to C_i\}_{i \in I}$ generating a sieve in $J(C)$. Recovering K is analogous. $\qquad\square$

3. The sheaf property and representability

Now, consider an affine $\mathbb{C}$-algebra R and the corresponding set-valued functor

$$\begin{aligned} \mathtt{Azu} &\longrightarrow \mathtt{Sets} \\ A &\longmapsto \mathtt{Alg}(R, A) \end{aligned},$$

which we will denote by $\mathtt{Alg}(R, -)$. In this section, we will show that this functor is in fact a sheaf with respect to the maximal flat topology as defined above. This will imply that it is also a sheaf for any coarser Grothendieck topology, e.g. the Grothendieck topologies J_K as in the previous section where J is the Zariski or étale topology, and where K is an arbitrary Grothendieck topology on $\mathbb{N}_+^\times$. It

immediately follows that for each Azumaya algebra A the set-valued functor

$$\mathtt{Comm}_C \longrightarrow \mathtt{Sets}$$

$$D \longmapsto \mathtt{Alg}(R, A \otimes_C D)$$

on the category of commutative C-algebras is also a sheaf for the flat topology. This sheaf turns out to be representable by a scheme, which we will call the *Azumaya representation scheme* of R associated to A. We will give a ring-theoretic description of the coordinate ring of this scheme and discuss its geometric structure.

Recall that, by definition, a sieve is a covering sieve for the flat topology if it contains a family $\{C \to C_i\}_{i \in I}$ with

$$C \to \prod_{i \in I} C_i$$

faithfully flat.

Lemma 3.1. *Let $A \to B$ be a morphism in $\mathtt{Azu}$. Then the following are equivalent:*

(1) $A \to B$ is left faithfully flat;
(2) $A \to B$ is right faithfully flat;
(3) $Z(A) \to Z(B)$ is faithfully flat;

Moreover, if any of the above is satisfied, then the sequence

$$0 \longrightarrow A \longrightarrow B \rightrightarrows B \otimes_A B$$

is exact.

Proof. $(1) \Leftrightarrow (3)$: by the double centralizer theorem, the functor $- \otimes_A B$ is equivalent to $- \otimes_{Z(A)} Z(B) \otimes_{Z(B)} B^A$. Because B^A is always faithfully flat over its center $Z(B)$, we get that $A \to B$ is left faithfully flat if and only if $Z(A) \to Z(B)$ is faithfully flat.

$(2) \Leftrightarrow (3)$: analogously.

The sequence in the lemma appeared in [2] and is a noncommutative version of the Amitsur complex. By faithfully flatness, it is enough to check that

$$(1) \qquad 0 \longrightarrow B \xrightarrow{b \mapsto b \otimes 1} B \otimes_A B \underset{b \otimes b' \mapsto b \otimes 1 \otimes b'}{\overset{b \otimes b' \mapsto b \otimes b' \otimes 1}{\rightrightarrows}} B \otimes_A B \otimes_A B$$

is exact. The morphism $B \to B \otimes_A B$ has a retraction given by the multiplication morphism. In particular it is injective. Further, suppose $\sum_i b_i \otimes b'_i \otimes 1 = \sum_i b_i \otimes 1 \otimes b'_i$. Applying multiplication to the first two tensor factors, we get that $\sum b_i b'_i \otimes 1 = \sum_i b_i \otimes b'_i$. But this means that $\sum_i b_i \otimes b'_i$ lies in the image of $B \to B \otimes_A B$. $\qquad \square$

Proposition 3.2. *The functor $\mathtt{Alg}(R, -)$ on $\mathtt{Azu}$ is a sheaf for the maximal flat topology on $\mathtt{Azu}^{\mathrm{op}}$ (and hence for any coarser Grothendieck topology).*

Proof. We need to prove that we can glue sections in a unique way whenever they agree locally. It is enough to show that

$$(2) \qquad 0 \longrightarrow \mathtt{Alg}(R, A) \longrightarrow \prod_{i \in I} \mathtt{Alg}(R, A_i) \rightrightarrows \prod_{i,j \in I} \mathtt{Alg}(R, A_i \otimes_A A_j)$$

is exact for every family of morphisms $\{A \to A_i\}_{i \in I}$ in $\mathtt{Azu}$ such that $A \to \prod_{i \in I} A_i$ is faithfully flat (note that $\prod_{i \in I} A_i$ is not necessarily Azumaya). We know that

$\mathtt{Alg}(R, -)$ commutes with limits of rings (in particular with categorical kernels, products and inverse limits), so it is enough to show that

$$(3) \qquad 0 \longrightarrow A \longrightarrow \prod_{i \in J} A_i \rightrightarrows \prod_{i,j \in J} A_i \otimes_A A_j$$

is exact, for every finite subset $J \subseteq I$ such that $A \to \prod_{i \in J} A_i$ is still faithfully flat. But this follows from Lemma 3.1. $\qquad\square$

If we fix the Azumaya algebra A we can consider the category $\mathtt{Azu}_A$ of Azumaya algebras B equipped with center-preserving algebra morphism $A \to B$. Morphisms in this category are algebra morphisms $B \to B'$ making the triangle

$$A \diagup \begin{matrix} B \\ \downarrow \\ B' \end{matrix}$$

commute. It is well known that any Grothendieck topology on $\mathtt{Azu}^{\mathrm{op}}$ restricts to a Grothendieck topology on the "comma category" $\mathtt{Azu}_A^{\mathrm{op}}$. But now we can consider the composition of geometric morphisms

$$(4) \qquad \mathtt{Sh}(\mathtt{Azu}^{\mathrm{op}}) \longrightarrow \mathtt{Sh}(\mathtt{Azu}_A^{\mathrm{op}}) \longrightarrow \mathtt{Sh}(\mathtt{Azu}_C^{\mathrm{op}}) \longrightarrow \mathtt{Sh}(\mathtt{Comm}_C^{\mathrm{op}}) \ ,$$

where the middle arrow is given by $F \mapsto F(A \otimes_C -)$ and the others are given by restriction. Here we work in the (maximal) flat topology. The image of $\mathtt{Alg}(R, -)$ along this composition is the functor

$$\mathtt{Comm}_C \longrightarrow \mathtt{Sets}$$
$$D \longmapsto \mathtt{Alg}(R, A \otimes_C D)$$

which is therefore also a sheaf for the flat topology. In the rest of the section, we will show that this sheaf is even representable by an affine scheme and describe its coordinate ring and basic properties.

For a $\mathbb{C}$-algebra S, Artin S-bimodules (see [2] or [14]) are vector spaces M equipped with compatible left and right S-action, and generated by invariants M^S as a two-sided S-module. Artin S-algebras are algebras R equipped with a structure morphism $\phi_R : S \to R$ making R into an Artin bimodule. Equivalently, ϕ_R is a Procesi extension [14]. We will denote by $\mathtt{Bimod}_S$ the category of Artin S-bimodules with morphisms that are S-linear on both sides. Similarly, $\mathtt{Alg}_S$ will denote the category of Artin S-algebras with S-linear algebra morphisms.

Now let C be a commutative algebra and A an Azumaya algebra over C. Note that this makes A into an Artin C-algebra. In [2] it is shown that there are equivalences of categories

$$(5) \qquad \mathtt{Bimod}_C \underset{(-)^A}{\overset{A \otimes_C -}{\rightleftarrows}} \mathtt{Bimod}_A \ ,$$

(6)
$$\mathtt{Alg}_C \underset{(-)^A}{\overset{A \otimes_C -}{\rightleftarrows}} \mathtt{Alg}_A \quad .$$

Observe that in the case of an Azumaya algebra A we can reformulate Artin's definition, by invoking the double centralizer theorem. For an Azumaya A with center C, Artin A-bimodules are the ones such that the induced C-action is symmetric. Similarly, Artin A-algebras are the algebras with structure morphism sending C into the center.

In order to describe the functor $\mathtt{Alg}(R, A \otimes_C -)$, we have to introduce a generalization of the root algebra $\sqrt[n]{R}$, used in studying n-dimensional representations of R, see [3] or [17]. Note that morphisms $R \to A$ with A Azumaya over C are the same as C-algebra morphisms $R \otimes C \to A$, so we may assume that R is a C-algebra.

Definition 3.3. *Let A be an Azumaya algebra with center C and let R be a C-algebra. Then the A-th root algebra of R, denoted $\sqrt[A]{R}$, is defined to be*

(7)
$$\sqrt[A]{R} = (R *_C A)^A.$$

Here $*_C$ denotes the coproduct of C-algebras, i.e. the pushout of the diagram

$$\begin{array}{ccc} C & \longrightarrow & R \\ \downarrow & & \\ A & & \end{array}$$

in the category of rings.

Proposition 3.4. *The functor $\sqrt[A]{-} : \mathtt{Alg}_C \to \mathtt{Alg}_C$ is left adjoint to tensoring $- \otimes_C A : \mathtt{Alg}_C \to \mathtt{Alg}_C$.*

Proof. Note that we can write the functor $A \otimes_C - : \mathtt{Alg}_C \to \mathtt{Alg}_C$ as a composition

(8)
$$\mathtt{Alg}_C \xrightarrow{A \otimes_C -} \mathtt{Alg}_A \longrightarrow \mathtt{Alg}_C,$$

where the first functor is the equivalence (6) and the second functor is the forgetful one. Being an equivalence, the first one has its quasi-inverse $(-)^A$ as left adjoint. Further, one can check that the second one has left adjoint $A *_C -$. The proposition follows from composition of adjunctions. $\qquad\square$

Theorem 3.5. *If A is a constant degree n Azumaya algebra with center C, then for every algebra R there is an affine C-scheme $\mathtt{rep}_A(R)$, which we call the Azumaya representation scheme of R with respect to A, representing the functor*

$$\mathtt{Comm}_C \longrightarrow \mathtt{Sets} \qquad D \mapsto \mathtt{Alg}_{\mathbb{C}}(R, A \otimes_C D).$$

Proof. Define the Azumaya representation scheme as

$$\mathtt{rep}_A(R) = \mathtt{Spec}(\sqrt[A]{R \otimes C})_{\mathrm{ab}}.$$

To check that this represents the given functor, use Proposition 3.4 and the fact that $- \otimes C$ and ab are both adjoint to the appropriate forgetful functors. $\qquad\square$

Proposition 3.6. *Let A and B be Azumaya algebras with center C. Let R be a C-algebra and S a $\mathbb{C}$-algebra.*

(1) *There are natural isomorphisms* $\sqrt[A]{\sqrt[B]{R}} \simeq \sqrt[A\otimes_C B]{R} \simeq \sqrt[B]{\sqrt[A]{R}}$ *of C-algebras.*
(2) *For any morphism of commutative algebras $C \to D$, we get natural isomorphisms* $\sqrt[A\otimes_C D]{R \otimes_C D} \simeq \sqrt[A]{R} \otimes_C D$.
(3) *Suppose that A is of constant degree n. Then $\sqrt[A]{S \otimes C}$ is, étale locally on C, isomorphic to $\sqrt[n]{S} \otimes C$.*
(4) *A C-linear morphism $A \to B$ induces a C-linear morphism $\sqrt[B]{R} \to \sqrt[A]{R}$, functorial in R.*

Proof. All statements follow by invoking the Yoneda Lemma and some computations. We prove (1) as an example. For any C-algebra S, we have

$$\mathtt{Alg}_C(\sqrt[A]{\sqrt[B]{R}}, S) \simeq \mathtt{Alg}_C(\sqrt[B]{R}, A \otimes_C S)$$
$$\simeq \mathtt{Alg}_C(R, B \otimes_C A \otimes_C S)$$
$$\simeq \mathtt{Alg}_C(\sqrt[A\otimes_C B]{R}, S),$$

so by the Yoneda Lemma we have $\sqrt[A]{\sqrt[B]{R}} \simeq \sqrt[A\otimes_C B]{R}$. Similarly for $\sqrt[B]{\sqrt[A]{R}}$. $\square$

Note that, by part (3), $\mathtt{rep}_A(R)$ is étale locally on C isomorphic to $\mathtt{rep}_n(R) \times \mathrm{Spec}(C)$. So Azumaya representation schemes are *twisted* versions of representation schemes, similarly to Azumaya algebras being twisted versions of matrix algebras.

Example 3.7. *Let A be an Azumaya algebra with center C. Then the C-linear automorphisms of A form a sheaf on $\mathrm{Spec}(C)$, which is represented by $\mathtt{rep}_A(A)$.*

Example 3.8 (Free algebras)**.** *Consider the diagram of adjunctions*

where the dashed arrows are right adjoint to the solid ones. The unlabeled functors are forgetful functors. It is obvious that the diagram of dashed arrows is commutative and by uniqueness of adjoint functors this implies that the diagram of solid arrows is commutative too. In particular we have

$$(9) \qquad \sqrt[A]{(\mathtt{T}_{\mathbb{C}}V) \otimes C} \simeq \mathtt{T}_C(V \otimes A^e)^A \simeq \mathtt{T}_C(V \otimes A^\vee)$$

for any vector space V. Here $A^\vee$ is the C-linear dual of A. More generally, for any C-module M we have

$$(10) \qquad \sqrt[A]{\mathtt{T}_C M} \simeq \mathtt{T}_C(M \otimes_C A^\vee)$$

4. Twisted quiver representations

In this section we will describe the α-dimensional representation stack

$$[\mathtt{rep}_\alpha(R)/\mathrm{PGL}(\alpha)]$$

where R is an affine $\mathbb{C}^k$-algebra and $\alpha = (d_1, \ldots, d_k)$ is a dimension vector of total dimension $n = \sum_i d_i$. We will always assume that $d_i \neq 0$ for all $i \in \{1, \ldots, k\}$. In

[11] it was shown that the pointed set $H^1_{et}(\mathtt{Spec}(C), \mathrm{PGL}(\alpha))$ classifies isomorphism classes of Azumaya algebras B with center C, together with a distinguished embedding $\mathbb{C}^k \hookrightarrow B$ having the property that for an étale extension $C \longrightarrow D$ splitting A this embedding is conjugated to the diagonal embedding of $\mathbb{C}^k$ in $M_n(D)$ with the i-th idempotent having rank d_i. We will then call B an Azumaya algebra of dimension vector $\alpha = (d_1, \ldots, d_k)$. The images of the standard idempotents of $\mathbb{C}^k$ in A will be called $e_1, \ldots, e_k$. By computing traces étale locally, we find that then $\mathrm{tr}(e_i) = d_i$ for each $i \in \{1, \ldots, k\}$. Conversely, any idempotent in A has locally constant trace and a complete orthogonal set of idempotents $e_1, \ldots, e_k$ makes A into an Azumaya algebra of dimension vector α if $\mathrm{tr}(e_i) = d_i$ for all $i \in \{1, \ldots, k\}$.

One way to construct Azumaya algebras of dimension vector $\alpha = (d_1, \ldots, d_k)$ is as follows. Take an Azumaya algebra A of degree m and take $P_1, \ldots, P_n$ projective C-modules of rank md_i, equipped with a C-linear right A-action. Then it is clear that

$$B = \mathrm{End}_{A^{\mathrm{op}}} \left(\bigoplus_{i=1}^k P_i \right)$$

is again an Azumaya algebra. It has a standard complete orthogonal set of idempotents corresponding to the projections onto $P_1, \ldots, P_k$, making it into an Azumaya algebra of dimension vector $\alpha = (d_1, \ldots, d_k)$. Moreover, this construction does not change the Brauer class: B is the endomorphism ring associated to the progenerator $P_1 \oplus \cdots \oplus P_k$ and consequently B is Morita equivalent to A. Similarly, for each $i \in \{1, \ldots, k\}$, there is a Morita equivalence from A to $\mathrm{End}_{A^{\mathrm{op}}}(P_i)$.

It is not difficult to see that every Azumaya algebra B of dimension vector $\alpha = (d_1, \ldots, d_k)$ is of the above form. Denote the idempotents in B by $e_1, \ldots, e_k$. Then

$$(11) \qquad\qquad B \simeq \mathrm{End}_{B^{\mathrm{op}}} \left(\bigoplus_{i=1}^k e_i B \right)$$

and the idempotents $e_1, \ldots, e_k$ on the left correspond to the standard idempotents on the right. Moreover, for each $i \in \{1, \ldots, k\}$, the rank of $e_i B$ as a C-module is nd_i (this can be checked étale locally). This proves the following result. Recall that the *period* of an Azumaya algebra is its order in the Brauer group.

Proposition 4.1. *Let B be an Azumaya algebra of dimension vector $\alpha = (d_1, \ldots, d_k)$ over a commutative algebra C. Then the period of B divides $d = \gcd(d_1, \ldots, d_k)$.*

Proof. For each $i \in \{1, \ldots, k\}$, there is a Morita equivalence from B to $e_i B e_i$ (given by $e_i B$). The period always divides the degree of any Azumaya in the Brauer class, so in this case it divides $\gcd(d_1, \ldots, d_k)$. $\qquad\square$

In particular, B has trivial Brauer class whenever $d = 1$. This reproves the fact that $\mathrm{PGL}(\alpha)$-torsors are Zariski locally trivial in this case [11].

For $d > 1$, the above discussion relates the study of idempotents to some natural questions regarding existence of Azumaya algebras with given degree and given Brauer class:

- In a given class of $\mathtt{Br}(C)$, does there exist an Azumaya algebra with degree equal to the index? Here the index is the greatest common divisor of the degrees of Azumaya algebras in the Brauer class. Antieau and Williams

constructed a counterexample with C regular, finitely generated and of Krull dimension 6 [1, Corollary 1.2].
- In a given class of $\mathtt{Br}(C)$, is the period equal to the index? This even fails for fields. However, de Jong proved in [8] that for fields of transcendence degree 2, the equality still holds. If the transcendence degree is 1, then there is nothing to prove by Tsen's Theorem.

There are other useful ways to write B as an endomorphism algebra. Take for example $A = e_1 B e_1$ and $P_i = e_i B e_1$ for $i \in \{1, \ldots, k\}$. Each P_i is projective as right A-module and of rank $d_1 d_i$ (this can again be checked étale locally). Moreover, we have an isomorphism

$$B \cong \mathrm{End}_{A^{\mathrm{op}}} \left(\bigoplus_{i=1}^{k} P_i \right).$$

Of course, the choice of the first idempotent is here irrelevant. The realization here is that any $\mathbb{C}^k$-linear automorphism of B is inner with respect to the induced automorphism on A, in a sense that we will make precise now.

Proposition 4.2. *Let B be an Azumaya algebra of dimension vector $\alpha = (d_1, \ldots, d_k)$ over a commutative algebra C. Take $A = e_1 B e_1$ and $P_i = e_i B e_1$ for all $i \in \{1, \ldots, k\}$, equipped with a right A-action given by multiplication in B. Then the $\mathbb{C}^k$-linear automorphisms of B correspond bijectively to tuples $(\sigma, \{\sigma_i\}_{i=2}^{k})$ with*

- *σ an algebra automorphism of A;*
- *σ_i a right σ-linear isomorphism $P_i \to P_i$, for all $i \in \{2, \ldots, k\}$, i.e. a C-linear isomorphism such that*

$$(12) \qquad \sigma_i(x \cdot a) = \sigma_i(x) \cdot \sigma(a) \quad \textit{for all } x \in P_i \textit{ and } a \in A.$$

Proof. Let ψ be a $\mathbb{C}^k$-linear automorphism of B. It restricts to an algebra automorphism σ of $A = e_1 B e_1$, because the idempotents $e_1, \ldots, e_k$ are preserved. Similarly, ψ restricts to C-linear isomorphism $\sigma_i : P_i \to P_i$ for $i \in \{2, \ldots, k\}$. By the multiplicativity of ψ, formula (12) holds.

The inverse construction is given as follows. For a tuple $(\sigma, \{\sigma_{i=2}^{k}\})$, set $P_1 = A$ and $\sigma_1 = \sigma$. Then we can construct a $\mathbb{C}^k$-linear isomorphism

$$\bigoplus_{i=1}^{k} P_i \xrightarrow{\phi} \bigoplus_{i=1}^{k} P_i$$

by applying σ_i component-wisely. This ϕ determines an automorphism ψ of

$$B \simeq \mathrm{End}_{A^{\mathrm{op}}} \left(\bigoplus_{i=1}^{k} P_i \right)$$

by conjugation, i.e. $\psi(b) = \phi b \phi^{-1}$ for all $b \in B$. Here both ϕ and b are interpreted as endomorphisms of $\bigoplus_{i=1}^{k} P_i$, and it is easy to see that $\phi b \phi^{-1}$ is indeed right A-linear. $\square$

Recall from [6, II.6] that to a C-linear automorphism $\sigma : A \to A$ of an Azumaya algebra, we can associate a rank 1 projective C-module

$$(A_\sigma)^A,$$

where A_σ is equal to A as left A-module, but with new right action

$$x \cdot a := x \sigma(a) \quad \text{for } x, a \in A.$$

Using this construction, one can identify the C-linear outer automorphisms of A with an n-torsion subgroup of the Picard group of C. Moreover, for Dedekind domains, the inclusion

$$\mathtt{Out}_C(A) \subseteq \mathtt{Pic}_n(C)$$

is an equality by the Steinitz Isomorphism Theorem (see [16, Lemma 2]). With this interpretation of the outer automorphisms, we can prove the following corollary of Proposition 4.2.

Proposition 4.3. *Let B be an Azumaya algebra of dimension vector $\alpha = (d_1, \ldots, d_k)$ over a commutative algebra C. Let ψ be a $\mathbb{C}^k$-linear automorphism of B. Then the order of ψ in $\mathtt{Out}_C(B)$ divides $d = \gcd(d_1, \ldots, d_k)$.*

Proof. Take $A = e_1 B e_1$ and $P_i = e_i B e_1$ for each $i \in \{1, \ldots, k\}$, as in Proposition 4.2. Moreover, for each $i \in \{1, \ldots, k\}$, set $B_i = e_i B e_i = \mathrm{End}_{A^{\mathrm{op}}}(P_i)$, in particular $B_1 = A$. We then have subgroups

$$\mathtt{Out}_C(B_1), \ldots, \mathtt{Out}_C(B_k), \mathtt{Out}_C(B) \subseteq \mathtt{Pic}(C).$$

Here $\mathtt{Out}_C(B_i)$ is d_i-torsion for each $i \in \{1, \ldots, k\}$. It suffices to prove that

$$\psi \in \bigcap_{i=1}^k \mathtt{Out}_C(B_i).$$

Consider the tuple $(\sigma, \{\sigma_i\}_{i=2}^k)$ as in Proposition 4.2 and take $\sigma_1 = \sigma$. Each σ_i is a σ-linear automorphism of P_i. The restriction of ψ to B_i is an algebra automorphism $\tilde{\sigma}_i$, which is given by conjugation with σ_i. We will show that the class of $\tilde{\sigma}_i$ in $\mathtt{Pic}(C)$ coincides with the class of σ. In other words, we want that

$$(A_\sigma)^A \cong (B_{i,\tilde{\sigma}_i})^{B_i}$$

as C-modules. Note that $A^{\mathrm{op}} \otimes_C B_i \simeq \mathrm{End}_C(P_i)$. An element $f \in (B_{i,\tilde{\sigma}_i})^{B_i}$ is a C-linear morphism such that $f\tilde{\sigma}_i(b) = bf$ for every $b \in B_i$. So $f\sigma_i b \sigma_i^{-1} = bf$, but this means that $f\sigma_i$ commutes with every element of B_i. By the Double Centralizer Theorem, $f\sigma_i$ is an element of A^{op}, and now it is easy to see that

$$f\sigma_i \in (A^{\mathrm{op}}_{\sigma^{-1}})^{A^{\mathrm{op}}} \cong (A_\sigma)^A.$$

This gives an isomorphism between $(B_{i,\tilde{\sigma}_i})^{B_i}$ and $(A_\sigma)^A$. An analogous computation shows that $(B_\psi)^B \cong (A_\sigma)^A$, so the class of ψ also agrees with the class of σ. This implies that $\psi \in \bigcap_{i=1}^k \mathtt{Out}_C(B_i)$. $\qquad\square$

In particular, if $d = 1$, then all $\mathbb{C}^k$-linear automorphisms of B are inner.

Let B be an Azumaya algebra of dimension vector $\alpha = (d_1, \ldots, d_k)$ over a commutative algebra C. As before, we associate to B the Azumaya algebra $A = e_1 B e_1$ and the projective right A-modules $P_i = e_i B e_1$, for $i \in \{2, \ldots, k\}$. For the sake of simplifying notation, we set $P_1 = A$. The isomorphism

$$B \cong \mathrm{End}_{A^{\mathrm{op}}} \left(\bigoplus_{i=1}^k P_i \right)$$

shows that B is completely determined by the tuple $(A, \{P_i\}_{i=2}^k)$, but it is possible that different tuples give rise to isomorphic Azumaya algebras. Such tuples are, however, strongly related, as we will show now.

Outer automorphisms will again play an important role. For an Azumaya algebra A and an automorphism σ, we already defined A_σ. For any right A-module M, we can similarly define M_σ as being equal to M but with new right action

$$m \cdot a := m\sigma(a) \quad \text{for } m \in M, a \in A.$$

Clearly, $M_\sigma \cong M \otimes_A A_\sigma$.

Proposition 4.4. *Let $\alpha = (d_1, \ldots, d_k)$ be a dimension vector. Take A and A' Azumaya algebras of degree d_1, and let P_i (resp. P_i') be projective C-modules of rank $d_1 d_i$ equipped with C-linear right A-action (resp. right A'-action), for $i \in \{2, \ldots, k\}$. We set $P_1 = A$ and $P_1' = A'$. Suppose that there is a $\mathbb{C}^k$-linear algebra isomorphism*

$$\text{End}_{A^{\text{op}}} \left(\bigoplus_{i=1}^k P_i \right) \xrightarrow{\ \psi\ } \text{End}_{A'^{\text{op}}} \left(\bigoplus_{i=1}^k P_i' \right) .$$

Then ψ restricts to an algebra isomorphism $\sigma : A \to A'$, so we can assume that $A = A'$ and that σ is an automorphism. But then $P_i' \cong P_{i,\sigma^{-1}}$ for all $i \in \{2, \ldots, k\}$.

Proof. The algebra automorphism ψ restricts to an algebra automorphism $\sigma : A \to A'$ and to right σ-linear isomorphisms $\sigma_i : P_i \to P_i'$ as in Proposition 4.2. Now assume that $A = A'$ and that σ is an automorphism. Then each σ_i can be interpreted as a right A-linear isomorphism from $P_{i,\sigma^{-1}}$ to P_i', so $P_i' \cong P_{i,\sigma^{-1}}$. $\qquad\square$

For the remaining part of this section, fix a quiver Q with k vertices and let $R = \mathbb{C}Q/I$ be the its path algebra modulo an ideal I. Every affine $\mathbb{C}^k$-algebra R can be written in this form. Consider a dimension vector $\alpha = (d_1, \ldots, d_k)$. Then we would like to study the points of the quotient stack

$$[\texttt{rep}_\alpha(R)/\text{PGL}(\alpha)].$$

We use the correspondence between $\text{PGL}(\alpha)$-torsors and Azumaya algebras of dimension vector α, as discussed above. It is now a straightforward extension of the results in [9] to give an algebraic description of this quotient stack. The C-points for a commutative algebra C are given by the $\mathbb{C}^k$-linear algebra morphisms $\phi : R \longrightarrow A$, where A varies over the Azumaya algebras with dimension vector α and center C. Isomorphisms between C-points ϕ and ϕ' are given by a $\mathbb{C}^k$-linear isomorphism $\psi : A \longrightarrow A'$ making the triangle

$$
\begin{array}{ccc}
 & & A \\
 & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle \psi} \\
R & & \\
 & \underset{\phi'}{\searrow} & A'
\end{array}
$$

commute.

The previous results in this section will allow us to give a more representation-theoretic description of the C-points in terms of what we will call *twisted representations*. Recall that an Azumaya algebra B with dimension vector $(d_1, \ldots, d_k)$ is determined by the tuple $(A, \{P_i\}_{i=2}^k)$, with $A = e_1 B e_1$ an Azumaya algebra of degree d_1 and $P_i = e_i B e_1$ a projective C-module of rank $d_1 d_i$, equipped with a right A-action. This motivates the following definition. We denote by Q^1 the set of arrows of the quiver Q.

Definition 4.5. *Let C be a commutative algebra and let $\alpha = (d_1, \ldots, d_k)$ be a dimension vector. We define a* twisted representation *of $\mathbb{C}Q/I$ over C with dimension vector α to be a triple $(A, \{P_i\}_{i=2}^k, \{\rho_a\}_{a \in Q_1})$, with*

- *(1) A an Azumaya algebra over C of degree d_1;*
- *(2) P_i a projective C-module of rank $d_1 d_i$ equipped with a right A-action, for all $i \in \{2, \ldots, k\}$ (we write $P_1 = A$);*
- *(3) $\rho_a : P_i \to P_j$ a right A-linear morphism for every arrow $a : i \to j$ in Q, satisfying*

$$f(\rho_{a_1}, \ldots, \rho_{a_l}) = 0$$

for every polynomial $f(a_1, \ldots, a_l) \in I$ in l arrows $a_1, \ldots, a_l \in Q^1$.

An isomorphism *between two twisted representations $(A, \{P_i\}_{i=2}^k, \{\rho_a\}_{a \in Q^1})$ and $(A', \{P_i'\}_{i=2}^k, \{\rho_a'\}_{a \in Q^1})$ is a tuple $(\sigma, \{\sigma_i\}_{i=2}^k)$, with*

- *(1) $\sigma : A \to A'$ an isomorphism of C-algebras;*
- *(2) $\sigma_i : P_i \to P_i'$ a σ-linear isomorphism for each $i \in \{2, \ldots, k\}$, i.e. such that*

$$\sigma_i(ma) = \sigma_i(m)\sigma(a) \text{ for all } m \in P_i \text{ and } a \in A,$$

and $\sigma_1 = \sigma$, making the diagram

$$
\begin{array}{ccc}
P_i & \xrightarrow{\sigma_i} & P_i' \\
\rho_a \downarrow & & \downarrow \rho_a' \\
P_j & \xrightarrow{\sigma_j} & P_j'
\end{array}
$$

commute, for each $a : i \to j$ in Q.

So a twisted representation can be seen as a certain representation of the quiver in the category of projective right A-modules, or equivalently in the category of $|A|$-twisted sheaves, where $|A|$ is the Brauer class of A (see [7], [4], [12]).

Twisted representations behave in a functorial way: if $(A, \{P_i\}_{i=1}^k, \{\rho_a\}_{a \in Q^1})$ is a twisted representation over C and $\phi : C \to D$ is a morphism of commutative algebras, then the base change $(A \otimes_C D, \{P_i \otimes_C D\}_{i=1}^k, \{\rho_a \otimes_C D\}_{a \in Q^1})$ is a twisted representation over D. Further, isomorphisms between twisted representations are still isomorphisms after extension of scalars. Therefore we can define a functor

$$F_R : \mathtt{Comm} \longrightarrow \mathtt{Groupoids}$$

by setting

$$(13) \qquad F_R(C) = \left\{ \begin{array}{c} \text{twisted } R\text{-representations over } C \\ \text{of dimension vector } \alpha \\ \text{and isomorphisms between them} \end{array} \right\}.$$

From a twisted representation $(A, \{P_i\}_{i=2}^k, \{\rho_a\}_{a \in Q^1})$, we can construct an algebra morphism

$$\phi : R \to \mathrm{End}_{A^{\mathrm{op}}} \left(\bigoplus_{i=1}^k P_i \right)$$

given by

$$\phi(a) = \rho_a \quad \text{for all } a \in Q^1.$$

Further, by the results of the previous section, any Azumaya algebra of dimension vector $(d_1, \ldots, d_k)$ can be written as such an endomorphism algebra for some tuple

$(A, \{P_i\}_{i=2}^k)$. It was moreover shown that any isomorphism of these corresponding endomorphism algebras is given by a tuple $(\sigma, \{\sigma_i\}_{i=2}^k)$. From this all we deduce the following theorem.

Theorem 4.6. *Let $R = \mathbb{C}Q/I$ be the path algebra of a quiver Q modulo an ideal I. Let $\alpha = (d_1, \ldots, d_k)$ be a dimension vector, with k the number of vertices of Q. The functor F_R of twisted representations (13) is equivalent to the functor of points of the quotient stack*

$$[\mathtt{rep}_\alpha(R)/\mathrm{PGL}(\alpha)].$$

If $(\sigma, \{\sigma_i\}_{i=2}^k)$ is an isomorphism of twisted representations between

$$(A, \{P_i\}_{i=2}^k, \{\rho_a\}_{a \in Q^1}) \quad \text{and} \quad (A', \{P'_i\}_{i=2}^k, \{\rho'_a\}_{a \in Q^1}),$$

then by Proposition 4.4 we can assume that $A' = A$ and $P'_i = P_{i,\sigma^{-1}}$ for each $i \in \{2, \ldots, k\}$. We then get

$$\rho'_a = \sigma_j \rho_a \sigma_i^{-1}$$

for each arrow $a \in Q^1$ going from vertex i to vertex j. This gives a group action on twisted representations by conjugation, similarly to the case of ordinary representations of quivers.

Example 4.7. *If R is a basic finite dimensional algebra, then we can write R as a path algebra of a quiver Q modulo an admissible ideal I, so $R \cong \mathbb{C}Q/I$. An important family of representations of R is now given by the θ-stable representations for a stability vector θ and dimension vector α.*

Let $\mathtt{Spec}(D) \subseteq \mathtt{rep}_\alpha(R)$ be a $\mathrm{PGL}(\alpha)$-stable nonempty affine open subset of θ-stable representations, for some stability vector θ. Then $\mathtt{Spec}(D)$ is a $\mathrm{PGL}(\alpha)$-torsor over the quotient scheme $\mathtt{Spec}(D^{\mathrm{PGL}(\alpha)})$. This corresponds to an Azumaya algebra $\mathrm{M}_n(D)^{\mathrm{PGL}(\alpha)}$ of dimension vector α with center $D^{\mathrm{PGL}(\alpha)}$ and a $\mathbb{C}^k$-linear algebra morphism

$$R \longrightarrow \mathrm{M}_n(D)^{\mathrm{PGL}(\alpha)}.$$

The diagonal idempotents $e_1, \ldots, e_k$ of $\mathrm{M}_n(D)$ are invariant under the action of $\mathrm{PGL}(\alpha)$, so these give the desired idempotents in $\mathrm{M}_n(D)^{\mathrm{PGL}(\alpha)}$. By Proposition 4.1, the period of this Azumaya algebra divides $\gcd(d_1, \ldots, d_k)$. In particular, if $\gcd(d_1, \ldots, d_k) = 1$, then this Azumaya algebra is Zariski locally trivial, see [11].

References

[1] Benjamin Antieau and Ben Williams, *Unramified division algebras do not always contain Azumaya maximal orders*, Invent. math. **197** (2014) 47-56

[2] Michael Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Algebra **11** (1969) 532-563

[3] George Mark Bergman, *Coproducts and some universal ring constructions*, Trans. Amer. Math. Soc. **200** (1974) 33-88

[4] Andrei Căldăraru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, Ph.D. thesis, Cornell University (2000)

[5] Alain Connes and Caterina Consani, *Geometry of the arithmetic site*, `arXiv:1502.05580` (2015)

[6] Frank De Meyer and Edward Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics, vol. 181, Springer-Verlag, Berlin-New York, 1971

[7] Jean Giraud, *Cohomologie non abélienne*, Grundlehren Math. Wiss., vol. 179, Springer-Verlag, Berlin-New York, 1971

[8] Aise Johan de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71-94

[9] Lieven Le Bruyn, *Representation stacks, D-branes and noncommutative geometry*, Comm. Alg. **40** (2012) 3636-3651

[10] Lieven Le Bruyn, *Covers of the arithmetic site*, `arXiv:1602.01627` (2016)

[11] Lieven Le Bruyn and Aidan Schofield, *Rational invariants of quivers and the ring of matrix-invariants*, Perspectives in Ring Theory, NATO-ASI Series C-233 (1988) 21-29

[12] Max Lieblich, *Moduli of twisted sheaves*, Duke Math. J. **138** (2007), no. 1, 23-118.

[13] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic, a first introduction to topos theory*, Universitext, Springer-Verlag (1992)

[14] Claudio Procesi, *Rings with polynomial identities*, Pure and Appl. Math **17**, Marcel Dekker (1973)

[15] Claudio Procesi, *A formal inverse to the Cayley-Hamilton theorem*, J. Algebra **107** (1987) 63-74

[16] Irving Reiner, *Integral Representations of Cyclic Groups of Prime Order*, Proc. Amer. Math. Soc. **8** (1957), 142-146

[17] Aidan Schofield, *Representations of rings over skew fields*, LMS Lect. Note Series **92**, Cambridge Univ. Press (1985)

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