BRAUER-SEVERI MOTIVES AND DONALDSON-THOMAS INVARIANTS OF QUANTIZED THREEFOLDS

LIEVEN LE BRUYN

Abstract. Motives of Brauer-Severi schemes of Cayley-smooth algebras associated to homogeneous superpotentials are used to compute inductively the motivic Donaldson-Thomas invariants of the corresponding Jacobian algebras. We use this approach to test some conjectural exponential expressions for these invariants, proposed in [3].

1. Introduction

We fix a homogeneous degree d superpotential W in m non-commuting variables $X_1, \ldots, X_m$. For every dimension $n \geq 1$, W defines a regular functions, sometimes called the Chern-Simons functional

$$Tr(W) \ : \ \mathbb{M}_{m,n} = \underbrace{M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})}_{m} \longrightarrow \mathbb{C}$$

obtained by replacing in W each occurrence of X_i by the $n \times n$ matrix n the i-th component, and taking traces.

We are interested in the (naive, equivariant) motives of the fibers of this functional which we denote by

$$\mathbb{M}_{m,n}^W(\lambda) = Tr(W)^{-1}(\lambda).$$

Recall that to each isomorphism class of a complex variety X (equipped with a good action of a finite group of roots of unity) we associate its naive equivariant motive $[X]$ which is an element in the ring $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1/2}]$ (see [4] or [3]) and is subject to the scissor- and product-relations

$$[X] - [Z] = [X - Z] \quad \text{and} \quad [X].[Y] = [X \times Y]$$

whenever Z is a Zariski closed subvariety of X. A special element is the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}, id]$ and we recall from [12, Lemma 4.1] that $[GL_n] = \prod_{k=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^k)$ and from [3, 2.2] that $[\mathbb{A}^n, \mu_k] = \mathbb{L}^n$ for a linear action of μ_k on $\mathbb{A}^n$. This ring is equipped with a plethystic exponential $\mathtt{Exp}$, see for example [2] and [4].

The representation theoretic interest of the degeneracy locus $Z = \{dTr(W) = 0\}$ of the Chern-Simons functional is that it coincides with the scheme of n-dimensional representations

$$Z = \mathtt{rep}_n(R_W) \quad \text{where} \quad R_W = \frac{\mathbb{C}\langle X_1, \ldots, X_m \rangle}{(\partial_{X_i}(W) : 1 \leq i \leq m)}$$

of the corresponding Jacobi algebra R_W where ∂_{X_i} is the cyclic derivative with respect to X_i. As W is homogeneous it follows from [4, Thm. 1.3] (or [1] if the superpotential allows 'a cut') that its virtual motive is equal to

$$[\texttt{rep}_n(R_W)]_{virt} = \mathbb{L}^{-\frac{mn^2}{2}}([\mathbb{M}^W_{m,n}(0)] - [\mathbb{M}^W_{m,n}(1)])$$

where $\hat{\mu}$ acts via μ_d on $\mathbb{M}^W_{m,n}(1)$ and trivially on $\mathbb{M}^W_{m,n}(0)$. These virtual motives can be packaged together into the motivic Donaldson-Thomas series

$$U_W(t) = \sum_{n=0}^{\infty} \mathbb{L}^{-\frac{(m-1)n^2}{2}} \frac{[\mathbb{M}^W_{m,n}(0)] - [\mathbb{M}^W_{m,n}(1)]}{[GL_n]} t^n$$

In [3] A. Cazzaniga, A. Morrison, B. Pym and B. Szendröi conjecture that this generating series has an exponential expression involving simple rational functions of virtual motives determined by representation theoretic information of the Jacobi algebra R_W

$$U_W(t) \overset{?}{=} \texttt{Exp}(-\sum_{i=1}^{k} \frac{M_i}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \frac{t^{m_i}}{1-t^{m_i}})$$

where $m_1 = 1, \dots, m_k$ are the dimensions of simple representations of R_W and $M_i \in \mathcal{M}_{\mathbb{C}}$ are motivic expressions without denominators, with M_1 the virtual motive of the scheme parametrizing (simple) 1-dimensional representations. Evidence for this conjecture comes from cases where the superpotential admits a cut and hence one can use dimensional reduction, introduced by A. Morrison in [12], as in the case of quantum affine three-space [3].

The purpose of this paper is to introduce an inductive procedure to test the conjectural exponential expressions given in [3] in other interesting cases such as the homogenized Weyl algebra and elliptic Sklyanin algebras. To this end we introduce the following quotient of the free necklace algebra on m variables

$$\mathbb{T}^W_m(\lambda) = \frac{\mathbb{C}\langle X_1, \dots, X_m\rangle \otimes \texttt{Sym}(V_m)}{(W-\lambda)}, \text{ where } V_m = \frac{\mathbb{C}\langle X_1, \dots, X_m\rangle}{[\mathbb{C}\langle X_1, \dots, X_m\rangle, \mathbb{C}\langle X_1, \dots, X_m\rangle]_{vect}}$$

is the vectorspace space having as a basis all cyclic words in $X_1, \dots, X_m$. Note that any superpotential is an element of $\texttt{Sym}(V_m)$. Substituting each X_k by a generic $n \times n$ matrix and each cyclic word by the corresponding trace we obtain a quotient of the trace ring of m generic $n \times n$ matrices

$$\mathbb{T}^W_{m,n}(\lambda) = \frac{\mathbb{T}_{m,n}}{(Tr(W) - \lambda)} \quad \text{with} \quad \mathbb{M}^W_{m,n}(\lambda) = \texttt{trep}_n(\mathbb{T}^W_{m,n})$$

such that its scheme of trace preserving n-dimensional representations is isomorphic to the fiber $\mathbb{M}^W_{m,n}(\lambda)$. We will see that if $\lambda \neq 0$ the algebra $\mathbb{T}^W_{m,n}(\lambda)$ shares many ringtheoretic properties of trace rings of generic matrices, in particular it is a Cayley-smooth algebra, see [10]. As such one might hope to describe $\mathbb{M}^W_{m,n}(\lambda)$ using the Luna stratification of the quotient and its fibers in terms of marked quiver settings given in [10]. However, all this is with respect to the étale topology and hence useless in computing motives.

For this reason we consider the Brauer-Severi scheme of $\mathbb{T}^W_{m,n}(\lambda)$, as introduced by M. Van den Bergh in [17] and further investigated by M. Reineke in [16], which are quotients of a principal GL_n-bundles and hence behave well with respect to motives. More precisely, the Brauer-Severi scheme of $\mathbb{T}^W_{m,n}(\lambda)$ is defined as

$$\texttt{BS}^W_{m,n}(\lambda) = \{(v, \phi) \in \mathbb{C}^n \times \texttt{trep}_n(\mathbb{T}^W_{m,n}(\lambda) \mid \phi(\mathbb{T}^W_{m,n}(\lambda))v = \mathbb{C}^n\}/GL_n$$

and their motives determine inductively the motives of the fibers $\mathbb{M}^W_{m,n}(1)$ and $\mathbb{M}^W_{m,n}(0)$ via

$$(\mathbb{L}^n - 1)[\mathbb{M}^W_{m,n}(1)] = [GL_n][\mathtt{BS}^W_{m,n}(1)] + \sum_{k=1}^{n-1} \mathbb{L}^{(m-1)k(n-k)} \frac{[GL_n]}{[GL_{n-k}]} \times$$

$$((\mathbb{L}-2)[\mathtt{BS}^W_{m,k}(1)][\mathbb{M}^W_{m,n-k}(1)] + [\mathtt{BS}^W_{m,k}(0)][\mathbb{M}^W_{m,n-k}(1)] + [\mathtt{BS}^W_{m,k}(1)][\mathbb{M}^W_{m,n-k}(0)])$$

and

$$(\mathbb{L}^n - 1)[\mathbb{M}^W_{m,n}(0)] = [GL_n][\mathtt{BS}^W_{m,n}(0)] + \sum_{k=1}^{n-1} \mathbb{L}^{(m-1)k(n-k)} \frac{[GL_n]}{[GL_{n-k}]} \times$$

$$((\mathbb{L}-1)[\mathtt{BS}^W_{m,k}(1)][\mathbb{M}^W_{m,n-k}(1)] + [\mathtt{BS}^W_{m,k}(0)][\mathbb{M}^W_{m,n-k}(0)]$$

which we will prove in Proposition 5. That is, if we can compute $[\mathtt{BS}^W_{m,i}(1)]$ and $[\mathtt{BS}^W_{m,k}(0)]$ for all $i \leq n$, we can compute the first n terms of the generating series $U_W(t)$ of the motivic Donaldson-Thomas invariants.

In section 4 we will compute the first two terms of $U_W(t)$ in the case of the quantized 3-space in a variety of ways. In the final section we repeat the computation for the homogenized Weyl algebra and compare it to the conjectured expression of [3]. In [11] we will compute the case of the elliptic Sklyanin algebras.

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2. Brauer-Severi motives

With $\mathbb{T}_{m,n}$ we will denote the *trace ring of m generic $n \times n$ matrices*. That is, $\mathbb{T}_{m,n}$ is the $\mathbb{C}$-subalgebra of the full matrix-algebra $M_n(\mathbb{C}[x_{ij}(k) \mid 1 \leq i,j \leq n, 1 \leq k \leq m])$ generated by the m generic matrices

$$X_k = \begin{bmatrix} x_{11}(k) & \dots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \dots & x_{nn}(k) \end{bmatrix}$$

together with all elements of the form $Tr(M)1_n$ where M runs over all monomials in the X_i. These algebras have been studied extensively by ringtheorists in the 80ties and some of the results are summarized in the following result

Proposition 1. *Let $\mathbb{T}_{m,n}$ be the trace ring of m generic $n \times n$ matrices, then*

1. *$\mathbb{T}_{m,n}$ is an affine Noetherian domain with center $Z(\mathbb{T}_{m,n})$ of dimension $(m-1)n^2+1$ and generated as $\mathbb{C}$-algebra by the $Tr(M)$ where M runs over all monomials in the generic matrices X_k.*
2. *$\mathbb{T}_{m,n}$ is a maximal order and a noncommutative UFD, that is all twosided prime ideals of height one are generated by a central element and $Z(\mathbb{T}_{m,n})$ is a commutative UFD which is a complete intersection if and only if $n = 1$ or $(m,n) = (2,2), (2,3)$ or $(3,2)$.*
3. *$\mathbb{T}_{m,n}$ is a reflexive Azumaya algebra unless $(m,n) = (2,2)$, that is, every localization at a central height one prime ideal is an Azumaya algebra.*

Proof. For (1) see for example [13] or [15]. For (2) see for example [8], for (3) for example [7]. □

A Cayley-Hamilton algebra of degree n is a $\mathbb{C}$-algebra A , equipped with a linear trace map $tr : A \longrightarrow A$ satisfying the following properties:

(1) $tr(a).b = b.tr(a)$
(2) $tr(a.b) = tr(b.a)$
(3) $tr(tr(a).b) = tr(a).tr(b)$
(4) $tr(a) = n$
(5) $\chi_a^{(n)}(a) = 0$ where $\chi_a^{(n)}(t)$ is the formal Cayley-Hamilton polynomial of degree n, see [14]

For a Cayley-Hamilton algebra A of degree n it is natural to look at the scheme $\mathtt{trep}_n(A)$ of all *trace preserving* n-dimensional representations of A, that is, all trace preserving algebra maps $A \longrightarrow M_n(\mathbb{C})$. A Cayley-Hamilton algebra A of degree n is said to be a *smooth Cayley-Hamilton algebra* if $\mathtt{trep}_n(A)$ is a smooth variety. Procesi has shown that these are precisely the algebras having the smoothness property of allowing lifts modulo nilpotent ideals in the category of all Cayley-Hamilton algebras of degree n, see [14]. The étale local structure of smooth Cayley-Hamilton algebras and their centers have been extensively studied in [10].

Proposition 2. *Let W be a homogeneous superpotential in m variables and define the algebra*

$$\mathbb{T}^W_{m,n}(\lambda) = \frac{\mathbb{T}_{m,n}}{(Tr(W) - \lambda)} \quad \textit{then} \quad \mathbb{M}^W_{m,n}(\lambda) = \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$$

If $Tr(W) - \lambda$ is irreducible in the UFD $Z(\mathbb{T}_{m,n})$, then for $\lambda \neq 0$

(1) *$\mathbb{T}^W_{m,n}(\lambda)$ is a reflexive Azumaya algebra.*
(2) *$\mathbb{T}^W_{m,n}(\lambda)$ is a smooth Cayley-Hamilton algebra of degree n and of Krull dimension $(m-1)n^2$.*
(3) *$\mathbb{T}^W_{m,n}(\lambda)$ is a domain.*
(4) *The central singular locus is the the non-Azumaya locus of $\mathbb{T}^W_{m,n}(\lambda)$ unless $(m,n) = (2,2)$.*

Proof. (1) : As $\mathbb{M}^W_{m,n}(\lambda) = \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$ is a smooth affine variety for $\lambda \neq 0$ (due to homogeneity of W) on which GL_n acts by automorphisms, we know that the ring of invariants,

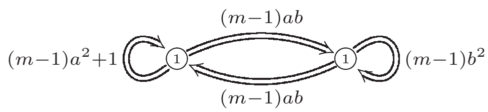
$$\mathbb{C}[\mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))]^{GL_n} = Z(\mathbb{T}^W_{m,n}(\lambda))$$

which coincides with the center of $\mathbb{T}^W_{m,n}(\lambda)$ by e.g. [10, Prop. 2.12], is a normal domain. Because the non-Azumaya locus of $\mathbb{T}_{m,n}$ has codimension at least 3 (if $(m,n) \neq (2,2)$) by [7], it follows that all localizations of $\mathbb{T}^W_{m,n}(\lambda)$ at height one prime ideals are Azumaya algebras. Alternatively, using (2) one can use the theory of local quivers as in [10].

(2) : That the Cayley-Hamilton degree of the quotient $\mathbb{T}^W_{m,n}(\lambda)$ remains n follows from the fact that $\mathbb{T}_{m,n}$ is a reflexive Azumaya algebra and irreducibility of $Tr(W) - \lambda$. Because $\mathbb{M}^W_{m,n}(\lambda) = \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$ is a smooth affine variety, $\mathbb{T}^W_{m,n}(\lambda)$ is a smooth Cayley-Hamilton algebra. The statement on Krull dimension follows from the fact that the Krull dimension of $\mathbb{T}_{m,n}$ is known to be $(m-1)n^2 + 1$.

(3) : After taking determinants, this follows from factoriality of $Z(\mathbb{T}_{m,n})$ and irreducibility of $Tr(W) - \lambda$.

(4) : This follows from the theory of local quivers as in [10]. The most general non-simple representations are of representation type $(1, a; 1, b)$ with the dimensions of the two simple representations a, b adding up to n. The corresponding local quiver is



and as $(m-1)ab \geq 2$ under the assumptions, it follows that the corresponding singular point is singular. $\square$

Let us define for all $k \leq n$ and all $\lambda \in \mathbb{C}$ the locally closed subscheme of $\mathbb{C}^n \times \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$

$$\mathtt{X}_{k,n,\lambda} = \{(v, \phi) \in \mathbb{C}^n \times \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda)) \mid dim_{\mathbb{C}}(\phi(\mathbb{T}^W_{m,n}(\lambda)).v) = k\}$$

Sending a point (v, ϕ) to the point in the Grassmannian $\mathtt{Gr}(k, n)$ determined by the k-dimensional subspace $V = \phi(\mathbb{T}^W_{m,n}(\lambda)).v \subset \mathbb{C}^n$ we get a Zariskian fibration as in [12]

$$\mathtt{X}_{k,n,\lambda} \twoheadrightarrow \mathtt{Gr}(k, n)$$

To compute the fiber over V we choose a basis of $\mathbb{C}^n$ such that the first k base vectors span $V = \phi(\mathbb{T}^W_{m,n}(\lambda)).v$. With respect to this basis, the images of the generic matrices X_i all are of the following block-form

$$\phi(X_i) = \begin{bmatrix} \phi_k(X_i) & \sigma(X_i) \\ 0 & \phi_{n-k}(X_i) \end{bmatrix} \quad \text{with} \quad \begin{cases} \phi_k(X_i) \in M_k(\mathbb{C}) \\ \phi_{n-k}(X_i) \in M_{n-k}(\mathbb{C}) \\ \sigma(X_i) \in M_{n-k \times k}(\mathbb{C}) \end{cases}$$

Using these matrix-form it is easy to see that

$$Tr(\phi(W(X_1, \ldots, X_m))) = Tr(\phi_k(W(X_1, \ldots, X_m))) + Tr(\phi_{n-k}(W(X_1, \ldots, X_m)))$$

That is, if $\phi_k \in \mathtt{trep}_k(\mathbb{T}^W_{m,k}(\mu))$ then $\phi_{n-k} \in \mathtt{trep}(\mathbb{T}^W_{m,n-k}(\lambda - \mu))$ and moreover we have that $(v, \phi_k) \in \mathtt{X}_{k,k,\mu}$. Further, the m matrices $\sigma(X_i) \in M_{n-k \times k}(\mathbb{C})$ can be taken arbitrary. Rephrasing this in motives we get

$$[\mathtt{X}_{k,n,\lambda}] = \mathbb{L}^{mk(n-k)}[\mathtt{Gr}(k, n)] \sum_{\mu \in \mathbb{C}} [\mathtt{X}_{k,k,\mu}][\mathtt{trep}_{n-k}(\mathbb{T}_{m,n-k}(\lambda - \mu))]$$

Further, we have

$$[\mathtt{Gr}(k, n)] = \frac{[GL_n]}{[GL_k][GL_{n-k}]\mathbb{L}^{k(n-k)}} \quad \text{and} \quad [\mathtt{X}_{k,k,\mu}] = [GL_k][\mathtt{BS}^W_{m,k}(\mu)]$$

and substituting this in the above, and recalling that $\mathbb{M}^W_{m,l}(\alpha) = \mathtt{trep}_l(\mathbb{T}^W_{m,l}(\alpha))$, we get

Proposition 3. *With notations as before we have for all $0 < k < n$ and all $\lambda \in \mathbb{C}$ that*

$$[\mathtt{X}_{k,n,\lambda}] = [GL_n]\mathbb{L}^{(m-1)k(n-k)} \sum_{\mu \in \mathbb{C}} [\mathtt{BS}^W_{m,k}(\mu)] \frac{[\mathbb{M}^W_{m,n-k}(\lambda - \mu)]}{[GL_{n-k}]}$$

Further, we have

$$[\mathtt{X}_{0,n,\lambda}] = [\mathbb{M}^W_{m,n}(\lambda)] \quad \textit{and} \quad [\mathtt{X}_{n,n,\lambda}] = [GL_n][\mathtt{BS}^W_{m,n}(\lambda)]$$

We can also express this in terms of generating series. Equip the commutative ring $\mathcal{M}_{\mathbb{C}}[[t]]$ with the modified product

$$t^a * t^b = \mathbb{L}^{(m-1)ab} t^{a+b}$$

and consider the following two generating series for all $\frac{1}{2} \neq \lambda \in \mathbb{C}$

$$\mathtt{B}_\lambda(t) = \sum_{n=1}^{\infty} [\mathtt{BS}^W_{m,n}(\lambda)] t^n \quad \text{and} \quad \mathtt{R}_\lambda(t) = \sum_{n=1}^{\infty} \frac{[\mathbb{M}^W_{m,n}(\lambda)]}{[GL_n]} t^n$$

$$\mathtt{B}_{\frac{1}{2}}(t) = \sum_{n=0}^{\infty} [\mathtt{BS}^W_{m,n}(\frac{1}{2})] t^n \quad \text{and} \quad \mathtt{R}_{\frac{1}{2}}(t) = \sum_{n=0}^{\infty} \frac{[\mathbb{M}^W_{m,n}(\frac{1}{2})]}{[GL_n]} t^n$$

Proposition 4. *With notations as before we have the functional equation*

$$1 + \mathtt{R}_1(\mathbb{L}t) = \sum_{\mu} \mathtt{B}_\mu(t) * \mathtt{R}_{1-\mu}(t)$$

Proof. The disjoint union of the strata of the dimension function on $\mathbb{C}^n \times \mathtt{trep}_n(\mathbb{T}^W_{m,n}(\lambda))$ gives

$$\mathbb{C}^n \times \mathbb{M}^W_{m,n}(\lambda) = \mathtt{X}_{0,n,\lambda} \sqcup \mathtt{X}_{1,n,\lambda} \sqcup \ldots \sqcup \mathtt{X}_{n,n,\lambda}$$

Rephrasing this in terms of motives gives

$$\mathbb{L}^n [\mathbb{M}^W_{m,n}(\lambda)] = [\mathbb{M}^W_{m,n}(\lambda)] + \sum_{k=1}^{n-1} [\mathtt{X}_{k,n,\lambda}] + [GL_n][\mathtt{BS}^W_{m,n}(\lambda)]$$

and substituting the formula of proposition 3 into this we get

$$\frac{[\mathbb{M}^W_{m,n}(\lambda)]}{[GL_n]} \mathbb{L}^n t^n = \frac{[\mathbb{M}^W_{m,n}(\lambda)]}{[GL_n]} t^n +$$

$$\sum_{k=1}^{n-1} \sum_{\mu \in \mathbb{C}} ([\mathtt{BS}^W_{m,k}(\mu)] t^k) * (\frac{[\mathbb{M}^W_{m,n-k}(\lambda-\mu)]}{[GL_{n-k}]} t^{n-k}) + [\mathtt{BS}^W_{m,n}(\lambda)] t^n$$

Now, take $\lambda = 1$ then on the left hand side we have the n-th term of the series $1 + \mathtt{R}_1(\mathbb{L}t)$ and on the right hand side we have the n-th factor of the series $\sum_\mu \mathtt{B}_\mu(t) * \mathtt{R}_{1-\mu}(t)$. The outer two terms arise from the product $\mathtt{B}_{\frac{1}{2}}(t) * \mathtt{R}_{\frac{1}{2}}(t)$, using that W is homogeneous whence for all $\lambda \neq 0$

$$\mathtt{BS}^W_{m,n}(\lambda) \simeq \mathtt{BS}^W_{m,n}(1) \quad \text{and} \quad \mathbb{M}^W_{m,n}(\lambda) \simeq \mathbb{M}^W_{m,n}(1)$$

This finishes the proof. $\square$

These formulas allow us to determine the motive $[\mathbb{M}^W_{m,n}(\lambda)]$ inductively from lower dimensional contributions and from the knowledge of the motive of the Brauer-Severi scheme $[\mathtt{BS}^W_{m,n}(\lambda)]$.

Proposition 5. *For all n we have the following inductive description of $[\mathbb{M}^W_{m,n}(1)]$*

$$(\mathbb{L}^n - 1)[\mathbb{M}^W_{m,n}(1)] = [GL_n][\mathtt{BS}^W_{m,n}(1)] + \sum_{k=1}^{n-1} \mathbb{L}^{(m-1)k(n-k)} \frac{[GL_n]}{[GL_{n-k}]} \times$$

$$((\mathbb{L}-2)[\mathtt{BS}^W_{m,k}(1)][\mathbb{M}^W_{m,n-k}(1)] + [\mathtt{BS}^W_{m,k}(0)][\mathbb{M}^W_{m,n-k}(1)] + [\mathtt{BS}^W_{m,k}(1)][\mathbb{M}^W_{m,n-k}(0)])$$

and for $[\mathbb{M}^W_{m,n}(0)]$ *we have*

$$(\mathbb{L}^n - 1)[\mathbb{M}^W_{m,n}(0)] = [GL_n][\mathtt{BS}^W_{m,n}(0)] + \sum_{k=1}^{n-1} \mathbb{L}^{(m-1)k(n-k)} \frac{[GL_n]}{[GL_{n-k}]} \times$$

$$((\mathbb{L}-1)[\mathtt{BS}^W_{m,k}(1)][\mathbb{M}^W_{m,n-k}(1)] + [\mathtt{BS}^W_{m,k}(0)][\mathbb{M}^W_{m,n-k}(0)]$$

Proof. Follows from Proposition 3 and the fact that for all $\mu \neq 0$ we have that $[\mathbb{M}^W_{m,k}(\mu)] = [\mathbb{M}^W_{m,k}(1)]$ and $[\mathtt{BS}^W_{m,k}(\mu)] = [\mathtt{BS}^W_{m,k}(1)]$. □

3. Deformations of affine 3-space

The commutative polynomial ring $\mathbb{C}[x,y,z]$ is the Jacobi algebra associated with the superpotential $W = XYZ - XZY$. For this reason we restrict in the rest of this paper to cases where the superpotential W is a cubic necklace in three non-commuting variables X, Y and Z, that is $m = 3$ from now on. As even in this case the calculations become quickly unmanageable we restrict to $n \leq 2$, that is we only will compute the coefficients of t and t^2 in $U_W(t)$. We will have to compute the motives of fibers of the Chern-Simons functional

$$M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \xrightarrow{Tr(W)} \mathbb{C}$$

so we want to express $Tr(W)$ as a function in the variables of the three generic 2×2 matrices

$$X = \begin{bmatrix} n & p \\ q & r \end{bmatrix}, \; Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \; Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

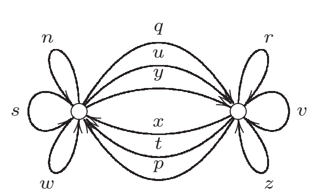
We will call $\{n,r,s,v,w,x\}$ (resp. $\{p,t,x\}$ and $\{q,u,y\}$) the diagonal- (resp. upper- and lower-) variables. We claim that

$$Tr(W) = C + Q_q.q + Q_u.u + Q_y.y$$

where C is a cubic in the diagonal variables and Q_q, Q_u and Q_y are bilinear in the diagonal and upper variables, that is, there are linear terms L_{ab} in the diagonal variables such that

$$\begin{cases} Q_q = L_{qp}.p + L_{qt}.t + L_{qx}.x \\ Q_u = L_{up}.p + L_{ut}.t + L_{ux}.x \\ Q_y = L_{yp}.p + L_{yt}.t + L_{yx}.x \end{cases}$$

This follows from considering the two diagonal entries of a 2×2 matrix as the vertices of a quiver and the variables as arrows connecting these vertices as follows



and observing that only an oriented path of length 3 starting and ending in the same vertex can contribute something non-zero to $Tr(W)$. Clearly these linear and cubic terms are fully determined by W. If we take

$$W = \alpha X^3 + \beta Y^3 + \gamma Z^3 + \delta XYZ + \epsilon XZY$$

then we have $C = W(n,s,w) + W(r,v,z)$ and

$$\begin{cases} L_{qp} &= 3\alpha(n+r) \\ L_{qt} &= \epsilon w + \delta z \\ L_{qx} &= \delta s + \epsilon v \end{cases} \quad \begin{cases} L_{up} &= \delta w + \epsilon z \\ L_{ut} &= 3\beta(s+v) \\ L_{ux} &= \epsilon n + \delta r \end{cases} \quad \begin{cases} L_{yp} &= \epsilon s + \delta v \\ L_{yt} &= \delta n + \epsilon r \\ L_{yx} &= 3\gamma(w+z) \end{cases}$$

By using the cellular decomposition of the Brauer-Severi scheme of $\mathbb{T}_{3,2}$ one can simplify the computations further by specializing certain variables. From [16] we deduce that $\mathtt{BS}_2(\mathbb{T}_{3,2})$ has a cellular decomposition as $\mathbb{A}^{10} \sqcup \mathbb{A}^8 \sqcup \mathbb{A}^8$ where the three cells have representatives

$$\begin{cases} \mathtt{cell}_1 \ : \ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & p \\ 1 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ \\ \mathtt{cell}_2 \ : \ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & t \\ 1 & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ \\ \mathtt{cell}_3 \ : \ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ 0 & v \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & x \\ 1 & z \end{bmatrix} \end{cases}$$

It follows that $\mathtt{BS}^W_{3,2}(1)$ decomposes as $\mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3}$ where the subschemes $\mathbf{S_i}$ of $\mathbb{A}^{11-i}$ have defining equations

$$\begin{cases} \mathbf{S_1} \ : \ (C + Q_u.u + Q_y.y + Q_q)|_{n=0} = 1 \\ \mathbf{S_2} \ : \ (C + Q_y.y + Q_u)|_{s=0} = 1 \\ \mathbf{S_3} \ : \ (C + Q_y)|_{w=0} = 1 \end{cases}$$

Note that in using the cellular decomposition, we set a variable equal to 1. So, in order to retain a homogeneous form we let $\mathbb{G}_m$ act on n, s, w, r, v, z with weight one, on q, u, y with weight two and on x, t, p with weight zero. Thus, we need a slight extension of [4, Thm. 1.3] as to allow $\mathbb{G}_m$ to act with weight two on certain variables.

From now on we will assume that W is as above with $\delta = 1$ and $\epsilon \neq 0$. In this generality we can prove:

Proposition 6. *With assumptions as above*

$$[\mathbf{S_3}] = \begin{cases} \mathbb{L}^7 - \mathbb{L}^4 + \mathbb{L}^3[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, 0) = 1]_{\mathbb{A}^2} & \textit{if } \gamma \neq 0 \\ \mathbb{L}^7 - \mathbb{L}^5 + \mathbb{L}^3[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, z) = 1]_{\mathbb{A}^3} & \textit{if } \gamma = 0 \end{cases}$$

Proof. $\mathbf{S_3}$: The defining equation in $\mathbb{A}^8$ is equal to

$$W(n,s,0) + W(r,v,z) + (\epsilon s + v)p + (n + \epsilon r)t + 3\gamma(z)x = 1$$

If $\epsilon s + v \neq 0$ we can eliminate p and get a contribution $\mathbb{L}^5(\mathbb{L}^2 - \mathbb{L})$. If $v = -\epsilon s$ but $n + \epsilon r \neq 0$ we can eliminate t and get a term $\mathbb{L}^4(\mathbb{L}^2 - \mathbb{L})$. From now on we may assume that $v = -\epsilon s$ and $r = -\epsilon^{-1}n$.
$\gamma \neq 0$: Assume first that $z \neq 0$ then we can eliminate x and get a contribution $\mathbb{L}^4(\mathbb{L} - 1)$. If $z = 0$ then we get a term

$$\mathbb{L}^3[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, 0) = 1]_{\mathbb{A}^2}$$

$\gamma = 0$: Then we have a remaining contribution

$$\mathbb{L}^3[W(n,s,0) + W(-\epsilon^{-1}n, -\epsilon s, z) = 1]_{\mathbb{A}^3}$$

Summing up all contributions gives the result. □

Calculating the motives of $\mathbf{S_2}$ and $\mathbf{S_1}$ in this generality quickly leads to a myriad of subcases to consider. For this reason we will defer the calculations in the cases of interest to the next sections. Specializing Proposition 5 to the case of $n = 2$ we get

Proposition 7. *For $n = 2$ we have the following relation*

$$[\mathbb{M}^W_{3,2}(1)] = \mathbb{L}(\mathbb{L}-1)[\mathbf{BS}^W_{3,2}(1)] + \mathbb{L}^3((\mathbb{L}-2)[\mathbb{M}^W_{3,1}(1)]^2 + 2[\mathbb{M}^W_{3,1}(0)][\mathbb{M}^W_{3,1}(1)])$$

Proof. From Proposition 5 we have that $[\mathbb{M}^W_{3,2}(1)]$ is equal to

$$\mathbb{L}(\mathbb{L}-1)[\mathbf{BS}^W_{3,2}(1)] + \mathbb{L}^3((\mathbb{L}-2)[\mathbf{BS}^W_{3,1}(1)][\mathbb{M}^W_{3,1}(1)]+$$
$$[\mathbf{BS}^W_{3,1}(0)][\mathbb{M}^W_{3,1}(1)] + [\mathbf{BS}^W_{3,1}(1)][\mathbb{M}^W_{3,1}(0)])$$

The result follows from this from the fact that $\mathbf{BS}^W_{3,1}(1) = \mathbb{M}^W_{3,1}(1)$ and $\mathbf{BS}^W_{3,1}(0) = \mathbb{M}^W_{3,1}(0)$. □

4. Quantum affine three-space

For $q \in \mathbb{C}^*$ consider the superpotential $W_q = XYZ - qXZY$, then the associated algebra R_{W_q} is the quantum affine 3-space

$$R_{W_q} = \frac{\mathbb{C}\langle X, Y, Z\rangle}{(XY - qYX, ZX - qXZ, YZ - qZY)}$$

It is well-known that R_{W_q} has finite dimensional simple representations of dimension n if and only if q is a primitive n-th root of unity. For other values of q the only finite dimensional simples are 1-dimensional and parametrized by $XYZ = 0$ in $\mathbb{A}^3$. In this case we have

$$\begin{cases} [\mathbb{M}^{W_q}_{3,1}(1)] = [(q-1)XYZ = 1]_{\mathbb{A}^3} = (\mathbb{L}-1)^2 \\ [\mathbb{M}^{W_q}_{3,1}(0)] = [(1-q)XYZ = 0]_{\mathbb{A}^3} = 3\mathbb{L}^2 - 3\mathbb{L} + 1 \end{cases}$$

That is, the coefficient of t in $U_{W_q}(t)$ is equal to

$$\mathbb{L}^{-1}\frac{[\mathbb{M}^{W_q}_{3,1}(0) - [\mathbb{M}^{W_q}_{3,1}(1)]}{[GL_1]} = \mathbb{L}^{-1}\frac{2\mathbb{L}^2 - \mathbb{L}}{\mathbb{L}-1} = \frac{2\mathbb{L}-1}{\mathbb{L}-1}$$

In [3, Thm. 3.1] it is shown that in case q is not a root of unity, then

$$U_{W_q}(t) = \mathtt{Exp}(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t})$$

and if q is a primitive n-th root of unity then

$$U_{W_q}(t) = \mathtt{Exp}(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t} + (\mathbb{L}-1)\frac{t^n}{1-t^n})$$

In [3, 3.4.1] a rather complicated attempt is made to explain the term $\mathbb{L} - 1$ in case q is an n-th root of unity in terms of certain simple n-dimensional representations of R_{W_q}. Note that the geometry of finite dimensional representations of the algebra R_{W_q} is studied extensively in [5] and note that there are additional simple n-dimensional representations not taken into account in [3, 3.4.1].

Perhaps a more conceptual explanation of the two terms in the exponential expression of $U_{W_q}(t)$ in case q is an n-th root of unity is as follows. As W_q admits a cut $W_q = X(YZ - qZY)$ it follows from [12] that for all dimensions m we have

$$[\mathbb{M}^{W_q}_{3,m}(0)] - [\mathbb{M}^{W_q}_{3,m}(1)] = \mathbb{L}^{m^2}[\mathtt{rep}_m(\mathbb{C}_q[Y,Z])]$$

where $\mathbb{C}_q[Y,Z] = \mathbb{C}\langle Y,Z\rangle/(YZ - qZY)$ is the quantum plane. If q is an n-th root of unity the only finite dimensional simple representations of $\mathbb{C}_q[Y,Z]$ are of dimension 1 or n. The 1-dimensional simples are parametrized by $YZ = 0$ in $\mathbb{A}^2$ having as motive $2\mathbb{L} - 1$ and as all have GL_1 as stabilizer group, this explains the term $(2\mathbb{L} - 1)/(\mathbb{L} - 1)$. The center of $\mathbb{C}_q[Y,Z]$ is equal to $\mathbb{C}[Y^n, Z^n]$ and the corresponding variety $\mathbb{A}^2 = \mathtt{Max}(\mathbb{C}[Y^n, Z^n])$ parametrizes n-dimensional semi-simple representations.The n-dimensional simples correspond to the Zariski open set $\mathbb{A}^2 - (Y^nZ^n = 0)$ which has as motive $(\mathbb{L} - 1)^2$. Again, as all these have as GL_2-stabilizer subgroup GL_1, this explains the term

$$\mathbb{L} - 1 = \frac{(\mathbb{L} - 1)^2}{[GL_1]}$$

As the superpotential allows a cut in this case we can use the full strength of [1]and can obtain $[\mathbb{M}^W_{3,2}(0)]$ from $[\mathbb{M}^W_{3,2}(1)]$ from the equality

$$\mathbb{L}^{12} = [\mathbb{M}^W_{3,2}(0)] + (\mathbb{L} - 1)[\mathbb{M}^W_{3,2}(1)]$$

To illustrate the inductive procedure using Brauer-Severi motives we will consider the case $n = 2$, that is $q = -1$ with superpotential $W = XYZ + XZY$. In this case we have from [3, Thm. 3.1] that

$$U_W(t) = \mathtt{Exp}(\frac{2\mathbb{L} - 1}{\mathbb{L} - 1}\frac{t}{1 - t} + (\mathbb{L} - 1)\frac{t^2}{1 - t^2}$$

The basic rules of the plethystic exponential on $\mathcal{M}_{\mathbb{C}}[[t]]$ are

$$\mathtt{Exp}(\sum_{n\geq 1}[A_n]t^n) = \prod_{n\geq 1}(1 - t^n)^{-[A_n]} \quad \text{where} \quad (1 - t)^{-\mathbb{L}^m} = (1 - \mathbb{L}^m t)^{-1}$$

and one has to extend all infinite products in t and $\mathbb{L}^{-1}$. One starts by rewriting $U_W(t)$ as a product

$$U_W(t) = \mathtt{Exp}(\frac{t}{1 - t})\mathtt{Exp}(\frac{\mathbb{L}}{\mathbb{L} - 1}\frac{t}{1 - t})\mathtt{Exp}(\frac{\mathbb{L}t^2}{1 - t^2})\mathtt{Exp}(\frac{t^2}{1 - t^2})^{-1}$$

where each of the four terms is an infinite product

$$\mathtt{Exp}(\frac{t}{1 - t}) = \prod_{m\geq 1}(1 - t^m)^{-1}, \qquad \mathtt{Exp}(\frac{\mathbb{L}}{\mathbb{L} - 1}\frac{t}{1 - t}) = \prod_{m\geq 1}\prod_{j\geq 0}(1 - \mathbb{L}^{-j}t^m)^{-1}$$

$$\mathtt{Exp}(\frac{\mathbb{L}t^2}{1 - t^2}) = \prod_{m\geq 1}(1 - \mathbb{L}t^{2m})^{-1}, \qquad \mathtt{Exp}(\frac{t^2}{1 - t^2})^{-1} = \prod_{m\geq 1}(1 - t^{2m}$$

That is, we have to work out the infinite product

$$\prod_{m\geq 1}((1 - t^{2m-1})^{-1}(1 - \mathbb{L}t^{2m})^{-1})\prod_{m\geq 1}\prod_{j\geq 0}(1 - \mathbb{L}^{-j}t^m)^{-1}$$

as a power series in t, at least up to quadratic terms. One obtains

$$U_W(t) = 1 + \frac{2\mathbb{L} - 1}{\mathbb{L} - 1}t + \frac{\mathbb{L}^4 + 3\mathbb{L}^3 - 2\mathbb{L}^2 - 2\mathbb{L} + 1}{(\mathbb{L}^2 - 1)(\mathbb{L} - 1)}t^2 + \ldots$$

That is, if $W = XYZ + XZY$ one must have the relation:

$$[\mathbb{M}^W_{3,2}(0)] - [\mathbb{M}^W_{3,2}(1)] = \mathbb{L}^5(\mathbb{L}^4 + 3\mathbb{L}^3 - 2\mathbb{L}^2 - 2\mathbb{L} + 1)$$

4.1. **Dimensional reduction.** It follows from the dimensional reduction argument of [12] that

$$[\mathbb{M}^W_{3,2}(0)] - [\mathbb{M}^W_{3,2}(1)] = \mathbb{L}^4[\mathtt{rep}_2\ \mathbb{C}_{-1}[X,Y]]$$

where $\mathbb{C}_{-1}[X,Y]$ is the quantum plane at $q = -1$, that is, $\mathbb{C}\langle X,Y\rangle/(XY + YX)$. The matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

gives us the following system of equations

$$\begin{cases} 2ae + bg + fc = 0 \\ 2hd + bg + fc = 0 \\ f(a+d) + b(e+h) = 0 \\ c(h+e) + g(a+d) = 0 \end{cases}$$

where the two first are equivalent to $ae = hd$ and $2ae + bg + fc = 0$. Changing variables

$$x = \frac{1}{2}(a+d), \quad y = \frac{1}{2}(a-d), \quad u = \frac{1}{2}(e+h), \quad v = \frac{1}{2}(e-h)$$

the equivalent system then becomes (in the variables b, c, f, g, u, v, x, y)

$$\begin{cases} xv + yu = 0 \\ xu + yv + bg + fc = 0 \\ fx + bu = 0 \\ cu + gx = 0 \end{cases}$$

Proposition 8. *The motive of $R_2 = \mathtt{rep}_2\ \mathbb{C}_{-1}[x,y]$ is equal to*

$$[R_2] = \mathbb{L}^5 + 3\mathbb{L}^4 - 2\mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}$$

Proof. If $x \neq 0$ we obtain

$$v = -\frac{yu}{x}, \quad f = -\frac{bu}{x}, \quad g = -\frac{cu}{x}$$

and substituting these in the remaining second equation we get the equation(s)

$$u(y^2 - x^2 + 2bc) = 0 \quad \text{and} \quad x \neq 0$$

If $u \neq 0$ then $y^2 - x^2 + 2bc = 0$. If in addition $b \neq 0$ then $c = \frac{x^2-y^2}{2b}$ and y is free. As x, u and b are non-zero this gives a contribution $(\mathbb{L}-1)^3\mathbb{L}$. If $b = 0$ then c is free and $x^2 - y^2 = 0$, so $y = \pm x$. This together with $x \neq 0 \neq u$ leads to a contribution of $2\mathbb{L}(\mathbb{L}-1)^2$. If $u = 0$ then y, b and c are free variables, and together with $x \neq 0$ this gives $(\mathbb{L}-1)\mathbb{L}^3$.

Remains the case that $x = 0$. Then the system reduces to

$$\begin{cases} yu = 0 \\ yv + bg + fc = 0 \\ bu = 0 \\ cu = 0 \end{cases}$$

If $u \neq 0$ then $y = 0, b = 0$ and $c = 0$ leaving c, g, v free. This gives $(\mathbb{L}-1)\mathbb{L}^3$. If $u = 0$ then the only remaining equation is $yv + bg + fc = 0$. That is, we get the cone in $\mathbb{A}^6$ of the Grassmannian $Gr(2,4)$ in $\mathbb{P}^5$. As the motive of $Gr(2,4)$ is

$$[Gr(2,4)] = (\mathbb{L}^2+1)(\mathbb{L}^2+\mathbb{L}+1)$$

we get a contribution of

$$(\mathbb{L}-1)(\mathbb{L}^2+1)(\mathbb{L}^2+\mathbb{L}+1)+1$$

Summing up all contributions gives the desired result. □

4.2. **Brauer-Severi motives.** In the three cells of the Brauer-Severi scheme of $\mathbb{T}_{3,2}$ of dimensions resp. $10, 9$ and 8 the superpotential $Tr(XYZ+XZY)$ induces the equations:

$$\begin{cases} \mathbf{S_1} \ : \ 2rvz + puz + pvy + rty + psy + rux + puw + tz + vx + sx + tw = 1 \\ \mathbf{S_2} \ : \ 2rvz + pvy + rty + nty + pz + rx + nx + pw = 1 \\ \mathbf{S_3} \ : \ 2rvz + pv + rt + nt + ps = 1 \end{cases}$$

Proposition 9. *With notations as above, the Brauer-Severi scheme of* $\mathbb{T}^W_{3,2}(1)$ *has a decomposition*

$$\mathbf{BS}^W_{3,2}(1) = \mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3}$$

where the schemes $\mathbf{S_i}$ *have motives*

$$\begin{cases} [\mathbf{S_1}] = \mathbb{L}^9 - \mathbb{L}^6 - 2\mathbb{L}^5 + 3\mathbb{L}^4 - \mathbb{L}^3 \\ [\mathbf{S_2}] = \mathbb{L}^8 - 2\mathbb{L}^5 + \mathbb{L}^4 \\ [\mathbf{S_3}] = \mathbb{L}^7 - 2\mathbb{L}^4 + \mathbb{L}^3 \end{cases}$$

Therefore, the Brauer-Severi scheme has motive

$$[\mathbf{BS}^W_{3,2}(1)] = \mathbb{L}^9 + \mathbb{L}^8 + \mathbb{L}^7 - \mathbb{L}^6 - 4\mathbb{L}^5 + 2\mathbb{L}^4$$

Proof. $\mathbf{S_1}$: From Proposition 6 we obtain

$$[\mathbf{S_3}] = \mathbb{L}^7 - \mathbb{L}^5 + \mathbb{L}^3[W(n,s,0) + W(-n,-s,z) = 1]_{\mathbb{A}^3}$$

and as $W(n,s,0) + W(-n,-s,z) = 2nsz$ we get $\mathbb{L}^7 - \mathbb{L}^5 + \mathbb{L}^3(\mathbb{L}-1)^2$.

$\mathbf{S_2}$: The defining equation is

$$2rvz + y(pv + (r+n)t) + p(z+w) + x(r+n) = 1$$

If $r + n \neq 0$ we can eliminate x and have a contribution $\mathbb{L}^6(\mathbb{L}^2 - \mathbb{L})$. If $r + n = 0$ we get the equation

$$2rvz + p(yv + z + w) = 1$$

If $yv + z + w \neq 0$ we can eliminate p and get a term $\mathbb{L}^3(\mathbb{L}^4 - \mathbb{L}^3)$. If $r + n = 0$ and $yv + z + w = 0$ we have $2rvz = 1$ so a term $\mathbb{L}^4(\mathbb{L}-1)^2$. Summing up gives us

$$[\mathbf{S_2}] = \mathbb{L}^4(\mathbb{L}-1)(\mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} - 1) = \mathbb{L}^8 - 2\mathbb{L}^5 + \mathbb{L}^4$$

$\mathbf{S_1}$: The defining equation is

$$2rvz + p(u(z+w) + y(v+s)) + t(z + w + ry) + x(v + s + ru) = 1$$

If $v+s+ru \neq 0$ we can eliminate x and get $\mathbb{L}^5(\mathbb{L}^4-\mathbb{L}^3)$. If $v+s+ru=0$ and $z+w+ry \neq 0$ we can eliminate t and have a term $\mathbb{L}^4(\mathbb{L}^4-\mathbb{L}^3)$. If $v+s+ru=0$ and $z+w+ry=0$, the equation becomes (in $\mathbb{A}^8$, with t,x free variables)

$$2r(vz-puy)=1$$

giving a term $\mathbb{L}^2(\mathbb{L}^5-[vz=puy])$. To compute $[vz=puy]_{\mathbb{A}^5}$ assume first that $v\neq 0$, then this gives $\mathbb{L}^3(\mathbb{L}-1)$ and if $v=0$ we get $\mathbb{L}(3\mathbb{L}^2-3\mathbb{L}+1)$. That is, $[vz=puy]_{\mathbb{A}^5}=\mathbb{L}^4+2\mathbb{L}^3-3\mathbb{L}^2+\mathbb{L}$. In total this gives us

$$[\mathbf{S}_1]=\mathbb{L}^3(\mathbb{L}-1)(\mathbb{L}^5+\mathbb{L}^4+\mathbb{L}^3-2\mathbb{L}+1)=\mathbb{L}^9-\mathbb{L}^6-2\mathbb{L}^5+3\mathbb{L}^4-\mathbb{L}^3$$

finishing the proof. □

Proposition 10. *From the Brauer-Severi motive we obtain*

$$\begin{cases}[\mathbb{M}^W_{3,2}(1)] & =\mathbb{L}^{11}-\mathbb{L}^8-3\mathbb{L}^7+2\mathbb{L}^6+2\mathbb{L}^5-\mathbb{L}^4\\ [\mathbb{M}^W_{3,2}(0)] & =\mathbb{L}^{11}+\mathbb{L}^9+2\mathbb{L}^8-5\mathbb{L}^7+3\mathbb{L}^5-\mathbb{L}^4\end{cases}$$

As a consequence we have,

$$[\mathbb{M}^W_{3,2}(0)]-[\mathbb{M}^W_{3,2}(1)]=\mathbb{L}^4(\mathbb{L}^5+3\mathbb{L}^4-2\mathbb{L}^3-2\mathbb{L}^2+\mathbb{L})$$

Proof. We have already seen that $\mathbb{M}^W_{3,1}(1)=\{(x,y,z)\mid 2xyz=1\}$ and $\mathbb{M}^W_{3,1}(0)=\{(x,y,z)\mid xyz=0\}$ whence

$$[\mathbb{M}^W_{3,1}(1)]=(\mathbb{L}-1)^2 \quad \text{and} \quad [\mathbb{M}^W_{3,1}(0)]=3\mathbb{L}^2-3\mathbb{L}+1$$

Plugging this and the obtained Brauer-Severi motive into Proposition 5 gives $[\mathbb{M}^W_{3,2}(1)]$. From this $[\mathbb{M}^W_{3,2}(0)]$ follows from the equation $\mathbb{L}^{12}=(\mathbb{L}-1)[\mathbb{M}^W_{3,2}(1)]+[\mathbb{M}^W_{3,2}(0)]$. □

5. The homogenized Weyl algebra

If we consider the superpotential $W=XYZ-XZY-\frac{1}{3}X^3$ then the associated algebra R_W is the homogenized Weyl algebra

$$R_W=\frac{\mathbb{C}\langle X,Y,Z\rangle}{(XZ-ZX,XY-YX,YZ-ZY-X^2)}$$

In this case we have $\mathbb{M}^W_{3,1}(1)=\{x^3=-3\}$ and $\mathbb{M}^W_{3,1}(0)=\{x^3=0\}$, whence

$$[\mathbb{M}^W_{3,1}(1)]=\mathbb{L}^2[\mu_3], \quad \text{and} \quad [\mathbb{M}^W_{3,1}(0)]=\mathbb{L}^2$$

where, as in [3, 3.1.3] we denote by $[\mu_3]$ the equivariant motivic class of $\{x^3=1\}\subset\mathbb{A}^1$ carrying the canonical action of μ_3. Therefore, the coefficient of t in $U_W(t)$ is equal to

$$\mathbb{L}^{-1}\frac{[\mathbb{M}^W_{3,1}(0)]-[\mathbb{M}^W_{3,1}(0)]}{[GL_1]}=\frac{\mathbb{L}(1-[\mu_3])}{\mathbb{L}-1}$$

As all finite dimensional simple representations of R_W are of dimension one, this leads to the conjectural expression [3, Conjecture 3.3]

$$U_W(t)\overset{?}{=}\mathtt{Exp}(\frac{\mathbb{L}(1-[\mu_3])}{\mathbb{L}-1}\frac{t}{1-t})$$

Balazs Szendröi kindly provided the calculation of the first two terms of this series. Denote with $\tilde{\mathbf{M}}=1-[\mu_3]$, then

$$U_W(t)\overset{?}{=}1+\frac{\mathbb{L}\tilde{\mathbf{M}}}{\mathbb{L}-1}t+\frac{\mathbb{L}^2\tilde{\mathbf{M}}^2+\mathbb{L}(\mathbb{L}^2-1)\tilde{\mathbf{M}}+\mathbb{L}^2(\mathbb{L}-1)\sigma_2(\tilde{\mathbf{M}})}{(\mathbb{L}^2-1)(\mathbb{L}-1)}t^2+\ldots$$

We will now compute the left-hand side using Brauer-Severi motives.

Recall that $\mathtt{BS}^W_{3,2}(i)$, for $i=0,1$, decomposes as $\mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3}$ where the subschemes $\mathbf{S_i}$ of $\mathbb{A}^{11-i}$ have defining equations

$$\begin{cases} \mathbf{S_1} \ : \ -\frac{1}{3}r^3 + ((w-z)p+rx)u + ((v-s)p-rt)y - rp + (z-w)t + (s-v)x = \delta_{i1} \\ \mathbf{S_2} \ : \ -\frac{1}{3}n^3 - \frac{1}{3}r^3 + (vp+(n-r)t)y + (w-z)p + (r-n)x = \delta_{i1} \\ \mathbf{S_3} \ : \ -\frac{1}{3}n^3 - \frac{1}{3}r^3 + (v-s)p + (n-r)t = \delta_{i1} \end{cases}$$

If we let the generator of μ_3 act with weight one on the variables n,s,w,r,v,z, with weight two on x,t,p and with weight zero on q,u,y we see that the schemes S_j for $i=1$ are indeed μ_3-varieties. We will now compute their equivariant motives:

Proposition 11. *With notations as above, the Brauer-Severi scheme of* $\mathbb{T}^W_{3,2}(1)$ *has a decomposition*

$$\mathbf{BS}^W_{3,2}(1) = \mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3}$$

where the schemes $\mathbf{S_i}$ *have equivariant motives*

$$\begin{cases} [\mathbf{S_1}] = \mathbb{L}^9 - \mathbb{L}^6 \\ [\mathbf{S_2}] = \mathbb{L}^8 + ([\mu_3]-1)\mathbb{L}^6 \\ [\mathbf{S_3}] = \mathbb{L}^7 + ([\mu_3]-1)\mathbb{L}^5 \end{cases}$$

Therefore, the Brauer-Severi scheme has equivariant motive

$$[\mathbf{BS}^W_{3,2}(1)] = \mathbb{L}^9 + \mathbb{L}^8 + \mathbb{L}^7 + ([\mu_3]-2)\mathbb{L}^6 + ([\mu_3]-1)\mathbb{L}^5$$

Proof. $\mathbf{S_3}$: If $v-s \neq 0$ we can eliminate p and obtain a contribution $\mathbb{L}^5(\mathbb{L}^2-\mathbb{L})$. If $v=s$ and $n-r \neq 0$ we can eliminate t and obtain a term $\mathbb{L}^4(\mathbb{L}^2-\mathbb{L})$. Finally, if $v=s$ and $n=r$ we have the identity $-\frac{2}{3}n^3 = 1$ and a contribution $\mathbb{L}^5[\mu_3]$.

$\mathbf{S_2}$: If $r-n \neq 0$ we can eliminate x and get a term $\mathbb{L}^6(\mathbb{L}^2-\mathbb{L})$. If $r-n=0$ we get the equation in $\mathbb{A}^8$

$$-\frac{2}{3}n^3 + p(vy+w-z) = 1$$

If $vy+w-z \neq 0$ we can eliminate p and get a contribution $\mathbb{L}^3(\mathbb{L}^4-\mathbb{L}^3)$. Finally, if $vy+w-z=0$ we get the equation $-\frac{2}{3}n^3=1$ and hence a term $\mathbb{L}^3.\mathbb{L}^3[\mu_3]$.

$\mathbf{S_1}$: If $(w-z)p+rx \neq 0$ then we can eliminate u and get a contribution

$$\mathbb{L}^4(\mathbb{L}^5 - [(w-z)p+rx=0]_{\mathbb{A}^5}) = \mathbb{L}^6(\mathbb{L}-1)(\mathbb{L}^2-1)$$

If $(w-z)p+rx=0$ but $(v-s)p-rt \neq 0$ we can eliminate y and get a term

$$\mathbb{L}.[(w-z)p+rx=0,(v-s)p-rt \neq 0]_{\mathbb{A}^8}$$

To compute the equivariant motive in $\mathbb{A}^8$ assume first that $r \neq 0$ then we can eliminate x from the equation and obtain

$$\mathbb{L}^2[r \neq 0,(v-s)p-rt \neq 0]_{\mathbb{A}^5} = \mathbb{L}^2(\mathbb{L}^4(\mathbb{L}-1)-[r\neq 0,(v-s)p-rt=0]_{\mathbb{A}^5}) = \mathbb{L}^5(\mathbb{L}-1)^2$$

If $r=0$ we have to compute $[(w-z)p=0,(v-s)p \neq 0]_{\mathbb{A}^7} = \mathbb{L}^2(\mathbb{L}-1)(\mathbb{L}^2-\mathbb{L})\mathbb{L} = \mathbb{L}^4(\mathbb{L}-1)^2$. So, in total this case gives a contribution

$$\mathbb{L}.[(w-z)p+rx=0,(v-s)p-rt \neq 0]_{\mathbb{A}^8} = \mathbb{L}^5(\mathbb{L}-1)(\mathbb{L}^2-1)$$

If $(w-z)p+rx=0$, $(v-s)p-rt=0$ and $r \neq 0$ we can eliminate x and t from the two equations and p from the defining equation of $\mathbf{S_1}$ and obtain a contribution

$\mathbb{L}^6(\mathbb{L}-1)$. Finally, if $(w-z)p+rx=0$, $(v-s)p-rt=0$ and $r=0$ we get the system of equations

$$\begin{cases} (w-z)p = 0 \\ (v-s)p = 0 \\ (z-w)t+(s-v)x = 1 \end{cases}$$

If $z-w \neq 0$ we have $p=0$ and can eliminate t to get a term $\mathbb{L}^5(\mathbb{L}^2-\mathbb{L})$. If $z-w=0$ then we must have $s-v \neq 0$ and hence $p=0$ and $x=1/(s-v)$ whence a contribution $\mathbb{L}^4(\mathbb{L}^2-\mathbb{L})$. So, this case gives a total contribution of $\mathbb{L}^5(\mathbb{L}^2-1)$. Summing up the contributions of all subcases gives us the claimed motive. □

Proposition 12. *With notations as above, the Brauer-Severi scheme of* $\mathbb{T}^W_{3,2}(0)$ *has a decomposition*

$$\mathbf{BS}^W_{3,2}(0) = \mathbf{S_1} \sqcup \mathbf{S_2} \sqcup \mathbf{S_3}$$

where the schemes $\mathbf{S_i}$ *have (equivariant) motives*

$$\begin{cases} [\mathbf{S_1}] = \mathbb{L}^9 + \mathbb{L}^7 - \mathbb{L}^6 \\ [\mathbf{S_2}] = \mathbb{L}^8 \\ [\mathbf{S_3}] = \mathbb{L}^7 \end{cases}$$

Therefore, the Brauer-Severi scheme has (equivariant) motive

$$[\mathbf{BS}^W_{3,2}(0)] = \mathbb{L}^9 + \mathbb{L}^8 + 2\mathbb{L}^7 - \mathbb{L}^6$$

Proof. $\mathbf{S_3}$: If $v-s \neq 0$ we can eliminate p and obtain a contribution $\mathbb{L}^5(\mathbb{L}^2-\mathbb{L})$. If $v=s$ and $n-r \neq 0$ we can eliminate t and obtain a term $\mathbb{L}^4(\mathbb{L}^2-\mathbb{L})$. Finally, if $v=s$ and $n=r$ we have the identity $n^3=0$ and a contribution $\mathbb{L}^5$.

$\mathbf{S_2}$: If $r-n \neq 0$ we can eliminate x and get a term $\mathbb{L}^6(\mathbb{L}^2-\mathbb{L})$. If $r-n=0$ we get the equation in $\mathbb{A}^8$

$$-\frac{2}{3}n^3 + p(vy+w-z) = 1$$

If $vy+w-z \neq 0$ we can eliminate p and get a contribution $\mathbb{L}^3(\mathbb{L}^4-\mathbb{L}^3)$. Finally, if $vy+w-z=0$ we get the equation $n^3=0$ and hence a term $\mathbb{L}^6$.

$\mathbf{S_1}$: If $(w-z)p+rx \neq 0$ we can eliminate u and obtain a term

$$\mathbb{L}^4(\mathbb{L}^5 - [(w-z)p+rx=0]_{\mathbb{A}^5}) = \mathbb{L}^6(\mathbb{L}-1)(\mathbb{L}^2-1)$$

If $(w-z)p+rx=0$ but $(v-s)p-rt \neq 0$ then we can eliminate y and obtain a contribution

$$\mathbb{L}[(w-z)p+rx=0,(v-s)p-rt \neq 0]_{\mathbb{A}^8} = \mathbb{L}^5(\mathbb{L}-1)(\mathbb{L}^2-1)$$

Now, assume that $(w-z)p+rx=0$ and $(v-s)p-rt=0$. If $r \neq 0$ then we can eliminate p,t and x and get a term $\mathbb{L}^6(\mathbb{L}-1)$. Finally, if $(w-z)p+rx=0$ and $(v-s)p-rt=0$ and $r=0$ we have the system of equations

$$\begin{cases} (w-z)p = 0 \\ (v-s)p = 0 \\ (z-w)t+(s-v)x = 0 \end{cases}$$

If $z-w \neq 0$ we have $p=0$ and can eliminate t to get a term $\mathbb{L}^5(\mathbb{L}^2-\mathbb{L})$. If $z-w=0$ then we get a contribution

$$\mathbb{L}^4[(v-s)p=0,(v-s)x=0]_{\mathbb{A}^4} = \mathbb{L}^4(\mathbb{L}^3+\mathbb{L}^2-\mathbb{L})$$

So, this case gives a total contribution of $2\mathbb{L}^7-\mathbb{L}^5$. □

Now, we have all the information to compute the equivariant motives of the 0- and 1-fibre of the superpotential map as

$$\begin{cases} [\mathbb{M}^W_{3,2}(1)] = \mathbb{L}(\mathbb{L}-1)[\mathbf{BS}^W_{3,2}(1)] + \mathbb{L}^3(\mathbb{L}-2)[\mathbb{M}^W_{3,1}(1)]^2 + 2\mathbb{L}^3[\mathbb{M}^W_{3,1}(1)][\mathbb{M}^W_{3,1}(0)] \\ [\mathbb{M}^W_{3,2}(0)] = \mathbb{L}(\mathbb{L}-1)[\mathbf{BS}^W_{3,2}(0)] + \mathbb{L}^3(\mathbb{L}-1)[\mathbb{M}^W_{3,1}(1)]^2 + \mathbb{L}^3[\mathbb{M}^W_{3,1}(0)]^2 \end{cases}$$

Theorem 1. *If we denote with $\tilde{\mathbf{M}} = 1-[\mu_3]$, then we obtain*

$$[\mathbb{M}^W_{3,2}(0)] - [\mathbb{M}^W_{3,2}(1)] = \mathbb{L}^7\tilde{\mathbf{M}}^2 + \mathbb{L}^6(\mathbb{L}^2-1)\tilde{\mathbf{M}} + 2\mathbb{L}^8 - 3\mathbb{L}^7 + \mathbb{L}^6$$

As a consequence, the second term of the Donaldson-Thomas series is equal to

$$\frac{\mathbb{L}^2\tilde{\mathbf{M}}^2 + \mathbb{L}(\mathbb{L}^2-1)\tilde{\mathbf{M}} + 2\mathbb{L}^3 - 3\mathbb{L}^2 + \mathbb{L}}{(\mathbb{L}-1)(\mathbb{L}^2-1)}$$

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Department of Mathematics, University of Antwerp, Middelheimlaan 1, B-2020 Antwerp (Belgium), lieven.lebruyn@uantwerpen.be