

Absolute geometry and the Habiro topology

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“One can postulate, of course, that $\mathrm{Spec}(\mathbb{F}_1)$ is the absolute point, but the real problem is to develop non-trivial consequences of this point of view.”

(Mikhail Kapranov and Alexander Smirnov in [20])

“Analogies, it is true, decide nothing, but they can make one feel more at home.”

(Sigmund Freud, The Essentials of Psycho-Analysis)

1. Manin’s plane

1.1. Prologue. Ever since the work of Richard Dedekind and Leopold Kronecker the striking similarities between number fields and function fields of smooth projective curves over finite fields have served as a powerful analogy to transport results and conjectures from number theory to these function fields. Using the powerful machinery of geometry one could prove results on the function field side for which the corresponding number theoretic result still remains conjectural. Most noteworthy are André Weil’s proof of the Riemann hypothesis for function fields in the 1940’s and, more recently, the proof of the ABC conjecture for function fields.

Since the mid 1980’s, attempts have been made to mimic Weil’s approach to the Riemann hypothesis by trying to imagine the integral prime spectrum $\text{Spec}(\mathbb{Z})$ to be a “curve” over the *absolute point*, that is, $\text{Spec}(\mathbb{F}_1)$, where \mathbb{F}_1 is the elusive field with one element, and subsequently to study intersection theory on the “surface” $\text{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \text{Spec}(\mathbb{Z})$, which is part of Weil’s approach to the Riemann hypothesis. Since he could not invent what this “surface” might be, Alexander Smirnov decided to study intersection theory on the easier “surface” $\mathbb{P}_{\mathbb{F}_1}^1 \times_{\mathbb{F}_1} \text{Spec}(\mathbb{Z})$, and in particular to investigate the graphs of “maps”

$$q: \text{Spec}(\mathbb{Z}) \longrightarrow \mathbb{P}_{\mathbb{F}_1}^1 \quad \text{where } q \in \mathbb{Q},$$

which should exist by analogy with the function field case, as \mathbb{Q} should be thought of as the function field for the “curve” $\text{Spec}(\mathbb{Z})$. Smirnov dreamed up the following definitions for both geometric objects and for the maps between them (see [38]):

- The *absolute projective line* $\mathbb{P}_{\mathbb{F}_1}^1$ should have as its schematic points the set

$$\{[0], [\infty]\} \cup \{[n] \mid n \in \mathbb{N}_0\}$$

where the degree of the point $[n]$ should be the Euler function $\phi(n)$ (formally defined later on) for $n \in \mathbb{N}_0 = \mathbb{N} \setminus \{0\}$ and equal to 1 for $[0]$ and $[\infty]$. The schematic points $[n]$ should be thought of as corresponding to the set of geometric points being the primitive n th roots of unity.

- The *completed prime spectrum* $\overline{\text{Spec}(\mathbb{Z})}$ should have as its schematic points the set

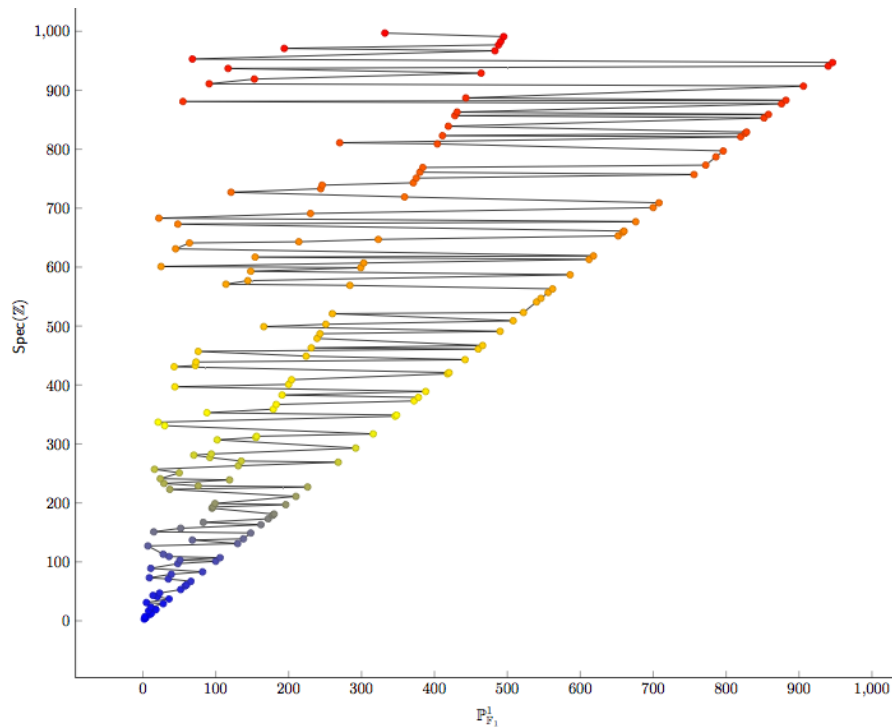
$$\{(p) \mid p \text{ a prime number}\} \cup \{\infty\},$$

where the degree of ∞ is equal to one and the degree of (p) should be $\log(p)$.

- If $q = \frac{a}{b} \in \mathbb{Q}$ with $(a, b) = 1$, the map

$$q: \overline{\text{Spec}(\mathbb{Z})} \longrightarrow \mathbb{P}_{\mathbb{F}_1}^1$$

should send the point (p) to $[0]$ if p is a prime factor of a , to $[\infty]$ if p is a prime factor of b and, in the remaining cases, to $[n]$ if n is the order of the image of $\frac{a}{b}$ in the finite group \mathbb{F}_p^* . Finally, ∞ should be sent to $[0]$ if $a < b$ and to $[\infty]$ if $a > b$.

Figure 1. The graph of $q = 2$.

In Figure 1 there is part of the graph of the map $q = 2$ (for primes $p < 1000$) in the “surface” $\mathbb{P}_{\mathbb{F}_1}^1 \times \overline{\text{Spec}(\mathbb{Z})}$. It is easy to verify that these maps are finite, but showing that they are actually covers for most values of q relies upon a result of Zsigmondy [45]. In [38] Alexander Smirnov was able to deduce the ABC conjecture for \mathbb{Z} provided one would be able to develop an absolute geometry, admitting suitable versions of $\overline{\text{Spec}(\mathbb{Z})}$ and $\mathbb{P}_{\mathbb{F}_1}^1$ and such that one can prove an analogue of the Riemann–Hurwitz formula for maps such as q . Since then, numerous proposals for a geometry over \mathbb{F}_1 have been made, all of them allowing objects such as $\mathbb{P}_{\mathbb{F}_1}^1$ and similar combinatorially defined varieties such as affine and projective spaces, Grassmannians etc., but almost none of them containing objects having the desired properties of $\overline{\text{Spec}(\mathbb{Z})}$. For an overview of these attempts and the connections between them we refer to [28]. Perhaps the most promising approach was put forward by Jim Borger, based on the notion of λ -rings, see [3] and Borger’s chapter in this monograph. For our purposes, a λ -ring is a \mathbb{Z} -algebra R without additive torsion and admitting a commuting family of endomorphisms $\{\Psi^n \mid n \in \mathbb{N}_0\}$ such that for prime numbers p the map Ψ^p is a lift of the Frobenius morphism on $R \otimes_{\mathbb{Z}} \mathbb{F}_p$. Borger interprets this family of endomorphisms as descent data from \mathbb{Z} to \mathbb{F}_1 , and conversely views the forgetful functor, stripping off the λ -structure, as base extension $- \otimes_{\mathbb{F}_1} \mathbb{Z}$. In this approach, $\mathbb{P}_{\mathbb{F}_1}^1$ would then be the usual integral

scheme $\mathbb{P}_{\mathbb{Z}}^1$ equipped with the toric λ -ring structure induced by the endomorphisms $\Psi^n(x) = x^n$ on $\mathbb{Z}[x]$, giving us the fanciful identity

$$\mathbb{P}_{\mathbb{F}_1}^1 \times_{\text{Spec}(\mathbb{F}_1)} \text{Spec}(\mathbb{Z}) = \mathbb{P}_{\mathbb{Z}}^1,$$

where on the right-hand side we forget the λ -structure on the integral projective line $\mathbb{P}_{\mathbb{Z}}^1$, giving us a concrete candidate for Smirnov’s proposal. Unlike other approaches, Borger’s proposal allows us to define how an integral scheme $X_{\mathbb{Z}}$ should be viewed over \mathbb{F}_1 . Indeed, the forgetful functor (that is, base extension $- \otimes_{\mathbb{F}_1} \mathbb{Z}$) has a right-adjoint functor

$$w: \mathbf{rings} \longrightarrow \mathbf{rings}_{\lambda}$$

assigning to a \mathbb{Z} -algebra A a close relative of the ring $w(A) = 1 + tA[[t]]$ of big Witt vectors, equipped with a new addition \oplus being the ordinary multiplication of power series, and a new multiplication \otimes induced functorially by the condition that

$$\left(\frac{1}{1 - a \cdot t} \right) \otimes \left(\frac{1}{1 - b \cdot t} \right) = \frac{1}{1 - ab \cdot t}$$

for all $a, b \in A$. This functor can then be viewed as Weil-restriction from integral schemes to \mathbb{F}_1 -schemes. Hence, in particular, this proposal allows us to define $\text{Spec}(\mathbb{Z})/\mathbb{F}_1$ as the \mathbb{F}_1 -geometric object corresponding to the ring $w(\mathbb{Z})$ which is isomorphic to the completed Burnside ring $\widehat{B}(C)$ of the infinite cyclic group C , by [14]. In these notes we will explore how Smirnov’s maps $q: \text{Spec}(\mathbb{Z}) \rightarrow \mathbb{P}_{\mathbb{F}_1}^1$ fit into Borger’s proposal.

A second theme of these notes is to explore the origins of a new topology on the roots of unity μ_{∞} introduced and studied by Kazuo Habiro in [17] in order to unify invariants of 3-dimensional homology spheres, introduced first by Edward Witten by means of path integrals and rigorously constructed by Reshitikhin and Turaev. Habiro calls two roots of unity *adjacent* to each other whenever their quotient is of pure prime-power order. For example, we depict in Figure 2 the adjacency relation on 60th roots of unity where we used different colors for different prime-powers (2-powers are colored yellow, 3- and 5-powers, respectively, blue and red). The *Habiro topology* on μ_{∞} is then defined by taking as open sets those subsets $U \subset \mu_{\infty}$ having the property that for every $\alpha \in U$ all but finitely many $\beta \in \mu_{\infty}$ that are adjacent to α also belong to U . The Galois action is continuous in this topology, which is in sharp contrast to the induced analytic topology. The Habiro topology is best understood by applying techniques from noncommutative algebraic geometry to objects like $\mathbb{P}_{\mathbb{F}_1}^1$. Recall that the schematic point $[n]$ of $\mathbb{P}_{\mathbb{F}_1}^1$ corresponds to the set of primitive n th roots of unity and hence corresponds to the closed subscheme of $\mathbb{P}_{\mathbb{Z}}^1$ defined by the n th cyclotomic polynomial $\Phi_n(x)$. For $n \neq m$ the corresponding ideals do not have to be co-maximal (that is, the closed subschemes can intersect over some prime numbers p) and, in fact, whenever $\frac{m}{n} = p^k$ for some prime number p there are non-split extensions of $\mathbb{Z}[x, x^{-1}]$ -modules

$$0 \longrightarrow \mathbb{Z}[\zeta_n] \longrightarrow E \longrightarrow \mathbb{Z}[\zeta_m] \longrightarrow 0.$$

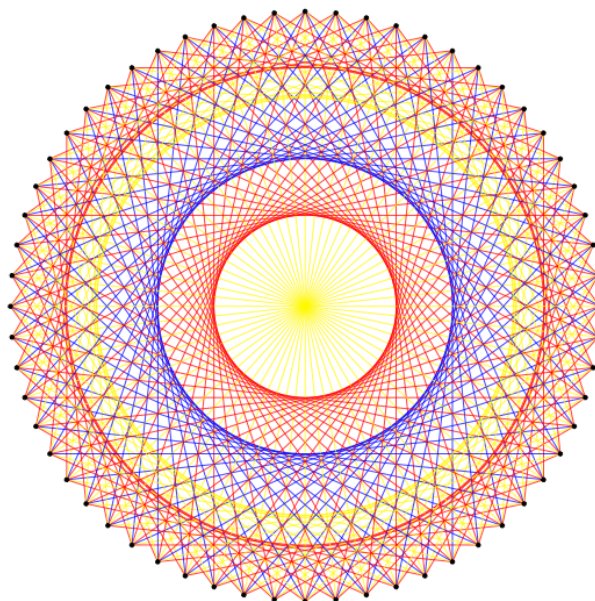


Figure 2. Adjacency relation on 60th roots of unity.

In noncommutative algebraic geometry such situations are interpreted as saying that the corresponding points $[m]$ and $[n]$ lie *infinitely close* to each other, as they share some common tangent information. This then is the origin of the Habiro topology on μ_∞ . So in Figure 2 one should view two roots of unity to be infinitely close whenever they are connected by a colored line, giving us a horrible topological space. The tools of noncommutative geometry allow us to study such bad spaces by associating noncommutative algebras to them; in this case, the Bost–Connes algebra Λ naturally arises from it. More generally, one assigns to a λ -ring A a noncommutative algebra, namely the skew-monoid algebra $A * \mathbb{N}_0^\times$ where the skew-action is determined by the family of endomorphisms Ψ^n . Therefore, one might argue that \mathbb{F}_1 -geometry is essentially of a noncommutative nature. In these notes we will explore this line of thoughts and show, in particular, that the Habiro topology on $\mathbb{P}_{\mathbb{Z}}^1$ is a proper refinement of the Zariski-topology (that is, the cofinite one) and is no longer compact. We can then also define an exotic new topology on $\mathrm{Spec}(\mathbb{Z})$ by demanding that all the Smirnov-maps $q: \mathrm{Spec}(\mathbb{Z}) \rightarrow \mathbb{P}_{\mathbb{F}_1}^1$ should be continuous with respect to the Habiro topology on $\mathbb{P}_{\mathbb{F}_1}^1$.

Acknowledgements. I thank Jim Borger and Jack Morava for several illuminating emails. These notes are based on a rather chaotic master course given in Antwerp in 2011–12. I thank the students for their patience and inspiring enthusiasm and, in particular, Pieter Belmans for pointing me to Mumford’s picture of

$\mathbb{P}_{\mathbb{Z}}^1$, as well as for generous help with Sage/TikZ in order to produce some of the pictures.

1.2. Mumford’s drawings of $\mathbb{A}_{\mathbb{Z}}^1$ and $\mathbb{P}_{\mathbb{Z}}^1$. Let us start with the iconic drawing in Figure 3 of the “arithmetic surface,” that is, of the prime spectrum $\mathbb{A}_{\mathbb{Z}}^1 = \text{Spec}(\mathbb{Z}[x])$, by David Mumford in the original version of his Red Book [34, p. 141]. Subsequent more polished versions of the drawing can be found in the reprinted Red Book [35, p. 75] and [36, p. 24] and [15, p. 85].

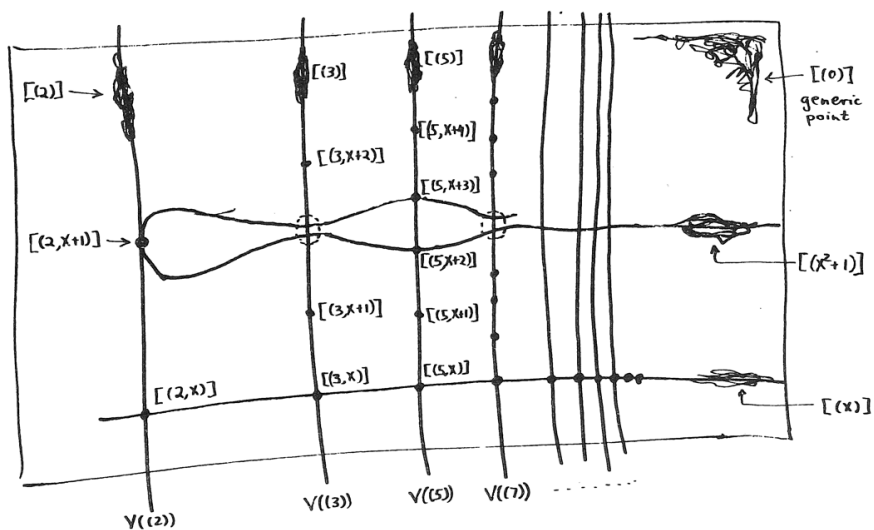


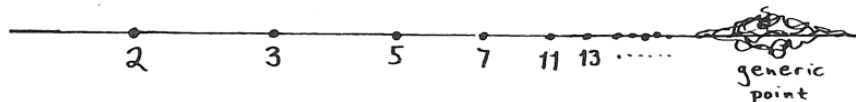
Figure 3. Mumford’s drawing of $\mathbb{A}_{\mathbb{Z}}^1$.

It was believed to be the first depiction of one of Grothendieck’s prime spectra having a real mixing of arithmetic and geometric properties, and as such was influential for generations of arithmetic geometers. Clearly, $\mathbb{A}_{\mathbb{Z}}^1 = \text{Spec}(\mathbb{Z}[x])$, that is, the set of all prime ideals of $\mathbb{Z}[x]$, contains the following elements:

- (0) depicted as the generic point $[(0)]$,
- principal prime ideals (f) , where f is either a prime number p (giving the vertical lines $V((p)) = \text{Spec}(\mathbb{F}_p[x])$) or a \mathbb{Q} -irreducible polynomial written so that its coefficients have greatest common divisor 1 (the horizontal “curves” in the picture such as $[(x^2 + 1)]$),
- maximal ideals (p, f) where p is a prime number and f is a monic polynomial which remains irreducible modulo p , the “points” in the picture.

Mumford’s drawing focuses on the vertical direction as the vertical lines $V((p))$ are the fibers of the projection $\text{Spec}(\mathbb{Z}[x]) \rightarrow \text{Spec}(\mathbb{Z})$ associated to the structural map

$\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. This is consistent with Mumford’s drawing of $\mathrm{Spec}(\mathbb{Z})$ in [34, p. 137] where he writes “ \mathbb{Z} is a principal ideal domain like $k[x]$, and $\mathrm{Spec}(\mathbb{Z})$ is *usually* visualized as a line:



There is one closed point for each prime number, plus a generic point $[(0)]$.” I’ve emphasized the word “usually,” as Mumford knew at the time he was writing the Red Book perfectly well that there were other, and potentially better, descriptions of $\mathrm{Spec}(\mathbb{Z})$ than this archaic prime number line.

In July 1964 David Mumford attended the Woods Hole conference, which became famous for producing the Atiyah–Bott fixed point theorem. On July 10th there were three talks on the hot topic of that time, emerging from Grothendieck’s Parisian seminar: Étale cohomology. Mike Artin spoke on “Étale cohomology of schemes” (see [1]), Jean-Louis Verdier on “A duality theorem in the étale cohomology of schemes” (see [42]) and John Tate on “Étale cohomology over number fields” (see [41]). Later in the conference, Mike Artin and Jean-Louis Verdier ran a “Seminar on étale cohomology of number fields” [2] in which they proved their famous duality result

$$H_{\text{ét}}^r(\mathrm{Spec}(\mathbb{Z}), \mathcal{F}) \times \mathrm{Ext}_{\mathrm{Spec}(\mathbb{Z})}^{3-r}(\mathcal{F}, \mathbb{G}_m) \longrightarrow H_{\text{ét}}^3(\mathrm{Spec}(\mathbb{Z}), \mathcal{F}) \cong \mathbb{Q}/\mathbb{Z}$$

for abelian constructible sheaves \mathcal{F} , suggesting a 3-dimensional picture of $\mathrm{Spec}(\mathbb{Z})$.

Combining this with the fact that the étale fundamental group of $\mathrm{Spec}(\mathbb{Z})$ is trivial (and that the étale fundamental group of $\mathrm{Spec}(\mathbb{F}_p)$ is the profinite completion of $\pi_1(\mathbb{S}^1) = \mathbb{Z}$), Mumford dreamed up the analogy between prime number and knots in \mathbb{S}^3 , see for example the opening paragraph of the unpublished preprint [31] by Barry Mazur: “Guided by the results of Artin and Tate applied to the calculation of the Grothendieck Cohomology Groups of the schemes

$$\mathrm{Spec}(\mathbb{Z}/p\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}),$$

Mumford has suggested a most elegant model as a geometric interpretation of the above situation: $\mathrm{Spec}(\mathbb{Z}/p\mathbb{Z})$ is like a one-dimensional knot in $\mathrm{Spec}(\mathbb{Z})$ which is like a simply connected three-manifold.” This analogy between prime numbers and knots has led in the past decades to the field of “Arithmetic Topology,” a good introduction to which can be found in the lecture notes by Masanori Miyashita [32].

However, the arithmetic plane wasn’t the first attempt by Mumford to draw an arithmetic scheme. In his lectures [33] there is, on page 28, the drawing of $\mathbb{P}_{\mathbb{Z}}^1 = \mathrm{Proj}(\mathbb{Z}[X, Y])$ reproduced in Figure 4. This drawing has at the same time a more classical touch to it, separating the different elements of $\mathrm{Proj}(\mathbb{Z}[X, Y])$ (that is, the graded prime ideals of $\mathbb{Z}[X, Y]$ not containing (X, Y)) according to

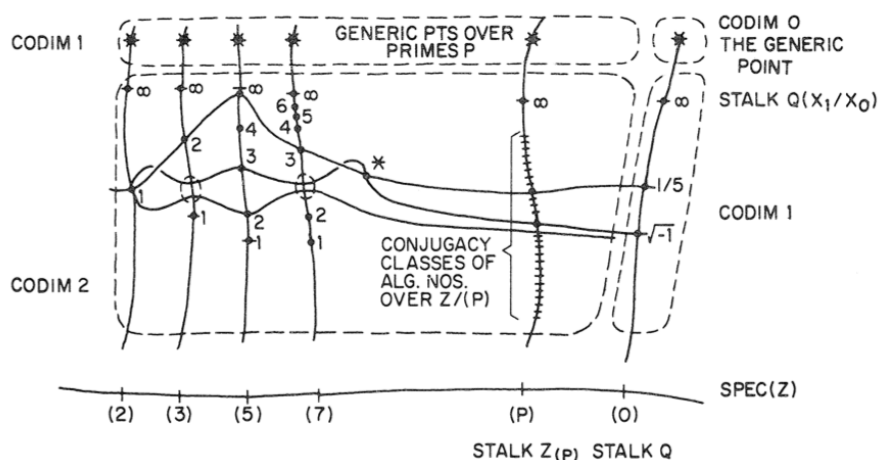


Figure 4. Mumford's drawing of $\mathbb{P}^1_{\mathbb{Z}}$.

codimension, as well as being more modern in that there is a 3-dimensional feel to it (the closed subschemes $\mathbb{V}(X^2 + Y^2)$ and $\mathbb{V}(5X - Y)$ have over- and undercrossings). The points of $\text{Proj}(\mathbb{Z}[X, Y])$ are

- the graded ideal 0 corresponding to the unique codimension-zero point—the generic point;
- the codimension-one points, which correspond to the graded prime ideals of height one which are either the vertical fibers $\mathbb{V}((p)) = \mathbb{P}^1_{\mathbb{F}_p} = \text{Proj}(\mathbb{F}_p[X, Y])$ or the horizontal subschemes corresponding to the homogenization (with respect to Y) of a \mathbb{Q} -irreducible polynomial in $\mathbb{Z}[X]$ written such that the greatest common divisor of its coefficients equals 1; and
- the codimension-two points, which correspond to the graded ideals (p, F) where F is a homogeneous element of $\mathbb{Z}[X, Y]$ such that its reduction modulo p remains irreducible.

For example, the point marked $*$ in Figure 4 is the point $(13, X - 8Y)$. This picture resembles that of $\mathbb{A}^1_{\mathbb{Z}}$ and is in fact the gluing of two such drawings over their intersection; the first is obtained by removing the ∞ -section (that is, $\mathbb{V}(Y)$) and is $\text{Spec}(\mathbb{Z}[x])$ with $x = \frac{X}{Y}$, whereas the second is obtained by removing the 0-section $\mathbb{V}(X)$ and is $\text{Spec}(\mathbb{Z}[x^{-1}])$. They are glued together over their intersection $\text{Spec}(\mathbb{Z}[x, x^{-1}])$.

Influential as these drawings have been, there are a couple of obvious problems with them which will lead us unavoidably to the concept of the *absolute point* $\text{Spec}(\mathbb{F}_1)$, that is, the geometric object associated to the elusive *field with one element* \mathbb{F}_1 .

- (1) **What is the vertical axis?** These drawings of $\mathbb{A}_{\mathbb{Z}}^1$ and $\mathbb{P}_{\mathbb{Z}}^1$ as arithmetic “planes” suggest that, apart from the “horizontal axis” $\text{Spec}(\mathbb{Z})$, coming from the structural morphisms $\mathbb{Z} \hookrightarrow \mathbb{Z}[x^{\pm}]$, there should also be a “vertical axis” and corresponding projection, so what is it?
- (2) **What is the correct topology?** The drawing of the horizontal curves suggests a natural identification between vertical fibers $\mathbb{P}_{\mathbb{F}_p}^1 \leftrightarrow \mathbb{P}_{\mathbb{F}_q}^1$ for primes $p \neq q$, so is there one? And what is the correct topology on these fibers, and on $\text{Spec}(\mathbb{Z})$?

1.3. The vertical axis $\mathbb{P}_{\mathbb{F}_1}^1$. We have seen that the “points” correspond to maximal ideals of $\mathbb{Z}[x]$ which are all of the form $\mathfrak{m} = (p, F)$, where p is a prime number and F is a monic irreducible polynomial such that its reduction $\overline{F} \in \mathbb{F}_p[x]$ remains irreducible. Clearly \mathfrak{m} lies on a unique vertical ruling $\mathbb{V}((p))$ and we wonder whether there exists an appropriate set of horizontal rulings containing all points \mathfrak{m} . We know that the quotient

$$\mathbb{Z}[x]/\mathfrak{m} \cong \mathbb{F}_p[x]/(\overline{F}) \cong \mathbb{F}_{p^d}$$

is the finite field \mathbb{F}_{p^d} , where d is the degree of F , and that its multiplicative group of units is the cyclic group C_{p^d-1} , and hence \mathbb{F}_{p^d} consists of roots of unity together with zero.

This observation led Yuri I. Manin in [30] to consider the ring $\mathbb{Z}[x]_S$, which is the localization of $\mathbb{Z}[x]$ at the multiplicative system S generated by the polynomial $\Phi_0(x) = x$ together with the *cyclotomic polynomials*

$$\Phi_n(x) = \prod_{i=1}^{\mu(n)} (x - \epsilon_i),$$

where ϵ_i runs over all primitive roots of unity of order n , of which there are exactly $\phi(n)$, where ϕ is the *Euler function* $\phi(n) = \#\{1 \leq j < n \mid (j, n) = 1\}$. It follows that the above point \mathfrak{m} lies on the “curve” $[(\Phi_{p^d-1}(x))]$ in the arithmetic plane $\text{Spec}(\mathbb{Z}[x])$. Hence, localizing at S removes all these curves $[(\Phi_n(x))]$ together with all the points lying on them. That is, $\text{Spec}(\mathbb{Z}[x]_S)$ has no height-two prime ideals and so consists of (0) and the remaining height-one prime ideals, all of which are principal. We conclude that the localized ring $\mathbb{Z}[x]_S$ is a principal ideal domain.

This is completely analogous to the more classical setting in which we localize $\mathbb{Z}[x]$ at the multiplicative system S' generated by all prime numbers p , thus getting the principal ideal domain $\mathbb{Q}[x]$. Here the localization removes all the vertical rulings $\mathbb{V}((p))$ from the arithmetic plane together with all the points \mathfrak{m} lying on them. Hence, this suggests to take the “curves” $[(\Phi_n(x))] = \mathbb{V}(\Phi_n(x))$ as a set of horizontal rulings, and then indeed the point \mathfrak{m} lies on the intersection of the vertical ruling $\mathbb{V}((p))$ and of the horizontal ruling $\mathbb{V}(\Phi_{p^d-1}(x))$.

Manin writes: “This suggests that the union of cyclotomic arithmetic curves $\Phi_n(x) = 0$ can be imagined as the union of closed fibers of the projection

$$\text{Spec}(\mathbb{Z}[x]) \longrightarrow \text{Spec}(\mathbb{F}_1[x]),$$

and the arithmetic plane itself as the product of two coordinate axes, an arithmetic one, $\text{Spec}(\mathbb{Z})$, and a geometric one, $\text{Spec}(\mathbb{F}_1[x])$, over the “absolute point” $\text{Spec}(\mathbb{F}_1)$.” Clearly we can repeat the same argument for $\text{Spec}(\mathbb{Z}[x^{-1}])$ and we obtain as Manin’s proposal for a set of horizontal rulings on $\mathbb{P}_{\mathbb{Z}}^1$ the set of codimension-one closed subschemes determined by the irreducible homogeneous polynomials

$$\{\Phi_0 = X, \Phi_\infty = Y\} \cup \left\{ \Phi_n = \prod_{i=0}^{\mu(n)} (X - \epsilon_i Y) \mid n \in \mathbb{N}_0 \right\}.$$

That is, we can extend in Figure 5 Mumford’s drawing of $\mathbb{P}_{\mathbb{Z}}^1$ with a horizontal projection such that every codimension-two point of $\mathbb{P}_{\mathbb{Z}}^1$ lies on the intersection of a vertical and a horizontal ruling. But, you may wonder, what is this elusive “field

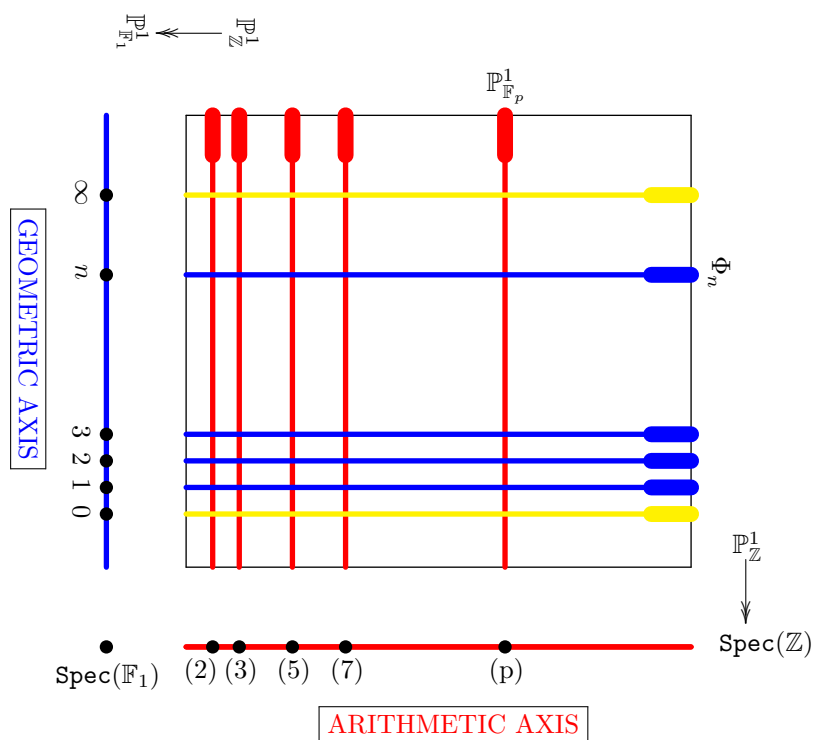


Figure 5. The extended Mumford drawing

with one element” \mathbb{F}_1 , and what do we mean by “geometric objects” defined over it?

Two papers mark the beginning of this subject—one is [38] by Alexander L. Smirnov and the other is [29] by Yuri I. Manin, both originating from the fall of

1991, containing the first constructions over a “field with one element.” Manin introduces the “absolute motive” and Smirnov “an object that partially replaces the projective line over the constant field” in the number field case. Information about the historical origins and motivation behind these two papers can be gleaned from the two letters [39] from A. L. Smirnov to Y. I. Manin and from the unpublished preprint [20] by Smirnov and Mikhail Kapranov. Also, more on Manin’s paper and absolute motives can be found in Thas’s second chapter of the present monograph.

It is an old idea to interpret the combinatorics of finite sets as the limit case of linear algebra over finite fields \mathbb{F}_q , when q goes to 1. This led to a folklore object, the “field with one element,” \mathbb{F}_1 , vector spaces over which are just sets V or pointed sets $V_\bullet = V \cup \{0\}$ —if we want to add a zero vector. The dimension of the vector space V is its cardinality $\#V$ and an \mathbb{F}_1 -linear map between vector spaces $V \rightarrow W$ is just a map of sets (or a map of pointed sets $V_\bullet \rightarrow W_\bullet$ mapping the distinguished element of V —the zero vector—to that of W). Consequently, one should interpret the general linear group $\mathbf{GL}_n(\mathbb{F}_1)$ as the group of all permutations on a set with n elements, that is $\mathbf{GL}_n(\mathbb{F}_1) = \mathbf{S}_n$. The analog of the usual determinant $\det: \mathbf{GL}_n(\mathbb{F}_1) \rightarrow \mathbb{F}_1^*$ is then the sign group morphism $\text{sgn}: \mathbf{S}_n \rightarrow \{\pm 1\}$ and hence one should view the alternating group \mathbf{A}_n as the special linear group $\mathbf{SL}_n(\mathbb{F}_1)$. As an example of the slogan that linear algebra over \mathbb{F}_1 is the same as the combinatorics of finite sets, consider an n -dimensional vector space V/\mathbb{F}_1 (that is, $\#V = n$); then the k -element subsets of V should be viewed as points in the Grassmannian $\mathbf{Grass}(k, n)(\mathbb{F}_1)$, whose cardinality is the limit of the cardinalities of the actual Grassmannians $\mathbf{Grass}(n, k)(\mathbb{F}_q)$ over actual finite fields:

$$\binom{n}{k} = \lim_{q \rightarrow 1} \# \mathbf{Grass}(k, n)(\mathbb{F}_q).$$

(The combinatorial side of the story is described in detail in the first chapter of this monograph.) One can play for some time exploring similar analogies, but quickly one feels that the setting lacks flexibility. In order to resolve this, Alexander L. Smirnov introduced finite field extensions of \mathbb{F}_1 . By analogy with the genuine finite field case, for every n one should have just one field extension \mathbb{F}_{1^n} of degree n up to isomorphism, and Smirnov proposed to define it as the monoid

$$\mathbb{F}_{1^n} = \{0\} \cup \mu_n,$$

where μ_n is the group of all roots of unity of order n . Vector spaces over \mathbb{F}_{1^n} can then be taken to be pointed sets $V_\bullet = \{0\} \cup V$, where V is a set having a free action of μ_n , the dimension being the number of μ_n -orbits. Linear maps are then maps of pointed sets which are also maps of μ_n -sets. Therefore, the Galois group $\text{Gal}(\mathbb{F}_{1^n}/\mathbb{F}_1)$ should be the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ consisting of the power maps $\mu \mapsto \mu^d$ for all $(d, n) = 1$. Taking limits, the algebraic closure of the field with one element should be considered as the pointed set, or monoid, $\overline{\mathbb{F}}_1 = \{0\} \cup \mu_\infty$ consisting of zero together with all roots of unity.

Smirnov and Kapranov remark in [20] that the idea of adjoining roots of unity as analog to extension of the base field goes back at least to a letter of André Weil to Emil Artin [43] in which Weil writes: “Our proof of the Riemann hypothesis

[*Author's note:* in the function field case] depended upon the extension of the function fields by roots of unity, i.e. by constants; the way in which the Galois group of such extensions operates on the classes of divisors in the original field and its extensions gives a linear operator, the characteristic roots (i.e. the eigenvalues) of which are the roots of the zeta function. On a number field, the nearest we can get to this is by adjunction of l th roots of unity, l being fixed; the Galois group of this infinite extension is cyclic, and defines a linear operator on the projective limit of the (absolute) class groups of those successive finite extensions; this should have something to do with the roots of the zeta function of the field. However, our extensions are ramified (but only at a finite number of places, viz. the prime divisors of l). Thus a preliminary study of similar problems in function fields might enable one to guess what will happen in number fields." Smirnov's proposal was then to take as the schematic points of $\mathbb{P}_{\mathbb{F}_1}^1$ the set

$$\{[0], [\infty]\} \cup \{[1], [2], [3], [4], [5], \dots\}$$

of all positive integers \mathbb{N} together with a point at infinity. He also declares the *degree* of the point $n \in \mathbb{N}_0$ to be the Euler function $\phi(n)$, whereas the points $[0]$ and $[\infty]$ have degree one. Here is why. The *geometric* points of $\mathbb{P}_{\mathbb{F}_p}^1$ are of course

$$\mathbb{P}_{\mathbb{F}_p}^1(\overline{\mathbb{F}_p}) = \{[0] = [0 : 1], [\infty] = [1 : 0]\} \cup \{[\alpha] = [\alpha : 1] \mid \alpha \in \overline{\mathbb{F}_p}^*\}.$$

The Galois group $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \widehat{\mathbb{Z}}_+$ acts on this set by fixing the points $[0]$ and $[\infty]$ and by $\sigma([\alpha]) = [\sigma(\alpha)]$. The *schematic* points of $\mathbb{P}_{\mathbb{F}_p}^1$ are then the Galois orbits for this action and the degree of a schematic point is the size of the corresponding orbit. If we assign to an orbit \mathcal{O} the polynomial

$$\prod_{[\alpha] \in \mathcal{O}} (x - \alpha),$$

we see that the schematic points of $\mathbb{P}_{\mathbb{F}_p}^1$ consist of $[\infty]$ together with the set of all monic irreducible polynomials in $\mathbb{F}_p[x]$ and that the notion of degree of the schematic point coincides with the usual degree of the corresponding polynomial.

Here is an alternative description. We claim that we can identify the multiplicative group of the non-zero elements of the algebraic closure

$$\overline{\mathbb{F}_p}^* \cong \mu^{(p)}$$

with the group $\mu^{(p)}$ of all roots of unity having order prime to p . Clearly, if $\alpha \in \overline{\mathbb{F}_p}^*$, then the order of α is a divisor of $p^n - 1$ and hence a number prime to p . Conversely, if $(m, p) = 1$ then the residue class $\bar{p} \in \mathbb{Z}/m\mathbb{Z}$ is a unit and therefore for some integer n we must have $p^n \equiv 1 \pmod{m}$. But then, $m \mid p^n - 1$ and the primitive m th roots of unity can be identified with a subgroup of the multiplicative group $\overline{\mathbb{F}_p}^*$. However, describing the correspondence $\overline{\mathbb{F}_p}^* \leftrightarrow \mu^{(p)}$ explicitly from a given construction of $\overline{\mathbb{F}_p}$ is very challenging and we will address it later.

By *analogy* we can therefore define the *geometric* points of $\mathbb{P}_{\mathbb{F}_1}^1$ as being the set

$$\mathbb{P}_{\mathbb{F}_1}^1(\overline{\mathbb{F}_1}) = \{[0], [\infty]\} \cup \mu^{(1)},$$

with $\mu^{(1)}$ the set of all roots of unity with order “prime to 1,” that is, the group μ_∞ of all roots of unity, leading to the proposal that

$$\overline{\mathbb{F}}_1 = \{0\} \cup \mu_\infty.$$

The *schematic* points of $\mathbb{P}_{\mathbb{F}_1}^1$ are then the orbits of this set under the action of the Galois group $\text{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) = \widehat{\mathbb{Z}}^*$. Clearly, these orbits are classified by

$$\{[0], [\infty]\} \cup \{[1], [2], [3], [4], [5], \dots\}$$

where $[n]$ is the orbit consisting of all primitive n th roots of unity, and hence the degree of the schematic point $[n]$ must be equal to the number of primitive n th roots of unity, that is, to $\phi(n)$.

2. Habiro topology

2.1. The Habiro topology on $\mathbb{P}_{\mathbb{F}_1}^1$. The additive structure of the *profinite integers* $\widehat{\mathbb{Z}}$ as well as its multiplicative group of units $\widehat{\mathbb{Z}}^*$ have already made their appearance as (absolute) Galois groups. As we will encounter them often, let us formally define these profinite integers following Hendrik Lenstra’s account in [27].

Recall that any positive integer n has a unique representation of the form

$$n = c_k \cdot k! + c_{k-1} \cdot (k-1)! + \dots + c_2 \cdot 2! + c_1 \cdot 1!$$

where the “digits” c_i are integers such that $0 \leq c_i \leq i$ for all $1 \leq i \leq k$ and $c_k \neq 0$. We then write n in the *factorial number system* as $n = (c_k c_{k-1} \dots c_2 c_1)_!$, so for example $25 = (1001)_!$. Profinite integers arise if we allow the sequences of digits to extend indefinitely to the left to get expressions such as $(\dots c_4 c_3 c_2 c_1)_!$. One can then identify the positive integers \mathbb{N} to be those profinite integers with $c_i = 0$ for $i \gg 0$. Also the negative integers $-\mathbb{N}_0$ can be characterized as those profinite integers such that $c_i = i$ for all but finitely many i . For example, $-1 = (\dots 654321)_!$, that is $c_i = i$ for all i .

To add two profinite integers, one adds them digitwise, proceeding from the right, and if the sum of the two i th digits is larger than i , one subtracts $i + 1$ from it and adds a carry of 1 to the sum of the $(i + 1)$ th digits. With this rule we have indeed that -1 (as above) $+1 = 0$. To multiply two profinite integers we use the rule that the first k digits of the product $s \times t$ depend only on the first k digits of s and t , thereby reducing the problem of computing products to the case of ordinary positive integers. These operations make the profinite integers $\widehat{\mathbb{Z}}$ into a commutative ring with unit element 1. Those in the know will have observed that all we did was to work out the ring rules for the projective limit $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n!\mathbb{Z}$.

But let us return to $\mathbb{P}_{\mathbb{F}_1}^1$. We have defined, following Smirnov, that the schematic points of $\mathbb{P}_{\mathbb{F}_1}^1$ are the orbits of μ_∞ under the action of the multiplicative group of profinite integers $\widehat{\mathbb{Z}}^*$, which is the abelian Galois group $\text{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$. We would now like to define a topology on μ_∞ compatible with this action, and clearly the induced analytic topology does not satisfy this condition.

In [18] Kazuo Habiro introduced a new topology on μ_∞ in order to unify invariants of 3-dimensional homology spheres, introduced first by Ed Witten by means of path integrals and rigorously constructed by Reshitikhin and Turaev. Two roots of unity $\alpha, \beta \in \mu_\infty$ are said to be *adjacent* if their quotient $\alpha\beta^{-1}$ is of pure prime-power order p^m for $m \in \mathbb{Z}$ and p a prime number, or equivalently, if the difference $\alpha - \beta$ is not a unit in the integral closure of \mathbb{Z} in $\mathbb{Q}(\alpha, \beta)$. Clearly, the action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and of $\widehat{\mathbb{Z}}^* = \text{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ preserves adjacency. In Figure 6 we indicated adjacency of 32nd roots of unity under increasing powers

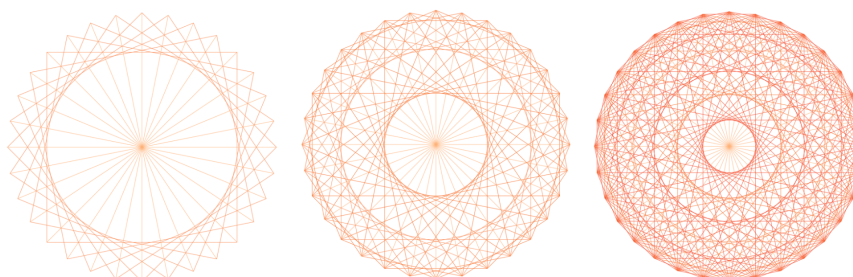


Figure 6. The relation of 2^k adjacency among 32nd roots of unity, for $k = 2, 3$ and 4 .

2^k , for $k \leq 2, 3$ and 4 ; finally, in Figure 7 we indicated adjacency under 5th powers of 2.

The *Habiro topology* on μ_∞ is then defined by taking as open sets those subsets $U \subset \mu_\infty$ having the property that for every $\alpha \in U$ all except finitely many $\beta \in \mu_\infty$ adjacent to α also belong to U . The Galois action is continuous in this topology, in sharp contrast to the case of the induced analytic topology. Further, any cofinite subset is clearly open in the Habiro topology, but we will soon see that there are plenty of other open subsets. In fact we will show that the Habiro topology is not locally compact on μ_∞ .

But let us first explain the \mathbb{F}_1 -origins of the Habiro topology. Our depiction of $\mathbb{P}_{\mathbb{Z}}^1$ in Figure 5 as the product of the arithmetic and the geometric axis was an over-simplification. Whereas the vertical fibers $\mathbb{V}(p)$ (the red lines) are clearly disjoint, this is not necessarily the case for the blue lines $\mathbb{V}(\Phi_n)$, because the height-one prime ideals $(\Phi_n(x))$ and $(\Phi_m(x))$ do not have to be comaximal, and whenever $(\Phi_n(x), \Phi_m(x)) \neq \mathbb{Z}[x]$, there will be a point lying on both curves, so the “lines” will intersect over certain prime numbers. This must happen, as we know that the prime spectrum of the integral group ring $\text{Spec}(\mathbb{Z}C_n)$ is connected, see for example [37]. Hence, its minimal prime ideals, which are all of the form $\mathbb{V}(\Phi_d(x))$ for $d \mid n$, will intersect. For cyclotomic polynomials we have complete information about potential comaximality:

- If $\frac{m}{n} \neq p^k$ for some prime number p , then $(\Phi_m(x), \Phi_n(x)) = \mathbb{Z}[x]$, so these cyclotomic prime ideals are comaximal and the corresponding blue lines will not intersect; however
- if $\frac{m}{n} = p^k$ for some prime number p , then $\Phi_m(x) \equiv \Phi_n(x)^d \pmod{p}$ for some

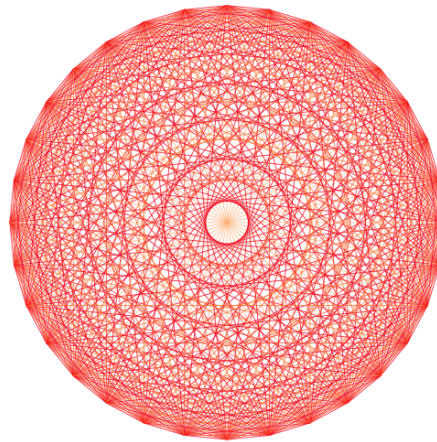


Figure 7. The relation of 2^5 adjacency among 32nd roots of unity.

integer d and hence these cyclotomic prime ideals are not comaximal and the corresponding blue lines will intersect over p .

In the second case we have non-split extensions as $\mathbb{Z}[x, x^{-1}]$ -modules

$$0 \longrightarrow \mathbb{Z}[\zeta_n] \longrightarrow E \longrightarrow \mathbb{Z}[\zeta_m] \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}[\zeta_m] \longrightarrow F \longrightarrow \mathbb{Z}[\zeta_n] \longrightarrow 0$$

In fact, Fritz-Erdmann Diederichsen, a student of Zassenhaus, calculated already in 1940 that there are exactly $p^{\mu(\min(m,n))}$ such extensions, in either direction, see [11] (see also [10, Theorem 25.26]).

In noncommutative algebraic geometry we are very familiar with such situations. If S and T are two finite-dimensional simple representations of a \mathbb{C} -algebra A such that $\text{Ext}_A^1(S, T) \neq 0$, we say that their annihilating maximal ideals belong to the same “clique” and know that we should think of S and T as two noncommutative points lying infinitely close together, or equivalently, that S and T share some tangent information. Using this noncommutative intuition we therefore define a clique relation—or an adjacency relation—on pairs of natural numbers, via

$$m \sim n \quad \text{if and only if} \quad \frac{m}{n} = p^{\pm a}$$

for some prime number p . It is this clique relation which lies behind the definition of the Habiro topology on μ_∞ . In Figure 8, we depict the inter-weaving patterns of the horizontal lines $\Phi_1, \Phi_p, \Phi_q, \Phi_{p^2}$ and Φ_{pq} for prime numbers $p < q$. The *Habiro topology* on $\mathbb{P}_{\mathbb{F}_1}^1 = \{[n] \mid n \in \mathbb{N}_0^\times\} \cup \{[0], [\infty]\}$ is therefore defined as follows. An *open set* is a subset of $\mathbb{P}_{\mathbb{F}_1}^1$ of the form

$$U \quad \text{or} \quad U^0 = U \cup \{[0]\} \quad \text{or} \quad U^\infty = U \cup \{[\infty]\} \quad \text{or} \quad U^{0\infty} = U \cup \{[0], [\infty]\},$$

2.2. The étale site of \mathbb{F}_1 . Until now our exposition has been pretty intuitive and we would like to have a solid framework to give formal definitions for these elusive objects as well as to perform actual calculations. Such a proposal was put forward by Jim Borger in 2009 in [3] and it streamlined a plethora of previous attempts, all giving more or less the same class of examples. For a map of \mathbb{F}_1 -land before Borger we refer to [28].

Borger’s approach is based upon the notion of λ -rings in the sense of Grothendieck’s Riemann–Roch theory [16]. For us, a λ -ring will be a commutative ring R with identity *without additive torsion*, equipped with maps $\lambda^n: R \rightarrow R$ for $n \in \mathbb{N}$ satisfying an unwieldy list of axioms, see for example [21, Chapter 1]. By [44], under our no-torsion assumption, R is a λ -ring if and only if there is a collection of *commuting* ring endomorphisms $\psi^p: R \rightarrow R$ for all prime numbers p such that ψ^p is a lift of the Frobenius automorphism on $R \otimes_{\mathbb{Z}} \mathbb{F}_p$, that is, we have a commuting diagram for all prime numbers p

$$\begin{array}{ccc} R & \xrightarrow{\psi^p} & R \\ \downarrow & & \downarrow \\ R/pR & \xrightarrow{\text{Frob}} & R/pR \end{array}$$

Borger’s proposal is to *define* \mathbb{F}_1 -algebras as \mathbb{Z} -algebras with a λ -ring structure, the idea being that one can interpret the collection of commuting endomorphisms $\{\psi^p\}$ as descent data from \mathbb{Z} -algebras to \mathbb{F}_1 -algebras. Maps of \mathbb{F}_1 -algebras are then morphisms of λ -rings, that is, ring morphisms commuting with the Frobenius lifts $\{\psi^p\}$. Accepting this proposal, it then follows that base ring extension $- \otimes_{\mathbb{F}_1} \mathbb{Z}$ can be viewed formally as the operation of forgetting the λ -ring structure, that is, of stripping off the \mathbb{F}_1 -structure.

In this way we can *define* \mathbb{F}_1 to be \mathbb{Z} with its unique λ -ring structure in which all $\psi^p = \text{id}_{\mathbb{Z}}$. Similarly, the polynomial ring $\mathbb{F}_1[x]$ can be defined to be the integral polynomial ring $\mathbb{Z}[x]$ equipped with the λ -ring structure defined by $\psi^p(x) = x^p$, which is indeed a \mathbb{Z} -endomorphism lift of the Frobenius automorphism on $\mathbb{F}_p[x]$ by little Fermat and the binomial theorem. The field \mathbb{F}_{1^n} should then be taken to be the integral group ring $\mathbb{Z}[\mu_n]$ with λ -ring structure induced by $\psi^p(\mu) = \mu^p$. This gives the fanciful identities

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[\mu_n], \quad \mathbb{F}_1[x] \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[x] \quad \text{and} \quad \mathbb{P}_{\mathbb{F}_1}^1 \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{P}_{\mathbb{Z}}^1,$$

where on the right-hand sides we forget about the λ -structures.

If G is a finite group and χ_1, \dots, χ_h are its irreducible characters, then the Grothendieck ring of finite-dimensional $\mathbb{C}[G]$ -representations

$$R(G) = \mathbb{Z}\chi_1 \oplus \dots \oplus \mathbb{Z}\chi_h$$

has a λ -ring structure defined by $\psi^p(\chi) = \chi'$, where χ' is the class function $\chi'(g) = \chi(g^p)$ for all $g \in G$, see for example [37, §9.1]. If p does not divide the order of G , then ψ^p is an automorphism permuting the irreducible characters. As the ψ^p

commute, we can define operations $\psi^n = (\psi^{p_1})^{\circ a_1} \circ \dots \circ (\psi^{p_k})^{\circ a_k}$ if $n = p_1^{a_1} \dots p_k^{a_k}$. The ψ^n are the Adams operations and they give an action of the multiplicative monoid \mathbb{N}_0^\times on $R(G)$. On the other hand, we have that

$$\mathbb{Q}(\mu_\infty) \otimes_{\mathbb{Z}} R(G) \cong \underbrace{\mathbb{Q}(\mu_\infty) \times \dots \times \mathbb{Q}(\mu_\infty)}_{h \text{ copies}}$$

and hence there is a Galois action of $\widehat{\mathbb{Z}}^* = \text{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ on $\mathbb{Q}(\mu_\infty) \otimes_{\mathbb{Z}} R(G)$. This Galois action is compatible with the action of Adams operations on $R(G)$ via the embedding $\mathbb{N}_0^\times \hookrightarrow \widehat{\mathbb{Z}}^*$, see for example [37, §12.4].

James Borger and Bart de Smit vastly generalized this example in [4] to include all reduced λ -rings R that are finite projective \mathbb{Z} -modules, allowing us to describe the Galois (or étale) site of \mathbb{F}_1 .

Let us first consider the case when $K = R \otimes_{\mathbb{Z}} \mathbb{Q}$ is a λ -ring and a finite étale \mathbb{Q} -algebra. By Grothendieck’s version of Galois theory, we have an anti-equivalence of categories between the category of finite étale \mathbb{Q} -algebras and the category of finite discrete sets equipped with a continuous action of the absolute Galois group $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, assigning to K the set

$$S = \text{alg}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$$

on which $\sigma \in G$ acts by left composition, that is, $\sigma \cdot s = \sigma \circ s$. If we have in addition a λ -ring structure, that is, a commuting family of \mathbb{Q} -endomorphisms Ψ^n on K , then we have also an action of the monoid \mathbb{N}_0^\times on S by composition on the right, that is, $n \cdot s = s \circ \Psi^n$. Consequently, the category of rational λ -algebras which are finite étale over \mathbb{Q} is the category of finite discrete sets equipped with a continuous action of the monoid $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{N}_0^\times$, where \mathbb{N}_0^\times is given the discrete topology.

The Galois action on S gives us a group morphism

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{perm}(S) \subset \text{maps}(S),$$

where $\text{perm}(S)$ are all permutations on S and $\text{maps}(S)$ are all set-maps from S to itself. The kernel N gives us a finite Galois extension $L = \overline{\mathbb{Q}}^N$ of \mathbb{Q} with Galois group $\overline{G} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/N$, and as $R \otimes_{\mathbb{Z}} \mathbb{Q} = K = L_1 \times \dots \times L_k$, all L_i are subfields of L and hence every $\sigma \in \overline{G}$ induces an automorphism on L_i . Let \mathcal{O}_L be the integral closure of \mathbb{Z} in L ; then by Galois theory we have

$$S = \text{alg}_{\mathbb{Q}}(K, \overline{\mathbb{Q}}) = \text{alg}_{\mathbb{Q}}(K, L) = \text{alg}_{\mathbb{Z}}(R, \mathcal{O}_L)$$

and we have an action by $(\sigma, n) \in \overline{G} \times \mathbb{N}_0^\times$ on S via

$$\begin{array}{ccc} R & \xrightarrow{s} & \mathcal{O}_L \\ \psi^n \uparrow & \searrow & \downarrow \sigma \\ R & & \mathcal{O}_L \end{array}$$

By Chebotarev’s theorem, there are infinitely many prime numbers p such that there exists a prime ideal $P \triangleleft \mathcal{O}_L$ lying over p with σ a lift of the Frobenius automorphism on \mathcal{O}_L/P , so we can choose a prime p not dividing the discriminant $\Delta(R)$. Note that if $p \nmid \Delta(R)$ there is a unique lift of the Frobenius map which is automatically an automorphism by the category equivalence between étale \mathbb{F}_p - and étale $\widehat{\mathbb{Z}}_p$ -algebras, where $\widehat{\mathbb{Z}}_p$ is the p -adic completion. But then, the restriction of σ on $R \otimes \widehat{\mathbb{Z}}_p$ to R via the embedding s is equal to Ψ^p and we have $\sigma \circ s = s \circ \Psi^p$ on R . As this holds for any $\sigma \in \overline{G}$ we have that the image of $\overline{G} \rightarrow \text{perm}(S) \subset \text{maps}(S)$ is contained in the image of $\mathbb{N}_0^\times \rightarrow \text{maps}(S)$. But, as \mathbb{N}_0^\times is an abelian monoid, it follows that the image in $\text{perm}(S)$ is abelian and hence that L/\mathbb{Q} is an abelian Galois extension!

But now we can invoke the Kronecker–Weber theorem asserting that $L \subset \mathbb{Q}(\mu_c)$ where c is only divisible by primes p which ramify in L , and as L is the common Galois extension of the components of $R \otimes_{\mathbb{Z}} \mathbb{Q}$, those p must also divide $\Delta(R)$. That is, there exists a $c \in \mathbb{N}$ with all its prime factors dividing the discriminant $\Delta(R)$ such that the Galois action on $S = \text{alg}_{\mathbb{Z}}(R, \mathcal{O}_L)$ factors through the cyclotomic character

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow (\mathbb{Z}/c\mathbb{Z})^* = \text{Gal}(\mathbb{Q}(\mu_c)/\mathbb{Q})$$

and for all $p \nmid \Delta(R)$ the action of $p \in \mathbb{N}_0^\times$ on S is equal to that of $p \pmod{c} \in \text{Gal}(\mathbb{Q}(\mu_c)/\mathbb{Q})$.

So far, we have factored the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{N}_0^\times$ -action on S via $\widehat{\mathbb{Z}}^* \times \mathbb{N}_0^\times$ and now we want to factor it further through the map $\widehat{\mathbb{Z}}^* \times \mathbb{N}_0^\times \longrightarrow \widehat{\mathbb{Z}}^\times$ via the natural embeddings of both factors and where $\widehat{\mathbb{Z}}^\times$ is the multiplicative monoid of the profinite integers $\widehat{\mathbb{Z}}$.

We can apply the foregoing for every $d \in \mathbb{N}_0^\times$ on the sub λ -ring $\Psi^d(R) \subset R$ with corresponding finite set $d \cdot S$. That is, there exists $c_d \in \mathbb{N}_0$ such that the Galois action on $d \cdot S$ factors through $(\mathbb{Z}/c_d\mathbb{Z})^*$ and such that the action of any n with $nd \cdot S = d \cdot S$ is the same as the action of $n \pmod{c_d}$. For every prime number p let a_p be the smallest power such that $p^{a_p+1} \cdot S = a^{a_p} \cdot S$; as $a_p > 0$ only for those $p \mid \Delta(R)$, we have a finite number

$$r_0 = \prod_{p \mid \Delta(R)} p^{a_p}$$

which satisfies the property that for all $n \in \mathbb{N}_0$ we have that $n \cdot S = \text{gcd}(n, r_0) \cdot S$. Now let r be the least common multiple of all $d \cdot c_d$ where $d \mid r_0$; then we claim that the above action factors through the multiplicative monoid $(\mathbb{Z}/c_d\mathbb{Z})^\times$, that is, we have to show that if $d_1 \equiv d_2 \pmod{r}$, then the actions of d_1 and d_2 on S coincide. As $r_0 \mid r$, we have $\text{gcd}(d_1, r_0) = \text{gcd}(d_2, r_0) = d$, whence $d_1 \cdot S = d \cdot S = d_2 \cdot S$. If we write $d_i = dd'_i$ this entails that $\text{gcd}(d'_i, c_d) = 1$, but then $d_1 = dd'_1 \equiv dd'_2 = d_2 \pmod{dc_d}$. But then, $d'_1 \equiv d'_2 \pmod{c_d}$ and so they act in the same way on $d \cdot S$ whence d_1 and d_2 act in the same way on S !

This then is the main result of Borger and de Smit [4]: that a necessary and sufficient condition for the existence of an integral λ -ring R which is finite and projective over \mathbb{Z} contained in the λ -ring $R \otimes_{\mathbb{Z}} \mathbb{Q} = K$ is that the action of the

monoid $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{N}_0^\times$ on the finite set S describing K , factors through an action of the monoid $\widehat{\mathbb{Z}}^\times$. It follows that the category of all such λ -rings is anti-equivalent to the category of finite discrete sets with a continuous action of the monoid $\widehat{\mathbb{Z}}^\times$ and that every such λ -ring is contained as λ -ring in a product of cyclotomic fields, generalizing the case of the Grothendieck ring $R(G)$ of a finite group G .

Motivated by Grothendieck’s interpretation of Galois theory, we have the fanciful picture of the *absolute Galois monoid* of the field with one element \mathbb{F}_1 :

$$\text{Gal}(\overline{\mathbb{F}}_1/\mathbb{F}_1) \cong \widehat{\mathbb{Z}}^\times.$$

Because the subset $0 \cdot S \subset S$ is Galois invariant, it corresponds to a factor \mathbb{Q} in K , so K can never be a field unless $K = \mathbb{Q}$. In particular, whereas $\mathbb{Z}[\mu_n] = \mathbb{Z}[x]/(x^n - 1)$ is an integral λ -ring, the subring $\mathbb{Z}[\zeta_n]$ of the cyclotomic field $\mathbb{Q}(\zeta_n)$ where ζ_n is a primitive n th root of unity is *not*.

For example, let us work out the λ -ring structure on $R(\mathbf{S}_3)$, the representation ring of the symmetric group \mathbf{S}_3 and its associated finite $\widehat{\mathbb{Z}}$ -set. For any finite group G let $X = X(G)$ be the set of conjugacy classes in G ; then we can identify this set with

$$S = \text{alg}_{\mathbb{Z}}(R(G), \mathbb{C}) = \{x: R(G) \rightarrow \mathbb{C} \mid x(V_i) = \chi_{V_i}(x) \forall V_i \in \text{irreps}(G)\}.$$

Moreover, one knows in general that the discriminant verifies

$$\Delta(R(G)) = \frac{(\#G)^{\#X}}{\prod_{x \in X} \#x}.$$

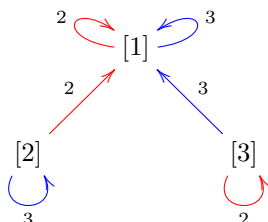
Specializing to the case when $G = \mathbf{S}_3$ we have that $\Delta(R(\mathbf{S}_3)) = 36$, and we recall that the character table of \mathbf{S}_3 is

x	[1]	[2]	[3]
	()	(1, 2)	(1, 2, 3)
V_1	1	1	1
V_2	1	-1	1
V_3	2	0	-1

The Frobenius lifts (a.k.a. Adams operators) send a class function χ to the class function $\Psi^n(\chi): g \mapsto \chi(g^n)$. Moreover, $()^n = ()$ for all n , $(1, 2)^n = ()$ for even n and $= (1, 2)$ for odd n , and $(1, 2, 3)^n = ()$ for n a multiple of 3, and is conjugated to $(1, 2, 3)$ otherwise. Therefore, if χ_i is the character function of V_i , one computes from the character table that for prime numbers p we have

- $\Psi^p(\chi_1) = \chi_1, \forall p$
- $\Psi^2(\chi_2) = \chi_1$ and $\Psi^p(\chi_2) = \chi_2, \forall p \neq 2$
- $\Psi^2(\chi_3) = \chi_1 + \chi_3 - \chi_2, \Psi^3(\chi_3) = \chi_1 + \chi_2$ and $\Psi^p(\chi_3) = \chi_3, \forall p \neq 2, 3$

which determines the λ -ring structure on $R(\mathbf{S}_3)$. The action of $n \in \mathbb{N}_0^\times$ on the algebra map $[i] \in \mathbf{alg}_{\mathbb{Z}}(R(\mathbf{S}_3), \mathbb{C})$ is given by $n \cdot [i] = [i] \circ \Psi^n$, and hence it follows from the above that $p \cdot [i] = [i]$ for all primes $p \neq 2, 3$ and one verifies that the action of 2 and 3 is given by the following maps on $S = \{[1], [2], [3]\}$;



From this it follows that $2 \cdot S = 2^2 \cdot S = \{[1], [3]\}$ and $3 \cdot S = 3^2 \cdot S = \{[1], [2]\}$, whereas $6 \cdot S = 12 \cdot S = 18 \cdot S = \{[1]\}$. Further, the Galois action on $R(G)$ and any of its sub λ -rings is trivial. With the notations used before we therefore get that $r_0 = 6$ and all $c_d = 1$, showing that the $\widehat{\mathbb{Z}}^\times$ -action on S factorizes through the multiplicative monoid action of $(\mathbb{Z}/6\mathbb{Z})^\times$ as indicated in the above colored graph.

2.3. What is $\mathbb{P}_{\mathbb{F}_1}^1$? Now that we have a formal definition of \mathbb{F}_1 -algebras, namely those \mathbb{Z} -rings without additive torsion which are λ -rings, it makes sense to define for any such λ -ring R its λ -spectrum, which is the collection of all kernels of λ -ring morphisms from R to reduced λ -rings

$$\mathbf{Spec}_\lambda(R) = \{\ker(R \xrightarrow{\phi} A) \mid A \text{ is a reduced } \lambda\text{-ring and } \phi \in \mathbf{alg}_\lambda(R, A)\},$$

which “is” clearly functorial.

The geometric or $\overline{\mathbb{F}}_1$ -points in the λ -spectrum then correspond to kernels of λ -ring morphisms $R \rightarrow A$, where A is one of the integral λ -rings described in the previous section, that is, a finite projective \mathbb{Z} -ring with λ -structure such that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an étale \mathbb{Q} -algebra. We have seen that such rings are of the form $A = A_S$, where S is a finite set with a continuous monoid action by $\widehat{\mathbb{Z}}^\times$. As these sets are ordered by inclusions $S \subset T$ compatible with the $\widehat{\mathbb{Z}}^\times$ -action and knowing that via the anti-equivalence this corresponds to λ -ring epimorphisms $A_T \twoheadrightarrow A_S$, it makes sense to define the *maximal λ -spectrum* of R to be

$$\mathbf{max}_\lambda(R) = \{\ker(R \xrightarrow{\phi} A) \mid A \otimes \mathbb{Q} \text{ is étale over } \mathbb{Q} \text{ and } \phi \in \mathbf{alg}_\lambda(R, A)\}.$$

As this space may still be too hard to compute in specific examples, we often reduce to the subset of all *cyclotomic points* (or in Manin parlance of [30], the *cyclotomic coordinates*), which is the set

$$\mathbf{max}_{\text{cycl}}(R) = \{\ker(R \xrightarrow{\phi} \mathbb{Z}[\mu_n]) \mid n \in \mathbb{N}, \phi \in \mathbf{alg}_\lambda(R, \mathbb{Z}[\mu_n])\}.$$

For the λ -ring $\mathbb{Z}[x]$ we have that

$$\mathbf{Spec}_\lambda(\mathbb{Z}[x]) = \{0\} \cup \mathbf{max}_\lambda(\mathbb{Z}[x]) \quad \text{and} \quad \mathbf{max}_{\text{cycl}}(\mathbb{Z}[x]) = \{(\Phi_n(x)) \mid n \in \mathbb{N}\},$$

since any λ -ring epimorphism $\mathbb{Z}[x] \rightarrow \mathbb{Z}[\mu_n]$ must map x to x^i with $(i, n) = 1$, that is, to a primitive n th root of unity. Hence, we finally have a formal definition of $\mathbb{P}_{\mathbb{F}_1}^1$: it is the set of cyclotomic points of $\mathbb{P}_{\mathbb{Z}}^1$, equipped with the Habiro topology.

One can again use methods from noncommutative algebraic geometry to obtain “geometric objects” and their associated “rings of functions” and apply this to the setting of \mathbb{F}_1 -geometry to arrive at a similar description.

In [22] Maxim Kontsevich and Yan Soibelman introduce a *noncommutative thin scheme* (over the complex numbers) as a covariant functor $\mathbb{X}: \mathbf{fd}\text{-alg}_{\mathbb{C}} \rightarrow \mathbf{sets}$ from finite-dimensional (not necessarily commutative) \mathbb{C} -algebras to sets, commuting with finite projective limits. They show that every such thin scheme is represented by a coalgebra $C_{\mathbb{X}}$ which they call the *coalgebra of distributions* on \mathbb{X} , and its dual algebra $C_{\mathbb{X}}^*$ (note that we use the superscript $*$ in this discussion to denote the full linear dual, and *not* the group of units) is then called the *algebra of functions* $\mathcal{O}(\mathbb{X})$ on \mathbb{X} . We will be interested in affine thin schemes obtained by associating to a \mathbb{C} -algebra A its *representation functor*

$$\mathbf{rep}_A: \mathbf{fd}\text{-alg}_{\mathbb{C}} \longrightarrow \mathbf{sets}, \quad B \longmapsto \mathbf{alg}_{\mathbb{C}}(A, B).$$

By Kostant duality (see for example [40, Chapter VI]), this thin scheme is represented by the *dual coalgebra* A° which consists of all linear functionals on A which factor through a finite-dimensional algebra quotient of A :

$$A^{\circ} = \{f \in A^* \mid \ker(f) \supset I \triangleleft A \text{ such that } \dim_{\mathbb{C}}(A/I) < \infty\},$$

and hence its corresponding ring of functions is $(A^{\circ})^*$. One can use the A_{∞} -structure on Yoneda-Ext algebras to describe the structure of the dual coalgebra A° for general A , see [24].

The motivating example being X a commutative (complex) affine variety, when the dual coalgebra $\mathbb{C}[X]^{\circ}$ decomposes over the points of X , we obtain—since distinct maximal ideals \mathfrak{m}_x are comaximal—that

$$C_X = \mathbb{C}[X]^{\circ} = \bigoplus_{x \in X} C_{X,x},$$

where $C_{X,x}$ is a subcoalgebra of the enveloping coalgebra $U(T_{X,x})$ of the abelian Lie algebra on the Zariski tangent space $T_{X,x} = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Consequently, the ring of functions also decomposes over the points

$$(\mathbb{C}[X]^{\circ})^* = \prod_{x \in X} \widehat{\mathcal{O}}_{\mathfrak{m}_x},$$

where $\widehat{\mathcal{O}}_{\mathfrak{m}_x}$ is the \mathfrak{m}_x -adic completion of the local ring $\mathcal{O}_{\mathfrak{m}_x}$. Hence, the dual coalgebra contains a lot of geometric information: the points of X can be recovered from it as the simple factors of the coradical $\mathbf{corad}(\mathbb{C}[X]^{\circ})$ and its dual algebra gives us the basics of the étale topology on X .

Let us illustrate this in the case of interest, that is, when $X = \mathbb{A}_{\mathbb{C}}^1$ with coordinate ring $\mathbb{C}[x]$. Every cofinite-dimensional ideal of $\mathbb{C}[x]$ is of the form

$I = ((x - \alpha_1)^{e_1} \cdots (x - \alpha_k)^{e_k})$, and since the different factors are comaximal, linear functionals on $\mathbb{C}[x]/I$ split over the distinct factors

$$\left(\frac{\mathbb{C}[x]}{I}\right)^* = \left(\frac{\mathbb{C}[x]}{(x - \alpha_1)^{e_1}}\right)^* \oplus \cdots \oplus \left(\frac{\mathbb{C}[x]}{(x - \alpha_k)^{e_k}}\right)^*.$$

Each of these factors is the dual coalgebra of a truncated polynomial ring and if we take z^i to be the basis dual to the y^i we have

$$\left(\frac{\mathbb{C}[y]}{y^n}\right)^* = \mathbb{C} \cdot 1 + \mathbb{C}z + \cdots + \mathbb{C}z^{n-1} \quad \text{with} \quad \begin{cases} \Delta(z^k) = \sum_{i+j=k} z^i \otimes z^j, \\ \epsilon(z^i) = \delta_{0i}, \end{cases}$$

which is the structure of the truncated enveloping algebra. Hence we have proved that

$$\mathbb{C}[x]^o = \bigoplus_{\alpha \in \mathbb{A}_{\mathbb{C}}^1} U(T_{\mathbb{A}_{\mathbb{C}}^1, \alpha}), \quad \text{and hence} \quad (\mathbb{C}[x]^o)^* = \prod_{\alpha \in \mathbb{A}_{\mathbb{C}}^1} \mathbb{C}[x - \alpha],$$

the natural inclusion $\mathbb{C}[x] \hookrightarrow (\mathbb{C}[x]^o)^*$ sending a polynomial to its Taylor series expansion in every point $\alpha \in \mathbb{A}_{\mathbb{C}}^1$.

An intermediate step in arriving at \mathbb{F}_1 -geometry would be to extend this complex coalgebra approach to integral schemes $\text{Spec}(R)$, where R is a finitely generated \mathbb{Z} -algebra, without additive torsion. In [23] it was shown that in this case we still have Kostant duality, which asserts that for all \mathbb{Z} -algebras R and all \mathbb{Z} -coalgebras C there is a natural one-to-one correspondence

$$\text{alg}_{\mathbb{Z}}(R, C^*) \longleftrightarrow \text{coalg}_{\mathbb{Z}}(C, R^o)$$

if we take as the modified dual coalgebra R^o the set of all $g \in R^* = \text{Hom}_{\mathbb{Z}}(R, \mathbb{Z})$ with the property that $\ker(g)$ contains a two-sided ideal $I \triangleleft R$ such that R/I is a finitely generated projective \mathbb{Z} -module.

The crucial difference with the complex case is that now the relevant ideals I no longer need to be comaximal and that there is no longer a decomposition of the dual coalgebra. In our example when $R = \mathbb{Z}[x]$ the relevant ideals are those generated by a monic polynomial f which can be decomposed in irreducible monic polynomials $f = g_1^{e_1} \cdots g_k^{e_k}$. But, as it may happen that $(g_i, g_j) \neq \mathbb{Z}[x]$, we have

$$\frac{\mathbb{Z}[x]}{f} \neq \frac{\mathbb{Z}[x]}{g_1^{e_1}} \oplus \cdots \oplus \frac{\mathbb{Z}[x]}{g_k^{e_k}},$$

and we can no longer decompose the dual coalgebra $\mathbb{Z}[x]^o$ over the codimension-one points $\mathbb{V}(g_i)$. Hence, we must recourse to describe the dual coalgebra as a direct limit

$$\mathbb{Z}[x]^o = \varinjlim \left(\frac{\mathbb{Z}[x]}{f}\right)^*,$$

where the limit is considered with respect to divisibility of monic polynomials, as there are canonical inclusions of \mathbb{Z} -coalgebras,

$$\left(\frac{\mathbb{Z}[x]}{f}\right)^* \hookrightarrow \left(\frac{\mathbb{Z}[x]}{g}\right)^* \quad \text{whenever} \quad f \mid g.$$

But then, also the \mathbb{Z} -algebra of distribution must be described as an inverse limit and we have a canonical ring morphism

$$\mathbb{Z}[x] \hookrightarrow (\mathbb{Z}[x]^o)^* = \varprojlim \frac{\mathbb{Z}[x]}{f}.$$

Finally, to get at \mathbb{F}_1 -geometry via this coalgebra approach we start with a λ -ring R and define the λ -dual coalgebra

$$R_\lambda^o = \{g \in R^* \mid \exists I \subset \ker(g) \text{ such that } R/I \text{ is a } \lambda\text{-ring finite over } \mathbb{Z} \},$$

which is indeed a coalgebra since the tensor product of λ -rings is again a λ -ring. Or specialize even further to the *cyclotomic dual coalgebra* R_{cycl}^o on the sub-coalgebra of R_λ^o spanned by the maps g having in their kernel an ideal I such that $R/I \cong \mathbb{Z}[x]/\phi_1 \oplus \dots \oplus \mathbb{Z}[x]/\phi_k$, where the ϕ_i are products of cyclotomic polynomials $\Phi_n(x)$.

For example, the (cyclotomic) coalgebra representing $\mathbb{P}_{\mathbb{F}_1}^1$ would then be

$$C_{\mathbb{P}_{\mathbb{F}_1}^1} = \mathbb{Z}[t_x] \oplus \varinjlim \left(\frac{\mathbb{Z}[x]}{\phi} \right)^* \oplus \mathbb{Z}[t_{x^{-1}}],$$

where the ϕ run in the multiplicative system generated by the cyclotomic polynomials $\Phi_n(x)$ with $n \in \mathbb{N}_0$, and the other two factors, which are the enveloping coalgebras of the one-dimensional Lie algebra, correspond to the points $[0]$ and $[\infty]$. Its corresponding algebra of distributions is then

$$\mathbb{Z}[[x]] \oplus \widehat{\mathbb{Z}[x]}_{\text{Hab}} \oplus \mathbb{Z}[[x^{-1}]],$$

where $\widehat{\mathbb{Z}[x]}_{\text{Hab}}$ is the *Habiro ring* or the *cyclotomic completion* of $\mathbb{Z}[x]$ introduced and studied by Kazuo Habiro in [17].

The Habiro ring is the straightforward generalization along the geometric axis of the profinite integers $\widehat{\mathbb{Z}}$ along the arithmetic axis. For we can write it as

$$\widehat{\mathbb{Z}[x]}_{\text{Hab}} = \varprojlim \frac{\mathbb{Z}[x, x^{-1}]}{[n!]_x} \quad \text{with} \quad [n!]_x = (x^n - 1)(x^{n-1} - 1) \dots (x - 1).$$

Its elements have a unique description as formal Laurent polynomials over \mathbb{Z} of the form

$$\sum_{n=0}^{\infty} a_n(x)[n!]_x \in \mathbb{Z}[[x]][x^{-1}] \quad \text{with} \quad \deg(a_n(x)) < n,$$

and hence can be evaluated at every root of unity (but possibly nowhere else).

Some of its elements had been discovered before. For example, during his investigations on Feynman integrals, Maxim Kontsevich observed that the formal power series $\sum_{n=0}^{\infty} (-1)^n [n!]_x$ is defined in all roots of unity, and Don Zagier subsequently proved the hilarious identity

$$\sum_{n=0}^{\infty} (-1)^n [n!]_x = -\frac{1}{2} \sum_{n=1}^{\infty} n \chi(n) x^{\frac{(n^2-1)}{24}},$$

where χ is the quadratic character of conductor 12, whereas the functions on both sides never makes sense simultaneously. The right-hand side converges only within the unit disc, but still if one approaches a root of unity radially, the limit of the function values on the right coincide with the values on the left. Such functions are said to “leak through” roots of unity.

The Habiro topology was introduced to describe the properties of the Habiro ring $\widehat{\mathbb{Z}[x]}_{\text{Hab}}$. For example, if U is an infinite set of roots of unity having $\alpha \in \mu_\infty$ as a limit point, meaning that U contains infinitely many elements adjacent to α , then if $f \in \widehat{\mathbb{Z}[x]}_{\text{Hab}}$ evaluates to zero in all roots $\beta \in U$, one has $f = 0$. For any subset $S \subset \mathbb{N}_0$ define the completion

$$\mathbb{Z}[x, x^{-1}]^S = \varprojlim_{\phi \in \Phi_S^*} \frac{\mathbb{Z}[x, x^{-1}]}{\phi}$$

where Φ_S^* is the multiplicative set of all monic polynomials generated by all cyclotomic polynomials $\Phi_n(x)$ for $n \in S$. Among the many precise results proved in [17] we mention the following two.

- If $S' \subset S$ has the property that every component of S with respect to the nearness relation contains an element of S' , then the natural map between the completions is an inclusion

$$\rho_{S'}^S : \mathbb{Z}[x, x^{-1}]^S \hookrightarrow \mathbb{Z}[x, x^{-1}]^{S'}$$

- If S is a *saturated subset* of \mathbb{N}_0 , which means that for all $n \in S$ also its divisor set $\langle n \rangle = \{m \mid n\}$ is contained in S , then we have

$$\mathbb{Z}[x, x^{-1}]^S = \bigcap_{n \in S} \mathbb{Z}[x, x^{-1}]^{\langle n \rangle} = \bigcap_{n \in S} \widehat{\mathbb{Z}[x, x^{-1}]_{(x^n - 1)}}$$

where the right-hand side terms are the I -adic completions of $\mathbb{Z}[x, x^{-1}]$ with respect to the ideals $I = (x^n - 1)$.

2.4. Conway’s big picture. In [9], John H. Conway investigates \mathbb{Q} -projectivity classes of lattices commensurable with the standard 2-dimensional lattice $L_1 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2$ and he shows that any such lattice has a unique form

$$L_{M, \frac{g}{h}} = \langle M\mathbf{e}_1 + \frac{g}{h}\mathbf{e}_2, \mathbf{e}_2 \rangle$$

with rational numbers $M > 0$ and $0 \leq \frac{g}{h} < 1$. Lattices $L_{M,0} = L_M$ are called *number-like* and if, in addition, $M \in \mathbb{N}_0$, we just call them *number lattices*.

We now define a metric on the set of (equivalence classes) of lattices. For two lattices $L = L_{M, \frac{g}{h}}$ and $L' = L_{N, \frac{i}{j}}$ consider the matrix

$$D_{LL'} = \begin{bmatrix} M & \frac{g}{h} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} N & \frac{i}{j} \\ 0 & 1 \end{bmatrix}^{-1},$$

where $g', h' \in \mathbb{Z}$ are such that $gg' - hh' = 1$. Dividing by hM we get the *reversed normalized* form for $L_{M, \frac{g}{h}}$

$$L = \left\langle \frac{1}{h^2 M} \mathbf{e}_2 + \frac{g'}{h} \mathbf{e}_1, \mathbf{e} \right\rangle.$$

So we get an involution on the vertices of the big picture

$$\left(M, \frac{g}{h}\right) \longleftrightarrow \left(\frac{1}{h^2 M}, \frac{g'}{h}\right),$$

where g' is the inverse of g modulo h .

The vertices of the big picture correspond to couples $(M, \frac{g}{h})$, so are elements of $\mathbb{Q}_0 \times \mathbb{Q}/\mathbb{Z}$, and we can identify each of the factors \mathbb{Q}/\mathbb{Z} (written additively) with μ_∞ (written multiplicatively). One quickly verifies that for the hyperdistance we have

$$\delta(L_M, L_{M, \frac{g}{h}}) = h^2.$$

So the cyclic subgroup μ_n corresponding to M is contained in a ball $B(L_M, n^2)$ around the lattice L_M with hyperdistance n^2 . In particular, the non-trivial elements of the cyclic group μ_p for p a prime number have hyperdistance p^2 from L_M and are the $p - 1$ vertices in the p -tree that are connected to L_{pM} .

The lattices L_n with $n \in \mathbb{N}_0$ form the big cell in this picture, which is the product of graphs of type A_∞^+ , one for each prime number p

$$A_\infty^+ : 1 \longrightarrow p \longrightarrow p^2 \longrightarrow \dots \longrightarrow p^k \longrightarrow \dots$$

and can be identified with $\mathbb{N}_0^\times = \mathbb{P}_{\mathbb{P}_1}^1 \setminus \{[0], [\infty]\}$. But then, we can extend the Habiro topology to Conway's big picture by calling two lattices *related* if their hyperdistance is a pure prime-power:

$$L \sim L' \iff \delta(L, L') = p^a.$$

An open set is then a subset U of vertices of the big picture having the property that for each $L \in U$, the set $\{L' \sim L \mid L' \notin U\}$ is finite. Clearly, the restriction of the extended Habiro topology yields the usual Habiro topology on the big cell \mathbb{N}_0^\times .

The free \mathbb{Z} -module on the vertices of \mathbb{B} , written $\mathbb{Z}\mathbb{B}$, is the playground of several operations on \mathbb{B} . Some well-known classical ones are the *Hecke operators* T_n which take the vertex representing the lattice L to the sum of all vertices corresponding to lattices L' with $\delta(L, L') = n$. That is, T_n replaces the center of each ball of hyperradius n by its periphery. For $a > 1$, these Hecke operators clearly satisfy the relation

$$T_p \circ T_{p^a} = pT_{p^{a-1}} + T_{p^{a+1}},$$

as the left-hand side takes a vertex to the sum of all neighbors of vertices at hyperdistance p^a from it, but in this sum each vertex of hyperdistance p^{a-1} occurs p times and each point of hyperdistance p^{a+1} just once, giving the right-hand side.

More operators come from the action of a certain noncommutative algebra on $\mathbb{Z}\mathbb{B}$, the *Bost–Connes algebra* Λ , see for example [7]. If Λ hadn't been constructed years before (in [5]), it would have arisen naturally from \mathbb{F}_1 -geometry by a construction which is well-known in noncommutative algebraic geometry.

If X is an affine \mathbb{C} -variety with a linear action by a finite group G , then the coordinate ring of its quotient variety $\mathbb{C}[X/G] = \mathbb{C}[X]^G$ is *Morita equivalent* to the skew group algebra $\mathbb{C}[X] * G$ (that is, they have equivalent module categories), which as a \mathbb{C} -vector space is the group algebra $\mathbb{C}[X]G$, but with multiplication induced by $f \cdot e_g = e_g \cdot \phi_g(f)$ where ϕ_g denotes the action by g on $\mathbb{C}[X]$. That is, one way to handle the descended algebra $\mathbb{C}[X]^G$ is by considering the noncommutative skew group algebra $\mathbb{C}[X] * G$.

In Borger's proposal for \mathbb{F}_1 -geometry this approach may be very helpful, as an \mathbb{F}_1 -algebra is a \mathbb{Z} -algebra R together with descent data given by the action of the monoid \mathbb{N}_0^\times by the endomorphisms $\{\Psi^n\}$. Now we cannot directly construct the invariant algebra $R^{\mathbb{N}_0^\times}$ (which would be our elusive \mathbb{F}_1 -algebra), but we can still construct the skew-monoid algebra $R * \mathbb{N}_0^\times$ which, as before, coincides as a \mathbb{Z} -module with $RG = \bigoplus_{n \in \mathbb{N}_0^\times} Re_n$ and has a noncommutative multiplication induced by $r \cdot e_n = e_n \cdot \Psi_n(r)$.

For example, let us try to understand the algebraic closure $\overline{\mathbb{F}_1}$ by considering the associated skew-monoid algebra. The λ -algebra corresponding to $\overline{\mathbb{F}_1}$ is the group algebra $\mathbb{Z}[\mu_\infty]$ with Frobenius lifts Ψ^n induced by sending a root of unity ω to ω^n . If we write the group law additively instead of multiplicatively we get the group algebra $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with $\Psi^n(e_{g/h}) = e_{(g/h \bmod 1)}$. The corresponding skew-monoid algebra is then

$$\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] * \mathbb{N}_0^\times = \bigoplus_{n \in \mathbb{N}_0^\times} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]e_n \quad \text{with} \quad e_{g/h} \cdot e_n = e_n \cdot \Psi^n(e_{g/h}).$$

Noncommutative algebraic geometers would then study properties of this ring to get insight into $\overline{\mathbb{F}_1}$. Noncommutative *differential* geometers however work with $*$ -algebras, therefore they need to construct the minimal $*$ -algebra generated by $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] * \mathbb{N}_0^\times$ and therefore consider the algebra

$$\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \bowtie \mathbb{N}_0^\times = \bigoplus_{m, n \in \mathbb{N}_0^\times, (m, n) = 1} e_m^* \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] e_n,$$

in which the generators e_m^* , e_n , and $e_{g/h}$ satisfy the following multiplication rules:

$$\begin{aligned} e_n \cdot e_{g/h} \cdot e_n^* &= \rho_n(e_{g/h}) & \text{where } \rho_n(e_{g/h}) &= \sum_{n \cdot \frac{i}{j} = \frac{g}{h}} e_{i/j} \\ e_n^* \cdot e_{g/h} &= \Psi^n(e_{g/h})e_n^* \\ e_{g/h} \cdot e_n &= e_n \cdot \Psi^n(e_{g/h}) \\ e_n \cdot e_m &= e_{nm} \\ e_n^* \cdot e_m^* &= e_{nm}^* \\ e_n^* \cdot e_n &= n \\ e_n \cdot e_m^* &= e_m^* \cdot e_n \text{ if } (m, n) = 1. \end{aligned}$$

This algebra $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}_0^\times$ is then the (integral version of the) Bost–Connes algebra Λ constructed in [7], where it is also shown that there is an action of Λ and of $\mathbb{Z}\mathbb{B}$ given by the rules

$$\begin{aligned} e_n \cdot L_{\frac{c}{d}, \frac{g}{h}} &= L_{\frac{nc}{d}, \rho_m(\frac{g}{h})} && \text{where } m = (n, d) \\ e_n^* \cdot L_{\frac{c}{d}, \frac{g}{h}} &= (n, c) L_{\frac{c}{nd}, \Psi^{\frac{n}{m}}(\frac{g}{h})} && \text{where } m = (n, c) \\ e_{a/b} \cdot L_{\frac{c}{d}, \frac{g}{h}} &= L_{\frac{c}{d}, \Psi^c(\frac{a}{b})\frac{g}{h}} \end{aligned}$$

with ρ_n and Ψ^n defined on $\frac{g}{h}$ as they were defined before on $e_{g/h}$. So far, we have identified $\mathbb{P}_{\mathbb{F}_1}^1$ (equipped with the adjacency relation among its schematic points) with the big cell in the Conway picture. It is believed that this bigger picture will play an ever increasing role of importance in future developments in \mathbb{F}_1 -geometry and will illuminate surprise appearances of the Bost–Connes algebra Λ as a generalized symmetry on geometric \mathbb{F}_1 -objects, see for example [6].

3. Smirnov’s proposal

3.1. Exotic topology on $\text{Spec}(\mathbb{Z})$. Now that we have a formal definition of $\mathbb{P}_{\mathbb{F}_1}^1$ let’s try to make sense of the ultimate question in \mathbb{F}_1 -geometry: what (if any) geometric object is $\overline{\text{Spec}(\mathbb{Z})}$ over \mathbb{F}_1 ? Again, we will start with an intuitive proposal due to A. L. Smirnov [38] and later try to formalize it using λ -rings.

Smirnov proposes to take as the set of *schematic points* of $\overline{\text{Spec}(\mathbb{Z})}$ the set

$$\{[2], [3], [5], [7], [11], [13], [17], \dots\} \cup \{[\infty]\}$$

of all prime numbers together with a point at infinity. The *degrees* of these schematic points are then taken to be

$$\text{deg}([p]) = \log(p) \quad \text{and} \quad \text{deg}([\infty]) = 1.$$

To understand this proposal, recall that if C is a smooth projective curve over \mathbb{F}_p , then a schematic point $P \in C$ corresponds to a discrete valuation ring \mathcal{O}_P in the function field $\mathbb{F}_p(C)$ with maximal ideal $\mathfrak{m}_P = (t_P)$, where t_P is a uniformizing parameter. The *degree* $\text{deg}(P)$ of the schematic point $P \in C$, is defined to be n if $\mathcal{O}_P/\mathfrak{m}_P = \mathbb{F}_{p^n}$. A rational function $f \in \mathbb{F}_p(C)$ is said to be *regular* in P if $f \in \mathcal{O}_P$ and the *order* of f in P is the valuation of f , that is, $\text{ord}_P(f) = k$ if $f \in \mathfrak{m}_P^k \setminus \mathfrak{m}_P^{k+1}$ for a unique $k \in \mathbb{Z}$. The *divisor* of the rational function $f \in \mathbb{F}_p(C)$,

$$\text{div}(f) = \sum_{P \in C} \text{ord}_P(f)P$$

then has degree zero, that is

$$0 = \text{deg}(\text{div}(f)) = \sum_{P \in C} \text{ord}_P(f)\text{deg}(P).$$

By analogy, we may take the schematic points of $\overline{\text{Spec}(\mathbb{Z})}$ to be the different discrete valuation rings in the corresponding “function field” \mathbb{Q} . By Ostrovski’s theorem, these are the *p-adic valuations*

$$v_p(q) = n \quad \text{if} \quad q = p^n \frac{r}{s} \quad \text{and} \quad p \nmid r \cdot s$$

for every prime number p , together with the *real valuation*

$$v_\infty(q) = -\log(|q|);$$

the minus sign arises because of the convention that $v_\infty(0) = \infty$. But then, if $q = \pm \frac{p_1^{e_1} \cdots p_r^{e_r}}{q_1^{f_1} \cdots q_s^{f_s}}$, its corresponding divisor must be

$$\text{div}(q) = \sum_{i=1}^r e_i [p_i] - \sum_{j=1}^s f_j [q_j] - \log(|q|)[\infty].$$

The proposal for the degrees of the schematic points of $\overline{\text{Spec}(\mathbb{Z})}$ is then the only possible one (up to a common multiple) such that the degrees of all these principal divisors are equal to zero. Any non-constant rational function $f \in \mathbb{F}_p(C)$ determines a *cover map* $f: C \rightarrow \mathbb{P}_{\mathbb{F}_p}^1$. Smirnov defines as the constant rational numbers the intersection $\mathbb{Q} \cap \overline{\mathbb{F}_1} = \{0\} \cup \{1, -1\} = \mathbb{F}_{1^2}$. Therefore, we would expect by analogy rational numbers $q = \frac{a}{b} \in \mathbb{Q}$ with $(a, b) = 1$ to determine a cover

$$q: \overline{\text{Spec}(\mathbb{Z})} \longrightarrow \mathbb{P}_{\mathbb{F}_1}^1.$$

Smirnov’s proposal in [38] is to *define* a map by

$$[p] \longmapsto \begin{cases} [0] & \text{if } p \mid a, \\ [\infty] & \text{if } p \mid b, \\ [n] & \text{if } p \nmid ab \text{ and } \bar{a}(\bar{b})^{-1} \text{ has order } n \text{ in } \mathbb{F}_p^*, \end{cases}$$

and by

$$[\infty] \longmapsto \begin{cases} [0] & \text{if } a < b, \\ [\infty] & \text{if } a > b. \end{cases}$$

To motivate this definition let us again look at the function field case. Any rational function $f \in \mathbb{F}_p(C)$ defines a map between the geometric points

$$C(\overline{\mathbb{F}_p}) \longrightarrow \mathbb{P}_{\mathbb{F}_p}^1(\overline{\mathbb{F}_p}) \quad P \longmapsto \begin{cases} [f(P) : 1] & \text{if } f \in \mathcal{O}_P, \\ [\infty] & \text{if } f \notin \mathcal{O}_P, \end{cases}$$

with $f(P) = \bar{f} \in \mathcal{O}_P/\mathfrak{m}_P \subset \overline{\mathbb{F}_p}$. Because $f \in \mathbb{F}_p(C)$, we have for all $P \in C(\overline{\mathbb{F}_p})$ and all $\sigma \in \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ that $\sigma(f(P)) = f(\sigma(P))$, and hence this map induces a map between the schematic points $C \rightarrow \mathbb{P}_{\mathbb{F}_p}^1$ sending a schematic point (a Galois

orbit of a geometric point P) to the Frobenius orbit of the root of unity $f(P)$ (or its corresponding monic irreducible polynomial in $\mathbb{F}_p[x]$). Returning to the above map given by a rational number $q = \frac{a}{b}$, it is clear that $q([p]) = [0]$ for all prime factors p of a and that $q([p]) = [\infty]$ for all factors of b . To understand the other images, note that if $\bar{a}(\bar{b})^{-1}$ has order n in \mathbb{F}_p^* , there exists a prime ideal P in the ring of cyclotomic integers $\mathbb{Z}[\epsilon]$ (for ϵ a primitive n th root of unity) lying over (p) , that is $P \cap \mathbb{Z} = (p)$, with corresponding discrete valuation ring \mathcal{O}_P such that

$$\frac{a}{b} - \epsilon \in P\mathcal{O}_P = \mathfrak{m}_P,$$

and therefore $\frac{a}{b}(P) = \epsilon(P)$, explaining why the schematic point $[p]$ is sent to the Galois orbit of ϵ which is precisely the schematic point $[n]$ of $\mathbb{P}_{\mathbb{F}_1}^1$.

In the function field case we have for every non-constant rational function $f \in \mathbb{F}_p(C) \setminus \mathbb{F}_p$ that the map $C \rightarrow \mathbb{P}_{\mathbb{F}_p}^1$ is surjective with finite fibers. Let us first verify finiteness for the map $q = \frac{a}{b}$, that is, for every $[n]$ we must show that there are only finitely many primes p for which

$$\left(\frac{\bar{a}}{\bar{b}}\right)^n = 1 \quad \text{in } \mathbb{F}_p^*.$$

This is clearly equivalent to $p \mid a^n - b^n$ and $p \nmid a^m - b^m$ for all $m < n$, so $q^{-1}([n])$ is a subset of the finite number of prime factors of $a^n - b^n$. Surjectivity of the map q is less clear, as there seems to be no reason why there should always be a prime factor of $a^n - b^n$ not dividing the number $a^m - b^m$ for all $m < n$. In fact, surjectivity is not always true. For example, the map $q = \frac{2}{1}$ has no prime mapping to $[6]$. Figure 9 gives a portion of the graph of the map 2 in the Smirnov plane $\mathbb{P}_{\mathbb{F}_1}^1 \times \text{Spec}(\mathbb{Z})$, where we have used a logarithmic scale on the prime number axis and determined the full fibers of all $[n] \in \mathbb{P}_{\mathbb{F}_1}^1$ for $n < 330$. The points on the “diagonal” are the first few Mersenne prime numbers, that is, primes p such that $M_p = 2^p - 1$ is again a prime number.

Perhaps surprisingly we can determine all rational numbers q for which the map $\text{Spec}(\mathbb{Z}) \rightarrow \mathbb{P}_{\mathbb{F}_1}^1$ fails to be surjective, as well as the schematic points $[n]$ of $\mathbb{P}_{\mathbb{F}_1}^1$ for which $q^{-1}([n]) = \emptyset$. The crucial result needed is *Zsigmondy’s Theorem* [45]. Consider positive integers $1 \leq b < a$ with $(a, b) = 1$. Then, for every $n > 1$ there exist prime numbers $p \mid a^n - b^n$ such that $p \nmid a^m - b^m$ for all $m < n$ unless we are in one of the following two cases:

- i) $a = 2, b = 1$ and $n = 6$; or
- ii) $a + b = 2^k$ and $n = 2$.

Smirnov’s interest in these maps is that the ABC conjecture would follow provided one can prove a suitable analogue of the *Riemann–Hurwitz formula* for the maps q . Recall that if $f: C \rightarrow \mathbb{P}_k^1$ is a non-constant cover from a smooth projective curve C over a field k , then the Riemann–Hurwitz formula asserts that

$$2g_C - 2 \geq -2 \deg(f) + \sum_{\text{points } P} (e_f(P) - 1) \deg(P),$$

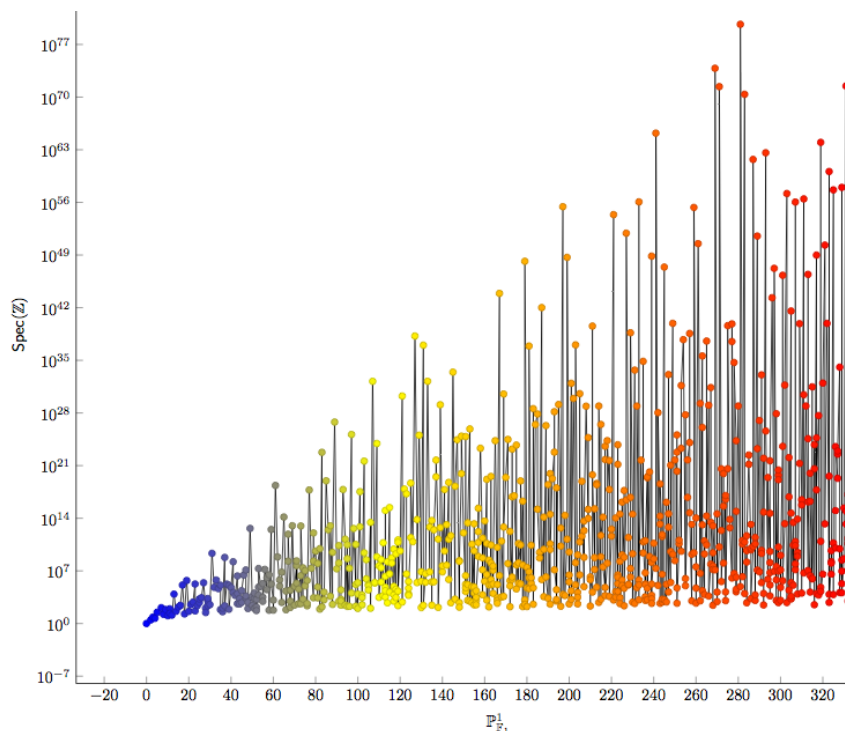


Figure 9. A portion of the graph of [2] inside the Smirnov plane.

where g_C is the genus of C and $e_f(P)$ is the ramification of a schematic point $P \in C$ of degree $\mathbf{deg}(P)$. If we define the *defect* δ_P of a schematic point $P \in C$ to be the number $\delta_P = \frac{(e_f(P)-1)\mathbf{deg}(P)}{\mathbf{deg}(f)} \geq 0$, then the Riemann-Hurwitz formula can be rewritten as

$$\sum_{\text{points } P} \delta_P \leq 2 - \frac{2 - 2g_C}{\mathbf{deg}(f)},$$

and we note that this inequality still holds if we sum over a sub-selection of the schematic points $P \in C$. Again, we want to define via analogy the *ramification index* $e_q(p)$ and the *arithmetic defect* $\delta(p)$ for any prime number p with respect to a cover $q: \overline{\text{Spec}(\mathbb{Z})} \rightarrow \mathbb{P}^1_{\mathbb{F}_1}$. If $q = \frac{a}{b}$, then Smirnov proposes to take for $e_q(p)$ the largest power of p dividing a (provided $p \in q^{-1}([0])$), the largest power dividing b (provided $p \in q^{-1}([\infty])$), and if $p \in q^{-1}([n])$ to take $e_q(p) = k$ if p^k is the largest power dividing $a^n - b^n$. With this definition of the ramification index, he then proposes to define the arithmetic defect by

$$\delta(p) = \frac{(e_q(p) - 1) \log(p)}{\log(a)},$$

which coincides with the classical definition (given our proposal for the degree of p)

provided we define the degree of the map q to be $\log(a)$. Let us try to motivate this proposal in the case when $a \gg b$. Take $[p] \in \mathbb{P}_{\mathbb{F}_1}^1$ for p a prime number: then the divisor of $q^{-1}([p]) = \sum_i n_i [q_i]$ if $\prod_i q_i^{n_i} = a^p - b^p$, whence $\sum_i n_i \log(q_i) \approx p \cdot \log(a)$, and as we took $\mathbf{deg}([p]) = \phi(p) = p - 1$ it follows that indeed

$$\frac{\sum_i n_i \mathbf{deg}([q_i])}{\mathbf{deg}([p])} \approx \log(a).$$

For any schematic point $[d] \in \mathbb{P}_{\mathbb{F}_1}^1$ let us define the *defect* of $[d]$ to be

$$\delta([d]) = \sum_{p \in q^{-1}([d])} \delta(p).$$

Now, if $a = p_1^{e_1} \cdots p_k^{e_k}$ and $b = q_1^{f_1} \cdots q_s^{f_s}$ we define $a_0 = p_1 \cdots p_k$ and $b_0 = q_1 \cdots q_s$ and $a_1 = a/a_0$, $b_1 = b/b_0$. Then, with the above proposals it is easy to work out that

$$\delta([0]) = \frac{\log(a_1)}{\log(a)}, \quad \delta([\infty]) = \frac{\log(b_1) + \log(q) - 1}{\log(a)}, \quad \delta([1]) = \frac{\log((a - b)_1)}{\log(a)},$$

as $q^{-1}([1]) = \{p \mid a - b\}$, and where in the middle term $\log(q) - 1$ is the contribution of ∞ to the defect. If we could prove a variant of the Riemann–Hurwitz formula in \mathbb{F}_1 -geometry for all covering maps $q: \overline{\mathbf{Spec}(\mathbb{Z})} \rightarrow \mathbb{P}_{\mathbb{F}_1}^1$, and if we assume the constant $\gamma = 2g_{\overline{\mathbf{Spec}(\mathbb{Z})}} - 2 \geq 0$, then it would follow that (limiting to points lying in the fibers of $[0], [1]$ and $[\infty]$) that

$$\begin{aligned} \delta([0]) + \delta([1]) + \delta([\infty]) &= \\ &= \frac{1}{\log(a)} (\log(a_1) + \log((a - b)_1) + \log(b_1) + \log(a) - \log(b) - 1) \\ &\leq 2 + \frac{\gamma}{\log(a)}. \end{aligned}$$

Now, let's turn to the ABC conjecture. Suppose $A + B = C$ with $\overline{(A, B, C)} = 1$ and take $a = C$ and $b = \min(A, B)$, and consider the cover $q = \frac{a}{b}: \overline{\mathbf{Spec}(\mathbb{Z})} \rightarrow \mathbb{P}_{\mathbb{F}_1}^1$; clearly we have $a - b \geq \frac{a}{2}$. Then, we can rewrite the above inequality as

$$1 \leq \frac{\log(a_0 \cdot b_0 \cdot (a - b)_0)}{\log(a)} + \frac{\log(C')}{\log(a)}$$

where $\log(C') = \gamma + \log(2) + 1$; but then $a \leq C'(a_0 \cdot b_0 \cdot (a - b)_0)$ or, in other words,

$$C \leq C'(\text{rad}(A \cdot B \cdot C))$$

which is (too strong) a formulation of the ABC conjecture.

Now that we have a family of non-constant covering maps for all $q \in \mathbb{Q}$

$$q: \mathbf{Spec}(\mathbb{Z}) \longrightarrow \mathbb{P}_{\mathbb{F}_1}^1$$

we can define the *exotic topology* on $\text{Spec}(\mathbb{Z})$ to be the coarsest topology with the property that all maps q are continuous with respect to the Habiro topology on $\mathbb{P}_{\mathbb{R}_1}^1$. As the covers are finite and the Habiro topology is finer than the cofinite topology, the exotic topology refines the usual—that is, cofinite—topology on $\text{Spec}(\mathbb{Z})$. Again, this topology is no longer compact.

3.2. Witt and Burnside rings. Surprisingly, the forgetful functor $f: \text{rings}_\lambda \rightarrow \text{rings}$ has a *right adjoint* (a left adjoint is the common situation), that is, there is a functor

$$w: \text{rings} \longrightarrow \text{rings}_\lambda \quad \text{such that} \quad \text{alg}_\lambda(A, w(B)) = \text{alg}(f(A), B)$$

for every λ -ring A and all rings B . We will recall the construction of this “*witty functor*” (it is closely related to the functor of big Witt vectors).

For any ring A let $w(A) = 1 + tA[[t]]$ be the set of all formal power series with coefficients in A and with constant term equal to 1. We will turn this set into a ring with an addition \oplus and a multiplication \otimes (to distinguish these operations from the usual ones on the formal power series ring $A[[t]]$). The addition \oplus on $w(A)$ will be the usual multiplication of formal power series, that is

$$u(t) \oplus v(t) = u(t) \cdot v(t), \quad \text{and hence} \quad \textcircled{0} = 1 \quad \text{and} \quad \ominus u(t) = u(t)^{-1}.$$

Multiplication is enforced by functoriality and the rule that for all $a, b \in A$ we demand that

$$\frac{1}{1-at} \otimes \frac{1}{1-bt} = \frac{1}{1-abt}, \quad \text{and hence} \quad \textcircled{1} = \frac{1}{1-t} = 1 + t + t^2 + \dots.$$

What we mean by this, at least if A is a domain in characteristic zero, is that for any $u(t) \in w(A)$ there exists unique $a_i \in A$ such that

$$u(t) = \prod_{i=1}^{\infty} \frac{1}{1-a_i t^i}.$$

For each n , denote $\alpha_n = \sqrt[n]{a_n}$ and let ζ_n be a primitive n th root of unity, so that for all n we have that $1 - a_n t^n = \prod_{i=0}^{n-1} (1 - \zeta_n^i \alpha_n t)$. But then, over the ring $A[\mu_\infty][\alpha_1, \alpha_2, \dots]$ we can write $u(t)$ as

$$u(t) = A_1 \oplus A_2 \oplus A_3 \oplus \dots \quad \text{with} \quad A_n = \frac{1}{1-\alpha_n t} \oplus \frac{1}{1-\zeta_n \alpha_n t} \oplus \dots \oplus \frac{1}{1-\zeta_n^{n-1} \alpha_n t}.$$

If we similarly write the power series $v(t) = B_1 \oplus B_2 \oplus B_3 \oplus \dots$, then the product must be

$$u(t) \otimes v(t) = C_1 \oplus C_2 \oplus \dots \quad \text{with} \quad C_{i+1} = \bigoplus_{j+k=i+1} A_j \otimes B_k,$$

and by construction and symmetric function theory one verifies that the formal power series $u(t) \otimes v(t)$ has all its coefficients in A . In this way we see that $w(A)$ is

a commutative ring whose zero element is the constant power series 1 and whose multiplicative unit is the power series $1 + t + t^2 + \dots$. In addition, $w(A)$ becomes a λ -ring with the Frobenius lifts induced by the rule that

$$\psi^p \left(\frac{1}{1-at} \right) = \frac{1}{1-a^p t}$$

and extended additively so that if an element can be written $u(t) = A_1 \oplus A_2 \oplus \dots$, then $\psi^p(u(t)) = \psi^p(A_1) \oplus \psi^p(A_2) \oplus \dots$. The Frobenius lifts are also multiplicative by functoriality and the calculation that

$$\psi^p \left(\frac{1}{1-at} \otimes \frac{1}{1-bt} \right) = \frac{1}{1-a^p b^p t} = \psi^p \left(\frac{1}{1-at} \right) \otimes \psi^p \left(\frac{1}{1-bt} \right).$$

Clearly, the endomorphisms $\psi^n, n \in \mathbb{N}$, commute with each other and ψ^p is a Frobenius lift because

$$\left(\frac{1}{1-a_1 t} \oplus \dots \oplus \frac{1}{1-a_k t} \right)^{\otimes p} = \left(\frac{1}{1-a_1^p t} \oplus \dots \oplus \frac{1}{1-a_k^p t} \right)$$

is divisible by p by the binomial formula. There is an additional family of additive group endomorphisms V_n on $w(A)$, the *Verschiebung operators* which are defined by $V_n(s(t)) = s(t^n)$, and finally there is the $[n]$ operator which maps $s(t)$ to $s(t)^n = s(t) \oplus \dots \oplus s(t)$ (n times). These maps satisfy the relations

$$V_n \circ V_m = V_m \circ V_n, \quad \Psi^m \circ \Psi^n = \Psi^n \circ \Psi^m \quad \text{and} \quad \Psi^n \circ V_n = [n]$$

and $\Psi^n \circ V_m = V_m \circ \Psi^n$ if $(m, n) = 1$. This witty construction is functorial because for any ring morphism $\phi: A \rightarrow B$ we have a ring morphism $\Phi: w(A) \rightarrow w(B)$ compatible with the Frobenius lifts, induced by the rule that

$$\Phi \left(\frac{1}{1-at} \right) = \frac{1}{1-\phi(a)t},$$

which gives us that $\Phi(1 + a_1 t + a_2 t^2 + \dots) = 1 + \phi(a_1)t + \phi(a_2)t^2 + \dots$. We now define maps $\gamma_n: w(A) \rightarrow A$ via the formula

$$\frac{tu'}{u} = \sum_{n=1}^{\infty} \gamma_n(u)t^n,$$

where we have used the *logarithmic derivative* $\frac{u'}{u}$ which transforms multiplication into addition. If we work this out for $u = \frac{1}{1-at}$, then $u' = \frac{a}{(1-at)^2}$ and hence

$$\frac{tu'}{u} = at + a^2 t^2 + a^3 t^3 + \dots,$$

whence $\gamma_n\left(\frac{1}{1-at}\right) = a^n$ and therefore all γ_n are multiplicative. Using functoriality it is also easy to conclude that all the maps $\gamma_n: w(A) \rightarrow A$ are in fact ring morphisms.

If A is in addition a λ -ring with commuting family of endomorphisms Ψ^n generated by the Frobenius lifts, then there is a λ -ring morphism σ_t making the diagram below commute

$$\begin{array}{ccc}
 & & w(A) = 1 + tA[[t]] \\
 & \nearrow \sigma_t & \downarrow (\gamma_1, \gamma_2, \gamma_3, \dots) \\
 A & \xrightarrow{\Psi = (\Psi^1, \Psi^2, \Psi^3, \dots)} & A^\omega = (A, A, A, \dots)
 \end{array}$$

where σ_t is defined via the formula

$$\sigma_t(a) = \exp \left(\int \frac{1}{t} \sum_{n=1}^{\infty} \Psi^n(a) t^n \right).$$

Again, it is easy to verify that $\sigma_t(a+b) = \sigma_t(a) \oplus \sigma_t(b)$ and slightly more difficult to prove that $\sigma_t(a \cdot b) = \sigma_t(a) \otimes \sigma_t(b)$, whence σ_t is a ring morphism and is compatible with the Ψ^n -endomorphisms, so it is a λ -ring morphism.

From these facts the right-adjointness of the witty functor with respect to the forgetful functor follows. If A is a λ -ring and ϕ is a ring morphism $f(A) = A \rightarrow B$, then we get a λ -ring morphism

$$A \xrightarrow{\sigma_t} w(A) \xrightarrow{\Phi} w(B).$$

Conversely, a λ -ring morphism $A \rightarrow w(B)$ composed with the ring morphism $\gamma_1: w(B) \rightarrow B$ gives a ring morphism $A \rightarrow B$ and one verifies that the two constructions are inverse of each other.

If one accepts Borger’s proposal that \mathbb{F}_1 -algebras are just λ -rings without additive torsion, where we interpret the commuting family of endomorphisms $\{\Psi^n \mid n \in \mathbb{N}_0\}$ as descent data from \mathbb{Z} to \mathbb{F}_1 , then the forgetful functor

$$f = - \otimes_{\mathbb{F}_1} \mathbb{Z}: \mathbf{alg}_{\mathbb{F}_1} = \mathbf{rings}_\lambda \longrightarrow \mathbf{rings}$$

that is stripping off the descent data, can be interpreted as the base extension functor from \mathbb{F}_1 to \mathbb{Z} . But then, as a right adjoint to base extension, the witty functor w can be interpreted as the Weil descent from \mathbb{Z} -rings to \mathbb{F}_1 -algebras. Hence, we finally know what $\mathbf{Spec}(\mathbb{Z})$ should be over the elusive field \mathbb{F}_1 : it must be the geometric object associated to the λ -algebra $w(\mathbb{Z})$!

We will now make the connection between the construction of $w(A)$ and the more classical notion of the ring of *big Witt vectors* $W(A)$. For much more details we refer to the lecture notes of Michiel Hazewinkel [19]. Let us take $W(A) = A^\omega = (A, A, A, \dots)$ and consider the diagram

$$\begin{array}{ccc}
 W(A) & \xrightarrow{\gamma} & w(A) \\
 \uparrow = & & \downarrow \frac{u'}{u} \\
 A^\omega & \xrightarrow{=} & tA[[t]]
 \end{array}$$

where γ is the map sending (a_1, a_2, a_3, \dots) to $\prod_i \frac{1}{1-a_i t^i}$ and which can be used to define a ring structure on the big Witt vectors $W(A)$ by transport of structure.

Before we describe the geometry, let us give a combinatorial interpretation of $w(\mathbb{Z})$ due to Andreas Dress and Christian Siebeneicher [14].

Let $C = C_\infty = \langle c \rangle$ be the infinite cyclic group, written multiplicatively. A C -set X is called *almost finite* if X has no infinite orbits and if the number of orbits of size n is finite for every $n \in \mathbb{N}_0$. A motivating example is the set of geometric points $X(\overline{\mathbb{F}_p})$ of an \mathbb{F}_p -variety X on which c acts as the Frobenius morphism.

If X and Y are almost finite C -sets, then so are their disjoint union $X \sqcup Y$ and Cartesian product $X \times Y$. These operations define an addition $+ = \sqcup$ and multiplication $\cdot = \times$ on the isomorphism classes $\widehat{B}(C)$ of all almost finite C -sets, as such obtaining the *Burnside ring*. For any almost finite C -set X and $n \in \mathbb{N}$, define

$$\phi_{C^n}(X) = \#\{x \in X \mid c^n \cdot x = x\}$$

that is, the number of elements lying in a C -orbit of size a divisor of n : this number is finite. Moreover, the ϕ_{C^n} take disjoint unions (respectively products) to sums (respectively products) of the corresponding numbers, and so all maps $\phi_{C^n} : \widehat{B}(C) \rightarrow \mathbb{Z}$ are ring morphisms. This gives us a collective ring morphism

$$\widehat{\phi} = \prod_n \phi_{C^n} : \widehat{B}(C) \longrightarrow \mathbb{Z}^\omega = \mathbf{gh}(C),$$

where $\mathbf{gh}(C)$ is the *ghost ring*, that is, all maps $\mathbb{N} \rightarrow \mathbb{Z}$ with componentwise addition and multiplication. One verifies that $\widehat{\phi}$ is injective, but not surjective. We can extend the diagram of the previous section to

$$\begin{array}{ccccc} & & Nr(\mathbb{Z}) & & \\ & & \downarrow \text{itp} & & \\ W(\mathbb{Z}) & \xrightarrow{\tau} & \widehat{B}(C) & \xrightarrow{st} & w(\mathbb{Z}) \\ \Phi \downarrow & & \widehat{\phi} \downarrow & & \downarrow L_z \\ \prod_n \mathbb{Z} & \xrightarrow{\text{obv}} & \mathbf{gh}(C) & \xrightarrow{\text{idn}} & t\mathbb{Z}[[t]] \end{array}$$

where $Nr(\mathbb{Z})$ is called the *necklace algebra*, that is, the set \mathbb{Z}^ω with componentwise addition but multiplication defined as follows: if $b = (b_1, b_2, \dots)$ and $b' = (b'_1, b'_2, \dots)$ then

$$(b \cdot b')_n = \sum_{\text{lcm}(i,j)=n} (i, j) b_i b'_j.$$

The interpretation map itp , which is a ring morphism, sends $b = (b_1, b_2, \dots)$ to the element of $\widehat{B}(C)$ given by $\sum_{n=1}^\infty b_n [C_n]$ (where C_n is the C -orbit of length n) and can thus be written as the difference $[X_+] - [X_-]$ of two almost finite C -sets, X_+ corresponding to the positive b_n and X_- to minus the negative b_n . The composition of the interpretation map with $\widehat{\phi}$ is the *ghost map* $\mathbf{gh}: b \mapsto d$, where

$d_n = \phi_{C^n}(X(b)) = \phi_{C^n}(X_+(b)) - \phi_{C^n}(X_-(b))$ and the sequence of integers d is related to that of b via the formula

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n} \right)^{b_n} = \exp \left(\int \sum_{n=1}^{\infty} d_n t^{n-1} dt \right),$$

that is, $\text{gh}(b) = \widehat{b}$, where $\widehat{b}_n = \sum_{i|n} ib_i$. If X is an almost finite C -set then so is its n th symmetric power

$$S^n X = \{g : X \rightarrow \mathbb{N} \mid g \text{ has finite support and } \sum_{x \in X} g(x) = n\},$$

on which C acts via $(c \cdot g)(x) = g(c^{-1} \cdot x)$. Dress and Siebeneicher prove in [14] that the map $s_t : \widehat{B}(C) \rightarrow w(\mathbb{Z})$ induced by

$$[X] \mapsto s_t(X) = 1 + \text{fix}(S^1(X))t + \text{fix}(S^2(X))t^2 + \dots$$

is an isomorphism of rings, and if $s_t(X(b)) = 1 + \sum_{n=1}^{\infty} a_n t^n = s_{\mathbf{a}}(t)$ then,

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n} \right)^{b_n} = 1 + \sum_{n=1}^{\infty} a_n t^n.$$

This allows us in the case of $w(\mathbb{Z})$ to compute the product combinatorially. If $s_t(X(b)) = s_{\mathbf{a}}(t)$ and $s_t(X(d)) = s_{\mathbf{c}}(t)$, then we have in $w(\mathbb{Z})$ that

$$s_{\mathbf{a}}(t) \otimes s_{\mathbf{c}}(t) = s_t(X(b) \times X(d)).$$

If $m \in \mathbb{N}$ define the congruence maps

$$m^{(C)} = \{g : C \rightarrow \{1, 2, \dots, m\} \text{ such that} \\ \text{there exists } n \in \mathbb{N} \mid z_1 z_2^{-1} \in C^n = \langle c^n \rangle \Rightarrow g(z_1) = g(z_2)\}.$$

Observe that $m^{(C)}$ is again an almost finite C -set under the action $c \cdot g(z) = g(c^{-1} \cdot z)$ and one verifies that $\phi_{C^n}(m^{(C)}) = m^n$. The map $m \mapsto m^{(C)}$ from \mathbb{N}_0 to $\widehat{B}(C)$ extends to a map $\mathbb{Z} \rightarrow \widehat{B}(C)$ and $\widehat{\phi}(m^{(C)}) = (m, m^2, m^3, \dots)$. We will now extend this map to a map $W(\mathbb{Z}) \rightarrow \widehat{B}(C)$. If X is an almost finite C -set and $n \in \mathbb{N}$, then we define its induction $\text{ind}_n(X)$ with respect to the n th power map $\sigma_n : C \rightarrow C$ given by $c \mapsto c^n$, as the set of C -orbits in $C \times X$ under the action $c \cdot (c', x) = (c'c^{-n}, c \cdot x)$. Again, $\text{ind}_n(X)$ becomes an almost finite C -set via the action $c \cdot \mathcal{O}(c', x) = \mathcal{O}(c \cdot c', x)$ and one verifies that

$$\text{ind}_n(X_1 \sqcup X_2) = \text{ind}_n(X_1) \sqcup \text{ind}_n(X_2) \quad \text{and} \quad \text{ind}_n(C_i) = C_{ni};$$

we also have that $\phi_{C^m}(\text{ind}_n(X)) = n \cdot \phi_{C^{m/n}}(X)$ if $n \mid m$, and zero otherwise. This then gives a ring isomorphism

$$\tau : W(\mathbb{Z}) \rightarrow \widehat{B}(C), \quad q = (q_1, q_2, \dots) \mapsto \sum_{n=1}^{\infty} \text{ind}_n(q_n^{(C)}).$$

If $\tau(q) = X(b)$, then the sequences of integers q and b are related via the formula

$$\prod_{n=1}^{\infty} \frac{1}{1 - q_n t^n} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - t^n} \right)^{b_n}.$$

If X is an almost finite C -set, then $\text{res}_n(X)$ is the restriction to the subgroup $\langle c^n \rangle$, that is, $X = \text{res}_n(X)$ but with a new action \odot defined by $c \odot x = c^n \cdot x$. Clearly, res_n is compatible with disjoint union and direct product and hence defines endomorphisms

$$\text{res}_n: \widehat{B}(C) \longrightarrow \widehat{B}(C)$$

which are the *Adams operations* on $\widehat{B}(C)$, and this family of commuting endomorphisms of $\widehat{B}(C)$ corresponds to the family of commuting endomorphisms Ψ^n on $w(\mathbb{Z})$. Similarly, the Verschiebung additive maps on $w(\mathbb{Z})$ are given by induction from the subgroup $\langle c^n \rangle$. Induction and restriction satisfy the following properties

- $\text{res}_n(C_m) = (n, m)C_{[n,m]/n}$, where $[n, m] = \text{lcm}(n, m)$
- $\text{ind}_n(C_m) = C_{nm}$
- $\ker(\text{res}_n) = \{x \in \widehat{B}(C) \mid \phi_{C^m}(x) = 0 \forall n \mid m\}$
- $\text{im}(\text{ind}_n) = \{x \in \widehat{B}(C) \mid \phi_{C^m}(x) = 0 \forall n \nmid m\}$

Similarly, one can make Frobenius and Verschiebung operators explicit on the necklace algebra $Nr(\mathbb{Z})$. Define the Frobenius ring morphisms $f_n: Nr(\mathbb{Z}) \rightarrow Nr(\mathbb{Z})$ by

$$f_n(b_1, b_2, \dots) = \left(\sum_{[n,i]=n} (n, i)b_i, \sum_{[n,i]=2n} (n, i)b_i, \dots \right)$$

and the Verschiebung additive morphisms $v_n: Nr(\mathbb{Z}) \rightarrow Nr(\mathbb{Z})$ via

$$v_n(b_1, b_2, \dots) = (\underbrace{0, \dots, 0}_{n-1}, b_1, \underbrace{0, \dots, 0}_{n-1}, b_2, \dots);$$

these Frobenius and Verschiebung operations f_n and v_n commute with the induction and restriction maps ind_n and res_n on $\widehat{B}(C)$.

In retrospect, the appearance of Burnside rings in \mathbb{F}_1 -geometry is not surprising. Recall from the Smirnov–Kapranov paper [20] (and the first chapter in this book) that $\mathbf{GL}_n(\mathbb{F}_1) \cong \mathbf{S}_n$, so for any group G an n -dimensional representation of G over \mathbb{F}_1 would be a group morphism $G \rightarrow \mathbf{S}_n$, that is, a permutation representation of G , or equivalently, a finite G -set (see also the first chapter). If G is an infinite discrete group, this says that any finite-dimensional \mathbb{F}_1 -representation of G factors as a permutation representation through a finite group quotient, and hence determines an element in the Burnside ring $B(\widehat{G})$ of the profinite completion of G . In the special case when $G = C$ we can write C additively (that is, $C = \mathbb{Z}$) and its \mathbb{C} -representations are of course all 1-dimensional and parametrized by \mathbb{C}^* . The

\mathbb{F}_1 -representations are then the representations of the profinite completion $\widehat{\mathbb{Z}}_+$ and its \mathbb{C} -points are precisely the roots of unity! Further, for completed Burnside rings we have $\widehat{B}(G) = \widehat{B}(\widehat{G})$, so in our case $w(\mathbb{Z}) = \widehat{B}(C) = \widehat{B}(\mathbb{Z}) = \widehat{B}(\widehat{\mathbb{Z}})$.

In [13] Andreas Dress and Christian Siebeneicher have extended the Witt construction to the profinite completion \widehat{G} of an arbitrary discrete group G (and in fact even to arbitrary profinite groups). Let $\text{cosg}(\widehat{G})$ be the set of conjugacy classes of open subgroups of \widehat{G} (that is, the conjugacy classes of subgroups of G of finite index); then one can consider the covariant functor

$$W_G: \mathbf{rings} \longrightarrow \mathbf{rings}, \quad A \longmapsto A^{\text{cosg}(\widehat{G})}$$

and they show that with respect to this functor we have an isomorphism between $W_G(\mathbb{Z})$ and the Burnside ring $\widehat{B}(\widehat{G})$ of almost finite G -sets. Moreover, the rings $W_G(R)$ all have Frobenius-like and Verschiebung-like morphisms to (and from) $W_U(R)$, for any subgroup U of G of finite index. The Frobenius and Verschiebung maps

$$W_G(R) \xrightarrow{\Psi^U} W_U(R) \quad \text{and} \quad W_U(R) \xrightarrow{V_U} W_G(R)$$

are defined by restriction, respectively induction. Clearly, in the case when $G = \mathbb{Z}$ all cofinite subgroups are isomorphic to \mathbb{Z} , giving rise to the Frobenius lift *endomorphisms* and corresponding Verschiebung operations on $w(R)$.

This raises the exciting prospect of extending or modifying Borger’s λ -rings approach to \mathbb{F}_1 -geometry to other categories \mathbf{rings}_G of commutative rings with suitable morphisms to/from a collection of rings (for any conjugacy class of a cofinite subgroup of G) such that the Dress–Siebeneicher–Witt functor W_G is a right adjoint functor to the forgetful functor $\mathbf{rings}_G \rightarrow \mathbf{rings}$. We expect such an approach to be fruitful when one starts with the braid group \mathbb{B}_3 or its quotient $\mathbf{PSL}_2(\mathbb{Z})$, which may also clarify the role of Conway’s big picture, which after all was intended to provide a better understanding of cofinite subgroups of the modular group $\mathbf{PSL}_2(\mathbb{Z})$.

3.3. What is $\text{Spec}(\mathbb{Z})$ over \mathbb{F}_1 ? So we can compute explicitly with $w(\mathbb{Z})$ and know that $\text{Spec}(\mathbb{Z})/\mathbb{F}_1$ is the geometric object associated to $w(\mathbb{Z})$, but what is this object and can we make sense of Smirnov’s covering maps $\text{Spec}(\mathbb{Z}) \twoheadrightarrow \mathbb{P}_{\mathbb{F}_1}^1$?

We have a candidate for the geometric object, namely the λ -spectrum of $w(\mathbb{Z})$

$$\text{Spec}(\mathbb{Z})_{\mathbb{F}_1} = \text{Spec}_{\lambda}(w(\mathbb{Z})) = \{ \ker(w(\mathbb{Z}) \longrightarrow A) \mid A \text{ is a reduced } \lambda\text{-ring} \}.$$

If $\text{Spec}(\mathbb{Z})$ would behave as a “curve” over \mathbb{F}_1 , one would expect the λ -spectrum to contain many geometric points over \mathbb{F}_1 . However, we will soon see that

$$\max_{\lambda}(w(\mathbb{Z})) = \emptyset = \max_{\text{cycl}}(w(\mathbb{Z})).$$

In fact a similar result holds for any $w(R)$.

The fact we will use is that the Verschiebung operators survive the action of taking λ -ring quotients $A = w(R)/I$ which have no additive torsion. I thank Jim Borger for communicating this to me. Clearly, there are additive maps

$$v_n: w(R) \xrightarrow{V_n} w(R) \longrightarrow A$$

and we have to show that $\ker(v_n) \subset I$. Because the Ψ^p are lifts of the Frobenius, there is a unique map d on $w(R)$ such that for all $s(t) \in w(R)$ we have the identity

$$(s(t))^{\otimes p} + [p]d(s(t)) = \Psi^p(s(t)),$$

and hence any λ -ideal I must be preserved by d . Assume $s(t) \in \ker(v_{np})$, that is, $V_{np}(s(t)) \in I$; then from the identities

$$V_{np}(s(t))^{\otimes p} + [p]d(V_{np}(s(t))) = \Psi^p(V_{np}(s(t))) = \Psi^p \circ V_p \circ V_n(s(t)) = [p]V_n(s(t))$$

it follows that the left-hand side is contained in I , and so must be the right-hand side. Since $A = w(R)/I$ has no additive torsion, it follows that $V_n(s(t)) \in I$, so $v_n(s(t)) \in \ker(v_n)$. As we can repeat this process for any prime factor p of $m = np$ it follows that if $s(p) \in \ker(v_m)$, then $s(t) \in \ker(v_1) = I$. Thus, if A is a λ -ring quotient of $w(R)$ without additive torsion, A is equipped not only with ring endomorphisms Ψ^n , but also with additive morphisms v_n satisfying all the properties the Frobenius and Verschiebung operators satisfy on $w(R)$, indicating that A must itself be close to a witty ring.

Now assume that A is étale over \mathbb{F}_1 and hence of finite rank over \mathbb{Z} . Recall from [26] that we can also define the ring structure of $w(R)$ as the inverse limit

$$w(R) = \varprojlim w_n(R) \quad \text{with} \quad w_n(R) = \ker\left(R[t]/(t^{n+1})^* \longrightarrow R^*\right).$$

As A is finite over \mathbb{Z} , the ring morphism $w(R) \rightarrow A$ factors through a $w_n(R)$ for some $n \in \mathbb{N}$. But this means that $V_n(w(R))$ is contained in the ideal I , in particular $v_n(1) \in I$, and then from the argument given before we conclude that $1 \in I$ and hence that $A = 0$. That is, witty rings $w(R)$ do not have torsion-free λ -ring quotients, finite over \mathbb{Z} .

That is, λ -spectra of witty rings do not have geometric points and hence behave very unlike \mathbb{F}_1 -geometric objects of finite dimension. Still, the λ -spectrum has many other points—in fact we can identify the usual prime spectrum $\text{Spec}(R)$ with a subset of witty points in $\text{Spec}_\lambda(w(R))$:

$$\text{Spec}(R) \cong \text{Spec}_w(w(R)) = \left\{ \ker\left(w(R) \longrightarrow w(Q(R/p))\right) \mid p \in \text{Spec}(R) \right\}$$

where $Q(R/p)$ denotes the field of fractions of the domain R/p .

Let us work out what the witty ring $w(F)$ of a field F is. If \overline{F} is algebraically closed, then by construction we have an inclusion of multiplicative groups $\overline{F}^* \hookrightarrow w(\overline{F})$ determined by $a \mapsto \frac{1}{1-at}$ which extends to a ring morphism on the group algebra of \overline{F} , $\mathbb{Z}[\overline{F}^*] \xrightarrow{L} w(\overline{F})$ with image the set of all rational formal power series

$$\frac{\prod_i (1 - \alpha_i t)^{e_i}}{\prod_j (1 - \beta_j t)^{f_j}} = L\left(\sum_j f_j \beta_j - \sum_i e_i \alpha_i\right).$$

In other words, we have a suitably dense subring of $w(\overline{F})$ isomorphic to the integral group algebra $\mathbb{Z}[\overline{F}^*]$. The absolute Galois group $G = \text{Gal}(\overline{F}/F)$ acts on both rings,

giving an inclusion of rings

$$\mathbb{Z}[\overline{F^*}]^G = \text{Div}(\mathbb{P}_F^1 \setminus \{0, \infty\}) \hookrightarrow w(F)$$

where $\text{Div}(\mathbb{P}_F^1 \setminus \{0, \infty\})$ is the abelian group of divisors on $\mathbb{P}_F^1 \setminus \{0, \infty\}$, that is, of all formal finite combinations $\sum n_i [f_i]$ with $n_i \in \mathbb{Z}$ and the f_i irreducible monic polynomials in $F[x, x^{-1}]$, which gets an induced multiplication (and even a λ -ring structure) from the ring structure of $\mathbb{Z}[\overline{F^*}]$. The Frobenius lifts on $\mathbb{Z}[\overline{F^*}]$ are the linearizations of the multiplicative group endomorphisms $a \mapsto a^n$ for $a \in \overline{F^*}$.

In the special case of \mathbb{F}_p we have seen before that we can identify the multiplicative group $\overline{\mathbb{F}_p^*}$ with the group of all roots of unity $\mu^{(p)}$ of order prime to p , and hence we get a dense subring

$$\mathbb{Z}[\mu^{(p)}]^{\widehat{\mathbb{Z}}_+} = \text{Div}(\mathbb{P}_{\mathbb{F}_p}^1 \setminus \{0, \infty\}) \hookrightarrow w(\mathbb{F}_p).$$

Thus we see a surprise guest re-appearance of the fiber $\mathbb{P}_{\mathbb{F}_p}^1$ of the structural map $\mathbb{P}_{\mathbb{Z}}^1 \rightarrow \text{Spec}(\mathbb{Z})$ in the description of the witty point in $\text{Spec}_\lambda(w(\mathbb{Z}))$ determined by the λ -ring morphism $w(\mathbb{Z}) \rightarrow w(\mathbb{F}_p)$, somewhat closing the circle of thoughts we began by looking at Mumford’s drawings!

Still, there’s the eternal problem of finding a natural identification between $\overline{\mathbb{F}_p^*}$ and $\mu^{(p)}$. We will briefly sketch how this can be done “in principle” using ordinal numbers. In [8] John H. Conway identified the algebraic closure of \mathbb{F}_2 with the set of all ordinal numbers smaller than ω^{ω^ω} equipped with nim-addition and multiplication. Later Joseph DiMuro extended this to identify the algebraic closure of \mathbb{F}_p with ω^{ω^ω} in [12]. We will recall the case of characteristic 2 and refer to [12] for the general case.

To distinguish the nim-rules from addition and multiplication of ordinal numbers, we will denote the latter ones enclosed in brackets. So, for example, $[\omega^2]$ will be the ordinal number, whereas ω^2 will be the square of the ordinal number $[\omega]$ in nim-arithmetic. These nim-rules can be defined on all ordinals as follows

$$\alpha + \beta = \text{mex}(\alpha' + \beta, \alpha + \beta') \quad \text{and} \quad \alpha \cdot \beta = \text{mex}(\alpha' \cdot \beta + \alpha \cdot \beta' - \alpha' \cdot \beta'),$$

where α' (respectively β') ranges over all ordinals less than α (respectively, than β) and mex stands for the “minimal excludent” of the given set, that is, the smallest ordinal not contained in the set. Observe that these definitions allow us to compute with ordinals inductively. Computing the sum of two ordinals is easy: write each one uniquely as a sum of ordinal numbers $\alpha = [2^{\alpha_0} + 2^{\alpha_1} + \dots + 2^{\alpha_k}]$, then to compute $\alpha + \beta$ we delete powers appearing in each factor and take the Cantor ordinal sum of the remaining sum (for finite ordinals this is the common nim-addition “adding binary expressions without carry”). To compute multiplication of ordinals, introduce the following special element

$$\kappa_{2^n} = [2^{2^{n-1}}]$$

and, for primes $p > 2$, the elements

$$\kappa_{p^n} = [2^{\omega^k \cdot p^{n-1}}],$$

where k is the number of primes strictly smaller than p . Because $[2^{\alpha_0} + \dots + 2^{\alpha_k}] = [2^{\alpha_0}] + \dots + [2^{\alpha_k}]$ we can multiply two ordinals smaller than $[\omega^{\omega^\omega}]$ if we know how to compute products $[2^\alpha] \cdot [2^\beta]$ with $\alpha, \beta < [\omega^\omega]$. Each such α or β can be expressed uniquely as

$$[\omega^t \cdot n_t + \omega^{t-1} \cdot n_{t-1} + \dots + \omega \cdot n_1 + n_0]$$

with t and all n_k natural numbers. If we write n_k in base p where p is the $(k+1)$ th prime number, that is, $n_k = [\sum_j p^j \cdot m(j, k)]$ for $0 \leq m(j, k) < p$, then we can write any 2-power smaller than $[\omega^{\omega^\omega}]$ as a decreasing finite product $[\prod_q \kappa_q^{m(q)}]$ with $0 \leq m(q) < p$ and q a power of p . Conway has shown that we have $[\prod_q \kappa_q^{m(q)}] = \prod_q \kappa_q^{m(q)}$, which allows us to compute all products except when $[m(q) + m'(q)] \geq p$ for some q . Thus it remains to specify the ordinals $(\kappa_q)^p$ and here Conway proved the following rules, depending on the still to be determined elements α_p ,

$$(\kappa_{2^n})^2 = \kappa_{2^n} + \prod_{1 \leq i \leq n} \kappa_{2^i}, \quad (\kappa_p)^p = \alpha_p, \quad \text{and} \quad (\kappa_{p^n})^p = \kappa_{p^{n-1}}$$

for p an odd prime and $n \geq 2$. Conway calculated the first few α_p , for example $\alpha_3 = 2, \alpha_5 = 4, \alpha_7 = [2^\omega] + 1$ etc. and then Hendrik Lenstra [25] gave an explicit algorithm to compute the α_p and managed to determine them for all $p \leq 43$. Today we know all α_p for $p \leq 293$ with only a few exceptions. In principle this allows us to determine the ordinal number corresponding to any realistic occurring element in $\overline{\mathbb{F}}_2$. Similarly, DiMuro proved that $\overline{\mathbb{F}}_p$ can be identified with $[\omega^{\omega^\omega}]$ and listed the values for the α_q in those cases for primes $q \leq 43$ and $p \leq 11$.

Using this correspondence we can now construct a one-to-one correspondence $\overline{\mathbb{F}}_p^* \leftrightarrow \mu^{(p)}$, which we will illustrate in the case $p = 2$. Conway showed that the ordinals $[2^{2^n}]$ form a subfield isomorphic to $\mathbb{F}_{2^{2^n}}$ and so there is a consistent embedding of the quadratic closure of \mathbb{F}_2 into roots of unity by starting with $[2]$ being the smallest ordinal generating the multiplicative group of the subfield $[2^2]$ (of order 3) and taking it to be $e^{2\pi i/3}$; for the next subfield $[2^{16}]$ we have to look for the smallest ordinal $[k]$ such that $[k]^5 = 2$, which turns out to be $[4]$ which then corresponds to $e^{2\pi i/15}$, and the correspondence between \mathbb{F}_{2^4} and μ_{15} is depicted in Figure 10 (together with the addition and multiplication tables of $[16]$ to verify the claims). We have indicated the different orbits under the Frobenius $x \mapsto x^2$ with different colors. There are two orbits of size one: $\{0\}$ corresponding to x , and $\{1\}$ corresponding to $x + 1$. One orbit $\{2, 3\}$ of size two corresponding to the irreducible polynomial $(x - 2)(x - 3) = x^2 + x + 1$, and three orbits of size four corresponding to the three irreducible monic polynomials in $\mathbb{F}_2[x]$ of degree 4, for example $\{4, 6, 5, 7\} \leftrightarrow x^4 + x + 1$. Iterating this procedure we get an explicit embedding of the quadratic closure of \mathbb{F}_2 into roots of unity (the relevant generators for the next stages are 32, then 1051, then 1361923 and 1127700028470). Having obtained an explicit identification of the quadratic closure of \mathbb{F}_2 inside the roots of unity, we can then proceed by associating to $[\omega]$ the root $e^{2\pi i/9}$ as $[\omega]^3 = [2]$, mapping $[\omega^\omega]$ to $e^{2\pi i/75}$ as $[\omega^\omega]^5 = [4]$, and so on until we have identified $\overline{\mathbb{F}}_2^*$ with $\mu^{(2)}$. This then allows us to associate to a schematic point of $\mathbb{A}_{\mathbb{F}_2}^1$, that is, to an irreducible monic polynomial in $\mathbb{F}_2[x]$, the root of unity corresponding to

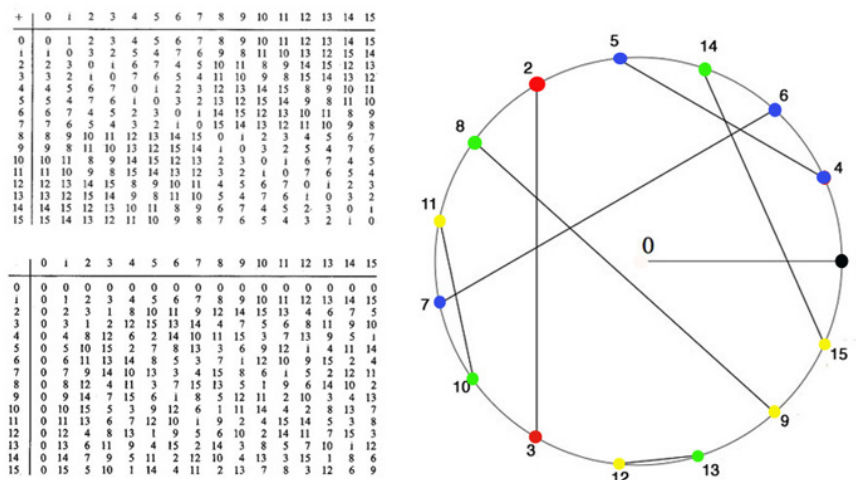


Figure 10. The multiplication tables of [16] and the correspondence $\mathbb{F}_{24} \leftrightarrow \mu_{15}$

the smallest ordinal in the Frobenius orbit associated to the polynomial. So, for example, to $x^4 + x^3 + x^2 + x + 1$ one assigns $e^{12\pi i/15}$, as the roots of the polynomial are the ordinals $\{[8], [10], [13], [14]\}$. Once again, one can repeat these arguments for the algebraic closures \mathbb{F}_p using the results from [12].

3.4. What is the map from $\overline{\text{Spec}(\mathbb{Z})}$ to $\mathbb{P}_{\mathbb{F}_1}^1$? In the foregoing sections we have recalled some of the successes of Borger’s approach to absolute geometry via λ -rings. For example, the identification of the étale site of \mathbb{F}_1 with the category of finite sets equipped with an action of the monoid $\widehat{\mathbb{Z}}^\times$ is one of the most convincing theories around vindicating Smirnov’s proposal that one should interpret μ_∞ as the algebraic closure of \mathbb{F}_1 . Further, with this λ -ring approach we obtain roughly the same class of examples provided by all other approaches to \mathbb{F}_1 -geometry, such as affine and projective spaces, Grassmannians, toric varieties, among others. In addition, we can associate a space of geometric points as well as a new topology to such an \mathbb{F}_1 -geometric object X . For, assume that X is locally controlled by a λ -ring R ; then locally its *geometric points* correspond to kernels of λ -ring morphisms $R \rightarrow S$ where S is étale over \mathbb{F}_1 , among which are the *cyclotomic coordinates* which are the special points obtained by taking $S = \mathbb{Z}[x]/(x^n - 1) = \mathbb{Z}[\mu_n]$. But, as S is finite projective over \mathbb{Z} , these kernels are *not* maximal ideals of the λ -ring R , but rather sub-maximal ones, entailing that two such kernels no longer have to be co-maximal. This then leads to an adjacency (or clique) relation among the corresponding geometric points which gives us the *Habiro topology* on $\max_\lambda(R)$. This new topological feature encodes the fact that the closed subschemes of the usual integral affine scheme $\text{Spec}(R)$ corresponding to the kernels of the two geometric points intersect over certain prime numbers p . As an example, we have seen that the cyclotomic points of $\mathbb{P}_{\mathbb{Z}}^1$ (for the toric λ -structure) give us indeed the proposal

that

$$\mathbb{P}_{\mathbb{F}_1}^1 = \{[0], \infty\} \cup \{[n] \mid n \in \mathbb{N}_0\},$$

where two cyclotomic points $[n]$ and $[m]$ are adjacent if and only if their quotient is a pure prime-power, leading to the Habiro topology on the roots of unity μ_∞ . Further, the λ -ring structure, that is, the commuting family of endomorphisms $\{\Psi^n \mid n \in \mathbb{N}_0\}$, can be viewed as descent data from $\mathbf{Spec}(\mathbb{Z})$ to $\mathbf{Spec}(\mathbb{F}_1)$, and hence conversely we can view the process of forgetting the λ -ring structure as the base extension functor $- \times_{\mathbf{Spec}(\mathbb{F}_1)} \mathbf{Spec}(\mathbb{Z})$. In particular we can now make sense of the identity

$$\mathbb{P}_{\mathbb{F}_1}^1 \times_{\mathbf{Spec}(\mathbb{F}_1)} \mathbf{Spec}(\mathbb{Z}) = \mathbb{P}_{\mathbb{Z}}^1,$$

where the right-hand side is the usual integral scheme $\mathbb{P}_{\mathbb{Z}}^1$, without emphasis on the toric λ -structure.

But Borger’s proposal really shines in that it allows us to make sense of what the Weil restriction to $\mathbf{Spec}(\mathbb{F}_1)$ is of *any* integral scheme. Indeed, the witty functor $w: \mathbf{rings} \rightarrow \mathbf{rings}_\lambda$ is the right adjoint of the forgetful functor (which we have seen is base extension), and hence if the integral scheme X is locally of the form $\mathbf{Spec}(R)$, then X/\mathbb{F}_1 is locally the geometric object corresponding to the λ -ring $w(R)$. However, such rings do not have geometric points as before, so we have a dichotomy among the geometric \mathbb{F}_1 -objects which resembles the dichotomy in noncommutative algebraic geometry between algebras having plenty of finite-dimensional representations, and algebras that have no such representations. Geometric \mathbb{F}_1 -objects are either the restricted class of combinatorial controlled integral schemes allowing a λ -structure, or the class of infinite-dimensional objects corresponding to Witt schemes of integral schemes. Still, the ordinary integral scheme structure survives this witty-fication, as $\mathbf{Spec}(R)$ can be embedded in the λ -spectrum $\mathbf{Spec}_\lambda(w(R))$ via the kernels of the λ -ring maps $w(R) \rightarrow w(R/P) \rightarrow w(Q(R/P))$ for any prime ideal P of R . As an example, $\mathbf{Spec}(\mathbb{Z})$ is the \mathbb{F}_1 -geometric object corresponding to the Burnside ring $w(\mathbb{Z}) = \widehat{B}(C)$ which does indeed contain the proposal that

$$\mathbf{Spec}(\mathbb{Z})/\mathbb{F}_1 = \{(p) \mid p \text{ a prime number} \},$$

where we view the prime number p as corresponding to the λ -ring morphism

$$w(\mathbb{Z}) \longrightarrow w(\overline{\mathbb{F}_p}) \approx \text{Div}(\mathbb{P}_{\mathbb{F}_p}^1 \setminus \{[0], [\infty]\}).$$

Although these two classes of geometric \mathbb{F}_1 -objects are quite different, we can still make sense of morphisms between them, as they have to be locally given by λ -ring morphisms. In particular, let us investigate whether we can make sense of Smirnov’s maps

$$q = \frac{a}{b}: \mathbf{Spec}(\mathbb{Z}) \longrightarrow \mathbb{P}_{\mathbb{F}_1}^1$$

in Borger’s λ -rings approach to \mathbb{F}_1 -geometry, that is, whether this map is locally determined by a λ -ring morphism. With $\mathbb{P}_{\mathbb{F}_1}^1$ we mean the cyclotomic points of the integral scheme $\mathbb{P}_{\mathbb{Z}}^1$ equipped with the toric λ -ring structure. Because $(a, b) = 1$,

we can cover $\mathbb{P}_{\mathbb{Z}}^1$ with the prime spectra of two λ -rings, namely $\mathbb{P}_{\mathbb{Z}}^1 = \text{Spec}(\mathbb{Z}_b[x]) \cup \text{Spec}(\mathbb{Z}_a[x^{-1}])$, and therefore

$$\mathbb{P}_{\mathbb{F}_1}^1 = \text{Spec}_{\text{cycl}}(\mathbb{Z}_b[x]) \cup \text{Spec}_{\text{cycl}}(\mathbb{Z}_a[x^{-1}]).$$

Further, we have seen that $\text{Spec}(\mathbb{Z})/\mathbb{F}_1$ should be viewed as $\text{Spec}_{\lambda}(w(\mathbb{Z}))$ which we can cover as $\text{Spec}_{\lambda}(w(\mathbb{Z}_b)) \cup \text{Spec}_{\lambda}(w(\mathbb{Z}_a))$. Now, consider the λ -ring morphisms

$$\mathbb{Z}_b[x] \longrightarrow w(\mathbb{Z}_b), \quad x \longmapsto \frac{1}{1 - \frac{a}{b}t},$$

and

$$\mathbb{Z}_a[x^{-1}] \longrightarrow w(\mathbb{Z}_a), \quad x^{-1} \longmapsto \frac{1}{1 - \frac{b}{a}t},$$

which coincide on the intersection with the λ -morphism $\mathbb{Z}_{ab}[x, x^{-1}] \rightarrow w(\mathbb{Z}_{ab})$ determined by $x \mapsto \frac{1}{1 - \frac{1}{ab}t}$. So, in order to investigate the associated geometric map

$$\text{Spec}(\mathbb{Z}_b) \cong \text{Spec}_w(w(\mathbb{Z}_b)) \longrightarrow \text{Spec}_{\text{cycl}}(\mathbb{Z}_b[x])$$

we have to look, for any prime p not dividing b , at the composition $\mathbb{Z}_b[x] \longrightarrow w(\mathbb{F}_p)$ which sends x to $\frac{1}{1 - \frac{a}{b}t}$ and hence is the map

$$x^n \longmapsto \frac{1}{1 - (\frac{a}{b})^n t} \quad \text{for } n \in \mathbb{N}_0.$$

If \bar{a}/\bar{b} has order n in \mathbb{F}_p this says that x^n maps to $1/(1 - t) = \textcircled{1} \in w(\mathbb{F}_p)$, and if $p \mid a$, then x is mapped to $\frac{1}{1-0t} = 1 = \textcircled{0} \in w(\mathbb{F}_p)$. Further, if $p \mid b$ we get in the composition $\mathbb{Z}_a[x^{-1}] \rightarrow w(\mathbb{F}_p)$ that x^{-1} is mapped to $\frac{1}{1-0t} = 1 = \textcircled{0} \in w(\mathbb{F}_p)$. So, if we write $[p]$ for the witt-point corresponding to the kernel of $w(\mathbb{Z}) \rightarrow w(\mathbb{F}_p)$ we get indeed that

$$[p] \longmapsto \begin{cases} [0] & \text{if } p \mid a \\ [\infty] & \text{if } p \mid b \\ [n] & \text{if } n \text{ is minimal such that } a^n - b^n \equiv 1 \pmod{p} \end{cases}$$

which coincides with Smirnov's proposal.

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