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Extending irreducible braid representations to the 3-component loop braid group

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Abstract

In a recent paper [3] it is shown that irreducible representations of the three string braid group B_3 of dimension ≤ 5 extend to representations of the three component loop braid group LB_3. Further, an explicit 6-dimensional irreducible B_3-representation is given not allowing such an extension.

In this note we give a necessary and sufficient condition, in all dimensions, on the components of irreducible representations of the modular group Γ such that sufficiently general representations extend to $\Gamma *_{C_3} S_3$. As a consequence, the corresponding irreducible B_3-representations do extend to LB_3.

1 The strategy

The 3-component loop braid group LB_3 encodes motions of 3 oriented circles in $\mathbb{R}^3$. The generator σ_i $(i = 1, 2)$ is interpreted as passing the i-th circle under and through the $i+1$-th circle ending with the two circles' positions interchanged. The generator s_i $(i = 1, 2)$ simply interchanges the circles i and $i + 1$. For physical motivation and graphics we refer to the paper by

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John Baez, Derek Wise and Alissa Crans [2]. The defining relations of LB_3 are:

1. $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

2. $s_1s_2s_1 = s_2s_1s_2$

3. $s_1^2 = s_2^2 = 1$

4. $s_1s_2\sigma_1 = \sigma_2s_1s_2$

5. $\sigma_1\sigma_2s_1 = s_2\sigma_1\sigma_2$

Note that (1) is the defining relation for the 3-string braid group B_3, (2) and (3) define the symmetric group S_3, therefore the first three relations describe the free group product $B_3 * S_3$.

Recall that the modular group $\Gamma = C_2 * C_3 = \langle s,t | s^2 = 1 = t^3 \rangle$ is a quotient of B_3 by dividing out the central element $c = (\sigma_1\sigma_2)^3$, so that we can take $t = \overline{\sigma}_1\overline{\sigma}_2$ and $s = \overline{\sigma}_1\overline{\sigma}_2\overline{\sigma}_1$. Hence, any irreducible n-dimensional representation $\phi : B_3 \to GL_n$ will be isomorphic to one of the form

$$\phi(\sigma_1) = \lambda\psi(\overline{\sigma}_1), \quad \text{and} \quad \phi(\sigma_2) = \lambda\psi(\overline{\sigma}_2)$$

for some $\lambda \in \mathbb{C}^*$ and $\psi : \Gamma \to GL_n$ an n-dimensional irreducible representation of $\Gamma = \langle s,t \rangle = \langle \overline{\sigma}_1, \overline{\sigma_2} \rangle$. With $S_3 = \langle s_1, s_2 | s_1s_2s_1 = s_2s_1s_2, s_1^2 = 1 = s_2^2 \rangle$, we consider the amalgamated free product

$$G = \Gamma *_{C_3} S_3$$

in which the generator of C_3 is equal to $t = \overline{\sigma}_1\overline{\sigma}_2$ in Γ and to s_1s_2 in S_3.

We will impose conditions on ψ such that it extends to a (necessarily irreducible) representations of G. Then, if this is possible, as $\psi(\overline{\sigma}_1\overline{\sigma}_2) = \psi(s_1s_2)$ and as the defining equations (1),(4) and (5) of LB_3 are homogeneous in the σ_i it will follow that

$$\phi(\sigma_i) = \lambda\psi(\overline{\sigma}_i), \quad \text{and} \quad \phi(s_i) = \psi(s_i)$$

is a representation of LB_3 extending the irreducible representation ϕ of B_3.

2 The result

Bruce Westbury has shown in [7] that the variety $\mathtt{iss}_n\ \Gamma$ classifying isomorphism classes of n-dimensional semi-simple Γ-representations decomposes as a disjoint union of irreducible components

$$\mathtt{iss}_n\ \Gamma = \bigsqcup_\alpha \mathtt{iss}_\alpha\ \Gamma$$

where $\alpha = (a, b; x, y, z) \in \mathbb{N}^{\oplus 5}$ satisfying $a + b = n = x + y + z$. Moreover, if $xyz \neq 0$ then the component $\mathtt{iss}_\alpha\ \Gamma$ contains a Zariski open and dense subset of irreducible Γ-representations if and only if $max(x, y, z) \leq min(a, b)$. In this case, the dimension of $\mathtt{iss}_\alpha\ \Gamma$ is equal to $1 + n^2 - (a^2 + b^2 + x^2 + y^2 + z^2)$. In going from irreducible Γ-representations to irreducible B_3-representations we multiply by $\lambda \in \mathbb{C}^*$. As a result, it is shown in [7] that there is a $\boldsymbol{\mu}_6$-action on the components $\mathtt{iss}_\alpha\ \Gamma$ leading to the same component of B_3-representations. That is, the variety $\mathtt{irr}_n\ B_3$ classifying isomorphism classes of irreducible n-dimensional B_3-representations decomposes into irreducible components

$$\mathtt{irr}_n\ B_3 = \bigcup_\alpha \mathtt{irr}_\alpha\ B_3$$

where $\alpha = (a, b; x, y, z)$ satisfies $a + b = n = x + y + z$, $a \geq b \geq x = max(x, y, z)$.

Theorem 2.1. *A Zariski open and dense subset of irreducible Γ-representations in $\mathtt{iss}_\alpha\ \Gamma$ extends to the group $G = \Gamma *_{C_3} S_3$ if and only if there are natural numbers u, v, w with $w \geq max(u, v)$ such that*

$$\alpha = (v + w, u + w; u + v, w, w)$$

As a consequence, a Zariski open and dense subset of irreducible B_3-representations in $\mathtt{irr}_\alpha\ B_3$ extends to the three-component loop braid group LB_3 if there are natural numbers $u \leq v \leq w$ such that $\alpha = (a, b; x, y, z)$ with $x = max(x, y, z)$ and

$$a = v + w,\ \ b = u + w,\ \ \{x, y, z\} = \{u + v, w, w\}$$

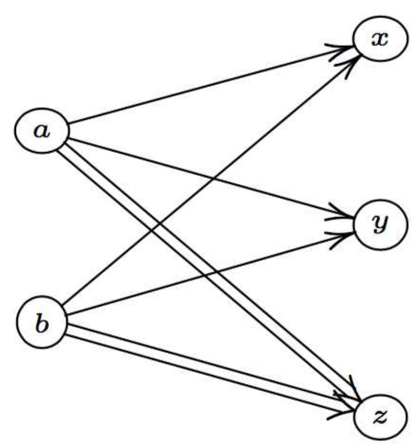
Observe that the first dimension n allowing an admissible 5-tuple not satisfying this condition is $n = 6$ with $\alpha = (3, 3; 3, 2, 1)$.

3 The proof

If V is an n-dimensional $G = \Gamma *_{C_3} S_3 \simeq C_2 * S_3$-representation, then by restricting it to the subgroups C_2 and S_3 we get decomposition of V into

$$S_+^{\oplus a} \oplus S_-^{\oplus b} = V \downarrow_{C_2} = V = V \downarrow_{S_3} = T^{\oplus x} \oplus S^{\oplus y} \oplus P^{\oplus z}$$

where $\{S_+, S_-\}$ are the 1-dimensional irreducibles of C_2, T is the trivial S_3-representation, S the sign representation and P the 2-dimensional irreducible S_3-representation. Clearly we must have $a + b = n = x + y + 2z$ and once we choose bases in each of these irreducibles we have that V itself determines a representation of the following quiver setting



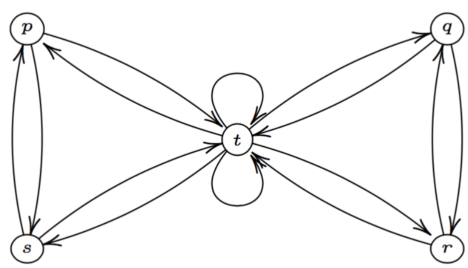
where the arrows give the block-decomposition of the base-change matrix B from the chosen basis of $V \downarrow_{C_2}$ to the chosen basis of $V \downarrow_{S_3}$. Isomorphism classes of irreducible G-representations correspond to isomorphism classes of θ-stable quiver representation of dimension vector $\beta = (a, b; x, y, z)$ for the stability structure $\theta = (-1, -1; 1, 1, 2)$. The minimal dimension vectors of θ-stable representations are

$$\begin{cases} \alpha_1 = (1, 0; 1, 0, 0) \\ \alpha_2 = (1, 0; 0, 1, 0) \\ \alpha_3 = (0, 1; 1, 0, 0) \\ \alpha_4 = (0, 1; 0, 1, 0) \\ \alpha_5 = (1, 1; 0, 0, 1) \end{cases}$$

which give us unique 1-dimensional G-representations S_1, S_2, S_3, S_4 and a 2-parameter family of 2-dimensional irreducible G-representations from which we choose S_5. By the results of [1], the local structure of the component $\mathtt{iss}_\beta\ G$ for $\beta = (p+q+t, r+s+t; p+r, q+s, t)$ in a neighborhood of the semi-simple G-representation

$$M = S_1^{\oplus p} \oplus S_2^{\oplus q} \oplus S_3^{\oplus r} \oplus S_4^{\oplus s} \oplus S_5^{\oplus t}$$

is étale equivalent to the local structure of the quiver-quotient variety of the setting below at the zero-representation

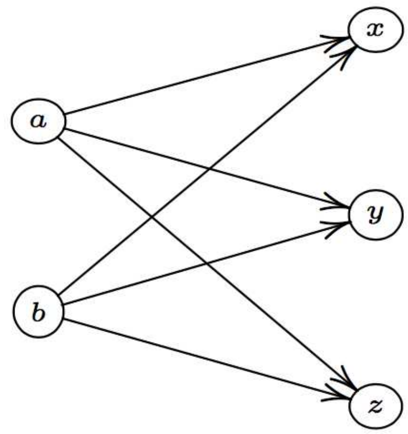


Hence, $\mathtt{iss}_\beta\ G$ will contain a Zariski open and dense subset of irreducible representations if and only if $\gamma = (p, q, r, s, t)$ is a simple dimension vector for this quiver, which by [5] is equivalent to γ being either $(1, 0, 0, 1, 0)$ or $(0, 1, 1, 0, 0)$ or satisfying the inequalities

$$p \leq s+t,\ q \leq r+t,\ r \leq q+t,\ s \leq p+t$$

Having determined the components containing irreducible G-representations, we have to determine those containing a Zariski open subset which remain irreducible when restricted to Γ.

As $\Gamma = C_2 * C_3$ any Γ-representation V corresponds to a semi-stable quiver represenation for the setting



when

$$V\downarrow_{C_2} = S_+^{\oplus a} \oplus S_-^{\oplus b} \quad \text{and} \quad V\downarrow_{C_3} = T_1^{\oplus x} \oplus T_\rho^{\oplus y} \oplus T_{\rho^2}^{\oplus z}$$

with $\{T_1, T_\rho, T_{\rho^2}\}$ the irreducible C_3-representations. Because $T\downarrow_{C_3} = T_1 = S\downarrow_{C_3}$ and $P\downarrow_{C_3} = T_\rho \oplus T_{\rho^2}$ we have that $M\downarrow_\Gamma$ has dimension vector

$$\alpha = (a, b; x, y, z) = (p + q + t, r + s + t; p + q + r + s, t, t)$$

which satisfies the condition that $max(x, y, z) \leq min(a, b)$ if and only if $t \geq r + s$ and $t \geq p + q$. Setting $u = r + s$, $v = p + q$ and $w = t$, the statement of Theorem 1 follows.

References

[1] Jan Adriaenssens, Lieven Le Bruyn, *Local quivers and stable representations*, Communications in Algebra, **31,** (2003), 1777-1797.

[2] John Baez, Derek Wise, Alissa Crans, *Exotic statistics for strings in 4D BF theory*, Adv. Theor. Math. Phys., **11** (2007), 707-749, `arXiv:gr-qc/0603085`.

[3] Paul Bruillard, Seung-Moon Hong, Julia Yael Plavnik, Eric C. Rowell, Liang Chang, Michael Yuan Sun, *Low-dimensional representations of the three component loop braid group*, `arXiv:1508.00005` (2015).

[4] Lieven Le Bruyn, *Dense families of B_3-representations and braid reversion*, Journal of Pure and Appl. Algebra, **215,** (2011) 1003-1014, `math.RA/1003.1610`.

[5] Lieven Le Bruyn, Claudio Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc., **317,** (1990), 585-598.

[6] Imre Tuba, Hans Wenzl, *Representations of the braid group B_3 and of $SL(2,\mathbb{Z})$*, Pacific J. Math., **197,** (2001).

[7] Bruce Westbury, *On the character varieties of the modular group*, preprint, Nottingham University, (1995).