Journal of Algebra and Its Applications
Vol. 15, No. 1 (2016) 1650006 (14 pages)

DOI: 10.1142/S0219498816500067

Bulk irreducibles of the modular group

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Received 28 May 2013
Accepted 26 December 2014
Published 31 August 2015

Communicated by C. M. Ringel

As the 3-string braid group B_3 and the modular group Γ are both of wild representation type one cannot expect a full classification of all their finite dimensional simple representations. Still, one can aim to describe 'most' irreducible representations by constructing for each d-dimensional irreducible component X of the variety $\mathtt{iss}_n(\Gamma)$ classifying the isomorphism classes of semi-simple n-dimensional representations of Γ an explicit minimal étale rational map $\mathbb{A}^d \to X$ having a Zariski dense image. Such rational dense parametrizations were obtained for all components when $n < 12$ in [5]. The aim of the present paper is to establish such parametrizations for all finite dimensions n.

Keywords: Modular group; quiver representations; linear dynamical systems.

Mathematics Subject Classification: 14L24, 16G20, 16R30

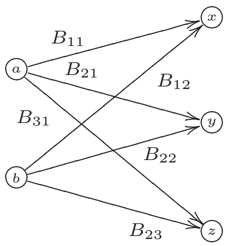
1. Introduction

The modular group $\Gamma = PSL_2(\mathbb{Z})$ is isomorphic to the free product $C_2 * C_3 = \langle \sigma, \tau \mid \sigma^2 = 1 = \tau^3 \rangle$. If V is an n-dimensional Γ-representation, we can decompose it into eigenspaces with respect to the actions of σ and τ

$$V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2},$$

where ρ is a primitive third root of unity. Denote the dimensions of these eigenspaces by $a = \dim(V_+), b = \dim(V_-)$ resp. $x = \dim(V_1), y = \dim(V_\rho)$ and $z = \dim(V_{\rho^2})$, then clearly $a + b = n = x + y + z$.

Choose a vector-space basis for V compatible with the decomposition $V_+ \oplus V_-$ and another basis of V compatible with the decomposition $V_1 \oplus V_\rho \oplus V_{\rho^2}$, then the associated base-change matrix $B \in GL_n(\mathbb{C})$ determines a quiver-representation V_B with dimension vector $\alpha = (a, b; x, y, z)$ for the indicated quiver Q_0



$$\text{with} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}. \tag{1}$$

A Q_0-representation W of dimension vector α is said to be θ-stable, (resp. θ-semi-stable), if for every proper sub-representations W', with dimension vector $\beta = (a', b'; x', y', z')$, we have that $x' + y' + z' > a' + b'$, (resp. $x' + y' + z' \geq a' + b'$).

Bruce Westbury proved in [9] V that an n-dimensional Γ-representation is irreducible if and only if the corresponding Q_0-representation V_B is θ-stable. Moreover, $V \simeq W$ as Γ-representations if and only if corresponding Q_0-representations V_B and $W_{B'}$ are isomorphic as quiver-representations.

Moreover, as the group-algebra $\mathbb{C}\Gamma$ is formally smooth, the affine GIT-quotient $\mathtt{iss}_n\ \Gamma = \mathtt{rep}_n\ \Gamma / PGL_n$, classifying isomorphism classes of n-dimensional semi-simple Γ-representations, decomposes into a disjoint union of irreducible components

$$\mathtt{iss}_n\ \Gamma = \bigsqcup_{\alpha} \mathtt{iss}_\alpha\ \Gamma$$

one component for every dimension vector $\alpha = (a, b; x, y, z)$ satisfying $a + b = n = x + y + z$. If $\alpha = (a, b; x, y, z)$ satisfies $x \cdot y \cdot z \neq 0$, then the component $\mathtt{iss}_\alpha\ \Gamma$ contains an open subset of simple representations if and only if $\max(x, y, z) \leq \min(a, b)$. In this case, the dimension of $\mathtt{iss}_\alpha\ \Gamma$ is equal to $d_\alpha = 1 + n^2 - (a^2 + b^2 + x^2 + y^2 + z^2)$. For proofs and extensions to the case of torus-groups we refer to [1, §6, 7].

The 3-string braid group $B_3 = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ has an infinite cyclic centre with generator $c = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$ and hence the corresponding quotient group (taking $\tilde{\sigma} = \sigma_1\sigma_2\sigma_1$ and $\tilde{\tau} = \sigma_1\sigma_2$) is isomorphic to Γ. By Schur's lemma, c acts via scalar multiplication with $\lambda \in \mathbb{C}^*$ on a finite-dimensional irreducible B_3-representation, hence their classification reduces to that of irreducible Γ-representations.

Working backwards, a θ-stable α-dimensional Q_0-representation V_B (with corresponding $n \times n$ matrix B) determines an irreducible n-dimensional Γ-representation and a 1-parameter family (parametrized by $\lambda \in \mathbb{C}^*$) of irreducible n-dimensional

B_3-representations determined by

$$\begin{cases} \sigma \mapsto \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \tau \mapsto B \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho 1_y & 0 \\ 0 & 0 & \rho^2 1_z \end{bmatrix} B^{-1} \end{cases} \quad \begin{cases} \sigma_1 \mapsto \lambda B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \sigma_2 \mapsto \lambda \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases}.$$

One would like to describe bulk irreducible representations of Γ (that is a Zariski open dense subset of $\mathtt{iss}_\alpha\ \Gamma$) by specifying suitable $n \times n$ matrices B having exactly d_α free matrix-entries. For $n \leq 5$ this was achieved in [8] (in fact, they gave a complete classification of low-dimensional irreducibles), and, for $n \leq 11$ in [5, §3]. The purpose of the present paper is to extend this to all finite dimensions n.

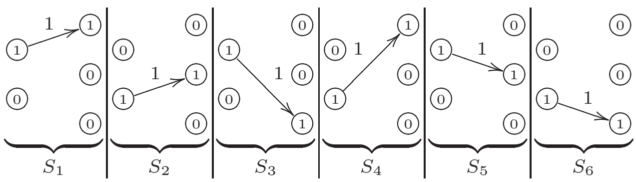
Theorem 1. *For every component* $\mathtt{iss}_\alpha\ \Gamma$ *of* $\mathtt{iss}_n\ \Gamma$ *containing irreducible representations, there exists an explicit étale rational map*

$$\mathbb{A}^{d_\alpha} \dashrightarrow \mathtt{iss}_\alpha\ \Gamma$$

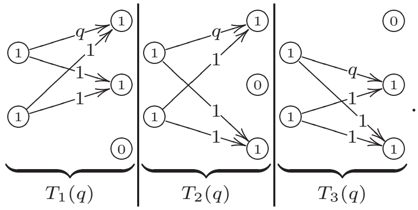
such that the image contains a Zariski open dense subset of $\mathtt{iss}_\alpha\ \Gamma$.

2. The Strategy

There are six one-dimensional irreducible Γ-representations, coresponding to the quiver-representations S_i for $1 \leq i \leq 6$:



and three one-parameter families of two-dimensional irreducibles corresponding to the quiver-representations $T_i(q)$ for $q \neq 0, 1$ and $1 \leq i \leq 3$



The semi-simple representation

$$M_0 = S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6} \oplus T_1(q)^{\oplus b_\alpha} \oplus T_2(q)^{\oplus b_\beta} \oplus T_3(q)^{\oplus b_\gamma}$$

clearly belongs to the component $\mathtt{iss}_\sigma\ \Gamma$ with dimension vector $\sigma = (a, b; x, y, z)$ where

$$\begin{cases} a = a_1 + a_3 + a_5 + b_\alpha + b_\beta \\ b = a_2 + a_4 + a_6 + b_\alpha + b_\gamma \\ x = a_1 + a_4 + b_\alpha + b_\beta \\ y = a_2 + a_5 + b_\alpha + b_\gamma \\ z = a_3 + a_6 + b_\beta + b_\gamma \end{cases}$$

and is fully determined by the base-change matrix B_0 with block-form as above

$$\left|\begin{array}{cccccc|cccccc}
1_{a_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{a_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q1_{b_\alpha} & 0 & 0 & 0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 \\
0 & 0 & 0 & 0 & q1_{b_\beta} & 0 & 0 & 0 & 0 & 0 & 1_{b_\beta} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1_{a_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_{a_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 & 0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q1_{b_\gamma} & 0 & 0 & 0 & 0 & 0 & 1_{b_\gamma} \\
\hline
0 & 1_{a_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{a_6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{b_\beta} & 0 & 0 & 0 & 0 & 0 & 1_{b_\beta} & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{b_\gamma} & 0 & 0 & 0 & 0 & 0 & 1_{b_\beta}
\end{array}\right| .$$

We will now determine the structure of the base-change matrices B of isoclasses of Γ-representations M in a Zariski open neighborhood of $[M_0]$ in $\mathtt{iss}_\sigma\ \Gamma$.

As M_0 is semi-simple, its isomorphism class forms a Zariski closed orbit $\mathcal{O}(M_0)$ in the smooth irreducible component $\mathtt{rep}_\sigma\ \Gamma$ under the action of $GL(\sigma) = GL_a \times GL_b \times GL_x \times GL_y \times GL_z$. The stabilizer subgroup $\mathrm{Stab}(M_0)$ is the automorphism group and is the subgroup of $GL(\sigma)$ we will denote by $GL(\tau) = GL_{a_1} \times GL_{a_2} \times GL_{a_3} \times GL_{a_4} \times GL_{a_5} \times GL_{a_6} \times GL_{b_\alpha} \times GL_{b_\beta} \times GL_{b_\gamma}$.

The normal space to the orbit $\mathcal{O}(M_0)$ can be identified as $GL(\tau)$-representation with the vector space of self-extensions $\mathrm{Ext}^1_{\mathbb{C}\Gamma}(M_0, M_0)$, see for example [3, II.2.7]. The Luna slice theorem, see for example [4, §4.2], asserts that the action map

$$GL(\sigma) \times^{GL(\tau)} \mathrm{Ext}^1_{\mathbb{C}\Gamma}(M_0, M_0) \longrightarrow \mathtt{rep}_\sigma\ \Gamma$$

sending the class of $(g, \vec{n})$ in the associated fibre bundle to the Γ-representation $g \cdot (M + \vec{n})$ is a $GL(\sigma)$-equivariant étale map whose image contains a Zariski open dense subset. Taking $GL(\sigma)$-quotients on both sides we obtain an étale map

$$\mathrm{Ext}^1_{\mathbb{C}\Gamma}(M_0, M_0)/GL(\tau) \longrightarrow \mathtt{iss}_\sigma\ \Gamma$$

whose image contains a Zariski open dense subset.

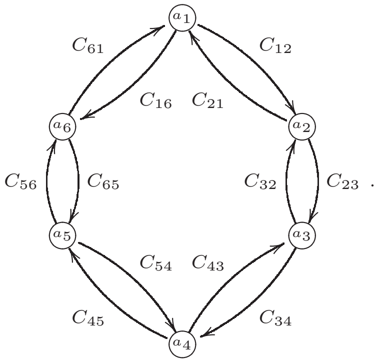
The crucial observation to make is that it follows from the theory of local quivers, [4, §4.2] or [1], that as a $GL(\tau)$-representation $\mathrm{Ext}^1_{\mathbb{C}\Gamma}(M_0, M_0)$ is isomorphic to

$\texttt{rep}_\tau\ Q$ for the quiver Q having 9 vertices (one for each of the distinct simple factors of M_0) and having as many directed arrows from the vertex corresponding to the simple factor S to that of the simple factor T as is the dimension of the space $\mathrm{Ext}^1_{\mathbb{C}\Gamma}(S,T)$.

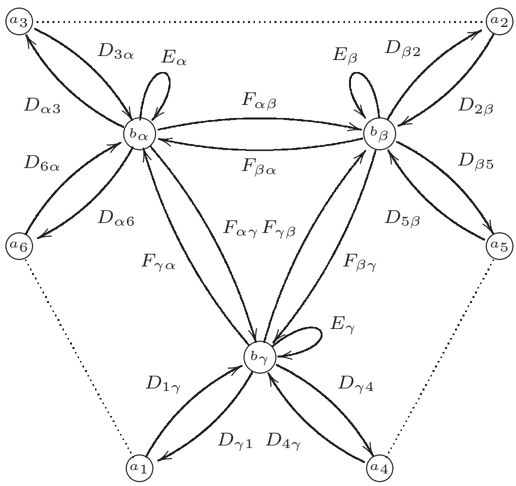
This then allows to identify the quotient variety $\mathrm{Ext}^1_{\mathbb{C}\Gamma}(M_0, M_0)/GL(\tau)$ with the affine variety $\texttt{iss}_\tau\ Q$ whose points are the isoclasses of semi-simple representations of Q of dimension-vector $\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma)$, and the action map induces an étale map with image containing a Zariski open dense subset

$$\texttt{iss}_\tau\ Q \longrightarrow \texttt{iss}_\sigma\ \Gamma.$$

Computing the normal space to the orbit $\mathcal{O}(M_0)$ as in the proof of [5, Theorem 4] but for the more complicated representation M_0 one obtains that the sub quiver of Q on the 6 vertices corresponding to the one-dimensional simple components $S_1, \ldots, S_6$ coincides with that of [5], that is corresponds to the quiver-setting



The additional quiver-setting depending on the three vertices corresponding to the two-dimensional simple factors $T_1(q)$, $T_2(q)$ and $T_3(q)$ can be verified to be which



concludes the proof of the following:

Theorem 2. *The étale action map $GL(\sigma) \times^{GL(\tau)} \mathtt{rep}_\tau\ Q \to \mathtt{rep}_\sigma\ \Gamma$ sends a τ-dimensional Q-representation to the Γ-representation determined by the base-change matrix B*

$$\left|\begin{array}{cccccc|cccccc}
1_{a_1} & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & C_{61} & 0 & 0 & D_{\gamma 1} \\
0 & C_{34} & C_{54} & 0 & 0 & D_{\gamma 4} & 0 & 1_{a_4} & 0 & 0 & 0 & 0 \\
0 & D_{3\alpha} & 0 & q1_{b_\alpha} + E_\alpha & 0 & 0 & 0 & 0 & D_{6\alpha} & 1_{b_\alpha} & 0 & F_{\gamma\alpha} \\
0 & 0 & 0 & 0 & q1_{b_\beta} & F_{\gamma\beta} & 0 & 0 & 0 & 0 & 1_{b_\beta} & 0 \\
\hline
C_{12} & C_{32} & 0 & 0 & D_{\beta 2} & 0 & 1_{a_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_{a_5} & 0 & 0 & 0 & 0 & C_{45} & C_{65} & 0 & D_{\beta 5} & 0 \\
0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 & 0 & 0 & 0 & 1_{b_\alpha} & F_{\beta\alpha} & 0 \\
D_{1\gamma} & 0 & 0 & 0 & F_{\beta\gamma} & q1_{b_\gamma} + E_\gamma & 0 & D_{4\gamma} & 0 & 0 & 0 & 1_{b_\gamma} \\
\hline
0 & 1_{a_3} & 0 & 0 & 0 & 0 & C_{23} & C_{43} & 0 & D_{\alpha 3} & 0 & 0 \\
C_{16} & 0 & C_{56} & D_{\alpha 6} & 0 & 0 & 0 & 0 & 1_{a_6} & 0 & 0 & 0 \\
0 & 0 & D_{5\beta} & 0 & 1_{b_\beta} + E_\beta & 0 & D_{2\beta} & 0 & 0 & F_{\alpha\beta} & 1_{b_\beta} & 0 \\
0 & 0 & 0 & F_{\alpha\gamma} & 0 & 1_{b_\gamma} & 0 & 0 & 0 & 0 & 0 & 1_{b_\gamma}
\end{array}\right| .$$

Under this map, simple Q-representations are mapped to irreducible Γ-representations, and if the coefficients of the block-matrices C_{ij}, D_{ij}, E_i and F_{ij} occurring in B give a parametrization of a Zariski open subset of the quotient variety $\mathtt{iss}_\tau\ Q$, then the corresponding n-dimensional representations of Γ contain a Zariski open dense set of irreducible Γ-representations in the component $\mathtt{iss}_\sigma\ \Gamma$ of $\mathtt{iss}_n\ \Gamma$ where $\sigma = (a, b; x, y, z)$ with

$$\begin{cases} a = a_1 + a_3 + a_5 + b_\alpha + b_\beta \\ b = a_2 + a_4 + a_6 + b_\alpha + b_\gamma \\ x = a_1 + a_4 + b_\alpha + b_\beta \\ y = a_2 + a_5 + b_\alpha + b_\gamma \\ z = a_3 + a_6 + b_\beta + b_\gamma. \end{cases}$$

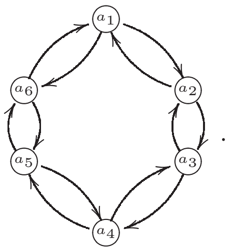
In view of the previous result and the symmetry of the quiver Q_0, it remains to find for each $\sigma = (a, b; x, y, z)$ satisfying

$$a + b = n = x + y + z \quad \text{and} \quad x = \max(x, y, z) \leq b = \min(a, b)$$

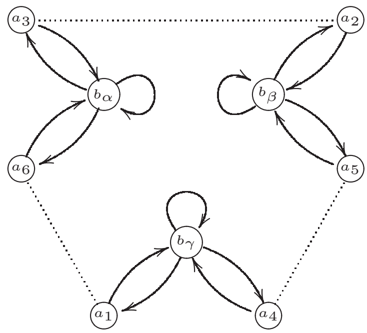
a judiciously chosen dimension-vector $\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma)$ of type σ together with an explicit rational parametrization of $\mathtt{iss}_\tau\ Q$.

We will separate this investigation in two cases, sharing the same underlying strategy.

First we choose $a_1, a_2, a_3, a_4, a_5, a_6$ such that $\sigma_1 = (a_1 + a_3 + a_5, a_2 + a_4 + a_6; a_1 + a_4, a_2 + a_5, a_3 + a_6)$ is a component containing simples and such that we have an explicit rational parametrization of the isoclasses of the quiver-setting



The upshot being that for a general representation the stabilizer subgroup reduces to $\mathbb{C}^*(1_{a_1} \times \cdots \times 1_{a_6})$. But then, the additional arrows D_{ij} and E_i, that is the quiver setting



give three settings corresponding, as we will see in the next section, to canonical linear control systems with $m = p = a_i + a_{i+3}$ and we will give a rational parametrization of the isoclasses which further reduces the stabilizer subgroup to $\mathbb{C}^*(1_{a_1} \times \cdots \times 1_{a_6} \times 1_{b_\alpha} \times 1_{b_\beta} \times 1_{b_\gamma})$. This then leaves the trivial action on the remaining arrows F_{ij} and hence these generic matrices conclude the desired rational parametrization.

3. The Proof

A linear control system Σ is determined by the system of linear differential equations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases}$$

where $\Sigma = (A, B, C) \in M_n(\mathbb{C}) \times M_{n\times m}(\mathbb{C}) \times M_{p\times n}(\mathbb{C})$ and $u(t) \in \mathbb{C}^m$ is the control at time t, $x(t) \in \mathbb{C}^n$ is the state of the system and $y(t) \in \mathbb{C}^p$ its output. Equivalent control systems differ only by a base change in the state space, that is $\Sigma' = (A', B', C')$ is equivalent to Σ if and only if there exists a $g \in GL_n(\mathbb{C})$ such that

$$A' = gAg^{-1}, \quad B' = gB \quad \text{and} \quad C' = Cg^{-1}$$

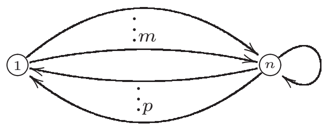
Σ is said to be *canonical* if the matrices

$$c_\Sigma = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \quad \text{and} \quad o_\Sigma = \begin{bmatrix} C & CA & CA^2 & \cdots & CA^{n-1} \end{bmatrix}$$

are of maximal rank.

Michiel Hazewinkel proved in [2] that the moduli space $\mathtt{sys}^c_{m,n,p}$ of all such canonical linear systems is a smooth rational quasi-affine variety of dimension $(m+p)n$. We will give another short proof of this result and draw some consequences from it (see also [7]).

Consider the quiver setting with m arrows $\{b_1, \ldots, b_m\}$ from left to right and p arrows $\{c_1, \ldots, c_p\}$ from right to left



To a system $\Sigma = (A, B, C)$ we associate the quiver-representation V_Σ by assigning to the arrow b_i the ith column B_i of the matrix B, to the arrow c_j the jth row C^j of C and the matrix A to the loop. As the base change group $\mathbb{C}^* \times GL_n$ acts on these quiver-representations by

$$(\lambda, g) \cdot V_\Sigma = (gAg^{-1}, gB_1\lambda^{-1}, \ldots, gB_m\lambda^{-1}, \lambda C^1 g^{-1}, \ldots, \lambda C^p g^{-1})$$

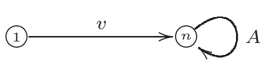
with the subgroup $\mathbb{C}^*(1, 1_n)$ acting trivially, there is a natural one-to-one correspondence between equivalence classes of linear systems Σ and isomorphism classes of quiver-representations V_Σ. Under this correspondence it is easy to see that canonical systems correspond to *simple* quiver-representations, see [7, Lemma 1]. Hence, the moduli-space $\mathtt{sys}^c_{m,n,p}$ is isomorphic to the Zariski-open subset of the affine quotient-variety classifying isomorphism classes of semi-simple quiver-representations, proving smoothness, quasi-affineness as well as determining the dimension by general results, see for example [4].

Lemma 1. *A generic canonical system Σ is equivalent to a triple $(A_n, B^\bullet_{nm}, C_{pn})$ with*

$$A_n = \begin{bmatrix} 0 & 0 & \ldots & & x_n \\ 1 & 0 & \ldots & & x_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & x_2 \\ & & & 1 & x_1 \end{bmatrix} \qquad B^\bullet_{nm} = \begin{bmatrix} 1 & b_{12} & \ldots & b_{1m} \\ 0 & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n2} & \ldots & b_{nm} \end{bmatrix}$$

that is, where A_n is a companion $n \times n$-matrix, $B^\bullet_{nm}$ is the generic $n \times m$-matrix with fixed first column and C_{pn} a generic $p \times n$-matrix.

Proof. A generic representation of the quiver-setting



will have the property that v is a cyclic-vector for the matrix A, that is, $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ are linearly independent. But then, performing a base-change we get a representation of the form

$$\begin{tikzcd} \textcircled{1} \arrow[r, "{\begin{bmatrix}1 & 0 & \ldots & 0\end{bmatrix}^{tr}}"] & \textcircled{n} \arrow[loop right, "A_n"] \end{tikzcd},$$

where A_n is a companion matrix whose nth column expresses the vector $-A^n v$ in the new basis. As the automorphism group of this representation is reduced to $\mathbb{C}^*(1, 1_n)$, any general representation V_Σ is isomorphic to one with $B_1 = \begin{bmatrix}1 & 0 & \ldots & 0\end{bmatrix}^{\mathrm{tr}}$, $A = A_n$ and the other columns of B and all rows of C generic vectors. $\square$

Lemma 2. *The following representations give a rational parametrization of the isomorphism classes of simple representations of these quiver-settings*

$$R_k : \begin{tikzcd} \textcircled{1} \arrow[r, bend left, "{\begin{bmatrix}1 & 0 & \ldots & 0\end{bmatrix}^{tr}}"] & \textcircled{k} \arrow[l, bend left, "{\begin{bmatrix}y_1 & y_2 & \ldots & y_k\end{bmatrix}}"] \arrow[r, bend left, "A_k"] & \textcircled{k} \arrow[l, bend left, "1_k"] \end{tikzcd}$$

and

$$S_k : \begin{tikzcd} \textcircled{1} \arrow[r, bend left, "{\begin{bmatrix}1 & 0 & \ldots & 0\end{bmatrix}^{tr}}"] & \textcircled{k} \arrow[l, bend left, "{\begin{bmatrix}y_1 & y_2 & \ldots & y_k\end{bmatrix}}"] \arrow[r, bend left, "A_k^\dagger"] & \textcircled{k-1} \arrow[l, bend left, "{\begin{bmatrix}\underline{0} \\ 1_{k-1}\end{bmatrix}}"] \end{tikzcd}$$

where A_k (resp. $A_k^\dagger$) is the generic $k \times k$ companion matrix (resp. the reduced $k-1 \times k$ companion matrix)

$$A_k = \begin{bmatrix} 0 & 0 & \ldots & & x_k \\ 1 & 0 & \ldots & & x_{k-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & x_2 \\ & & & 1 & x_1 \end{bmatrix} \quad \textit{and} \quad A_k^\dagger = \begin{bmatrix} 1 & 0 & \ldots & & x_{k-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & x_2 \\ & & & 1 & x_1 \end{bmatrix}.$$

Proof. By invoking the first fundamental theorem of GL_n-invariants (see for example [3, Theorem II.4.1]) we can in case R_k eliminate the base-change action in the right-most vertex, giving a natural one-to-one correspondence between isoclasses of representations

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left, "v"] & \textcircled{k} \arrow[l, bend left, "w^\tau"] \arrow[r, bend left, "X"] & \textcircled{k} \arrow[l, bend left, "Y"] \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} \textcircled{1} \arrow[r, bend left, "v"] & \textcircled{k} \arrow[l, bend left, "w^\tau"] \arrow[loop right, "Y.X"] \end{tikzcd}$$

and hence the claim follows from the previous lemma. As for case S_k we can again apply the first fundamental theorem for GL_n-invariants, now with respect to the base-change action in the middle vertex, to obtain a natural one-to-one correspondence between isoclasses of representations

$$\begin{tikzcd} 1 \arrow[r, shift left, "v"] & k \arrow[l, shift left, "w^\tau"] \arrow[r, shift left, "X"] & k-1 \arrow[l, shift left, "Y"] \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} 1 \arrow[loop left, "w^\tau.v"] \arrow[r, shift left, "X.v"] & k-1 \arrow[l, shift left, "w^\tau.Y"] \arrow[loop right, "X.Y"] \end{tikzcd}$$

and again the claim follows from the previous lemma, taking into account the extra free loop in the left-most vertex, which corresponds to y_1. □

Lemma 3. *The following representations give a rational parametrization for the isomorphism classes of simple representations of the quiver-setting*

$$\begin{tikzcd}[column sep=huge] 1 \arrow[r, shift left, "1"] & 1 \arrow[l, shift left, "z"] \arrow[r, shift left, "{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^{tr}}"] & k \arrow[l, shift left, "{\begin{bmatrix} y_1 & y_2 & \dots & y_k \end{bmatrix}}"] \arrow[r, shift left, "A_k^\dagger"] & k-1 \arrow[l, shift left, "{\begin{bmatrix} \underline{0} \\ 1_{k-1} \end{bmatrix}}"] \arrow[r, shift left, "1_{k-1}"] & k-1 \arrow[l, shift left, "B"] \end{tikzcd},$$

where B is a generic $k-1 \times k-1$ matrix and, as before, $A_k^\dagger$ is a reduced generic companion matrix.

Proof. Forgetting the end-vertices (and maps to and from them) we are in the situation of the previous lemma. For general values these are simple quiver-representations and hence the automorphism group is reduced to $\mathbb{C}^*(1, 1_k, 1_{k-1})$. If we now add the end vertices we can use base-change in them to force one of the two arrows to be the identity map, leaving the remaining map generic. Alternatively, we can use the first fundamental theorem of GL_n-invariants as before, to obtain the claimed result. □

After these preliminaries, we follow the strategy laid out in the previous section for a dimension-vector $\alpha = (a, b; x, y, z)$ satisfying

$$a + b = n = x + y + z \quad \text{and} \quad x = \max(x, y, z) \leq b = \min(a, b).$$

That is, such that there are θ-stable Q_0-representations of dimension-vector α.

3.1. *Case* 1: $a > b$

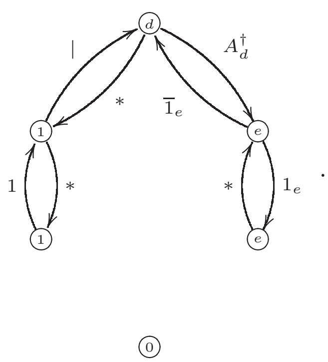
Define $d = a - b$, $e = d - 1$, $f = b - z$, $g = b - y$ and $h = b - x$, then the dimension-vector

$$\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (d, e, e, 0, 1, 1, f, g, h)$$

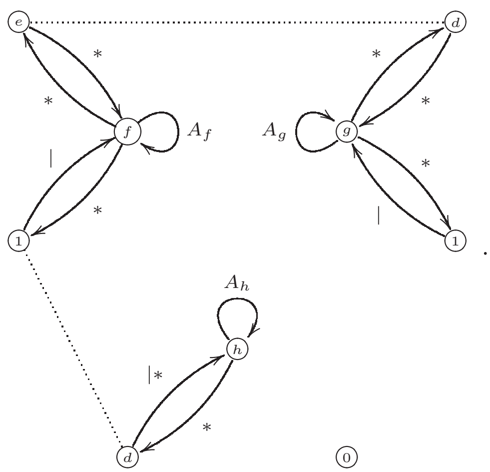
is of type α. If we denote by

$$\begin{cases} * & \text{a generic matrix} \\ | & \text{the column vector} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ \overline{1}_n & \text{the } n+1 \times n \text{ matrix} \left[\begin{array}{c} 0 \\ \hline 1_n \end{array}\right] \end{cases}$$

and the (reduced) companion matrices as in Lemma 2, then using Lemma 3 a rational parametrization of the first stage is given by the representations



By Lemma 1 a rational parametrization of the second stage is then given by the representations



This concludes the proof of the following result.

Theorem 3. *If $\alpha = (a, b; x, y, z)$ with $a > b$ is the dimension vector of θ-stable Q_0-representations, then there is an étale rational map*

$$\mathbb{A}^{d_\alpha} \dashrightarrow \mathtt{iss}_\alpha\ \Gamma$$

given by

$$\begin{cases} \sigma \mapsto \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \tau \mapsto B \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho 1_y & 0 \\ 0 & 0 & \rho^2 1_z \end{bmatrix} B^{-1} \end{cases}$$

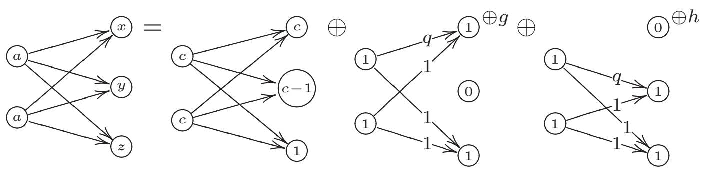
for all $n \times n$ matrices B of the form

$$\left|\begin{array}{cccccc|ccccc} 1_d & 0 & 0 & 0 & 0 & 0 & \overline{1}_e & \vert & 0 & 0 & * \\ 0 & * & 0 & q1_f + A_f & 0 & 0 & 0 & \vert & 1_f & 0 & * \\ 0 & 0 & 0 & 0 & q1_g & * & 0 & 0 & 0 & 1_g & 0 \\ \hline A_d^\dagger & * & 0 & 0 & * & 0 & 1_e & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1_f & 0 & 0 & 0 & 0 & 1_f & * & 0 \\ \vert * & 0 & 0 & 0 & * & q1_h + A_h & 0 & 0 & 0 & 0 & 1_h \\ \hline 0 & 1_e & 0 & 0 & 0 & 0 & 1_e & 0 & * & 0 & 0 \\ * & 0 & 1 & * & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \vert & 0 & 1_g + A_g & 0 & * & 0 & * & 1_g & 0 \\ 0 & 0 & 0 & * & 0 & 1_h & 0 & 0 & 0 & 0 & 1_g \end{array}\right|,$$

where $d = a - b$, $e = d - 1$, $f = b - z$, $g = b - y$ and $h = b - x$.

3.2. *Case 2:* $a = b$

Define $c = x + y + 1 - a$, $g = a - y - 1$ and $h = a - x$, which corresponds to the decomposition

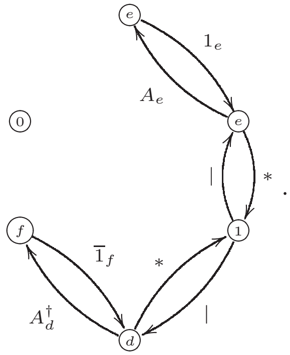


.

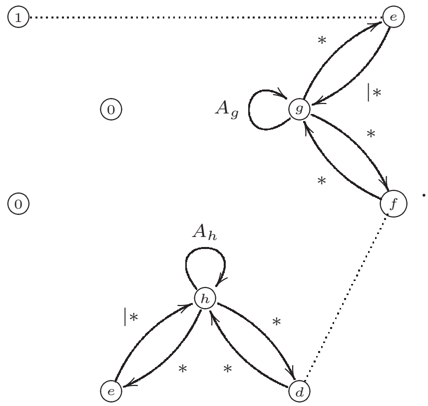
If c is odd, define $c = 2d + 1$, $e = d + 1$ and $f = d - 1$, then the dimension vector

$$\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (e, e, 1, d, f, 0, 0, g, h)$$

is of type α. Then, using Lemma 2 a rational parametrization for the first stage is given by the representations



Using Lemma 1 we then get that a rational parametrization of the second stage is given by the following representations



If c is even, we can define $c = 2e$ and $f = e - 1$ in which case the dimension vector

$$\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (e, e, 1, e, f, 0, 0, g, h)$$

is of type α and exactly the same representations give a rational parametrization of both stages if we replace all occurrences of d by e. This then concludes the proof of the next theorem.

Theorem 4. *If $\alpha = (a, b; x, y, z)$ with $a = b$ is the dimension vector of θ-stable Q_0-representations, then there is an étale rational map*

$$\mathbb{A}^{d_\alpha} \dashrightarrow \mathtt{iss}_\alpha\, \Gamma$$

given by

$$\begin{cases} \sigma \mapsto \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \tau \mapsto B \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho 1_y & 0 \\ 0 & 0 & \rho^2 1_z \end{bmatrix} B^{-1} \end{cases}$$

for all $n \times n$ matrices B of the form

$$\left|\begin{array}{ccccc|cccc}
1_e\ 0 & 0 & 0 & 0 & & A_e & 0 & 0 & * \\
0\ \ | & \overline{1}_f & 0 & * & & 0 & 1_d & 0 & 0 \\
0\ 0 & 0 & q1_g & * & & 0 & 0 & 1_g & 0 \\
\hline
1_e\ | & 0 & * & 0 & & 1_e & 0 & 0 & 0 \\
0\ 0 & 1_f & 0 & 0 & & 0 & A_d^{\dagger} & * & 0 \\
|{*}\ 0 & 0 & * & q1_h + A_h & & 0 & * & 0 & 1_h \\
\hline
0\ 1 & 0 & 0 & 0 & & * & * & 0 & 0 \\
0\ 0 & * & 1_g + A_g & 0 & & |{*} & 0 & 1_g & 0 \\
0\ 0 & 0 & 0 & 1_h & & 0 & 0 & 0 & 1_h
\end{array}\right|,$$

where $g = a - y - 1$, $h = a - x$ and if $c = x + y + 1 - a$ is odd we take $c = 2d + 1$, $e = d + 1$ and $f = d - 1$ whereas if $c = x + y + 1 - a$ is even we take $c = 2e$ and $f = e - 1$ and we replace all occurrences of d in the matrix to e.

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