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The sieve topology on the arithmetic site

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The induced topology on the points of the Connes–Consani "arithmetic site," which are the finite adèle classes, is family coarse. In this note we define another topology on this set having properties one might expect of the mythical object $\overline{\mathtt{Spec}(\mathbb{Z})}/\mathbb{F}_1$.

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1. Introduction

Connes and Consani introduced and studied in [3, 4] the *arithmetic site* as the topos $\widehat{\mathbb{N}_+^\times}$ of sheaves of sets on the small category corresponding to the multiplicative monoid $\mathbb{N}_+^\times$, equipped with the chaotic topology. They proved the remarkable result that the isomorphism classes of points of this topos are in canonical bijection with the finite adèle classes

$$\mathtt{points}(\widehat{\mathbb{N}_+^\times}) = [\mathbb{A}_\mathbb{Q}^f] = \mathbb{Q}_+^* \backslash \mathbb{A}_\mathbb{Q}^f / \hat{\mathbb{Z}}^*.$$

The *standard topology* on this set, induced from the locally compact topology on the finite adèle ring $\mathbb{A}_\mathbb{Q}^f$, is rather coarse, and therefore this "space" is best studied by the tools of noncommutative geometry, as was achieved in [3].

In this paper, we will define another topology on this set of points. Even though the SGA4-*topology*, see [1, IV.8], on the arithmetic site is trivial, we can define for any sieve $\mathtt{C}$ in $\mathbb{N}_+^\times$

$$\mathbb{X}(\mathtt{C}) = \mathtt{points}(\widehat{\mathbb{N}_+^\times}) \cap \mathtt{points}(\widehat{\mathtt{C}})$$

(see Sec. 3 for relevant definitions). This corresponding *sieve-topology* on the points $[\mathbb{A}_\mathbb{Q}^f] = \mathbb{Q}_+^* \backslash \mathbb{A}_\mathbb{Q}^f / \hat{\mathbb{Z}}^*$ shares many properties one might expect the mythical object

$\overline{\mathtt{Spec}(\mathbb{Z})}/\mathbb{F}_1$ to have: it is compact, does not have a countable basis of opens, each nonempty open being dense, and, it satisfies the T_1 separation property for incomparable points.

2. The Standard Topology on $[\mathbb{A}_{\mathbb{Q}}^{f}] = \mathbb{Q}_+^* \backslash \mathbb{A}_{\mathbb{Q}}^{f}/\hat{\mathbb{Z}}^*$

In this section, we will introduce an equivalence relation on the set $\mathbb{S}$ of supernatural numbers such that its set of equivalence classes $[\mathbb{S}]$ is in natural bijection with the set of finite adèle classes $[\mathbb{A}_{\mathbb{Q}}^{f}] = \mathbb{Q}_+^* \backslash \mathbb{A}_{\mathbb{Q}}^{f}/\hat{\mathbb{Z}}^*$. We will investigate the standard topology on $[\mathbb{A}_{\mathbb{Q}}^{f}]$ induced by the locally compact topology on $\mathbb{A}_{\mathbb{Q}}^{f}$ and recall three settings in which $[\mathbb{A}_{\mathbb{Q}}^{f}]$ appears naturally.

A *supernatural number* (also called a *Steinitz number*) is a formal product $s = \prod_{p \in \mathbb{P}} p^{s_p}$ where p runs over all prime numbers $\mathbb{P}$ and each $s_p \in \mathbb{N} \cup \{\infty\}$. The set $\mathbb{S}$ of all supernatural numbers forms a multiplicative semigroup with multiplication defined by exponent addition and the multiplicative semigroup $\mathbb{N}_+^{\times}$ embeds in $\mathbb{S}$ via unique factorization.

If $s \in \mathbb{S}$, we will write s_p for the exponent of $p \in \mathbb{P}$ appearing in s. The *support* of s is the set of prime numbers p for which $s_p > 0$. For two supernatural numbers s and s' we say that s divides s', written $s \mid s'$ if $s' = s \cdot s''$ for some supernatural number s'', or equivalently, if for all primes p we have $s_p \le s'_p$. There is an equivalence relation on $\mathbb{S}$ defined by

$$s \sim s' \text{ if and only if } \begin{cases} s_p = \infty \Leftrightarrow s'_p = \infty, \\ s_p = s'_p \text{ for all but at most finitely many } p. \end{cases}$$

With $[s]$ we will denote the equivalence class of $s \in \mathbb{S}$.

The set of equivalence classes $[\mathbb{S}]$ is in canonical bijection with the set of finite adèle classes $[\mathbb{A}_{\mathbb{Q}}^{f}] = \mathbb{Q}_+^* \backslash \mathbb{A}_{\mathbb{Q}}^{f}/\hat{\mathbb{Z}}^*$. Indeed, for the profinite integers $\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \widehat{\mathbb{Z}}_p$ and the finite adèle ring $\mathbb{A}_{\mathbb{Q}}^{f} = \mathbb{Q} \otimes \widehat{\mathbb{Z}}$ we have a canonical bijection between $\mathbb{S} \leftrightarrow \widehat{\mathbb{Z}}/\widehat{\mathbb{Z}}^*$ and $\mathbb{A}_{\mathbb{Q}}^{f}/\widehat{\mathbb{Z}}^* \leftrightarrow \mathbb{S}_f$ where $\mathbb{S}_f$ are the fractional supernatural numbers, that is, formal products $f = \prod p^{f_p}$ where we allow finitely many f_p to be strictly negative integers. If we take $a = \prod_{p: f_p < 0} p^{-f_p}$ then f corresponds to $(a, s) \in \mathbb{N}_+ \times \mathbb{S}$ with $s = a \cdot f$ and we have that a is coprime with the support of s. The inclusion of semigroups $\mathbb{N}_+^{\times} \subset \mathbb{S}$ defines the point $[1] \in [\mathbb{A}_{\mathbb{Q}}^{f}]$.

We can equip the supernatural numbers $\mathbb{S}$ with the structure of a compact Hausdorff topological space. View $\mathbb{N} \cup \{\infty\}$ as the one-point compactification of the discrete topology on $\mathbb{N}$, that is, a basis of open sets in $\mathbb{N} \cup \{\infty\}$ is given by the singletons $\{n\}$ for all $n \in \mathbb{N}$ and the sets $\{m \ge n\} \cup \{\infty\}$. Identify $\mathbb{S}$ with $\prod_{p \in \mathbb{P}} (\mathbb{N} \cup \{\infty\})$ equipped with the product topology, that is a basis of open sets are of the form $U = \prod U_p$ with each U_p open in $\mathbb{N} \cup \{\infty\}$ and with $U_p = \mathbb{N} \cup \{\infty\}$ for all p except possibly a finite number of primes. This is the induced topology on $\mathbb{S}$ coming from the compact (respectively, locally compact) topologies on $\widehat{\mathbb{Z}}$ and $\mathbb{A}_{\mathbb{Q}}^{f}$ via the identification $\mathbb{S} \leftrightarrow \widehat{\mathbb{Z}}/\widehat{\mathbb{Z}}^*$.

Theorem 1. *The induced topology on $[\mathbb{S}] = [\mathbb{A}_{\mathbb{Q}}^f]$ has a countable basis of opens corresponding to the open sets of the form*

$$\mathbb{B}(P) = \{[s] \in [\mathbb{S}] \,|\, \forall\, p \in P : s_p \neq \infty\},$$

where $P = \{p_1, \dots, p_k\}$ is any finite subset of the set of all prime numbers $\mathbb{P}$.

Proof. A basic open set $U = \prod U_p$ in $\mathbb{S}$ is stable under the equivalence relation if and only if all U_p are either equal to $\mathbb{N}$ or $\mathbb{N} \cup \{\infty\}$. As there is only a finite set of indices $P = \{p_1, \dots, p_k\}$ such that $U_p \neq \mathbb{N} \cup \{\infty\}$, this finishes the proof. □

A general open set U in the standard topology on $[\mathbb{S}]$ is thus of the form $\bigcup_i \mathbb{B}(P_i)$ and determines (and is determined by) the hereditary subset

$$\mathcal{H}_U = \{Q \subset \mathbb{P} \text{ finite} \,|\, \exists\, i : P_i \subset Q\} \in 2^{(\mathbb{P})},$$

where $2^{(\mathbb{P})}$ is the set of all finite subsets of $\mathbb{P}$. Conversely, one associates to each hereditary subset $\mathcal{H} \subset 2^{(\mathbb{P})}$ the open set

$$U_{\mathcal{H}} = \bigcup_{Q \in \mathcal{H}} \mathbb{B}(Q).$$

It is well known that the space $[\mathbb{S}] = [\mathbb{A}_{\mathbb{Q}}^f]$ appears naturally in quite different mathematical settings:

2.1. *Isomorphism classes of additive subgroups of $\mathbb{Q}$*

In [2, Theorems 1 and 2] Beaumont and Zuckerman characterize all additive subgroups of $\mathbb{Q}$ as being determined by a couple

$$(a, s) \in \mathbb{N}_+ \times \mathbb{S}$$

with a being coprime to all primes p in the support of s. The corresponding additive subgroup is then

$$\mathbb{Q}(a, s) = \left\{ \frac{a \cdot n}{m} | n \in \mathbb{Z}, m \in \mathbb{N}_+, m | s \right\}$$

with positive part $\mathbb{Q}_+(a, s) = \mathbb{Q}(a, s) \cap \mathbb{Q}_+$. Two additive subgroups are isomorphic if there is a positive rational number $q \in \mathbb{Q}_+^*$ such that $q \cdot \mathbb{Q}_+(a, s) = \mathbb{Q}_+(a', s')$.

That is, up to isomorphism, we only need to consider the positive parts $\mathbb{Q}_+(s) = \mathbb{Q}_+(1, s)$, and we have $\mathbb{Q}_+(s) \simeq \mathbb{Q}_+(s')$ if and only if s and s' are equivalent under the equivalence relation. Thus, $\mathbb{A}_{\mathbb{Q}}^f / \widehat{\mathbb{Z}}^*$ classifies additive subgroups of $\mathbb{Q}$ and the $\mathbb{Q}^*$-action corresponds to taking isomorphism classes. Hence $[\mathbb{S}] = [\mathbb{A}_{\mathbb{Q}}^f]$ classifies isomorphism classes of additive subgroups of $\mathbb{Q}$.

2.2. *Morita equivalence classes of UHF-algebras*

The equivalence relation on supernatural numbers also appears in the noncommutative geometry of C^*-algebras, as follows from the work of Glimm [7], later generalized by Dixmier [5] as explained by Connes in his closing remarks of [4].

Recall that a uniformly hyperfinite (UHF), algebra A is a C^*-algebra that can be written as the closure, in the norm topology, of an increasing union of finite-dimensional full matrix algebras

$$M_{c_1}(\mathbb{C}) \hookrightarrow M_{c_2}(\mathbb{C}) \hookrightarrow M_{c_3}(\mathbb{C}) \hookrightarrow \cdots = A.$$

By the double centralizer result it follows that $c_1|c_2|c_3\cdots$. Glimm proved in [7] that the supernatural number $s = \prod_i s_i$, where $s_1 = c_1$ and $s_i = c_i/c_{i-1}$ for $i > 1$, is an isomorphism invariant for A among UHF-algebras. The algebra A_∞ corresponding to $s_\infty = \prod_{p\in\mathbb{P}} p^\infty$ is often called the universal UHF-algebra.

A *noncommutative space* is a Morita equivalence class of C^*-algebras. In [6] it is shown that two UHF-algebras A and A' with corresponding invariants $s, s' \in \mathbb{S}$ are Morita equivalent if and only if $s \sim s'$, and that the additive subgroup $\mathbb{Q}(s)$ is the Grothendieck group $K_0(A)$, so its positive part $\mathbb{Q}_+(s)$ can be seen as the positive cone of $K_0(A)$. Hence, we can view the set of equivalence classes

$$[\mathbb{S}] = [\mathbb{A}^f_{\mathbb{Q}}] \simeq \texttt{moduli(UHF)}$$

as the moduli space for noncommutative spaces corresponding to UHF-algebras.

All strictly positive integers n lie in the equivalence class [1], expressing the fact that $M_n(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$, or that the Brauer group $\mathrm{Br}(\mathbb{C})$ is trivial. In this way we can view $[\mathbb{S}] = [\mathbb{A}^f_{\mathbb{Q}}]$ with the induced (well defined) multiplication as the *Brauer-monoid* $\mathrm{Br}_\infty(\mathbb{C})$ of UHF-algebras.

2.3. *Stable isomorphism classes of algebraic* $\mathbb{F}_p$*-extensions*

Steinitz' original motivation to introduce the supernatural numbers was to study algebraic extensions of finite fields. Fix an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Steinitz showed that there is a natural bijection between intermediate fields $\mathbb{F}_p \subset \mathbb{F} \subset \overline{\mathbb{F}}_p$ and supernatural numbers $s \in \mathbb{S}$ under the correspondence

$$s \leftrightarrow \mathbb{F}_{p^s} = \{x \in \overline{\mathbb{F}}_p \,|\, [\mathbb{F}_p(x) : \mathbb{F}_p] \text{ divides } s\}.$$

Two such algebraic $\mathbb{F}_p$-extensions $\mathbb{F}_{p^s}$ and $\mathbb{F}_{p^{s'}}$ are said to be *stably isomorphic* if there are finite $\mathbb{F}_p$-extensions $\mathbb{F}_{p^n}$ and $\mathbb{F}_{p^m}$ (with $n, m \in \mathbb{N}_+$) such that

$$\mathbb{F}_{p^s} \otimes \mathbb{F}_{p^n} \simeq \mathbb{F}_{p^{s'}} \otimes \mathbb{F}_{p^m}$$

and therefore $[\mathbb{S}] = [\mathbb{A}^f_{\mathbb{Q}}]$ can be identified with the set of stable isomorphism classes of algebraic $\mathbb{F}_p$-extensions.

3. The Points of Arithmetic Sites

In this section, we will use topos-theoretic results in order to classify the points of toposes determined by sieves in the original Connes–Consani arithmetic site. These results will then allow us in the next section to refine the standard topology on $[\mathbb{A}_{\mathbb{Q}}^{f}] = \mathbb{Q}_{+}^{*} \backslash \mathbb{A}_{\mathbb{Q}}^{f} / \hat{\mathbb{Z}}^{*}$.

We will call a multiplicative sub-monoid $\mathtt{C}$ of $\mathbb{N}_{+}^{\times}$ an *s-monoid* if for all $c \in \mathtt{C}$ also all additive multiples $c \cdot n \in \mathtt{C}$. Every s-monoid has a presentation

$$\mathtt{C} = \mathtt{C}(c_1, c_2, \ldots) = c_1\mathbb{N}_{+}^{\times} \cup c_2\mathbb{N}_{+}^{\times} \cup \cdots$$

for possibly infinitely many elements c_i from $\mathtt{C}$. An s-monoid which is also additively closed, that is, for all $c, c' \in \mathtt{C}$ we have $c + c' \in \mathtt{C}$ will be called an S-monoid. Every S-monoid has a presentation

$$\mathtt{C} = \mathtt{C}_{+}(c_1, c_2, \ldots, c_k) = (c_1 \cdot \mathbb{N} + c_2 \cdot \mathbb{N} + \cdots + c_k \cdot \mathbb{N}) \cap \mathbb{N}_{+}^{\times}$$

for finitely many elements c_i from $\mathtt{C}$. To an s-monoid $\mathtt{C}$ we associate a small category, also denoted $\mathtt{C}$, which has just one object $\bullet$ with its monoid of endomorphisms, $\mathtt{C}(\bullet, \bullet)$, under composition isomorphic, as monoid, to the multiplicative sub-monoid $\mathtt{C} \cup \{1\}$ of $\mathbb{N}_{+}^{\times}$ (here, 1 corresponds to the obligatory identity morphism $\mathrm{id}_{\bullet}$). If, for $\mathtt{C} \neq \mathbb{N}_{+}^{\times}$, this small distinction may cause confusion, we will use the adjectives “monoid” or “category” to distinguish between the two uses of $\mathtt{C}$.

From [8, p. 37] we recall that a *sieve* in the category $\mathtt{C}$ is a collection $\mathtt{S}$ of arrows in $\mathtt{C}$ with the property that if $f \in \mathtt{S}$ and if the composition $f \circ h$ exists in $\mathtt{C}$ then also $f \circ h \in \mathtt{S}$. That is, a sieve S is of the form

$$\mathtt{S} = x_1 \cdot \mathtt{C} \cup x_2 \cdot \mathtt{C} \cup \cdots = \mathtt{S}(x_1, x_2, \ldots)$$

The set $\Omega_{\mathtt{C}}$ of all sieves on $\mathtt{C}$ is a lattice (under $\cap$ and $\cup$) and is ordered (under $\subset$) having a unique maximal element $\mathtt{C}$.

Observe that the set of arrows in a sieve $\mathtt{S}$ of $\mathbb{N}_{+}^{\times}$ is the same thing as an s-monoid in $\mathbb{N}_{+}^{\times}$. Moreover, the same holds for any sieve $\mathtt{S}$ of the category $\mathtt{C}$ corresponding to an s-monoid $\mathtt{C}$.

The set $\Omega_{\mathtt{C}}$ has a right action by $\mathtt{C} \cup \{1\}$ by $S \odot c = c^{-1} \cdot S \cap (\mathtt{C} \cup \{1\})$. A *Grothendieck topology*, see [8, III.2], on the category $\mathtt{C}$ is a subset $\mathcal{J} \subset \Omega_{\mathtt{C}}$ satisfying the following properties:

(1) $\mathtt{C} \in \mathcal{J}$,
(2) (stability) if $\mathtt{S} \in \mathcal{J}$, then $\mathtt{S} \odot \mathtt{C} = \{\mathtt{S} \odot c \,|\, c \in \mathtt{C}\} \subset \mathcal{J}$,
(3) (transitivity) if $\mathtt{S} \in \mathcal{J}$ and if $\mathtt{R} \in \Omega_{\mathtt{C}}$ such that $\mathtt{R} \odot \mathtt{S} = \{\mathtt{R} \odot s \,|\, s \in \mathtt{S}\} \subset \mathcal{J}$, then $\mathtt{R} \in \mathcal{J}$.

Observe that it follows that if $\mathtt{S} \in \mathcal{J}$ and $\mathtt{S} \subset \mathtt{S}'$ in $\mathcal{S}$, then also $\mathtt{S}' \in \mathcal{J}$. The coarsest Grothendieck topology on $\mathtt{C}$, with $\mathcal{J}_{\mathrm{ch}} = \{\mathtt{C}\}$, is called the *chaotic topology* on the category $\mathtt{C}$, see [1, II.1.1.4].

A *presheaf* on the category $\mathtt{C}$, equipped with a Grothendieck topology $\mathcal{J}$, is a contravariant functor

$$P : \mathtt{C} \longrightarrow \mathtt{Sets}$$

and hence corresponds to a set $R = P(\bullet)$, equipped with a *right*-action of the monoid $\mathtt{C}$. We will then denote the pre-sheaf by P_R, and we have an equivalence of categories between the category of all presheaves $\mathtt{PreSh}(\mathtt{C}, \mathcal{J})$, with natural transformations as morphisms, and, the category $\mathtt{Sets} - \mathtt{C}$ of all right $\mathtt{C}$-sets with $\mathtt{C}$-maps as morphisms.

The *Yoneda embedding* $\mathtt{y} : \mathtt{C} \longrightarrow \mathtt{PreSh}(\mathtt{C}, \mathcal{J})$ sends $\bullet$ to the functor $y(\bullet) = \mathtt{C}(-, \bullet)$, that is, to the pre-sheaf $P_{\mathtt{C}}$ where the monoid $\mathtt{C}$ has the right-action by multiplication on itself. As such, sieves can be viewed as sub-functors of $\mathtt{y}(\bullet)$.

A presheaf $P = P_R \in \mathtt{PreSh}(\mathtt{C}, \mathcal{J})$ is said to be a *sheaf* if and only if for every sieve $\mathtt{S} \in \mathcal{J}$, the sub-functor $\mathtt{S} \hookrightarrow \mathtt{y}(\bullet)$ induces an isomorphism between the natural transformations $\mathtt{Nat}(\mathtt{S}, P_R) \simeq \mathtt{Nat}(\mathtt{y}(\bullet), P_R)$, see [8, III.4].

In terms of the right $\mathtt{C}$-set the sheaf property says that there is a natural isomorphism

$$\mathtt{Maps}_{\mathtt{C}}(\mathtt{S}, R) = \mathtt{Maps}_{\mathtt{C}}(\mathtt{C}, R).$$

That is, every right $\mathtt{C}$-map $\mathtt{S} \longrightarrow R$ extends uniquely to a right $\mathtt{C}$-map $\mathtt{C} \longrightarrow R$.

This can be reformulated in terms of *matching families*, see [8, III.4] as follows: if we have a family of elements $r_x \in R$ for all $x \in \mathtt{S}$ such that $r_{x \cdot c} = r_x \cdot c$ for all $c \in \mathtt{C}$, then there is a unique element $r \in R$ such that each $r_x = r \cdot x$.

The category of all sheaves on $\mathtt{C}$ for the Grothendieck topology $\mathcal{J}$ will be denoted by $\mathtt{Sh}(\mathtt{C}, \mathcal{J})$. Categories equivalent to sheaf categories are called *toposes*.

If we take the chaotic topology $\mathcal{J}_{\mathrm{ch}} = \{\mathtt{C}\}$, then the sheaf-condition is void whence $\mathtt{PreSh}(\mathtt{C}, \mathcal{J}_{\mathrm{ch}}) \simeq \mathtt{Sh}(\mathtt{C}, \mathcal{J}_{\mathrm{ch}}) \simeq \mathtt{Sets} - \mathtt{C}$.

Definition 1. The *arithmetic site* corresponding to the s-monoid $\mathtt{C}$ is the topos of all sheaves on the category $\mathtt{C}$, equipped with the chaotic topology, and will be denoted by

$$\widehat{\mathtt{C}} \simeq \mathtt{Sh}(\mathtt{C}, \mathcal{J}_{\mathrm{ch}}) \simeq \mathtt{Sets} - \mathtt{C}.$$

By definition, a *point* p of the arithmetic site $\widehat{\mathtt{C}}$ is a *geometric morphism*, that is, a pair of functors

$$\begin{tikzcd} \mathtt{Sets} \arrow[r, bend left, "f_*"] & \widehat{\mathtt{C}} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

such that f^* is a left adjoint to f_* and f^* is left-exact. Here, the functor f^* can be viewed as taking the stalk of a sheaf at p, and, the functor f_* assigns to a set, the corresponding "skyscraper sheaf" at p.

The functor f^* is determined up to isomorphism by its composition with the Yoneda embedding $\mathtt{y}$, that is, by the functor

$$\mathtt{C} \xrightarrow{\mathtt{y}} \widehat{\mathtt{C}} \xrightarrow{f^*} \mathtt{Sets}$$

and hence we have to consider suitable *covariant* functors $A : \mathtt{C} \longrightarrow \mathtt{Sets}$. Such functors correspond to sets $L = A(\bullet)$, this time with a *left* $\mathtt{C}$-action and we will denote the functor A_L. The functor $f^* : \widehat{\mathtt{N}} \longrightarrow \mathtt{Sets}$ corresponding to A_L is given by

$$\widehat{\mathtt{C}} \xrightarrow{f^*} \mathtt{Sets} \quad P_R \mapsto R \otimes L,$$

where $R \otimes L$ is the quotient-set of the product $R \times L$ modulo the equivalence relation induced by all relations $(r \cdot c, l) = (r, c \cdot l)$ for all $r \in R, l \in L$ and all $c \in \mathtt{C}$, see [8, p. 379]. Such functors have an adjoint functor $f_* : \mathtt{Sets} \longrightarrow \widehat{\mathtt{C}}$ determined by sending a set S to the $\mathtt{C}$-set

$$S \mapsto \mathtt{Maps}(L, S),$$

where the right action is given by $\phi \cdot c = \phi \circ (c \cdot -)$. This leaves us to determines the left $\mathtt{C}$-sets L which are *flat*, that is, such that $f^* = - \otimes L$ is left exact.

Theorem 2. *A point p of the arithmetic site $\widehat{\mathtt{C}}$ corresponds to a nonempty left $\mathtt{C}$-set L satisfying the following properties*:

(1) $\mathtt{C}$ *acts freely on L, that is, for all $a \in L$ and all $c, c' \in \mathtt{C}$, if $c \cdot a = c' \cdot a$ then $c = c'$.*
(2) *L is of rank one, that is, for all $a, a' \in L$ there is an element $b \in L$ and elements $c, c' \in \mathtt{C}$ such that $a = c \cdot b$ and $a' = c' \cdot b$.*

Proof. This follows by applying Grothendieck's construction of filtering functors to characterize flat functors, see [8, VII.6, Theorem 3]. □

We are interested in the isomorphism classes of points of $\widehat{\mathtt{C}}$, that is, in left $\mathtt{C}$-sets L satisfying the requirements of the theorem, upto isomorphism as left $\mathtt{C}$-set. The next result shows that we can always realize every point upto isomorphism as a specific subset of the strictly positive rational numbers $\mathbb{Q}_+$:

Theorem 3. *Every point p of the arithmetic site $\widehat{\mathtt{C}}$ for an s-monoid $\mathtt{C}$ is isomorphic to a subset $L \subset \mathbb{Q}_+$ of the form*

$$L = L(c_1, c_2, \ldots) = \mathtt{C} \cup \bigcup_{i=1}^{\infty} \mathtt{C} \cdot \frac{1}{c_i}$$

for certain elements $c_i \in \mathtt{C}$ satisfying the divisibility condition $c_1 | c_2 | c_3 | \cdots$. Conversely, any such subset is indeed a point of $\widehat{\mathtt{C}}$.

Proof. Let p correspond to the left $\mathtt{C}$-set L satisfying the properties of the previous theorem. Take $l_0 \in L$ and send $l_0 \mapsto 1$, then we have an isomorphism as $\mathtt{C}$-sets between $\mathtt{C} \cdot l_0$ and $\mathtt{C} \subset \mathbb{Q}_+$. If $l_0' \in L - \mathtt{C} \cdot l_0$, then there exists an element $l_1 \in L$ and elements $c_1, c_1' \in \mathtt{C}$ such that $l_0 = c_1 \cdot l_1$ and $l_0' = c_1' \cdot l_1$ and send $l_1 \mapsto \frac{1}{c_1}$, then the left $\mathtt{C}$-subset of L spanned by l_1 (which contains l_0, l_0') is isomorphic as $\mathtt{C}$-set to $C \cdot \frac{1}{c_1}$.

If there is an element $l'_1 \in L - \mathtt{C} \cdot l_1$, take an element l_2 and elements $d_2, d'_2 \in \mathtt{C}$ such that $l_1 = d_2 \cdot l_2$ and $l'_1 = d'_2 \cdot l_2$ and send $l_2 \mapsto \frac{1}{c_2}$ where $c_2 = c_1 \cdot d_2$ and the $\mathtt{C}$-subset of L spanned by l_2 (which contains l_0, l'_0, l_1, l'_1) is isomorphic to $\mathtt{C} \cdot \frac{1}{c_2}$. Iterating this process we obtain the claimed isomorphism.

Conversely, $\mathtt{C}$ acts clearly freely on any such subset, and if $\frac{a}{c_i}$ and $\frac{b}{c_j}$ are arbitrary elements of this set (with $a, b \in \mathtt{C}$) and if $i \leq j$, then we can consider the element $\frac{1}{c_j}$ and have elements b and $b' = a \cdot \frac{c_j}{c_i}$ (both in $\mathtt{C}$ because it is an s-monoid) to satisfy the rank one condition. □

Corollary 1. *If* $\mathtt{C}$ *is an S-monoid, every point in* $\widehat{\mathtt{C}}$ *is isomorphic to the positive part of an additive subgroup of* $\mathbb{Q}$.

Proof. If $\mathtt{C}$ is an S-monoid, it is additively closed, and hence, so are each of the subsets $\mathtt{C} \cdot \frac{1}{c_i}$ of $\mathbb{Q}_+$, proving the claim. □

We will now characterize which positive cones $\mathbb{Q}_+(s)$ are points of the topos $\widehat{\mathtt{C}}$ when $\mathtt{C}$ is an s-monoid. If $n \in \mathbb{N}^\times_+$, then $n \sim 1$ and as $\mathbb{Q}_+(1) = \mathbb{N}^\times_+$, the only s-monoid for which $\mathbb{Q}_+(1)$ satisfies the rank one condition is $\mathbb{N}^\times_+$. Hence, the only topos for which $\mathbb{Q}_+(n)$ is a point is $\widehat{\mathbb{N}^\times_+}$. Remains to deal with the $s \in \mathbb{S} - \mathbb{N}^\times_+$:

Theorem 4. *For* $\mathtt{C}$ *an s-monoid properly contained in* $\mathbb{N}^\times_+$, *the positive cone* $\mathbb{Q}_+(s)$, *for* $s \in \mathbb{S} - \mathbb{N}^\times_+$ *is a point of the topos* $\widehat{\mathtt{C}}$ *if and only if there exist elements* $c_1, c_2, \ldots \in \mathtt{C}$ *such that the supernatural number* $\prod_{i=1}^\infty c_i$ *divides* s.

Proof. Assume $\prod_{i=1}^\infty c_i \,|\, s$, we have to verify that $\mathbb{Q}_+(s)$ satisfies the rank one condition with respect to $\mathtt{C}$. Again, because $\mathtt{C}$ is an s-monoid, it suffices to verify this for all additive generators $\frac{1}{n}$ and $\frac{1}{m}$ of $\mathbb{Q}_+(s)$ with natural numbers n and m dividing s. But then, also $lcm(m,n) \,|\, s$. As $\prod_{i=1}^\infty c_i \,|\, s$ there must be an element $c_i \in \mathtt{C}$ such that also $N = c_i \cdot lcm(m,n) \,|\, s$. But then $\frac{1}{N} \in \mathbb{Q}_+(s)$ and we have, if $lcm(m,n) = a \cdot n = b \cdot m$ that

$$\frac{c_i \cdot a}{N} = \frac{1}{n} \quad \text{and} \quad \frac{c_i \cdot b}{N} = \frac{1}{m}.$$

Conversely, assume that $\mathbb{Q}_+(s)$ is a point of $\widehat{\mathtt{C}}$ and let n be a natural number dividing s. Let $a \in \mathbb{N}^\times_+ - \mathtt{C}$, then because $\frac{1}{n}$ and $\frac{a}{n}$ belong to $\mathbb{Q}_+(s)$ there must be a natural number $N \,|\, s$ and elements $c_1, c'_1 \in \mathtt{C}$ such that

$$\frac{c_1}{N} = \frac{1}{n} \quad \text{and} \quad \frac{c'_1}{N} = \frac{a}{n}$$

but then $c_1 \cdot n = N$ and so $c_1 \,|\, s$. Start again with the elements $\frac{1}{c_1}$ and $\frac{a}{c_1}$ in $\mathbb{Q}_+(s)$ to get $M \,|\, s$ and elements c_2, c'_2 giving $\frac{c_2}{M} = \frac{1}{c_1}$, to obtain $c_1 \cdot c_2 = M \,|\, s$. Iterating this procedure proves the claim. □

If $\mathtt{C}$ is an S-monoid, it follows from Corollary 1 that every point in the topos $\widehat{\mathtt{C}}$ is isomorphic to some positive cone $\mathbb{Q}_+(s)$ where s satisfies the condition of the above theorem. However, if $\mathtt{C}$ is only an s-monoid, other "exotic" points exists and are described in Theorem 3.

The previous theorem excludes the case of prime interest, that of $\mathbb{N}_+^\times$, but we easily deduce [3, Theorem 2.2.(ii)]:

Theorem 5 (Connes–Consani). *The set of isomorphism classes of the arithmetic site* $\widehat{\mathbb{N}_+^\times}$ *is in natural bijection with the finite adèle classes*

$$[\mathbb{A}_\mathbb{Q}^f] = \mathbb{Q}_+^* \backslash \mathbb{A}_\mathbb{Q}^f / \hat{\mathbb{Z}}^*.$$

Proof. For any $s \in \mathbb{S}$ and all natural numbers n and m dividing s, $lcm(m,n)$ also divides s. This means that the rank one condition for $\mathbb{Q}_+(s)$ is always satisfied when the S-monoid is $\mathbb{N}_+^\times$. Therefore, isomorphism classes of points of $\widehat{\mathbb{N}_+^\times}$ are in natural bijection to the set of all equivalence classes $[\mathbb{S}] = [\mathbb{A}_\mathbb{Q}^f]$. □

4. The Sieve Topology on $[\mathbb{A}_\mathbb{Q}^f] = \mathbb{Q}_+^* \backslash \mathbb{A}_\mathbb{Q}^f / \hat{\mathbb{Z}}^*$

Note that the SGA4-topology on $\widehat{\mathbb{N}_+^\times}$ is trivial, see [1, IV.8.4.3]. Here, open sets correspond to subobjects of the terminal object $\mathtt{1}$. Note that there is no terminal object in the small category $\mathbb{N}_+^\times$, but there is a terminal object in the presheaf topos $\widehat{\mathbb{N}_+^\times}$, given by the functor

$$\mathtt{1} : \mathbb{N}_+^\times \longrightarrow \mathtt{Sets} \quad \bullet \mapsto \{*\},$$

where $\{*\}$ is a singleton with trivial right $\mathbb{N}_+^\times$-action. As the functor S corresponding to the right $\mathbb{N}_+^\times$-set $\Omega_{\mathbb{N}_+^\times}$ is the sub-object classifier in $\widehat{\mathbb{N}_+^\times}$, see [8, I.4], sub-objects of $\mathtt{1}$ correspond to natural transformations $1 \longrightarrow S$, that is, to right $\mathbb{N}_+^\times$-maps

$$\{*\} \longrightarrow \Omega_{\mathbb{N}_+^\times}$$

so the image must be a sieve which is an $\mathbb{N}_+^\times$-fix point. There are precisely two such sieves: $\emptyset$ and $\mathbb{N}_+^\times$.

Definition 2. The *sieve-topology* on $\mathtt{points}(\widehat{\mathbb{N}_+^\times})$ is the topology having as a basis of open sets the subsets

$$\mathbb{X}(\mathtt{S}) = \mathtt{points}(\widehat{\mathbb{N}_+^\times}) \cap \mathtt{points}(\widehat{\mathtt{S}}),$$

where $\mathtt{S} \in \Omega_{\mathbb{N}_+^\times}$.

This indeed defines a topology, for if $\mathtt{S} = c_1 \cdot \mathbb{N}_+^\times \cup c_2 \cdot \mathbb{N}_+^\times \cup \cdots$ and $\mathtt{S}' = d_1 \cdot \mathbb{N}_+^\times \cup d_2 \cdot \mathbb{N}_+^\times \cup \cdots$, we obtain from Theorem 4 that

$$\mathbb{X}(\mathtt{S}) \cap \mathbb{X}(\mathtt{S}') = \mathbb{X}(\mathtt{S} \cdot \mathtt{S}')$$

where $\mathtt{S} \cdot \mathtt{S}' = \bigcup_{i,j} lcm(c_i, d_j)\mathbb{N}_+^\times$, whence, the sets $\mathbb{X}(S)$ form a basis of opens.

Even though every sieve is of the form $\mathtt{S} = n_1 \cdot \mathbb{N}_+^\times \cup n_2 \cdot \mathbb{N}_+^\times \cup \cdots$, it is *not* true that the opens $\mathbb{X}(n\mathbb{N}_+^\times)$ form a basis for the sieve-topology. In general we have for infinite unions

$$\bigcup_{i=1}^{\infty} \mathbb{X}(n_i\mathbb{N}_+^\times) \subsetneq \mathbb{X}(n_1\mathbb{N}_+^\times \cup n_2\mathbb{N}_+^\times \cup \cdots).$$

For example, take $n_i = p_i^{e_i}$ for distinct primes p_i, then with $s = \prod_i p_i^{e_i}$ we have that $[\mathbb{Q}_+(s)]$ belongs to the open set on the right-hand side, but not to the union on the left-hand side. That is, the topology does not have a countable basis of open sets.

If $P = \{p_1, \ldots, p_k\}$ is a finite set of primes, then the basic open set $\mathbb{B}(P) = \{[s] \in [\mathbb{S}] \mid \forall\, p \in P : s_p \neq \infty\}$ in the standard topology is by Theorem 4 of the form

$$\mathbb{B}(P) = \mathbb{S} - \mathbb{X}(p_1\mathbb{N}_+^\times \cup \cdots \cup p_k\mathbb{N}_+^\times).$$

That is, basic opens in the standard topology on $[\mathbb{A}_{\mathbb{Q}}^f]$ are closed in the sieve topology.

Remark 1. If $[\mathbb{Q}_+(s)] \in \mathbb{X}(\mathtt{S})$, and if $s \mid s'$, then also $[\mathbb{Q}_+(s')] \in \mathbb{X}(\mathtt{S})$, so we will not be able to separate points corresponding to positive cones whenever the corresponding supernatural numbers divide each other, or even, if they *weakly divide* each other. By this we mean

$$s\|s' \quad \text{if there exist } t, t' \in \mathbb{S} \text{ such that } s \sim t,\ s' \sim t' \text{ and } t \mid t'.$$

In particular $[\mathbb{Q}_+(s_\infty)] = [\mathbb{Q}_+]$ for $s_\infty = \prod_{p\in\mathbb{P}} p^\infty$ belongs to each of the open sets $\mathbb{X}(\mathtt{S})$.

For this reason, we will say that the points $[\mathbb{Q}_+(s)]$ and $[\mathbb{Q}_+(s')]$ are *incomparable* if and only neither $s\|s'$ nor $s'\|s$. This is equivalent to saying that, with $\infty(t) = \{p : p^\infty \mid t\}$, either the sets $\infty(s)$ and $\infty(s')$ are incomparable, or that the subsets of primes $I = \{p \in \mathbb{P} : \infty > s_p > s'_p\}$ and $J = \{q \in \mathbb{P} : \infty > s'_q > s_q\}$ are both infinite. In the first case we can easily separate s and s', in the second case we take the sieves (or s-monoids)

$$\mathtt{S} = \bigcup_{p\in I} p^{s_p} \cdot \mathbb{N}_+^\times \quad \text{and} \quad \mathtt{S}' = \bigcup_{q\in J} q^{s'_q} \cdot \mathbb{N}_+^\times$$

then we have that

$$[\mathbb{Q}_+(s)] \in \mathbb{X}(\mathtt{S}) - \mathbb{X}(\mathtt{S}') \quad \text{and} \quad [\mathbb{Q}_+(s')] \in \mathbb{X}(\mathtt{S}') - \mathbb{X}(\mathtt{S}).$$

That is, the sieve-topology on the finite adèle classes satisfies the T_1 separation property with respect to incomparable points.

Theorem 6. *The sieve-topology on the finite adèle classes*

$$\mathtt{points}(\widehat{\mathbb{N}_+^\times}) \simeq \mathbb{Q}_+^* \backslash \mathbb{A}_{\mathbb{Q}}^f / \hat{\mathbb{Z}}^* = [\mathbb{A}_{\mathbb{Q}}^f] = [\mathbb{S}]$$

satisfies the following properties:

(1) *It is compact.*
(2) *It does not admit a countable basis of open sets.*

(3) *Every nonempty open set is dense.*
(4) *It satisfies the T_1 separation property for incomparable points.*

Proof. For (1) recall that the only s-monoid $\mathtt{S}$ for which $[\mathbb{Q}_+(1)] = [\mathbb{N}_+^\times]$ is a point is $\mathbb{N}_+^\times$ itself. For (3) note that the point $[\mathbb{Q}_+(s_\infty)] = [\mathbb{Q}_+]$ lies in each and every of the open sets $\mathbb{X}(\mathtt{S})$, so any two opens have a nonempty intersection. (2) and (4) were shown above. □

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