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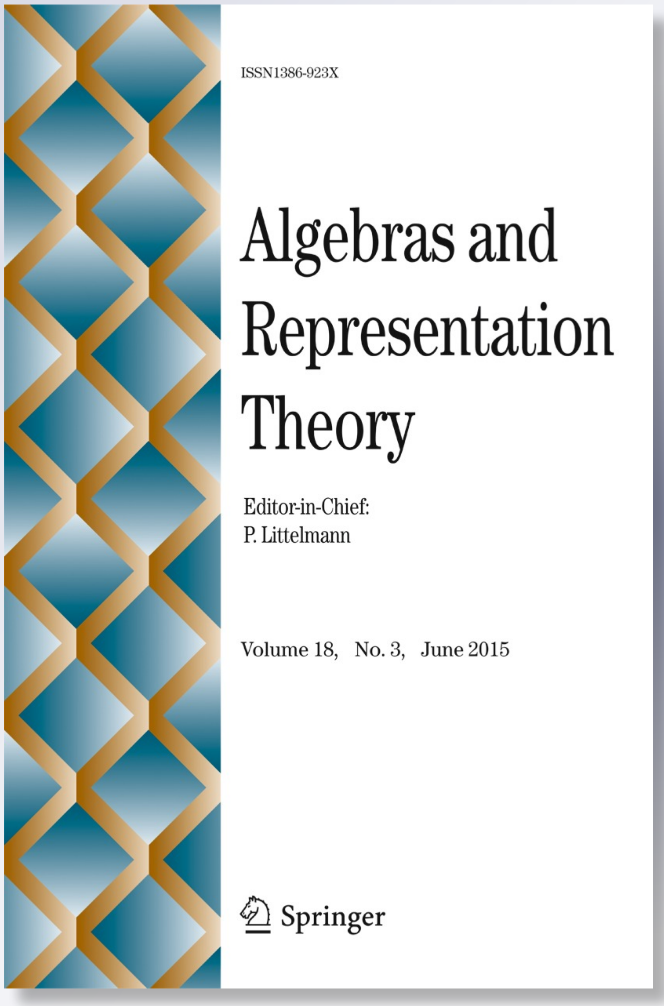
Algebras and Representation Theory

ISSN 1386-923X
Volume 18
Number 3

Algebr Represent Theor (2015)
18:761-776
DOI 10.1007/s10468-014-9515-6



Springer

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The Geometry of Representations of 3-Dimensional Sklyanin Algebras

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Received: 9 May 2014 / Accepted: 17 December 2014 / Published online: 30 January 2015

Abstract The representation scheme $\mathsf{rep}_n\ A$ of the 3-dimensional Sklyanin algebra A associated to a plane elliptic curve and n-torsion point contains singularities over the augmentation ideal $\mathfrak{m}$. We investigate the semi-stable representations of the noncommutative blow-up algebra $B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \ldots$ to obtain a partial resolution of the central singularity

$$\mathtt{proj}\ Z(B) \longrightarrow\!\!\!\!\!\rightarrow \mathtt{spec}\ Z(A)$$

such that the remaining singularities in the exceptional fiber determine an elliptic curve and are all of type $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$.

Keywords Sklyanin algebras · Noncommutative geometry · Representation schemes · Superpotentials

Mathematics Subject Classification (2010) 16G99

1 Introduction

Three dimensional Sklyanin algebras appear in the classification by M. Artin and W. Schelter [2] of graded algebras of global dimension 3. In the early 90ties this class of algebras was studied extensively by means of noncommutative projective algebraic geometry, see for example [3–5, 9] and [15]. Renewed interest in this class of algebras arose recently as they are superpotential algebras and as such relevant in supersymmetric quantum field theories, see [6] and [16].

Presented by Paul Smith.

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Consider a smooth elliptic curve E in Hesse normal form $\mathbb{V}((a^3+b^3+c^3)XYZ - abc(X^3+Y^3+Z^3)) \hookrightarrow \mathbb{P}^2$ and the point $p = [a : b : c]$ on E. The 3-dimensional Sklyanin algebra A corresponding to the pair (E, p) is the noncommutative algebra with defining equations

$$\begin{cases} axy + byx + cz^2 = 0, \\ ayz + bzy + cx^2 = 0, \\ azx + bxz + cy^2 = 0. \end{cases}$$

The connection comes from the fact that the multi-linearization of these equations defines a closed subscheme in $\mathbb{P}^2 \times \mathbb{P}^2$ which is the graph of translation by p on the elliptic curve E, see [4]. Alternatively, one obtains the defining equations of A from the superpotential $W = axyz + byxz + \frac{c}{3}\left(x^3 + y^3 + z^3\right)$, see [16, Ex. 2.3].

The algebra A has a central element of degree 3, found by a computer calculation in [2]

$$c_3 = c\left(a^3 - c^3\right)x^3 + a\left(b^3 - c^3\right)xyz + b\left(c^3 - a^3\right)yxz + c\left(c^3 - b^3\right)y^3,$$

with the property that $A/(c_3)$ is the twisted coordinate ring of the elliptic curve E with respect to the automorphism given by translation by p, see [4]. We will prove an intrinsic description of this central element, answering a MathOverflow question from 2013 (see [8]).

Theorem 1 *The central element c_3 of the 3-dimensional Sklyanin algebra A corresponding to the pair (E, p) can be written as*

$$a\left(b^3 - c^3\right)(xyz + yzx + zxy) + b\left(c^3 - a^3\right)(yxz + xzy + zyx) + c\left(a^3 - b^3\right)\left(x^3 + y^3 + z^3\right)$$

and is the superpotential of the 3-dimensional Sklyanin algebra A' corresponding to the pair $(E, [-2]p)$.

Next, we turn to the geometry of finite dimensional representations of A in the special case when A is a finite module over its center. This setting is important in physics in order to understand the Calabi-Yau geometry of deformed N=4 SYM theories. We refer the interested reader to the introduction of [16] for more details.

It is well known that A is a finite module over its center $Z(A)$ and a maximal order in a central simple algebra of dimension n^2 if and only if the point p is of finite order n, see [5, Thm II,]. We will further assume that $(n, 3) = 1$ in which case J. Tate and P. Smith proved in [15, Thm. 4.7,] that the center $Z(A)$ is generated by c_3 and the reduced norms of x, y and z (which are three degree n elements, say x', y', z') satisfying one relation of the form

$$c_3^n = cubic(x', y', z').$$

It is also known that $\mathsf{proj}\ Z(A) \simeq \mathbb{P}^2$ with coordinates $[x' : y' : z']$ in which the $cubic(x', y', z')$ defines the isogenous elliptic curve $E' = E/\langle p \rangle$, see [9, Section 2]. We will use these facts to give explicit matrices for the simple n-dimensional representations of A and show that A is an Azumaya algebra away from the isolated central singularity.

However, the scheme $\mathsf{rep}_n\ A$ of all (trace preserving) n-dimensional representations of A contains singularities in the nullcone. We then try to resolve these representation singularities by considering the noncommutative analogue of a blow-up algebra

$$B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \ldots \subset A[t, t^{-1}]$$

where $\mathfrak{m} = (x, y, z)$ is the augmentation ideal of A. We will prove

Theorem 2 *The scheme* $\mathsf{rep}_n^{ss}\ B$ *of all semi-stable n-dimensional representations of the blow-up algebra B is a smooth variety.*

This allows us to compute all the (graded) local quivers in the closed orbits of $\mathsf{rep}_n^{ss}\ B$ as in [10] and [7]. This information then leads to the main result of this paper which gives a partial resolution of the central isolated singularity.

Theorem 3 *The exceptional fiber* $\mathbb{P}^2$ *of the canonical map*

$$\texttt{proj}\ Z(B) \longrightarrow\!\!\!\!\rightarrow \texttt{spec}\ Z(A)$$

contains $E' = E/\langle p \rangle$ *as the singular locus of* $\mathsf{proj}\ Z(B)$. *Moreover, all these singularities are of type* $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$ *with* $\mathbb{C}^2/\mathbb{Z}_n$ *an Abelian quotient surface singularity.*

2 Central Elements and Superpotentials

The finite Heisenberg group of order 27

$$\langle\ u, v, w \mid [u, v] = w,\ \ [u, w] = [v, w] = 1,\ u^3 = v^3 = w^3 = 1\ \rangle$$

has a 3-dimensional irreducible representation $V = \underset{\sim}{x} + \underset{\sim}{y} + \underset{\sim}{z}$ given by the action

$$u \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad v \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{bmatrix}, \quad w \mapsto \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{bmatrix},$$

with $\rho^3 = 1$ a primitive 3rd root of unity. One verifies that $V \otimes V$ decomposes as three copies of V^*, that is,

$$V \otimes V \simeq \wedge^2(V) \oplus S^2(V) \simeq V^* \oplus (V^* \oplus V^*),$$

where the three copies can be taken to be the subspaces

$$\begin{cases} V_1 = \mathbb{C}(yz - zy) + \mathbb{C}(zx - xz) + \mathbb{C}(xy - yx), \\ V_2 = \mathbb{C}(yz + zy) + \mathbb{C}(zx + xz) + \mathbb{C}(xy + yx), \\ V_3 = \underset{\sim}{x}^2 + \underset{\sim}{y}^2 + \underset{\sim}{z}^2. \end{cases}$$

Taking the quotient of $\mathbb{C}\langle x, y, z\rangle$ modulo the ideal generated by $V_1 = \wedge^2 V$ gives the commutative polynomial ring $\mathbb{C}[x, y, z]$. Hence we can find analogues of the polynomial ring in three variables by dividing $\mathbb{C}\langle x, y, z\rangle$ modulo the ideal generated by another copy of V^* in $V \otimes V$ and the resulting algebra will inherit an action by H_3. Such a copy of V^* exists for all $[A : B : C] \in \mathbb{P}^2$ and is spanned by the three vectors

$$\begin{cases} A(yz - zy) + B(yz + zy) + Cx^2, \\ A(zx - xz) + B(zx + xz) + Cy^2, \\ A(xy - yx) + B(xy + yx) + Cz^2. \end{cases}$$

and by taking $a = A + B$, $b = B - A$ and $c = C$ we obtain the defining relations of the 3-dimensional Sklyanin algebra. In particular there is an H_3-action on A and the canonical central element c_3 of degree 3 must be a 1-dimensional representation of H_3. It is obvious that c_3 is fixed by the action of v and a minor calculation shows that c_3 is also fixed by u. Therefore, the central element c_3 given above, or rather $3c_3$, can also be represented as

$$a \quad \left(b^3 - c^3\right)(xyz + yzx + zxy) + b\left(c^3 - a^3\right)(yxz + xzy + zyx)$$
$$+c\left(a^3 - b^3\right)\left(x^3 + y^3 + z^3\right).$$

Now, let us reconsider the superpotential $W = axyz + byxz + \frac{c}{3}\left(x^3 + y^3 + z^3\right)$ for a $[a : b : c] \in \mathbb{P}^2$. This superpotential gives us three quadratic relations by taking cyclic derivatives with respect to the variables

$$\begin{cases} \partial_x W = ayz + bzy + cx^2, \\ \partial_y W = azx + bxz + cy^2, \\ \partial_z W = axy + byx + cz^2, \end{cases}$$

giving us the defining relations of the 3-dimensional Sklyanin algebra. We obtain the same equations by considering a more symmetric form of W, or rather of $3W$

$$a(xyz + yzx + zxy) + b(yxz + xzy + zyx) + c\left(x^3 + y^3 + z^3\right).$$

We see that the form of the central degree 3 element and of the superpotential are similar but with different coefficients. This means that the central element is the superpotential defining another 3-dimensional Sklyanin algebra and theorem 1 clarifies this connection.

Proof of Theorem 1 : The 3-dimensional Sklyanin algebra associated to the superpotential $3c_3$ is determined by the point $[a_1 : b_1 : c_1] = [a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)] \in \mathbb{P}^2$ (instead of $[a : b : c]$ for the original). Therefore, the associated elliptic curve has defining Hesse equation

$$\mathbb{V}(\alpha(x^3 + y^3 + z^3) - \beta xyz) \hookrightarrow \mathbb{P}^2$$

where

$$\begin{cases} \alpha = a_1 b_1 c_1 = a(b^3 - c^3)b(c^3 - a^3)c(a^3 - b^3), \\ \beta = a_1^3 + b_1^3 + c_1^3 = (a(b^3 - c^3))^3 + (b(c^3 - a^3))^3 + (c(a^3 - b^3))^3, \end{cases}$$

but by a Maple computation one verifies that, up to a scalar, this is the original curve

$$E = \mathbb{V}(abc(x^3 + y^3 + c^3) - (a^3 + b^3 + c^3)xyz).$$

The tangent line to E in the point $p = [a : b : c]$ has equation

$$(2a^3bc - b^4c - bc^4)(x - a) + (2ab^3c - a^4c - ac^4)(y - b) + (2abc^3 - a^4b - ab^4)(z - c) = 0$$

and so the third point of intersection is

$$[-2]p = [a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)],$$

which are the parameters of the algebra associated to the the superpotential $3c_3$. □

3 Resolving Representation Singularities

Let R be a graded $\mathbb{C}$-algebra, generated by finitely many elements $x_1, \ldots, x_m$ where $deg(x_i) = d_i \geq 0$, which is a finite module over its center $Z(R)$. Following [14, 2.3] we say that R is a Cayley-Hamilton algebra of degree n if there is a $Z(R)$-linear gradation preserving trace map $tr : R \longrightarrow Z(R)$ such that for all $a, b \in R$ we have

- $tr(ab) = tr(ba)$
- $tr(1) = n$
- $\chi_{n,a}(a) = 0$

where $\chi_{n,a}(t)$ is the n-th Cayley-Hamilton identity expressed in the traces of powers of a. Maximal orders in a central simple algebra of dimension n^2 are examples of Cayley-Hamilton algebras of degree n.

In particular, a 3-dimensional Sklyanin algebra A associated to a couple (E, p) where p is a torsion point of order n, and the corresponding blow-up algebra $B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2 t^2 \oplus \ldots$ are finitely generated graded Cayley-Hamilton algebras of degree n equipped with the (gradation preserving) reduced trace map.

If R is an finitely generated graded Cayley-Hamilton algebra of degree n we define $\mathsf{rep}_n\ R$ to be the affine scheme of all n-dimensional trace preserving representations, that is of

$$R \xrightarrow{\phi} M_n(\mathbb{C}) \qquad \text{such that} \qquad \forall a \in R\ :\ \phi(tr(a)) = Tr(\phi(a)),$$

where Tr is the usual trace map on $M_n(\mathbb{C})$. Isomorphism of representations defines a GL_n-action of $\mathsf{rep}_n\ R$ and a result of Artin's [1, 12.6], asserts that the closed orbits under this action, that is the points of the GIT-quotient scheme $\mathsf{rep}_n\ R//\mathsf{GL}_n$, are precisely the isomorphism classes of n-dimensional trace preserving semi-simple representations of R. The reconstruction result of Procesi [14, Thm. 2.6] asserts that in this setting

$$\mathsf{spec}\ Z(R) \simeq \mathsf{rep}_n\ R/\mathsf{GL}_n.$$

The gradation on R defines an additional $\mathbb{C}^*$-action on $\mathsf{rep}_n\ R$ commuting with the GL_n-action. With $\mathsf{rep}_n^{ss}\ R$ we denote the Zariski open subset of all semi-stable trace preserving representations $\phi : R \longrightarrow M_n(\mathbb{C})$, that is, such that there is an homogeneous central element c of positive degree such that $\phi(c) \neq 0$. We have the following graded version of Procesi's reconstruction result, see amongst others [7, Section 8],

$$\mathsf{proj}\ Z(R) \simeq \mathsf{rep}_n^{ss}\ R//\mathsf{GL}_n \times \mathbb{C}^*.$$

As a $\mathsf{GL}_n \times \mathbb{C}^*$-orbit is closed in $\mathsf{rep}_n^{ss}\ R$ if and only if the GL_n-orbit is closed we see that points of $\mathsf{proj}\ Z(R)$ classify one-parameter families of isoclasses of trace-preserving n-dimensional semi-simple representations of R. In case of a simple representation such a one-parameter family determines a graded algebra morphism

$$R \longrightarrow M_n(\mathbb{C}[t, t^{-1}])(\underbrace{0, \ldots, 0}_{m_0}, \underbrace{1, \ldots, 1}_{m_1}, \ldots, \underbrace{e-1, \ldots, e-1}_{m_{e-1}})$$

where e is the degree of t, the m_i are natural numbers with $\sum_{i=0}^{e-1} m_i = n$ and where we follow [13] in defining the shifted graded matrix algebra $M_n(S)(a_1, \ldots, a_n)$ by taking its homogeneous part of degree i to be

$$\begin{bmatrix} S_i & S_{i-a_1+a_2} & \ldots & S_{i-a_1+a_n} \\ S_{i-a_2+a_1} & S_i & \ldots & S_{i-a_2+a_n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{i-a_n+a_1} & S_{i-a_n+a_2} & \ldots & S_i \end{bmatrix}.$$

The $\mathsf{GL}_n \times \mathbb{C}^*$-stabilizer subgroup of any of the simples ϕ in this family is then isomorphic to $\mathbb{C}^* \times \boldsymbol{\mu}_e$ where the cyclic group $\boldsymbol{\mu}_e$ has generator $(g_\zeta, \zeta) \in \mathsf{GL}_n \times \mathbb{C}^*$ where ζ is a primitive e-th root of unity and

$$g_\zeta = \mathsf{diag}(\underbrace{1, \ldots, 1}_{m_0}, \underbrace{\zeta, \ldots, \zeta}_{m_1}, \ldots, \underbrace{\zeta^{e-1}, \ldots, \zeta^{e-1}}_{m_{e-1}}),$$

see [7, lemma 4]. If, in addition, ϕ is a smooth point of $\mathsf{rep}_n^{ss}\ R$ then the normal space

$$N(\phi) = T_\phi \mathsf{rep}_n^{ss} R / T_\phi \mathsf{GL}_n.\phi$$

to the GL_n-orbit decomposes as a $\boldsymbol{\mu}_e$-representation into a direct sum of 1-dimensional simples

$$N(\phi) = \mathbb{C}_{i_1} \oplus \ldots \oplus \mathbb{C}_{i_d}$$

where the action of the generator on $\mathbb{C}_k$ is by multiplication with ζ^k. Alternatively, ϕ determines a (necessarily smooth) point $[\phi] \in \mathsf{spec}\ Z(R)$ and because $N(\phi)$ is equal to $Ext^1_R(S_\phi, S_\phi)$ and because R is Azumaya in $[\phi]$ it coincides with $Ext^1_{Z(R)}(S_{[\phi]}, S_{[\phi]})$ (where $S_{[\phi]}$ is the simple 1-dimensional representation of $Z(R)$ determined by $[\phi]$) which is identical to the tangent space $T_{[\phi]}\mathsf{spec} Z(R)$. The action of the stabilizer subgroup $\boldsymbol{\mu}_e$ on $Ext^1_R(S_\phi, S_\phi)$ carries over to that on $T_{[\phi]}\mathsf{spec} Z(R)$.

The one-parameter family of simple representations also determines a point $\overline{\phi} \in \mathsf{proj} Z(R)$ and an application of the Luna slice theorem [12] asserts that for all $t \in \mathbb{C}$ there is a neighborhood of $(\overline{\phi}, t) \in \mathsf{proj} Z(R) \times \mathbb{C}$ which is étale isomorphic to a neighborhood of 0 in $N(\phi)//\boldsymbol{\mu}_e$, see [7, Thm. 5].

3.1 From $\mathsf{Proj}(A)$ to $\mathsf{rep}_n A$

In noncommutative projective algebraic geometry, see for example [4, 5] and [3], one studies the Grothendieck category $\mathsf{Proj}(A)$ which is the quotient category of all graded left A-modules modulo the subcategory of torsion modules. In the case of 3-dimensional Sklyanin algebras the linear modules, that is those with Hilbert series $(1-t)^{-1}$ (point modules) or $(1-t)^{-2}$ (line modules) were classified in [5, Section 6]. Identify $\mathbb{P}^2$ with $\mathbb{P}^2_{nc} = \mathbb{P}(A_1^*)$, then

- point modules correspond to points on the elliptic curve $E \hookrightarrow \mathbb{P}^2_{nc}$,
- line modules correspond to lines in $\mathbb{P}^2_{nc}$.

In the case of interest to us, when A corresponds to a couple (E, p) with p a torsion point of order n also fat modules are important which are critical cyclic graded left A-modules with Hilbert series $n.(1-t)^{-1}$. They were classified by M. Artin [3, Section 3] and are relevant in the study of $\mathsf{proj} Z(A) = \mathbb{P}^2_c = \mathbb{P}(Z(A)^*_n)$. Observe that the reduced norm map N relates the different manifestations of $\mathbb{P}^2$ and the elliptic curve E with $E/\langle p \rangle$

$$\begin{array}{ccc} \mathbb{P}^2_{nc} = \mathbb{P}(A_1^*) & \xrightarrow{\quad N \quad} & \mathbb{P}^2_c = \mathbb{P}(Z(A)^*_n) \\ \uparrow & & \uparrow \\ E & \xrightarrow{\quad ./\langle p \rangle \quad} & E' = E/\langle p \rangle \end{array}$$

Points $\pi \in \mathbb{P}^2_c - E'$ determine fat points F_π with graded endomorphism ring isomorphic to $M_n(\mathbb{C}[t, t^{-1}])$ with $deg(t) = 1$, and hence determine a one-parameter family of simple

n-dimensional representations in $\mathsf{rep}_n^{ss} A$ with $\mathsf{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\mathbb{C}^* 1_n \times 1$. There is an effective method to construct F_π, see [9, Section 3]. Write π as the i $\mathbb{V}(z) \cap \mathbb{V}(z')$ and let $\mathbb{V}(z') \cap E' = \{q_1, q_2, q_3\}$ be the intersection with the elliptic curve E'. Then by lifting the q_i through the isogeny to n points $p_{ij} \in E$ we see that we can lift the line $\mathbb{V}(z')$ to n^2 lines in $\mathbb{P}^2_{nc} = \mathbb{P}(A_1^*)$, that is, there are n^2 one-dimensional subspaces ļ $\subset A_1$ with the property that N̦$(l) =$ z̦$'$. The fat point corresponding to π is then the shifted quotient of a line module determined by l

$$F_\rho \simeq \frac{A}{A.l + A.z}[n].$$

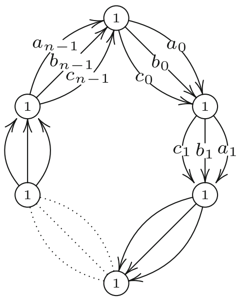
On the other hand, if q is a point on E', then lifting q through the isogeny results in an orbit of n points of E, $\{r, r+p, r+[2]p, \ldots, r+[n-1]p\}$. If P is the point module corresponding to $r \in E$, then the fat point module corresponding to q is

$$F_q = P \oplus P[1] \oplus P[2] \oplus \ldots \oplus P[n-1]$$

and the corresponding graded endomorphism ring is isomorphic to $M_n(\mathbb{C}[t, t^{-1}])$ $(0, 1, 2, \ldots, n-1)$ where $deg(t) = n$ and hence corresponds to a one-parameter family of simple n-dimensional representations in $\mathsf{rep}_n^{ss} A$ with $\mathsf{GL}_n \times \mathbb{C}^*$-stabilizer subgroup generated by $\mathbb{C}^* \times 1$ and a cyclic group of order n

$$\boldsymbol{\mu}_n = \left\langle \left(\begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{bmatrix}, \zeta \right) \right\rangle$$

with ζ a primitive n-th root of unity. In fact, we can give a concrete matrix-representation of these simple modules. Assume that $r - [i]p = [a_i : b_i : c_i] \in \mathbb{P}^2_{nc}$, then the fat point module F_q corresponds to the quiver-representation



and the map $A \longrightarrow M_n(\mathbb{C}[t, t^{-1}])(0, 1, 2, \ldots, n-1)$ sends the generators x, y and z to the degree one matrices

$$\begin{bmatrix} 0 & 0 & \ldots\ldots & a_{n-1}t \\ a_0 & 0 & \ldots\ldots & 0 \\ 0 & a_1 & \ddots & \vdots \\ \vdots & & \ddots\ \ddots & \vdots \\ 0 & 0 & \ldots\ a_{n-2} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \ldots\ldots & b_{n-1}t \\ b_0 & 0 & \ldots\ldots & 0 \\ 0 & b_1 & \ddots & \vdots \\ \vdots & & \ddots\ \ddots & \vdots \\ 0 & 0 & \ldots\ b_{n-2} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \ldots\ldots & c_{n-1}t \\ c_0 & 0 & \ldots\ldots & 0 \\ 0 & c_1 & \ddots & \vdots \\ \vdots & & \ddots\ \ddots & \vdots \\ 0 & 0 & \ldots\ c_{n-2} & 0 \end{bmatrix}.$$

Lemma 1 *The three matrices define a simple n-dimensional representation of A for each choice of $t \in \mathbb{C}^*$.*

Proof The point modules of A are given by the elliptic curve E and the automorphism is determined by summation with the point $p = [a : b : c]$. Choose $r \in E$ and let $r_i = r - [i]p = [a_i : b_i : c_i] \in \mathbb{P}^2_{nc}$. Then by definition of point modules and the associated

$$\begin{cases} aa_{i+1}b_i + bb_{i+1}a_i + cc_{i+1}c_i = 0, \\ ab_{i+1}c_i + bc_{i+1}b_i + ca_{i+1}a_i = 0, \\ ac_{i+1}a_i + ba_{i+1}c_i + cb_{i+1}b_i = 0. \end{cases}$$

Therefore, a quick calculation shows that for each $t \in \mathbb{C}^*$, these 3 matrices define a n-dimensional representation of A. This representation is not an extension of the trivial representation for the center should then be mapped to 0, which is not the case. Using [16, Thm. 3.7], we conclude that this representation is indeed simple. □

Theorem 4 *Let A be a 3-dimensional Sklyanin algebra corresponding to a couple (E, p) where p is a torsion point of order n and assume that $(n, 3) = 1$. Consider the GIT-quotient*

$$\mathtt{rep}_n A \xrightarrow{\pi} \mathtt{spec} Z(A) = \mathtt{rep}_n A // \mathtt{GL}_n.$$

Then we have

(1) *$\mathsf{rep}^{ss}_n A$ is a smooth variety of dimension $n^2 + 2$,*
(2) *A is an Azumaya algebra away from the isolated singularity $\tau \in \mathsf{spec} Z(A)$,*
(3) *the nullcone $\pi^{-1}(\tau)$ contains singularities.*

Proof We know that $\mathbb{P}^2_c = \mathsf{proj} Z(A) = \mathsf{rep}^{ss}_n A // \mathsf{GL}_n \times \mathbb{C}^*$ classifies one-parameter families of semi-stable n-dimensional semi-simple representations of A. To every point $\rho \in \mathbb{P}^2_c$ we have associated a one-parameter family of simples, so all semi-stable A-representations are in fact simple as the semi-simplification M^{ss} of a semi-stable representation still belongs to $\mathsf{rep}^{ss}_n A$. But then, all non-trivial semi-simple A-representations are simple and therefore the GIT-quotient

$$\mathtt{rep}^{ss}_n A \longrightarrow\!\!\!\!\!\rightarrow \mathtt{spec} Z(A) - \{\tau\} = \mathtt{rep}^{ss}_n A // \mathtt{GL}_n$$

is a principal PGL_n-fibration in the étale topology. This proves (1).

The second assertion follows as principal PGL_n-fibrations in the étale topology correspond to Azumaya algebras. For (3), if $\mathsf{rep}_n A$ would be smooth, the algebra A would be Cayley-smooth as in [10]. There it is shown that the only type of central singularity that can arise for Cayley-smooth algebras with a 3-dimensional center is the conifold singularity. This is not the case as for the conifold singularity there need to be at least 2 simple representations lying above τ, but there is only one. □

In general, if R is Cayley-smooth in $\mathfrak{m} \in \mathsf{spec} Z(R)$ and if $M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$ is an isotypical decomposition of the corresponding semi-simple representation M, then we know that the tangent space $T_M(\mathsf{rep}_n R)$ is the vector space of all (trace preserving) algebra maps ψ

$$R \xrightarrow{\psi} M_n(\mathbb{C}[\epsilon]) \longrightarrow\!\!\!\!\!\rightarrow M_n(\mathbb{C})$$

such that the composition with the canonical epimorphism to $M_n(\mathbb{C})$ is the representation ϕ_M determined by M. Likewise, the normal space N_M to the GL_n-orbit coincides with the vector space of all trace preserving extensions $Ext^{tr}_1(M, M)$. From [10, §4.2] we recall that this vector space, together with the natural action of $Stab(M) = (\mathsf{GL}_{e_1} \times \ldots \times$

$\mathsf{GL}_{e_k})/\mathbb{C}^*(1_{e_1}, \ldots, 1_{e_k})$, is given by the representation space $\mathsf{rep}(Q^\bullet, \alpha_M)$ of a (marked) quiver setting $(Q^\bullet, \alpha_M)$ where $Q^\bullet$ is a directed graph on k vertices, corresponding to the distinct simple components S_i of M where some of the loops may be marked, the dimension vector $\alpha_M = (e_1, \ldots, e_k)$ encodes the multiplicities of the simple components in M and the representation space is the usual quiver-representation space modulo the requirement that matrices corresponding to marked loops are required to have trace zero. This allows us to compute a defect against R being Cayley-smooth in $\mathfrak{m}$. With notations as before, this defect is

$$\mathsf{defect}_{\mathfrak{m}}(R) = dim_{\mathbb{C}} Ext^{tr}_R(M, M) + \left(n^2 - \sum_{i=1}^{k} e_i^2\right) - dim\ \mathsf{rep}_n R.$$

For example, if $R_{\mathfrak{m}}$ is an Azumaya algebra over $Z(R)_{\mathfrak{m}}$, then R is Cayley-smooth in $\mathfrak{m}$ if and only if $\mathfrak{m}$ is a smooth point of $\mathsf{spec} Z(R)$.

In the previous section we have seen that there are two different types of simple n-dimensional representations of A corresponding to whether or not the maximal ideal $\mathfrak{m}$ lies over a point of $E' \subset \mathbb{P}^2_c$ or not. Still, their marked quiver-settings are the same (as A is Azumaya in this point and $\mathfrak{m}$ is a smooth point of the center). In order to distinguish between the two types we have to bring in the extra $\mathbb{C}^*$-action coming from the gradation and turn these (marked) quiver-settings into weighted quiver-settings as in [7].

If the $\mathsf{GL}_n \times \mathbb{C}^*$-orbit of the simple representation M corresponding to $\mathfrak{m}$ determines a point *not* lying on the elliptic curve E', its stabiliser subgroup is reduced to $\mathbb{C}^* 1_n \times 1$, whereas if it determines a point in E' the stabiliser subgroup is generated by $\mathbb{C}^* 1_n \times 1$ together with a cyclic group of order n

$$\boldsymbol{\mu}_n = \left\langle \left(\begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{bmatrix}, \zeta \right) \right\rangle$$

where ζ is a primitive n-th root of unity. This can be easily verified using the quiver-representation description of the matrices given before.

As a consequence, this finite group acts on the normal-space to the orbit and hence all three loops correspond to a one-dimensional eigenspace for the $\boldsymbol{\mu}_n$-action with eigenvalue ζ^i for some i. To encode this extra information we will weight the corresponding loop by i. Let us work though the special case of quaternionic Sklyanin algebras:

Example 1 (Quaternionic Sklyanin algebras) It is easy to see that 3-dimensional Sklyanin algebras A_λ determined by a point τ of order 2 have defining equations

$$\begin{cases} xy + yx = \lambda z^2 \\ yz + zy = \lambda x^2 \\ zx + xz = \lambda y^2 \end{cases}$$

with $-27\lambda^3 \neq \left(2 - \lambda^3\right)^3$, $\lambda \neq 0$. An alternative description of A_λ is as a Clifford algebra over $\mathbb{C}[u, v, w]$ (with $u = x^2$, $v = y^2$ and $w = z^2$) associated to the rank 3 bilinear form determined by the symmetric matrix

$$\begin{bmatrix} 2u & \lambda w & \lambda v \\ \lambda w & 2v & \lambda u \\ \lambda v & \lambda u & 2w \end{bmatrix}$$

The center $Z(A_\lambda)$ is generated by u, v, w and the determinant of the matrix which gives the equation of the elliptic curve E' in $\mathbb{P}^2_c$

$$\lambda^2\left(u^3+v^3+w^3\right)-\left(4+\lambda^3\right)uvw=0$$

A 2-dimensional simple representation M_1 of A_λ corresponding to the point $[0:0:1] \in \mathbb{P}^2 - E'$ is given by the matrices

$$x \mapsto \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A simple representation M_2 corresponding to the point $[1:-1:0] \in E'$ is given by the matrices

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix}$$

Note that M_2 corresponds to the orbit $\{[1:-1:0],[1:1:-\lambda]\}$ of points on the elliptic curve $E \subset \mathbb{P}^2_{nc}$ given by the equation

$$\lambda\left(x^3+y^3+z^3\right)-\left(\lambda^3-2\right)xyz=0$$

The tangent space in M_2 to $\mathsf{rep}_2(A_\lambda)$ is determined by trace-preserving maps

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+\epsilon\begin{bmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}+\epsilon\begin{bmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix}+\epsilon\begin{bmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{bmatrix}$$

satisfying the three quadratic defining relations of A_λ. The ϵ-terms of these equalities give the independent linear relations

$$\begin{cases} -\lambda^2 c_2 = a_2 - a_3 + b_2 + b_3, \\ \lambda(a_2+a_3+b_2) = c_2 - c_3, \\ c_2 + c_3 - \lambda a_2 = \lambda(b_2 - b_3). \end{cases}$$

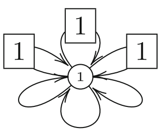
which implies that this tangent space is indeed (as required) 6-dimensional. The additional $\boldsymbol{\mu}_2$-stabilizer for the $\mathsf{PGL}_2 \times \mathbb{C}^*$-action is generated by

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, -1\right)$$

which acts on a trace zero matrix by sending it to

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -a & -b \\ -c & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -a & b \\ c & a \end{bmatrix}.$$

Observe that the three linear equations above are fixed under this action so correspond to eigenspaces of weight 0. Hence we can encode the tangent space together with the action of the stabiliser subgroup by the weighted quiver setting



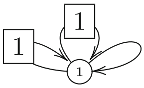
where unadorned loops correspond to weight zero. To compute the tangent space to the GL_n-orbit we have to determine the subspace of the tangent space given by the ϵ-terms of

$$\left(1_2 + \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) (\phi_{M_2}(x), \phi_{M_2}(y), \phi_{M_2}(z)) \left(1_2 - \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

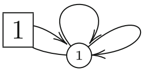
which gives us the three dimensional subspace consisting of matrix triples

$$\left(\begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix}, \begin{bmatrix} b+c & d-a \\ d-a & -b-c \end{bmatrix}, \begin{bmatrix} -\lambda b & 0 \\ \lambda(a-d) & \lambda b \end{bmatrix} \right)$$

and under the action of $\boldsymbol{\mu_2}$ this space is spanned by one eigenvector of weight zero $a-d$ and two of weight one b and c, whence the tangent space to the orbit can be represented by the weighted quiver-setting

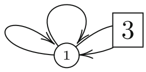


and hence the weighted quiver-setting corresponding to the normal space is represented by

$N \leftrightarrow$ 

Having a precise description of the center $Z(A)$ we can shortcut such tangent space computations, even for general order n Sklyanin algebras:

Lemma 2 *If S is a simple A-representation with $\mathsf{GL}_n \times \mathbb{C}^*$-orbit determining a fat point F_q with $q \in E'$, then the normal space $N(S)$ to the GL_n-orbit decomposes as representation over the $\mathsf{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\boldsymbol{\mu}_n$ as $\mathbb{C}_0 \oplus \mathbb{C}_0 \oplus \mathbb{C}_3$, or in the terminology of* [7], *the associated local weighted quiver is*



Proof From [15, Thm. 4.7] we know that the center $Z(A)$ can be represented as

$$Z(A) = \frac{\mathbb{C}[x', y', z', c_3]}{(c_3^n - cubic(x', y', z'))}$$

where x', y', z' are of degree n (the reduced norms of x, y, z) and c_3 is the canonical central element of degree 3. The simple A-representation S determines a point $s \in \mathsf{spec} Z(A)$ such that $c_3(s) = 0$. Again, as A is Azumaya over s we have that $N(S) = Ext^1_A(S, S)$ coincides with the tangent space $T_s \mathsf{spec} Z(A)$. Gradation defines a $\boldsymbol{\mu}_n$-action on $Z(A)$ leaving x', y', z' invariant and sending c_3 to $\zeta^3 c_3$. The stabilizer subgroup of this action in s is clearly $\boldsymbol{\mu}_n$ and computing the tangent space gives the required decomposition. □

3.2 $\mathcal{A}$ is Cayley-Smooth

Because A is a finitely generated module over $Z(A)$, it defines a coherent sheaf of algebras $\mathcal{A}$ over $\mathsf{proj} Z(A) = \mathbb{P}^2$. In this subsection we will show that $\mathcal{A}$ is a sheaf of Cayley-smooth algebras of degree n.

As $(n, 3) = 1$ it follows that the graded localisation $Q^g_{x'}(A)$ at the multiplicative set of central elements $\{1, x', x'2, \dots\}$ contains central elements t of degree one and hence is isomorphic as a graded algebra to

$$Q^g_{x'}(A) = \left(Q^g_{x'}(A)\right)_0 \left[t, t^{-1}\right].$$

For $u \in Z(A)$, let $\mathbb{X}(u) = \{u \neq 0\} \subset \mathbb{P}^2$. By definition $\Gamma(\mathbb{X}(x'), \mathcal{A}) = (Q^g_{x'}(A))_0$ and by the above isomorphism it follows that $\Gamma(\mathbb{X}(x'), \mathcal{A})$ is a Cayley-Hamilton domain of degree n and is Auslander regular of dimension two and consequently a maximal order. Repeating this argument for the other standard opens $\mathbb{X}(y')$ and $\mathbb{X}(z')$ we deduce

Proposition 1 *$\mathcal{A}$ is a coherent sheaf of Cayley-Hamilton maximal orders of degree n which are Auslander regular domains of dimension 2 over* $\mathsf{proj}\, Z(A) = \mathbb{P}^2_c$.

Thus, $\mathcal{A}$ is a maximal order over $\mathbb{P}^2$ in a division algebra Σ over $\mathbb{C}\left(\mathbb{P}^2\right)$ of degree n. By the Artin-Mumford exact sequence (see for example [10, Thm. 3.11]) describing the Brauer group of $\mathbb{C}(\mathbb{P}^2)$ we know that Σ is determined by the ramification locus of $\mathcal{A}$ together with a cyclic $\mathbb{Z}_n$-cover over it.

Again using the above local description of A as a graded algebra over $Z(A)$ we see that the fat point module corresponding to a point $p \notin E'$ determines a simple n-dimensional representation of $\mathcal{A}$ and therefore $\mathcal{A}$ is Azumaya in p. However, if $p \in E'$, then the corresponding fat point is of the form $P \oplus P[1] \oplus \dots \oplus P[n-1]$ and this corresponds to a semi-simple n-dimensional representation which is the direct sum of n distinct one-dimensional $\mathcal{A}$-representations, one component for each point of E lying over p. Hence, we see that the ramification divisor of $\mathcal{A}$ coincides with E' and, naturally, the division algebra Σ is the one coresponding to the cyclic $\mathbb{Z}_n$-cover $E \twoheadrightarrow E' = E/\langle\tau\rangle$.

Because $\mathcal{A}$ is a maximal order with smooth ramification locus, we deduce from [10, §5.4]

Proposition 2 *$\mathcal{A}$ is a sheaf of Cayley-smooth algebras over $\mathbb{P}^2_c$ and hence $\mathsf{rep}_n(\mathcal{A})$ is a smooth variety of dimension $n^2 + 1$ with GIT-quotient*

$$\mathtt{rep}_n(\mathcal{A}) \overset{\pi}{\twoheadrightarrow} \mathbb{P}^2_c = \mathtt{rep}_n(\mathcal{A})//\mathtt{GL}_n$$

and is a principal PGL_n-fibration over $\mathbb{P}^2_c - E'$.

3.3 The Non-Commutative Blow-Up

Consider the augmentation ideal $\mathfrak{m} = (x, y, z)$ of the 3-dimensional Sklyanin algebra A corresponding to a couple (E, p) with p a torsion point of order n. Define the non-commutative blow-up algebra to be the graded algebra

$$B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2 t^2 \oplus \dots \subset A[t]$$

with degree zero part A and where the commuting variable t is given degree 1. Note that B is a graded subalgebra of $A[t]$ and therefore is again a Cayley-Hamilton algebra of degree n. Moreover, B is a finite module over its center $Z(B)$ which is a graded subalgebra of $Z(A)[t]$. Observe that B is generated by the degree zero elements x, y, z and by the degree one elements $X = xt$, $Y = yt$ and $Z = zt$. Apart from the Sklyanin relations among x, y, z and among X, Y, Z these generators also satisfy commutation relations such as $Xx = xX$, $Xy = xY$, $Xz = xZ$ and so on.

With $\mathsf{rep}_n^{ss} B$ we will denote again the Zariski open subset of $\mathsf{rep}_n B$ consisting of all trace-preserving n-dimensional semi-stable representations, that is, those on which some central homogeneous element of $Z(B)$ of strictly positive degree does not vanish. Theorem 2 asserts that $\mathsf{rep}_n^{ss} B$ is a smooth variety of dimension $n^2 + 3$.

Proof of Theorem 2 : As before, we have a $\mathsf{GL}_n \times \mathbb{C}^*$-action on $\mathsf{rep}_n^{ss} B$ with corresponding GIT-quotient

$$\mathsf{proj} Z(B) \simeq \mathsf{rep}_n^{ss} B // \mathsf{GL}_n \times \mathbb{C}^*$$

Composing the GIT-quotient map with the canonical morphism (taking the degree zero part) $\mathsf{proj} Z(B) \twoheadrightarrow \mathsf{spec} Z(A)$ we have a projection

$$\gamma \ : \ \mathtt{rep}_n^{ss} B \twoheadrightarrow \mathtt{spec} Z(A).$$

Let $\mathfrak{p}$ be a maximal ideal of $Z(A)$ corresponding to a smooth point, then the graded localization of B at the degree zero multiplicative subset $Z(A) - \mathfrak{p}$ gives

$$B_{\mathfrak{p}} \simeq A_{\mathfrak{p}}[t, t^{-1}]$$

whence $B_{\mathfrak{p}}$ is an Azumaya algebra over $Z(A)[t, t^{-1}]$ and therefore over $\mathsf{spec} Z(A) - \{\tau\}$ the projection γ is a principal $\mathsf{PGL}_n \times \mathbb{C}^*$-fibration and in particular the dimension of $\mathsf{rep}_n^{ss} B$ is equal to $n^2 + 3$.

This further shows that possible singularities of $\mathsf{rep}_n^{ss} B$ must lie in $\gamma^{-1}(\tau)$ and as the singular locus is Zariski closed we only have to prove smoothness in points of closed GL_n-orbits in $\gamma^{-1}(\tau)$. Such a point ϕ must be of the form

$$x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M.$$

By semi-stability, (K, L, M) defines a simple n-dimensional representation of A and its $\mathsf{GL}_n \times \mathbb{C}^*$-orbit defines the point $[det(K) : det(L) : det(M)] \in \mathbb{P}^2_c$. hence we may assume for instance that K is invertible.

The tangent space $T_\phi \mathsf{rep}_n^{ss} B$ is the linear space of all trace-preserving algebra maps $B \longrightarrow M_n(\mathbb{C}[\epsilon])$ of the form

$$x \mapsto 0 + \epsilon U, y \mapsto 0 + \epsilon V, z \mapsto 0 + \epsilon W, X \mapsto K + \epsilon R, Y \mapsto L + \epsilon S, Z \mapsto M + \epsilon T$$

and we have to use the relations in B to show that the dimension of this space is at most $n^2 + 3$. As (K, L, M) is a simple n-dimensional representation of the Sklyanin algebra, we know already that (R, S, T) depend on at most $n^2 + 2$ parameters. Further, from the commutation relations in B we deduce the following equalities (using the assumption that K is invertible)

- $xX = Xx \Rightarrow UK = KU$,
- $xY = Xy \Rightarrow UL = KV \Rightarrow K^{-1}UL = V$,
- $xZ = Xz \Rightarrow UM = KW \Rightarrow K^{-1}UM = W$,
- $Yx = yX \Rightarrow LU = VK \Rightarrow LK^{-1}U = V$,
- $Zx = zX \Rightarrow MU = WK \Rightarrow MK^{-1}U = W$.

These equalities imply that $K^{-1}U$ commutes with K, L and M and as (K, L, M) is a simple representation and hence generate $M_n(\mathbb{C})$ it follows that $K^{-1}U = \lambda 1_n$ for some $\lambda \in \mathbb{C}$. But then it follows that

$$U = \lambda K, \quad V = \lambda L, \quad W = \lambda M$$

and so the triple (U, V, W) depends on at most one extra parameter, showing that $T_\phi \mathsf{rep}_n^{ss} B$ has dimension at most $n^2 + 3$, finishing the proof. □

Remark 1 *The statement of the previous theorem holds in a more general setting, that is,* $\mathsf{rep}_n^{ss} B$ *is smooth whenever* $B = A \oplus A^+ t \oplus (A^+)^2 t^2 \oplus \dots$ *with* A *a positively graded algebra that is Azumaya away from the maximal ideal* A^+ *and* $Z(A)$ *smooth away from the origin.*

Unfortunately this does not imply that $\mathsf{proj} Z(B) = \mathsf{rep}_n^{ss} B // \mathsf{GL}_n \times \mathbb{C}^*$ is smooth as there are closed $\mathsf{GL}_n \times \mathbb{C}^*$ orbits with stabilizer subgroups strictly larger than $\mathbb{C}^* \times 1$. This happens precisely in semi-stable representations ϕ determined by

$$x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M$$

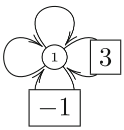
with $[det(K) : det(L) : det(M)] \in E'$. In which case the matrices (K, L, M) can be brought into the form

$$\begin{bmatrix} 0 & 0 & \dots & \dots & a_{n-1}t \\ a_0 & 0 & \dots & \dots & 0 \\ & & \ddots & & \vdots \\ 0 & a_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & \dots & b_{n-1}t \\ b_0 & 0 & \dots & \dots & 0 \\ & & \ddots & & \vdots \\ 0 & b_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b_{n-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & \dots & c_{n-1}t \\ c_0 & 0 & \dots & \dots & 0 \\ & & \ddots & & \vdots \\ 0 & c_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & c_{n-2} & 0 \end{bmatrix}$$

and the stabilizer subgroup is generated by $\mathbb{C}^* \times 1$ together with the cyclic group of order n

$$\boldsymbol{\mu}_n = \left\langle \left(\begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{bmatrix}, \zeta \right) \right\rangle.$$

Lemma 3 *If* ϕ *is a representation as above, then the normal space* $N(\phi)$ *to the* GL_n*-orbit decomposes as a representation over the* $\mathsf{GL}_n \times \mathbb{C}^*$*-stabilizer subgroup* $\boldsymbol{\mu}_n$ *as* $\mathbb{C}_0 \oplus \mathbb{C}_0 \oplus \mathbb{C}_3 \oplus \mathbb{C}_{-1}$*, that is, the associated local weighted quiver is*



Proof The extra tangential coordinate λ determines the tangent vectors of the three degree zero generators

$$x \mapsto 0 + \epsilon\lambda K, \qquad y \mapsto 0 + \epsilon\lambda L, \qquad z \mapsto 0 + \epsilon\lambda M$$

and so the generator of $\boldsymbol{\mu}_n$ acts as follows

$$\begin{bmatrix} 1 & & & \\ & \zeta^{n-1} & & \\ & & \ddots & \\ & & & \zeta \end{bmatrix} .(\epsilon\lambda(K, L, M)). \begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{bmatrix} = \epsilon\zeta^{n-1}\lambda(K, L, M)$$

and hence accounts for the extra component $\mathbb{C}_{-1}$. □

We have now all information to prove Theorem 3 which asserts that the canonical map

$$\mathtt{proj}\,Z(B) \twoheadrightarrow \mathtt{spec}\,Z(A)$$

is a partial resolution of singularities, with singular locus $E' = E/\langle p \rangle$ in the exceptional fiber, all singularities of type $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$. In other words, the isolated singularity of $\mathsf{spec}\,Z(A)$ 'sees' the elliptic curve E' and the isogeny $E \twoheadrightarrow E'$ defining the 3-dimensional Sklyanin algebra A.

*Proof of Theorem 3 :*The GIT-quotient map

$$\mathtt{rep}_n^{ss} B \twoheadrightarrow \mathtt{proj}\,Z(B)$$

is a principal $\mathsf{PGL}_n \times \mathbb{C}^*$-bundle away from the elliptic curve E' in the exceptional fiber whence $\mathsf{proj}\,Z(B) - E'$ is smooth. The application to the Luna slice theorem of [7, Thm. 5] asserts that for any point $\overline{\phi} \in E' \hookrightarrow \mathsf{proj}\,Z(B)$ and all $t \in \mathbb{C}$ there is a neighborhood of $(\overline{\phi}, t) \in \mathsf{proj}\,Z(B) \times \mathbb{C}$ which is étale isomorphic to a neighborhood of 0 in $N(\phi)//\boldsymbol{\mu}_n$. From the previous lemma we deduce that

$$N(\phi)//\boldsymbol{\mu}_n \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$$

where $\mathbb{C}\left[\mathbb{C}^2/\mathbb{Z}_n\right] \simeq \mathbb{C}[u, v, w]/(w^n - uv^3)$, finishing the proof.

As B is a finite module over its center, it defines a coherent sheaf of algebras over $\mathtt{proj}\,Z(B)$. From Theorem 3 we obtain

Corollary 1 *The sheaf of Cayley-Hamilton algebras $\mathcal{B}$ on $\mathtt{proj}\,Z(B)$ is Azumaya away from the elliptic curve E' in the exceptional fiber $\pi^{-1}(\mathfrak{m}) = \mathbb{P}^2$ and hence is Cayley-smooth on this open set. However, $\mathcal{B}$ is not Cayley-smooth.*

Proof For a point p in the exceptional fiber $\pi^{-1}(\mathfrak{m}) - E'$ we already know that $\mathsf{proj}\,Z(B)$ is smooth and that B is Azumaya, which implies that $\mathsf{rep}_n^{ss} B$ is smooth in the corresponding orbit. However, for a point $p \in E'$ we know that $\mathsf{proj}\,Z(B)$ has a non-isolated singularity in p. Therefore, $\mathsf{rep}_n^{ss} \mathcal{B}$ can not be smooth in the corresponding orbit, as the only central singularity possible for a Cayley-smooth order over a center of dimension 3 is the conifold singularity, which is isolated. □

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