HIGH-DIMENSIONAL REPRESENTATIONS OF THE 3-COMPONENT LOOP BRAID GROUP

LIEVEN LE BRUYN

Abstract. In a recent paper [4] it is shown that irreducible representations of the three string braid group B_3 of dimension ≤ 5 extend to representations of the three component loop braid group LB_3. Further, an explicit 6-dimensional irreducible B_3-representation is given not allowing does such an extension.

In this note we give a necessary and sufficient condition on the components of irreducible B_3-representations, in arbitrary dimensions, such that sufficiently general representations in that component extend to LB_3.

1. The result

The 3-component loop braid group LB_3 encodes motions of 3 oriented circles in $\mathbb{R}^3$. The generator σ_i $(i = 1, 2)$ is interpreted as passing the i-th circle under and through the $i+1$-th circle ending with the two circles' positions interchanged. The generator s_i $(i = 1, 2)$ simply interchanges the circles i and $i+1$. For physical motivation and graphics we refer to the paper by John Baez, Derek Wise and Alissa Crans [3]. The defining relations of LB_3 are:

(1) $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$
(2) $s_1s_2s_1 = s_2s_1s_2$
(3) $s_1^2 = s_2^2 = 1$
(4) $s_1s_2\sigma_1 = \sigma_2s_1s_2$
(5) $\sigma_1\sigma_2s_1 = s_2\sigma_1\sigma_2$

Note that (1) is the defining relation for the 3-string braid group B_3, (2) and (3) define the symmetric group S_3, therefore the first three relations describe the free group product $B_3 * S_3$.

In [10] it is shown that the distinct irreducible components in the moduli space $\mathtt{irr}_n(B_3) = \cup_\alpha \mathtt{irr}_\alpha(B_3)$ of isomorphism classes of n-dimensional irreducible B_3-representations correspond to 5-tuples $\alpha = (a, b; x, y, z) \in \mathbb{N}^5$ satisfying the following conditions

$$a + b = n = x + y + z, \quad a \geq b, \quad \text{and} \quad x = max(x, y, z) \leq b$$

The main result of this note is :

Theorem 1. *There is a Zariski open subset of irreducible n-dimensional representations of B_3 which extend to LB_3 in the component of $\mathtt{irr}_n(B_3)$ corresponding to $\alpha = (a, b; x, y, z)$ if and only if there are positive integers $u, v, w \in \mathbb{N}$ such that*

$$u \leq v \leq w, \quad a = v + w. \quad b = u + w, \quad \{x, y, z\} = \{u + v, w, w\}$$

Observe that the first dimension n allowing an admissible 5-tuple not satisfying this condition is $n = 6$ with $\alpha = (3, 3; 3, 2, 1)$, so there are indeed 6-dimensional irreducible B_3-representations which do not extend to LB_3.

2. The strategy

In [6] and [7] we gave explicit representations of a Zariski open subset of $\mathtt{irr}_\alpha(B_3)$ for all dimensions n by reducing to the study of irreducible representations of the modular group Γ and using the local quiver approach to the étale local structure of quiver moduli spaces proved in [1]. Here we will follow a similar strategy.

Recall that the modular group $\Gamma = C_2 * C_3 = \langle s,t | s^2 = 1 = t^3 \rangle$ is a quotient of B_3 by dividing out the central element $c = (\sigma_1\sigma_2)^3$, so that we can take $t = \overline{\sigma}_1\overline{\sigma}_2$ and $s = \overline{\sigma}_1\overline{\sigma}_2\overline{\sigma}_1$.

Hence, any irreducible n-dimensional representation $\phi : B_3 \longrightarrow GL_n$ will be isomorphic to one of the form

$$\phi(\sigma_1) = \lambda\psi(\overline{\sigma}_1), \quad \text{and} \quad \phi(\sigma_2) = \lambda\psi(\overline{\sigma}_2)$$

for some $\lambda \in \mathbb{C}^*$ and $\psi : \Gamma \longrightarrow GL_n$ an n-dimensional irreducible representation of $\Gamma = \langle s,t \rangle = \langle \overline{\sigma}_1, \overline{\sigma_2} \rangle$.

With $S_3 = \langle s_1, s_2 | s_1s_2s_1 = s_2s_1s_2, s_1^2 = 1 = s_2^2 \rangle$, we consider the amalgamated free product

$$G = \Gamma *_{C_3} S_3$$

in which the generator of C_3 is equal to $t = \overline{\sigma}_1\overline{\sigma}_2$ in Γ and to s_1s_2 in S_3.

We will impose conditions on ψ such that it extends to a (necessarily irreducible) representations of G. Then, if this is possible, as $\psi(\overline{\sigma}_1\overline{\sigma}_2) = \psi(s_1s_2)$ and as the defining equations (1),(4) and (5) of LB_3 are homogeneous in the σ_i it will follow that

$$\phi(\sigma_i) = \lambda\psi(\overline{\sigma}_i), \quad \text{and} \quad \phi(s_i) = \psi(s_i)$$

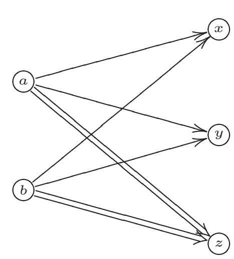
is a representation of LB_3 extending the irreducible representation ϕ of B_3.

3. Irreducible representations of $G = \Gamma *_{C_3} S_3$

To get at the irreducibles of G one can observe that it is a free group product $G \simeq C_2 * S_3$ and use the approach via stable representations and local quivers of [2]. If V is an n-dimensional G-representation, then by restricting it to the subgroups C_2 and S_3 we get decomposition of V into

$$S_+^{\oplus a} \oplus S_-^{\oplus b} = V \downarrow_{C_2} = V = V \downarrow_{S_3} = T^{\oplus x} \oplus S^{\oplus y} \oplus P^{\oplus z}$$

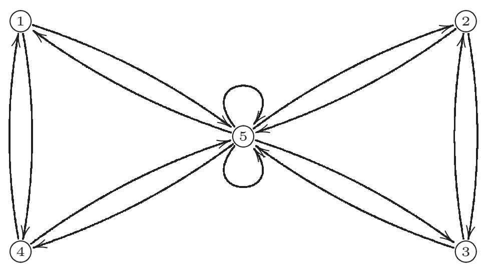
where $\{S_+, S_-\}$ are the 1-dimensional irreducibles of C_2, T is the trivial S_3-representation, S the sign representation and P the 2-dimensional irreducible S_3-representation. Clearly we must have $a + b = n = x + y + 2z$ and once we choose bases in each of these irreducibles we have that V itself determines a representation of the following quiver setting



where the arrows give the block-decomposition of the base-change matrix B from the chosen basis of $V \downarrow_{C_2}$ to the chosen basis of $V \downarrow_{S_3}$. Isomorphism classes of irreducible G-representations correspond to isomorphism classes of θ-stable quiver representation of dimension vector $\beta = (a, b; x, y, z)$ for the stability structure (see [5] for full details) $\theta = (-1, -1; 1, 1, 2)$. To determine which dimension vectors allow for θ-stable representation we follow the approach of local quivers from [1]. The minimal dimension vectors of θ-stable representations are

$$\begin{cases} \alpha_1 = (1, 0; 1, 0, 0) \\ \alpha_2 = (1, 0; 0, 1, 0) \\ \alpha_3 = (0, 1; 1, 0, 0) \\ \alpha_4 = (0, 1; 0, 1, 0) \\ \alpha_5 = (1, 1; 0, 0, 1) \end{cases}$$

The corresponding local quiver is



with corresponding Ringel-bilinear form

$$\chi = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

From the results of [1] and the classification of the dimension vectors of simple quiver representations given in [8] we obtain the following result:

Theorem 2. *The moduli space of isomorphism classes of irreducible n-dimensional representations of $G = \Gamma *_{C_3} S_3$ decomposes as*

$$\mathtt{irr}_n(G) = \sqcup_\alpha \mathtt{irr}_\alpha(G)$$

where $\alpha = (a, b; x, y, z)$ with $a + b = n = x + y + 2z$ such that there exist natural numbers $\gamma = (p, q, r, s, t) \in \mathbb{N}^5$ with

$$\alpha = (a, b; x, y, z) = (p + q + t, r + s + t, p + r, q + s, t)$$

such that $\mathtt{supp}(\gamma)$ is of type $\tilde{A}_1$ and γ is either $(1, 0, 0, 1, 0)$ or $(0, 1, 1, 0, 0)$ or such that for all $1 \leq i \leq 5$ we have that

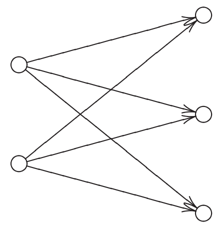
$$\chi(\alpha, \epsilon_i) \leq 0 \quad \textit{and} \quad \chi(\epsilon_i, \alpha) \leq 0$$

where $\epsilon_i = (\delta_{ij})_j$.

In fact, one can give explicit matrix-representations of general irreducible G-representations for every component by using similar methods as used in [7]. We leave this as a suggestion for further research.

4. Proof of Theorem 1

We now have to control when a G-representian V is an irreducible Γ-representation. Recall that $\Gamma = C_2 * C_3$ so that the corresponding quiver is



With on the left hand sides the irreducible components $\{S_+, S_-\}$ of C_2 and on the right hand side the irreducible C_3-representation $\{T_1, T_\omega, T_{\omega^2}\}$. So, we have to know the restrictions of the irreducible S_3-representations to C_3

$$T \downarrow_{C_3} = T_1, \quad S \downarrow_{C_3} = T_1, \quad \text{and} \quad P \downarrow_{C_3} = T_\omega \oplus T_{\omega^2}$$

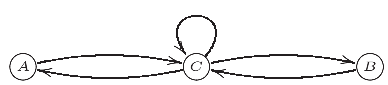
This then implies that the Γ-dimension vectors of the five minimal G-irreducibles are

$$\begin{cases} \alpha_1 \rightarrow (1,0;1,0,0) \\ \alpha_2 \rightarrow (1,0,1,0,0) \\ \alpha_3 \rightarrow (0,1;1,0,0) \\ \alpha_4 \rightarrow (0,1;1,0,0) \\ \alpha_5 \rightarrow (1,1;0,1,1) \end{cases}$$

So, we have three cases to consider

$$A = (1,0;1,0,0), \quad B = (0,1;1,0,0), \quad \text{and} \quad C = (1,1;0,1,1)$$

which gives us the local quiver



By [1] as before, this tells us that the only irreducible G-representations which restricted to Γ are still irreducible will correspond to dimension vectors (p,q,r) of the above quiver allowing simple dimension vectors. As the Ringel-bilinear form in this case is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

we obtain that these simple dimension vectors are either $(1,0,0), (0,1,0), (0,1,0)$ or dimension vectors (v,u,w) such that

$$u \leq v \leq w$$

That is, leading to Γ-dimension vectors like

$$(v+w, u+w; u+v, w, w)$$

and as we still have an action left of μ_6 on the dimension vectors for B_3-irreducibles, we can still take $x = max(u+v, w, w)$ and adjust accordingly.

References

[1] Jan Adriaenssens and Lieven Le Bruyn, *Local quivers and stable representations*, Communications in Algebra, 31 (2003) 1777-1797

[2] Jan Adriaenssens and Lieven Le Bruyn, *Trees of semi-simple algebras*, `arXiv:math/0507503` (2005)

[3] John Baez, Derek Wise and Alissa Crans, *Exotic statistics for strings in 4D BF theory*, Adv. Theor. Math. Phys. **11** (2007) 707-749, `arXiv:gr-qc/0603085`

[4] Paul Bruillard, Seung-Moon Hong, Julia Yael Plavnik, Eric C. Rowell, Liang Chang and Michael Yuan Sun, *Low-dimensional representations of the three component loop braid group*, `arXiv:1508.00005` (2015)

[5] Alastair King, *Moduli of representations of finite dimensional algebras*, Quat. J. Math. Oxford Ser. (2) **45** (1994) 515-530

[6] Lieven Le Bruyn, *Dense families of B_3-representations and braid reversion*, Journal of Pure and Appl. Algebra 215 (2011) 1003-1014, `math.RA/1003.1610`

[7] Lieven Le Bruyn, *Most irreducible representation of the 3-string braid group*, `arXiv:1303.4907` (2013)

[8] Lieven Le Bruyn and Claudio Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990) 585-598

[9] Imre Tuba and Hans Wenzl, *Representations of the braid group B_3 and of $SL(2,\mathbb{Z})$*, Pacific J. Math. 197 (2001)

[10] Bruce Westbury, *On the character varieties of the modular group*, preprint Nottingham University (1995)

Department of Mathematics, University of Antwerp, Middelheimlaan 1, B-2020 Antwerp (Belgium), `lieven.lebruyn@uantwerpen.be`