POINT MODULES OF QUANTUM PROJECTIVE SPACES

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Abstract. In this note we give an explicit description of the irreducible components of the reduced point varieties of quantum polynomial algebras.

Consider the quantum polynomial algebra on $k+1$-variables with quantum commutation relations

$$A_Q = \mathbb{C}\langle u_o, u_1, \ldots, u_k \rangle / (u_i u_j - q_{ij} u_j u_i, 0 \leq i, j \leq k)$$

where the entries of the $k+1 \times k+1$ matrix $Q = (q_{ij})_{i,j}$ are all non-zero and satisfy $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$.

As A_Q is a graded connected, iterated Ore-extension, it is Auslander-regular of dimension $k+1$ and we can consider the corresponding non-commutative projective space $\mathbb{P}^k_Q = \mathtt{Proj}(A_Q)$ in the sense of [1].

Recall from [1] that a *point module* $P = P_0 \oplus P_1 \oplus \ldots$ of A_Q is a graded left A_Q-module which is cyclic (that is, generated by one element in degree 0), critical (implying that normalizing elements of A_Q act on it either as zero or a non-zero divisor) with Hilbert-series $(1-t)^{-1}$.

Hence, a point module P is necessarily of the form $A_Q/(A_Q l_1 + \ldots + A_Q l_k)$ where the l_i are linearly independent degree one elements in A_Q, and hence determines a unique point $x_P = \mathbb{V}(l_1, \ldots, l_k)$ in commutative k-dimensional projective space $\mathbb{P}^n = \mathtt{Proj}((A_Q)_1^*)$.

In this note we will describe the reduced subvariety of $\mathbb{P}^k$, which is called *the point-variety of* A_Q

$$\mathtt{pts}(A_Q) = \{ x_P \in \mathbb{P}^k \mid P \text{ a point module of } A_Q \}.$$

We can approach this problem inductively.

Proposition 1. *For each of the generators u_i of A_Q we have*

$$\mathtt{pts}(A_Q) = (\mathtt{pts}(A_Q) \cap \mathbb{V}(u_i^*)) \sqcup (\mathtt{pts}(A_Q) \cap \mathbb{X}(u_i^*))$$

and these pieces can be described as follows:

(1) $\mathtt{pts}(A_Q) \cap \mathbb{V}(u_i^*) = \mathtt{pts}(A_{\overline{Q}})$ *where* $\overline{Q}$ *is the $k \times k$ matrix obtained from Q after deleting the i-th row and column.*

(2) $\mathtt{pts}(A_Q) \cap \mathbb{X}(u_i^*)$ *is the affine variety*

$$\bigcap_{j,l \neq i} \mathbb{V}((r_{jl} - 1) v_j^* v_l^*)$$

for the polynomial functions on $\mathbb{X}(u_i^*)$: $v_j^* = u_j^*(u_i^*)^{-1}$ and with $r_{jl} = q_{ij} q_{jl} q_{il}^{-1}$

(3) *In particular,* $\mathtt{pts}(A_Q) = \mathbb{P}^k$ *if and only if the rank of the matrix Q is equal to one.*

Proof. As u_i is normalizing in A_Q it acts either as zero on a point module P or as a non-zero divisor. The point modules on which u_i acts as zero are $\mathtt{pts}(A_Q) \cap \mathbb{V}(u_i^*)$, correspond to point modules of the quantum polynomial algebra $A_Q/(u_i) \simeq A_{\overline{Q}}$ and are contained in the projective subspace $\mathbb{P}^{k-1} = \mathbb{V}(u_i^*) = \mathtt{Proj}((A_{\overline{Q}})_1^*)$. This proves (1).

If u_i acts as a non-zero divisor on the point module P, it extends to a graded module over the localization of A_Q at the multiplicative system of homogeneous elements $\{1, u_i, u_i^2, \ldots\}$ and as this localization has an invertible degree one element it is a *strongly graded algebra*, see [3, §I.3], and hence is a skew Laurent-polynomial extension

$$A_Q[u_i^{-1}] \simeq (A_Q[u_i^{-1}])_0[u_i, u_i^{-1}, \sigma]$$

where $(A_Q[u_i^{-1}])_0$ is the degree zero part of the localization and σ the automorphism on it induced by conjugation with u_i.

The algebra $(A_Q[u_i^{-1}])_0$ is generated by the k elements $v_j = u_j u_i^{-1}$ and as we have the commuting relations $u_j u_i^{-1} = q_{ij} u_i^{-1} u_j$ we have

$$v_j v_l = q_{ij} u_i^{-1} u_j u_l u_i^{-1} = q_{ij} q_{jl} u_i^{-1} u_l u_j u_i^{-1} = q_{ij} q_{jl} q_{il}^{-1} u_l u_i^{-1} u_j u_i^{-1} = q_{ij} q_{jl} q_{il}^{-1} v_l v_j$$

Therefore, $(A_Q[u_i^{-1}])_0$ is again a quantum polynomial algebra of the form A_R where $R = (r_{jl})_{j,l}$ is the $k \times k$ matrix with entries

$$r_{jl} = q_{ij} q_{jl} q_{il}^{-1}$$

Because $(A_Q[u_i^{-1}])_0$ is strongly graded, the localization $P[u_i^{-1}]$ (and hence the point module P) is fully determined by the one-dimensional representation $P[u_i^{-1}]_0$ of $(A_Q[u_i^{-1}])_0$, see [3, §I.3] or [1, Proposition 7.5].

One-dimensional representations of A_R correspond to points $(a_j)_j \in \mathbb{A}^k$ (via the association $v_j \mapsto a_j$ for all $j \neq i$) satisfying all the defining relations of A_R, that is, they must satisfy the relations

$$(1 - r_{jl}) a_j a_l = 0$$

which proves (2).

As for (3), observe that $\mathtt{pts}(A_Q) \cap \mathbb{X}(u_i^*) = \mathbb{A}^k$ if and only if all the $r_{jl} = 1$. This in turn is equivalent, by the definition of the r_{jl} to

$$\forall j, l \neq i \ : \ q_{jl} = q_{il} q_{ij}^{-1}$$

but then, any 2×2 minor of Q has determinant zero as

$$\begin{bmatrix} q_{ju} & q_{jv} \\ q_{lu} & q_{lv} \end{bmatrix} = \begin{bmatrix} \frac{q_{iu}}{q_{ij}} & \frac{q_{iv}}{q_{ij}} \\ \frac{q_{iu}}{q_{il}} & \frac{q_{iv}}{q_{il}} \end{bmatrix}$$

and the same applies to minors involving the i-th row or column of Q. Hence, Q is of rank one (and so is $\overline{Q}$) finishing the proof. □

This result also allows us to describe the irreducible components of the point-varieties of quantum polynomial algebras directly. Take a quantum polynomial algebra on $n+1$ variables

$$A_M = \mathbb{C}\langle x_0, x_1, \ldots, x_n \rangle / (x_i x_j - m_{ij} x_j x_i, \ 0 \leq i, j \leq n)$$

Consider the points $\delta_i = [\delta_{i0} : \ldots : \delta_{in}]$ in $\mathbb{P}^n = \mathtt{Proj}((A_M)_0^*)$. For any $k+1$-tuple $(i_0, i_1, \ldots, i_k)$ with $0 \leq i_0 < i_1 < \ldots < i_k \leq n$ consider the k-dimensional projective subspace $\mathbb{P}(i_0,\ldots,i_k) \subset \mathbb{P}^n$ spanned by the points δ_{i_j}. Also denote the $k+1 \times k+1$ minor of M by

$$M(i_0,\ldots,i_k) = \begin{bmatrix} 1 & m_{i_0 i_1} & \ldots & m_{i_0 i_k} \\ m_{i_1 i_0} & 1 & \ldots & m_{i_1 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{i_k i_o} & m_{i_k i_1} & \ldots & 1 \end{bmatrix}$$

With these notations, we deduce from Proposition 1:

Theorem 2. *The reduced point-variety of the quantum polynomial algebra A_M is equal to*

$$\mathtt{pts}(A_M) = \bigcup_{rk(M(i_0,i_1,\ldots,i_k))=1} \mathbb{P}(i_0,i_1,\ldots,i_k) \subset \mathbb{P}^n$$

As a consequence, the union of all lines $\cup_{i,j}\mathbb{P}(i,j)$ is always contained in $\mathtt{pts}(A_M)$ and will be equal to it for generic M.

Proof. As $\mathbb{P}(i_0,\ldots,i_k) = \mathtt{Proj}((A_Q)^*_0)$ with $A_Q = A_M/(x_{j_1},\ldots,x_{j_{n-k}})$ where $\{0,1,\ldots,n+1\} = \{i_0,i_1,\ldots,i_k\} \sqcup \{j_1,\ldots,j_{n-k}\}$, the description of $\mathtt{pts}(A_M)$ follows from Proposition 1.

As for the generic statement, consider the matrix M with $m_{ij} = -1$ if $i \neq j$. Clearly, as any 3×3 minor

$$M(i,j,k) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

has rank 3, we have that $\mathtt{pts}(A_M) = \cup_{i,j}\mathbb{P}(i,j)$. Alternatively, one can use the results of [2]. In this case, the algebra A_M is a Clifford algebra over $\mathbb{C}[x_0^2,\ldots,x_n^2]$ with associated quadratic form $D = \mathtt{diag}(x_0^2,\ldots,x_n^2)$. In [2] it was shown that for a Clifford algebra, the point-variety is a double cover $\mathtt{pts}(A_M) \twoheadrightarrow \mathbb{V}(minor(3,D)) \subset \mathbb{P}^n_{[x_0^2:\ldots:x_n^2]}$, ramified over $\mathbb{V}(minor(2,D))$. For the given M one easily checks that $\mathbb{V}(minor(3,D))$ is one-dimensional, hence so is $\mathtt{pts}(A_M)$. □

We leave the combinatorial problem of determining which subvarieties of $\mathbb{P}^n$ can actually arise as a suggestion for further research. Not all unions as above can occur.

Example 3. *In $\mathbb{P}^3$ only two of the $\mathbb{P}^2$ (out of four possible) can arise in a proper subvariety $\mathtt{pts}(A_M) \subsetneq \mathbb{P}^3$. For example, take*

$$M = \begin{bmatrix} 1 & a & b & x \\ a^{-1} & 1 & a^{-1}b & c \\ b^{-1} & ab^{-1} & 1 & bca^{-1} \\ x^{-1} & c^{-1} & ba^{-1}c^{-1} & 1 \end{bmatrix}$$

then, for generic x we have

$$\mathtt{pts}(A_M) = \mathbb{P}(0,1,2) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(0,3)$$

but once we want to include another $\mathbb{P}^2$, for example, $\mathbb{P}(0,1,3)$ we need the relation $x = ac$ in which case M becomes of rank one, whence $\mathtt{pts}(A_M) = \mathbb{P}^3$.

References

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