THE SINGULARITIES OF NONCOMMUTATIVE MANIFOLDS

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Abstract. We present a faster method to determine all singularities of quiver moduli spaces up to smooth equivalence. We show that every quiver controls a large family of noncommutative compact manifolds.

1. The problem

Let Q be a quiver on a finite set of vertices $Q_v = \{1, 2, \ldots, k\}$ having a finite set of arrows Q_a. The quiver may contain loops and oriented cycles. The structure of Q is fully encoded in the Ringel bilinear form χ_Q on $\mathbb{Z}^{\oplus k}$ determined by

$$\chi_Q(\epsilon_i, \epsilon_j) = \delta_{ij} - \#\{a \in Q_a \mid \circ_i \xrightarrow{a} \circ_j \}$$

The path algebra $\mathbb{C}Q$ has as $\mathbb{C}$-basis the set of all oriented paths in Q including those of length zero which correspond to the vertices. Multiplication in $\mathbb{C}Q$ is induced by concatenation of paths. The path algebras $\mathbb{C}Q$ are very special cases of quasi-free, or formally smooth, algebras as in [6] or [9] and can be seen as corresponding to all possible noncommutative affine spaces among all noncommutative manifolds, see for example [13]. The study of the additive category of finite dimensional representations of $\mathbb{C}Q$ reduces to that of quiver-representations of Q. Each such representation has a dimension vector $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}^{\oplus k}$ (giving the dimensions of the vertex-spaces) of total dimension $d(\alpha) = \sum_i a_i$. The set of all Q-representations of dimension vector α is an affine space $\mathtt{rep}_\alpha(Q)$ on which the group $GL(\alpha) = \prod_i GL_{a_i}$ acts via base-change in the vertex-spaces. It is well known that the corresponding GIT-quotient $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ classifies isomorphism classes of semi-simple Q-representation of dimension vector α and that its coordinate ring, the ring of polynomial quiver invariants, is generated by traces along loops and oriented cycles in Q, see [10].

Consider a stability structure $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{Z}^{\oplus k}$, then we call a Q-representation V of dimension vector α a θ-semistable representation if $\theta.\alpha = \sum_i \theta_i a_i = 0$ and if for every proper subrepresentation W of V we have $\theta.\beta_W \geq 0$ where β_W is the dimension vector of W, and, if all $\theta.\beta_W > 0$ we say that V is a θ-stable representation. The corresponding moduli space $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ of θ-semistable representations of dimension α was introduced and studied in [8]. Its points correspond to isomorphism classes of α-dimensional representations of the form

$$M = N_1^{\oplus e_1} \oplus \ldots \oplus N_u^{\oplus e_u}$$

such that all factors N_i are θ-stable of dimension vector β_i and occur in M with multiplicity $e_i \geq 0$, so that $\alpha = \sum_i e_i \beta_i$. It is well known that most of these quiver moduli spaces $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ are singular. In fact, in [7] it is shown that the only quivers Q having all their quiver moduli spaces smooth are the Dynkin or extended Dynkin quivers.

Hence, we would like to determine the types of singularities that occur in $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ up to smooth equivalence. Recall that points $x \in X$ and $y \in Y$ in two varieties X and Y are called smoothly equivalent, if there are natural numbers k and l and an isomorphism of complete local rings

$$\widehat{\mathcal{O}}_{X,x}[[x_1,\ldots,x_k]] \simeq \widehat{\mathcal{O}}_{Y,y}[[y_1,\ldots,y_l]]$$

In principle, we can determine these finite number of types by combining the method of local quivers from [1] with Bocklandt's reduction steps [3] and [4]. In [1] the étale local structure of $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ near M was described as the quiver quotient variety of a local quiver setting (Q_M,α_M). Here, Q_M is a quiver on k vertices corresponding to the distinct stable factors of M. The number of directed arrows (or loops) from vertex $\circ_i$ to vertex $\circ_j$ is equal to $\delta_{ij} - \chi_Q(\beta_i,\beta_j)$. The main result of [1] asserts that there is an étale isomorphism between a neighborhood of M in $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ and a neighborhood of $\overline{0}$ in the quiver quotient variety $\mathtt{rep}_\alpha(Q_M)/GL(\alpha_M)$. In [4] Bocklandt's eduction steps from [3] were used to to classify such quiver quotient singularities up to smooth equivalence.

However, in all but the more trivial situations, this is a very time consuming method. Whereas Bocklandt's reduction steps are fairly efficient, the determination of all possible representation types of M and the calculation of all local quiver settings is not. In this paper we introduce two concepts to speed up this process. First, we introduce an auxiliary quiver Q_θ, depending only on the stability structure θ and not on the particular dimension vector α, controlling all possible local quivers (Q_M,α_M). The quiver Q_θ will allow us to quickly determine the 'worst' singularity types in $\mathtt{mod}^{ss}_\alpha(Q,\theta)$. Next, we introduce a partial ordering on the possible types of quiver quotients, which can in any concrete situation be efficiently constructed inductively, and, which we use to characterize all other singularity types in $\mathtt{mod}^{ss}_\alpha(Q,\theta)$, starting from the 'worst' ones.

2. The controlling quiver Q_θ

We fix a quiver Q with vertices $Q_v = \{1,\ldots,k\}$ and fix a stability structure $\theta \in \mathbb{N}^{\oplus k}$. Let Σ_θ be the additive sub-monoid of $\mathbb{N}^{\oplus k}$ consisting of all dimension vectors α such that $\theta.\alpha = 0$. As the direct sum of two θ-semistable representations is again θ-semistable, we can consider in Σ_θ the additive sub-monoid V_θ consisting of those $\alpha \in \Sigma_\theta$ such that there exist θ-semistable representations of Q of dimension vector α. In [16] an inductive procedure is given to determine V_θ. Let $\{\gamma_1,\ldots,\gamma_l\}$ be a minimal set of additive monoid generators of V_θ. Such generating dimension vectors γ have special properties:

Lemma 1. *With notations as above we have:*

1. *γ is in a minimal generator set of V_θ if and only if every θ-semistable Q-representation is actually θ-stable.*
2. *γ is in a minimal generator set of V_θ if and only if the GIT-quotient*

$$\mathtt{rep}^{ss}_\alpha(Q) \twoheadrightarrow \mathtt{mod}^{ss}_\alpha(Q,\theta)$$

   is a principal PGL_n-fibration in the étale topology, where $\mathtt{rep}^{ss}_\alpha(Q)$ is the Zariski open subset of $\mathtt{rep}_\alpha(Q)$ consisting of θ-semistable representations.
3. *If γ is in a minimal generator set of V_θ then the moduli space $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ is smooth of dimension $1-\chi_Q(\gamma,\gamma)$.*

Proof. Every θ-semistable representation M has a filtration by θ-semistable subrepresentations such that all filtration quotients $N_i = Fi+1/F_i$ are θ-stable. It follows that the θ-semistable representation $N = \oplus_i N_i$ lies in the closure of the $GL(\alpha)$-orbit of M and if the dimension vector of the θ-stable factor N_i is β_i then clearly $\alpha = \sum_i \beta_i$. (1) follows from this, as does (2) using the fact that the stabilizer subgroup of a θ-stable representation is $\mathbb{C}^*$. (3) follows from (2) and the fact that $\mathtt{rep}^{ss}_\alpha(Q)$ is a non-empty Zariski open subset of the affine space $\mathtt{rep}_\alpha(Q)$ is smooth. □

Q_θ will then be the quiver with vertices $\{1,\ldots,l\}$ where vertex $\circ_i$ corresponds to the generator γ_i. In Q_θ the number of directed arrows (or loops) from vertex $\circ_i$ to vertex $\circ_j$ will be equal to $\delta_{ij} - \chi_Q(\gamma_i,\gamma_j)$. A first use of Q_θ lies in the characterization of θ-stable dimension vectors:

Lemma 2. *The following are equivalent*

1. *There exists a θ-stable representation of dimension vector α*
2. *We can write $\alpha = \sum_{i=1}^l c_i\gamma_i$ where $\gamma = (c_1,\ldots,c_l)$ is the dimension vector of a simple representation of Q_θ.*

Moreover, in this case we have $\chi_Q(\alpha,\alpha) = \chi_{Q_\theta}(\gamma,\gamma)$.

Proof. This is a direct consequence of [1, Thm. 5.1] and the fact that $\{\gamma_1,\ldots,\gamma_l\}$ generate V_θ. The conclusion follows because the dimension of $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ is in this case equal to $1-\chi_Q(\alpha,\alpha)$ and, on the other hand, the quiver setting (Q_θ,γ) is the local quiver encoding the étale local structure of $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ near a point

$$N = N_1^{\oplus c_1} \oplus \ldots \oplus N_l^{\oplus l}$$

where N_i is a θ-stable representation of dimension vector γ_i. As γ is a simple dimension vector for Q_θ the dimension of the quiver quotient variety $\mathtt{rep}_\gamma(Q_\theta)/GL(\gamma)$ is equal to $1-\chi_{Q_\theta}(\gamma,\gamma)$. □

The main purpose of the auxiliary quiver Q_θ is that it controls all local quiver settings (Q_M,α_M) describing the étale local structure of all moduli spaces $\mathtt{mod}^{ss}_\alpha(Q,\theta)$:

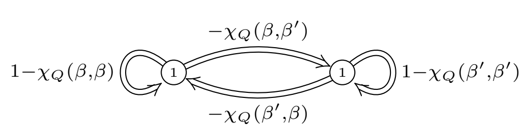
Theorem 1. *The quiver Q_θ contains enough information to construct the local quiver setting (Q_M,α_M) describing the étale local structure of the quiver moduli space $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ near the point corresponding to*

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_u^{\oplus e_u}$$

where the S_i are non-isomorphic θ-stable representations of dimension vector β_i. More precisely, if N and N' are θ-stable representations of dimension vectors β and β' and if we can write $\beta = \sum_i c_i\gamma_i$ and $\beta' = \sum_i c_i'\gamma_i$ with $\gamma = (c_1,\ldots,c_l)$ and $\gamma' = (c_1',\ldots,c_l')$ simple dimension vectors of Q_θ then we have

$$\chi_Q(\beta,\beta') = \chi_{Q_\theta}(\gamma,\gamma')$$

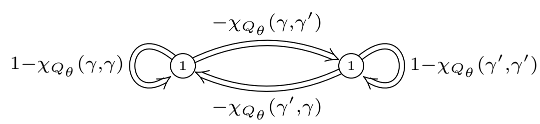
Proof. If $N \simeq N'$ (and hence $\beta = \beta'$ and $\gamma = \gamma'$) the claim follows from the previous lemma. So, assume that $N \not\simeq N'$, then the local quiver setting describing the étale local neighborhood of $\mathtt{mod}^{ss}_{\beta+\beta'}(Q,\theta)$ near $N \oplus N'$ is



On the other hand, $(Q_\theta, \gamma+\gamma')$ is the local quiver describing the étale local structure of $\mathtt{mod}^{ss}_{\beta+\beta'}(Q,\theta)$ near $P \oplus Q$ where

$$P = N_1^{\oplus c_1} \oplus \ldots \oplus N_l^{\oplus c_l} \quad \text{and} \quad Q = N_1^{\oplus c'_1} \oplus \ldots \oplus N_l^{\oplus c'_l}$$

By the previous lemma there are simple Q_θ-representations S and T of dimension vector γ and γ' such that the semi-simple Q_θ-representation $S \oplus T$ lies in a Zariski neighborhood of $\overline{0}$ in $\mathtt{rep}_{\gamma+\gamma'}(Q_\theta)/GL(\gamma+\gamma')$ and hence corresponds to a point in $\mathtt{mod}^{ss}_{\beta+\beta'}(Q,\theta)$ corresponding to a representation $N_1 \oplus N'_1$ with N_1 and N'_1 both θ-stable and of dimension vectors β and β'. By the theory of local quivers of semi-simple quiver representations as in [10] the étale local structure of the quiver quotient variety near $S \oplus T$ is determined by the local quiver setting



As local quivers-settings only depend on the representation type, this quiver must be the same as the one of $N \oplus N'$ finishing the proof. □

From now on we will restrict attention to the study of moduli spaces $\mathtt{mod}^{ss}_\theta(Q,\theta)$ for dimension vectors α allowing θ-stable representations. The general case reduces to this by the theory of general representations developed in [16].

3. Bocklandt's reduction steps

With `simps` we denote the set of all simple quiver settings, that is, all couples (Q,α) consisting of a quiver Q and dimension vector $\alpha = (a_1,\ldots,a_k)$ with all $a_i \neq 0$ (that is, the support *supp*(α) of α contains all vertices of Q), which satisfy (see [10]):

- Q must be strongly connected, meaning that there exist directed paths connecting any two of its vertices, and,

  $$\chi_Q(\alpha,\epsilon_i) \leq 0 \quad \text{and} \quad \chi_Q(\epsilon_i,\alpha) \leq 0$$

  for all vertex dimension vectors ϵ_i. That is, in every vertex $\circ_i$ the total number of incoming (and outgoing) dimensions is greater than or equal to the vertex-dimension.
- If however $Q = \tilde{A}_k$ with cyclic orientation, then only $\alpha = (1,\ldots,1)$ is allowed.

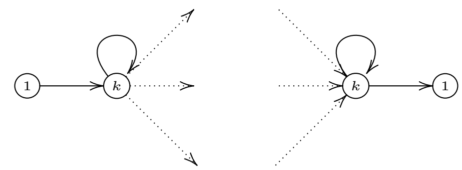
In verifying the numerical conditions it is practical to label each vertex with two numbers ≤ 0, giving the differences of the total incoming (resp outgoing) dimensions with the vertex-dimension. This allows us to spot quickly whether one of the reductions steps $(Q,\alpha) \twoheadrightarrow (Q',\alpha')$, discovered by Raf Bocklandt in his characterization of smooth quiver quotient varieties, can be applied [3]:

- If there is a vertex $\circ_i$ with vertex-dimension $a_i = 1$ having loops, remove the loops to get Q' and keep $\alpha' = \alpha$.
- If one of the two numbers for $\circ_i$ is zero, remove the vertex $\circ_i$ and cable all arrows through, that is, any situation

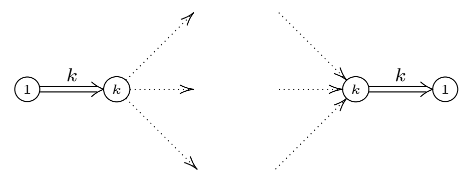
  $$\circ_k \xRightarrow{a} \circ_i \xRightarrow{b} \circ_l \quad \text{becomes} \quad \circ_k \xRightarrow{a \times b} \circ_l$$

  to obtain Q' and let α' be α with the i-th component removed.

- If there is a vertex $\circ_i$ having a unique loop and such that one of the two numbers is -1, then we are in one of the following local situations in $\circ_i$



we replace the loop in $\circ_i$ by a bunch of k arrows to or from ①, that is, locally Q' looks like



and keep $\alpha' = \alpha$.

In every reduction step we either decrease the number of vertices or the number of loops. So, after a finite number of moves we arrive at a simple quiver setting (Q^t, α^t) which cannot be reduced further. As there is an element of choice in the reduction steps we can perform, there is a priori no reason that any two reduction procedures should result in the same final setting. Still, surprisingly, this is the case as was proved in [4, §4]. We will call this unique irreducible simple quiver setting (Q^t, α^t) the type of (Q, α).

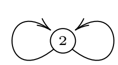
The upshot of this reduction process is the following result which follows from Bocklandt's work [3]:

Theorem 2. *For any* $(Q, \alpha) \in$ `simps` *with (unique) type* (Q^t, α^t) *there is an isomorphism of varieties*

$$\mathtt{rep}_\alpha(Q)/GL(\alpha) \simeq \mathtt{rep}_{\alpha^t}(Q^t)/GL(\alpha^t) \times \mathbb{C}^d$$

where $d = \chi_{Q^t}(\alpha^t, \alpha^t) - \chi_Q(\alpha, \alpha)$. *In particular, the corresponding quiver-quotient singularities are smoothly equivalent.*

In [3] Bocklandt proves that the quiver quotient variety $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ is smooth if and only if its type (Q^t, α^t) is either

or


4. The partially ordered set of `types`

We will put a partial order on the set `types` of all types of simple quiver settings. Take an irreducible simple quiver setting (Q, α) and look for loops $I = \{i\}$ or minimal oriented cycles $I = \{i_1, \ldots, i_v\}$ in it. Consider the dimension vector $\beta_I = (\delta_{1I}, \delta_{2I}, \ldots, \delta_{kI})$ then there exists a simple Q-representation S_I of dimension vector β_I. Now, consider the semi-simple Q-representation M of dimension vector $\alpha = (a_1, \ldots, a_k)$

$$M = S_I \oplus S_1^{\oplus a_1 - \delta_{1I}} \oplus S_2^{\oplus a_2 - \delta_{2I}} \oplus \ldots \oplus S_n^{\oplus a_n - \delta_{nI}}$$

where S_i is the 1-dimensional simple vertex-representation in $\circ_i$. Let (Q_M, α_M) be the local quiver setting associated to M as defined in [11]. In this case, Q_M is a quiver on $k+1$ vertices $\{0, 1, 2, \ldots, k\}$, with $\circ_0$ corresponding to the simple component S_I, such that $Q_M|\{1, \ldots, k\} \simeq Q$ and the number of loops in $\circ_0$ is given by $1 - \chi_Q(\beta_I, \beta_I)$ and the number of arrows from $\circ_0$ to $\circ_i$ (resp. from $\circ_i$ to $\circ_0$) is equal to $-\chi_Q(\beta_I, \epsilon_i)$ (resp. to $\chi_Q(\epsilon_i, \beta_I)$. The dimension vector $\alpha_M \in \mathbb{N}^{\oplus k+1}$ is determined by the multiplicities of the distinct simple factors in M, that is,

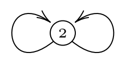
$$\alpha_M = (1, a_1 - \delta_{1I}, \ldots, a_k - \delta_{kI})$$

Next, let (Q'_M, α'_M) be the (necessarily simple) quiver setting obtained by restricting (Q_M, α_M) to the support of α_M. Finally, let (Q_I, α_I) be the type of (Q'_M, α'_M), then we say that (Q_I, α_I) is a direct successor of (Q, α) in `types` determined by the oriented cycle I and we denote this by an arrow

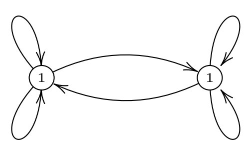
$$(Q, \alpha) \longrightarrow (Q_I, \alpha_I)$$

Composing such arrows then defines a partial order on `types`.

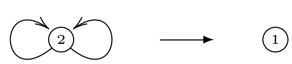
Example 1. *There is just one type of cycle (loop) $I = \{1\}$ for the type*



and the corresponding semi-simple representation M is the direct sum $S_I \oplus S_1$ of two distinct simple 1-dimensional representations, S_I has one of the loops non-zero, S_1 not. The corresponding local quiver setting is then



which has corresponding type $\textcircled{1}$*, that is, in* `types` *we have an arrow*



In fact, soon it will become apparent that $\textcircled{1}$ *is the unique minimal object in* `types`*.*

The upshot of this ordering is that it simplifies the singularity type as we move away from the worst singularity $\overline{0}$ in $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ to singularities at points in the first deformed strata.

Recall that points of the quiver quotient variety correspond to semi-simple representations of total dimension α, that is, representations of the form

$$N = T_1^{\oplus f_1} \oplus \ldots \oplus T_l^{\oplus f_l}$$

where all T_i are simple Q-representations of dimension vector β_i and such that $\sum_i f_i \beta_i = \alpha$. We then say that N is of representation type $\sigma(N) = (f_1, \beta_1; \ldots; f_l, \beta_l)$. The Luna stratification of $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ consists of strata $\mathtt{strata}(\sigma)$, consisting of points of the same representation type σ, which are all locally closed subvarieties. In fact, one can show that $\mathtt{strata}(\sigma)$ is contained in the Zariski closure of $\mathtt{strata}(\sigma')$ if and only if the stabilizer subgroup

$Stab(N_{\sigma'})$ is conjugated to a subgroup of $Stab(N_\sigma)$in $GL(\alpha)$ for $N_\sigma \in$ $\mathtt{strata}(\sigma)$ and $N_{\sigma'} \in$ $\mathtt{strata}(\sigma')$.

The point $\overline{0}$ is contained in the most degenerate stratum of representation type $\sigma_0 = (a_1, \epsilon_1; \ldots; a_n, \epsilon_n)$ if $\alpha = (a_1, \ldots, a_n)$. In the Hasse diagram of Luna strata, the strata of minimal dimension $\mathtt{strata}(\sigma)$ containing $\mathtt{strata}(\sigma_0)$ in their Zariski closure are exactly those with representation type of the form

$$\sigma_I = (1, \beta_U; a_1 - \delta_{1I}, \epsilon_1; \ldots; a_n - \delta_{nI}, \epsilon_n)$$

corresponding to a loop or a minimal proper oriented cycle I in Q. The theory of local quivers, see for example [11], then asserts that the étale local structure of $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ in a neighborhood of $M \in$ $\mathtt{strata}(\sigma)$ is isomorphic to an étale local neighborhood of $\overline{0}$ in the quiver quotient variety $\mathtt{rep}_{\alpha_M}(Q_M)/GL(\alpha_M)$.

Combining this with theorem 2 we get the first assertion of the following result:

Theorem 3. *For* $(Q, \alpha) \in$ $\mathtt{types}$ *we have:*

1. *If* $(Q, \alpha) \longrightarrow (Q', \alpha')$ *then any Zariski neighborhood of* $\overline{0}$ *in the quotient variety* $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ *contains points smoothly equivalent with type* (Q', α').
2. *Every singularity of* $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ *not of type* (Q, α) *is smoothly equivalent to a singularity contained in* $\mathtt{rep}_{\alpha'}(Q')/GL(\alpha')$ *for some type* (Q', α') *such that* $(Q, \alpha) \geq (Q', \alpha')$.

Proof. As for the second assertion, recall that étale singularity types of quiver quotient varieties on it depend on their representation type and as $\overline{0}$ lies in the Zariski closure of any representation stratum, we have that any Zariski neighborhood of $\overline{0}$ contains points of all occurring étale singularity types. Now, take such a singularity type τ with associated representation type σ and consider a representation type σ' having a stratum of minimal dimension such that

$$\mathtt{strata}(\sigma_0) \subsetneq \mathtt{strata}(\sigma') \subset \overline{\mathtt{strata}(\sigma)}$$

then $\sigma' = \sigma_I$ for some loop or minimal oriented cycle I in Q. Any point in $\mathtt{strata}(\sigma)$ is of type τ and as $\mathtt{strata}(\sigma_I)$ is contained in its Zariski closure, the theory of local quivers entails that any Zariski neighborhood of $\overline{0}$ in $\mathtt{rep}_{\alpha_M}(Q_M)/GL(\alpha_M)$, for M of type σ_I, contains points étale of type τ. As $\mathtt{rep}_{\alpha_M}(Q_M)/GL(\alpha_M)$ and $\mathtt{rep}_{\alpha'}(Q')/GL(\alpha')$ are smoothly isomorphic there are singularities in $\mathtt{rep}_{\alpha'}(Q')$ étale smoothly equivalent to τ, finishing the proof. □

This then gives an algorithm to determine all singularity types of quiver quotient varieties up to smooth equivalence.

Theorem 4. *Let* $(Q, \alpha) \in$ $\mathtt{simps}$ *and apply reduction steps to determine is type* $(Q^t, \alpha^t) \in$ $\mathtt{types}$. *Then, the singularity types of points in the quiver quotient variety* $\mathtt{rep}_\alpha(Q)/GL(\alpha)$ *are, up to smooth equivalence, exactly those* $(Q', \alpha') \in$ $\mathtt{types}$ *such that* $(Q^t, \alpha^t) \geq (Q', \alpha')$.

Note that different types may still be smoothly equivalent. For example, in [4] we showed that types 5_{3a} and type 5_{4c} have isomorphic rings of invariants. Further, the locus of all points in $\mathtt{rep}^{ss}_\alpha(Q)/GL(\alpha)$ consisting of points of a specific type may consist of several strata $\mathtt{strata}(\sigma)$, even of varying dimensions.

5. Hitchhiker's guide to `types`

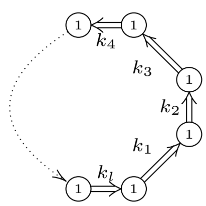
In principle one can build a map of the partial ordered set `types`, inductively by dimension of the quotient variety, and by number of the vertices in the quiver. If the dimension is D and the number of vertices is n we will enumerate all possible types (Q,α) as $D_{na}, D_{nb}, \ldots$. The quiver Q is determined by the integral matrix M_Q describing its Ringel bilinear form χ_Q, and the condition that $(Q,\alpha) \in$ `types` of dimension D can then be expressed as a system of equations and inequalities involving the entries of M_Q and α, starting with

$$1 - \chi_Q(\alpha,\alpha) = D \qquad \chi_Q(\alpha,\epsilon_i) \leq 0 \qquad \chi_Q(\epsilon_i,\alpha) \leq 0$$

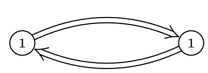
followed by relations expressing that none of the Bocklandt's reduction steps are possible for (Q,α). This then produces a list of all types of dimension D and we have to relate them to the already constructed poset of types of dimension $< D$.

This involves computing local quiver settings for representation types σ_I corresponding to loops or minimal oriented cycles I in the quiver Q. As there will be at least one loop in this local quiver in the vertex $\circ_0$, having vertex-dimension 1, we see that by going to its associated type we drop he dimension of the quotient-variety by at least one. That is, we will only have to draw red arrows connecting the new types of dimension D to types already constructed before.

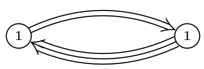
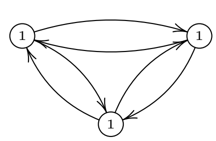
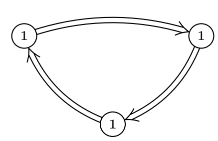
We can also describe easily, for all dimensions D, the types having only an arrow to the unique minimal element ①. However, we will not draw this arrow so that these types become sinks in the map. This happens when the corresponding quotient variety is an isolated singularity and those quiver-quotients have been classified in [5] to be of the form



where Q has l vertices and all $k_i \geq 2$. The resulting dimension is then $D = \sum_i k_i + l - 1$. In [4] all types of dimension $D \leq 6$ have been classified. The first dimension allowing a quiver-quotient singularity is $D = 3$ and there is just one such type, corresponding to the conifold singularity 3_c



In dimension $D = 4$ there are exactly three types

4_2 :  4_{3a} :  4_{3b} : 

Dimension $D = 5$ adds 11 types and in dimension $D = 6$ we get an additional 54 types, see [4] for all details. In [4] these types where then classified up to isomorphism by a method called 'fingerprinting' singularities of which our partial

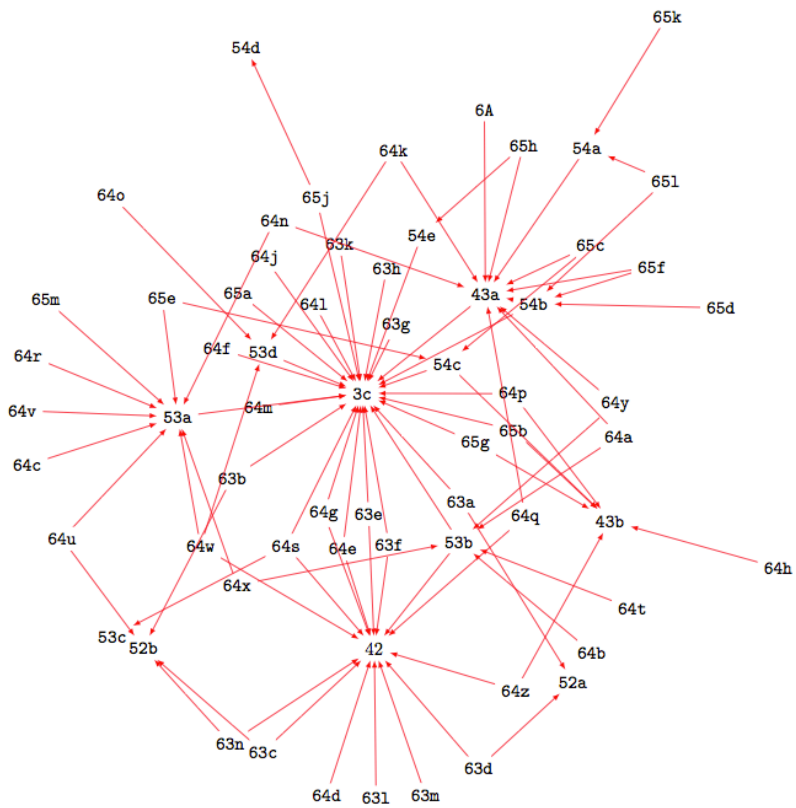


Figure 1. Hitchhiker's guide to `types`

order on types is a scaled down version. Using the enumeration of types as given in [4] we can then draw the poset `types` up to dimension $D = 6$ as in Figure 1. The isolated quiver-singularities in dimension 6 are not included.

6. Singularities of quiver moduli spaces

We are now in a position to describe more efficiently the types of singularities that occur in $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ up to smooth equivalence:

- For fixed dimension vector α there is a limited number of possible simple dimension vectors $\{\gamma_1,\ldots,\gamma_v\}$ of the controlling quiver Q_θ as in lemma 2.
- For $1 \leq i \leq v$ apply Bocklandt's reduction steps to obtain the type (Q^t_θ, γ^t_i) of $(Q_\theta | supp(\gamma_i), \gamma_i)$.
- Up to smooth equivalence, the types of singularities that occur in $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ are precisely those (Q',α') such that

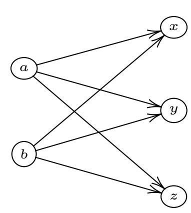
$$(Q^t_\theta, \gamma^t_i) \geq (Q',\alpha')$$

  for at least one $1 \leq i \leq v$.

In particular, this allows us to characterize for a given quiver Q and stability structure θ all the moduli spaces $\mathtt{mod}^{ss}_\alpha(Q,\theta)$ which are smooth, as those for which all types (Q^t_θ, γ^t_i) are either

We will illustrate the above by some examples. First, we will determine the controlling quiver Q_θ relevant in the study of representations of the modular group. Then we will illustrate how one can inductively extend on Figure 1, and, finally we will give a short proof of the classification of all smooth quiver moduli spaces in this case.

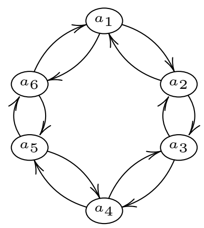
Example 2. *Consider the quiver Q below and stability structure $\theta = (-1,-1;1,1,1)$ that appears naturally in the study of character varieties of the modular group $\Gamma = PSL_2(\mathbb{Z})$, see* [14].



then simple Γ-representations of dimension $n = a+b = x+y+z$ determine θ-stable representations of dimension vector $\alpha = (a,b;x,y,z)$, which then must satisfy $min(a,b) \geq max(x,y,z)$. In this case, the monoid V_θ is generated by the six dimension vectors

$$\begin{cases} \gamma_1 = (1,0;1,0,0) & \gamma_2 = (0,1;0,1,0) & \gamma_3 = (1,0;0,0,1) \\ \gamma_4 = (0,1;1,0,0) & \gamma_5 = (1,0;0,1,0) & \gamma_6 = (0,1;0,0,1) \end{cases}$$

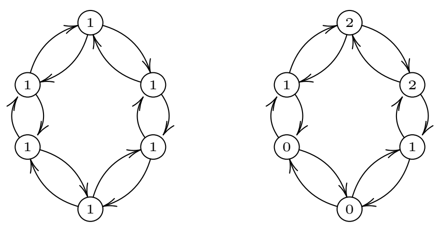
which correspond to the six one-dimensional representations of $\Gamma = C_2 * C_3$*. In this case the quiver* Q_θ *is*



and a dimension vector $\alpha_\theta = (a_1, \ldots, a_6)$ *corresponds to* α *if and only if*

$$\begin{cases} a = a_1 + a_3 + a_5 \quad b = a_2 + a_4 + a_6 \\ x = a_1 + a_4 \quad y = a_2 + a_5 \quad z = a_3 + a_6 \end{cases}$$

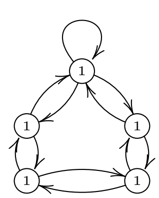
Example 3. *We will determine the singularity types occurring in the (unique) singular moduli space of smallest possible dimension, which is* 7*, corresponding to dimension vector* $\alpha = (3, 3; 2, 2, 2)$*. Up to hexagonal symmetry there are two corresponding simple dimension vectors* $\alpha_\theta^{(1)}, \alpha_\theta^{(2)}$



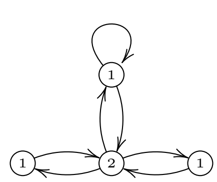
Neither of these quiver settings can be further reduced so they determine two new types, let us call them resp. 7_{6a} *and* 7_{4a} *(recall that the first subindex gives the number of vertices of the quiver). Next, we have to connect them to the map of Figure 1.*

We have to determine the local quivers associated to minimal oriented cycles in these two quivers. For type 7_{6a} *there is up to symmetry just one such cycle, namely between two consecutive vertices. For type* 7_{4a} *we have up to symmetry two possible cycles, either containing an ending vertex or between the two middle vertices.*

The corresponding local quiver for type 7_{6a} *and the second possibility for type* 7_{4a} *are both of the form*



and after reducing the loop we obtain type 6_{5k} *of the classification from* [4]*. The first possibility for type* 7_{4a} *gives as local quiver*

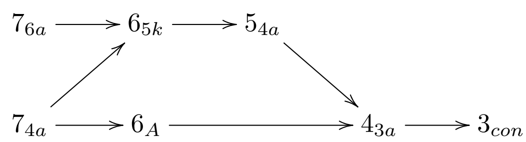


and, after reducing the loop, we obtain type 6_A *from* [4].

Both type 6_{5k} *and* 6_A *are already on our map (upper right hand), so we obtain that up to smooth equivalence the singularities of* $\mathtt{mod}^{ss}_{\alpha}(Q,\theta)$ *are exactly of the types*

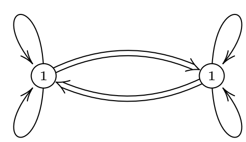
$$7_{6a},\ 7_{4a},\ 6_{5k},\ 6_A,\ 5_{4a},\ 4_{3a},\ 3_{con}$$

with degeneration diagram



This calculation illustrates the iterative process involved to extend on Figure 1 in a specific problem.

Example 4. *Next, let us see how to apply the foregoing in order to classify the smooth quiver moduli spaces. Before, we have seen that for* $\alpha = (3,3;2,2,2)$ *in the moduli space there should be singularities smoothly equivalent to the conifold singularity. Indeed, if we take representations of the form* $M = N \oplus N'$ *where* N *(resp.* N'*) is* θ*-stable of dimension vector* $(2,1;1,1,1)$ *(resp.* $(1,2;1,1,1)$*) then the corresponding local quiver is*



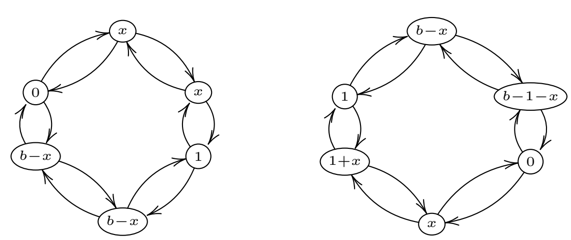
Now, if β *is a strictly larger dimension-vector (meaning that all its vertex dimensions are greater or equal than those of* α*) we can write*

$$\beta = \alpha + \sum_{i=1}^{6} n_i \gamma_i$$

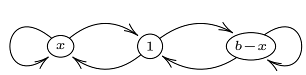
and calculating the local quiver of this representation type shows that we can never reduce to a smooth setting. Hence, all moduli spaces for dimension vectors larger than α *will be singular. The remaining dimension vectors are either Dynkin or extended Dynkin (and hence have smooth moduli spaces) or of the form*

$$(b+1,b;b,b,1), \quad (b,b;b,b-1,1) \quad \text{or} \quad (4,2;2,2,2)$$

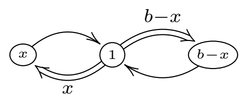
In the first case, there are (up to symmetry) two possible families of simple dimension vectors for Q_θ*, namely*



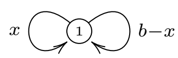
The first one we can reduce to



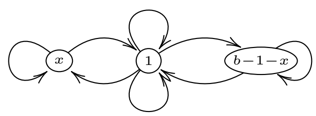
and subsequently to



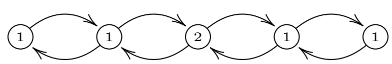
and finally to



which is of type $\textcircled{1}$ *whence smooth. The second one first reduces to*



which after reducing the two loops in the middle vertex is of the same type as the first reduction of the first case, so is again smooth. For the dimension vectors $(b, b; b, b-1, 1)$ *the argument is similar. As for the special dimension vector* $\alpha = (4, 2; 2, 2, 2)$ *here the corresponding quiver-setting* $(Q_\theta, \alpha_\theta)$ *is*



which is easily seen to reduce to $\textcircled{1}$*, whence is smooth.*

Concluding, we have the following characterization of all smooth moduli spaces for the above quiver Q and stability structure $\theta = (-1, -1; 1, 1, 1)$, see also [2]:

Theorem 5. $\texttt{mod}^{ss}_{\alpha}(Q, \theta)$ *is smooth unless all vertex-dimensions of* β *are greater or equal than those of* $\beta = (3, 3; 2, 2, 2)$*.*

Observe from [15] that these are exactly the components on which transposition induces the identity.

7. Compact noncommutative manifolds

If the quiver Q has no oriented cycles all moduli spaces $\texttt{mod}^{ss}_{\theta}(Q, \theta)$ are projective varieties and in [12] it is argued that one can view the family of projective varieties

$$(\bigsqcup_{d(\alpha)=n} \texttt{mod}^{ss}_{\alpha}(Q, \theta) \ )_n$$

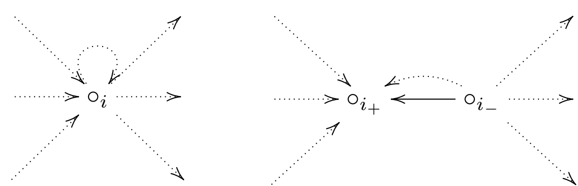
as a noncommutative compact manifold. That is, the additive category $\texttt{rep}^{ss}(Q, \theta)$ of all finite dimensional θ-stable representations of Q can be covered by representation categories $\texttt{rep}(A)$ consisting of all finite dimensional representations of formally smooth algebras A. We have seen that the singularities of the noncommutative compact manifold, as well as all local quiver settings describing its étale local quiver, are controlled by the quiver Q_θ. Conversely, we have

Theorem 6. *For every quiver $Q^\dagger$ there exist noncommutative compact manifolds of the form*

$$(\bigsqcup_{d(\alpha)=n} \mathtt{mod}^{ss}_{\alpha}(Q,\theta) \ )_n$$

with Q having no oriented cycles such that all local quiver settings are controlled by $Q^\dagger$.

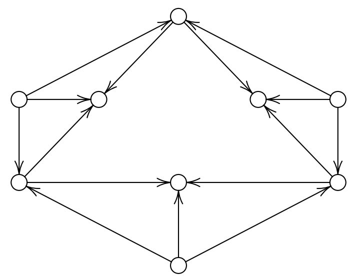
Proof. We start with $Q_0 = Q^\dagger$ and stability structure $\theta_0 = (0,\ldots,0)$. We use the trick of iterating the procedure of doubling vertices, see [7, §2], to remove all loops and oriented cycles in and to modify the stability structure accordingly. That is, if after k steps we have arrived at a situation (Q_k,θ_k) such that all local quiver settings for moduli spaces of θ_k-semistable representations are controlled by $Q^\dagger$ and if we still have a vertex $\circ_i$ in Q_k having loops or oriented cycles passing through it, and if the i-th θ_k-component is t_i, then we modify the situation by splitting the vertex in two vertices $\circ_{i_-}$ and $\circ_{i_+}$ and adjusting loops and arrows starting or ending in $\circ_i$ as indicated below



to get a new quiver Q_{k+1} and new stability structure θ_{k+1} which coincides with θ_k in all non-modified vertices and is equal to $-n$ in $\circ_{i_-}$ and equal to t_i+n in $\circ_{i_+}$, where n is chosen large enough to ensure that all local quiver settings for moduli spaces of θ_{k+1}-semistable representations of Q_{k+1} are controlled by $Q^\dagger$. Note that if we have a dimension vector α_k allowing θ_k-semistable representation, then the dimension vector α_{k+1}, which coincides with α_k in all non-modified vertices and with the α_k-component in $\circ_i$ in the new vertices $\circ_{i_-}$ and $\circ_{i_+}$, will allow θ_{k+1}-semistable representations. By [7, Thm. 2.2] this is always possible. In fact, we can even choose n such that the local quiver settings for $\mathtt{mod}^{ss}_{\alpha_k}(Q_k,\theta_k)$ are exactly local quiver settings for $\mathtt{mod}^{ss}_{\alpha_{k+1}}(Q_{k+1},\theta_{k+1})$. After a finite number of steps we obtain a quiver Q having no loops nor oriented cycles and a stability structure θ such that the corresponding noncommutative compact manifold is controlled by $Q^\dagger$. □

As is clear from the foregoing proof, the same quiver $Q^\dagger$ can control a large array of compact noncommutative manifolds, which can be quite different.

Example 5. *Take as quiver $Q^\dagger = Q_\theta$ of example 2 controlling the noncommutative compactification of the modular group. This quiver also controls the noncommutative compact manifold defined by the quiver Q' below*



and stability structure (with cyclic ordering op vertices of $Q^\dagger$ and split vertices as consecutive entries)

$$\theta' = (0; -p, p; 0; -q, q; 0; -r, r)$$

where p, q and r are sufficiently large primes. To a simple dimension vector $\alpha_\dagger = (a_1, a_2, a_3, a_4, a_5, a_6)$ of $Q^\dagger$ there is a unique dimension vector

$$\alpha' = (a_1; a_2, a_2; a_3; a_4, a_4; a_5; a_6, a_6)$$

of Q' allowing θ'-stable representations. By [7, Thm. 2.2] *the local quiver settings for the moduli space $\texttt{mod}^{ss}_{\alpha'}(Q', \theta')$ are exactly the same as those of the quiver quotient-variety $\texttt{rep}_{\alpha_\dagger}(Q^\dagger)/GL(\alpha_\dagger)$.*

On the other hand, for the quiver Q of example 2 and stability structure $\theta = (-1, -1; 1, 1, 1)$ the local quivers for the moduli space $\texttt{mod}^{ss}_{\alpha}(Q, \theta)$ for a dimension vector α allowing θ-stable representations are, in general, determined by those of several simple dimension vectors $\alpha_\dagger$ of $Q^\dagger$ as example 3 illustrates.

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