THE GEOMETRY OF REPRESENTATIONS OF 3-DIMENSIONAL SKLYANIN ALGEBRAS

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Abstract. The representation scheme $\mathtt{rep}_n\ A$ of the 3-dimensional Sklyanin algebra A associated to a plane elliptic curve and n-torsion point contains singularities over the augmentation ideal $\mathfrak{m}$. We investigate the semi-stable representations of the noncommutative blow-up algebra $B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2 t^2 \oplus \ldots$ to obtain a partial resolution of the central singularity

$$\mathtt{proj}\ Z(B) \twoheadrightarrow \mathtt{spec}\ Z(A)$$

such that the remaining singularities in the exceptional fiber determine an elliptic curve and are all of type $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$.

1. Introduction

Three dimensional Sklyanin algebras appear in the classification by M. Artin and W. Schelter [2] of graded algebras of global dimension 3. In the early 90ties this class of algebras was studied extensively by means of noncommutative projective algebraic geometry, see a.o. [3], [4], [5], [9] and [15]. Renewed interest in this class of algebras arose recently as they are superpotential algebras and as such relevant in supersymmetric quantum field theories, see a.o. [6] and [16].

Consider a smooth elliptic curve E in Hesse normal form $\mathbb{V}((a^3+b^3+c^3)XYZ - abc(X^3+Y^3+Z^3)) \hookrightarrow \mathbb{P}^2$ and the point $p = [a:b:c]$ on E. The 3-dimensional Sklyanin algebra A corresponding to the pair (E,p) is the noncommutative algebra with defining equations

$$\begin{cases} axy + byx + cz^2 = 0 \\ ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \end{cases}$$

The connection comes from the fact that the multi-linearization of these equations defines a closed subscheme in $\mathbb{P}^2 \times \mathbb{P}^2$ which is the graph of translation by p on the elliptic curve E, see [4]. Alternatively, one obtains the defining equations of A from the superpotential $W = axyz + byzx + \frac{c}{3}(x^3+y^3+z^3)$, see [16].

The algebra A has a central element of degree 3, found by computer search in [2]

$$c_3 = c(a^3-c^3)x^3 + a(b^3-c^3)xyz + b(c^3-a^3)yxz + c(c^3-b^3)y^3$$

with the property that $A/(c_3)$ is the twisted coordinate ring of the elliptic curve E with respect to the automorphism given by translation by p, see [4]. We will prove an intrinsic description of this central element, answering a MathOverflow question [8].

Theorem 1. *The central element c_3 of the 3-dimensional Sklyanin algebra A corresponding to the pair (E,p) can be written as*

$$c(a^3-b^3)(xyz+yzx+zxy)+b(c^3-a^3)(yxz+xzy+zyx)+c(a^3-b^3)(x^3+y^3+zx^3)$$

and is the superpotential of the 3-dimensional Sklyanin algebra A' corresponding to the pair $(E,[-2]p)$.

Next, we turn to the study of finite dimensional representations of A which is important in supersymmetric gauge theory as they correspond to the vacua states. It is well known that A is a finite module over its center $Z(A)$ and a maximal order in a central simple algebra of dimension n^2 if and only if the point p is of finite order n, see [4]. We will further assume that $(n,3)=1$ in which case J. Tate and P. Smith proved in [15] that the center $Z(A)$ is generated by c_3 and the reduced norms of x,y and z (which are three degree n elements, say x',y',z') satisfying one relation of the form

$$c_3^n = cubic(x',y',z')$$

It is also known that $\texttt{proj}\ Z(A) \simeq \mathbb{P}^2$ with coordinates $[x':y':z']$ in which the $cubic(x',y',z')$ defines the isogenous elliptic curve $E'=E/\langle p\rangle$, see a.o. [9]. We will use these facts to give explicit matrices for the simple n-dimensional representations of A and show that A is an Azumaya algebra away from the isolated central singularity.

However, the scheme $\texttt{rep}_n\ A$ of all (trace preserving) n-dimensional representations of A contains singularities in the nullcone. We then try to resolve these representation singularities by considering the noncommutative analogue of a blow-up algebra

$$B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \ldots \subset A[t,t^{-1}]$$

where $\mathfrak{m}=(x,y,z)$ is the augmentation ideal of A. We will prove

Theorem 2. *The scheme $\texttt{rep}_n^{ss}\ B$ of all semi-stable n-dimensional representations of the blow-up algebra B is a smooth variety.*

This allows us to compute all the (graded) local quivers in the closed orbits of $\texttt{rep}_n^{ss}\ B$ as in [10] and [7]. This information then leads to the main result of this paper which gives a partial resolution of the central isolated singularity.

Theorem 3. *The exceptional fiber $\mathbb{P}^2$ of the canonical map*

$$\texttt{proj}\ Z(B) \twoheadrightarrow \texttt{spec}\ Z(A)$$

contains $E'=E/\langle p\rangle$ as the singular locus of $\texttt{proj}\ Z(B)$. Moreover, all these singularities are of type $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_n$ with $\mathbb{C}^2/\mathbb{Z}_n$ an Abelian quotient surface singularity.

2. Central elements and superpotentials

The finite Heisenberg group of order 27

$$\langle\ u,v,w \mid [u,v]=w,\ [u,w]=[v,w]=1,\ u^3=v^3=w^3=1\ \rangle$$

has a 3-dimensional irreducible representation $V=\mathbb{C}x+\mathbb{C}y+\mathbb{C}z$ given by the action

$$u \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad v \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{bmatrix} \quad w \mapsto \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{bmatrix}$$

One verifies that $V \otimes V$ decomposes as three copies of V^*, that is,

$$V \otimes V \simeq \wedge^2(V) \oplus S^2(V) \simeq V^* \oplus (V^* \oplus V^*)$$

where the three copies can be taken to be the subspaces

$$\begin{cases} V_1 = \mathbb{C}(yz - zy) + \mathbb{C}(zx - xz) + \mathbb{C}(xy - yx) \\ V_2 = \mathbb{C}(yz + zy) + \mathbb{C}(zx + xz) + \mathbb{C}(xy + yx) \\ V_3 = \mathbb{C}x^2 + \mathbb{C}y^2 + \mathbb{C}z^2 \end{cases}$$

Taking the quotient of $\mathbb{C}\langle x, y, z\rangle$ modulo the ideal generated by $V_1 = \wedge^2 V$ gives the commutative polynomial ring $\mathbb{C}[x, y, z]$. Hence we can find analogues of the polynomial ring in three variables by dividing $\mathbb{C}\langle x, y, z\rangle$ modulo the ideal generated by another copy of V^* in $V \otimes V$ and the resulting algebra will inherit an action by H_3. Such a copy of V^* exists for all $[A : B : C] \in \mathbb{P}^2$ and is spanned by the three vectors

$$\begin{cases} A(yz - zy) + B(yz + zy) + Cx^2 \\ A(zx - xz) + B(zx + xz) + Cy^2 \\ A(xy - yx) + B(xy + yx) + Cz^2 \end{cases}$$

and by taking $a = A + B, b = B - A$ and $c = C$ we obtain the defining relations of the 3-dimensional Sklyanin algebra. In particular there is an H_3-action on A and the canonical central element c_3 of degree 3 must be a 1-dimensional representation of H_3. It is obvious that c_3 is fixed by the action of v and a minor calculation shows that c_3 is also fixed by u. Therefore, the central element c_3 given above, or rather $3c_3$, can also be represented as

$$a(b^3 - c^3)(xyz + yzx + zxy) + b(c^3 - a^3)(yxz + xzy + zyx) + c(a^3 - b^3)(x^3 + y^3 + z^3)$$

Now, let us reconsider the superpotential $W = axyz + byxz + \frac{c}{3}(x^3 + y^3 + z^3)$ for a $[a : b : c] \in \mathbb{P}^2$. This superpotential gives us three quadratic relations by taking cyclic derivatives with respect to the variables

$$\begin{cases} \partial_x W = ayz + bzy + cx^2 \\ \partial_y W = azx + bxz + cy^2 \\ \partial_z W = axy + byx + cz^2 \end{cases}$$

giving us the defining relations of the 3-dimensional Sklyanin algebra. We obtain the same equations by considering a more symmetric form of W, or rather of $3W$

$$a(xyz + yzx + zxy) + b(yxz + xzy + zyx) + c(x^3 + y^3 + z^3)$$

We see that the form of the central degree 3 element and of the superpotential are similar but with different coefficients. This means that the central element is the superpotential defining another 3-dimensional Sklyanin algebra and theorem 1 clarifies this connection.

Proof of Theorem 1 : The 3-dimensional Sklyanin algebra determined by the superpotential $3c_3$ is determined by $[a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)]$ (instead of $[a : b : c]$ for the original). Therefore, the associated elliptic curve has defining Hesse equation

$$\mathbb{V}(\alpha(x^3 + y^3 + z^3) - \beta xyz) \hookrightarrow \mathbb{P}^2$$

where

$$\begin{cases} \alpha = a(b^3 - c^3)b(c^3 - a^3)c(a^3 - b^3) \\ \beta = (a(b^3 - c^3))^3 + (b(c^3 - a^3))^3 + (c(a^3 - b^3))^3 \end{cases}$$

but this is the same equation, upto a scalar, as the original curve

$$E = \mathbb{V}(abc(x^3 + y^3 + c^3) - (a^3 + b^3 + c^3)xyz)$$

The tangent line to E in the point $p = [a : b : c]$ has equation

$$\mathbb{V}((2a^3bc - b^4c - bc^4)(x - a) + (2ab^3c - a^4c - ac^4)(y - b) + (2abc^3 - a^4b - ab^4)(z - c))$$

and so the third point of intersection is

$$[-2]p = [a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)]$$

which are the parameters of the algebra. □

3. Resolving representation singularities

Let R be a graded $\mathbb{C}$-algebra, generated by finitely many elements $x_1, \ldots, x_m$ where $deg(x_i) = d_i \geq 0$, which is a finite module over its center $Z(R)$. Following [14] we say that R is a Cayley-Hamilton algebra of degree n if there is a $Z(R)$-linear gradation preserving trace map $tr : R \longrightarrow Z(R)$ such that for all $a, b \in R$ we have

- $tr(ab) = tr(ba)$
- $tr(1) = n$
- $\chi_{n,a}(a) = 0$

where $\chi_{n,a}(t)$ is the n-th Cayley-Hamilton identity expressed in the traces of powers of a. Maximal orders in a central simple algebra of dimension n^2 are examples of Cayley-Hamilton algebras of degree n.

In particular, a 3-dimensional Sklyanin algebra A associated to a couple (E, p) where p is a torsion point of order n, and the corresponding blow-up algebra $B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \ldots$ are affine graded Cayley-Hamilton algebras of degree n equipped with the (gradation preserving) reduced trace map.

If R is an affine graded Cayley-Hamilton algebra of degree n we define $\mathtt{rep}_n\ R$ to be the affine scheme of all n-dimensional trace preserving representations, that is of all algebra morphisms

$$R \xrightarrow{\phi} M_n(\mathbb{C}) \qquad \text{such that} \qquad \forall a \in R \ : \ \phi(tr(a)) = Tr(\phi(a))$$

where Tr is the usual trace map on $M_n(\mathbb{C})$. Isomorphism of representations defines a $\mathtt{GL}_n$-action of $\mathtt{rep}_n\ R$ and a result of Artin's [1] asserts that the closed orbits under this action, that is the points of the GIT-quotient scheme $\mathtt{rep}_n\ R//\mathtt{GL}_n$, are precisely the isomorphism classes of n-dimensional trace preserving semi-simple representations of R. The reconstruction result of Procesi [14] asserts that in this setting

$$\mathtt{spec}\ Z(R) \simeq \mathtt{rep}_n\ R//\mathtt{GL}_n$$

The gradation on R defines an additional $\mathbb{C}^*$-action on $\mathtt{rep}_n\ R$ commuting with the $\mathtt{GL}_n$-action. With $\mathtt{rep}_n^{ss}\ R$ we denote the Zariski open subset of all semi-stable trace preserving representations $\phi : R \longrightarrow M_n(\mathbb{C})$, that is, such that there is an homogeneous central element c of positive degree such that $c(\phi) \neq 0$. We have the following graded version of Procesi's reconstruction result, see a.o. [7]

$$\mathtt{proj}\ Z(R) \simeq \mathtt{rep}_n^{ss}\ R//\mathtt{GL}_n \times \mathbb{C}^*$$

As a $\mathtt{GL}_n \times \mathbb{C}^*$-orbit is closed in $\mathtt{rep}_n^{ss}\ R$ if and only if the $\mathtt{GL}_n$-orbit is closed we see that points of $\mathtt{proj}\ Z(R)$ classify one-parameter families of isoclasses of trace-preserving n-dimensional semi-simple representations of R. In case of a simple representation such a one-parameter family determines a graded algebra morphism

$$R \longrightarrow M_n(\mathbb{C}[t,t^{-1}])(\underbrace{0,\ldots,0}_{m_0},\underbrace{1,\ldots,1}_{m_1},\ldots,\underbrace{e-1,\ldots,e-1}_{m_{e-1}})$$

where e is the degree of t and where we follow [13] in defining the shifted graded matrix algebra $M_n(S)(a_1,\ldots,a_n)$ by taking is homogeneous part of degree i to be

$$\begin{bmatrix} S_i & S_{i-a_1+a_2} & \ldots & S_{i-a_1+a_n} \\ S_{i-a_2+a_1} & S_i & \ldots & S_{i-a_2+a_n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{i-a_n+a_1} & S_{i-a_n+a_2} & \ldots & S_i \end{bmatrix}$$

The $\mathtt{GL}_n \times \mathbb{C}^*$-stabilizer subgroup of any of the simples ϕ in this family is then isomorphic to $\mathbb{C}^* \times \boldsymbol{\mu}_e$ where the cyclic group $\boldsymbol{\mu}_e$ has generator $(g_\zeta,\zeta) \in \mathtt{GL}_n \times \mathbb{C}^*$ where ζ is a primitive e-th root of unity and

$$g_\zeta = \mathtt{diag}(\underbrace{1,\ldots,1}_{m_0},\underbrace{\zeta,\ldots,\zeta}_{m_1},\ldots,\underbrace{\zeta^{e-1},\ldots,\zeta^{e-1}}_{m_{e-1}})$$

see [7, lemma 4]. If, in addition, ϕ is a smooth point of $\mathtt{rep}_n^{ss}\ R$ then the normal space

$$N(\phi) = T_\phi \mathtt{rep}_n^{ss} R / T_\phi \mathtt{GL}_n.\phi$$

to the $\mathtt{GL}_n$-orbit decomposes as a $\boldsymbol{\mu}_e$-representation into a direct sum of 1-dimensional simples

$$N(\phi) = \mathbb{C}_{i_1} \oplus \ldots \oplus \mathbb{C}_{i_d}$$

where the action of the generator on $\mathbb{C}_k$ is by multiplication with ζ^k. Alternatively, ϕ determines a (necessarily smooth) point $[\phi] \in \mathtt{spec}\ Z(R)$ and because $N(\phi)$ is equal to $Ext^1_R(S_\phi,S_\phi)$ and because R is Azumaya in $[\phi]$ it coincides with $Ext^1_{Z(R)}(S_{[\phi]},S_{[\phi]})$ (where $S_{[\phi]}$ is the simple 1-dimensional representation of $Z(R)$ determined by $[\phi]$) which is identical to the tangent space $T_{[\phi]}\mathtt{spec}Z(R)$. The action of the stabilizer subgroup $\boldsymbol{\mu}_e$ on $Ext^1_R(S_\phi,S_\phi)$ carries over to that on $T_{[\phi]}\mathtt{spec}Z(R)$.

The one-parameter family of simple representations also determines a point $\overline{\phi} \in \mathtt{proj}Z(R)$ and an application of the Luna slice theorem [12] asserts that for all $t \in \mathbb{C}$ there is a neighborhood of $(\overline{\phi},t) \in \mathtt{proj}Z(R) \times \mathbb{C}$ which is étale isomorphic to a neighborhood of 0 in $N(\phi)//\boldsymbol{\mu}_e$, see [7, Thm. 5].

3.1. **From $\mathtt{Proj}(A)$ to $\mathtt{rep}_n A$.** In noncommutative projective algebraic geometry, see a.o. [4],[5] and [3], one studies the Grothendieck category $\mathtt{Proj}(A)$ which is the quotient category of all graded left A-modules modulo the subcategory of torsion modules. In the case of 3-dimensional Sklyanin algebras the linear modules, that is those with Hilbert series $(1-t)^{-1}$ (point modules) or $(1-t)^{-2}$ (line modules) were classified in [5]. Identify $\mathbb{P}^2$ with $\mathbb{P}^2_{nc} = \mathbb{P}(A_1^*)$, then

- point modules correspond to points on the elliptic curve $E \hookrightarrow \mathbb{P}^2_{nc}$
- line modules correspond to lines in $\mathbb{P}^2_{nc}$

In the case of interest to us, when A corresponds to a couple (E, p) with p a torsion point of order n also fat modules are important which are critical cyclic graded left A-modules with Hilbert series $n.(1-t)^{-1}$. They were classified by M. Artin [3] and are relevant in the study of $\mathtt{proj} Z(A) = \mathbb{P}^2_c = \mathbb{P}(Z(A)^*_n)$. Observe that the reduced norm map N relates the different manifestations of $\mathbb{P}^2$ and the elliptic curve E with its isogenous curve $E/\langle p \rangle$

$$\begin{array}{ccc} \mathbb{P}^2_{nc} = \mathbb{P}(A_1^*) & \xrightarrow{N} & \mathbb{P}^2_c = \mathbb{P}(Z(A)^*_n) \\ \uparrow & & \uparrow \\ E & \xrightarrow{./\langle p \rangle} & E' = E/\langle p \rangle \end{array}$$

Points $\rho \in \mathbb{P}^2_c - E'$ determine fat points F_π with graded endomorphism ring isomorphic to $M_n(\mathbb{C}[t, t^{-1}])$ with $deg(t) = 1$, and hence determine a one-parameter family of simple n-dimensional representations in $\mathtt{rep}^{ss}_n A$ with $\mathtt{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\mathbb{C}^* \times 1$. There is an effective method to construct F_π, see [9]. Write ρ as the intersection of two lines $\mathbb{V}(z) \cap \mathbb{V}(z')$ and let $\mathbb{V}(z') \cap E' = \{q_1, q_2, q_3\}$ be the intersection with the elliptic curve E'. Then by lifting the q_i through the isogeny to n points $p_{ij} \in E$ we see that we can lift the line $\mathbb{V}(z')$ to n^2 lines in $\mathbb{P}^2_{nc} = \mathbb{P}(A_1^*)$, that is, there are n^2 one-dimensional subspaces $\mathbb{C}l \subset A_1$ with the property that $\mathbb{C}N(l) = \mathbb{C}z'$. The fat point corresponding to π is then the shifted quotient of a line module determined by l

$$F_\rho \simeq \frac{A}{A.l + A.z}[n]$$

On the other hand, if q is a point on E', then lifting q through the isogeny results in an orbit of n points of E, $\{r, r+p, r+[2]p, \ldots, r+[n-1]p\}$. If P is the point module corresponding to $r \in E$, then the fat point module corresponding to q is

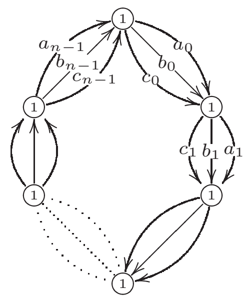
$$F_q = P \oplus P[1] \oplus P[2] \oplus \ldots \oplus P[n-1]$$

and the corresponding graded endomorphism ring is isomorphic to $M_n(\mathbb{C}[t, t^{-1}])(0, 1, 2, \ldots, n-1)$ where $deg(t) = n$ and hence corresponds to a one-parameter family of simple n-dimensional representations in $\mathtt{rep}^{ss}_n A$ with $\mathtt{GL}_n \times \mathbb{C}^*$-stabilizer subgroup generated by $\mathbb{C}^* \times 1$ and a cyclic group of order n

$$\boldsymbol{\mu}_n = \langle (\begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{bmatrix}, \zeta) \rangle$$

with ζ a primitive n-th root of unity. In fact, we can give a concrete matrix-representation of these simple modules. Assume that $r + [i]p = [a_i : b_i : c_i] \in \mathbb{P}^2_{nc}$

then the fat point module F_q corresponds to the quiver-representation



and the map $A \longrightarrow M_n(\mathbb{C}[t,t^{-1}])(0,1,2,\ldots,n-1)$ sends the generators x, y and z to the degree one matrices

$$\begin{bmatrix} 0 & 0 & \ldots & \ldots & a_{n-1}t \\ a_0 & 0 & \ldots & \ldots & 0 \\ & & \ddots & & \vdots \\ 0 & a_1 & & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{n-2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \ldots & \ldots & b_{n-1}t \\ b_0 & 0 & \ldots & \ldots & 0 \\ & & \ddots & & \vdots \\ 0 & b_1 & & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & b_{n-2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \ldots & \ldots & c_{n-1}t \\ c_0 & 0 & \ldots & \ldots & 0 \\ & & \ddots & & \vdots \\ 0 & c_1 & & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{n-2} & 0 \end{bmatrix}$$

Theorem 4. *Let A be a 3-dimensional Sklyanin algebra corresponding to a couple (E,p) where p is a torsion point of order n and assume that $(n,3)=1$. Consider the GIT-quotient*

$$\mathtt{rep}_n A \xrightarrow{\pi} \mathtt{spec} Z(A) = \mathtt{rep}_n A // \mathtt{GL}_n$$

Then we have

1. *$\mathtt{rep}_n^{ss} A$ is a smooth variety of dimension n^2+2*
2. *A is an Azumaya algebra away from the isolated singularity $\tau \in \mathtt{spec} Z(A)$*
3. *the nullcone $\pi^{-1}(\tau)$ contains singularities*

Proof. We know that $\mathbb{P}^2_c = \mathtt{proj} Z(A) = \mathtt{rep}_n^{ss} A // \mathtt{GL}_n \times \mathbb{C}^*$ classifies one-parameter families of semi-stable n-dimensional semi-simple representations of A. To every point $\rho \in \mathbb{P}^2_c$ we have associated a one-parameter family of simples, so all semi-stable A-representations are in fact simple as the semi-simplification M^{ss} of a semi-stable representation still belongs to $\mathtt{rep}_n^{ss} A$. But then, all non-trivial semi-simple A-representations are simple and therefore the GIT-quotient

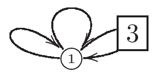
$$\mathtt{rep}_n^{ss} A \longrightarrow \mathtt{spec} Z(A) - \{\tau\} = \mathtt{rep}_n^{ss} A // \mathtt{GL}_n$$

is a principal $\mathtt{PGL}_n$-fibration in the étale topology. This proves (1).

The second assertion follows as principal $\mathtt{PGL}_n$-fibrations in the étale topology correspond to Azumaya algebras. For (3), if $\mathtt{rep}_n A$ would be smooth, the algebra A would be Cayley-smooth as in [10]. There it is shown that the only type of central singularity that can arise for Cayley-smooth algebras with a 3-dimensional center is the conifold singularity. □

If we want to distinguish between the two types of simple representations, we have to consider the $\mathtt{GL}_n \times \mathbb{C}^*$-action.

Lemma 1. *If S is a simple A-representation with $\mathtt{GL}_n \times \mathbb{C}^*$-orbit determining a fat point F_q with $q \in E'$, then the normal space $N(S)$ to the $\mathtt{GL}_n$-orbit decomposes* *as representation over the* $\mathtt{GL}_n \times \mathbb{C}^*$*-stabilizer subgroup* $\boldsymbol{\mu}_n$ *as* $\mathbb{C}_0 \oplus \mathbb{C}_0 \oplus \mathbb{C}_3$*, or in the terminology of* [7]*, the associated local weighted quiver is*



Proof. From [15] we know that the center $Z(A)$ can be represented as

$$Z(A) = \frac{\mathbb{C}[x', y', z', c_3]}{(c_3^n - cubic(x', y', z'))}$$

where x', y', z' are of degree n (the reduced norms of x, y, z) and c_3 is the canonical central element of degree 3. The simple A-representation S determines a point $s \in \mathtt{spec} Z(A)$ such that $c_3(s) = 0$. Again, as A is Azumaya over s we have that $N(S) = Ext^1_A(S, S)$ coincides with the tangent space $T_s \mathtt{spec} Z(A)$. Gradation defines a $\boldsymbol{\mu}_n$-action on $Z(A)$ leaving x', y', z' invariant and sending c_3 to $\zeta^3 c_3$. The stabilizer subgroup of this action in s is clearly $\boldsymbol{\mu}_n$ and computing the tangent space gives the required decomposition. □

3.2. $\mathcal{A}$ **is Cayley-smooth.** Because A is a finitely generated module over $Z(A)$, it defines a coherent sheaf of algebras $\mathcal{A}$ over $\mathtt{proj} Z(A) = \mathbb{P}^2$. In this subsection we will show that $\mathcal{A}$ is a sheaf of Cayley-smooth algebras of degree n.

As $(n, 3) = 1$ it follows that the graded localisation $Q^g_{x'}(A)$ at the multiplicative set of central elements $\{1, x', x'^2, \ldots\}$ contains central elements t of degree one and hence is isomorphic as a graded algebra to

$$Q^g_{x'}(A) = (Q^g_{x'}(A))_0[t, t^{-1}]$$

By definition $\Gamma(\mathbb{X}(x'), \mathcal{A}) = (Q^g_{x'}(A))_0$ and by the above isomorphism it follows that $\Gamma(\mathbb{X}(x'), \mathcal{A})$ is a Cayley-Hamilton domain of degree n and is Auslander regular of dimension two and consequently a maximal order. Repeating this argument for the other standard opens $\mathbb{X}(y')$ and $\mathbb{X}(z')$ we deduce

Proposition 1. $\mathcal{A}$ *is a coherent sheaf of Cayley-Hamilton maximal orders of degree* n *which are Auslander regular domains of dimension* 2 *over* $\mathtt{proj} Z(A) = \mathbb{P}^2_c$.

Thus, $\mathcal{A}$ is a maximal order over $\mathbb{P}^2$ in a division algebra Σ over $\mathbb{C}(\mathbb{P}^2)$ of degree n. By the Artin-Mumford exact sequence (see for example [10, 3.6]) describing the Brauer group of $\mathbb{C}(\mathbb{P}^2)$ we know that Σ is determined by the ramification locus of $\mathcal{A}$ together with a cyclic $\mathbb{Z}_n$-cover over it.

Again using the above local description of A as a graded algebra over $Z(A)$ we see that the fat point module corresponding to a point $p \notin E'$ determines a simple n-dimensional representation of $\mathcal{A}$ and therefore $\mathcal{A}$ is Azumaya in p. However, if $p \in E'$, then the corresponding fat point is of the form $P \oplus P[1] \oplus \ldots \oplus P[n-1]$ and this corresponds to a semi-simple n-dimensional representation which is the direct sum of n distinct one-dimensional $\mathcal{A}$-representations, one component for each point of E lying over p. Hence, we see that the ramification divisor of $\mathcal{A}$ coincides with E' and, naturally, the division algebra Σ is the one coresponding to the cyclic $\mathbb{Z}_n$-cover $E \twoheadrightarrow E' = E/\langle \tau \rangle$.

Because $\mathcal{A}$ is a maximal order with smooth ramification locus, we deduce from [10, §5.4]

Proposition 2. *$\mathcal{A}$ is a sheaf of Cayley-smooth algebras over $\mathbb{P}^2_c$ and hence $\mathtt{rep}_n(\mathcal{A})$ is a smooth variety of dimension n^2+1 with GIT-quotient*

$$\mathtt{rep}_n(\mathcal{A}) \xrightarrow{\pi} \mathbb{P}^2_c = \mathtt{rep}_n(\mathcal{A})//\mathtt{GL}_n$$

and is a principal $\mathtt{PGL}_n$-fibration over $\mathbb{P}^2_c - E'$.

3.3. **The non-commutative blow-up.** Consider the augmentation ideal $\mathfrak{m} = (x,y,z)$ of the 3-dimensional Sklyanin algebra A corresponding to a couple (E,p) with p a torsion point of order n. Define the non-commutative blow-up algebra to be the graded algebra

$$B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \ldots \subset A[t]$$

with degree zero part A and where the commuting variable t is given degree 1. Note that B is a graded subalgebra of $A[t]$ and therefore is again a Cayley-Hamilton algebra of degree n. Moreover, B is a finite module over its center $Z(B)$ which is a graded subalgebra of $Z(A)[t]$. Observe that B is generated by the degree zero elements x,y,z and by the degree one elements $X = xt, Y = yt$ and $Z = zt$. Apart from the Sklyanin relations among x,y,z and among X,Y,Z these generators also satisfy commutation relations such as $Xx = xX, Xy = xY, Xz = xZ$ and so on.

With $\mathtt{rep}^{ss}_n B$ we will denote again the Zariski open subset of $\mathtt{rep}_n B$ consisting of all trace-preserving n-dimensional semi-stable representations, that is, those on which some central homogeneous element of $Z(B)$ of strictly positive degree does not vanish. Theorem 2 asserts that $\mathtt{rep}^{ss}_n B$ is a smooth variety of dimension n^2+3.

Proof of Theorem 2 : As before, we have a $\mathtt{GL}_n \times \mathbb{C}^*$-action on $\mathtt{rep}^{ss}_n B$ with corresponding GIT-quotient

$$\mathtt{proj} Z(B) \simeq \mathtt{rep}^{ss}_n B // \mathtt{GL}_n \times \mathbb{C}^*$$

Composing the GIT-quotient map with the canonical morphism (taking the degree zero part) $\mathtt{proj} Z(B) \twoheadrightarrow \mathtt{spec} Z(A)$ we have a projection

$$\gamma \ : \ \mathtt{rep}^{ss}_n B \twoheadrightarrow \mathtt{spec} Z(A)$$

Let $\mathfrak{p}$ be a maximal ideal of $Z(A)$ corresponding to a smooth point, then the graded localization of B at the degree zero multiplicative subset $Z(A) - \mathfrak{p}$ gives

$$B_{\mathfrak{p}} \simeq A_{\mathfrak{p}}[t,t^{-1}]$$

whence $B_{\mathfrak{p}}$ is an Azumaya algebra over $Z(A)[t,t^{-1}]$ and therefore over $\mathtt{spec} Z(A) - \{\tau\}$ the projection γ is a principal $\mathtt{PGL}_n \times \mathbb{C}^*$-fibration and in particular the dimension of $\mathtt{rep}^{ss}_n B$ is equal to n^2+3.

This further shows that possible singularities of $\mathtt{rep}^{ss}_n B$ must lie in $\gamma^{-1}(\tau)$ and as the singular locus is Zariski closed we only have to prove smoothness in points of closed $\mathtt{GL}_n$-orbits in $\gamma^{-1}(\tau)$. Such a point ϕ must be of the form

$$x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M$$

By semi-stability, (K,L,M) defines a simple n-dimensional representation of A and its $\mathtt{GL}_n \times \mathbb{C}^*$-orbit defines the point $[det(K) : det(L) : det(M)] \in \mathbb{P}^2_c$. hence we may assume for instance that K is invertible.

The tangent space $T_\phi \mathtt{rep}^{ss}_n B$ is the linear space of all trace-preserving algebra maps $B \longrightarrow M_n(\mathbb{C}[\epsilon])$ of the form

$$x \mapsto 0 + \epsilon U, y \mapsto 0 + \epsilon V, z \mapsto 0 + \epsilon W, X \mapsto K + \epsilon R, Y \mapsto L + \epsilon S, Z \mapsto M + \epsilon T$$

and we have to use the relations in B to show that the dimension of this space is at most n^2+3. As (K,L,M) is a simple n-dimensional representation of the Sklyanin algebra, we know already that (R,S,T) depend on at most n^2+2 parameters. Further, from the commutation relations in B we deduce the following equalities (using the assumption that K is invertible)

- $xX = Xx \Rightarrow UK = KU$
- $xY = Xy \Rightarrow UL = KV \Rightarrow K^{-1}UL = V$
- $xZ = Xz \Rightarrow UM = KW \Rightarrow K^{-1}UM = W$
- $Yx = yX \Rightarrow LU = VK \Rightarrow LK^{-1}U = V$
- $Zx = zX \Rightarrow MU = WK \Rightarrow MK^{-1}U = W$

These equalities imply that $K^{-1}U$ commutes with K, L and M and as (K,L,M) is a simple representation and hence generate $M_n(\mathbb{C})$ it follows that $K^{-1}U = \lambda 1_n$ for some $\lambda \in \mathbb{C}$. But then it follows that

$$U = \lambda K, \quad V = \lambda L, \quad W = \lambda M$$

and so the triple (U,V,W) depends on at most one extra parameter, showing that $T_\phi \mathtt{rep}_n^{ss} B$ has dimension at most n^2+3, finishing the proof. $\square$

Remark 1. *The statement of the previous theorem holds in a more general setting, that is, $\mathtt{rep}_n^{ss} B$ is smooth whenever $B = A \oplus A^+ t \oplus (A^+)^2 t^2 \oplus \dots$ with A a positively graded algebra that is Azumaya away from the maximal ideal A^+ and $Z(A)$ smooth away from the origin.*

Unfortunately this does not imply that $\mathtt{proj} Z(B) = \mathtt{rep}_n^{ss} B // \mathtt{GL}_n \times \mathbb{C}^*$ is smooth as there are closed $\mathtt{GL}_n \times \mathbb{C}^*$ orbits with stabilizer subgroups strictly larger than $\mathbb{C}^* \times 1$. This happens precisely in semi-stable representations ϕ determined by

$$x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M$$

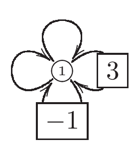
with $[det(K) : det(L) : det(M)] \in E'$. In which case the matrices (K,L,M) can be brought into the form

$$\begin{bmatrix} 0 & 0 & \dots & \dots & a_{n-1}t \\ a_0 & 0 & \dots & \dots & 0 \\ & & \ddots & & \vdots \\ 0 & a_1 & & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & \dots & b_{n-1}t \\ b_0 & 0 & \dots & \dots & 0 \\ & & \ddots & & \vdots \\ 0 & b_1 & & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b_{n-2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & \dots & c_{n-1}t \\ c_0 & 0 & \dots & \dots & 0 \\ & & \ddots & & \vdots \\ 0 & c_1 & & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & c_{n-2} & 0 \end{bmatrix}$$

and the stabilizer subgroup is generated by $\mathbb{C}^* \times 1$ together with the cyclic group of order n

$$\boldsymbol{\mu}_n = \langle (\begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{bmatrix}, \zeta) \rangle$$

Lemma 2. *If ϕ is a representation as above, then the normal space $N(\phi)$ to the $\mathtt{GL}_n$-orbit decomposes as a representation over the $\mathtt{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\boldsymbol{\mu}_n$ as $\mathbb{C}_0 \oplus \mathbb{C}_0 \oplus \mathbb{C}_3 \oplus \mathbb{C}_{-1}$, that is, the associated local weighted quiver is*



Proof. The extra tangential coordinate λ determines the tangent-vectors of the three degree zero generators

$$x \mapsto 0 + \epsilon\lambda K, \qquad y \mapsto 0 + \epsilon\lambda L, \qquad z \mapsto 0 + \epsilon\lambda M$$

and so the generator of $\boldsymbol{\mu}_n$ acts as follows

$$\begin{bmatrix} 1 & & & \\ & \zeta^{n-1} & & \\ & & \ddots & \\ & & & \zeta \end{bmatrix} .(\epsilon\lambda(K,L,M)). \begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{bmatrix} = \epsilon\zeta^{n-1}\lambda(K,L,M)$$

and hence accounts for the extra component $\mathbb{C}_{-1}$. □

We have now all information to prove Theorem 3 which asserts that the canonical map

$$\mathtt{proj}Z(B) \twoheadrightarrow \mathtt{spec}Z(A)$$

is a partial resolution of singularities, with singular locus $E' = E/\langle p \rangle$ in the exceptional fiber, all singularities of type $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$. In other words, the isolated singularity of $\mathtt{spec}Z(A)$ 'sees' the elliptic curve E' and the isogeny $E \twoheadrightarrow E'$ defining the 3-dimensional Sklyanin algebra A.

Proof of Theorem 3 : The GIT-quotient map

$$\mathtt{rep}_n^{ss} B \twoheadrightarrow \mathtt{proj}Z(B)$$

is a principal $\mathtt{PGL}_n \times \mathbb{C}^*$-bundle away from the elliptic curve E' in the exceptional fiber whence $\mathtt{proj}Z(B) - E'$ is smooth. The application to the Luna slice theorem of [7, Thm. 5] asserts that for any point $\overline{\phi} \in E' \hookrightarrow \mathtt{proj}Z(B)$ and all $t \in \mathbb{C}$ there is a neighborhood of $(\overline{\phi}, t) \in \mathtt{proj}Z(B) \times \mathbb{C}$ which is étale isomorphic to a neighborhood of 0 in $N(\phi)//\boldsymbol{\mu}_n$. From the previous lemma we deduce that

$$N(\phi)//\boldsymbol{\mu}_n \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$$

where $\mathbb{C}[\mathbb{C}^2/\mathbb{Z}_n] \simeq \mathbb{C}[u,v,w]/(w^n - uv^3)$, finishing the proof.

As B is a finite module over its center, it defines a coherent sheaf of algebras over $\mathtt{proj}Z(B)$. From Theorem 3 we obtain

Corollary 1. *The sheaf of Cayley-Hamilton algebras $\mathcal{B}$ on $\mathtt{proj}Z(B)$ is Azumaya away from the elliptic curve E' in the exceptional fiber $\pi^{-1}(\mathfrak{m}) = \mathbb{P}^2$ and hence is Cayley-smooth on this open set. However, $\mathcal{B}$ is not Cayley-smooth.*

Proof. For a point p in the exceptional fiber $\pi^{-1}(\mathfrak{m}) - E'$ we already know that $\mathtt{proj}Z(B)$ is smooth and that B is Azumaya, which implies that $\mathtt{rep}_n^{ss} B$ is smooth in the corresponding orbit. However, for a point $p \in E'$ we know that $\mathtt{proj}Z(B)$ has a non-isolated singularity in p. Therefore, $\mathtt{rep}_n^{ss} \mathcal{B}$ can not be smooth in the corresponding orbit, as the only central singularity possible for a Cayley-smooth order over a center of dimension 3 is the conifold singularity, which is isolated. □

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