MOST IRREDUCIBLE REPRESENTATIONS OF THE 3-STRING BRAID GROUP

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1. Introduction

With $\mathtt{iss}_n\ B_3$ we denote the affine variety of all isomorphism classes of semi-simple n-dimensional representations of the 3-string braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

It is well-known, see for example [8], [4] and [5], that any irreducible components X_σ of $\mathtt{iss}_n\ B_3$ containing a Zariski open subset of irreducible representations is determined by a dimension-vector $\sigma = (a, b; x, y, z)$ satisfying

$$n = a + b = x + y + z \quad \text{and} \quad x = \mathtt{max}(x, y, z) \leq b = \mathtt{min}(a, b)$$

with $dim\ X_\sigma = n_\sigma = 1+n^2-(a^2+b^2+x^2+y^2+z^2)$. As B_3 is of wild representation type one cannot expect a full classification of all its finite dimensional irreducible representations. In fact, such a classification is only known for $n \leq 5$ by work of Imre Tuba and Hans Wenzl [7]. Still, one can aim to describe 'most' irreducible representations by constructing for each component X_σ an explicit minimal (étale) rational map

$$f_\sigma\ :\ \mathbb{A}^{n_\sigma} \dashrightarrow X_\sigma \hookrightarrow \mathtt{iss}_n\ B_3$$

having a Zariski dense image. Such rational dense parametrizations were constructed in [4] for all components when $n < 12$. The purpose of the present paper is to extend this to all finite dimensions n.

2. Linear systems and some rational quiver settings

A linear control system Σ is determined by the system of linear differential equations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where $\Sigma = (A, B, C) \in M_n(\mathbb{C}) \times M_{n\times m}(\mathbb{C}) \times M_{p\times n}(\mathbb{C})$ and $u(t) \in \mathbb{C}^m$ is the control at time t, $x(t) \in \mathbb{C}^n$ is the state of the system and $y(t) \in \mathbb{C}^p$ its output. Equivalent control systems differ only by a base change in the state space, that is $\Sigma' = (A', B', C')$ is equivalent to Σ if and only if there exists a $g \in GL_n(\mathbb{C})$ such that

$$A' = gAg^{-1}, \qquad B' = gB \quad \text{and} \quad C' = Cg^{-1}$$

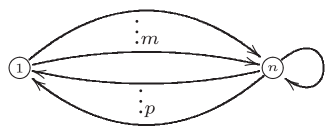
Σ is said to be *canonical* if the matrices

$$c_\Sigma = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \quad \text{and} \quad o_\Sigma = \begin{bmatrix} C & CA & CA^2 & \dots & CA^{n-1} \end{bmatrix}$$

are of maximal rank.

Michiel Hazewinkel proved in [1] that the moduli space $\mathtt{sys}^c_{m,n,p}$ of all such canonical linear systems is a smooth rational quasi-affine variety of dimension $(m+p)n$. We will give another short proof of this result and draw some consequences from it (see also [6]).

Consider the quiver setting with m arrows $\{b_1,\dots,b_m\}$ from left to right and p arrows $\{c_1,\dots,c_p\}$ from right to left



To a system $\Sigma = (A,B,C)$ we associate the quiver-representation V_Σ by assigning to the arrow b_i the i-th column B_i of the matrix B, to the arrow c_j the j-th row C^j of C and the matrix A to the loop. As the base change group $\mathbb{C}^* \times GL_n$ acts on these quiver-representations by

$$(\lambda, g).V_\Sigma = (gAg^{-1}, gB_1\lambda^{-1}, \dots, gB_m\lambda^{-1}, \lambda C^1 g^{-1}, \dots, \lambda C^p g^{-1})$$

with the subgroup $\mathbb{C}^*(1, 1_n)$ acting trivially, there is a natural one-to-one correspondence between equivalence classes of linear systems Σ and isomorphism classes of quiver-representations V_Σ. Under this correspondence it is easy to see that canonical systems correspond to *simple* quiver-representations, see [6, Lemma 1]. Hence, the moduli-space $\mathtt{sys}^c_{m,n,p}$ is isomorphic to the Zariski-open subset of the affine quotient-variety classifying isomorphism classes of semi-simple quiver-representations, proving smoothness, quasi-affineness as well as determining the dimension by general results, see for example [3].

Lemma 1. *A generic canonical system Σ is equivalent to a triple $(A_n, B^\bullet_{nm}, C_{pn})$ with*

$$A_n = \begin{bmatrix} 0 & 0 & \dots & & x_n \\ 1 & 0 & \dots & & x_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & x_2 \\ & & & 1 & x_1 \end{bmatrix} \qquad B^\bullet_{nm} = \begin{bmatrix} 1 & b_{12} & \dots & b_{1m} \\ 0 & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n2} & \dots & b_{nm} \end{bmatrix}$$

that is, where A_n is a companion $n \times n$-matrix, $B^\bullet_{nm}$ is the generic $n \times m$-matrix with fixed first column and C_{pn} a generic $p \times n$-matrix.

Proof. A generic representation of the quiver-setting

$$\begin{tikzcd} \textcircled{1} \arrow[r, "v"] & \textcircled{n} \arrow[loop right, "A"] \end{tikzcd}$$

will have the property that v is a cyclic-vector for the matrix A, that is, $\{v, Av, A^2v, \dots, A^{n-1}v\}$ are linearly independent. But then, performing a base-change we get a representation of the form

$$\begin{tikzcd} \textcircled{1} \arrow[r, "\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^{tr}"] & \textcircled{n} \arrow[loop right, "A_n"] \end{tikzcd}$$

where A_n is a companion matrix whose n-th column expresses the vector $-A^n v$ in the new basis. As the automorphism group of this representation is reduced to $\mathbb{C}^*(1, 1_n)$, any general representation V_Σ is isomorphic to one with

$B_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^{tr}$, $A = A_n$ and the other columns of B and all rows of C generic vectors. $\square$

Lemma 2. *The following representations give a rational parametrization of the isomorphism classes of simple representations of these quiver-settings*

$$R_k \quad : \quad \begin{tikzcd} \textcircled{1} \arrow[rr, bend left, "{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^{tr}}"] & & \textcircled{k} \arrow[ll, bend left, "{\begin{bmatrix} y_1 & y_2 & \dots & y_k \end{bmatrix}}"] \arrow[r, bend left, "A_k"] & \textcircled{k} \arrow[l, bend left, "1_k"] \end{tikzcd}$$

and

$$S_k \quad : \quad \begin{tikzcd} \textcircled{1} \arrow[rr, bend left, "{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^{tr}}"] & & \textcircled{k} \arrow[ll, bend left, "{\begin{bmatrix} y_1 & y_2 & \dots & y_k \end{bmatrix}}"] \arrow[r, bend left, "A_k^\dagger"] & \textcircled{k-1} \arrow[l, bend left, "{\begin{bmatrix} \underline{0} \\ 1_{k-1} \end{bmatrix}}"] \end{tikzcd}$$

where A_k (reps. $A_k^\dagger$) is the generic $k \times k$ companion matrix (resp. the reduced $k-1 \times k$ companion matrix)

$$A_k = \begin{bmatrix} 0 & 0 & \dots & & x_k \\ 1 & 0 & \dots & & x_{k-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & x_2 \\ & & & 1 & x_1 \end{bmatrix} \quad \text{and} \quad A_k^\dagger = \begin{bmatrix} 1 & 0 & \dots & & x_{k-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & x_2 \\ & & & 1 & x_1 \end{bmatrix}$$

Proof. By invoking the first fundamental theorem of GL_n-invariants (see for example [2, Thm. II.4.1]) we can in case R_k eliminate the base-change action in the right-most vertex, giving a natural one-to-one correspondence between isoclasses of representations

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left, "v"] & \textcircled{k} \arrow[l, bend left, "w^\tau"] \arrow[r, bend left, "X"] & \textcircled{k} \arrow[l, bend left, "Y"] \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} \textcircled{1} \arrow[r, bend left, "v"] & \textcircled{k} \arrow[l, bend left, "w^\tau"] \arrow[loop right, "Y.X"] \end{tikzcd}$$

and hence the claim follows from the previous lemma. As for case S_k we can again apply the first fundamental theorem for GL_n-invariants, now with respect to the base-change action in the middle vertex, to obtain a natural one-to-one correspondence between isoclasses of representations

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left, "v"] & \textcircled{k} \arrow[l, bend left, "w^\tau"] \arrow[r, bend left, "X"] & \textcircled{k-1} \arrow[l, bend left, "Y"] \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} \textcircled{1} \arrow[loop left, "w^\tau.v"] \arrow[r, bend left, "X.v"] & \textcircled{k-1} \arrow[l, bend left, "w^\tau.Y"] \arrow[loop right, "X.Y"] \end{tikzcd}$$

and again the claim follows from the previous lemma, taking into account the extra free loop in the left-most vertex, which corresponds to y_1. $\square$

Lemma 3. *The following representations give a rational parametrization for the isomorphism classes of simple representations of the quiver-setting*

$$\begin{tikzcd} 1 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "z"] \arrow[r, bend left, "{\begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}^{tr}}"] & k \arrow[l, bend left, "{\begin{bmatrix}y_1 & y_2 & \dots & y_k\end{bmatrix}}"] \arrow[r, bend left, "A_k^{\dagger}"] & k-1 \arrow[l, bend left, "{\begin{bmatrix}\underline{0} \\ 1_{k-1}\end{bmatrix}}"] \arrow[r, bend left, "1_{k-1}"] & k-1 \arrow[l, bend left, "B"] \end{tikzcd}$$

where B is a generic $k-1 \times k-1$ matrix and, as before, $A_k^{\dagger}$ is a reduced generic companion matrix.

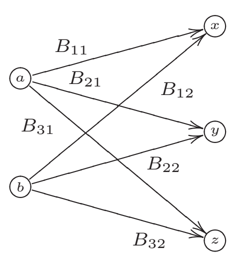
Proof. Forgetting the end-vertices (and maps to and from them) we are in the situation of the previous lemma. For general values these are simple quiver-representations and hence the automorphism group is reduced to $\mathbb{C}^*(1, 1_k, 1_{k-1})$. If we now add the end vertices we can use base-change in them to force one of the two arrows to be the identity map, leaving the remaining map generic. Alternatively, we can use the first fundamental theorem of GL_n-invariants as before, to obtain the claimed result. $\square$

3. Luna slices and the action map

We quickly recall the basic strategy of [4]. As the central generator $c = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$ of B_3 acts via a scalar $\lambda \in \mathbb{C}^*$ on any irreducible B_3-representation it suffices to study irreducible representations of the quotient group $B_3/\langle c \rangle \simeq C_2 * C_3 = \langle s, t \mid s^2 = e = t^3 \rangle$ where s is the class of $\sigma_1\sigma_2\sigma_1$ and t that of $\sigma_1\sigma_2$. Note that this quotient-group is isomorphic to the modular group $PSL_2(\mathbb{Z})$. The action of s and t on a finite dimensional $C_2 * C_3$-representation V induce two decompositions of V into eigen-spaces

$$V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2}$$

where ρ is a primitive 3-rd root of unity. Hence V is fully determined by a base-change matrix $B = (B_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 2}$ from a fixed basis compatible with the first decomposition to a fixed basis compatible with the second, that is by a representation of the quiver-setting



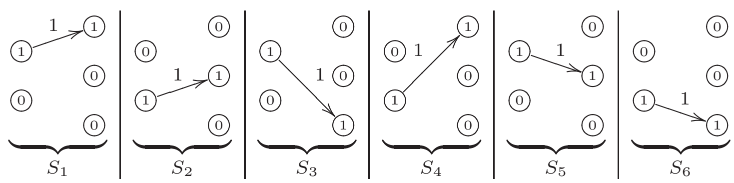
Bruce Westbury observed in [8] that under this correspondence isoclasses of $C_2 * C_3$-representations coincide with isoclasses of quiver-representations, and that irreducible group-representations correspond to stable quiver-representations wrt. the stability structure $\theta = (-1, -1; 1, 1, 1)$. It then follows from this stability condition that the dimension-vectors $\sigma = (a, b; x, y, z)$ containing a Zariski open subset of irreducible n-dimensional $C_2 * C_3$-representations must satisfy $a + b = n = x + y + z$ as well as $max(x, y, z) \leq min(a, b)$.

Working backwards, we obtain for each $\lambda \in \mathbb{C}^*$ an irreducible B_3-representation determined by the above base-change matrix B via

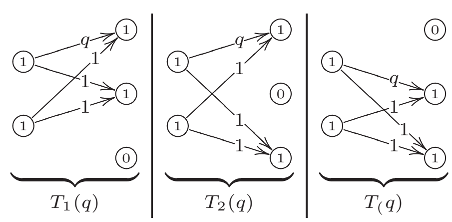
$$(*) \begin{cases} \sigma_1 \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \sigma_2 \mapsto \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases}$$

Observe that in lifting irreducibles from $C_2 * C_3$ to B_3 we get an action by multiplication of 6-th roots of unity on the components which contain irreducibles, which accounts for the fact that the irreducible components X_σ containing irreducible B_3-representations are classified by the dimension vectors $\sigma = (a, b; x, y, z)$ as above with the extra condition that $b = min(a, b)$ and $x = max(x, y, z)$. We will now construct special semi-simple $C_2 * C_3$-representations M_0 in every component, with all its irreducible factors being 1- or 2-dimensional.

There are 6 one-dimensional irreducible $C_2 * C_3$-representations, coresponding to the quiver-representations S_i for $1 \leq i \leq 6$:



and three one-parameter families of two-dimensional irreducibles corresponding to the quiver-representations $T_i(q)$ for $q \neq 0, 1$ and $1 \leq i \leq 3$



The semi-simple representation

$$M_0 = S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6} \oplus T_1(q)^{\oplus b_\alpha} \oplus T_2(q)^{\oplus b_\beta} \oplus T_3(q)^{\oplus b_\gamma}$$

clearly belongs to the component X_σ with dimension vector $\sigma = (a, b; x, y, z)$ where

$$\begin{cases} a = a_1 + a_3 + a_5 + b_\alpha + b_\beta \\ b = a_2 + a_4 + a_6 + b_\alpha + b_\gamma \\ x = a_1 + a_4 + b_\alpha + b_\beta \\ y = a_2 + a_5 + b_\alpha + b_\gamma \\ z = a_3 + a_6 + b_\beta + b_\gamma \end{cases}$$

and is fully determined by the base-change matrix B_0 with block-form as above

$$
\left|\begin{array}{cccccc|cccccc}
1_{a_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{a_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q1_{b_\alpha} & 0 & 0 & 0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 \\
0 & 0 & 0 & 0 & q1_{b_\beta} & 0 & 0 & 0 & 0 & 0 & 1_{b_\beta} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1_{a_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_{a_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 & 0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q1_{b_\gamma} & 0 & 0 & 0 & 0 & 0 & 1_{b_\gamma} \\
\hline
0 & 1_{a_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{a_6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{b_\beta} & 0 & 0 & 0 & 0 & 0 & 1_{b_\beta} & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{b_\gamma} & 0 & 0 & 0 & 0 & 0 & 1_{b_\beta}
\end{array}\right|
$$

We will now determine the structure of the base-change matrices B of isoclasses of $C_2 * C_3$-representations M in a Zariski open neighborhood of $[M_0]$ in $\mathtt{iss}_\sigma\ C_2 * C_3$.

As M_0 is semi-simple, its isomorphism class forms a Zariski closed orbit $\mathcal{O}(M_0)$ in the smooth irreducible component $\mathtt{rep}_\sigma\ C_2 * C_3$ under the action of $GL(\sigma) = GL_a \times GL_b \times GL_x \times GL_y \times GL_z$. The stabilizer subgroup $Stab(M_0)$ is the automorphism group and is the subgroup of $GL(\sigma)$ we will denote by $GL(\tau) = GL_{a_1} \times GL_{a_2} \times GL_{a_3} \times GL_{a_4} \times GL_{a_5} \times GL_{a_6} \times GL_{b_\alpha} \times GL_{b_\beta} \times GL_{b_\gamma}$. The normal space to the orbit $\mathcal{O}(M_0)$ can be identified as $GL(\tau)$-representation with the vectorspace of self-extensions $Ext^1_{C_2*C_3}(M_0, M_0)$, see for example [2, II.2.7]. The Luna slice theorem, see for example [3, §4.2], asserts that the action map

$$GL(\sigma) \times^{GL(\tau)} Ext^1_{C_2*C_3}(M_0, M_0) \longrightarrow \mathtt{rep}_\sigma\ C_2 * C_3$$

sending the class of $(g, \vec{n})$ in the associated fibre bundle to the C_2*C_3-representation $g.(M + \vec{n})$ is a $GL(\sigma)$-equivariant étale map with a Zariski dense image. Taking $GL(\sigma)$-quotients on both sides we obtain an étale map

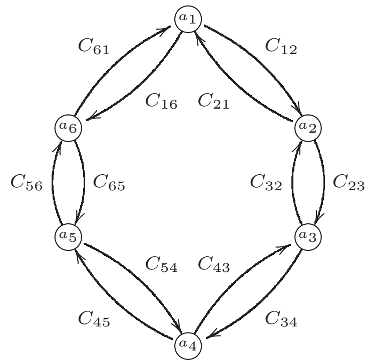
$$Ext^1_{C_2*C_3}(M_0, M_0)/GL(\tau) \longrightarrow \mathtt{iss}_\sigma\ C_2 * C_3$$

with a Zariski dense image. The crucial observation to make is that it follows from the theory of local quivers, [3, §4.2], that as a $GL(\tau)$-representation $Ext^1_{C_2*C_3}(M_0, M_0)$ is isomorphic to $\mathtt{rep}_\tau\ Q$ for the quiver Q having 9 vertices (one for each of the distinct simple factors of M_0) and having as many directed arrows from the vertex corresponding to the simple factor S to that of the simple factor T as is the dimension of the space $Ext^1_{C_2*C_3}(S, T)$. This then allows to identify the quotient variety $Ext^1_{C_2*C_3}(M_0, M_0)/GL(\tau)$ with the affine variety $\mathtt{iss}_\tau\ Q$ whose points are the isoclasses of semi-simple representations of Q of dimension-vector $\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma)$, and the action map induces an étale map with dense image

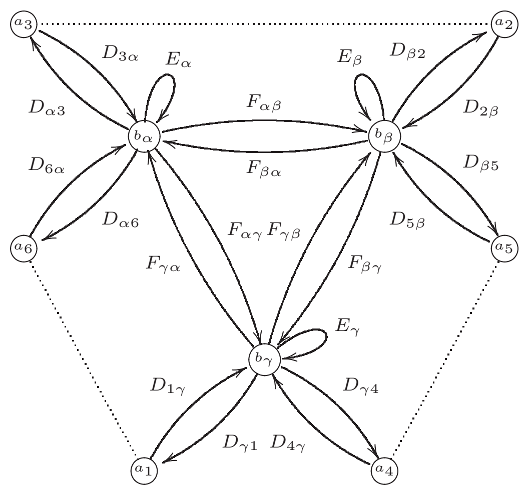
$$\mathtt{iss}_\tau\ Q \longrightarrow \mathtt{iss}_\sigma\ C_2 * C_3$$

Computing the normal space to the orbit $\mathcal{O}(M_0)$ as in the proof of [4, Thm. 4] but for the more complicated representation M_0 one obtains that the sub quiver of Q on the 6 vertices corresponding to the 1-dimensional simple components $S_1, \ldots, S_6$

coincides with that of [4], that is corresponds to the quiver-setting



The additional quiver-setting depending on the 3 vertices corresponding to the 2-dimensional simple factors $T_1(q)$, $T_2(q)$ and $T_3(q)$ can be verified to be



which concludes the proof of the following:

Theorem 1. *The étale action map $GL(\sigma) \times^{GL(\tau)} \mathtt{rep}_\tau\ Q \longrightarrow \mathtt{rep}_\sigma\ C_2 * C_3$ sends a τ-dimensional Q-representation to the $C_2 * C_3$-representation determined by the base-change matrix B*

$$
\left|\begin{array}{cccccc|cccccc}
1_{a_1} & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & C_{61} & 0 & 0 & D_{\gamma 1} \\
0 & C_{34} & C_{54} & 0 & 0 & D_{\gamma 4} & 0 & 1_{a_4} & 0 & 0 & 0 & 0 \\
0 & D_{3\alpha} & 0 & q1_{b_\alpha} + E_\alpha & 0 & 0 & 0 & 0 & D_{6\alpha} & 1_{b_\alpha} & 0 & F_{\gamma\alpha} \\
0 & 0 & 0 & 0 & q1_{b_\beta} & F_{\gamma\beta} & 0 & 0 & 0 & 0 & 1_{b_\beta} & 0 \\
\hline
C_{12} & C_{32} & 0 & 0 & D_{\beta 2} & 0 & 1_{a_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_{a_5} & 0 & 0 & 0 & 0 & C_{45} & C_{65} & 0 & D_{\beta 5} & 0 \\
0 & 0 & 0 & 1_{b_\alpha} & 0 & 0 & 0 & 0 & 0 & 1_{b_\alpha} & F_{\beta\alpha} & 0 \\
D_{1\gamma} & 0 & 0 & 0 & F_{\beta\gamma} & q1_{b_\gamma} + E_\gamma & 0 & D_{4\gamma} & 0 & 0 & 0 & 1_{b_\gamma} \\
\hline
0 & 1_{a_3} & 0 & 0 & 0 & 0 & C_{23} & C_{43} & 0 & D_{\alpha 3} & 0 & 0 \\
C_{16} & 0 & C_{56} & D_{\alpha 6} & 0 & 0 & 0 & 0 & 1_{a_6} & 0 & 0 & 0 \\
0 & 0 & D_{5\beta} & 0 & 1_{b_\beta} + E_\beta & 0 & D_{2\beta} & 0 & 0 & F_{\alpha\beta} & 1_{b_\beta} & 0 \\
0 & 0 & 0 & F_{\alpha\gamma} & 0 & 1_{b_\gamma} & 0 & 0 & 0 & 0 & 0 & 1_{b_\gamma}
\end{array}\right|
$$

Under this map, simple Q-representations are mapped to irreducible $C_2 * C_3$-representations, and if the coefficients of the block-matrices C_{ij}, D_{ij}, E_i and F_{ij} occurring in B give a parametrization of a Zariski open subset of the quotient variety $\mathtt{iss}_\tau\ Q$, then the n-dimensional representations of the 3-string braid group B_3

given by

$$\begin{cases} \sigma_1 \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \sigma_2 \mapsto \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases} \qquad \begin{cases} a = a_1 + a_3 + a_5 + b_\alpha + b_\beta \\ b = a_2 + a_4 + a_6 + b_\alpha + b_\gamma \\ x = a_1 + a_4 + b_\alpha + b_\beta \\ y = a_2 + a_5 + b_\alpha + b_\gamma \\ z = a_3 + a_6 + b_\beta + b_\gamma \end{cases}$$

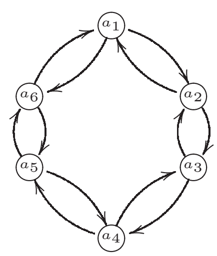
contain a Zariski dense set of irreducible B_3-representations in the component X_σ of $\mathtt{iss}_n\ B_3$.

4. The main result

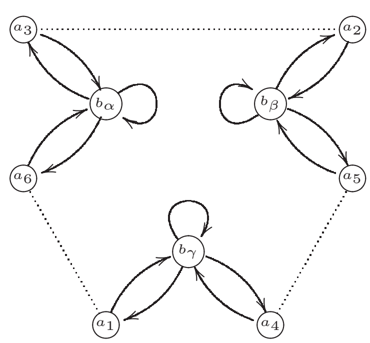
In view of the previous section it remains to find for each $\sigma = (a, b; x, y, z)$ satisfying

$$a + b = n = x + y + z \quad \text{and} \quad x = max(x, y, z) \leq b = min(a, b)$$

a judiciously chosen dimension-vector $\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma)$ of type σ together with an explicit rational parametrization of $\mathtt{iss}_\tau\ Q$. We will separate this investigation in two cases, sharing the same underlying strategy. First we choose $a_1, a_2, a_3, a_4, a_5, a_6$ such that $\sigma_1 = (a_1 + a_3 + a_5, a_2 + a_4 + a_6; a_1 + a_4, a_2 + a_5, a_3 + a_6)$ is a component containing simples and such that we have an explicit rational parametrization of the isoclasses of the quiver-setting



The upshot being that for a general representation the stabilizer subgroup reduces to $\mathbb{C}^*(1_{a_1} \times \ldots \times 1_{a_6})$. But then, the additional arrows D_{ij} and E_i, that is the quiver setting



give three settings corresponding to quiver settings of canonical linear systems with $m = p = a_i + a_{i+3}$ and the results of section 2 give a rational parametrization of the isoclasses and further reduces the stabilizer subgroup to $\mathbb{C}^*(1_{a_1} \times \ldots \times 1_{a_6} \times$

$1_{b_\alpha} \times 1_{b_\beta} \times 1_{b_\gamma}$). This then leaves the trivial action on the remaining arrows F_{ij} and hence these generic matrices conclude the desired rational parametrization.

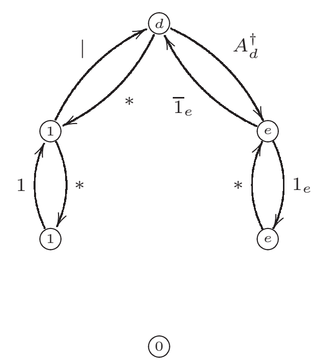
4.1. **Case 1 : $a > b$.** Define $d = a-b$, $e = d-1$, $f = b-z$, $g = b-y$ and $h = b-x$, then the dimension-vector

$$\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (d, e, e, 0, 1, 1, f, g, h)$$

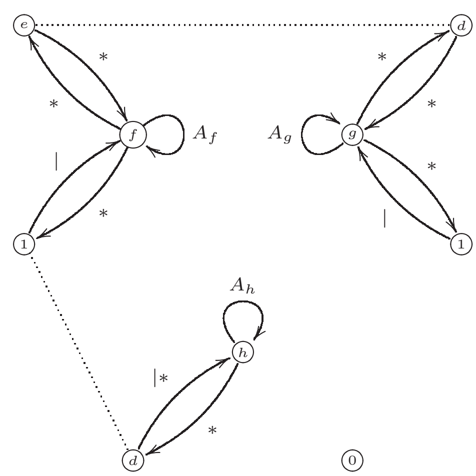
is of type σ. If we denote by

$$\begin{cases} * & \text{a generic matrix} \\ | & \text{the column vector } \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ \overline{1}_n & \text{the } n+1 \times n \text{ matrix } \begin{bmatrix} \underline{0} \\ 1_n \end{bmatrix} \end{cases}$$

and the (reduced) companion matrices as in lemma 2, then using lemma 3 a rational parametrization of the first stage is given by the representations



By lemma 1 a rational parametrization of the second stage is then given by the representations



This concludes the proof of

Theorem 2. *A Zariski dense rational parametrization of the component X_σ of $\mathtt{iss}_n B_3$ where $\sigma = (a,b;x,y,z)$ with $a > b$ is given by the representations*

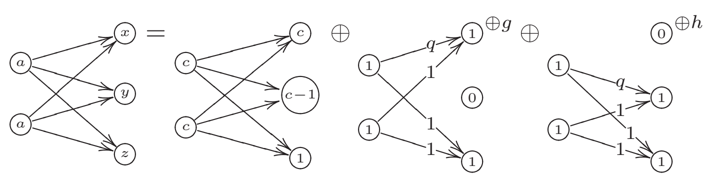
$$\begin{cases} \sigma_1 \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \sigma_2 \mapsto \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases}$$

for all $n \times n$ matrices B of the form

$$\left|\begin{array}{cccccc|ccccc} 1_d & 0 & 0 & 0 & 0 & 0 & \overline{1}_e & | & 0 & 0 & * \\ 0 & * & 0 & q1_f + A_f & 0 & 0 & 0 & | & 1_f & 0 & * \\ 0 & 0 & 0 & 0 & q1_g & * & 0 & 0 & 0 & 1_g & 0 \\ \hline A_d^\dagger & * & 0 & 0 & * & 0 & 1_e & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1_f & 0 & 0 & 0 & 0 & 1_f & * & 0 \\ |* & 0 & 0 & 0 & * & q1_h + A_h & 0 & 0 & 0 & 0 & 1_h \\ \hline 0 & 1_e & 0 & 0 & 0 & 0 & 1_e & 0 & * & 0 & 0 \\ * & 0 & 1 & * & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & | & 0 & 1_g + A_g & 0 & * & 0 & * & 1_g & 0 \\ 0 & 0 & 0 & * & 0 & 1_h & 0 & 0 & 0 & 0 & 1_g \end{array}\right|$$

where $d = a - b$, $e = d - 1$, $f = b - z$, $g = b - y$ and $h = b - x$.

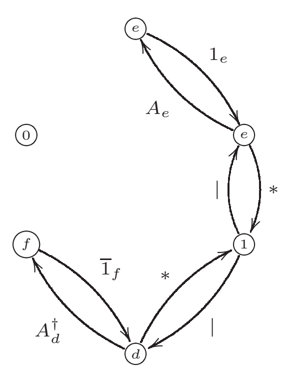
4.2. **Case 2 :** $a = b$**.** Define $c = x + y + 1 - a$, $g = a - y - 1$ and $h = a - x$, which corresponds to the decomposition



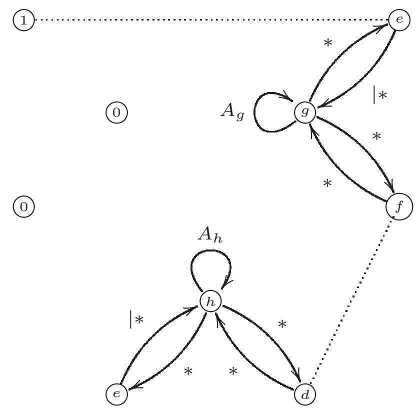
If c is odd, define $c = 2d + 1$, $e = d + 1$ and $f = d - 1$, then the dimension vector

$$\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (e, e, 1, d, f, 0, 0, g, h)$$

is of type σ. Then, using lemma 2 a rational parametrization for the first stage is given by the representations



Using lemma 1 we then get that a rational parametrization of the second stage is given by the following representations



If c is even, we can define $c = 2e$ and $f = e - 1$ in which case the dimension vector

$$\tau = (a_1, a_2, a_3, a_4, a_5, a_6, b_\alpha, b_\beta, b_\gamma) = (e, e, 1, e, f, 0, 0, g, h)$$

is of type σ and exactly the same representations give a rational parametrization of both stages if we replace all occurrences of d by e. This then concludes the proof of

Theorem 3. *A Zariski dense rational parametrization of the component X_σ of $\mathtt{iss}_n B_3$ where $\sigma = (a, b; x, y, z)$ with $a = b$ is given by the representations*

$$\begin{cases} \sigma_1 \mapsto \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \sigma_2 \mapsto \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases}$$

for all $n \times n$ matrices B of the form

$$\left|\begin{array}{ccccc|cccc} 1_e & 0 & 0 & 0 & 0 & A_e & 0 & 0 & * \\ 0 & | & \overline{1}_f & 0 & * & 0 & 1_d & 0 & 0 \\ 0 & 0 & 0 & q1_g & * & 0 & 0 & 1_g & 0 \\ \hline 1_e & | & 0 & * & 0 & 1_e & 0 & 0 & 0 \\ 0 & 0 & 1_f & 0 & 0 & 0 & A_d^\dagger & * & 0 \\ |* & 0 & 0 & * & q1_h + A_h & 0 & * & 0 & 1_h \\ \hline 0 & 1 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 1_g + A_g & 0 & |* & 0 & 1_g & 0 \\ 0 & 0 & 0 & 0 & 1_h & 0 & 0 & 0 & 1_h \end{array}\right|$$

where $g = a - y - 1$, $h = a - x$ and if $c = x + y + 1 - a$ is odd we take $c = 2d + 1$, $e = d + 1$ and $f = d - 1$ whereas if $c = x + y + 1 - a$ is even we take $c = 2e$ and $f = e - 1$ and we replace all occurrences of d in the matrix to e.

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