MATRIX TRANSPOSITION AND BRAID REVERSION

LIEVEN LE BRUYN

Abstract. Matrix transposition induces an involution τ on the equivalence classes of semi-simple n-dimensional complex representations of the three string braid group B_3. We show that a connected component of this variety can detect braid-reversion or that τ acts as the identity on it. We classify the fixed-point components.

1. Introduction

If $\phi = (X_1, X_2)$ is an n-dimensional complex representation of the three string braid group $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$, then so is the pair of transposed matrices $\tau(\phi) = (X_1^{tr}, X_2^{tr})$. In this paper we investigate when ϕ is equivalent to $\tau(\phi)$.

This problem is relevant to detect braid- and knot-reversion. Recall that a knot is said to be invertible if it can be deformed continuously to itself, but with the orientation reversed. There do exist non-invertible knots, the unique one with a minimal number of crossings is knot 8_{17}, see the Knot Atlas [4], which is the closure of the three string braid $b = \sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2$. Proving that 8_{17} is not invertible comes down to separating the conjugacy class of the braid b from that of its reversed braid $b' = \sigma_2^2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-2}$. Now, observe that a B_3-representation ϕ can separate b from b', via $Tr_\phi(b) \neq Tr_\phi(b')$, only if ϕ is not equivalent to the transposed representation $\tau(\phi)$.

The involution τ on the affine variety $\mathtt{rep}_n B_3$ of all n-dimensional representations passes to an involution τ on the quotient variety $\mathtt{rep}_n B_3/PGL_n = \mathtt{iss}_n B_3$, classifying equivalence classes of semi-simple n-dimensional representations. Recall from [7] and [9] that $\mathtt{iss}_n B_3$ decomposes as a disjoint union of its irreducible components

$$\mathtt{iss}_n B_3 = \sqcup_\alpha \ \mathtt{iss}_\alpha B_3$$

and the components containing a Zariski open subset of simple B_3-representations are classified by $\alpha = (a, b; x, y, z) \in \mathbb{N}^5$ satisfying $n = a + b = x + y + z$, $a \geq b$ and $x = max(x, y, z) \leq b$.

Theorem 1. *If the component $\mathtt{iss}_\alpha B_3$ contains simple representations, then τ acts as the identity on it if and only if α is equal to*

- $(1, 0; 1, 0, 0)$, *or* $(4, 2; 2, 2, 2)$, *or*
- $(k, k; k, k-1, 1)$, *or* $(k, k; k, 1, k-1)$, *or*
- $(k+1, k; k, k, 1)$, *or* $(k+1, k; k, 1, k)$

for some $k \geq 1$. In all these cases, dim $\mathtt{iss}_\alpha B_3 = n$. In all other cases, $\mathtt{iss}_\alpha B_3$ contains simple representations ϕ such that $Tr_\phi(b) \neq Tr_\phi(b')$, that is, $\mathtt{iss}_\alpha B_3$ can detect braid-reversion.

Note that this result generalizes [7] where it was proved that there is a unique component of $\mathtt{iss}_6 B_3$, namely $\mathtt{iss}_\alpha B_3$ for $\alpha = (3,3;2,2,2)$, containing representations ϕ such that $Tr_\phi(b) \neq Tr_\phi(b')$.

2. The involution τ and stable quiver representations

In this section we follow Bruce Westbury [9] reducing the study of simple B_3-representations to specific stable quiver representations, and, we will describe the involution τ in terms of these representations.

If $\phi = (X_1, X_2)$ is a simple n-dimensional B_3-representation, then the central element $c = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$ acts via a scalar matrix λI_n for some $\lambda \neq 0$. Hence, $\phi' = \lambda^{-1/6}\phi = (X_1', X_2')$ is a simple representation of the quotient group

$$B_3/\langle c \rangle = \langle s, t \mid s^2 = t^3 = e \rangle \simeq C_2 * C_3 \simeq \Gamma$$

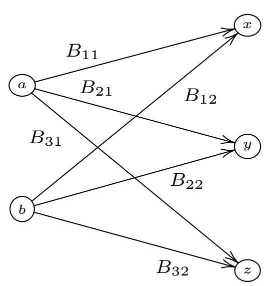
which is the free product of cyclic groups of order two and three (and thus isomorphic the modular group $\Gamma = PSL_2(\mathbb{Z})$) where s and t are the images of $\sigma_1\sigma_2\sigma_1$ and $\sigma_1\sigma_2$. Decompose the underlying n-dimensional space $V = \mathbb{C}^n_{\phi'}$ into eigenspaces for the actions of s and t

$$V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2}$$

with ρ a primitive 3rd root of unity. For $a = dim(V_+), b = dim(V_-)$, $x = dim(V_1), y = dim(V_\rho)$ and $z = dim(V_{\rho^2})$, clearly $a + b = n = x + y + z$. Choose a vector-space basis for V compatible with the decomposition $V_+ \oplus V_-$ and another basis of V compatible with the decomposition $V_1 \oplus V_\rho \oplus V_{\rho^2}$, then the corresponding base change block matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \in GL_n(\mathbb{C})$$

determines the quiver representation V_B with dimension vector $\alpha = (a, b; x, y, z)$



The quiver representation V_B is semi-stable in the sense of [3], meaning that for every proper sub-representations W, with dimension vector $\beta = (a', b'; x', y', z')$ we have $x' + y' + z' \geq a' + b'$. If this inequality is strict for all proper subrepresentations W, we call V_B a stable representation, which is equivalent to the Γ-representation $V = \mathbb{C}^n_{\phi'}$ being simple. Westbury [9] showed that two Γ-representations are equivalent if and only if the corresponding quiver representations are isomorphic $V_B \simeq V'_{B'}$, that is, there exist base changes in the eigenspaces

$$(M_1, M_2, N_1, N_2, N_3) \in GL(\alpha) = GL_a \times GL_b \times GL_x \times GL_y \times GL_z$$

such that

$$\begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \\ B'_{31} & B'_{32} \end{bmatrix} \begin{bmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

Working backwards, we recover the B_3-representation $\phi = (X_1, X_2)$ from the invertible matrix B via

$$(*) \begin{cases} X_1 = \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ X_2 = \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases}$$

Proposition 1. *If the n-dimensional simple B_3-representation $\phi = (X_1, X_2)$ is determined by $\lambda \in \mathbb{C}^*$ and the stable quiver representation V_B, then $\tau(\phi) = (X_1^{tr}, X_2^{tr})$ is isomorphic to the representation determined by λ and the stable quiver representation $V_{(B^{-1})^{tr}}$.*

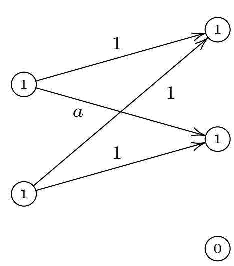
Proof. Taking transposes of the formulas $(*)$ for the X_i we get

$$\begin{cases} X_1^{tr} = \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{tr} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} (B^{-1})^{tr} \\ X_2^{tr} = \lambda^{1/6} B^{tr} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} (B^{-1})^{tr} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \end{cases}$$

Conjugating these with the matrix $\begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix}$ (which is also a base change action in $GL(\alpha)$) we obtain again a matrix-pair in standard-form $(*)$, this time replacing the matrix B by the matrix $(B^{-1})^{tr}$. □

That is, we have reduced the original problem of verifying whether or not $\phi \simeq \tau(\phi)$ as B_3-representations to the problem of verifying whether or not the two stable representations V_B and $V_{(B^{-1})^{tr}}$ lie in the same $GL(\alpha)$-orbit.

Example 1. *The two components $\mathtt{iss}_\alpha B_3$ containing simple 2-dimensional B_3-representations for $\alpha = (1,1;1,1,0)$ or $(1,1;1,0,1)$ are fixed-point components for the involution τ. A general stable $\alpha = (1,1;1,1,0)$ dimensional representation V_B is isomorphic to one of the form*



with $a \neq 1$*. Hence, we can take* $B = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$*. But then,* V_B *and* $V_{(B^{-1})^{tr}}$ *lie in the same* $GL(\alpha) = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$*-orbit because*

$$(B^{-1})^{tr} = \frac{1}{1-a} \begin{bmatrix} 1 & -a \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{a} \end{bmatrix} B \begin{bmatrix} \frac{1}{1-a} & 0 \\ 0 & \frac{-a}{1-a} \end{bmatrix}$$

3. The stratification and potential fixed-point components

In this section we will show that a component $\mathtt{iss}_\alpha B_3$ containing n-dimensional simple B_3-representations is a fixed-point component for the involution τ only if α is among the list of theorem 1.

Because the group algebra $\mathbb{C}\Gamma = \mathbb{C}C_2 * \mathbb{C}C_3$ is a formally smooth algebra, we have a Luna stratification of $\mathtt{iss}_\alpha\Gamma$ by representation types, see [6, §5.1]. A point p in $\mathtt{iss}_\alpha\Gamma$ determines the isomorphism class of a semi-simple Γ-representation

$$V_p = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

with all S_i distinct simple Γ-representations with corresponding dimension vectors $\beta_i = (a_i, b_i; x_i, y_i, z_i)$. We say that p (or V_p) is of representation type

$$\tau = (e_1, \beta_1; \ldots; e_k, \beta_k) \quad \text{and clearly} \quad \alpha = \sum_i e_i \beta_i$$

With $\mathtt{iss}_\alpha\Gamma(\tau)$ we denote the subset of all points of representation type τ. Recall that β_i is the dimension vector of a simple Γ-representation if and only if $a_i + b_i = x_i + y_i + z_i$ and $max(x_i, y_i, z_i) \leq min(a_i, b_i)$ if $x_i y_i z_i \neq 0$ (the remaining cases being the 1- and 2-dimensional components). It follows from Luna's results [8] that every $\mathtt{iss}_\alpha\Gamma(\tau)$ is a locally closed smooth irreducible subvariety of $\mathtt{iss}_\alpha\Gamma$ of dimension $\sum_i (1 + 2a_i b_i - (x_i^2 + y_i^2 + z_i^2)$ and that

$$\mathtt{iss}_\alpha\Gamma = \bigsqcup_\tau \mathtt{iss}_\alpha\Gamma(\tau)$$

is a finite smooth stratification of $\mathtt{iss}_\alpha\Gamma$. Degeneration of representation types, see [6, p. 247], defines an ordering $\leq$ on representation types and by [6, Prop. 5.3] we have that $\mathtt{iss}_\alpha\Gamma(\tau')$ lies in the Zariski closure of $\mathtt{iss}_\alpha\Gamma(\tau)$ if and only if $\tau' \leq \tau$.

Observe that the involution τ on $\mathtt{iss}_\alpha\Gamma$ induced by $\tau(V_B) = V_{(B^{-1})^{tr}}$ preserves the strata and its restriction to $\mathtt{iss}_\alpha\Gamma(\tau)$ is induced by the involutions τ on the components $\mathtt{iss}_{\beta_i}\Gamma$. As the fixed-point set of τ is a closed subvariety of $\mathtt{iss}_\alpha\Gamma$ we deduce immediately :

Lemma 1. *If* τ *is the identity on a Zariski open subset of* $\mathtt{iss}_\alpha\Gamma(\tau)$*, then* $\tau = id$ *on all strata* $\mathtt{iss}_\alpha\Gamma(\tau')$ *with* $\tau' \leq \tau$*. Conversely, if* $\tau = (e_1, \beta_1; \ldots; e_k, \beta_k)$ *and* $\tau \neq id$ *on one of the components* $\mathtt{iss}_{\beta_i}\Gamma$*, then* $\tau \neq id$ *on all strata* $\mathtt{iss}_\alpha\Gamma(\tau')$ *with* $\tau \leq \tau'$*.*

In [7] we have shown that for $\beta = (3, 3; 2, 2, 2)$ there are simple B_3-representations able to separate the braid b from the introduction from its reversed braid b'. In particular, τ does not act as the identity on $\mathtt{iss}_\beta\Gamma$. We proved this by parametrizing the matrices B for a dense open subset of $\mathtt{iss}_\beta \Gamma$ by

$$B = \begin{bmatrix} 1 & 0 & 0 & a & 0 & f \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & d & e \\ 0 & 1 & 0 & b & c & 0 \\ g & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

for free parameters $a, \ldots, g$. We then computed the matrix-pair $\phi = (X_1, X_2)$ from $(*)$ with generic values of the parameters in $\mathbb{Z}[\rho]$ and checked that $Tr_\phi(b) \neq Tr_\phi(b')$.

Proposition 2. *If α is the dimension vector of a simple Γ-representation such that $\alpha \geq \beta = (3,3;2,2,2)$, then $\tau \neq id$ on $\mathtt{iss}_\alpha \Gamma$ and there are simple representations $\phi \in \mathtt{iss}_\alpha \Gamma$ such that $Tr_\phi(b) \neq Tr_\phi(b')$.*

Proof. The unique open stratum of $\mathtt{iss}_\alpha \Gamma$ corresponds to the unique maximal representation type $\tau_{gen} = (1, \alpha)$, that is, $\mathtt{iss}_\alpha \Gamma(\tau_{gen})$ is the open set of simple Γ-representations.

If $\alpha - \beta$ is the dimension vector of a simple Γ-representation, then we have a representation type $\tau = (1, \beta; 1, \alpha - \beta)$ such that $\tau \neq id$ and $Tr(b) \neq Tr(b')$ on $\mathtt{iss}_\alpha \Gamma(\tau)$. But then, by the previous lemma, these facts also hold for $\mathtt{iss}_\alpha \Gamma(\tau_{gen})$.

If $\alpha - \beta$ is not the dimension vector of a simple Γ-representation, we consider the generic (maximal) representation type $\tau' = (e_1, \beta_1; \ldots; e_k, \beta_k)$ in $\mathtt{iss}_{\alpha-\beta} \Gamma$. But then, $\tau = (1, \beta; e_1, \beta_1; \ldots; e_k, \beta_k)$ is a representation type for $\mathtt{iss}_\alpha \Gamma$ and we can repeat the argument above. □

Proposition 3. *If $\alpha = (a, b; x, y, z)$ is a simple dimension vector such that τ acts trivially on $\mathtt{iss}_\alpha B_3$, then*

$$dim\ \mathtt{iss}_\alpha B_3 = n = a + b = x + y + z$$

Proof. By the previous result we must have $\beta \not\leq \alpha$ and hence either $n \leq 5$ or $min(x, y, z) = 1$. For a simple B_3-dimension vector we may assume that $a \geq b$ and $x = max(x, y, z)$, which leaves us with the following list of potential fixed-point components

n	α	$dim\ \mathtt{iss}_\alpha B_3$
1	$(1,0;1,0,0)$	1
2	$(1,1;1,1,0)$	2
	$(1,1;1,0,1)$	2
3	$(2,1;1,1,1)$	3
4	$(2,2;2,1,1)$	4
5	$(3,2;2,2,1)$	5
	$(3,2;2,1,2)$	5
6	$(3,3;3,2,1)$	6
	$(3,3;3,1,2)$	6
	$(4,2;2,2,2)$	6
$2k$	$(k,k;k,k-1,1)$	$2k$
	$(k,k;k,1,k-1)$	$2k$
$2k+1$	$(k+1,k;k,k,1)$	$2k+1$
	$(k+1,k;k,1,k)$	$2k+1$

□

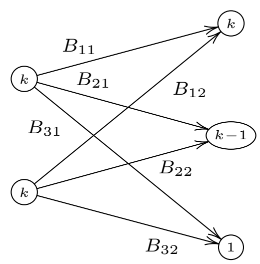
By example 1 we know that the 1- and 2-dimensional components are fixed-point components. All other potential fixed-point components belong to the infinite families, with one exception: $(4,2;2,2,2)$. In the following sections we will prove that all of these are indeed fixed-point components.

4. The infinite families

In this section we will prove that for $\alpha = (k,k;k,k-1,1)$ (the even case) and $\alpha = (k+1,k;k,k,1)$ (the odd case), $\mathtt{iss}_\alpha B_3$ is a fixed-point component. We will prove the even case by direct matrix calculations and deduce the odd case from it by a degeneration argument.

Proposition 4. *For all $k \in \mathbb{N}_+$ and $\alpha = (k,k;k,k-1,1)$, $\mathtt{iss}_\alpha B_3$ is a fixed-point component.*

Proof. A general representation in $\mathtt{iss}_\alpha \Gamma$ corresponds to an invertible $2m \times 2m$ matrix B and quiver representation



After a base change in the lower-left hand vertex, we may assume that the modified matrix blocks are such that
$$\begin{bmatrix} B'_{22} \\ B'_{32} \end{bmatrix} = I_k$$
The block B_{12} is modified to an invertible $k \times k$ matrix B'_{12} which becomes the identity matrix I_k after performing a base change in the top-right hand vertex. This changes the block B_{11} to an invertible matrix B'_{11} which becomes the identity matrix I_k after a base change in the top-left hand vertex. Hence, we may assume that, up to isomorphism, the matrix B has the following block form
$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} = \begin{bmatrix} I_k & I_k \\ A & I_k \end{bmatrix}$$
with A an invertible matrix such that B is invertible. One verifies that
$$(B^{-1})^{tr} = \begin{bmatrix} -C & I_k + C \\ C & -C \end{bmatrix} \qquad \text{with} \qquad C = (A - I_k)^{-1}$$
and performing the base change
$$(AC^{-1}, -C, -A^{-1}, I_{k-1}, I_1) \in GL_k \times GL_k \times GL_k \times GL_{k-1} \times GL_1$$
we obtain
$$B = \begin{bmatrix} I_k & I_k \\ A & I_k \end{bmatrix} = \begin{bmatrix} -A^{-1} & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} -C & I_k + C \\ C & -C \end{bmatrix} \begin{bmatrix} C^{-1}A & 0 \\ 0 & -C^{-1} \end{bmatrix}$$
Therefore, the Γ-representations determined by the matrices B and $(B^{-1})^{tr}$ are equivalent and hence the involution τ is the identity map on the component $\mathtt{iss}_\alpha B_3$. $\square$

Proposition 5. *For all $k \in \mathbb{N}_+$ and $\alpha = (k+1, k; k, k, 1)$, $\mathtt{iss}_\alpha B_3$ is a fixed-point component.*

Proof. Let $\alpha_+ = (k+1, k+1; k+1, k, 1)$, then the stratum $\tau = (1, \alpha; 1, (0, 1; 1, 0, 0))$ lies in the closure of the generic stratum $\tau_{gen} = (1, \alpha_+)$ in $\mathtt{iss}_{\alpha_+}\Gamma$. The result follows from the proposition above and lemma 1. $\square$

5. The exceptional component and vector bundles on $\mathbb{P}_2$

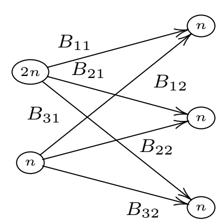
To finish the proof of theorem 1, it suffices to show that $\mathtt{iss}_\beta B_3$ is a fixed-point component for $\beta = (4, 2; 2, 2, 2)$. In [7] we have given a parametrization of the matrices B for a dense open subset of $\mathtt{iss}_\beta\Gamma$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & a & 0 \\ 0 & 1 & e & 1 & 0 & 1 \\ 1 & c & d & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & b \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

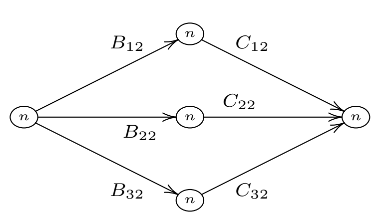
One can attempt to show that B and $(B^{-1})^{tr}$ belong to the same $GL(\beta)$-orbit by explicit computation. We follow a different approach, allowing us to connect this component to the study of stable vector bundles on $\mathbb{P}_2$.

Proposition 6. *For $\alpha = (2n, n; n, n, n)$, the component $\mathtt{iss}_\alpha\Gamma$ is birational to $M_{\mathbb{P}^2}(n; 0, n)$, the moduli space of semi-stable rank n bundles on $\mathbb{P}_2$ with Chern classes $c_1 = 0$ and $c_2 = n$.*

Proof. A representation in $\mathtt{rep}_\alpha\Gamma$ in general position



is such that $\psi : \mathbb{C}^{2n} \xrightarrow{B_{11} \oplus B_{21} \oplus B_{31}} \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}^n$ in injective, whence its cokernel defines maps $Cok(\psi) : \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}^n \xrightarrow{(C_{12}, C_{22}, C_{32})} \mathbb{C}^n$ and therefore a representation for the quiver setting



By the general theory of reflection functors, isomorphism classes of representations are preserved under this construction. By the fundamental theorem of GL_n-invariants [5, Thm. II.4.1] we can eliminate the base change action in the middle vertices and obtain a representation of the quiver setting

$$\begin{tikzcd} \circled{n} \arrow[r, bend left, "C_{12}B_{12}"] \arrow[r, "C_{22}B_{22}"] \arrow[r, bend right, "C_{32}B_{32}"'] & \circled{n} \end{tikzcd}$$

By results of Klaus Hulek [2], the corresponding moduli space of semi-stable quiver representations (as in [3] for the stability structure $(-1,1)$) is birational to $M_{\mathbb{P}_2}(n;0,n)$. $\square$

Proposition 7. $\mathtt{iss}_\beta B_3$ *is a fixed point component.*

Proof. By results of Wolf Barth [1], we know that a stable rank 2 bundle $\mathcal{E}$ on the projective plane with Chern-classes $c_1 = 0$ and $c_2 = 2$ is determined up to isomorphism by its curve of jumping lines, that is the collection of those lines $L \subset \mathbb{P}_2$ such that $\mathcal{E}|L \not\simeq \mathcal{O}_L^{\oplus 2}$. If $\mathcal{E}$ is determined by the quiver representation as in the previous proposition and if x, y, z are projective coordinates of the dual plane $\mathbb{P}_2^*$, then the equation of this curve of jumping lines is given by

$$det(C_{12}B_{12}x + C_{22}B_{22}y + C_{32}B_{32}z) = 0$$

In terms of the matrix B and its inverse B^{-1} these 2×2 matrices are given as

$$\underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ C_{12} & C_{22} & C_{32} \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} * & * & B_{12} \\ * & * & B_{22} \\ * & * & B_{32} \end{bmatrix}}_{B} = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

But then, the bundle $\mathcal{F}$ corresponding to the matrix $(B^{-1})^{tr}$ is determined by the 2×2 matrices B_{ij} and C_{ij} such that

$$\underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ B_{12}^{tr} & B_{22}^{tr} & B_{32}^{tr} \end{bmatrix}}_{B^{tr}} \underbrace{\begin{bmatrix} * & * & C_{12}^{tr} \\ * & * & C_{22}^{tr} \\ * & * & C_{32}^{tr} \end{bmatrix}}_{(B^{-1})^{tr}} = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

and hence its curve of jumping lines

$$det(B_{12}^{tr}C_{12}^{tr}x + B_{22}^{tr}C_{22}^{tr}y + B_{32}^{tr}C_{32}^{tr}z)$$

is the same as that for $\mathcal{E}$ and hence by Barth's result $\mathcal{E} \simeq \mathcal{F}$. $\square$

Remark 1. *One can repeat the above argument verbatim for $\alpha = (2n, n; n, n, n)$. However, if $n > 2$, the bundle $\mathcal{E}$ corresponding to the matrix B is determined by its curve of jumping lines (defined as above by the $n \times n$ matrices B_{ij} and C_{ij}) together with a half-canonical divisor on it, see* [2]. *Whereas the curve of jumping lines Y of the bundle $\mathcal{F}$ corresponding to the matrix $(B^{-1})^{tr}$ coincides with that of $\mathcal{E}$, the involution τ acts non-trivially on the Jacobian Pic_Y^d where $d = \frac{1}{2}n(n-1)$.*

References

[1] Wolf Barth, *Moduli of vector bundles on the projective plane*, Invent. Math. 42 (1977) 63-91

[2] Klaus Hulek, *On the classification of stable rank r vector bundles on the projective plane*, Birkhäuser PM 7 (1980) 113-144

[3] Alastair King, *Moduli of representations of finite-dimensional algebras*, Quat. J. Math. Oxford Ser. (2) 45 (1994) 515-530

[4] The Knot Atlas, `http://katlas.math.toronto.edu/wiki/`

[5] Hanspeter Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics D1, Vieweg (1985)
[6] Lieven Le Bruyn, *Noncommutative geometry and Cayley-smooth orders*, Pure and applied mathematics **290**, Chapman & Hall (2008)
[7] Lieven Le Bruyn, *Dense families of B_3-representations and braid reversion*, Journal of Pure and Appl. Algebra 215 (2011) 1003-1014, math.RA/1003.1610
[8] Domingo Luna, *Slices étales* Bull. Soc. Math. France Mémoire 33 (1973) 81-105
[9] Bruce Westbury, *On the character varieties of the modular group*, preprint Nottingham (1995)

Department of Mathematics, University of Antwerp, Middelheimlaan 1, B-2020 Antwerp (Belgium), lieven.lebruyn@ua.ac.be