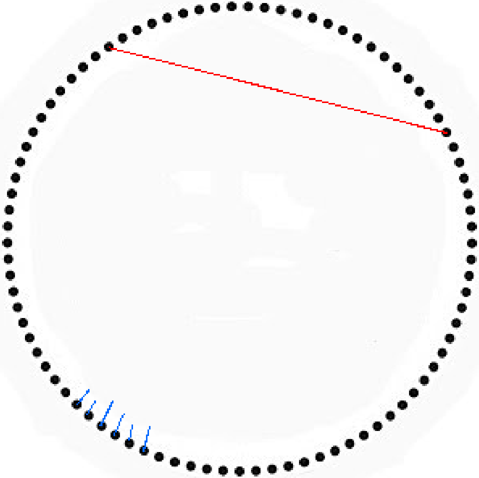
noncommutative $\mathbb{F}_1$-geometry

lieven.lebruyn@ua.ac.be

krakow, july 2012

new topology on μ_∞



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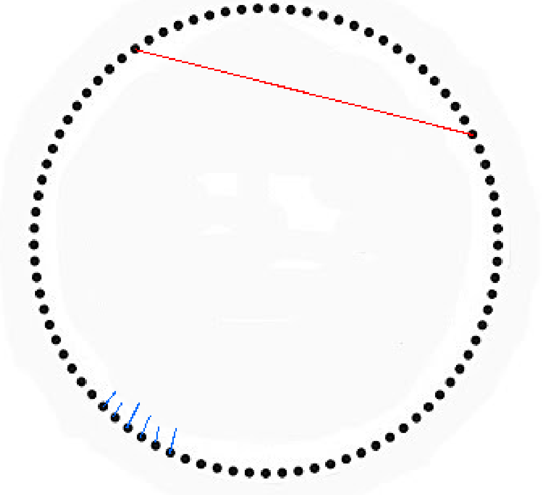
$\alpha \leftrightarrow \beta$ iff $ord(\frac{\alpha}{\beta}) = p^k$

stable under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$



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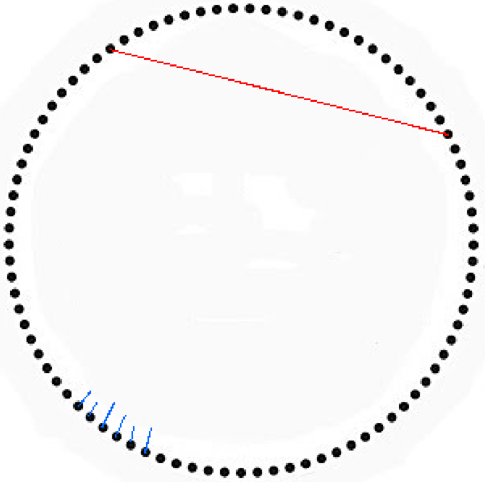

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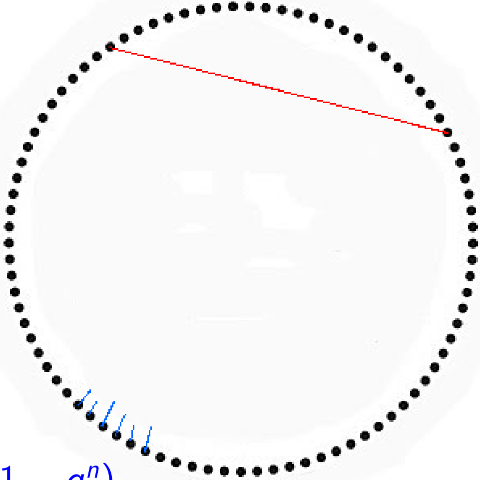

$\boxed{\text{new functions on } \boldsymbol{\mu}_\infty}$

$\lim\limits_{\leftarrow} \mathbb{Z}[q, q^{-1}]/([n]!)$

$[n]! = (1-q)(1-q^2)\dots(1-q^n)$

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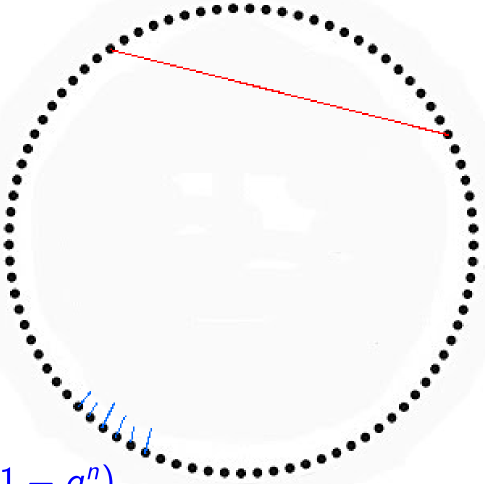

new functions on $\boldsymbol{\mu}_\infty$

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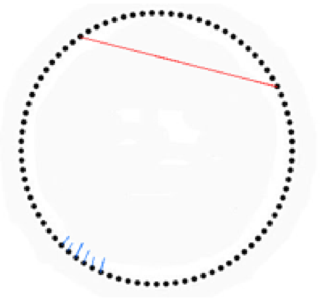
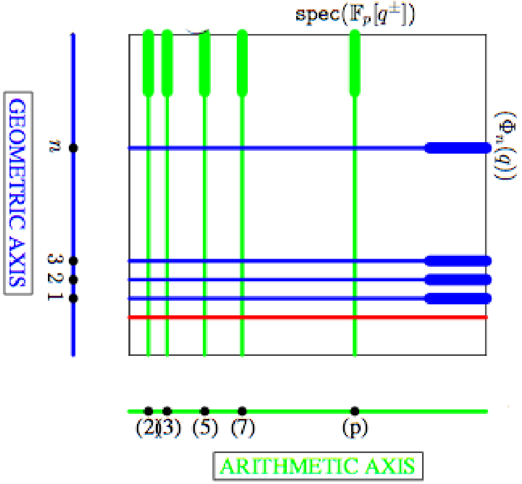
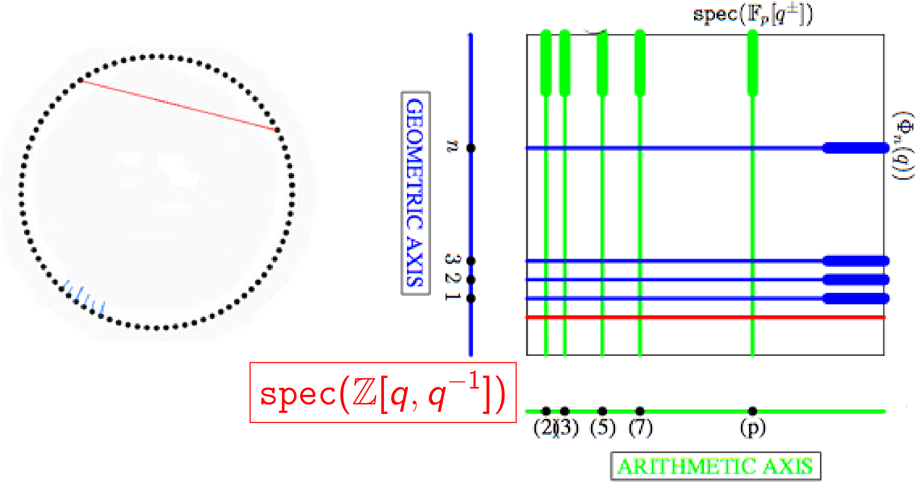
$1 + [1]! + [2]! + \dots + [n]! + \dots = -\frac{1}{2}\sum_{n=1}^{\infty} n\chi_{12}(n) q^{\frac{n^2-1}{24}}$

"functions leaking out of roots of unity" (Zagier)





spec($\mathbb{F}_p[q^{\pm}]$)
GEOMETRIC AXIS
n
3 2 1
$(\Phi_n(q))$
(2) (3) (5) (7) (p)
ARITHMETIC AXIS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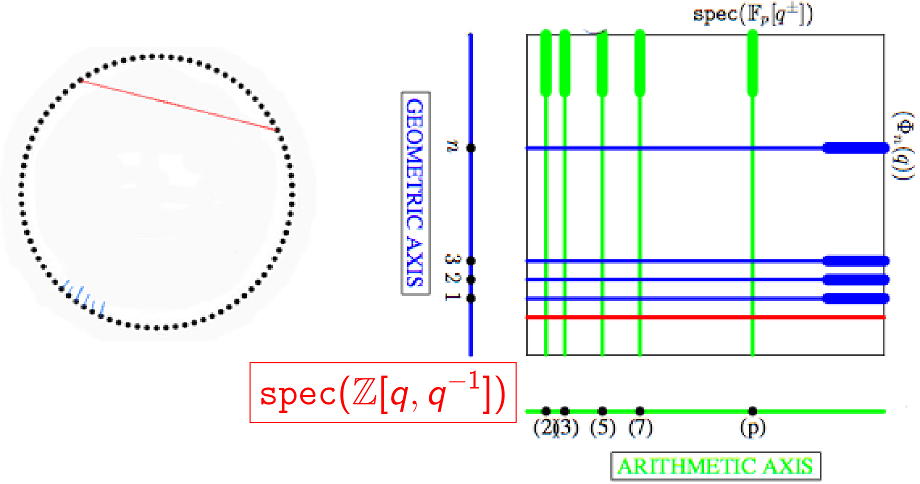

$$(\Phi_n(q), \Phi_m(q)) \neq \mathbb{Z}[q, q^{-1}] \Leftrightarrow \frac{m}{n} = p^k$$

arXiv.org > math > arXiv:math/0606241

Mathematics > Rings and Algebras

Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I

Maxim Kontsevich, Yan Soibelman

(Submitted on 11 Jun 2006 (v1), last revised 14 Aug 2006 (this version, v2))

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$\boxed{\text{commutative case}}$

$A = \mathbb{C}[X],\ A^o = \oplus_{x \in X} \varinjlim (A/\mathfrak{m}_x^n)^*$

$A \to \mathcal{O}(\mathrm{geo}_{\mathbb{C}}(A)) = (A^o)^* = \prod_{x \in X} \widehat{A}_{\mathfrak{m}_x}$

(non)commutative f-un geometry

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arXiv.org > math > arXiv:0909.2522

Mathematics > Rings and Algebras

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Borger's commutative approach via λ-rings

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$$\underbrace{A/I\ \lambda-ring}_{\text{f.g. free}/\mathbb{Z}} \begin{cases} \mathtt{geo}_{\mathbb{F}_1}(A) = \lim\limits_{\rightarrow} \mathtt{min}(A/I) \\ A^o_\lambda = \lim\limits_{\rightarrow} (A/I)^* \\ \mathcal{O}(\mathtt{geo}_{\mathbb{F}_1}(A)) = \lim\limits_{\leftarrow} A/I \end{cases}$$

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noncommutative example : discrete group Γ

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$$\Gamma \longrightarrow\!\!\!\!\!\rightarrow G \;\; \text{finite} \begin{cases} \mathtt{geo}_{\mathbb{F}_1}(\mathbb{Z}\Gamma) = \varinjlim \mathtt{min}(\mathbb{Z}G) \\ \mathcal{O}(\mathtt{geo}_{\mathbb{F}_1}(\mathbb{Z}\Gamma)) = \varprojlim \mathbb{Z}\Gamma / P_{i_1}^{e_1} P_{i_2}^{e_2} \dots P_{i_k}^{e_k} \end{cases}$$

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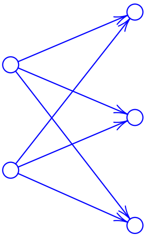

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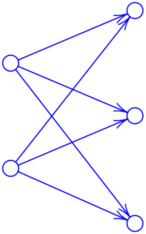

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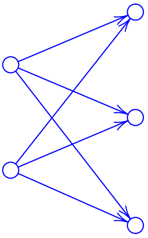

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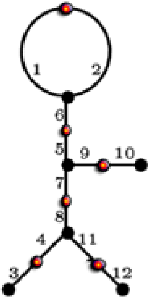
"the truly unimaginable richness of $\Gamma = PSL_2(\mathbb{Z})$, doubtless the most remarkable discrete infinite group ever encountered. All algebraic curves defined over number fields are 'contained' in the profinite completion $\widehat{\Gamma}$"
A. Grothendieck "Sketch of a programme"


1
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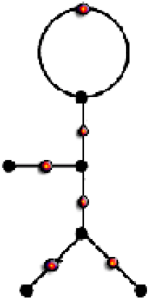
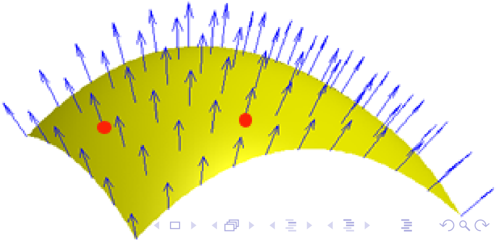
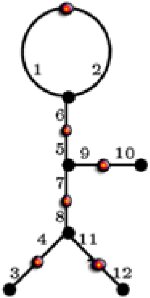
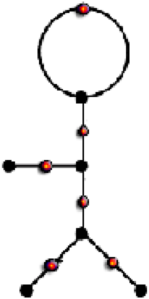


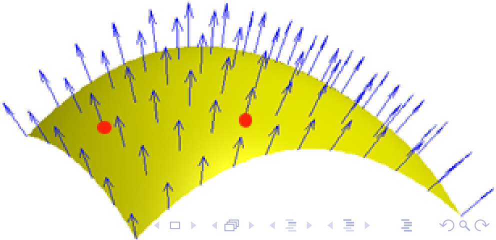


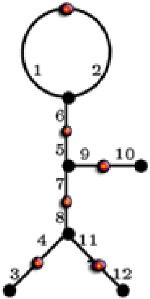


$$a = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$$

$$b = (1,6,2)(5,7,9)(8,4,11)$$







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$\langle a, b \rangle = M_{12} \qquad \mathbb{C}M_{12}/M_{11} = T \oplus X$

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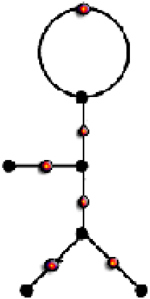

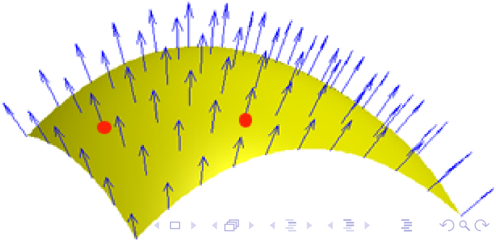

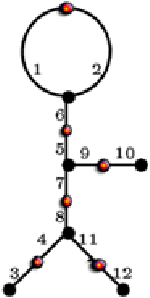

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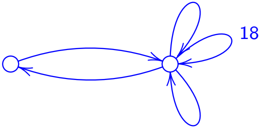

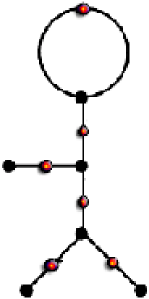

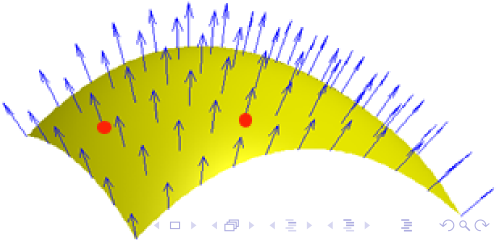

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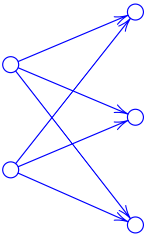

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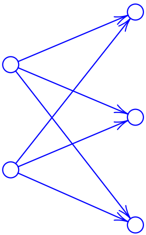

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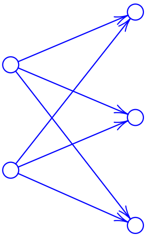

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$$\boxed{\text{permutation representations } \Gamma \leftrightarrow \text{'dessins d'enfant'}}$$

$$V = \mathbb{C}G/H \Rightarrow \overrightarrow{\mathtt{ext}} \mid \mathtt{irr}(V) \text{ is Galois invariant}$$

"the truly unimaginable richness of $\Gamma = PSL_2(\mathbb{Z})$, doubtless the most remarkable discrete infinite group ever encountered. All algebraic curves defined over number fields are 'contained' in the profinite completion $\widehat{\Gamma}$"
A. Grothendieck "Sketch of a programme"