

# MATRIX TRANSPOSITION AND BRAID REVERSION

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ABSTRACT. Matrix transposition induces an involution  $\tau$  on the isomorphism classes of semi-simple  $n$ -dimensional representations of the three string braid group  $B_3$ . We show that a connected component of this variety can detect braid-reversion or that  $\tau$  acts as the identity on it. We classify the fixed-point components.

## 1. INTRODUCTION

If  $\phi = (X_1, X_2)$  is an  $n$ -dimensional representation of the three string braid group  $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ , then so is the pair of transposed matrices  $\tau(\phi) = (X_1^{tr}, X_2^{tr})$ . In this paper we investigate when  $\phi \simeq \tau(\phi)$ .

This problem is relevant to detect braid- and knot-reversion. Recall that a knot is said to be invertible if it can be deformed continuously to itself, but with the orientation reversed. There do exist non-invertible knots, the unique one with a minimal number of crossings is knot  $8_{17}$ , see the Knot Atlas [4], which is the closure of the three string braid  $b = \sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2$ . Proving knot-vertibility of  $8_{17}$  comes down to separating the conjugacy class of the braid  $b$  from that of its reversed braid  $b' = \sigma_2^2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-2}$ . Now, observe that a  $B_3$ -representation  $\phi$  can separate  $b$  from  $b'$ , via  $Tr_\phi(b) \neq Tr_\phi(b')$ , only if  $\phi$  is not isomorphic to the transposed representation  $\tau(\phi)$ .

The involution  $\tau$  on the affine variety  $\mathbf{rep}_n B_3$  of all  $n$ -dimensional representations passes to an involution  $\tau$  on the quotient variety  $\mathbf{rep}_n B_3 / PGL_n = \mathbf{iss}_n B_3$ , classifying isomorphism classes of semi-simple  $n$ -dimensional representations. Recall from [7] and [9] that  $\mathbf{iss}_n B_3$  decomposes as a disjoint union of its irreducible components

$$\mathbf{iss}_n B_3 = \sqcup_\alpha \mathbf{iss}_\alpha B_3$$

and the components containing a Zariski open subset of simple  $B_3$ -representations are classified by  $\alpha = (a, b; x, y, z) \in \mathbb{N}^5$  satisfying  $n = a + b = x + y + z$ ,  $a \geq b$  and  $x = \max(x, y, z) \leq b$ .

**Theorem 1.** *If the component  $\mathbf{iss}_\alpha B_3$  contains simple representations, then  $\tau$  acts as the identity on it if and only if  $\alpha$  is equal to*

- $(1, 0; 1, 0, 0)$ , or  $(4, 2; 2, 2, 2)$ , or
- $(k, k; k, k - 1, 1)$ , or  $(k, k; k, 1, k - 1)$ , or
- $(k + 1, k; k, k, 1)$ , or  $(k + 1, k; k, 1, k)$

for some  $k \geq 1$ . In all these cases,  $\dim \mathbf{iss}_\alpha B_3 = n$ . In all other cases,  $\mathbf{iss}_\alpha B_3$  contains simple representations  $\phi$  such that  $Tr_\phi(b) \neq Tr_\phi(b')$ , that is,  $\mathbf{iss}_\alpha B_3$  can detect braid-reversion.

Note that this result generalizes [7] where it was proved that there is a unique component of  $\mathbf{iss}_6 B_3$ , namely  $\mathbf{iss}_\alpha B_3$  for  $\alpha = (3, 3; 2, 2, 2)$ , containing representations  $\phi$  such that  $Tr_\phi(b) \neq Tr_\phi(b')$ .

## 2. THE INVOLUTION $\tau$ AND STABLE QUIVER REPRESENTATIONS

In this section we follow Bruce Westbury [9] reducing the study of simple  $B_3$ -representations to specific stable quiver-representations, and, we will describe the involution  $\tau$  in terms of these representations.

If  $\phi = (X_1, X_2)$  is a simple  $n$ -dimensional  $B_3$ -representation, then the central element  $c = (\sigma_1 \sigma_2)^3 = (\sigma_1 \sigma_2 \sigma_1)^2$  acts via a scalar matrix  $\lambda I_n$  for some  $\lambda \neq 0$ . Hence,  $\phi' = \lambda^{-1/6} \phi = (X'_1, X'_2)$  is a simple representation of the quotient group

$$B_3 / \langle c \rangle = \langle s, t \mid s^2 = t^3 = e \rangle \simeq C_2 * C_3 \simeq \Gamma$$

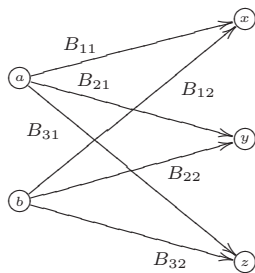
which is the free product of cyclic groups of order two and three (and thus isomorphic to the modular group  $\Gamma = PSL_2(\mathbb{Z})$ ) where  $s$  and  $t$  are the images of  $\sigma_1 \sigma_2 \sigma_1$  and  $\sigma_1 \sigma_2$ . Decompose the  $n$ -dimensional space  $V = \mathbb{C}_\phi^n$  into eigenspaces for the actions of  $s$  and  $t$

$$V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2}$$

with  $\rho$  a primitive 3rd root of unity. For  $a = \dim(V_+)$ ,  $b = \dim(V_-)$ ,  $x = \dim(V_1)$ ,  $y = \dim(V_\rho)$  and  $z = \dim(V_{\rho^2})$ , clearly  $a + b = n = x + y + z$ . Choose a vector-space basis for  $V$  compatible with the decomposition  $V_+ \oplus V_-$  and another basis of  $V$  compatible with the decomposition  $V_1 \oplus V_\rho \oplus V_{\rho^2}$ , then the corresponding base-change matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \in GL_n(\mathbb{C})$$

determines the quiver-representation  $V_B$  with dimension vector  $\alpha = (a, b; x, y, z)$



For  $B$  invertible, the representation  $V_B$  is semi-stable in the sense of [3], meaning that for every proper sub-representations  $W$ , with dimension vector  $\beta = (a', b'; x', y', z')$  we have  $x' + y' + z' \geq a' + b'$ . If all these inequalities are strict, we call  $V_B$  a stable representation, which is equivalent to the  $\Gamma$ -representation  $V = \mathbb{C}_\phi^n$  being simple. Westbury [9] showed that two  $\Gamma$ -representations  $V \simeq V'$  if and only if  $V_B \simeq V'_B$  as quiver-representations, that is, there exist base-changes in the eigenspaces

$$(M_1, M_2; N_1, N_2, N_3) \in GL_a \times GL_b \times GL_x \times GL_y \times GL_z$$

such that

$$\begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \\ B'_{31} & B'_{32} \end{bmatrix} \begin{bmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

Working backwards, we recover the  $B_3$ -representation  $\phi = (X_1, X_2)$  from the invertible matrix  $B$  via

$$(*) \begin{cases} X_1 = \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ X_2 = \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases}$$

**Proposition 1.** *If the  $n$ -dimensional simple  $B_3$ -representation  $\phi = (X_1, X_2)$  is determined by  $\lambda \in \mathbb{C}^*$  and the stable quiver-representation  $V_B$ , then  $\tau(\phi) = (X_1^{tr}, X_2^{tr})$  is isomorphic to the representation determined by  $\lambda$  and the stable quiver-representation  $V_{(B^{-1})^{tr}}$ .*

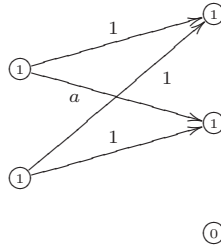
*Proof.* Taking transposes of the formulas (\*) for the  $X_i$  we get

$$\begin{cases} X_1^{tr} = \lambda^{1/6} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{tr} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} (B^{-1})^{tr} \\ X_2^{tr} = \lambda^{1/6} B^{tr} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} (B^{-1})^{tr} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \end{cases}$$

Conjugating these with the matrix  $\begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix}$  (which is also a base-change action in  $GL(\alpha)$ ) we obtain again a matrix-pair in standard-form (\*), this time replacing the matrix  $B$  by the matrix  $(B^{-1})^{tr}$ .  $\square$

That is, we have reduced the original problem of verifying whether or not  $\phi \simeq \tau(\phi)$  as  $B_3$ -representations to the problem of verifying whether or not the two stable representations  $V_B$  and  $V_{(B^{-1})^{tr}}$  lie in the same  $GL(\alpha)$ -orbit.

**Example 1.** *The two components  $\text{iss}_\alpha B_3$  containing simple 2-dimensional  $B_3$ -representations for  $\alpha = (1, 1; 1, 1, 0)$  or  $(1, 1; 1, 0, 1)$  are fixed-point components for the involution  $\tau$ . A general stable  $\alpha = (1, 1; 1, 1, 0)$  dimensional representation  $V_B$  is isomorphic to one of the form*



with  $a \neq 1$ . Hence, we can take  $B = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$ . But then,  $V_B$  and  $V_{(B^{-1})^{tr}}$  lie in the same  $GL(\alpha) = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ -orbit because

$$(B^{-1})^{tr} = \frac{1}{1-a} \begin{bmatrix} 1 & -a \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{a} \end{bmatrix} B \begin{bmatrix} \frac{1}{1-a} & 0 \\ 0 & \frac{-a}{1-a} \end{bmatrix}$$

### 3. THE STRATIFICATION AND POTENTIAL FIXED-POINT COMPONENTS

In this section we will show that a component  $\mathbf{iss}_\alpha B_3$  containing  $n$ -dimensional simple  $B_3$ -representations is a fixed-point component for the involution  $\tau$  only if  $\alpha$  is among the list of theorem 1.

Because the group algebra  $\mathbb{C}\Gamma = \mathbb{C}C_2 * \mathbb{C}C_3$  is a formally smooth algebra, we have a Luna stratification of  $\mathbf{iss}_\alpha \Gamma$  by representation types, see [6, §5.1]. A point  $p$  in  $\mathbf{iss}_\alpha \Gamma$  determines the isomorphism class of a semi-simple  $\Gamma$ -representation

$$V_p = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

with all  $S_i$  distinct simple  $\Gamma$ -representations with corresponding dimension vectors  $\beta_i = (a_i, b_i; x_i, y_i, z_i)$ . We say that  $p$  (or  $V_p$ ) is of representation type

$$\tau = (e_1, \beta_1; \dots; e_k, \beta_k) \quad \text{and clearly} \quad \alpha = \sum_i e_i \beta_i$$

With  $\mathbf{iss}_\alpha \Gamma(\tau)$  we denote the subset of all points of representation type  $\tau$ . Recall that  $\beta_i$  is the dimension vector of a simple  $\Gamma$ -representation if and only if  $a_i + b_i = x_i + y_i + z_i$  and  $\max(x_i, y_i, z_i) \leq \min(a_i, b_i)$  if  $x_i y_i z_i \neq 0$  (the remaining cases being the 1- and 2-dimensional components). It follows from Luna's results [8] that every  $\mathbf{iss}_\alpha \Gamma(\tau)$  is a locally closed smooth irreducible subvariety of  $\mathbf{iss}_\alpha \Gamma$  of dimension  $\sum_i (1 + 2a_i b_i - (x_i^2 + y_i^2 + z_i^2))$  and that

$$\mathbf{iss}_\alpha \Gamma = \bigsqcup_{\tau} \mathbf{iss}_\alpha \Gamma(\tau)$$

is a finite smooth stratification of  $\mathbf{iss}_\alpha \Gamma$ . Degeneration of representation types, see [6, p. 247], defines an ordering  $\leq$  on representation types and by [6, Prop. 5.3] we have that  $\mathbf{iss}_\alpha \Gamma(\tau')$  lies in the Zariski closure of  $\mathbf{iss}_\alpha \Gamma(\tau)$  if and only if  $\tau' \leq \tau$ .

Observe that the involution  $\tau$  on  $\mathbf{iss}_\alpha \Gamma$  induced by  $\tau(V_B) = V_{(B^{-1})^{tr}}$  preserves the strata and its restriction to  $\mathbf{iss}_\alpha \Gamma(\tau)$  is induced by the involutions  $\tau$  on the components  $\mathbf{iss}_{\beta_i} \Gamma$ . As the fixed-point set of  $\tau$  is a closed subvariety of  $\mathbf{iss}_\alpha \Gamma$  we deduce immediately :

**Lemma 1.** *If  $\tau$  is the identity on a Zariski open subset of  $\mathbf{iss}_\alpha \Gamma(\tau)$ , then  $\tau = id$  on all strata  $\mathbf{iss}_\alpha \Gamma(\tau')$  with  $\tau' \leq \tau$ . Conversely, if  $\tau = (e_1, \beta_1; \dots; e_k, \beta_k)$  and  $\tau \neq id$  on one of the components  $\mathbf{iss}_{\beta_i} \Gamma$ , then  $\tau \neq id$  on all strata  $\mathbf{iss}_\alpha \Gamma(\tau')$  with  $\tau \leq \tau'$ .*

In [7] we have shown that for  $\beta = (3, 3; 2, 2, 2)$  there are simple  $B_3$ -representations able to separate the braid  $b$  from the introduction from its reversed braid  $b'$ . In particular,  $\tau$  does not act as the identity on  $\mathbf{iss}_\beta \Gamma$ . We proved this by parametrizing

the matrices  $B$  for a dense open subset of  $\mathbf{iss}_\beta \Gamma$  by

$$B = \begin{bmatrix} 1 & 0 & 0 & a & 0 & f \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & d & e \\ 0 & 1 & 0 & b & c & 0 \\ g & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

for free parameters  $a, \dots, g$ . We then computed the matrix-pair  $\phi = (X_1, X_2)$  from (\*) with generic values of the parameters in  $\mathbb{Z}[\rho]$  and checked that  $Tr_\phi(b) \neq Tr_\phi(b')$ .

**Proposition 2.** *If  $\alpha$  is the dimension vector of a simple  $\Gamma$ -representation such that  $\alpha \geq \beta = (3, 3; 2, 2, 2)$ , then  $\tau \neq id$  on  $\mathbf{iss}_\alpha \Gamma$  and there are simple representations  $\phi \in \mathbf{iss}_\alpha \Gamma$  such that  $Tr_\phi(b) \neq Tr_\phi(b')$ .*

*Proof.* The unique open stratum of  $\mathbf{iss}_\alpha \Gamma$  corresponds to the unique maximal representation type  $\tau_{gen} = (1, \alpha)$ , that is,  $\mathbf{iss}_\alpha \Gamma(\tau_{gen})$  is the open set of simple  $\Gamma$ -representations.

If  $\alpha - \beta$  is the dimension vector of a simple  $\Gamma$ -representation, then we have a representation type  $\tau = (1, \beta; 1, \alpha - \beta)$  such that  $\tau \neq id$  and  $Tr(b) \neq Tr(b')$  on  $\mathbf{iss}_\alpha \Gamma(\tau)$ . But then, by the previous lemma, these facts also hold for  $\mathbf{iss}_\alpha \Gamma(\tau_{gen})$ .

If  $\alpha - \beta$  is not the dimension vector of a simple  $\Gamma$ -representation, we consider the generic (maximal) representation type  $\tau' = (e_1, \beta_1; \dots; e_k, \beta_k)$  in  $\mathbf{iss}_{\alpha - \beta} \Gamma$ . But then,  $\tau = (1, \beta; e_1, \beta_1; \dots; e_k, \beta_k)$  is a representation type for  $\mathbf{iss}_\alpha \Gamma$  and we can repeat the foregoing argument.  $\square$

**Proposition 3.** *If  $\alpha = (a, b; x, y, z)$  is a simple dimension vector such that  $\tau$  acts trivially on  $\mathbf{iss}_\alpha B_3$ , then*

$$\dim \mathbf{iss}_\alpha B_3 = n = a + b = x + y + z$$

*Proof.* By the previous result we must have  $\beta \not\leq \alpha$  and hence either  $n \leq 5$  or  $\min(x, y, z) = 1$ . For a simple  $B_3$ -dimension vector we may assume that  $a \geq b$  and  $x = \max(x, y, z)$ , which leaves us with the following list of potential fixed-point components

$n$	$\alpha$	$\dim \mathbf{iss}_\alpha B_3$
1	(1, 0; 1, 0, 0)	1
2	(1, 1; 1, 1, 0)	2
	(1, 1; 1, 0, 1)	2
3	(2, 1; 1, 1, 1)	3
	(2, 2; 2, 1, 1)	4
4	(3, 2; 2, 2, 1)	5
	(3, 2; 2, 1, 2)	5
6	(3, 3; 3, 2, 1)	6
	(3, 3; 3, 1, 2)	6
	(4, 2; 2, 2, 2)	6
$2k$	( $k, k; k, k - 1, 1$ )	$2k$
	( $k, k; k, 1, k - 1$ )	$2k$
$2k + 1$	( $k + 1, k; k, k, 1$ )	$2k + 1$
	( $k + 1, k; k, 1, k$ )	$2k + 1$

$\square$

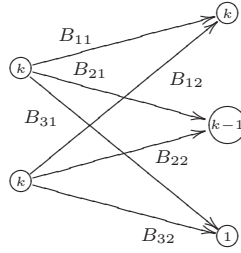
By example 1 we know that the 1- and 2-dimensional components are fixed-point components. All other potential fixed-point components belong to the infinite families, with one exception:  $(4, 2; 2, 2, 2)$ . In the following sections we will prove that all of these are indeed fixed-point components.

#### 4. THE INFINITE FAMILIES

In this section we will prove that for  $\alpha = (k, k; k, k - 1, 1)$  (the even case) and  $\alpha = (k + 1, k; k, k, 1)$  (the odd case),  $\mathbf{iss}_\alpha B_3$  is a fixed-point component. We will prove the even case by direct matrix calculations and deduce the odd case from it by a degeneration argument.

**Proposition 4.** *For all  $k \in \mathbb{N}_+$  and  $\alpha = (k, k; k, k - 1, 1)$ ,  $\mathbf{iss}_\alpha B_3$  is a fixed-point component.*

*Proof.* A general representation in  $\mathbf{iss}_\alpha \Gamma$  corresponds to an invertible  $2m \times 2m$  matrix  $B$  and quiver-representation



After a base-change in the lower-left hand vertex, we may assume that the modified matrix-blocks become

$$\begin{bmatrix} B'_{22} \\ B'_{32} \end{bmatrix} = I_k$$

The block  $B_{12}$  is modified to an invertible  $k \times k$  matrix  $B'_{12}$  which becomes the identity matrix  $I_k$  after performing a base-change in the top-right hand vertex. This changes the block  $B_{11}$  to an invertible matrix  $B'_{11}$  which becomes the identity matrix  $I_k$  after a base-change in the top-left hand vertex. Hence, we may assume that, up-to isomorphism, the matrix  $B$  has the following block-form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} = \begin{bmatrix} I_k & I_k \\ A & I_k \end{bmatrix}$$

with all blocks of sizes  $k \times k$  and  $A$  an invertible matrix such that  $B$  is invertible. One verifies that

$$(B^{-1})^{tr} = \begin{bmatrix} -C & I_k + C \\ C & -C \end{bmatrix} \quad \text{with} \quad C = (A - I_k)^{-1}$$

and performing the base-change

$$(AC^{-1}, -C; -A^{-1}, I_{k-1}, I_1) \in GL_k \times GL_k \times GL_k \times GL_{k-1} \times GL_1$$

we obtain

$$B = \begin{bmatrix} I_k & I_k \\ A & I_k \end{bmatrix} = \begin{bmatrix} -A^{-1} & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} -C & I_k + C \\ C & -C \end{bmatrix} \begin{bmatrix} C^{-1}A & 0 \\ 0 & -C^{-1} \end{bmatrix}$$

Therefore, the  $\Gamma$ -representations determined by the matrices  $B$  and  $(B^{-1})^{tr}$  are isomorphic and hence the involution  $\tau$  is the identity map on the component  $\mathbf{iss}_\alpha B_3$ .  $\square$

**Proposition 5.** *For all  $k \in \mathbb{N}_+$  and  $\alpha = (k + 1, k; k, k, 1)$ ,  $\mathbf{iss}_\alpha B_3$  is a fixed-point component.*

*Proof.* Let  $\alpha_+ = (k + 1, k + 1; k + 1, k, 1)$ , then the stratum  $\tau = (1, \alpha; 1, (0, 1; 1, 0, 0))$  lies in the closure of the generic stratum  $\tau_{gen} = (1, \alpha_+)$  in  $\mathbf{iss}_{\alpha_+} \Gamma$ . The result follows from the foregoing proposition and lemma 1.  $\square$

5. THE EXCEPTIONAL COMPONENT AND VECTORBUNDLES ON  $\mathbb{P}_2$

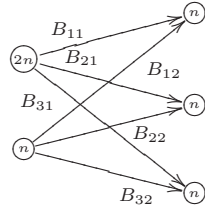
To finish the proof of theorem 1, it suffices to show that  $\mathbf{iss}_\beta B_3$  is a fixed-point component for  $\beta = (4, 2; 2, 2, 2)$ . In [7] we have given a parametrization of the matrices  $B$  for a dense open subset of  $\mathbf{iss}_\beta \Gamma$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & a & 0 \\ 0 & 1 & e & 1 & 0 & 1 \\ 1 & c & d & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & b \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

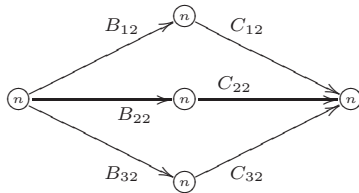
One can attempt to show that  $B$  and  $(B^{-1})^{tr}$  belong to the same  $GL(\beta)$ -orbit by explicit computation. We follow a different approach, allowing us to connect this component to the study of stable vectorbundles on  $\mathbb{P}_2$ .

**Proposition 6.** *For  $\alpha = (2n, n; n, n, n)$ , the component  $\mathbf{iss}_\alpha \Gamma$  is birational to  $M_{\mathbb{P}^2}(n; 0, n)$ , the moduli space of semi-stable rank  $n$  bundles on  $\mathbb{P}_2$  with Chern classes  $c_1 = 0$  and  $c_2 = n$ .*

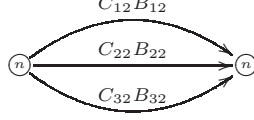
*Proof.* A representation in  $\mathbf{rep}_\alpha \Gamma$  in general position



is such that  $\psi : \mathbb{C}^{2n} \xrightarrow{B_{11} \oplus B_{21} \oplus B_{31}} \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}^n$  is injective, whence its cokernel defines maps  $Cok(\psi) : \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}^n \xrightarrow{(C_{12}, C_{22}, C_{32})} \mathbb{C}^n$  and therefore a representation for the quiver-setting



By the general theory of reflection functors, isomorphism classes of representations are preserved under this construction. By the fundamental theorem of  $GL_n$ -invariants [5, Thm. II.4.1] we can eliminate the base-change action in the middle vertices and obtain a representation of the quiver-setting



By results of Klaus Hulek [2], the corresponding moduli space of semi-stable quiver-representations (as in [3] for the stability structure  $(-1, 1)$ ) is birational to  $M_{\mathbb{P}_2}(n; 0, n)$ .  $\square$

**Proposition 7.**  $\text{iss}_\beta B_3$  is a fixed point component.

*Proof.* By results of Wolf Barth [1], we know that a stable rank 2 bundle  $\mathcal{E}$  on the projective plane with Chern-classes  $c_1 = 0$  and  $c_2 = 2$  is determined up to isomorphism by its curve of jumping lines, that is the collection of those lines  $L \subset \mathbb{P}_2$  such that  $\mathcal{E}|_L \not\cong \mathcal{O}_L^{\oplus 2}$ . If  $\mathcal{E}$  is determined by the quiver-representation as in the previous proposition and if  $x, y, z$  are projective coordinates of the dual plane  $\mathbb{P}_2^*$ , then the equation of this curve of jumping lines is given by

$$\det(C_{12}B_{12}x + C_{22}B_{22}y + C_{32}B_{32}z) = 0$$

In terms of the matrix  $B$  and its inverse  $B^{-1}$  these  $2 \times 2$  matrices are given as

$$\underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ C_{12} & C_{22} & C_{32} \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} * & * & B_{12} \\ * & * & B_{22} \\ * & * & B_{32} \end{bmatrix}}_B = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

But then, the bundle  $\mathcal{F}$  corresponding to the matrix  $(B^{-1})^{tr}$  is determined by the  $2 \times 2$  matrices

$$\underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ B_{12}^{tr} & B_{22}^{tr} & B_{32}^{tr} \end{bmatrix}}_{B^{tr}} \underbrace{\begin{bmatrix} * & * & C_{12}^{tr} \\ * & * & C_{22}^{tr} \\ * & * & C_{32}^{tr} \end{bmatrix}}_{(B^{-1})^{tr}} = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

and hence its curve of jumping lines

$$\det(B_{12}^{tr}C_{12}^{tr}x + B_{22}^{tr}C_{22}^{tr}y + B_{32}^{tr}C_{32}^{tr}z)$$

is the same as that for  $\mathcal{E}$  and hence by Barth's result  $\mathcal{E} \simeq \mathcal{F}$ .  $\square$

**Remark 1.** One can repeat the above argument verbatim for  $\alpha = (2n, n; n, n, n)$ . However, if  $n > 2$ , the bundle  $\mathcal{E}$  corresponding to the matrix  $B$  is determined by its curve of jumping lines (defined as above by the  $n \times n$  matrices  $B_{ij}$  and  $C_{ij}$ ) together with a half-canonical divisor on it, see [2]. Whereas the curve of jumping lines  $Y$  of the bundle  $\mathcal{F}$  corresponding to the matrix  $(B^{-1})^{tr}$  coincides with that of  $\mathcal{E}$ , the involution  $\tau$  acts non-trivially on the Jacobian  $\text{Pic}_Y^d$  where  $d = \frac{1}{2}n(n-1)$ .



## REFERENCES

- [1] Wolf Barth, *Moduli of vector bundles on the projective plane*, Invent. Math. 42 (1977) 63-91
- [2] Klaus Hulek, *On the classification of stable rank  $r$  vector bundles on the projective plane*, Birkhäuser PM 7 (1980) 113-144
- [3] Alastair King, *Moduli of representations of finite-dimensional algebras*, Quat. J. Math. Oxford Ser. (2) 45 (1994) 515-530
- [4] The Knot Atlas, <http://katlas.math.toronto.edu/wiki/>
- [5] Hanspeter Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics D1, Vieweg (1985)
- [6] Lieven Le Bruyn, *Noncommutative geometry and Cayley-smooth orders*, Pure and applied mathematics **290**, Chapman & Hall (2008)
- [7] Lieven Le Bruyn, *Dense families of  $B_3$ -representations and braid reversion*, Journal of Pure and Appl. Algebra 215 (2011) 1003-1014, math.RA/1003.1610
- [8] Domingo Luna, *Slices étales* Bull. Soc. Math. France Mémoire 33 (1973) 81-105
- [9] Bruce Westbury, *On the character varieties of the modular group*, preprint Nottingham (1995)

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