Algebraic D-brane

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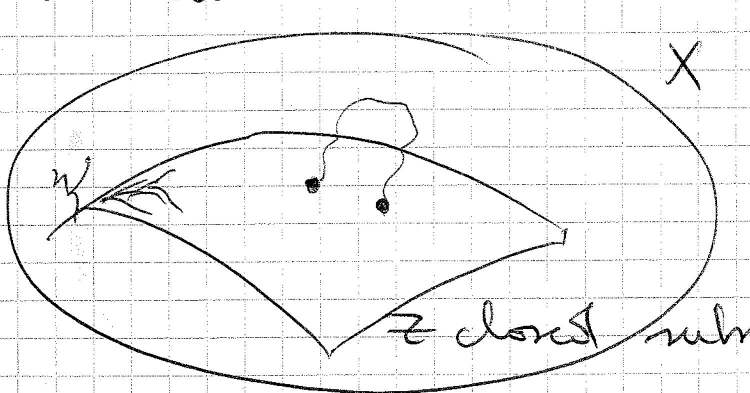
Dfn an stack of algebraic D-branes of size n is a $\mathbb{C}$-algebra map φ

$$R \xrightarrow{\varphi} A_n$$

R is an affine $\mathbb{C}$-algebra ((nc)-space time)
A_n is an affine Azumaya algebra with center $\mathbb{C}$ of degree n. (the brane) and φ encodes "embedding" of brane in space/time.

"Physicsy" motivation



one brane wrapped around Z
$\Rightarrow$ $\mathbb{C}^*$-symmetry (U(1)-bundle)
$\Rightarrow$ "embedding of brane" $\mathbb{C}[X] \longrightarrow \mathbb{C}[Z]$

n stack of branes wrapped around Z
$\Rightarrow$ $GL_n(\mathbb{C})$-symmetry (U(n)-bundle)
$\Rightarrow$ "embedding of brane" $\mathbb{C}[X] \longrightarrow \mathbb{C}[Z]$

$$\begin{tikzcd} & \mathbb{C}[Z] \arrow[d, hook] & A_n \\ \mathbb{C}[X] \arrow[r, dashed] & M_n(\mathbb{C}[Z]) \arrow[ur, leftrightarrow] & \end{tikzcd}$$

<u>locally</u>: in étale topology

but: branes have dynamics can split up and blend together

Higgsing $\Rightarrow$ symmetry breaking $\Leftrightarrow$ big $\rightsquigarrow$ smaller symmetry group

$$\begin{tikzcd} \mathbb{C}[X] \arrow[r, two heads] \arrow[dr, dashed, "\varphi_0"'] & \mathbb{C}[t] \arrow[d] \\ & A_n \end{tikzcd}$$

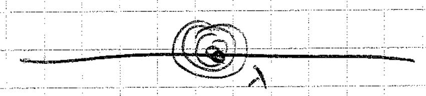
"maximal symmetry"

deform φ_0 in $\underbrace{\mathrm{Alg}_{\mathbb{C}}(\mathbb{C}[X], A_n)}_{\text{elements are alg D-branes.}}$

$\boxed{\text{Example}}$ (cartoon) one-dim space-time

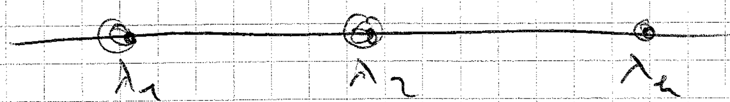
$$\mathbb{C}[X] \longrightarrow M_n(\mathbb{C})$$

$$X \longmapsto \text{matrix } M$$



n stack situation $M = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$ diagonal

more general matrices



different eigenvalues + also stack size distributes over points

seems to be the "expected" physical behaviour of D-branes.

Main problem: Describe "embeddology"

$$R \xrightarrow{\varphi} A_n \quad \text{geometrically}$$

i.e. make NC geometric gadgets containing enough info to recover φ.

$\boxed{\text{Last time}}$

explained Liu-Yau's noncommutative cloud

||

functoriality of $\longleftrightarrow$ Procesi extension bimodule structure sheaf (Alan & Fred)

$\boxed{\text{Today}}$

- explain Artin's theory of stacks and relate stack of D-branes to points of a certain Artin-stack.

- formalize Higgsing and impose natural compatibility condition on families of D-branes $n \to \infty$.

(8) n- $\boxed{\text{categorical geometry}}$ (algebraic geom = 1-cat geom / stacky geom = 2-cat geom) (4)

$$\left(\text{affine schemes}\right) = \left(\text{affine comm. } \mathbb{C}\text{-algebras}\right)^{op}$$

a) Grothendieck functor of points

$X \in$ (affine) determines contravariant functor

$$h_X : (\text{affines}) \longrightarrow (\text{sets})$$
$$Y \longmapsto \mathrm{Hom}(Y, X) \left(= \mathrm{Alg}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y]) \right)$$

so replace scheme X by functor h_X

what is morphism btw affine scheme?
morphism btw functors is natural transformation

$h_X \Rightarrow h_Y$ natural transformation.

does this coincide with usual notion of morphism?

YONEDA LEMMA

$$\mathrm{Hom}(h_X, h_Y) \Leftrightarrow h_Y(X) = \mathrm{Hom}(\cancel{X}, \cancel{Y})$$

$\mathrm{Alg}_{\mathbb{C}}(\mathbb{C}[\cancel{Y}], \mathbb{C}[\cancel{X}])$ ||

($\Rightarrow$) $h_X \overset{N}{\Rightarrow} h_Y$ gives $\forall Z \in$ (affines) a map

$$N(Z) : h_X(Z) \to h_Y(Z)$$

$$N(X)(id_X) \in h_Y(X)$$

$$\begin{tikzcd} X \arrow[r, "\text{given}"] & Y \\ Z \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

(⇐)

$h_X \Rightarrow h_Y \quad N(Z): h_X(Z) \longrightarrow h_Y(Z)$

via composition. ☒

The functor h_X satisfy strong properties, in particular satisfy the "sheaf-property" wrt étale topology
⇒ can extend (affine) to allow more "spaces" as those contravariant ftors satisfying sheaf property

will now consider another generalization by considering 2-categories

__dfn__ A 2-category is a category such that all its Hom-sets are again categories. I.e. 2-category consists of

- objects $\qquad A \,,\, B$
- morphisms between objects $\qquad A \xrightarrow{f} B$
- 2-morphisms (morphisms btw morph. $\qquad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$

__Examples__: (2-affine) (or similar trick to turn every cat. in a 2-category)
$Hom(X,Y)$ becomes smallest possible category (i.e. objects are $X \to Y$ and only allow identity 2-morphism

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow id"] & Y \end{tikzcd}$$

non-trivial example

(groupoids)

- objects are groupoids (i.e. cats s.t. every morphism is invertible)
- morfisms are functors btw groupoids
- 2-morfisms are natural transfos btw. fts btw gp.

Dfn ⊙ a stack is a contravariant 2-functor

$$\mathcal{G} : (2\text{-affines}) \longrightarrow (\text{groupoids})$$

That is assigns to every affine X a groupoid $\mathcal{G}(X)$

to each morfism $Y \to X$ a functor $\mathcal{G}(X) \to \mathcal{G}(Y)$

and to every 2-morfi

$$\begin{tikzcd} Y \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow id"] & X \end{tikzcd} \quad \text{a natural transfo} \quad \begin{tikzcd} \mathcal{G}(X) \arrow[r, bend left, "\mathcal{G}(f)"] \arrow[r, bend right, "\mathcal{G}(f)"'] \arrow[r, phantom, "\Downarrow \mathcal{G}(id)"] & \mathcal{G}(Y) \end{tikzcd}$$

clue: $\mathcal{G}(id)$ does NOT have to be identity natural iso!

⊙ a morphism between stacks is a nat. transfo

$$\begin{tikzcd} (2\text{-affines}) \arrow[r, bend left, "\mathcal{G}"] \arrow[r, bend right, "\mathcal{G}'"'] \arrow[r, phantom, "\Downarrow N"] & (\text{groupoids}) \end{tikzcd}$$

⊙ affines are stacks

$X \rightsquigarrow h_X$ ftor of points + trivi extn to 2-cat.

7bis

$$\begin{array}{c} E \\ \downarrow \\ Z \end{array} \quad \text{principal } G \text{ bundle iff}$$

① G acts on E fixing pts on Z

so $\exists$ morphism

$$E \times_Z G \xrightarrow{\varphi} E$$

②

$$E \times_Z ~~E~~ \xrightarrow{id_E \otimes \varphi} E \times_Z E$$

is isomorphism.

interpretatie in Hopf-terminologie ?
zie Waterhouse

H_Galois object / $\mathbb{C}[Z]$ with

$H = \mathbb{C}[G]$ morphism.

2 - Yoneda lemma

X affine
$\mathcal{G}$ stack

$\Leftarrow$ $\mathrm{Hom}(X, \mathcal{G}) = \mathrm{Hom}(h_X, \mathcal{G}) \Leftrightarrow \mathcal{G}(X)$

"pf": $h_X \overset{N}{\Rightarrow} \mathcal{G} \rightsquigarrow N\left(\begin{array}{c} X \\ \downarrow \mathrm{id} \\ X \end{array}\right) \in \mathcal{G}(X)$ $\boxtimes$

so can view elements of $\mathcal{G}(X)$ as morphism $X \to \mathcal{G}$

$$\boxed{\text{MAIN = MOTIVATIONAL EXAMPLE : QUOTIENT STACK}}$$

$\begin{cases} X \text{ affine scheme} \\ G \text{ reductive group acting on } X \end{cases}$

e.g. finite e.g. alg. group

$\mathbb{C}[X]$ H-comodule algebra

$H = \mathbb{C}[G]$

$\mathcal{G} = [X/G] : (\overset{2-}{\text{affine}}) \longrightarrow (\text{groupoids})$

$Y \longmapsto \underbrace{[X/G](Y)}$

- category with objects

$$\begin{array}{ccc} E & \xrightarrow{f} & X \\ \pi \downarrow & & \\ Y & & \end{array} \leftarrow *$$

when E is principal G-bundle over Y i.e. all fibers $\simeq G$ over local triv in ét. top. and f is G-equivariant map (commute with action of G on E and X)

- morphism

$$\begin{tikzcd} & X & \\ E \arrow[ur, "f"] \arrow[rr, dashed, "g"] \arrow[dr, "\pi"'] & & E' \arrow[ul, "f'"'] \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd}$$

g G-equiv
$\Rightarrow$ G-isomorphism (but not necess identity).

• morphism

$Y' \xrightarrow{f} Y$ gives functor $[X/G](Y) \longrightarrow [X/G](Y')$

fibre product

$$\begin{tikzcd} Y' \times_Y E \arrow[r, dashed] \arrow[d, dashed] & E \arrow[r, "f"] \arrow[d] & X \\ Y' \arrow[r, "f"'] & Y & \end{tikzcd}$$

$$\begin{pmatrix} E \to X \\ \downarrow \\ Y \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y' \times_Y E \to X \\ \downarrow \\ Y' \end{pmatrix}$$

$\boxed{\text{WTF is the advantage of } [X/G] \text{ ?}}$

(1) what are $\mathbb{C}$-points of $[X/G]$?

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & X \\ \bullet & \end{tikzcd}$$

$\exists!$ G-bundle $= G$

$$(G\text{-equiv morph } G \to X) \longleftrightarrow (G\text{-orbits in } X)$$

so have a "geometric" object classifying all orbits. In ordinary affine geometry have a quotient scheme X/G but this only classifies closed orbits

Example GL_n acting on $M_n(\mathbb{C})$ by conjugation $\Rightarrow$ closed orbits are diagonalizable matrices, so quotient map does not detect Jordan blocks.

OPM

$(g\cdot\alpha(z), z) \mapsfrom (z, g)$

$(\alpha(z), z)$

$$\begin{tikzcd}
 & E_f \times_Y Z \arrow[dl] \arrow[dr] & & Z \times G \arrow[ll, "\beta_\alpha"'] \arrow[dl] \\
E_f \arrow[dr, "f"'] & & Z \arrow[ll, "\alpha"'] \arrow[ul, bend right] \arrow[dl] & \\
 & Y & &
\end{tikzcd}$$

representable i $\left(X \xrightarrow{\text{affine}} Y\right)$ by

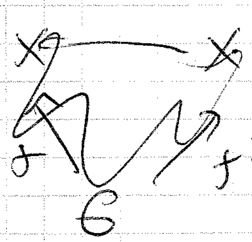
E_f

$[X/G](\mathbb{C})$

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d] & X \\
\bullet &
\end{tikzcd}$$

determined by $f(1) = x$

$\Rightarrow \operatorname{Im} f = O(x)$

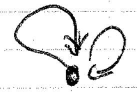


$$\begin{tikzcd}
 & X & \\
G \arrow[ur, "f"] \arrow[rr, dashed, "g\cdot"'] \arrow[dr, no head] & & G \arrow[ul, "f'"'] \arrow[dl, no head] \\
 & \bullet &
\end{tikzcd}$$

$\forall g$ od

von $g \in$ Stab

one loop for



x

every $\in$ Stab

② even when all orbits are closed (e.g. G finite)

X/G has lots of singular points.

But: stacky quotient map

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ {[X/G]} \end{tikzcd}$$

is smooth morphism!

i.e. if X is smooth $\Rightarrow$ also $[X/G]$ is smooth.

ⓐ what is π?

$[X/G](X)$ contains special element

$$\begin{tikzcd} X \times G \arrow[r, "\text{action}", "\text{map}"'] \arrow[d, "pr_1"] & X \\ X \end{tikzcd}$$

π is morphism corresponding to that element.

ⓑ π is "representable" meaning that for any affine Y and any map $Y \xrightarrow{\varphi} [X/G]$

$$\begin{tikzcd} Y \times_{[X/G]} X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ Y \arrow[r, "\varphi"] & {[X/G]} \end{tikzcd}$$

stacky fiber product is represented by affine scheme i.e.

$$Y \times_{[X/G]} X \simeq h_{E_\varphi}$$

affine scheme.

$Y \times X$; $(2\text{-affine})_{X,Y} \longrightarrow (\text{groupoids})$

$[X/G]$

$$\begin{tikzcd}[column sep=small] & Z \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} \quad \rightsquigarrow \quad \left(\begin{tikzcd}[column sep=small] & E_\varphi \times_Y Z \arrow[r, "\alpha"', "\text{iso}"] \arrow[dl] \arrow[dr, dashed] & Z \times_X G\times X \arrow[dl] \arrow[dr] & \\ E_\varphi \arrow[d] & & Z \arrow[dl] \arrow[dr] & G\times X \arrow[d] \\ Y & & & X \end{tikzcd} \right)$$

(Z, α)

$\longleftarrow (*)$

<u>CLAIM:</u> $\{(Z,\alpha)\} \overset{1\text{-}1}{\longleftrightarrow} h_{E_\varphi}(Z) = \{ Z \to E_\varphi \}$

$p_X(z) = g \cdot x$

$(z, g) \rightsquigarrow (z, (g, g^{-1} p_X(z)))$

$(z, (\longleftarrow (z, g)$

Γ

$$\begin{tikzcd} E_\varphi \times_Y Z \arrow[r, "\alpha", "\approx"'] \arrow[d] \arrow[dr] & Z \times_X (G\times X) \cong Z \times G \arrow[dl] \arrow[dr] \arrow[l, bend right, "\alpha^{-1}"'] & \\ E_\varphi \arrow[d] & Z \arrow[l, dashed] \arrow[dl] \arrow[dr, "p_X"] \arrow[u, bend right, "\text{split}"'] & G \times X \arrow[d] \\ Y & & X \end{tikzcd}$$

⊠

(c) properties of $X \xrightarrow{\pi} [X/G]$ are now by definition the COMMON properties of ALL morphism $E_\varphi \xrightarrow{\pi_\varphi} Y$ coming from representability. I.e.

π has property $\Longleftrightarrow$ all $E_\varphi \xrightarrow{\pi_\varphi} Y$ have
$\ast$ $\ast$

<u>Thm:</u> $X \xrightarrow{\pi} [X/G]$ is

- <u>étale</u> when G is finite, then $[X/G]$ is a DM-stack
- <u>smooth</u> when G is alg. group, then $[X/G]$ is Artin-stack.

in either case good props of X carry over to $[X/G]$.

§2 $\boxed{\text{BACK TO ALGEBRAIC D-branes}}$

want to construct quotient stack and relate it to n-stacks of alg branes

$$\begin{array}{c} R \xrightarrow{f} A_n \\ | \\ C \end{array}$$

$$\underbrace{X_n = \mathrm{rep}_n R} \qquad G_n = PGL_n$$

acts via sim conjugation

$$R = \frac{\mathbb{C}\langle x_1, \dots, x_k \rangle}{I}$$

$$\mathbb{V}(f(x_1, \dots, x_k) : \forall f \in I) \hookrightarrow \underbrace{\overset{\text{in } x_1}{\widetilde{M_n(\mathbb{C})}} \oplus \dots \oplus \overset{\text{in } x_k}{\widetilde{M_n(\mathbb{C})}}}_{\mathbb{A}^{n^2 k}}$$

$$\|$$

$\mathrm{rep}_n R$ n-representation scheme

$$\left(PGL_n\text{-orbits in } \mathrm{rep}_n R \right) = \left(\begin{array}{c} \text{isomorphism classes} \\ \text{of } n\text{-diml reps of } R \end{array} \right)$$

quotient scheme $= \mathrm{rep}_n R / PGL_n$

classifies iso of n-dml <u>semi simple</u> rep.

quotient stack $= [\mathrm{rep}_n R \,/\, PGL_n] = [X_n / G_n]$

<u>Theorem</u> :

$$\left(\begin{array}{c} \text{algebraic} \\ \text{D-brans} \\ \text{size } n \\ \begin{tikzcd} R \arrow[r] & A_n \\ & C \arrow[u] \end{tikzcd} \end{array} \right) \overset{1\text{-}1}{\longleftrightarrow} \left(\begin{array}{c} [\mathrm{rep}_n R / PGL_n](Z) \\ \\ Z = \mathrm{spec}(C) \\ \begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{rep}_n R \\ Z \end{tikzcd} \end{array} \right)$$

<u>proof</u> $[\mathrm{rep}_n R / PGL_n](Z)$ is by definition.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & \mathrm{rep}_n R \\ Z = \mathrm{Spec}(C) \end{tikzcd}$$

PGL_n-equivariant

principal PGL_n-bundle is $\in H^1_{et}(Z, PGL_n) =$ isoclasses of degree n Azumaya algebras / C

$$E = \mathrm{rep}_n A_n \quad \text{for some} \quad \begin{array}{c} A_n \\ \big| \text{ azu.} \\ C \end{array}$$

(*) $\mathrm{rep}_n A_n \xrightarrow{f} \mathrm{rep}_n R$ PGL_n - equivariant map.

$\boxed{\text{Procesi's reconstruction result}}$

S affine $\mathbb{C}$ algebra, $\exists$ functor

$$\begin{array}{c} S \\ {\scriptstyle u_n}\downarrow \\ \int_n S \end{array} = \frac{\oint S = S \otimes \mathrm{Sym}\left(\frac{S}{[S,S]_{vect}}\right)}{(tr(1) = n,\ \chi_a(a) = 0 \quad \forall a \in S)}$$

algebra with trace map

χ_a: formal CH-polynomial of degree n

⊙ $\int_n S = \{ \mathrm{rep}_n S \xrightarrow{\varphi} M_n(\mathbb{C}) : \varphi\ PGL_n\text{-equiv} \}$

$M_n(\mathbb{C}) = \mathbb{A}^{n^2}$ with action by conj

⊙ if S is azumaya of degree n over C then S is trace algebra and $S \cong \int_n S$

$\mathrm{rep}_n A_n \xrightarrow{f} \mathrm{rep}_n R$

$\mathrm{rep}_n R \to M_n(\mathbb{C})$ composition gives algebra map

$$R \xrightarrow{u_n} \int_n R \longrightarrow \int_n A_n = A_n$$

is D-brane. ⊠

Importance

Expect good properties for n-stack alg moduli if $[\mathrm{rep}_n R / PGL_n]$ is smooth stack, i.e. when $\mathrm{rep}_n R$ is smooth affine variety

$$\Updownarrow$$

$\int_n R$ smooth Cayley-Smith order.

In physics want $n \to \infty$ so good properties for all n

$$R \text{ formally smooth} \Rightarrow \forall n:\ \mathrm{rep}_n R \text{ smooth}$$

$$\Downarrow$$

$$[\mathrm{rep}_n R / PGL_n] \text{ smooth}$$

For such R's one can assign combinatorial gadgets to any orbit in $\mathrm{rep}_n R$ so to any point in $[\mathrm{rep}_n R / PGL_n]$ and these are important when studying (co)flipping or deform/degenerations.

NOTE

Liu and Yau only consider
trivial Azumaya algebras

That is they consider the quotient stack

$$\left[\mathrm{rep}_n R \,/\, GL_n \right]$$

GL_n - principal bundles

$H^1_{et}(Z, GL_n) =$ isos of projective rank n
modules over $\mathbb{C}[Z]$

P rk n bundle $\Rightarrow$? $\cancel{\mathrm{rep}_n \mathrm{End}(P)}$

$$\cancel{\begin{array}{c} \mathrm{rep}_n \mathrm{End}(P) \\ \downarrow \; \text{principal } GL_n\text{-bundle} \\ Z \end{array}}$$

This P is what Liu and Yau call the
Chan-Paton bundle.

NOG DOEN

§3 Higgsing and families $\left(1\tfrac{1}{2}\text{-alg}\right)$

Can make (alg) into a $1\frac{1}{2}$ category by defining 2-morphism

$$\begin{tikzcd} R \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

where α is $Z(A)$-monomorphism

$$C_A(\mathrm{im} f) \xrightarrow{\alpha} C_A(\mathrm{im}\, g)$$

Dfn: f is Higgsing of g $\Leftrightarrow$ $\exists \alpha$ $\begin{tikzcd} R \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A_n \end{tikzcd}$

intuition: $C_A(\mathrm{im} f)$ can be viewed as Lie-algebra of gauge-group, so smaller $\Rightarrow$ symmetry break $\Rightarrow$ Higgsing

Note: (alg) is NOT a 2-category. Have only vertical compositions of 2-morphisms, not horizontal ones.

What properties should family of algebraic D-maps have if we let $n \to \infty$

$$\begin{tikzcd} & A_n \arrow[r, dash] & C \\ R \arrow[ur, "\varphi_n"] \arrow[dr, "\varphi_m"'] & & \\ & A_m \arrow[uu, dashed, hook] \arrow[r, dash] & C \end{tikzcd}$$

$n \geq m$

expect $\mathbb{C}$-algebra monomorphm $A_m \to A_n$

but then we have double centralizer theorem of Azumaya algebras

$$A_n \cong A_m \otimes C_{A_n}(A_m)$$

is again an Azumaya algebra

So this can only happen if $m \mid n$

<u>Alternative proof 1</u>
$$\int_k A_n \cong \begin{cases} A_n & \text{if } n \mid k \\ 0 & \text{otherwise} \end{cases}$$

all k-dml reps of A_n must be $\oplus$ n dml simples

via diagonal embedding in $k \times k$

$A_m \to A_n$ gives $\quad \mathrm{rep}_n A_n \to \mathrm{rep}_n A_m$

$$\int_n A_m \to \int_n A_n = A_n$$

so impose existence of C-monos

$$A_m \xhookrightarrow{\varphi_{m,n}} A_n$$

whenever $m \mid n$

and moreover

$$\begin{tikzcd} & A_m \arrow[d, hook, "\varphi_{m,n}"] \\ R \arrow[ur, "\varphi_m"] \arrow[dr, "\varphi_n"'] \arrow[r, phantom, "\Uparrow\alpha"] & A_n \end{tikzcd}$$

do NOT assume commutativity, but assume flipping

that is want φ_n to be a deformation of $\varphi_{m,n} \circ \varphi_m$

So: families of D-mono for $n \to \infty$ have <u>multiplicative</u> compatibility restrictions.

Special case of families $(A_n, \varphi_{m,n})_{m|n}$ is rel

$$\mathbb{N}_\times \longrightarrow \mathrm{azu}(C) \text{ is monoidal functor.}$$

Such families are determined by A_p, p prime #.

Example of non-trivial family of D-branes

(i.e. such that not matrix-algebras, i.e. by Tsen must have dim $C \geq 2$)

$C = \mathbb{C}[s^{\pm}, t^{\pm}]$

$q_n = \sqrt[n]{1}$ primitive n-th root of unity

A_n is quantum torus $= \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$ with

$$V_n U_n = q_n U_n V_n \qquad U_n^n = s \qquad V_n^n = t$$

form family of Azumaya algebras over C

$$\varphi_{m,n} : \mathbb{C}_{q_m}[U_m^{\pm}, V_m^{\pm}] \longrightarrow \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$$

$n = k.m$

$$U_m \mapsto U_n^k$$
$$V_m \mapsto V_n^k$$

~~family 1 of D-branes on $\mathbb{A}^2$~~ ~~nopple~~

$$\cancel{\mathbb{C}[x,y] \xrightarrow{\varphi_n} \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]}$$

$$\mathbb{T}^2 \hookrightarrow \underset{\overset{\Psi}{z}}{X} \times \underset{(x,y)}{\mathbb{A}^2} \qquad \mathbb{C}[X \times \mathbb{A}^2] \xrightarrow{\pi} \mathbb{C}$$

can form family of D-branes wrapped around $\mathbb{T}^2$ torus

$$\mathbb{C}[X \times \mathbb{A}^2] \xrightarrow{\alpha_n} \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$$

$$\alpha_n \begin{cases} x \longmapsto s \\ y \longmapsto t \\ z \longmapsto \pi(z) \end{cases}$$

is minimal D-bran

want $C(\mathrm{Im}\,\alpha_n) = \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$

$$\mathbb{C}[X \times \mathbb{A}^2] \xrightarrow{\beta_n} \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$$

$$\beta_n \begin{cases} x \longrightarrow s \\ y \longrightarrow V_n \\ z \longrightarrow \pi(z) \end{cases}$$

~~$\mathrm{Im}(\alpha_n) \subseteq \mathbb{C}[s^{\pm}$~~

$\mathrm{Im}(\beta_n) = \mathbb{C}[s^{\pm}, V_n^{\pm}]$ en centr$(\quad) = \mathbb{C}[s^{\pm}, V_n^{\pm}]$

$\cup$ $\qquad\qquad \Downarrow \nVdash$

$\mathrm{Im}(\varphi_{m,n} \circ \beta_m) = \mathbb{C}[s^{\pm}, V_n^{\pm k}]$

en cent$(\quad) = \mathbb{C}[s^{\pm}, V_n^{\pm k}] \otimes_{\mathbb{C}[s^{\pm}, t^{\pm}]} \mathbb{C}_{q_k}[U_m^{\pm m}, V_n^{\pm m}]$