REPRESENTATION STACKS, D-BRANES AND NONCOMMUTATIVE GEOMETRY

LIEVEN LE BRUYN

Abstract. In this note we prove that the $\mathtt{spec}(C)$-points of the representation Artin-stack $[\mathtt{rep}_n R/PGL_n]$ of n-dimensional representations of an affine $\mathbb{C}$-algebra R correspond to $\mathbb{C}$-algebra morphisms $R \longrightarrow A_n$ where A_n is an Azumaya algebra of degree n over C. We connect this to the theory of D-branes and Azumaya noncommutative geometry, developed by Chien-Hao Liu and Shing-Tung Yau in a series of papers [9]-[14].

1. Representation stacks

Throughout, all algebras will be associative, unital $\mathbb{C}$-algebras which are finitely generated, that is, have a presentation

$$R = \frac{\mathbb{C}\langle x_1, \ldots, x_k \rangle}{(p_i(x_1, \ldots, x_k) \mid i \in I)}$$

With $\mathtt{rep}_n\ R$ we will denote the affine scheme of all n-dimensional representations of R. Its coordinate ring is the quotient of the polynomial algebra in all entries of k generic $n \times n$ matrices

$$X_i = \begin{bmatrix} x_{11}(i) & \ldots & x_{1n}(i) \\ \vdots & & \vdots \\ x_{n1}(i) & \ldots & x_{nn}(i) \end{bmatrix}$$

modulo the ideal generated by all the matrix-entries of the $n \times n$ matrices $p_i(X_1, \ldots, X_k)$ for all $i \in I$. That is,

$$\mathcal{O}(\mathtt{rep}_n\ R) = \frac{\mathbb{C}[x_{ij}(l)\ :\ 1 \leq i,j \leq n, 1 \leq l \leq k]}{(p_i(X_1, \ldots, X_k)_{uv}\ :\ 1 \leq u,v \leq n,\ i \in I)}$$

The affine group scheme GL_n (or rather PGL_n) acts via simultaneous conjugation on the generic matrices X_i and hence on the affine scheme $\mathtt{rep}_n\ R$, its orbits corresponding to isomorphism classes of n-dimensional representations of R. It is well-known that the GIT-quotient $\mathtt{rep}_n R/PGL_n$, that is the affine scheme corresponding to the ring of polynomial invariants $\mathcal{O}(\mathtt{rep}_n R)^{PGL_n}$, classifies isomorphism classes of *semi-simple* n-dimensional representations of R, see [18] or [7, Chp. 2].

In order to investigate all orbits, one considers the n-dimensional *representation stack* $[\mathtt{rep}_n R/PGL_n]$. By this we mean (following [2]) the category with objects all triples (Z, P, ϕ) where Z a scheme over $\mathbb{C}$, $\pi_P : P \longrightarrow\!\!\!\!\!\rightarrow Z$ a PGL_n-torsor in the étale topology

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\pi_P"] & \mathtt{rep}_n R \\ Z & \end{tikzcd}$$

and with ϕ a PGL_n-equivariant map. Morphisms in this category are pairs $(f, h) : (Z, P, \phi) \longrightarrow (Z', P', \phi')$ where $f : Z \longrightarrow Z'$ is a morphism

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[d, "\pi_P"] \arrow[dr, "h"] & & \mathtt{rep}_n R \\ Z \arrow[dr, "f"] & P' \arrow[ur, "\phi'"] \arrow[d, "\pi_{P'}"] & \\ & Z' & \end{tikzcd}$$

and h is a PGL_n-equivariant map such that $P \simeq Z \times_{Z'} P'$ and $\phi = \phi' \circ h$.

The full subcategory consisting of all triples (Z, P, ϕ) with fixed Z are called the Z-points of the stack and are denoted by $[\mathtt{rep}_n R/PGL_n](Z)$. Observe that this subcategory is a groupoid, that is, all its morphisms are isomorphisms.

Theorem 1. *For a commutative affine $\mathbb{C}$-algebra C, the $\mathtt{spec}(C)$-points of the representation stack $[\mathtt{rep}_n R/PGL_n]$ are in natural one-to-one correspondence with $\mathbb{C}$-algebra morphisms $R \longrightarrow A_n$ where A_n is an Azumaya algebra of degree n with center C.*

Before we can prove this result, we need to recall some facts on Cayley-Hamilton algebras, see [18] and [7], and on Azumaya algebras, see [3] and [5].

Every $\mathbb{C}$-algebra R has a universal trace map $n_R : R \longrightarrow R/[R, R]_v$ where $[R, R]_v$ is the sub-vectorspace of R spanned by all commutators $[r, s] = rs - sr$ in R. It allows us to define the *necklace functor*

$$\oint \; : \; \mathtt{alg} \longrightarrow \mathtt{commalg}$$

which assign to any $\mathbb{C}$-algebra R its necklace algebra $\oint R = Sym(\frac{R}{[R,R]_v})$ where for any $\mathbb{C}$-vectorspace V we denote by $Sym(V)$ the symmetric algebra on V.

With $\mathtt{alg@}$ we denote the category of $\mathbb{C}$-algebras *with trace*. That is, a $\mathbb{C}$-algebra R belongs to $\mathtt{alg@}$ if it has a linear map $tr : R \longrightarrow R$ satisfying the following properties for all $r, s \in R$

$$\begin{cases} tr(r)s = str(r) \\ tr(rs) = tr(sr) \\ tr(tr(r)s) = tr(r)tr(s) \end{cases}$$

In particular, it follows that the image of the trace map is contained in the center $Z(R)$. Morphisms in $\mathtt{alg@}$ are trace preserving $\mathbb{C}$-algebra morphisms. The forgetful functor $\mathtt{alg@} \hookrightarrow \mathtt{alg}$ has a left adjoint, called the *trace algebra functor*

$$\int \; : \; \mathtt{alg} \longrightarrow \mathtt{alg@} \qquad R \mapsto \int R = \oint R \otimes R$$

with the trace map on $\int R$ defined by $tr(c \otimes r) = cn_R(r) \otimes 1$, see [7]. That is, for any $\mathbb{C}$-algebra R and any $\mathbb{C}$-algebra with trace A, there is a natural one-to-one correspondence

$$Hom_{\mathtt{alg@}}(\int R, A) \longrightarrow Hom_{\mathtt{alg}}(R, A) \qquad \text{given by} \qquad f \mapsto f \circ n_R$$

Let $M \in M_n(\mathbb{C})$ be a diagonalizable matrix with distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, then the characteristic polynomial has as coefficients the elementary symmetric functions $\sigma_i = \sigma_i(\lambda_1, \ldots, \lambda_n)$. Another generating set of the ring of symmetric functions is given by the power sums $s_i = \lambda_1^i + \ldots + \lambda_n^i$, hence there exist uniquely determined polynomials p_i such that

$$\sigma_i(\lambda_1, \ldots, \lambda_n) = p_i(s_1, \ldots, s_n) = p_i(Tr(M), \ldots, Tr(M^i))$$

because $s_i = Tr(M^i)$.

This allows us to define for an algebra with trace $(R, tr_R) \in \mathtt{alg@}$ the formal Cayley-Hamilton polynomial of degree n for all elements $a \in R$ by

$$\chi_a^{(n)}(t) = t^n + p_1(tr_R(a))t^{n-1} + p_2(tr_R(a), tr_R(a^2))t^{n-2} + \ldots + p_n(tr_R(a), \ldots, tr_R(a^n))$$

With $\mathtt{alg@n}$ we denote the category of all Cayley-Hamilton algebras of degree n, that is, having as its objects algebras with trace map (R, tr_R) satisfying $tr_R(1) = n$ and $\chi_a^{(n)}(a) = 0$ for all $a \in R$, and, trace preserving $\mathbb{C}$-algebra maps as morphisms. We have functors

$$\begin{cases} \int_n \ : \ \mathtt{alg} \longrightarrow \mathtt{alg@n} & R \mapsto \frac{\int R}{(tr_R(1) - n, \chi_a^{(n)}(a) \ \forall a \in \int R)} \\ \oint_n \ : \ \mathtt{alg} \longrightarrow \mathtt{commalg} & R \mapsto tr_{\int_n R}(\int_n R) \end{cases}$$

The main structural results on the trace algebras $\int_n R$ and $\oint_n R$ and their invariant theoretic interpretation are summarized in the next theorem, due to Claudio Procesi. For a proof and more details the reader is referred to [7, Chp. 2] or to the original paper [18].

Theorem 2 (Procesi)**.** *Let R be a finitely generated $\mathbb{C}$-algebra. Then, with notations as before*

1. *$\oint_n R$ is an affine commutative $\mathbb{C}$-algebra*
2. *$\int_n R$ is a finitely generated module over $\oint_n R$*
3. *$\oint_n R$ is the coordinate ring of the GIT-quotient scheme $\mathtt{rep}_n\ R/PGL_n$*
4. *$\int_n R$ is the ring of all PGL_n-equivariant maps $\mathtt{rep}_n\ R \longrightarrow M_n(\mathbb{C})$*

Azumaya algebras form an important class of Cayley-Hamilton algebras. For a C-algebra R denote its enveloping algebra $R^e = R \otimes_C R^{op}$, where R^{op} is the C-algebra with opposite multiplication map. There is a natural C-algebra morphism

$$j_R \ : \ R^e = R \otimes_C R^{op} \longrightarrow End_C(R) \qquad j_R(a \otimes b)r = arb$$

Definition 1. *A C-algebra A is called an* Azumaya algebra *iff A is a finitely generated projective C-module and the map j_A is an isomorphism of C-algebras.*

It follows that the center $Z(A)$ of A is equal to C and that $A/\mathfrak{m}A \simeq M_n(\mathbb{C})$ for every maximal ideal $\mathfrak{m}$ of C where n^2 is the local rank of A at $\mathfrak{m}$. If the local rank is constant and equal to n^2 we say that A is an Azumaya algebra *of degree n*. We recall the geometry associated to C-Azumaya algebras of degree n.

Lemma 1. *C-Azumaya algebras of degree n are classified up to C-algebra isomorphism by the étale cohomology group $H^1_{et}(\mathtt{spec}(C), PGL_n)$, that is, C-Azumaya algebras of degree n correspond to principal PGL_n-fibrations over $\mathtt{spec}(C)$.*

Proof. It is well known (see for example [5]) that A is a C-Azumaya algebra of degree n if and only if there exist étale extensions $C \longrightarrow D$ splitting A, that is, such that $A \otimes_C D \simeq M_n(D)$. As $Aut(M_n(D)) = PGL_n(D)$, the isoclasses of such algebras are classified by the claimed cohomology group, see for example [16]. □

The principal PGL_n-fibration corresponding to the degree n C-Azumaya algebra A is the representation scheme

$$\mathtt{rep}_n A \twoheadrightarrow \mathtt{rep}_n A/PGL_n = \mathtt{spec}(C)$$

as all finite dimensional simple A-representation have to be n-dimensional because $A/\mathfrak{m}A = M_n(\mathbb{C})$. By étale descent, A has a reduced trace map with $tr(A) = C$, and hence we deduce from theorem 2 that $A \in \mathtt{alg@n}$ and

$$\int_n A = A \qquad \text{and} \qquad \oint_n A = C$$

We have now all the tools needed to prove theorem 1.

Proof of theorem 1 : Every $\mathbb{C}$-algebra morphism $\beta : R \longrightarrow A_n$ induces a PGL_n-equivariant map

$$\beta^* \ : \ \mathtt{rep}_n A_n \longrightarrow \mathtt{rep}_n R$$

by composition. By the above remarks we know that the GIT-quotient

$$\mathtt{rep}_n A_n \overset{\pi}{\twoheadrightarrow} \mathtt{rep}_n\ A_n/PGL_n = \mathtt{spec}(C)$$

is a PGL_n-torsor. Therefore $\beta : R \longrightarrow A_n$ determines the $\mathtt{spec}(C)$-point of the representation stack

$$\begin{tikzcd} \mathtt{rep}_n A_n \arrow[r, "\beta^*"] \arrow[d, "\pi"] & \mathtt{rep}_n R \\ \mathtt{spec}(C) & \end{tikzcd}$$

Conversely, PGL_n-torsors over $\mathtt{spec}(C)$ are classified by the pointed set

$$H^1_{et}(\mathtt{spec}(C), PGL_n)$$

which also classifies the isomorphism classes of Azumaya algebras of degree n with center C, see for example [16, p. 134]. Hence, any $\mathtt{spec}(C)$-point of the representation stack $[\mathtt{rep}_n R/PGL_n]$ is of the form

$$\begin{tikzcd} \mathtt{rep}_n A_n \arrow[r, "\phi"] \arrow[d, "\pi"] & \mathtt{rep}_n R \\ \mathtt{spec}(C) & \end{tikzcd}$$

Taking PGL_n-equivariant maps to $M_n(\mathbb{C})$ on both sides of the PGL_n-equivariant map ϕ gives us by theorem 2 and the remarks above a trace preserving algebra morphism

$$\int_n R \xrightarrow{\phi_*} \int_n A_n = A_n$$

and composing this with the universal morphism $R \longrightarrow \int_n R$ we obtain the desired $\mathbb{C}$-algebra morphism $R \xrightarrow{\beta} A_n$ which induces $\phi = \beta^*$.

2. Algebraic D-branes

In this section we relate the above to the description of D-branes via Azumaya noncommutative geometry as developed by Chian-Hao Liu and Shing-Tung Yau in [9]-[14].

Let the affine variety X be an affine open piece of (a spatial slice of) space-time and let Z be a closed subscheme giving the boundary conditions for the endpoints of open strings (a D-brane). Wrapping one brane around Z gives a $\mathbb{C}^*$-bundle (or a $U(1)$-symmetry) and the embedding of the D-brane in space-time X corresponds to the quotient map of the corresponding coordinate rings

$$\beta_1 : \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(Z)$$

However, if there are n branes wrapped around Z, then Z comes equipped with a GL_n-bundle P (or a $U(n)$-symmetry) and the so called *Polchinski Ansatz*, see for example [9], asserts that the embedding of the n-stack of branes in space time is now governed by a $\mathbb{C}$-algebra morphism

$$\beta : \mathcal{O}(X) \longrightarrow End_{\mathcal{O}(Z)}(P)$$

where P is the so-called Chan-Patton bundle on Z. Observe that $End_{\mathcal{O}(Z)}(P)$ is a trivial Azumaya algebra of degree n with center $\mathcal{O}(Z)$ and hence the embedding morphism β gives a Z-point of the representation stack $[\mathtt{rep}_n \mathcal{O}(X)/PGL_n]$.

In more general situations, for example when the B-field is turned on, one may replace the trivial Azumaya algebra $End_{\mathcal{O}(Z)}(P)$ by a non-trivial Azumaya algebra A_n of degree n with center $\mathcal{O}(Z)$. Sometimes, one even allows for a noncommutative space time R. Motivated by this, we formalize D-branes in a purely algebraic context.

Definition 2. *For R an affine $\mathbb{C}$-algebra and Z an affine scheme we will call a Z-point of the representation stack*

$$\beta \in [\mathtt{rep}_n R/PGL_n](Z)$$

an algebraic D-brane *of degree n on Z.*

The upshot of this interpretation of a D-brane embedding as a Z-point in the representation stack $[\mathtt{rep}_n \mathcal{O}(X)/PGL_n]$ (or more generally $[\mathtt{rep}_n R/PGL_n]$) is that one expects good properties of D-branes whenever the representation stack is smooth, in particular when $\mathtt{rep}_n R$ is a smooth affine variety.

This connects the study of D-branes to that of Cayley-smooth orders and formally smooth algebras as described in detail in the book [7]. Observe that several of the examples worked out by Liu and Yau, such as the case of curves in [9, §4] and [10], or the case of the conifold algebra [12] (see also [8]) fall in this framework.

The dynamical aspect of D-branes, that is their Higgsing and un-Higgsing behavior as in [9] can also be formalized purely algebraically by enhancing the category $\mathtt{alg}$ of all $\mathbb{C}$-algebras with a specific class of 2-morphisms.

Definition 3. *A* 2-morphism *between two $\mathbb{C}$-algebra morphisms f and g*

$$\begin{tikzcd} R \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & S \end{tikzcd}$$

is a $Z(S)$-algebra mono-morphism between the centralizers in S of the images

$$\alpha \ : \ C_S(Im(f)) \hookrightarrow C_S(Im(g))$$

If such a 2-morphism exists we say that f degenerates to g or, equivalently, that g deforms to f.

In the case of algebraic D-branes, that is when $S = A_n$ is an Azumaya algebra of degree n with center $C = \mathcal{O}(Z)$, these 2-morphisms correspond to the notions of 'Higgsing' and 'un-Higgsing' as in [9]. Here, the idea is that the Lie-algebra structure of the centralizer $C_A(Im(\beta))$ can be interpreted as the gauge-symmetry group of the D-brane. Higgsing corresponds to symmetry-breaking, that is the gauge-group becomes smaller or, in algebraic terms, a 2-morphism deformation. Likewise, un-Higgsing corresponds to a 2-morphism degeneration of the algebraic D-brane.

Example 1. *There are two extremal cases of algebraic D-branes. The 'maximal' ones coming from epimorphisms*

$$R \overset{\beta}{\twoheadrightarrow} A$$

If C is the center of A such a D-brane determines (and is determined by) a $\mathtt{spec}(C)$-family of simple n-dimensional representations of R. At the other extreme, 'minimal' algebraic D-branes are given by a composition

$$\beta \ : \ R \overset{\pi}{\twoheadrightarrow} C \overset{i}{\hookrightarrow} A$$

where C is a commutative quotient of R and A is an Azumaya algebra over C. Such D-branes essentially determine a $\mathtt{spec}(C)$-family of one-dimensional representations of R.

For a maximal D-brane $\beta_{max} : R \twoheadrightarrow A$ we have $C_A(Im(\beta_{max})) = Z(A)$ whence for any other algebraic D-brane $g : R \longrightarrow A$ we have a two-cell and hence a degeneration

$$\begin{tikzcd} R \arrow[r, bend left, "\beta_{max}"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

Likewise, if $\beta_{min} : R \twoheadrightarrow C \hookrightarrow A$ is a minimal D-brane we have $C_A(Im(\beta_{min})) = A$ whence for any other D-brane we have a two-cell and corresponding deformation

$$\begin{tikzcd} R \arrow[r, bend left, "g"] \arrow[r, bend right, "\beta_{min}"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

Note however that these 2-morphisms do *not* equip $\mathtt{alg}$ with a 2-category structure as defined for example in [15, XII.3]. Indeed, whereas we have an obvious vertical composition of 2-morphisms there is in general no horizontal composition of 2-morphisms.

Still, these 2-morphisms impose a natural compatibility structure on families of algebraic D-branes. In string-theory one often considers the limit $n \to \infty$ of n stacks of D-branes located at a subscheme $Z \hookrightarrow X$, that is, a family of algebra morphisms $\beta_n : \mathbb{C}[X] \longrightarrow A_n$ where A_n is a degree n Azumaya algebra with center $\mathbb{C}[Z]$. Our ringtheoretical considerations suggest one needs to impose a multiplicative compatibility requirement on such families.

Definition 4. *A family of algebraic D-branes*

$$\beta_n \ : \ R \longrightarrow A_n$$

such that $Z(A_n) = C$ for all n is said to be compatible *if for every $n|m$ we have a C-monomorphism $i_{nm} : A_n \hookrightarrow A_m$ and a corresponding 2-morphism*

$$\begin{tikzcd} R \arrow[r, bend left, "\beta_m"{name=U}] \arrow[r, bend right, "i_{nm}\circ\beta_n"'{name=D}] & A_m \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

That is, the algebraic D-brane β_m corresponding to the m stack of D-branes at $\mathtt{spec}(C)$ *is a Higgsing of all D-branes $R \longrightarrow A_n \hookrightarrow A_m$ for all divisors n of m.*

We will give an example of a non-trivial example of a family of algebraic D-branes which are all neither minimal nor maximal. Consider the 2-dimensional torus $\mathbb{C}[s^{\pm}, t^{\pm}]$ and a primitive n-th root of unity $q_n = \sqrt[n]{1}$. The *quantum torus* is the non-commutative algebra generated by two elements U_n and V_n (and their inverses) satisfying the relations

$$V_n U_n = q_n U_n V_n \quad U_n^n = s \quad V_n^n = t$$

and we will denote this algebra as $\mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$. The center of this algebra is easily seen to be $\mathbb{C}[s^{\pm}, t^{\pm}]$ and in fact $\mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$ is an Azumaya algebra over it of degree n. When $m = n.k$ there are obvious embeddings of $\mathbb{C}[s^{\pm}, t^{\pm}]$-algebras

$$i_{m,n} \ : \ \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}] \hookrightarrow \mathbb{C}_{q_m}[U_m^{\pm}, V_m^{\pm}] \qquad \begin{cases} U_n \longrightarrow U_m^k \\ V_n \longrightarrow V_m^k \end{cases}$$

Lemma 2. *Consider the coordinate ring of* $GL_2 = \begin{bmatrix} s & u \\ v & t \end{bmatrix}$

$$\mathcal{O}(GL_2) = \mathbb{C}[s, t, u, v, (st - uv)^{-1}]$$

then the $\mathbb{C}$-algebra morphisms $\beta_n : \mathcal{O}(GL_2) \longrightarrow \mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$ defined by

$$\beta = (\beta_n) \ : \ \begin{cases} u \mapsto 0 \\ v \mapsto 0 \\ s \mapsto s \\ t \mapsto V_n \end{cases}$$

is a compatible family of algebraic D-branes on the maximal torus $T_2 \hookrightarrow GL_2$.

Proof. Clearly the maps β_n are $\mathbb{C}$-algebra morphisms from $\mathcal{O}(GL_2)$ to the Azumaya algebra $\mathbb{C}_{q_n}[U_n^{\pm}, V_n^{\pm}]$ whence they are algebraic D-branes. Remains for every $m = n.k$ to compare the centralizers of the images at level m with those of the image with the inclusion $i_{m,n}$. One easily verifies that

$$Im(\beta_m) = \mathbb{C}[s^{\pm}, V_m^{\pm}] \quad \text{whereas} \quad Im(i_{m,n} \circ \beta_n) = \mathbb{C}[s^{\pm}, V_m^{\pm k}]$$

The centralizer of $Im(\beta_m)$ is equal to itself, whereas

$$C(Im(i_{m,n} \circ \beta_n) = \mathbb{C}[s^{\pm}, V_m^{\pm k}] \otimes_{\mathbb{C}[s^{\pm},t^{\pm}]} \mathbb{C}_{q_k}[U_m^{\pm n}, V_m^{\pm n}]$$

and so these form indeed a compatible family of algebraic D-branes on T_2. □

3. Azumaya noncommutative geometry

The main problem studied in the papers [9]-[14] by Chien-Hao Liu and Shing-Tung Yau is to associate some noncommutative geometry to the morphism $\beta : \mathcal{O}(X) \longrightarrow A_n$ (or more generally $\beta : R \longrightarrow A_n$) describing the embedding of the n-stack of D-branes in space-time. That is, one wants to associate geometric objects (such as varieties, topological spaces and sheaves on these) that will allow us to reconstruct β from the geometric data.

3.1. **Noncommutative structure sheaves.** We will rephrase Liu and Yau's Azumaya noncommutative geometry, based on 'surrogates' and the 'noncommutative cloud', in classical noncommutative algebraic geometry of pi-algebras, as developed by Fred Van Oystaeyen and Alain Verschoren [23] in the early 80-ties.

Let S be a finitely generated Noetherian $\mathbb{C}$-algebra satisfying all polynomial identities of degree n. Observe that $\int_n R$, A_n and its subalgebras

$$A_c = Im(\beta)C \qquad \text{and} \qquad A_e = Im(\beta)C_A(Im(\beta))$$

all satify these requirements. With $\texttt{spec}(S)$ we denote the set of all twosided prime ideals of S which becomes a topological space by equipping it with the Zariski topology having as typical closed sets $\mathbb{V}(I) = \{P \in \texttt{spec}(S) \mid I \subset P\}$ for I a twosided ideal of S. However, this twosided prime spectrum is not necessarily functorial, that is, if $S \xrightarrow{f} T$ is a $\mathbb{C}$-algebra morphism between suitable algebras, then $f^{-1}(P)$ for a twosided prime ideal $P \triangleleft T$ does not have to be a prime ideal in S. Still, the twosided prime spectrum is functorial whenever $S \xrightarrow{f} T$ is an *extension*, that is, if $T = Im(f)C_T(Im(f))$ by [17, Thm. II.6.5].

In [23, V.3] Van Oystaeyen and Verschoren constructed a noncommutative structure sheaf $\mathcal{O}_S^{bi}$ on $\texttt{spec}(S)$ by means of localization in the category of S-bimodules. The main properties of this construction are [23, Thm. V.3.17 and Thm. V.3.36]

- One recovers the algebra back from the global sections $S = \Gamma(\texttt{spec}(S), \mathcal{O}_S^{bi})$
- Every extension $S \xrightarrow{f} T$ defines a morphism of ringed spaces

$$(\texttt{spec}(T), \mathcal{O}_T^{bi}) \longrightarrow (\texttt{spec}(S), \mathcal{O}_S^{bi})$$

In fact, the results are stated there only for prime Noetherian rings satisfying all polynomial identities of degree n but extend to the case of interest here as our rings are finite modules over an affine center and hence we have enough central localizations. For more details, see [24].

An algebraic D-brane $R \xrightarrow{\beta} A$ of degree n is not necessarily an extension, but the morphisms below defined by it are :

$$\int_n R \xrightarrow{\beta} A_c = Im(\beta)C \quad \text{and} \quad \int_n R \xrightarrow{\beta} A_e = Im(\beta)C_A(Im(\beta))$$

Theorem 3. *Associate to the D-brane either of these noncommutative geometric data*

$$(\texttt{spec}(A_*), \mathcal{O}_{A_*}^{bi}) \longrightarrow (\texttt{spec}(\int_n R), \mathcal{O}_{\int_n R}^{bi})$$

It contains enough information to reconstruct $\beta \; : \; \int_n R \longrightarrow A$ *by taking global sections, and hence by compositing with the universal morphism* $R \longrightarrow \int_n R$ *also the D-brane.*

The fact that noncommutative prime spectra and bimodule structure sheaves behave only functorial with respect to $\mathbb{C}$-algebra morphisms which are extensions, explains the notion of a 'noncommutative cloud' in [9]. In our language, the noncommutative cloud of a C-Azumaya algebra A is the set

$$\mathtt{cloud}(A) = \bigsqcup_{A_*} \mathtt{spec}(A_*)$$

where the disjoint union of noncommutative prime spectra is taken over all subalgebras A_* satisfying

$$C \subset Z(A_*) \subset A_* \subset A$$

and such subalgebras are called 'surrogates' in [9]. The point being that we can now associate to an algebraic D-brane of degree n, $R \xrightarrow{\beta} A$ the partially defined map which is well defined because $\int_n R \longrightarrow A_e$ is an extension

$$\mathtt{cloud}(A) \hookleftarrow \mathtt{spec}(A_e) \longrightarrow \mathtt{spec}(\int_n R)$$

and by adorning $\mathtt{cloud}(A)$ componentswise with noncommutative structure sheaves $\mathcal{O}^{bi}_{A_*}$, this partially defined map contains enough information to reconstruct the D-brane.

3.2. **Noncommutative thin schemes.** In [6, §I.2] Maxim Kontsevich and Yan Soibelman define a *noncommutative thin scheme* to be a covariant functor commuting with finite projective limits

$$\mathtt{X} \; : \; \mathtt{fd\text{-}alg} \longrightarrow \mathtt{sets}$$

from the category `fd-alg` of all finite dimensional $\mathbb{C}$-algebras to the category `sets` of all sets. They prove [6, Thm. 2.1.1] that every noncommutative thin scheme is represented by a $\mathbb{C}$-coalgebra. That is, there is a $\mathbb{C}$-coalgebra $C_{\mathtt{X}}$ associated to the noncommutative thin scheme `X` having the property that there is a natural one-to-one correspondence

$$\mathtt{X}(B) = \mathtt{alg}(B, C^*_{\mathtt{X}})$$

for every finite dimensional $\mathbb{C}$-algebra B. Here, $C^*_{\mathtt{X}}$ is the dual algebra of all linear functionals on $C_{\mathtt{X}}$. $C_{\mathtt{X}}$ is called the *coalgebra of distributions* on `X` and the *noncommutative algebra of functions* on the thin scheme `X` is defined to be $\mathbb{C}[\mathtt{X}] = C^*_{\mathtt{X}}$. For a $\mathbb{C}$-algebra R, it is not true in general that the linear functionals R^* are a coalgebra, but the *dual coalgebra* R^o is, where

$$R^0 = \{ f \in R^* \mid Ker(f) \text{ contains a twosided ideal of finite codimension } \}$$

Kostant duality, see for example [21, Thm. 6.0.5], asserts that the functors

$$\begin{tikzcd} \mathtt{alg} \arrow[r, bend left, "\circ"] & \mathtt{coalg} \arrow[l, bend left, "*"] \end{tikzcd}$$

are adjoint. That is, for every $\mathbb{C}$-algebra R and $\mathbb{C}$-coalgebra C there is a natural one-to-one correspondence between the homomorphisms

$$\mathtt{alg}(R, C^*) = \mathtt{coalg}(C, A^\circ)$$

For an affine $\mathbb{C}$-algebra R we define the contravariant functor

$$\mathtt{rep}_R \;:\; \mathtt{fd\text{-}coalg} \longrightarrow \mathtt{sets} \qquad C \mapsto \mathtt{alg}(R, C^*)$$

describing the finite dimensional representations of R, see [6, Example 2.1.9]. As taking the linear dual restricts Koszul duality to an aniti-equivalence between the categories $\mathtt{fd\text{-}alg}$ and $\mathtt{fd\text{-}coalg}$, we can describe $\mathtt{rep}_R$ as the noncommutative thin scheme represented by the dual coalgebra R° as

$$\mathtt{rep}_R \;:\; \mathtt{fd\text{-}alg} \longrightarrow \mathtt{sets} \qquad B = C^* \mapsto \mathtt{alg}(R, B) = \mathtt{coalg}(C, R^\circ)$$

Definition 5. *The* noncommutative affine scheme $\mathtt{rep}_R$ *is the noncommutative thin scheme represented by the dual coalgebra R° of the affine $\mathbb{C}$-algebra R.*

By [21, Lemma 6.0.1] for every $\mathbb{C}$-algebra morphism $f \in \mathtt{alg}(R, B)$, the dual map determines a $\mathbb{C}$-coalgebra morphism $f^* \in \mathtt{coalg}(B^\circ, R^\circ)$. In particular, to an algebraic D-brane $\beta : R \longrightarrow A$ we can associate a morphism between their noncommutative affine schemes

$$\beta^* \;:\; \mathtt{rep}_A \longrightarrow \mathtt{rep}_R$$

determined by the coalgebra map $\beta^* : A^\circ \longrightarrow R^\circ$.

Theorem 4. *The morphism $\beta^* : \mathtt{rep}_A \longrightarrow \mathtt{rep}_R$ between the noncommutative thin schemes allows to reconstruct the algebraic D-brane $\beta : R \longrightarrow A$.*

In order to prove this we need to describe the dual coalgebras A^o and R^o. Recall that a coalgebra D is said to be simple if it has no proper nontrivial sub-coalgebras. Every simple $\mathbb{C}$-coalgebra is finite dimensional and as D^* is a simple $\mathbb{C}$-algebra, we have that $D \simeq M_n(\mathbb{C})^*$, the full matrix coalgebra that is, $\sum_{i,j} \mathbb{C}e_{ij}$ with

$$\Delta(e_{ij}) = \sum_{k=1}^{n} e_{ik} \otimes e_{kj} \quad \text{and} \quad \epsilon(e_{ij}) = \delta_{ij}$$

The coradical $corad(C)$ of a coalgebra C is the direct sum of all simple subcoalgebras of C. It follows from Kostant duality that for any affine $\mathbb{C}$-algebra R we have

$$corad(R^o) = \oplus_{S \in \mathtt{simp}R} M_n(\mathbb{C})^*_S$$

where $\mathtt{simp}R$ is the set of isomorphism classes of finite dimensional simple R-representations and the factor of $corad(R^o)$ corresponding to a simple representation S is isomorphic to the matrix coalgebra $M_n(\mathbb{C})^*$ if $dim(S) = n$.

In algebra, one can resize idempotents by Morita-theory and hence replace full matrices by the basefield. In coalgebra-theory there is an analogous duality known as *Takeuchi equivalence*, see [22]. The isotypical decomposition of $corad(R^o)$ as an R^o-comodule is of the form $\oplus_S C_S^{\oplus n_S}$, the sum again taken over all simple finite dimensional R-representations. Take the R^o-comodule $E = \oplus_S C_S$ and its *coendomorphism coalgebra*

$$R^\dagger = coend^{R^o}(E)$$

then Takeuchi-equivalence (see for example [1, §4, §5] and the references contained in this paper for more details) asserts that R^o is Takeuchi-equivalent to the coalgebra $R^\dagger$ which is *pointed*, that is, $corad(R^\dagger) = \mathbb{C}\ \mathtt{simp}(R) = \oplus_S \mathbb{C} g_S$ with one *group-like* element g_S for every simple finite dimensional R-representation. Remains to describe the structure of the full basic coalgebra $R^\dagger$.

Example 2. *For the affine Azumaya algebra A of degree n over its affine center C, we know that all finite dimensional simple A-representations are n-dimensional and are parametrized by the maximal ideals $\mathfrak{m} \in \mathtt{max}(C)$. That is, $corad(A^o) = \oplus_{\mathfrak{m}} M_n(\mathbb{C})^*$ and A^o is Takeuchi-equivalent to the pointed coalgebra*

$$A^\dagger = C^o \quad \textit{with} \quad corad(A^\dagger) = \oplus_{\mathfrak{m}} \mathbb{C} g_{\mathfrak{m}}$$

By [21, Prop. 8.0.7] *we know that any cocommutative pointed coalgebra is the direct sum of its pointed irreducible components (at the algebra level this says that a commutative semi-local algebra is the direct sum of local algebras). Therefore,*

$$A^\dagger = C^0 = \oplus_{\mathfrak{m}} C^0_{\mathfrak{m}}$$

where $C^0_{\mathfrak{m}}$ is a pointed irreducible cocommutative coalgebra and as such is a sub-coalgebra of the enveloping coalgebra of the abelian Lie algebra on the Zariski tangent space $(\mathfrak{m}/\mathfrak{m}^2)^*$. That is, we recover the maximal spectrum $\mathtt{max}(C)$ of the center C from $A^\dagger$. But then, the dual algebra

$$A^{\dagger *} = C^{o*} = \prod_{\mathfrak{m}} \hat{C}_{\mathfrak{m}}$$

the direct sum of the completions of C at all maximal ideals $\mathfrak{m}$. Also, the double dual algebra

$$A^{o*} = \prod_{\mathfrak{m}} M_n(\hat{C}_{\mathfrak{m}})$$

For a general affine noncommutative $\mathbb{C}$-algebra R, the description of the pointed coalgebra $R^\dagger$ is more complicated as there can be non-trivial extensions between non-isomorphic finite dimensional simple R-representations (note that this does not happen for Azumaya algebras).

For a (possibly infinite) quiver $\vec{Q}$ we define the *path coalgebra* $\mathbb{C}\vec{Q}$ to be the vectorspace $\oplus_p \mathbb{C}p$ with basis all oriented paths p in the quiver $\vec{Q}$ (including those of length zero, corresponding to the vertices) and with structural maps induced by

$$\Delta(p) = \sum_{p=p'p"} p' \otimes p" \qquad \text{and} \qquad \epsilon(p) = \delta_{0,l(p)}$$

where $p'p"$ denotes the concatenation of the oriented paths p' and $p"$ and where $l(p)$ denotes the length of the path p. Hence, every vertex v is a group-like element and for an arrow $v \xrightarrow{a} w$ we have $\Delta(a) = v \otimes a + a \otimes w$ and $\epsilon(a) = 0$, that is, arrows are skew-primitive elements.

For every natural number i, we define the *ext^i-quiver* $\overrightarrow{\mathtt{ext}}^i_R$ to have one vertex v_S for every $S \in \mathtt{simp}(R)$ and such that the number of arrows from v_S to v_T is equal to the dimension of the space $Ext^i_R(S,T)$. With $\mathtt{ext}^i_R$ we denote the $\mathbb{C}$-vectorspace on the arrows of $\overrightarrow{\mathtt{ext}}^i_R$.

The *Yoneda-space* $\mathtt{ext}^\bullet_R = \oplus \mathtt{ext}^i_R$ is endowed with a natural A_∞-structure [4], defining a linear map (the *homotopy Maurer-Cartan map*, [20])

$$\mu = \oplus_i m_i \ : \ \mathbb{C}\overrightarrow{\mathtt{ext}}^1_R \longrightarrow \mathtt{ext}^2_R$$

from the path coalgebra $\mathbb{C}\overrightarrow{\mathtt{ext}}^1_R$ of the ext^1-quiver to the vectorspace $\mathtt{ext}^2_R$, see [4, §2.2] and [20].

Theorem 5. *The dual coalgebra R^o is Takeuchi-equivalent to the pointed coalgebra $R^\dagger$ which is the sum of all subcoalgebras contained in the kernel of the linear map*

$$\mu = \oplus_i m_i \; : \; \mathbb{C}\overrightarrow{\mathtt{ext}}^1_R \longrightarrow \mathtt{ext}^2_R$$

determined by the A_∞-structure on the Yoneda-space $\mathtt{ext}^\bullet_R$.

Proof. We can reduce to finite subquivers as any subcoalgebra is the limit of finite dimensional subcoalgebras and because any finite dimensional R-representation involves only finitely many simples. Hence, the statement is a global version of the result on finite dimensional algebras due to B. Keller [4, §2.2].

Alternatively, we can use the results of E. Segal [20]. Let $S_1, \ldots, S_r$ be distinct simple finite dimensional R-representations and consider the semi-simple module $M = S_1 \oplus \ldots \oplus S_r$ which determines an algebra epimorphism

$$\pi_M \; : \; R \twoheadrightarrow M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C}) = B$$

If $\mathfrak{m} = Ker(\pi_M)$, then the $\mathfrak{m}$-adic completion $\hat{R}_\mathfrak{m} = \underset{\leftarrow}{lim}\; R/\mathfrak{m}^n$ is an augmented B-algebra and we are done if we can describe its finite dimensional (nilpotent) representations. Again, consider the A_∞-structure on the Yoneda-algebra $Ext^\bullet_R(M,M)$ and the quiver on r-vertices $\overrightarrow{\mathtt{ext}}^1_R(M,M)$ and the homotopy Mauer-Cartan map

$$\mu_M = \oplus_i m_i \; : \; \mathbb{C}\overrightarrow{\mathtt{ext}}^1_R(M,M) \longrightarrow Ext^2_R(M,M)$$

Dualizing we get a subspace $Im(\mu_M^*)$ in the path-*algebra* $\mathbb{C}\overrightarrow{\mathtt{ext}}^1_R(M,M)^*$ of the dual quiver. Ed Segal's main result [20, Thm 2.12] now asserts that $\hat{R}_\mathfrak{m}$ is Morita-equivalent to

$$\hat{R}_\mathfrak{m} \underset{M}{\sim} \frac{(\mathbb{C}\overrightarrow{\mathtt{ext}}^1_R(M,M)^*)\hat{}}{(Im(\mu_M^*))}$$

where $(\mathbb{C}\overrightarrow{\mathtt{ext}}^1_R(M,M)^*)\hat{}$ is the completion of the path-algebra at the ideals generated by the paths of positive length. The statement above is the dual coalgebra version of this. □

Proof of theorem 4 : The morphism between the thin noncommutative schemes

$$\beta^* : \mathtt{rep}_A \longrightarrow \mathtt{rep}_R$$

corresponds to the coalgebra-map dual to β

$$\beta^o : A^o \longrightarrow R^o$$

Dualizing this map again we obtain a $\mathbb{C}$-algebra map, and composing with the natural map $R \longrightarrow R^{o*}$ and the observations of example 2 we obtain an algebra map

$$R \longrightarrow R^{o*} \xrightarrow{\beta^{o*}} A^{o*} = \prod_\mathfrak{m} M_n(\hat{C}_\mathfrak{m})$$

the components of which are the maps from the global sections of the structure sheaf of R in the étale topology to the stalks of the structure sheaf of A in the étale topology, induced by β. By étale descent we can therefore reconstruct the algebraic D-brane $\beta : R \longrightarrow A$ from them.

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Department of Mathematics, University of Antwerp, Middelheimlaan 1, B-2020 Antwerp (Belgium), `lieven.lebruyn@ua.ac.be`