RATIONALITY AND DENSE FAMILIES OF B_3 REPRESENTATIONS

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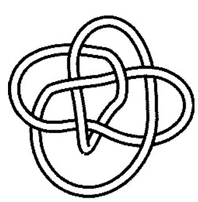
Abstract. Every irreducible component $\mathtt{iss}_\beta\ \Gamma_0$ of semi-simple n-dimensional representations of the modular group $\Gamma_0 = PSL_2(\mathbb{Z})$ has a Zariski dense subset contained in the image of an étale map

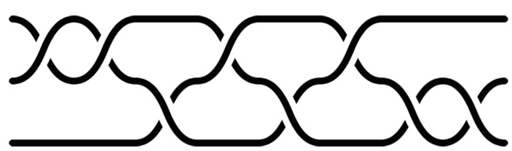
$$\mathtt{iss}(Q,\alpha) \longrightarrow \mathtt{iss}_\beta\ \Gamma_0$$

from the quotient variety $\mathtt{iss}(Q,\alpha)$ of representations of a fixed quiver Q and a dimension vector α such that $\mathtt{iss}(Q,\alpha)$ is a rational variety. As an application we will prove that there is a unique component of 6-dimensional simple representations of the three string braid group B_3 detecting braid-reversion. Further, we give explicit rational parametrizations of dense families of simple B_3-representations of all dimensions < 12.

A knot is said to be invertible if it can be deformed continuously to itself, but with the orientation reversed. There exist non-invertible knots, the unique one with a minimal number of crossings is knot 8_{17} which is the closure of the three string braid $b = \sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2$



8_{17}



b

Proving knot-invertibility of 8_{17} essentially comes down to separating the conjugacy class of the braid b from that of its reversed braid $b' = \sigma_2^2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-2}$. Recall that knot invariants derived from quantum groups cannot detect non-invertibility, see [5]. On the other hand, $Tr_V(b) \neq Tr_V(b')$ for a sufficiently large B_3-representation V. In fact, Bruce Westbury discovered such 12-dimensional representations, and asked for the minimal dimension of a B_3-representation able to detect a braid from its reversed braid, [15] .

Imre Tuba and Hans Wenzl have given a complete classification of all simple B_3-representations in dimension ≤ 5, [13]. By inspection, one verifies that none of these representation can detect invertibility, whence the minimal dimension must be 6. Unfortunately, no complete classification is known of simple B_3-representations of dimension ≥ 6. In this note, we propose a general method to solve this and similar separation problems for three string braids.

Let $\mathtt{rep}_n\ B_3$ be the affine variety of all n-dimensional representations of the three string braid group B_3. There is a base change action of GL_n on this variety having as its orbits the isomorphism classes of n-dimensional representations. The

GIT-quotient of this action, that is, the variety classifying closed orbits

$$\mathtt{rep}_n\ B_3\ //\ GL_n = \mathtt{iss}_n\ B_3$$

is the affine variety $\mathtt{iss}_n\ B_3$ whose points correspond to the isomorphism classes of semi-simple n-dimensional B_3-representations. In general, $\mathtt{iss}_n\ B_3$ will have several irreducible components

$$\mathtt{iss}_n\ B_3 = \bigcup_{\alpha} \mathtt{iss}_\alpha\ B_3$$

If we can prove that $Tr_W(b_1) = Tr_W(b_2)$ for all representations W in a Zariski-dense subset $Y_\alpha \subset \mathtt{iss}_\alpha\ B_3$, then no representation V in that component will be able to separate b_1 from b_2. In order to facilitate the calculations, we would like to parametrize the dense family Y_α by a minimal number of free parameters. That is, if $dim\ \mathtt{iss}_\alpha\ B_3 = d$ we would like to construct explicitly a morphism $X_\alpha \longrightarrow \mathtt{iss}_\alpha\ B_3$ from a rational affine variety of dimension d, having a Zariski dense image in $\mathtt{iss}_\alpha\ B_3$.

The theory of Luna-slices in geometric invariant theory, see [8] and [9], will provide us with a supply of affine varieties X_α and specific étale 'action'-maps $X_\alpha \longrightarrow \mathtt{iss}_\alpha\ B_3$. Rationality results on quiver representations, see [2] and [12], will then allow us to prove rationality of some specific of these varieties X_α. As the modular group $\Gamma_0 = B_3/ < c >$ is a central quotient of B_3 it suffices to obtain these results for Γ_0. In the first two sections we will prove

Theorem 1. *The affine variety classifying n-dimensional semi-simple representations of the modular group decomposes into a disjoint union of irreducible components*

$$\mathtt{iss}_n\ \Gamma_0 = \bigsqcup_{\beta}\ \mathtt{iss}_\beta\ \Gamma_0$$

There exists a fixed quiver Q having the following property. For every component $\mathtt{iss}_\beta\ \Gamma_0$ containing a simple representation, there is a Q-dimension vector α with rational quotient variety $\mathtt{iss}(Q,\alpha)$ and an étale action map

$$\mathtt{iss}(Q,\alpha) \longrightarrow \mathtt{iss}_\beta\ \Gamma_0$$

having a Zariski dense image in $\mathtt{iss}_\beta\ \Gamma_0$.

We apply this general method to solve Westbury's separation problem. Of the four irreducible components of $\mathtt{iss}_6\ B_3$ the three of dimension 6 cannot detect invertibility, whereas the component of dimension 8 can. A specific representation in that component is given by the matrices

$$\sigma_1 = \begin{bmatrix} \rho+1 & \rho-1 & \rho-1 & \rho-1 & -\rho+1 & -\rho+1 \\ -2\rho-1 & -1 & -2\rho-1 & 2\rho+1 & -2\rho-1 & 2\rho+1 \\ \rho+2 & \rho+2 & -\rho & \rho-2 & -\rho-2 & \rho+2 \\ -\rho-2 & -3\rho & \rho+2 & -\rho+2 & 3\rho & -\rho-2 \\ \rho-1 & -\rho+1 & 3\rho+3 & -\rho+1 & 3\rho+1 & -3\rho-3 \\ -3 & -2\rho-1 & 2\rho+1 & 3 & 2\rho+1 & -2\rho-3 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} \rho+1 & \rho-1 & \rho-1 & -\rho+1 & \rho-1 & \rho-1 \\ -2\rho-1 & -1 & -2\rho-1 & -2\rho-1 & 2\rho+1 & -2\rho-1 \\ \rho+2 & \rho+2 & -\rho & \rho+2 & \rho+2 & -\rho-2 \\ \rho+2 & 3\rho & -\rho-2 & \rho+2 & 3\rho & -\rho-2 \\ -\rho+1 & \rho-1 & -3\rho-3 & -\rho+1 & 3\rho+1 & -3\rho-3 \\ 3 & 2\rho+1 & -2\rho-1 & 3 & 2\rho+1 & -2\rho-3 \end{bmatrix}$$

where ρ is a primitive third root of unity. One verifies that $Tr(b) = -7128\rho - 1092$ for the braid b describing knot 8_{17}, whereas $Tr(b') = 7128\rho + 6036$ for the reversed braid b'.

In order to facilitate the application of his method to other separation problems of three string braids, we provide in section three explicit rational parametrizations of dense families of simple n-dimensional B_3-representations for all components and all $n \leq 11$. This can be viewed as a first step towards extending the Tuba-Wenzl classification [13].

1. Luna slices for representations of the modular group

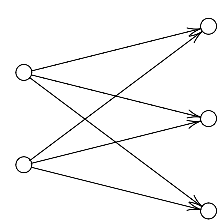
In this section we recall the reduction, due to Bruce Westbury [14], of the study of finite dimensional simple representations of the three string braid group $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ to those of a particular quiver Q_0. We will then identify the Luna slices at specific Q_0-representations to representation spaces of corresponding local quivers as introduced and studied in [1].

Recall that the center of B_3 is infinite cyclic with generator $c = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$ and hence that the corresponding quotient group (taking $S = \sigma_1\sigma_2\sigma_1$ and $T = \sigma_1\sigma_2$)

$$B_3/\langle c \rangle = \langle \overline{S}, \overline{T} \mid \overline{S}^2 = \overline{T}^3 = e \rangle \simeq C_2 * C_3$$

is the free product of cyclic groups of order two and three and therefore isomorphic to the modular group $\Gamma_0 = PSL_2(\mathbb{Z})$.

By Schur's lemma, c acts via scalar multiplication with $\lambda \in \mathbb{C}^*$ on any finite dimensional irreducible B_3-representation, hence it suffices to study the irreducible representations of the modular group Γ_0. Bruce Westbury [14] established the following connection between irreducible representations of Γ_0 and specific stable representations of the directed quiver Q_0 :



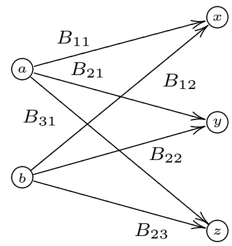
For V an n-dimensional representation of Γ_0, decompose V into eigenspaces with respect to the actions of $\overline{S}$ and $\overline{T}$

$$V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2}$$

(ρ a primitive 3rd root of unity). Denote the dimensions of these eigenspaces by $a = dim(V_+), b = dim(V_-)$ resp. $x = dim(V_1), y = dim(V_\rho)$ and $z = dim(V_{\rho^2})$, then clearly $a + b = n = x + y + z$.

Choose a vector-space basis for V compatible with the decomposition $V_+ \oplus V_-$ and another basis of V compatible with the decomposition $V_1 \oplus V_\rho \oplus V_{\rho^2}$, then the associated base-change matrix $B \in GL_n(\mathbb{C})$ determines a Q_0-representation V_B of dimension vector $\alpha = (a, b; x, y, z)$

(1)



with $$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

A Q_0-representation W of dimension vector α is said to be θ-stable, resp. θ-semi-stable if for every proper sub-representations W', with dimension vector $\beta = (a', b'; x;, y', z')$, we have that $x' + y' + z' > a' + b'$, resp. $x' + y' + z' \geq a' + b'$.

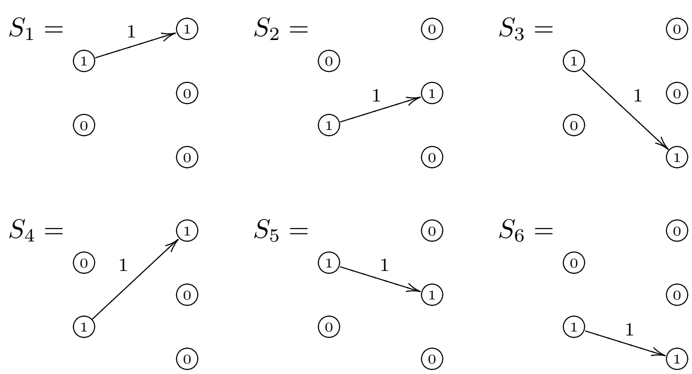
Theorem 2 (Westbury, [14]). *V is an n-dimensional simple Γ_0-representation if and only if the corresponding Q_0-representation V_B is θ-stable. Moreover, $V \simeq W$ as Γ_0-representations if and only if corresponding Q_0-representations V_B and $W_{B'}$ are isomorphic as quiver-representations.*

The affine GIT-quotient $\mathtt{iss}_n\ \Gamma_0 = \mathtt{rep}_n\ \Gamma_n / GL_n$ classifying isomorphism classes of n-dimensional semi-simple Γ_0-representations decomposes into a disjoint union of irreducible components

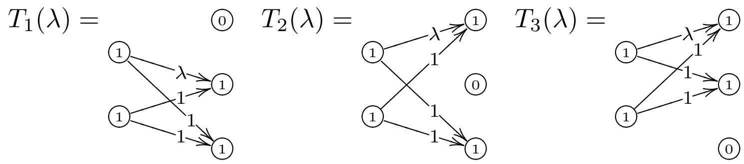
$$\mathtt{iss}_n\ \Gamma_0 = \bigsqcup_{\alpha} \mathtt{iss}_\alpha\ \Gamma_0$$

one component for every dimension vector $\alpha = (a, b; x, y, z)$ satisfying $a + b = n = x + y + z$. If $\alpha = (a, b; x, y, z)$ satisfies $x.y.z \neq 0$, then the component $\mathtt{iss}_\alpha\ \Gamma_0$ contains an open subset of simple representations if and only if $max(x, y, z) \leq min(a, b)$. In this case, the dimension of $\mathtt{iss}_\alpha\ \Gamma_0$ is equal to $1 + n^2 - (a^2 + b^2 + x^2 + y^2 + z^2)$.

The remaining simple Γ_0-representations (that is, those of dimension vector $\alpha = (a, b; x, y, z)$ with $x.y.z = 0$) are of dimension one or two. There are 6 one-dimensional simples (of the abelianization $\Gamma_{0,ab} = C_2 \times C_3$) corresponding to the Q_0-representations



and three one-parameter families of two-dimensional simple Γ_0-representations corresponding to the Q_0-representations



satisfying $\lambda \neq 1$. With $\mathcal{S}_1$ we will denote the set $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ of all one-dimensional Γ_0-representations.

Consider a finite set $\mathcal{S} = \{V_1, \ldots, V_k\}$ of simple Γ_0-representations and identify V_i with the corresponding Q_0-representation of dimension vector α_i. Consider the semi-simple Γ_0-representation

$$M = V_1^{\oplus m_1} \oplus \ldots \oplus V_k^{\oplus m_k}$$

The theory of Luna slices allows us to describe the étale local structure of the component $\mathtt{iss}_\beta\ \Gamma_0$, where $\beta = \sum_i m_i \alpha_i$, in a neighborhood of the point corresponding to M. We will assume throughout that $\beta = (a, b; x, y, z)$ is the dimension vector of a θ-stable representation, that is, that $max(x, y, z) \leq min(a, b)$.

Let $\mathcal{O}(M)$ be the $GL(\beta) = GL_a \times GL_b \times GL_x \times GL_y \times GL_z$-orbit of M in the representation space $\mathtt{rep}(Q_0, \beta)$, then the normal space to the orbit

$$N_M = \frac{T_M(\mathtt{rep}(Q_0, \beta))}{T_M(\mathcal{O}(M))} \simeq Ext^1_{Q_0}(M, M)$$

is the extension space, see for example [4, II.2.7]. Because

$$Ext^1_{Q_0}(M, M) = \begin{bmatrix} M_{m_1}(Ext^1_{Q_0}(V_1, V_1)) & \ldots & M_{m_1 \times m_k}(Ext^1_{Q_0}(V_1, V_k)) \\ \vdots & & \vdots \\ M_{m_k \times m_1}(Ext^1_{Q_0}(V_k, V_1)) & \ldots & M_{m_k}(Ext^1_{Q_0}(V_k, V_k)) \end{bmatrix}$$

we can identify the vectorspace $Ext^1_{Q_0}(M, M)$ to the representation space $\mathtt{rep}(Q_{\mathcal{S}}, \alpha_M)$ of the quiver $Q_{\mathcal{S}}$ on k vertices $\{v_1, \ldots, v_k\}$ (vertex v_i corresponding to the simple Γ_0-representation V_i) such that the number of directed arrows from vertex v_i to vertex v_j is equal to

$$\#\{ (i) \longrightarrow (j) \} = dim_{\mathbb{C}}\ Ext^1_{Q_0}(V_i, V_j)$$

and the dimension vector α_M of $Q_{\mathcal{S}}$ is given by the multiplicities of the simple factors in M, that is, $\alpha_M = (m_1, \ldots, m_k)$. Observe that the stabilizer subgroup of M is equal to $GL(\alpha_M) = GL_{m_1} \times \ldots \times GL_{m_k}$.

The quiver $Q_{\mathcal{S}}$ is called the local quiver of M, see for example [1] or [7], and can be determined from the Euler form of Q_0 which is the bilinear map

$$\chi_{Q_0} \ :\ \mathbb{Z}^5 \times \mathbb{Z}^5 \longrightarrow \mathbb{Z} \qquad \chi_{Q_0}(\alpha, \beta) = \alpha.M_{Q_0}.\beta^{tr}$$

determined by the matrix

$$M_{Q_0} = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For V and W Q_0-representation of dimension vector α and β, we have

$$dim_{\mathbb{C}}\ Hom_{Q_0}(V,W) - dim_{\mathbb{C}}\ Ext^1_{Q_0}(V,W) = \chi_{Q_0}(\alpha,\beta)$$

Recall that $Hom_{Q_0}(V,W) = \delta_{VW}\mathbb{C}$ whenever V and W are θ-stable quiver representations. From the Luna slice theorem we obtain, see for example [7, §4.2].

Theorem 3. *Let $\mathcal{S} = \{V_1,\ldots,V_k\}$ be a finite set of simple Γ_0-representations with corresponding Q_0-dimension vectors α_i. Consider the semi-simple Γ_0-representation*

$$M = V_1^{\oplus m_1} \oplus \ldots \oplus V_k^{\oplus m_k}$$

with Q_0-dimension vector $\beta = \sum_i m_i\alpha_i$. Let $Q_{\mathcal{S}}$ be the local quiver described above and let $\alpha_M = (m_1,\ldots,m_k)$ be the $Q_{\mathcal{S}}$-dimension vector determined by the multiplicities. Then, the action map

$$GL(\beta) \times^{GL(\alpha_M)} \mathtt{rep}(Q_{\mathcal{S}},\alpha_M) \longrightarrow \mathtt{rep}_\beta\ \Gamma_0$$

sending the class of (g,N) in the associated fibre bundle to the representation $g.(M+N)$ where $M+N$ is the representation in the normal space to the orbit $\mathcal{O}(M)$ corresponding to the $Q_{\mathcal{S}}$-representation N, is a $GL(\beta)$-equivariant étale map with a Zariski dense image. Taking $GL(\beta)$ quotients on both sides, we obtain an étale action map

$$\mathtt{iss}(Q_{\mathcal{S}},\alpha_M) \longrightarrow \mathtt{iss}_\beta\ \Gamma_0$$

with a Zariski dense image.

2. The action maps for $\mathcal{S}_1$ and rationality

In order to apply theorem 3 we need to consider a family $\mathcal{S}$ of simple Γ_0-representations generating a semi-simple representation M in every component $\mathtt{iss}_\beta\ \Gamma_0$, and, we need to make the action map explicit, that is, we need to identify representations of the quiver $Q_{\mathcal{S}}$ with representations in the normal space to the orbit $\mathcal{O}(M)$.

We consider the set $\mathcal{S}_1 = \{S_1,S_2,S_3,S_4,S_5,S_6\}$ of all one-dimensional Γ_0-representations, using the notations as before. Consider the semi-simple Γ_0-representation of dimension $n = \sum_i a_i$

$$M = S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6}$$

Then M corresponds to a θ-semistable Q_0-representation of dimension vector

$$\beta_M = (a_1+a_3+a_5, a_2+a_4+a_6, a_1+a_4, a_2+a_5, a_3+a_6)$$

Under the identification (1) the $n\times n$ matrix $B = (B_{ij})_{ij}$ determined by M consists of the following block-matrices B_{ij} containing themselves blocks of sizes $a_u \times a_v$ for the appropriate u and v

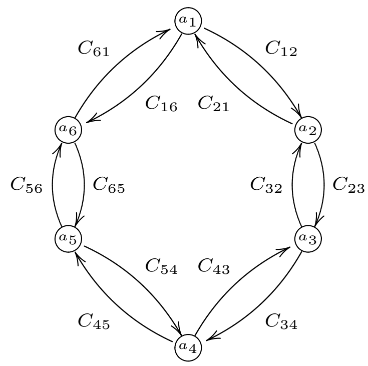
$$B_{11} = \begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{a_5} \end{bmatrix} \qquad B_{31} = \begin{bmatrix} 0 & 1_{a_3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1_{a_4} & 0 \end{bmatrix} \qquad B_{22} = \begin{bmatrix} 1_{a_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{a_6} \end{bmatrix}$$

Theorem 4. *With $\alpha_M = (a_1, \ldots, a_6)$, the étale action map*

$$\mathtt{iss}(Q_{\mathcal{S}_1}, \alpha_M) \longrightarrow \mathtt{iss}_{\beta_M}\ \Gamma_0$$

is induced by sending a representation in $\mathtt{rep}(Q_{\mathcal{S}_1}, \alpha_M)$ defined by the matrices



to the representation in $\mathtt{rep}(Q_0, \beta_M)$ corresponding to the $n \times n$ matrix (via identification (1))

$$B = \begin{bmatrix} 1_{a_1} & 0 & 0 & C_{21} & 0 & C_{61} \\ 0 & C_{34} & C_{54} & 0 & 1_{a_4} & 0 \\ C_{12} & C_{32} & 0 & 1_{a_2} & 0 & 0 \\ 0 & 0 & 1_{a_5} & 0 & C_{45} & C_{65} \\ 0 & 1_{a_3} & 0 & C_{23} & C_{43} & 0 \\ C_{16} & 0 & C_{56} & 0 & 0 & 1_{a_6} \end{bmatrix}$$

Under this map, simple $Q_{\mathcal{S}_1}$-representations with invertible matrix B are mapped to irreducible n-dimensional Γ_0-representations.

Hence, if the coefficients in the matrices C_{ij} give a parametrization of (an open set of) the quotient variety $\mathtt{iss}(Q_{\mathcal{S}_1}, \alpha_M)$, the n-dimensional representations of the three string braid group B_3 given by

$$\begin{cases} \sigma_1 \mapsto \lambda B^{-1} \begin{bmatrix} 1_{a_1+a_4} & 0 & 0 \\ 0 & \rho^2 1_{a_2+a_5} & 0 \\ 0 & 0 & \rho 1_{a_3+a_6} \end{bmatrix} B \begin{bmatrix} 1_{a_1+a_3+a_5} & 0 \\ 0 & -1_{a_2+a_4+a_6} \end{bmatrix} \\ \sigma_2 \mapsto \lambda \begin{bmatrix} 1_{a_1+a_3+a_5} & 0 \\ 0 & -1_{a_2+a_4+a_6} \end{bmatrix} B^{-1} \begin{bmatrix} 1_{a_1+a_4} & 0 & 0 \\ 0 & \rho^2 1_{a_2+a_5} & 0 \\ 0 & 0 & \rho 1_{a_3+a_6} \end{bmatrix} B \end{cases}$$

contain a Zariski dense set of the simple B_3-representations in $\mathtt{iss}_{\beta_M}\ B_3$.

Proof. Using the Euler-form of the quiver Q_0 and the Q_0-dimension vectors of the representations S_i one verifies that the quiver $Q_{\mathcal{S}_1}$ is the one given in the statement of the theorem. To compute the components of the tangent space in M to the orbit, take $Lie(GL(\beta_M))$ as the set of matrices (in block-matrices of sizes $a_u \times a_v$)

$$\begin{bmatrix} A_1 & A_{13} & A_{15} \\ A_{31} & A_3 & A_{35} \\ A_{51} & A_{53} & A_5 \end{bmatrix} \oplus \begin{bmatrix} A_2 & A_{24} & A_{26} \\ A_{42} & A_4 & A_{46} \\ A_{62} & A_{64} & A_6 \end{bmatrix} \oplus \begin{bmatrix} A'_1 & A_{14} \\ A_{41} & A'_4 \end{bmatrix} \oplus \begin{bmatrix} A'_2 & A_{25} \\ A_{52} & A'_5 \end{bmatrix} \oplus \begin{bmatrix} A'_3 & A_{36} \\ A_{63} & A'_6 \end{bmatrix}$$

and hence the tangent space to the orbit is computed using the action of $GL(\beta_M)$ on the quiver-representations, giving for example for the B_{11}-arrow

$$(\begin{bmatrix} 1_{a_1} & 0 \\ 0 & 1_{a_4} \end{bmatrix} + \epsilon \begin{bmatrix} A'_1 & A_{14} \\ A_{41} & A'_4 \end{bmatrix}). \begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .(\begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 1_{a_3} & 0 \\ 0 & 0 & 1_{a_5} \end{bmatrix} - \epsilon \begin{bmatrix} A_1 & A_{13} & A_{15} \\ A_{31} & A_3 & A_{35} \\ A_{51} & A_{53} & A_5 \end{bmatrix})$$

which is equal to

$$\begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} A_1 - A'_1 & -A_{13} & -A_{15} \\ A_{41} & 0 & 0 \end{bmatrix}$$

and, similarly, the ϵ-components of $B_{21}, B_{31}, B_{12}, B_{22}$ resp. B_{23} are calculated to be

$$B_{21} : \begin{bmatrix} 0 & 0 & A_{25} \\ -A_{51} & -A_{53} & A_5 - A'_5 \end{bmatrix} \qquad B_{31} : \begin{bmatrix} -A_{31} & A_3 - A'_3 & -A_{35} \\ 0 & A_{63} & 0 \end{bmatrix}$$

$$B_{12} : \begin{bmatrix} 0 & A_{41} & 0 \\ -A_{42} & A_4 - A'_4 & -A_{46} \end{bmatrix} \qquad B_{22} : \begin{bmatrix} A_2 - A'_2 & -A_{24} & -A_{26} \\ A_{52} & 0 & 0 \end{bmatrix}$$

$$B_{32} : \begin{bmatrix} 0 & 0 & A_{36} \\ -A_{62} & -A_{64} & A_6 - A'_6 \end{bmatrix}$$

Here the zero blocks correspond precisely to the matrices C_{ij} describing a representation in $\mathtt{rep}(Q_{\mathcal{S}_1}, \alpha_M)$ which can therefore be identified with the normal space in M to the orbit $\mathcal{O}(M)$. Here we use the inproduct on $T_M \ \mathtt{rep}_{\beta_M} \ \Gamma_0 = \mathtt{rep}_{\beta_M} \ \Gamma_0$ defined for all $B = (B_{ij})$ and $B' = (B'_{ij})$

$$\langle B, B' \rangle = \sum_{ij} B_{ij} \overline{B}_{ij}^{tr}$$

Hence, the representation $M + N$ is determined by the matrices

$$B_{11} = \begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & C_{34} & C_{54} \end{bmatrix} \qquad B_{21} = \begin{bmatrix} C_{12} & C_{32} & 0 \\ 0 & 0 & 1_{a_5} \end{bmatrix}$$

$$B_{31} = \begin{bmatrix} 0 & 1_{a_3} & 0 \\ C_{16} & 0 & C_{56} \end{bmatrix} \qquad B_{12} = \begin{bmatrix} C_{21} & 0 & C_{61} \\ 0 & 1_{a_4} & 0 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} 1_{a_2} & 0 & 0 \\ 0 & C_{45} & C_{65} \end{bmatrix} \qquad B_{32} = \begin{bmatrix} C_{23} & C_{43} & 0 \\ 0 & 0 & 1_{a_6} \end{bmatrix}$$

To any Q_0-representation V_B of dimension vector $\alpha = (a, b; x, y, z)$, with invertible $n \times n$ matrix B corresponds, via the identifications given in the previous section, the n-dimensional representation of the modular group Γ_0 defined by

$$\begin{cases} \overline{S} \mapsto \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \overline{T} \mapsto B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho 1_y & 0 \\ 0 & 0 & \rho^2 1_z \end{bmatrix} B \end{cases}$$

As $S = \sigma_1\sigma_2\sigma_1$ and $T = \sigma_1\sigma_2$ it follows that $\sigma_1 = T^{-1}S$ and $\sigma_2 = ST^{-1}$. Therefore, when V_B is θ-stable it determines a $\mathbb{C}^*$-family of n-dimensional representations of the three string braid group B_3 given by

$$\begin{cases} \sigma_1 \mapsto \lambda B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} \\ \sigma_2 \mapsto \lambda \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \end{bmatrix} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 1_y & 0 \\ 0 & 0 & \rho 1_z \end{bmatrix} B \end{cases}$$

Using the dimension vector β_M and the matrices B_{ij} found before, we obtain the required B_3-representations. The final statements follow from theorem 3. $\square$

The method of proof indicates how one can make the action maps explicit for any given finite family of irreducible Γ_0-representations. Note that the condition for β_M to be the dimension vector of a θ-stable representation is equivalent to the condition on α_M

$$a_i \leq a_{i-1} + a_{i+1} \qquad \text{for all } i \text{ } mod \text{ } 6$$

which is the condition for $\alpha_M = (a_1, \ldots, a_6)$ to be the dimension vector of a simple $Q_{\mathcal{S}_1}$-representation, by [6].

Theorem 5. *For any Q_0-dimension vector $\beta = (a, b; x, y, z)$ admitting a θ-stable representation, that is satisfying $max(x, y, z) \leq min(a, b)$ there exist semi-simple representations $M \in \mathtt{iss}_\beta\ \Gamma_0$*

$$M_\beta = S_1^{\oplus a_1} \oplus S_2^{\oplus a2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6}$$

such that for $\alpha_M = (a_1, \ldots, a_6)$ the quotient variety $\mathtt{iss}(Q_{\mathcal{S}_1}, \alpha_M)$ is rational. As a consequence, the étale action map

$$\mathtt{iss}(Q_{\mathcal{S}_1}, \alpha_M) \longrightarrow \mathtt{iss}_\beta\ \Gamma_0$$

given by theorem 4 determines a rational parametrization of a Zariski dense subset of $\mathtt{iss}_\beta\ B_3$.

Proof. The condition on β ensures that there are simple α_M-dimensional representations of $Q_{\mathcal{S}_1}$, and hence that α_M is a Schur root. By a result of Aidan Schofield [12] this implies that the quotient variety $\mathtt{iss}(Q_{\mathcal{S}_1}, \alpha_M)$ is birational to the quotient variety of p-tuples of $h \times h$ matrices under simultaneous conjugation, where

$$h = gcd(a_1, \ldots, a_l) \quad \text{and} \quad p = 1 - \chi_{Q_{\mathcal{S}_1}}(\frac{\alpha_M}{h}, \frac{\alpha_M}{h}) = 1 + \frac{1}{h^2}(2\sum_{i=1}^{6} a_i a_{i+1} - \sum_{i=1}^{6} a_i^2)$$

By Procesi's result [10, Prop. IV.6.4] and the known rationality results (see for example [2]) the result follows when we can find such an α_M satisfying $gcd(a_1, \ldots, a_6) \leq 4$. For small dimensions one verifies this by hand, and, for larger dimensions having found a representation M with $gcd(a_1, \ldots, a_6) > 4$, one can modify the multiplicities of the simple components by

$$
\begin{array}{ccccccccccc}
 & & 2 & & & \text{or} & & & 3 & & \\
1 & & & & 1 & & 1 & & & & 2 \\
-1 & & & & -1 & & -2 & & & & -1 \\
 & & -2 & & & & & & -3 & &
\end{array}
$$

(or a cyclic permutation of these) to obtain another semi-simple representation M' lying in the same component, but having $gcd(\alpha_{M'}) = 1$. $\square$

3. Rational dense families for $n < 12$

In this section we will determine for every irreducible component $\mathtt{iss}_\beta\ \Gamma_0$, with $\beta = (a, b; x, y, z)$ such that $a + b = x + y + z = n < 12$ satisfying $max(x, y, z) \leq min(a, b)$, a semi-simple Γ_0-representation

$$M_\beta = S_1^{\oplus a_1} \oplus S_2^{\oplus a2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6}$$

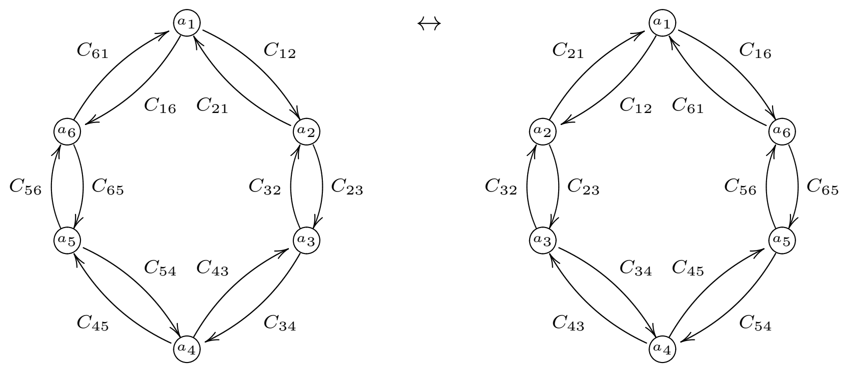
contained in $\mathtt{iss}_\beta\ \Gamma_0$. Then, we will explicitly determine a rational family of representations in $\mathtt{rep}(Q_{\mathcal{S}_1}, \alpha_M)$ for $\alpha_M = (a_1, \dots, a_6)$.

As we are only interested in the corresponding B_3-representations we may assume that $\beta = (a, b; x, y, z)$ is such that $a \geq b$ and $x \geq y \geq z$. Observe that in going from Γ_0-representations to B_3-representations we multiply with $\lambda \in \mathbb{C}^*$. Hence, there is a μ_6-action on the dimension vectors of Q_0-representations giving the same component of B_3-representations. That is, we have to consider only one dimension vector from the μ_6-orbit

$$
\begin{tikzcd}
(a,b;x,y,z) \arrow[r] & (b,a;z,x,y) \arrow[r] & (a,b;y,z,x) \arrow[d] \\
(b,a;y,z,x) \arrow[u] & (a,b;z,x,y) \arrow[l] & (b,a;x,y,z) \arrow[l]
\end{tikzcd}
$$

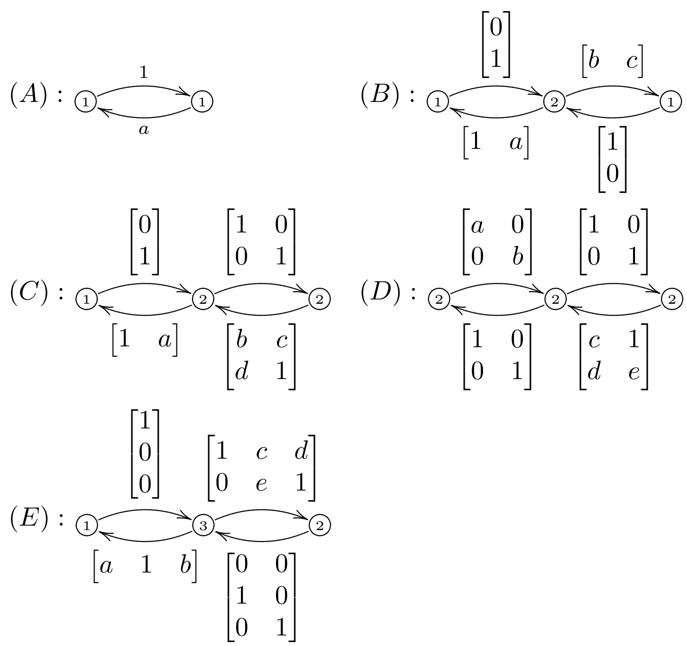
Hence we may assume that $a \geq b$ and that $x = max(x, y, z)$. If we find a module M_β and corresponding rational parametrization for $\beta = (a, b; x, y, z)$ by representations in $\mathtt{rep}(Q_{\mathcal{S}_1}, \alpha_M)$, then we can mirror that description to obtain similar data for

$\beta' = (a, b; x, z, y)$ via



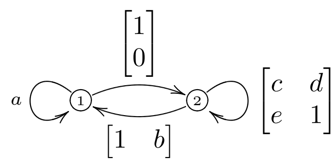
proving that we may indeed restrict to $\beta = (a, b; x, y, z)$ with $a \geq b$ and $x \geq y \geq z$.

Lemma 1. *The following representations provide a rational family determining a dense subset of the quotient varieties* $\mathtt{iss}(Q_{\mathcal{S}_1}, \alpha)$ *for these (minimalistic) dimension vectors* α.



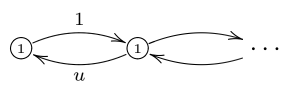
Proof. Recall from [6] that the rings of polynomial quiver invariants are generated by traces along oriented cycles in the quiver. It also follows from [6] that these dimension vectors are the dimension vectors of simple representations and that their quotient varieties are rational of transcendence degrees : 1(A), 3(B), 4(C), 5(D) and 5(E). This proves (A). Cases (B),(C) and (D) are easily seen (focus on the middle vertex) to be equivalent to the problem of classifying couples of 2×2 matrices (A, B) up to simultaneous conjugation in case (D). For (C) the matrix A needs to have rank one and for (B) both A and B have rank one. It is classical that the corresponding rings of polynomial invariants are : (B) $\mathbb{C}[Tr(A), Tr(B), Tr(AB)]$, (C) $\mathbb{C}[Tr(A), Tr(B), Det(B), Tr(AB)]$ and

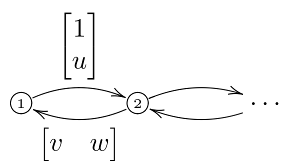
(D) $\mathbb{C}[Tr(A), Det(A), Tr(B), Det(B), Tr(AB)]$. In case (E), as $3 \geq 1, 2$ we can invoke the first fundamental theorem for GL_n-invariants (see [4, Thm. II.4.1]) to eliminate the GL_3-action by composing arrows through the 3-vertex. This reduces the study to the quiver-representations below, of which the indicated representations form a rational family by case (D)

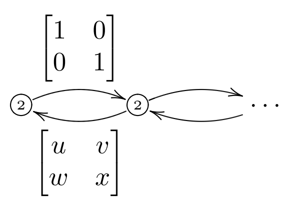


and one verifies that the indicated family of (E) representations does lead to these matrices. □

Lemma 2. *Adding one vertex at a time (until one has at most* 5 *vertices) to either end of the full subquivers of* $Q_{\mathcal{S}_1}$ *of lemma 1 one can extend the dimension vector and the rational dense family of representations as follows in these allowed cases*

- 

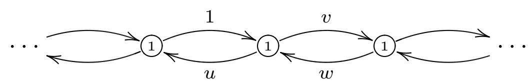
- 

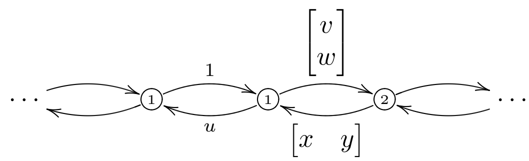
- 

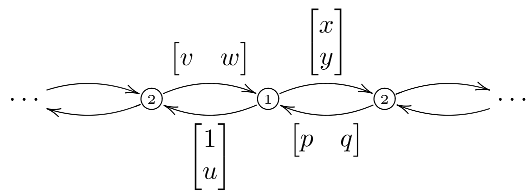
Proof. Before we add an extra vertex, the family of representations is simple and hence the stabilizer subgroup at the end-vertex is reduced to $\mathbb{C}^*$. Adding the vertex and arrows, we can quotient out the base-change action at the new end-vertex by composing the two new arrows (the 'first fundamental theorem for GL_n-invariant theory, see [4, Thm. II.4.1]). As a consequence, we have to classifying resp. a 1×1 matrix, a rank one 2×2 matrix or a 2×2 matrix with trivial action. The given representations do this. □

Assuming the total number of vertices is ≤ 5 we can 'glue' two such subquivers and families of representations at a common end-vertex having dimension 1. Indeed, the ring of polynomial invariants of the glued quivers is the tensor product of those of the two subquivers as any oriented cycle passing through the glue-vertex can be decomposed into the product of two oriented cycles, one belonging to each component. When the two subquivers Q and Q' are glued along a 1-vertex located at vertex i we will denote the new subquiver $Q \bullet_i Q'$. Remains only the problem of 'closing-the-circle' by adding the 6-th vertex to one of these families of representations on the remaining five vertices.

Lemma 3. *Assume we have one of these dimension vectors on the full subquiver of $Q_{\mathcal{S}_1}$ on five vertices and the corresponding family of generically simple representations. Then, we can extend the dimension vector and the rational family of representations to the full quiver $Q_{\mathcal{S}_1}$*

- 

- 

- 

Proof. Forgetting the right-hand arrows, the left-hand representations are added as in the previous lemma. Then, the action on the right-hand arrows is trivial. $\square$

For any of the obtained $Q_{\mathcal{S}_1}$-dimension vectors, we can now describe a rational family of generically simple representations via a code containing the following ingredients

- A,B,C,D,E will be representations as in lemma 1, $\overline{C}$ and $\overline{E}$ will denote the mirror images of C and E
- we add 1's or 2's when we add these dimensions to the appropriate side as in lemma 2
- we add $\bullet_i$ when we glue along a 1-vertex placed at spot i, and
- we add 1_j if we close-up the family at a 1-vertex placed at spot j as in lemma 3

Theorem 6. *In Figure 1 we list rational families for all irreducible components of* $\mathtt{iss}_n$ B_3 *for dimensions $n < 12$. A $+$-sign in column two indicates there are two such components, mirroring each other, β_M indicates the Q_0-dimension vector and α_M gives the multiplicities of the $\mathcal{S}_1$-simples of a semi-simple Γ_0-representation in the component. The last column gives the dimension of this* $\mathtt{iss}_n$ B_3*-component.*

Proof. We are looking for Q_0-dimension vectors $\beta = (a, b; x, y, z)$ satisfying $a + b = x + y + z = n$, $max(x, y, z) \leq min(a, b)$ and $a \geq b$, $x \geq y \geq z$ (if $y \neq z$ there is a mirror component). One verifies that one only obtains the given 5-tuples and that the indicated $Q_{\mathcal{S}_1}$-dimension vectors α_M are compatible with β and that the code is allowed by the previous lemmas. Only in the final entry, type $11e$ we are forced to close-up in a 2-vertex, but by the argument given in lemma 3 it follows that in

type		β_M	α_M	*code*	*dim*
$6a$	$+$	$(3,3;3,2,1)$	$(1,1,1,2,1,0)$	$11B$	6
$6b$		$(3,3;2,2,2)$	$(1,1,1,1,1,1)$	1_1A111	8
$6c$		$(4,2;2,2,2)$	$(1,1,2,1,1,0)$	$1B1$	6
$7a$	$+$	$(4,3;3,3,1)$	$(2,2,1,1,1,0)$	$\overline{C}11$	7
$7b$		$(4,3;3,2,2)$	$(2,1,1,1,1,1)$	1_41B1	9
$8a$		$(4,4;4,2,2)$	$(2,2,2,2,0,0)$	$D2$	10
$8b$	$+$	$(4,4;4,3,1)$	$(2,2,1,2,1,0)$	$\overline{C} \bullet_3 B$	8
$8c$	$+$	$(4,4;3,3,2)$	$(2,2,1,1,1,1)$	$1_51\overline{C}1$	12
$8d$	$+$	$(5,3;3,3,2)$	$(2,1,1,1,2,1)$	$1_3B \bullet_6 B$	10
$9a$	$+$	$(5,4;4,4,1)$	$(2,2,1,2,2,0)$	$\overline{C} \bullet_3 C$	9
$9b$	$+$	$(5,4;4,3,2)$	$(2,1,1,2,2,1)$	$1_3\overline{C} \bullet_6 B$	13
$9c$		$(5,4;3,3,3)$	$(2,2,2,1,1,1)$	1_51D1	15
$9d$		$(6,3;3,3,3)$	$(3,2,2,0,1,1)$	$1E2$	11
$10a$	$+$	$(5,5;5,4,1)$	$(2,2,1,3,2,0)$	$\overline{C} \bullet_3 \overline{F}$	10
$10b$	$+$	$(5,5;5,3,2)$	$(3,2,2,2,1,1)$	1_5E22	14
$10c$	$+$	$(5,5;4,4,2)$	$(2,2,1,2,2,1)$	$1_3\overline{C} \bullet_6 C$	16
$10d$		$(5,5;4,3,3)$	$(3,2,2,1,1,1)$	1_5E21	18
$10e$	$+$	$(6,4;4,4,2)$	$(2,2,2,2,2,0)$	$D22$	14
$10f$		$(6,4;4,3,3)$	$(3,2,2,1,1,1)$	1_41E2	16
$11a$	$+$	$(6,5;5,5,1)$	$(3,2,0,2,3,1)$	$\overline{E} \bullet_6 E$	11
$11b$	$+$	$(6,5;5,4,2)$	$(3,2,1,2,2,1)$	$1_3\overline{C} \bullet_6 E$	17
$11c$		$(6,5;5,3,3)$	$(3,2,2,2,1,1)$	1_5E22	19
$11d$	$+$	$(6,5;4,4,3)$	$(2,2,2,2,2,1)$	1_6D22	21
$11e$	$+$	$(7,4;4,4,3)$	$(3,2,2,1,2,1)$	$2_3B \bullet_6 E$	17

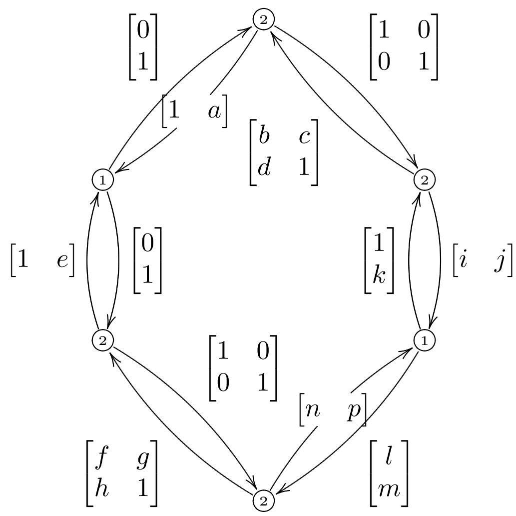
FIGURE 1. Rational dense families of B_3-representations

this case the following representations

$$\begin{tikzcd} \cdots \arrow[r, bend left] & \textcircled{2} \arrow[l, bend left] \arrow[r, bend left, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & \textcircled{2} \arrow[l, bend left, "\begin{bmatrix} u & v \\ w & x \end{bmatrix}"] \arrow[r, bend left, "\begin{bmatrix} y & z \end{bmatrix}"] & \textcircled{1} \arrow[l, bend left, "\begin{bmatrix} p \\ q \end{bmatrix}"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \end{tikzcd}$$

will extend the family to a rational dense family in $\mathtt{iss}(Q_{\mathcal{S}_1}, \alpha_M)$. $\square$

For example, the code $1_3\overline{C} \bullet_6 C$ of component $10c$ will denote the following rational family of $Q_{\mathcal{S}_1}$-representations



Corresponding to these representations are the B-matrices of theorem 4

$$B = \left[\begin{array}{cc|c|cc|cc|cc|c}
1 & 0 & 0 & 0 & 0 & b & c & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & d & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & l & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & m & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & f & g & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & h & 1 & 1 \\
\hline
0 & 0 & 1 & 0 & 0 & i & j & n & p & 0 \\
\hline
1 & a & 0 & 1 & e & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

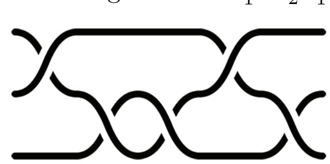
which give us a rational $16 = 15 + 1$-dimensional family of B_3-representations, Zariski dense in their component.

4. Detecting knot invertibility

The rational Zariski dense families of B_3-representations can be used to separate conjugacy classes of three string braids by taking traces. Recall that a *flype* is a braid of the form

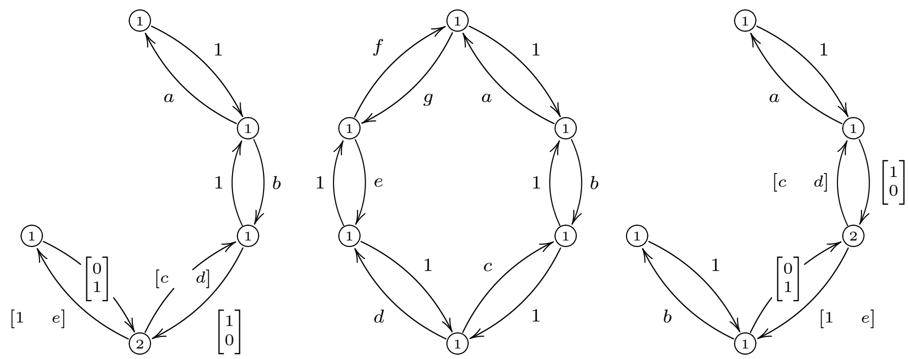
$$b = \sigma_1^u \sigma_2^v \sigma_1^w \sigma_2^\epsilon$$

where $u, v, w \in \mathbb{Z}$ and $\epsilon = \pm 1$. A flype is said to be non-degenerate if b and the reversed braid $b' = \sigma_2^\epsilon \sigma_1^w \sigma_2^v \sigma_1^u$ are in distinct conjugacy classes. An example of a non-degenerate flype of minimal length is $b = \sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_2$



Hence we can ask whether a 6-dimensional B_3-representation can detect that b lies in another conjugacy class than its reversed braid $b' = \sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1^{-1}$

Note that from the classification of all simple B_3-representation of dimension ≤ 5 by Tuba and Wenzl [13] no such representation can separate b from b'. Let us consider 6-dimensional B_3-representations. From the previous section we retain that there are 4 irreducible components in $\mathtt{iss}_6\ B_3$ (two being mirror images of each other) and that rational parametrizations for the corresponding Γ_0-components are given by the following representations of type 6a,6b and 6c



As a consequence we obtain a rational Zariski dense subset of $\mathtt{iss}_6\ B_3$ from theorem 4 using the following matrices B for the three components

$$\begin{bmatrix} 1 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & e \\ 0 & 1 & 0 & b & c & d \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & a & 0 & f \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & d & e \\ 0 & 1 & 0 & b & c & 0 \\ g & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & a & 0 \\ 0 & 1 & e & 1 & 0 & 1 \\ 1 & c & d & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & b \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Using a computer algebra system, for example SAGE [11], one verifies that $Tr(b) = Tr(b')$ for all 6-dimensional B_3-representations belonging to components $6a$ and $6c$. However, $Tr(b) \neq Tr(b')$ for an open subset of representations in component $6b$ and hence 6-dimensional B_3-representations can detect non-degeneracy of flypes. The specific representation given in the introduction (obtained by specializing $a = c = e = g = 1$ and $b = d = f = \lambda = -1$ gives $Tr(b) = 648\rho - 228$ whereas $Tr(b') = -648\rho - 876$. In order to use the family of 6-dimensional B_3-representations as an efficient test to separate 3-braids from their reversed braids it is best to specialize the variables to random integers in $\mathbb{Z}[\rho]$ to obtain an 8-parameter family of B_3-representations over $\mathbb{Q}(\rho)$. In SAGE [11] one can do this as follows

```
K.<l>=NumberField(x^2+x+1);
a=randint(1,1000)*l+randint(1,1000);
b=randint(1,1000)*l+randint(1,1000);
```

```
c=randint(1,1000)*l+randint(1,1000);
d=randint(1,1000)*l+randint(1,1000);
e=randint(1,1000)*l+randint(1,1000);
f=randint(1,1000)*l+randint(1,1000);
g=randint(1,1000)*l+randint(1,1000);
h=randint(1,1000)*l+randint(1,1000);
B=matrix(K,[[1,0,0,a,0,f],[0,1,1,0,1,0],[1,1,0,1,0,0],[0,0,1,0,d,e],
[0,1,0,b,c,0],[g,0,1,0,0,1]]);
Binv=B.inverse();
mat2=matrix(K,[[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0],[0,0,0,-1,0,0],
[0,0,0,0,-1,0],[0,0,0,0,0,-1]]);
mat3=matrix(K,[[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,l^2,0,0,0],
[0,0,0,l^2,0,0],[0,0,0,0,1,0],[0,0,0,0,0,l]]);
s1=h*Binv*mat3*B*mat2;
s2=h*mat2*Binv*mat3*B;
s1inv=s1.inverse();
s2inv=s2.inverse();
```

One can then test the ability of this family to separate braids from their reversed braid on the list of all knots having at most 8 crossings and being closures of three string braids, as provided by the Knot Atlas [3]. It turns out that the braids of the following knots can be separated from their reversed braids : 6_3, 7_5, 8_7, 8_9, 8_{10} (all flypes), as well as the smallest non-invertible knot 8_{17} as mentioned in the introduction.

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