(H)

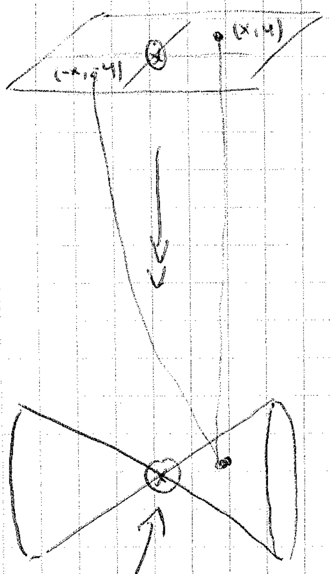
From zeroes to stacks in 2 hours

2009 c

(x,y)

$\downarrow$

(x^2,y^2,xy)



singularity

reason: bigger stab. subgroup

$X = \mathbb{C}^2$

$G = \mathbb{Z}/2\mathbb{Z}$

$\mathbb{C}[X]^G \subset \mathbb{C}[X]$

<u>quotient variety</u>

X/G

x^2, y^2, xy
$\overset{\shortparallel}{u} \quad \overset{\shortparallel}{v} \quad \overset{\shortparallel}{z}$

$\mathbb{V}(uv - z^2) \overset{\shortparallel}{} X/G$

in general:

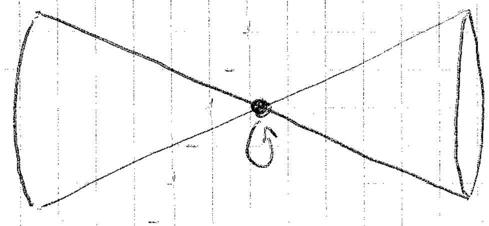
$\mathbb{C}^2$ Kleinian singularities

$\mathbb{C}^3$ Calabi-Yau orbifolds

<u>quotient stack</u>

$[X/G]$

extra info about stabilizer subgroups



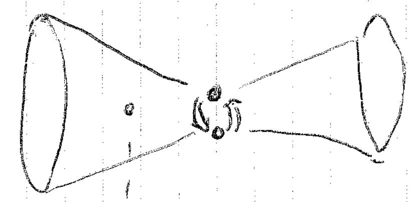
becomes <u>smooth</u> as stack

<u>noncommutative approach</u>

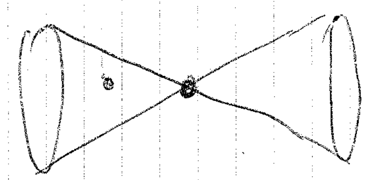
$\mathbb{C}[X] \# G$ smash product

$(f \# g)(f' \# g') = f\,g(f') \# g g'$

center $= \mathbb{C}[X/G]$



nonc. version $\mathbb{C}[X]\#G$



preproj. alg. of tame quiver

center

$\mathbb{C}[X]\#G$ is "smooth"

$\mathrm{gldim} < \infty$ + max

CY-algebra

(c)

CATEGORY THEORY 101

A $\boxed{\text{category}}$ consists of
- objects
- morphisms btw objects allowing composition and identity map

Example • (fg algs) has

objects: all fg comm $\mathbb{C}$-algebras

$$\mathbb{C}[X] = \frac{\mathbb{C}[x_1, \ldots x_n]}{(f_1, \ldots f_m)}$$

morphisms: all $\mathbb{C}$-algebra morphisms

$$\mathbb{C}[X] \xrightarrow{\varphi} \mathbb{C}[Y]$$

• (sets) ~~an~~ have objects sets
morph maps.

• (groups) objects groups
morph groupmorphisms.

A $\boxed{\text{FUNCTOR}}$ is a "map" between categories

$$F : (\text{cat}_1) \longrightarrow (\text{cat}_2)$$

sending objects to objects and morphisms to morphisms in a compatible way.

$$\text{cat}_1 \text{ in } \quad \begin{array}{ccc} A & & F(A) \\ \varphi\downarrow & & \downarrow F(\varphi) \\ B & & F(B) \end{array} \quad \text{in } \text{cat}_2$$

<u>example</u>: ⊙ $\mathbb{G}_m$: (fg algs) $\longrightarrow$ (groups)

$$\mathbb{C}[X] \longmapsto \mathbb{C}[X]^* \quad \text{units of } \mathbb{C}[X]$$

⊙ $\underline{GL_n}$: (fg algs) $\longrightarrow$ (groups)

$$\mathbb{C}[X] \longmapsto GL_n(\mathbb{C}[X])$$

A $\boxed{\text{NATURAL TRANSFORMATION}}$ is a map between two functors btw some categories

$$\begin{tikzcd}
(\mathrm{Cat}_1) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "N \Downarrow"] & (\mathrm{Cat}_2)
\end{tikzcd}$$

giving a map in (cat2) for every object in (cat1)

$$F(A) \xrightarrow[N(A)]{} G(A)$$

such that for all maps $A \xrightarrow{\varphi} B$ in (cat1) have commuting diagram in (cat2)

$$\begin{tikzcd}
F(A) \arrow[r, "N(A)"] \arrow[d, "F(\varphi)"'] & G(A) \arrow[d, "G(\varphi)"] \\
F(B) \arrow[r, "N(B)"] & G(B)
\end{tikzcd}$$

<u>example</u>: $\det : \underline{GL_n} \Longrightarrow \mathbb{G}_m$ is natural transfo.

$$(*)\quad \begin{cases} p_1(x_1, \dots, x_n) = 0 \\ \quad\vdots \\ p_k(x_1, \dots, x_n) = 0 \end{cases}$$

$I = (p_1, \dots, p_k) \lhd \mathbb{C}[x_1, \dots, x_n]$

$X = \mathbb{V}(I)$

$\mathbb{C}[X] = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(I)}$

$$\mathbb{V}(I) \longleftrightarrow \begin{cases} \text{max ideals of } \mathbb{C}[X] \\ \quad \shortparallel \\ \text{alg maps } \mathbb{C}[X] \to \mathbb{C} \end{cases}$$

$$\boxed{(\text{affine}) \quad \overset{?}{\cong} \quad (\text{fg algebras})^{\circ}}$$

<u>objects</u>: zero set

$\mathbb{V}(p_1, \dots, p_k) \subset \mathbb{A}^n$
$\quad\shortparallel$
$\quad X$

<u>objects</u>: f.g. $\mathbb{C}$-algebra

$(p_1, \dots, p_k) \to \mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow R$

$\mathbb{C}[X] = R \cong \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(p_1, \dots, p_k)}$

<u>morphism</u>

$X \longleftarrow Y$

<u>morphism</u>: $\mathbb{C}$-algebra morph.

$$\mathbb{C}[X] \longrightarrow \mathbb{C}[Y] \\ \quad\quad\quad\quad\quad \downarrow \\ \quad\quad\quad\quad\quad \mathbb{C}$$

<u>problem</u>: Hilbert Nullstellensatz

different algebras same zero set

<u>ex</u>: $\dfrac{\mathbb{C}[x_1, \dots, x_n]}{(p(x_1, \dots, x_n))}$ $\quad \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(p(x_1, \dots, x_n)^{\ell})}$ have same zeroes.

this is only restriction.

If we want $(\text{affine}) \cong (\text{fg-algs})^{\circ}$ have to add extra data to zero set X

3 approach:

(Serre): On X put "structure sheaf" $\mathcal{O}_X$
having global sections $\mathbb{C}[X] = \Gamma(X, \mathcal{O}_X)$

(Grothendieck): "functor of points"

$$h_X : (\text{affine}) \longrightarrow (\text{sets}) \qquad \text{contravariant ftor}$$
$$Y \longmapsto \mathrm{Hom}(Y, X)$$
$$= \mathrm{Hom}_{alg}(\mathbb{C}[X], \mathbb{C}[Y])$$

$$h_X(\mathbb{C}) = \mathrm{Hom}_{alg}(\mathbb{C}[X], \mathbb{C}) = \text{pts of } X$$
$$h_X\left(\frac{\mathbb{C}[t]}{(t^2)}\right) = \mathrm{Hom}_{alg}\left(\mathbb{C}[X], \frac{\mathbb{C}[t]}{(t^2)}\right) = \text{pts + tangent vecs}$$
$$= \text{pts of } TX$$

$\vdots$

(affine) **objects**: functors h_X
corresponding to f.g. alg $\mathbb{C}[X]$

How to relate with morphisms in (fg-alg) ? } **morphism** should be

$$h_X \longrightarrow h_Y \qquad \text{natural transformation}$$
$$\updownarrow \; 1\text{-}1$$

$\boxed{\text{YONEDA}}$

$$\mathrm{Hom}(h_X, h_Y) = h_Y(X) = \mathrm{Hom}_{alg}(\mathbb{C}[X], \mathbb{C}[Y])$$

proof ($\Leftarrow$)

$$\begin{tikzcd} \mathbb{C}[X] \arrow[r, "\varphi"] \arrow[dr, leftrightarrow] & \mathbb{C}[Y] \arrow[d, leftrightarrow] \\ & \mathbb{C}[Z] \end{tikzcd}$$

$\forall\, \mathbb{C}[Z] \in \mathrm{Ob}(\mathrm{fg\ alg})$

$$\mathrm{Hom}_{alg}(\mathbb{C}[Z], \mathbb{C}[X]) \longrightarrow \mathrm{Hom}_{alg}(\mathbb{C}[Z], \mathbb{C}[Y])$$

$$\begin{array}{ccc} \| & & \| \\ h_X(Z) & \longrightarrow & h_Y(Z) \end{array}$$

so φ determines ~~functor~~ nat. transfo $h_X \to h_Y$

($\Rightarrow$) given natural transfo

$$h_X \xrightarrow{N} h_Y$$

$$\forall Z: \quad h_X(Z) \xrightarrow{N(Z)} h_Y(Z)$$

in particular

$$id_X \in h_X(X) \longrightarrow N(X)(id_X) \in h_Y(X) = \mathrm{Hom}(X, Y)$$

$\boxtimes$

(Kontsevich): dual coalgebra

take as geometric object

$$\mathbb{C}[X]^\circ = \{ \lambda : \mathbb{C}[X] \xrightarrow{lin} \mathbb{C} \mid \mathrm{Ker}(\lambda) \text{ contains cofinite ideal} \}$$

is coalgebra

coradical recovers pts of X and $\mathbb{C}[X]^{\circ *} = \prod_{x \in X} \hat{\mathcal{O}}_{X,x}$

Grothendieck's categorical geometry

what is special about functors

$$h_X : (\text{affine}) \longrightarrow (\text{sets})$$

They are SHEAVES for the ETALE TOPOLOGY

Etale extension $Y \xrightarrow{\text{ét}} X$ iff

$$\mathbb{C}[X] \to \mathbb{C}[Y] = \mathbb{C}[X]\frac{[z_1, \dots, z_k]}{(f_1, \dots, f_k)}$$

$\boxed{\text{or direct sum of them}}$

with Jacobian condition

étale morphism required for failure of inverse F-n th in alg geo

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_k}{\partial z_1} \\ \vdots & & \\ \frac{\partial f_k}{\partial z_1} & \cdots & \frac{\partial f_k}{\partial z_k} \end{bmatrix} \in \mathbb{C}[Y]^*$$

Basic example

$$\mathbb{C}[X] \longrightarrow \mathbb{C}[X]\frac{[z]}{(z^n - a)} \qquad a \in \mathbb{C}[X]^*$$

should think of $Y \xrightarrow{\text{ét}} X$ as substitute for Zariski open $U \hookrightarrow X$

étale for Grothendieck topology

open for topology use intersections.

intersection replaced by ~~coproduct~~ fiber product

$$\begin{tikzcd} Y \times_X Y' \arrow[r] \arrow[d] & Y \arrow[d, "f", "\text{ét}"'] \\ Y' \arrow[r, "g", "\text{ét}"'] & X \end{tikzcd}$$

$$Y \times_X Y' = \{(y, y') : f(y) = g(y') \text{ in } X\}$$

$$\mathbb{C}[Y \times_X Y'] = \mathbb{C}[Y] \otimes_{\mathbb{C}[X]} \mathbb{C}[Y']$$

$$= \mathbb{C}[X]_{s_1} \frac{[z_1, \dots, z_n]}{(f_1, \dots, f_n)} \otimes_{\mathbb{C}[X]} \mathbb{C}[X]_{s_2} \frac{[v_1, \dots, v_\ell]}{(g_1, \dots, g_\ell)}$$

$$= \mathbb{C}[X]_{s_1 s_2} \frac{[z_1, \dots, z_n, v_1, \dots, v_\ell]}{(f_1, \dots, f_n, g_1, \dots, g_\ell)} \qquad J_{ac} = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right]$$

So again étale.

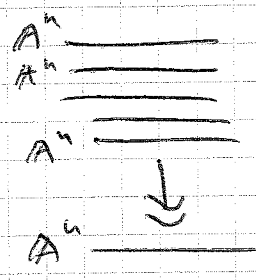
$$\begin{tikzcd} Y \times_X Y' \arrow[r, "\text{ét}"] \arrow[d, "\text{ét}"'] \arrow[dr, "\text{ét}"] & Y \arrow[d, "\text{ét}"] \\ Y' \arrow[r, "\text{ét}"'] & X \end{tikzcd}$$

<u>Étale cover</u> is étale map surjective on points.

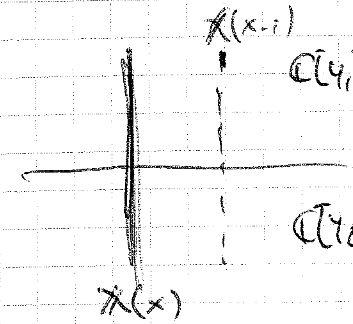
<u>Example</u> $X = \mathbb{A}^n$ $\quad \mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n]$ has only units $\mathbb{C}^*$

so has no global étale map

$$\mathbb{C}[x_1, \dots, x_n] \frac{[z]}{(z^n - \lambda)} = \prod_{i=1}^{n} \mathbb{C}[x_1, \dots, x_n]$$

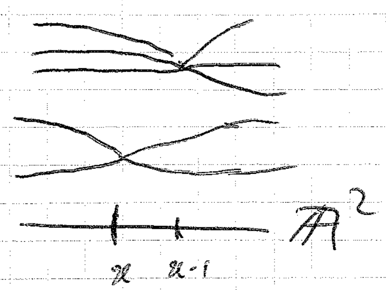


But can look at étale maps over affine open subsets



$\mathbb{C}[Y_1] = \mathbb{C}[x, x^{-1}, y] \frac{[z]}{(z^2 - x)}$ is étale ext of $\mathbb{C}[x, x^{-1}, y]$

$\mathbb{C}[Y_2] = \mathbb{C}[x, (x-1)^{-1}, y] \frac{[v]}{(z^3 - (x-1))}$ " $\mathbb{C}[x, (x-1)^{-1}, y]$



$Y = Y_1 \sqcup Y_2$

$\mathbb{C}[Y] = \mathbb{C}[Y_1] \oplus \mathbb{C}[Y_2]$

is étale ext of $\mathbb{C}[x,y]$

so there are plenty of étale covers of affine schemes and these covers can be refined by taking ~~coproducts~~ fibre products

SHEAF on ETALE TOPOLOGY

<u>Thm</u> (Grothendieck descent) $Y \xrightarrow{\text{ét}} X$ étale cover

Then for all $Z \in \mathrm{Ob}(\text{affine})$

$$\mathrm{Hom}(X, Z) \longrightarrow \mathrm{Hom}(Y, Z) \overset{i_1}{\underset{i_2}{\rightrightarrows}} \mathrm{Hom}(Y \times_X Y, Z)$$

$$\begin{tikzcd} & Y \times_X Y \arrow[r] \arrow[d] \arrow[dl, dashed, "i_2"'] \arrow[rr, dashed, bend left, "i_1"] & Y \arrow[d] & Z \arrow[l, phantom] \\ Z & Y \arrow[r] \arrow[l] & X & \end{tikzcd}$$

we say that h_Z ~~has~~ is a sheaf on (affine)

So far: only considered affine varieties

want to "glue" these together to get more interesting geometrical objects (like projective varieties, moduli spaces etc.)

<u>Dfn</u> A <u>SPACE</u> is a contravariant functor

$$\mathcal{F} : (\text{affine}) \longrightarrow (\text{sets})$$

enjoying the sheaf property for the étale topology i.e. $\forall$ $Y \xrightarrow{\text{ét}} X$ étale covers in (affine) have

$$\mathcal{F}(X) \to \mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y \times_X Y)$$

This is still too loose a condition. We want to restrict to spaces admitting an ATLAS like in diff. geometry but now with charts $\Longleftrightarrow$ affine schemes.

Space $\mathcal{F}$ has a set of points $|\mathcal{F}| = \mathcal{F}(\mathbb{C})$ and want to cover this with (points) of affine sch.

$$U = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k \longrightarrow\!\!\!\!\!\rightarrow |\mathcal{F}|$$

$$\mathbb{C}[U] = \mathbb{C}[U_1] \oplus \cdots \oplus \mathbb{C}[U_k]$$

but need to say when points in different charts are mapped to the same point in the space $\mathcal{F}$

$$\begin{array}{ccc} R = U \times_{|\mathcal{F}|} U & \xrightarrow{pr_1} & U \\ {\scriptstyle pr_2}\downarrow & & \downarrow \\ U & \longrightarrow & |\mathcal{F}| \end{array}$$

R closed subscheme of $U \times U$

R is equivalence relation.

<u>dfn</u> an ALGEBRAIC SPACE $\mathcal{L}$ is a SPACE $\mathcal{L}$ having an étale ATLAS

That is, $\exists$ affine scheme U and closed subsch $R \subset U \times U$ (i.e. $\mathbb{C}[R] \twoheadleftarrow \mathbb{C}[U] \otimes \mathbb{C}[U]$)

s.t. ① R is equivalence relation

② two projections $R \overset{pr_i}{\rightrightarrows} U$ are étale morph.

③ $|\mathcal{L}| = \dfrac{|U|}{|R|}$

we say $U \downarrow \mathcal{L}$ is étale atlas.

<u>dfn</u> a SCHEME is an ALGEBRAIC SPACE such that the restriction of R to each connected component of U is the trivial diagonal equivalence relation.

$\boxed{\text{CATEGORY THEORY 201}}$

A $\boxed{\text{2-CATEGORY}}$ is a category such that for all $A, B \in Ob(\mathcal{C})$ $Hom(A,B)$ is a <u>category</u>

That is, $\mathcal{C}$ consists of

- objects
- morphisms (which are a priori objects in a cat)
- 2-morphisms (~~ftors~~ morph between morphisms as elements in the cat)

satisfying "natural" conditions.

<u>Example</u> • (cat) { objects: categories; morph: functors; 2-morph: natural transfos

• (groupoids)
  - objects: groupoids (i.e. cat. in which all morph. are isomorph)
  - morphism: functors
  - 2-morph: natural transfor.

• Every category $\mathcal{C}$ becomes 2-category in obvious way: make $Hom(A,B)$ a category with
  objects the elements of $Hom(A,B)$
  every objects has identity map (and no other)

• (affine) as 2-category

• (affine/X) as 2-category
  ↑ all Y with "structural map" $Y \to X$

Recall: <u>Space</u> is contravariant functor

$$\mathcal{F} : (\text{affine}) \longrightarrow (\text{sets})$$

which is sheaf in étale top on (affine)

<u>Dfn</u>

a $\boxed{\text{STACK}}$ is a contravariant 2-functor

$$\mathcal{F} : \underset{\text{as 2-cat}}{(\text{affine})} \longrightarrow (\text{groupoids})$$

which is a sheaf in étale top on (affine).

So: what is a $\overset{\text{contra}}{}$ 2-functor?

sends $\left\{ \begin{array}{l} \text{objects} \to \text{objects} \\ \text{morph} \to \text{morphism} \\ \text{2-morp} \to \text{2-morphism} \end{array} \right.$

So: a stack associates to an affine scheme X a groupoid $\mathcal{F}(X)$

To a morphism of affine schemes $X \to Y$ (an alg. map $\mathbb{C}[Y] \to \mathbb{C}[X]$) a functor

$$\mathcal{F}(X) \longleftarrow \mathcal{F}(Y)$$

and to each 2-morphism in (affine)

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow id"] & Y \end{tikzcd}$$

a natural transfo

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, bend left, "\mathcal{F}(f)"] \arrow[r, bend right, "\mathcal{F}(f)"'] \arrow[r, phantom, "\Downarrow \mathcal{F}(id)"] & \mathcal{F}(Y) \end{tikzcd}$$

but <u>note</u>: this does not have to be the identity nat iso only natural iso.

<u>Example</u> X affine G finite gp acting on X

quotient stack

$[X/G]$: (affine) $\longrightarrow$ (groupoids) contravariant

$$Y \longmapsto [X/G](Y)$$

$[X/G](Y)$: <u>objects</u> are tuples

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"', two heads] & X \\ Y & \end{tikzcd}$$

π defines principal G-bundle in étale top

f is G-equivariant

<u>morphisms</u> in $[X/G](Y)$ are

$$\begin{tikzcd} & X & \\ E \arrow[ur, "f"] \arrow[rr, dashed, "\varphi"'] \arrow[dr, "\pi"', two heads] & & E' \arrow[ul, "f'"'] \arrow[dl, "\pi'", two heads] \\ & Y & \end{tikzcd}$$

where φ must be G-equiv isomorph.

morphisms acted on by $[X/G]$

$$Y' \xrightarrow{\ f\ } Y$$

gives functor $[X/G](f)$

$$[X/G](Y) \longrightarrow [X/G](Y')$$

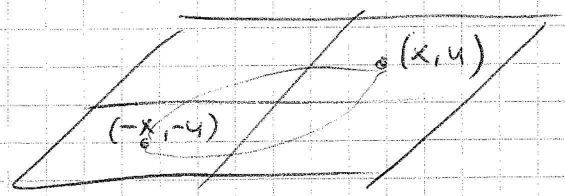
$$\begin{tikzcd} & & X \\ Y' \times_Y E \overset{\exists!}{=} E' \arrow[r, dashed] \arrow[d, dashed] \arrow[urr, dashed, bend left] & E \arrow[ur] \arrow[d] & \\ Y' \arrow[r] & Y & \end{tikzcd}$$

2-morphisms become

<u>Demystification</u> $X = \mathbb{A}^2$ $\quad \mathbb{C}[X] = \mathbb{C}[x,y]$

$$G = \mathbb{Z}/2\mathbb{Z}$$

action: point-reflexion in origin



X

what is $[X/G](Y)$?

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ Y & \end{tikzcd}$$

⊙ G-torsors are precisely

$$\mathbb{C}[E] = \frac{\mathbb{C}[Y][t]}{(t^2-a)} \qquad a \in \mathbb{C}[Y]^*$$

so are in particular étale

⊙ $\mathbb{C}[X] \longrightarrow \mathbb{C}[E] = \mathbb{C}[Y] + \mathbb{C}[Y]\sqrt{a}$

$$\begin{cases} x \mapsto \lambda\sqrt{a} \\ y \mapsto \mu\sqrt{a} \end{cases}$$

what are G-morphisms

$$\begin{tikzcd} & X & \\ E \arrow[ur, "f"] \arrow[rr, dashed, shift left, "\varphi"] \arrow[dr] & & E' \arrow[ul, "f'"'] \arrow[ll, dashed, shift left] \arrow[dl] \\ & Y & \end{tikzcd}$$

$\varphi : \mathbb{C}[E'] \longrightarrow \mathbb{C}[E]$

$\qquad \|$

$\mathbb{C}[Y] \oplus \mathbb{C}[Y]\sqrt{a'} \longrightarrow \mathbb{C}[Y] + \mathbb{C}[Y]\sqrt{a}$

φ is $\mathbb{C}[Y]$-morphism so

$$\sqrt{a'} \mapsto b\sqrt{a}$$

so that

φ: $a' = b^2 a$ $\quad$ so $b \in \mathbb{C}[Y]^*$

φ^{-1}: $a = b^{-2} a'$ $\quad$ so isomorphism

Morphisms btwn stacks are natural transfos

$$\begin{tikzcd} (\text{affine}) \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow"] & (\text{groupoids}) \end{tikzcd}$$

$\boxed{\text{YONEDA for stacks}}$

X affine as stack $\mathcal{F}$ stack

$$\mathrm{Hom}(X, \mathcal{F}) \cong \mathcal{F}(X)$$

<u>Proof:</u> "$X \overset{N}{\Rightarrow} \mathcal{F}$" $\longleftrightarrow$ $N\left(\begin{smallmatrix} X \\ \downarrow \mathrm{id} \\ X \end{smallmatrix}\right)$

Importance: can view elements of $\mathcal{F}(X)$ as maps $X \to \mathcal{F}$

<u>Example:</u> There is a special element in $[X/G](X)$

$$\begin{tikzcd} X \times G \arrow[r, "\text{action}"] \arrow[d, "\pi_1"'] & X \\ X & \end{tikzcd}$$

by Yoneda corresponds to map

$$\begin{tikzcd} X \arrow[d, "u"] \\ {[X/G]} \end{tikzcd}$$

$\boxed{\text{MAIN THEOREM ON QUOTIENTS STACKS}}$

$X \xrightarrow{u} [X/G]$ is representable <u>surjective</u> <u><u>étale</u></u> map giving atlas for the stack which then becomes an "algebraic" stack. Moreover X is a universal family for the quotient problem.

X affine, $\mathcal{F}$ stack

Dfn ⊙ $X \to \mathcal{F}$ is $\boxed{\text{representable}}$ iff $\forall\, Y$ affine $\forall\, Y \to \mathcal{F}$

$$\begin{tikzcd} U \cong Y \times_{\mathcal{F}} X \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \mathcal{F} \end{tikzcd}$$

form fibre product

$Y \times_{\mathcal{F}} X$ which is stack

(affine) $\longrightarrow$ (groupoids)

$Z \longmapsto \{(Z, \alpha) : \alpha$ iso in $\mathcal{F}(Z)$ between objects determined by (affine)

with structure maps to X & Y

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \arrow[d, bend left=60, "\mathrm{obj}_1"] \\ Y \arrow[r] \arrow[r, bend right=60, "\mathrm{obj}_2"'] & \mathcal{F} \end{tikzcd}$$

This fibre-product stack is representable by a scheme U

⊙ when map $X \to \mathcal{F}$ is representable the properties of the map hold if and only if the property holds for all maps $U \to Y$ coming from representability

e.g. surjective means all $U \to Y$ surject
étale " étale etc.

PROOF of main thm: claim:

$$\begin{tikzcd} E \arrow[r, "G\text{-eq}"] \arrow[d, "\text{princ } G\text{-bn}"'] & X \arrow[d] \\ Y \arrow[r] \arrow[u, bend right, "\text{defn}"'] & {[X/G]} \end{tikzcd}$$

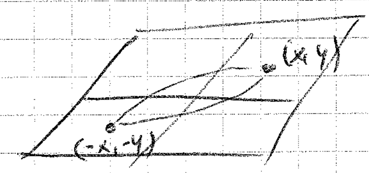
$Y \times_{[X/G]} X \cong E$

and so map will be surj + étale because $E \to Y$ is.

$$\begin{tikzcd}
Z\times(X\times G)\cong Z\times_X G \arrow[rr] \arrow[dr, dashed, "\cong"] \arrow[dd] & & X\times G \arrow[dd] \arrow[dd, bend left] \\
& Z\times E \arrow[dl] \arrow[dd] & \\
Z \arrow[uu, dashed, bend left] \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& E \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & {[X/G]}
\end{tikzcd}$$

$\square$

<u>Example</u>: $X = \mathbb{A}^2$
with $G = \mathbb{Z}/2\mathbb{Z}$ action



What are points of $[X/G]$, that is, $[X/G](\mathbb{C})$

There is only one G-fiber over $\mathbb{C}$ nl. $\overset{t\mapsto 1}{\mathbb{C}} \times \overset{t\mapsto -1}{\mathbb{C}} = \frac{\mathbb{C}[t]}{(t^2-1)}$

with action $\mathbb{C} \rightleftarrows \mathbb{C}$ (on $\mathbb{C}\times\mathbb{C}$). So element of $[X/G](\mathbb{C})$ is commut algebra diagram

$$\begin{array}{ccc} & \mathbb{C}\times\mathbb{C} & \xleftarrow{\varphi} \mathbb{C}[x,y] \\ (1,1) & \uparrow & \\ \uparrow & & \\ 1 & \mathbb{C} & \end{array}$$

$x \mapsto (\alpha,\beta)$
$-x \mapsto (-\alpha,-\beta) = (\beta,\alpha)$

so $\begin{cases} x \mapsto (\alpha,-\alpha) \\ y \mapsto (\beta,-\beta) \end{cases}$ are only possibilities

That is $\mathbb{C}[x,y] \xrightarrow{\varphi} \mathbb{C}\times\mathbb{C} \xrightarrow{m_1} \mathbb{C}$ determine the
$\mathbb{C}\times\mathbb{C} \xrightarrow{pr_2} \mathbb{C}$
two points (α,β), $(-\alpha,-\beta)$ in $\mathbb{A}^2$

so ~~pts~~ points of $[X/G]$ are precisely the orbits.

But: not just points, also have to compute autos in $[X/G](\mathbb{C})$

$$\begin{tikzcd} & \mathbb{C}[x,y] \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\ \mathbb{C}\times\mathbb{C} \arrow[rr, dashed, "?"] & & \mathbb{C}\times\mathbb{C} \\ & \mathbb{C} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

If α or $\beta \neq 0$ then $\nexists$ non-trivial automorphism making comm.

If α and $\beta = 0$ then have $\left\{ \xrightarrow{id} \atop \xrightarrow[\text{twist}]{} \right\}$ as automorphisms.

So: $[X/G](\mathbb{C})$ is