

noncommutative algebraic geometry over  $\mathbb{F}$

recall commutative case  $\mathbb{F}, \mathbb{Z} \subset \mathbb{C}$

- ① functor  $F: \text{fin abelian groups} \rightarrow \text{sets}$
- ②  $F$  complex affine variety and an evaluation for  $F$   
i.e.  $\exists A \mathbb{C}$ -algebra and ~~functor~~ (natural transfo)

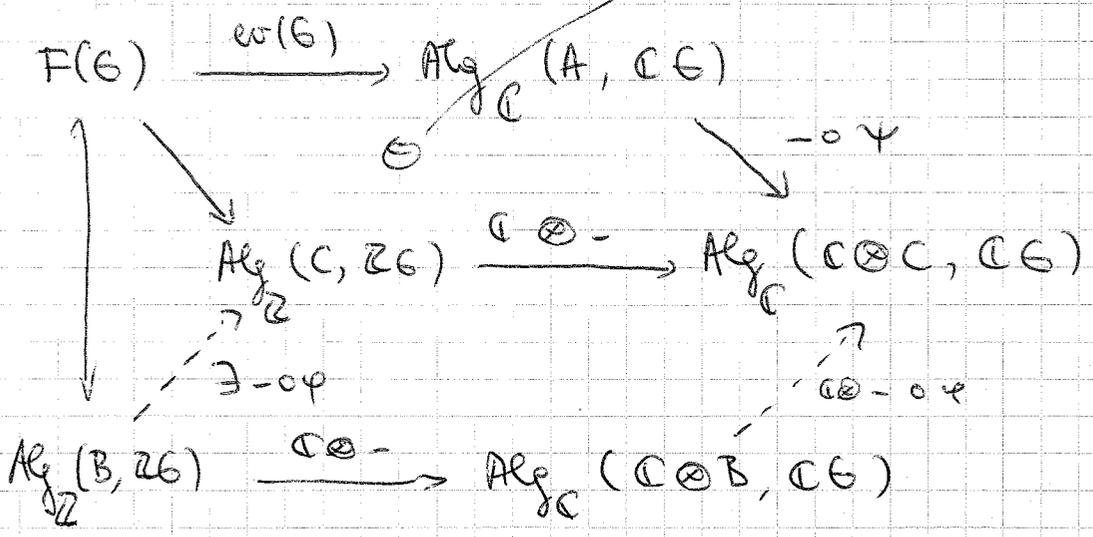
$$\text{ev}(\mathbb{C}) : F(\mathbb{C}) \rightarrow \text{Alg}_{\mathbb{C}}(A, \mathbb{C}\mathbb{C})$$

$$\text{ev} : F \rightarrow \text{Alg}_{\mathbb{C}}(A, -)$$

- ③  $F$  "best" integral scheme approximation of  $F$  w.r.t. the evaluation map.  
 $\exists \mathbb{Z}$ -algebra  $B$  and  $\forall \mathbb{C}$  inclusion

$$F(\mathbb{C}) \hookrightarrow \text{Alg}_{\mathbb{Z}}(B, \mathbb{Z}\mathbb{C})$$

with  $F \hookrightarrow \text{Alg}_{\mathbb{Z}}(B, -)$  natural transfo. "Best" means  
 $\forall \mathbb{Z}$ -alg  $C$   $\forall F \rightarrow \text{Alg}_{\mathbb{Z}}(C, -)$   $\forall \mathbb{C} \otimes C \xrightarrow{\psi} A$   
st.  $C$  is commutative  $\mathbb{C}$ -alg



$\exists \varphi : C \rightarrow B$   $\mathbb{Z}$ -alg morphism making everything commute.

Example

$$F: \text{abelian grp} \longrightarrow \text{sets} \quad \text{forgetful functor}$$

$$G \longrightarrow G$$

$$A = \mathbb{C}[x, x^{-1}] = \mathbb{C}\mathbb{Z} \quad ; \quad B = \mathbb{Z}[x, x^{-1}] = \mathbb{Z}\mathbb{Z}$$

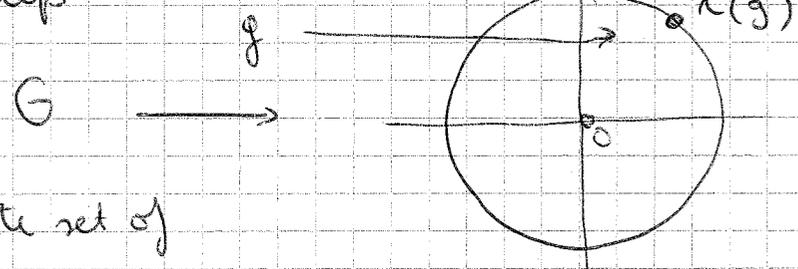
"evaluation":  $F(G) = G \longrightarrow \text{Alg}_{\mathbb{C}}(\mathbb{C}[x, x^{-1}], \mathbb{C}G)$

$x \mapsto g$

"best" property follows from fact that  $\mathbb{Z}$  has enough subgroups of finite index

what's geometry?  $\text{var } \mathbb{C}[x, x^{-1}] = \mathbb{C}^*$

evaluation map



so gives finite set of roots of unity.

for irreducible representation  $\chi$  (character)

$$\underbrace{\mathbb{M}_{\infty}}_{\widehat{\text{Sierp } \mathbb{Z}}} \subset \underbrace{\mathbb{C}^*}_{\widehat{\text{Sierp } \mathbb{Z}}}$$

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$$

profinite completion

integral picture: relevant  $\mathbb{Z}$ -schem are permutation representations  $\text{Spec}(\mathbb{Z}[x, x^{-1}]/(x^n - 1))$

what are analytic  $\mathbb{F}_1$ -functions on this  $\mathbb{F}_1$ -variety?

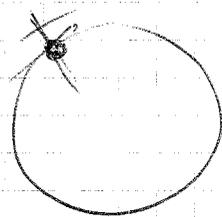
$$\mathbb{F}_1[x, x^{-1}]^{\text{an}} \otimes \mathbb{Z} = \bigcap \mathbb{Z}[x, x^{-1}] \quad \text{HABIRO RING}$$

$\mathbb{Z}[x, x^{-1}]^{\text{Hab}}$  are power series, evaluate well in all  $\mathbb{P}_\infty$  but possibly nowhere else. ③

importance:  $\left\{ \begin{array}{l} \textcircled{1} \exists \text{ Galois action } \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \text{ on } \text{Hab. ro. by} \\ \textcircled{2} \text{ Hab. ro. by "feels" inclusion } \mathbb{P}_\infty \subset \mathbb{C}^{**} \end{array} \right.$

$\text{rep}(\hat{\mathbb{Z}})$  is discrete semi-simple category

$\downarrow$   
 $\text{rep}(\mathbb{Z})$



$\text{rep}(\hat{\mathbb{Z}})$ -pt has "tangent up" via inclusion

(This tangent up is Galois inv.)

generalize to noncommutative case

Dfn: A noncommutative affine variety over  $\mathbb{F}$ , is

- ① functor  $F: \mu\text{-groups} \rightarrow \text{sets}$
- ②  $\exists \mathbb{C}$ -alg  $A$  and evaluation  $F \xrightarrow{\text{ev}} \text{Alg}_{\mathbb{C}}(A, -)$
- ③  $\exists \mathbb{Z}$ -alg  $B$ ,  $F \hookrightarrow \text{Alg}_{\mathbb{Z}}(B, -)$  and "best" prop.

Today:

④ are there interesting examples to motivate this extension?

④ real importance of  $\mathbb{F}_1$ -idea becomes clear when we

# Examples of noncommutative affine $\mathbb{F}_q$ -varieties

Two examples

- Grothendieck's theory of dessins d'enfants
- Topological quantum field theory

## DESSINS

projective Klein quartic  $X = X^3 + Y^3 + Z^3 \subset \mathbb{P}^2$

$X = \mathbb{R}/\Gamma(7) \leftarrow$  congruence subgroup

$\uparrow$   
hyperbolic plane

$\rightarrow$  group  $L_2(7)$  is simple order 168

part of  $\mathbb{R}$  consisting of 168 triangles. Some vertices are of degree 3 some of degree 2 some of degree 7

dessin = graph on degree 2 and 3 pts.

general

$X \quad X_0$  Riemann surface defined over  $\overline{\mathbb{Q}}$   
degree  $d$   $\mathbb{F}$   $\downarrow$  Belyi map

$\mathbb{P}^1$  ramified only in  $\{0, 1, \infty\}$



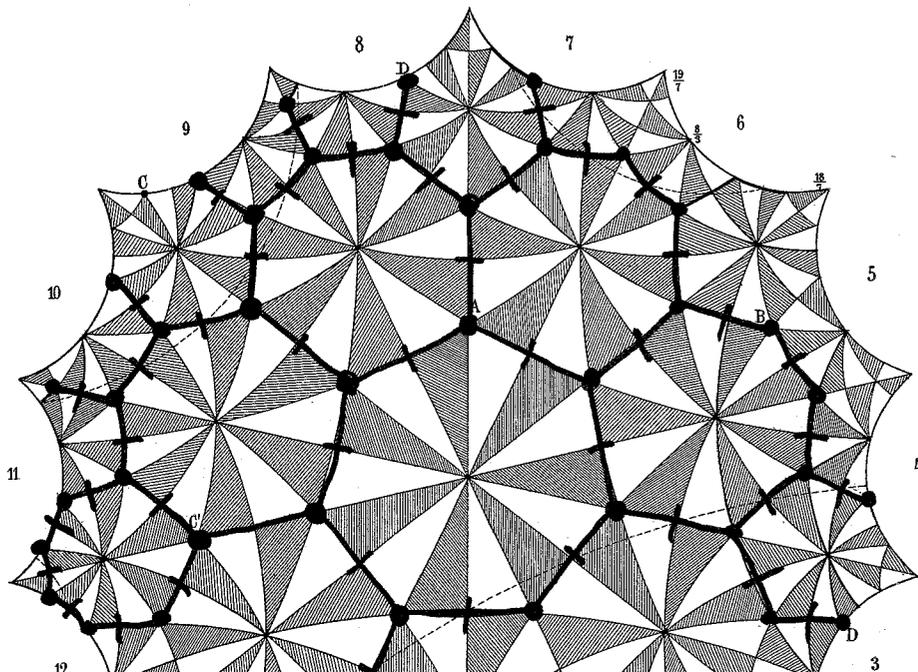
so  $[0, 1]$  lifts to  $d$  intervals in  $X$  but edgepoints of different lifts come together, indicating how different sheets should be glued <sup>in what way</sup> along their ramification pts. This graph is the dessin.

and  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts on these covers, so  $\mathbb{F}$  acts on dessin (the dessin itself is NOT Galois-invariant but subinvariant w.r.t.  $\mathbb{F}$ ).

## From the History of a Simple Group

JEREMY GRAY

The attractive pattern of 168 shaded and 168 unshaded triangles shown in Figure 1 has an interesting history. Since its discovery by Klein in 1878 (see

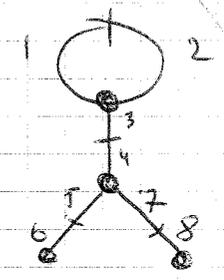


are interested in modular forms, i.e. modular  
 $X = \mathbb{H}/\Lambda$  for some subgroup  $\Lambda \subset \text{PSL}_2(\mathbb{Z}) = \Gamma$ . This  
 means that  $\Lambda$  has two types of points

- of degree 1 or 3
- of degree 2 or 2

graph gives the a permutation representation of  
 $\Gamma = C_2 * C_3$   
 $x \mapsto$  permutation determined by -  
 $y \mapsto$  " " " "

example



$$\begin{aligned} x &\mapsto (1,2)(3,4)(5,6)(7,8) \\ y &\mapsto ((1,3,2)(4,5,7)) \end{aligned} \left. \vphantom{\begin{aligned} x \\ y \end{aligned}} \right\} \begin{array}{l} \text{generate} \\ L_2(7) \end{array}$$

gives  $\Gamma \rightarrow \Gamma \rightarrow L_2(7)$

so for: permutation reps give subgroups of  $\Gamma$  of  
 finite index and hence  $\mathbb{H}/\Lambda$  surfaces (possibly  
 with cusps)

what is functor?

$F$ : groups  $\rightarrow$  sets

$G \mapsto$  { permut reps of  $\Gamma$  determined  
 by elements of  $G$  }

$$G \rightarrow G_{(2)} \times G_{(3)}$$

algebras?  $A = \mathbb{C} \text{PSL}_2(\mathbb{Z})$  and  $B = \mathbb{Z} \text{PSL}_2(\mathbb{Z})$

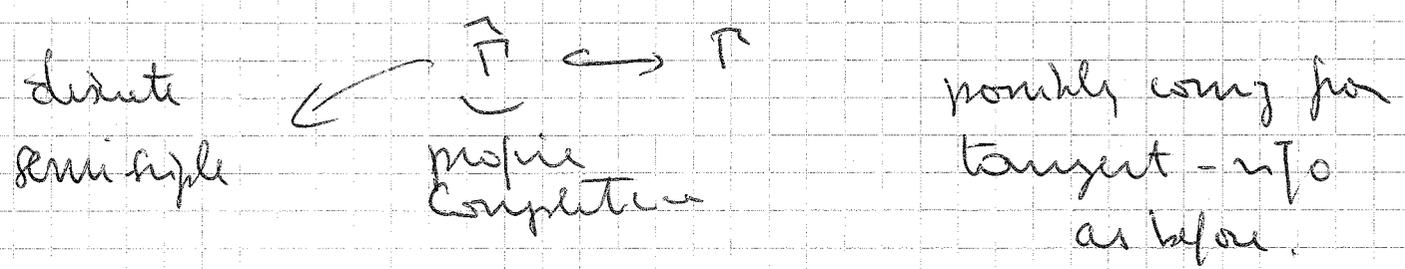
# evaluation map

$$G_{(2)} \times G_{(3)} \longrightarrow \text{Alg}_- \left( \begin{matrix} A \\ B \end{matrix}, -G \right)$$

$$(g_2, g_3) \quad \left\{ \begin{array}{l} x = g_2 \\ y = g_3 \end{array} \right.$$

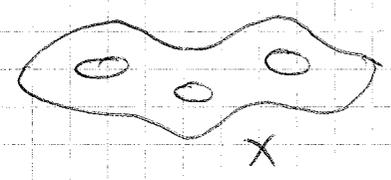
again  $B = \mathbb{C}^T$  is "best" because  $\mathbb{C}^T$  has enough subtypes of fin index.

aim: get new Galois invariants for densities  
 coming from geometrical embeddings



TQFT

want to associate to stuff  
 (Joachim Kock "Fibered algebras and 2D TQFT")



$\longrightarrow Z(X)$  expectation value

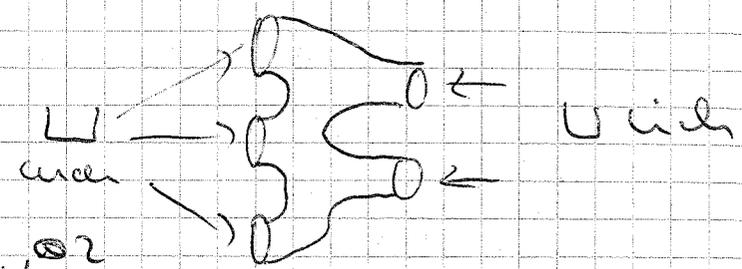
time  $\longrightarrow$

depending only on topology

idea of TQFT: associate to  $O = S^1$  of vector space  $V$

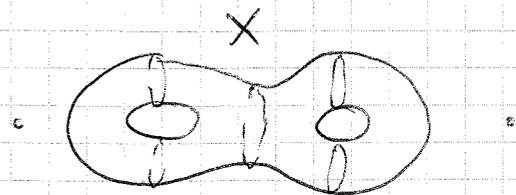
$$\begin{matrix} \text{to } \cup \text{ circle} & \iff & \otimes V \\ \downarrow & & \\ \emptyset & \iff & \mathbb{C} \end{matrix}$$

to "cobordism"



linear map  $V^{\otimes 1} \longrightarrow V^{\otimes 2}$

Then break up string in such pieces

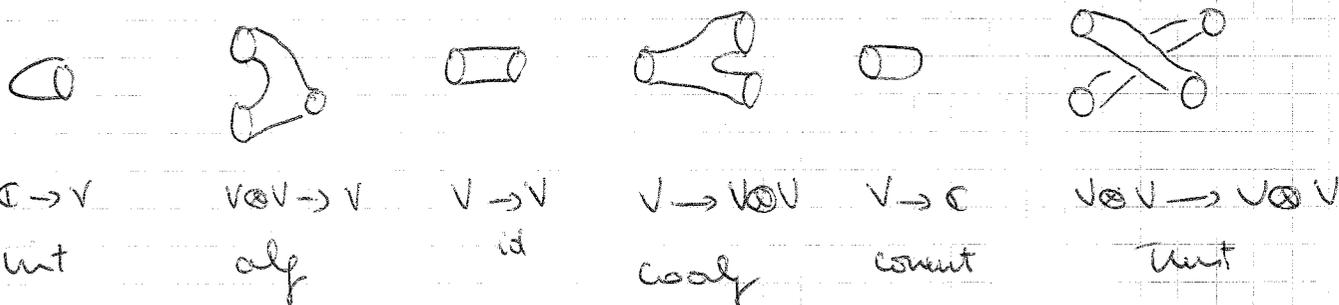


$$\mathbb{C} \rightarrow V \otimes V \rightarrow V \rightarrow V \otimes V \rightarrow \mathbb{C}$$



linear maps will give  $Z(X) \in \mathbb{C}$

To have topological invariance, must have string conditions on linear maps defining basic building blocks.



Thm: TQFT  $\leftrightarrow$  commutative fin dim Frobenius algs.  
(in particular Hopf algs)

example  $\mathbb{C}G$  is (noncom) Frobenius alg

$Z(\mathbb{C}G)$  (character of) is comm Frobenius alg.

What is  $Z(X)$  in the setting.

$L_2(7)$  example a genus 4  
817.404.081 fields

$$Z(X) = \# \{ \text{fields on } X \text{ with gauge group } G \}$$

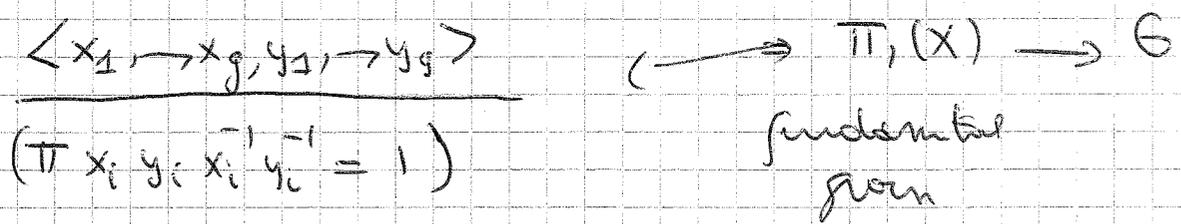
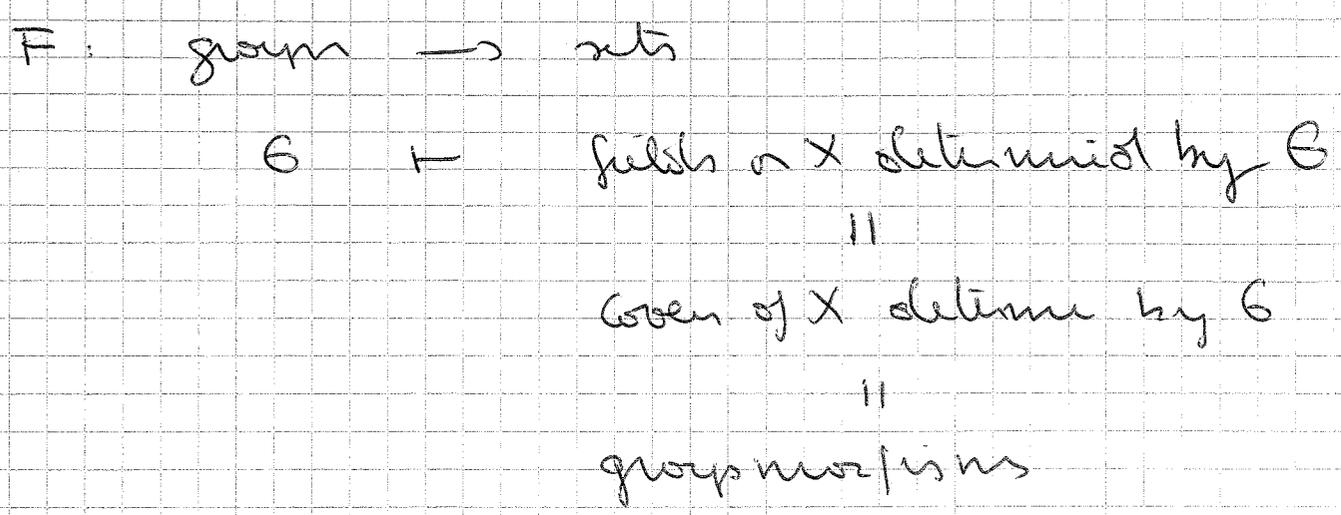
$$= \# \{ G\text{-covers of } X \}$$

Frobenius  
Schur

$$= \sum_{\chi} \left( \frac{161}{\dim \chi} \right)^{2g-2}$$

depends only on genus  $g$   
and dim of reps of  $G$

What is functor?



$\parallel$

$\{ (g_1 \mapsto g_g, g'_1 \mapsto g'_g) \in G^{2g} :$

$\pi_1(g_i g'_i g_i^{-1} g'_i^{-1}) = 1 \}$

What are algebras?

$A = \mathbb{C} \pi_1(X)$  and  $B = \mathbb{Z} \pi_1(X)$

again "best" as  $\pi_1(X)$  has enough cofinite subgroups.

$\mathbb{H}_c\text{-info} = \widehat{\pi_1(X)}$  profinite completion

gives étale (i.e. algebraic) covers of  $X$

So if  $X$  is étale over  $\mathbb{Q}$  the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $\widehat{\pi_1(X)}$ , Again want new Galois invariants coming from  $n \in \mathbb{C}$  on  $\widehat{\pi_1(X)} \hookrightarrow \pi_1(X)$ .

but: what is "noncomm geometry" for a  $\mathbb{C}$ -algebra  $A$  ?

(9)

A commutative  $\mathbb{C}$ -algebra  $\Rightarrow$  "geometry" = Topological Space + coordinate ring of functions.  $\text{Var}(A)$

A noncommutative  $\mathbb{C}$ -algebra  $\Rightarrow$  "geometry proposal"

Topological Space

dual coalgebra



$$A^\circ = \{ \lambda: A \rightarrow \mathbb{C} \}$$

$K(\lambda) \supset \text{col} \{ \text{rows} \}$

coordinate ring of functions

vector space (Lie algebra)



$$\mathfrak{g}_A = \frac{A}{[A, A]_{\text{vect}}}$$

Explanation in case  $A = \mathbb{C}\mathbb{G}$

$$A^\circ = M_n(\mathbb{C})^* \oplus \dots \oplus M_n(\mathbb{C})^*$$

$M_n(\mathbb{C})^*$  is matrix coalgebra =  $\sum \mathbb{C} e_{ij}$

gives  $\text{insep}(\mathbb{G})$

$$\left\{ \begin{aligned} \Delta e_{ij} &= \sum_k e_{ik} \otimes e_{kj} \\ \varepsilon e_{ij} &= \delta_{ij} \end{aligned} \right.$$

$$\frac{A}{[A, A]_{\text{vect}}} = \frac{\mathbb{C}G}{[\mathbb{C}G, \mathbb{C}G]_{\text{vect}}} = \text{trace maps on } \mathbb{C}G$$

(10)

" character of  $G$ .  
enough to separate the maps

general  $A^\circ$  is not always co-semi-simple

$$\oplus M_n(\mathbb{C})^* = \text{corad}(A^\circ) \subset A^\circ$$

simple n-dim rep of  $A$        $\oplus$  of (co)simple coalgebras      in com case =  $\uparrow$  var( $A$ )

so,  $A^\circ$  is rather the cocoradical  $\leftrightarrow$  is classes of simple rep.

Functorial properties:

$$A \xrightarrow{\varphi} A' \Rightarrow (A')^\circ \xrightarrow{\varphi^\circ} A^\circ$$

but NOT always

$$\text{corad}(A'^\circ) \not\rightarrow \text{corad}(A^\circ)$$

is TRUE when co-commutative  
explain good functorial prop for var(-) in commutative case.

"space" of  $A =$  simples of  $A = \text{corad}(A^\circ)$

$$\text{rep}(A) \leftrightarrow A^\circ$$

Cocoradical function

+ Poincaré  $A/[A, A] =$  trace maps separate simples

but: ① can we describe  $A^o$  explicitly?  
 ② what info does it contain?

If  $Q$  is (infinite) quiver,  $\mathbb{C}Q$  vector space of all paths  $\in Q$

$\mathbb{C}Q$  is coalgebra (path coalgebra)

$$\Delta P = \sum_{P_1 \circ P_2 = P} P_1 \otimes P_2 \quad ; \quad \varepsilon P = \begin{cases} 1 & \text{if } P \text{ is a vertex} \\ 0 & \text{otherwise} \end{cases}$$

Concatenation of paths

will make a huge quiver associated to  $A$

$$Q_A \begin{cases} \text{vertices} = \text{isoclasses of simple f.d. } A\text{-rep. } S \\ \text{arrows} = \#(S \rightarrow S') = \dim_{\mathbb{C}} \text{Ext}_A^1(S, S') \end{cases}$$

will describe subcoalgebra of path coalgebra  $\mathbb{C}Q_A$ .  
 any coalgebra is  $\varinjlim$  for dual subcoalgebra so  
 can restrict to finite # of simples (vertices)

$$M = S_1 \oplus S_2 \oplus \dots \oplus S_k$$

and consider subcoalgebra  $\mathbb{C}Q_A/M$ .

Xi Wei has explained canonical  $A_\infty$ -structure on Yoneda Ext-algebra

$$\text{Ext}_A^0(M, M)$$

one lots of maps  $m_i$  from tensors of  $\text{Ext}^i$   
in particular have multiplication maps

$$m_i : \underbrace{\text{Ext}_A^1(M, M) \otimes \dots \otimes \text{Ext}_A^1(M, M)} \rightarrow \text{Ext}_A^2(M, M)$$

so get a linear map (homotopy Maurer-Cartan map)

$$\oplus m_i : \mathbb{C}(\mathbb{Q}_A | M) \rightarrow \text{Ext}_A^2(M, M)$$

if we do this  $\forall M$

$$\oplus m_i : \mathbb{C} \mathbb{Q}_A \rightarrow \text{Ext}_A^2\text{-space}$$

Thm (based on Ed Segal, Bernard Keller ...)

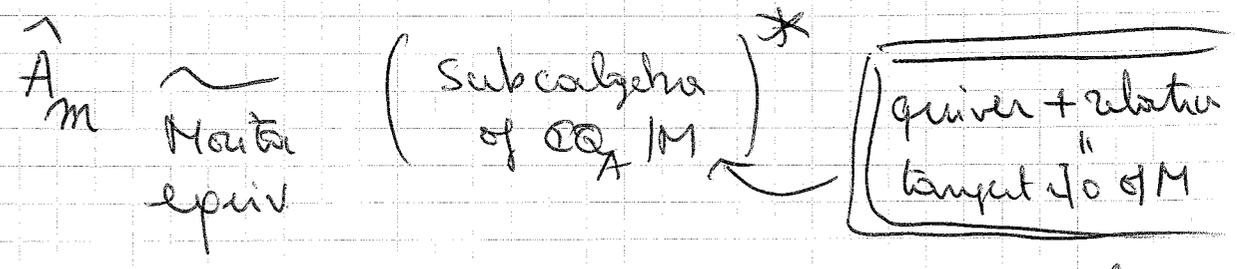
$A^\circ$   
Morita  
Tateuchi  
equivalent

largest subalgebra of  $\mathbb{C} \mathbb{Q}_A$   
contained in kernel of  
HMC

what is algebraic content of this?

$$M = S_1 \oplus \dots \oplus S_k$$

$$m = \ker(A \rightarrow M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C}))$$



so if we know the  $\hat{A}_m$ -structure  $\Rightarrow$  can compute all  $m$ -adic completions.

$A^0$  contains  $\left\{ \begin{array}{l} \text{space} = \text{simplicial} \\ \text{"local rings" in \acute{e}tale topology} \end{array} \right.$

even in commutative case hard to compute all completions, except when  $A$  is "smooth" then are just formal power series.

What are noncommutative smooth algebras of dim  $n$ ?

Dfn:  $A$  is called  $n$ -Calabi Yau iff

- ①  $\text{gldim } A = n$
- ②  $\forall x, y$  f.d.  $A$ -rem, exists identification  $\text{Ext}_A^i(x, y) \cong (\text{Ext}_A^{n-i}(y, x))^*$  "duality"

giving pairing  $\langle -, - \rangle_{x, y}^i : \text{Ext}_A^i(x, y) \times \text{Ext}_A^{n-i}(y, x) \rightarrow \mathbb{C}$

$$\textcircled{3} \langle f, g \rangle_{x, y}^i = \langle 1_x, g \circ f \rangle_{x, x}^0 = (-1)^{i(n-i)} \langle 1_y, f \circ g \rangle_{y, y}^0$$

Can we determine  $A^0$  better if  $A$  is n-CY?  
That is can we determine the relations given  
by  $A_\infty$  structure on  $\mathbb{Q}_A M$  to give  $\widehat{A}_M$ ?

"Curves"

dim 1 1 - Calabi-Yau = "formally smooth"

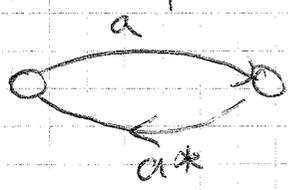
$\text{Ext}^2(M, M) = 0$  so are no relations

$\widehat{A}_M \sim$  Monte completion of path algebra  $\mathbb{Q}_A M$

"Surfaces"

dim 2 2 - Calabi-Yau

(Raf)  $\mathbb{Q}_A M$  is double quiver



relations are given by preprojective relation i.e.

$$\sum [a, a^*] = 0$$

"3-folds"

dim 3 3 - Calabi-Yau (Raf + Ext Sepal)

relations come from "SUPERPOTENTIALS"

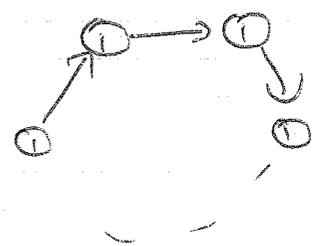
i.e.  $\sum$  circuits in quiver  $\Rightarrow$  nc. partial derivatives

so at least in dimension 1 and 2 know  
 no tangent info = quiver + relations  
 once we can compute the Ext between simply  
 one remaining problem to "know"  $A^0$ . can we  
 determine all simples ???

$\exists$  inductive procedure to find all simples  
 (at least for 1- and 2-cy's)

dim 1 (+ Procesi)

have already part of  $Q_A \Rightarrow$  new minimal  
 simples arise for all circuits in quiver

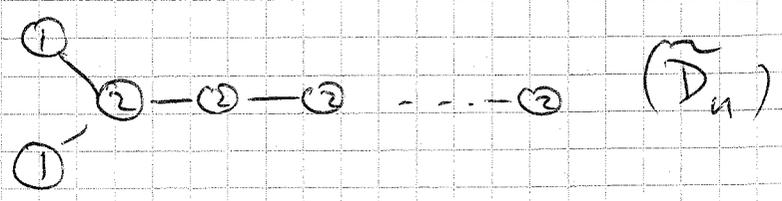
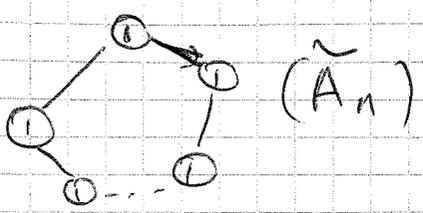


simple of dim  $\neq$  vertices  
 $\times$  cycles  
 of  $n$

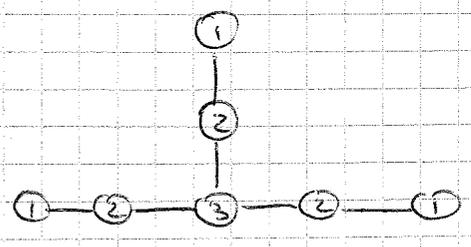
and also can compute Ext for this new simple  
 from Euler-form of the quiver.

dim 2 (+ Crawly-Boevey)

new minimal simples arise from  $\text{Voloboukhis}$ <sup>or</sup>  
 of ~~the~~ extended Dynkin  
 subquivers where



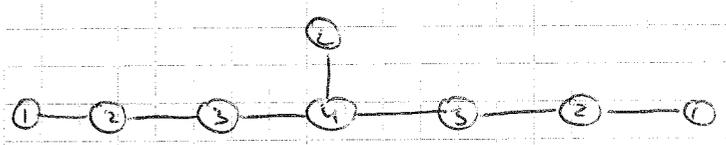
$\tilde{E}_6$



$\tilde{E}_8$



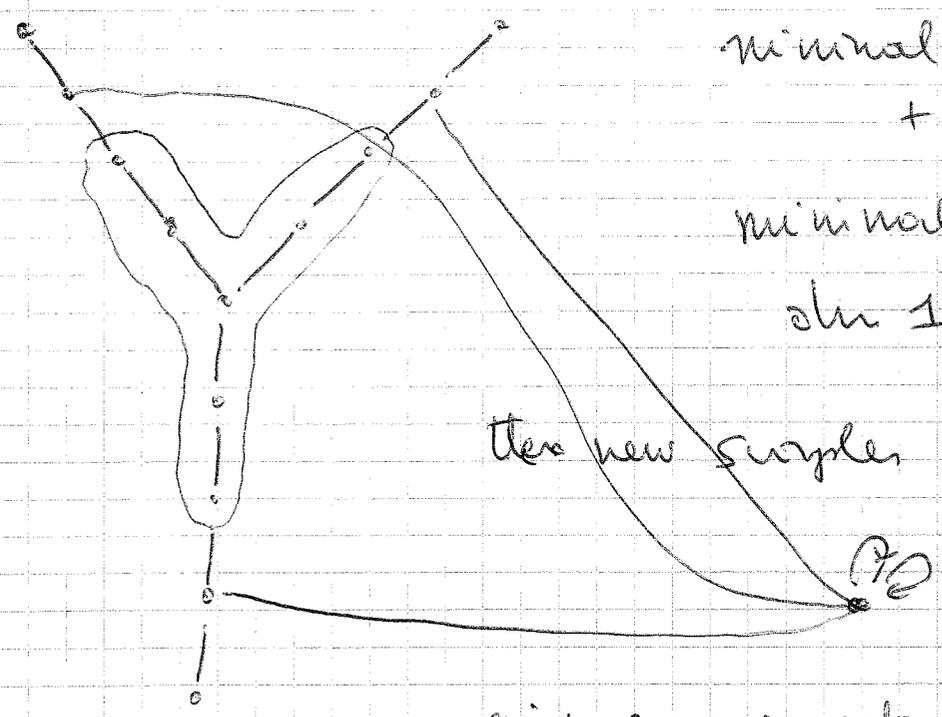
$\tilde{E}_7$



$\Rightarrow$  give new simple of  $\tilde{d}_n \quad \sum (\text{vector} \times \text{diagram})$

and can compute all  $\text{Ext}^i$  from the new simple to existing ones from Euler-form of smaller quiver.

example:  $\tilde{E}_6$ -quiver (from monster)



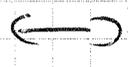
minimal simple 1 dual + ext's give by quiver

minimal new simple has dim 12 come from  $\tilde{E}_6$

then new simple add to quiver

giving new extend  $P_{\text{ext}}$  subyn etc etc.

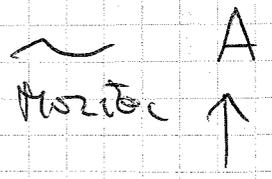
etalest pchir  
memojet real



Kleinian  
singularities



$\mathbb{C}[x, y] \neq \mathbb{C}$



2-CY

Michel Crepant resol.

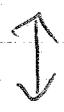
CONJECTURE: general n-CY A

quiver +  
relations  
Simples +

$A^\circ$  can be constructed

inductively starting from

minimal-mutations



Michel's n.c crepant res. i.e.

n-CY which are End(M) ← reflex Azum

↓  
isotyped  
Surf.

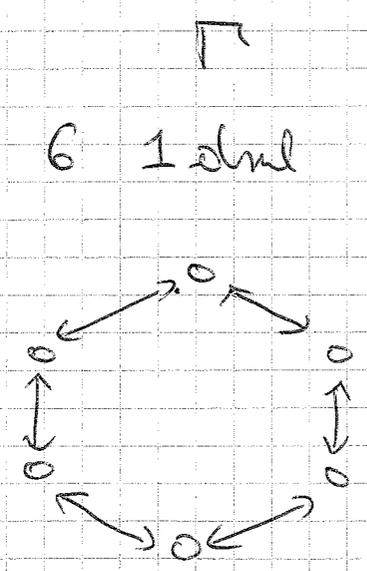
But

: what has all this to do with our  
examples of  $\mathbb{F}_1$  - n.c. varieties ???

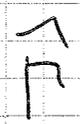
**dessins**  $\leftrightarrow A = \mathbb{C} \text{PSL}_2(\mathbb{Z}) = \mathbb{C} C_2 * C_3$   
 is 1-Calabi Yau

**TQFT**  $\leftrightarrow A = \mathbb{C} \pi_1(X)$   
 is 2-Calabi Yau (Kontsevich)

How do we start building our quiver + relations from the 1-representation.



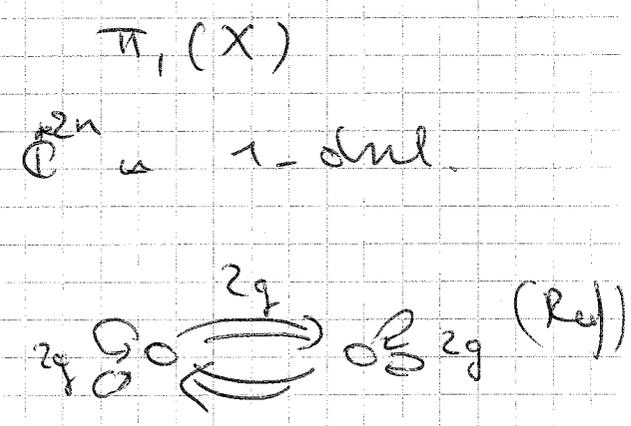
and go from there



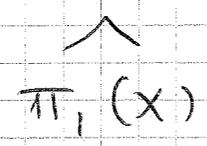
use  $X'$  for order 2 and order 3

can determine for any rep of  $\Gamma$

← n.c. tangent up to if we have character table →



and go from there

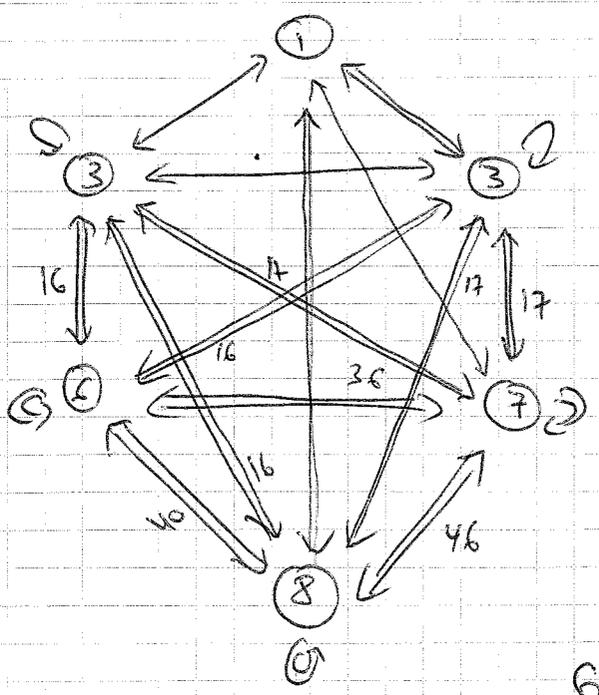


use only dims of irreducibles

Example  $G = L_2(7)$  simple of order 168

	1A	2A	3A	4A	7A	7B
$\chi_1$	1	1	1	1	1	1
$\chi_3$	3	-1	0	1	$\alpha$	$\bar{\alpha}$
$\bar{\chi}_3$	3	-1	0	1	$\bar{\alpha}$	$\alpha$
$\chi_6$	6	2	0	0	-1	-1
$\chi_7$	7	-1	1	-1	0	0
$\chi_8$	8	0	-1	0	1	1

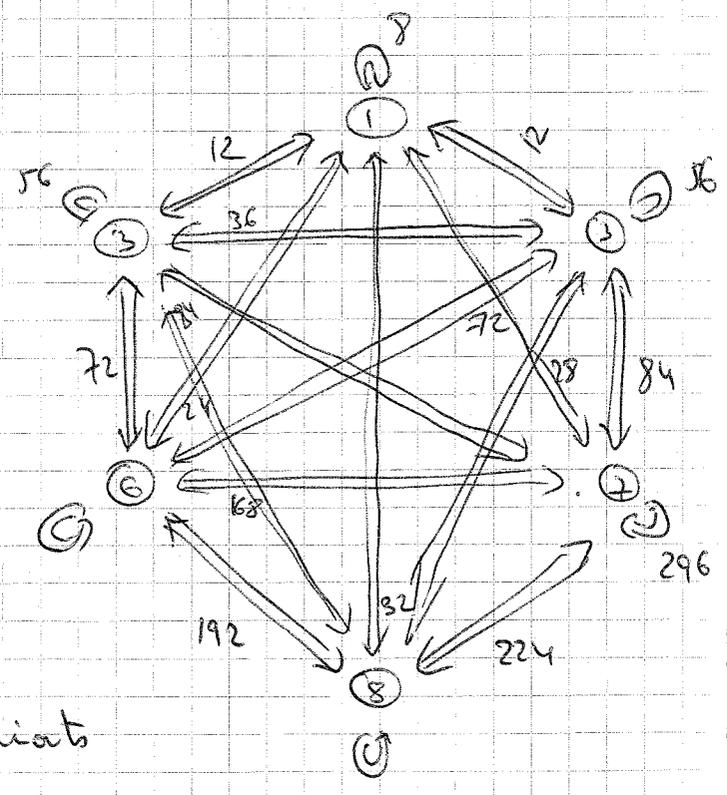
$\hat{\Gamma} \hookrightarrow \Gamma$   
 target  $u/p$



Galois invariants

deformation of  $CL_2(7)$  as  $\Gamma$ -rep  
 along a curve

$\hat{\Pi}_1(X) \hookrightarrow \Pi_1(X)$   
 target  $v/p$



deformation of  $CL_2(7)$  as  $\Pi_1(X)$ -rep  
 along a surface.