(1)

$\boxed{\text{affine } \mathbb{F}_1 \text{-geometry}}$ $\quad \mathbb{F}_1 \dashleftarrow\dashrightarrow \mathbb{Z} \longleftrightarrow \mathbb{C}$

<u>mantras</u>: ① $\mathbb{F}_1$-stuff is multiplicative, not additive
② $\mathbb{F}_1$-stuff becomes visible after extension of scalars

Affine $\mathbb{F}_1$-scheme will only be seen when it becomes an $\mathbb{Z}$-affine scheme or $\mathbb{C}$-complex variety.

<u>Grothendieck ('60)</u>

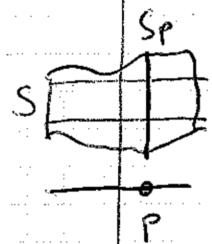
$$\begin{pmatrix}\text{commutative}\\ \text{rings } / R\end{pmatrix} \xleftrightarrow{\text{anti-equiv}} \begin{pmatrix}\text{affine } R\text{-schemes}\end{pmatrix}$$

prime ideals

$$\begin{array}{ccc} S & \longmapsto & (\mathrm{spec}(S), \mathcal{O}_S) \\ \varphi \uparrow & \longleftarrow \text{structure maps} \longrightarrow & \downarrow \\ R & & (\mathrm{Spec}(R), \mathcal{O}_R) \end{array}$$

structure sheaf

$$\Gamma(X, \mathcal{O}_X) \longleftarrow\!\!| \ (X, \mathcal{O}_X)$$



(I) $\boxed{\text{Complex affine varieties}}$

Use $\{A, B, \ldots$ for $\mathbb{C}$-algebras. Typical example:
$\{X, X', \ldots$ for complex affine schemes

$$A = \frac{\mathbb{C}[x_1, \ldots, x_n]}{I} = \mathbb{C}[X] \quad \text{coordinate ring of affine } \mathbb{C}\text{-scheme}$$

$$\mathbb{A}^n_{\mathbb{C}} \hookleftarrow X = \mathbb{V}(I) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n : f(\alpha_1, \ldots, \alpha_n) = 0 \ \forall f \in I\}$$

$\mathrm{Spec}(A) = \{$ all prime ideals of $A\}$

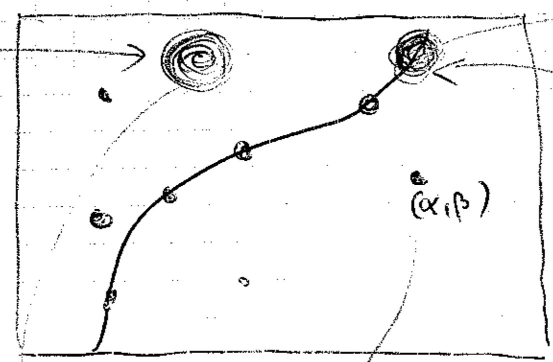
also contains info about other closed subvarieties

$\mathrm{Max}(A) = \{$ all maximal ideal of $A\} \longleftrightarrow$ points of $\mathbb{V}(I) = X$

Example $A = \mathbb{C}[x,y]$ $X = \mathbb{A}^2$

"generic pt of plane"

"generic point" of curve



Mumford's text book picture

prime ideals of $\mathbb{C}[x,y]$:

I $(x-\alpha, y-\beta)$ maximal ideals : points

II $(f(x,y))$ $f(x,y)$ irreducible pol : curves

III 0

$\boxed{\text{Grothendieck's functorial description}}$

affine $\mathbb{C}$ scheme X with "coordinate ring" $A = \mathbb{C}[X]$
determines (and is determined by) "FUNCTOR OF POINTS"

$$h_A = h_X : (\mathbb{C}\text{-algs}) \longrightarrow (\text{Sets})$$
$$B \longmapsto \mathrm{Alg}_{\mathbb{C}}(A, B)$$

Examples : $B = \mathbb{C}$ $\mathrm{Alg}_{\mathbb{C}}(A, \mathbb{C}) = \mathrm{Max}(A) =$ pts of X

$B = \mathbb{C}[\varepsilon]$ $\mathrm{Alg}_{\mathbb{C}}(A, \mathbb{C}[\varepsilon]) =$ pts + tangent vectors

$B = \mathbb{C}[x]/(x^n)$ pts + n-order infin. info

$\vdots$

$B = \mathbb{C}[t]$ $\mathrm{Alg}_{\mathbb{C}}(A, \mathbb{C}[t])$ lines in X

$\vdots$

functor of points contains A LOT of info

Morphisms of affine $\mathbb{C}$-varieties

$$\updownarrow$$

algebra morphism between coordinate rings

$$\begin{array}{ccccc}
A & X_A & & h_A = h_{X_A} & Alg_{\mathbb{C}}(A,C) \\
\varphi\downarrow & \uparrow & \text{natural transf} & \uparrow & \uparrow -\circ\varphi \\
B & X_B & & h_B = h_{X_B} & Alg_{\mathbb{C}}(B,C)
\end{array}$$

conversely $id \in Alg_{\mathbb{C}}(B,B)$

$$\downarrow$$

$\varphi \in Alg_{\mathbb{C}}(A,B)$

(II) $\boxed{\text{integral affine schemes}}$

use $\begin{cases} \Lambda, \Gamma, \dots & \text{for } \mathbb{Z}\text{-rings} \\ Y, Y_\Lambda, \dots & \text{for corresponding } \mathbb{Z}\text{-schemes} \end{cases}$

$Y_\Lambda = (\operatorname{spec}(\Lambda), \mathcal{O}_\Lambda)$ determines <u>functor of points</u>

$$h_\Lambda = h_Y : \underset{\substack{\| \\ (\text{comm})}}{(\mathbb{Z}\text{-rings})} \longrightarrow (\text{sets})$$

$$\Gamma \longmapsto Alg_{\mathbb{Z}}(\Lambda, \Gamma)$$

and again: morphism of schemes $\longleftrightarrow$ ring morphism

$$\begin{array}{cccc}
\Lambda & Y_\Lambda & h_\Lambda = h_{Y_\Lambda} & Alg_{\mathbb{Z}}(\Lambda, \Sigma) \\
\varphi\downarrow & \uparrow & \uparrow & \uparrow -\circ\varphi \\
\Gamma & Y_\Gamma & h_\Gamma = h_{Y_\Gamma} & Alg_{\mathbb{Z}}(\Gamma, \Sigma)
\end{array}$$

Example Spec $\mathbb{Z}[X]$ "ARITHMETIC SURFACE"

prime ideals of $\mathbb{Z}[X]$

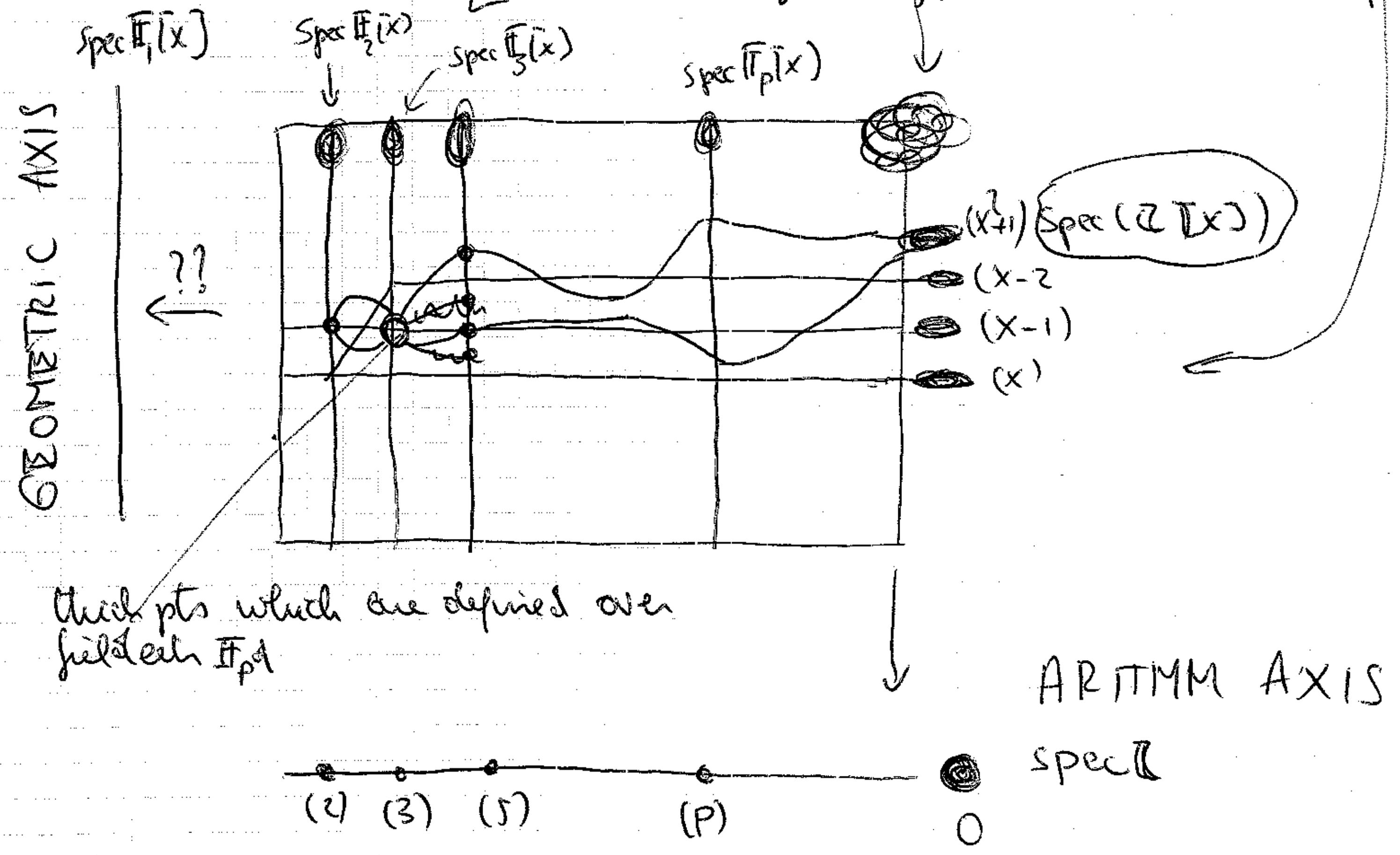
"horizontal"

- (0)
- principal ideals $(f(X))$ $f(x)$ irreducible poly
  (p) p prime ~~ideal~~ int

"vertical"

- maximal ideals

$$m = (p, f(x))$$ p prime nmbr, $f(x)$ remain irr in $\mathbb{F}_p[x]$



$$(X^2+1) = (x+1)(x+1) \quad \text{in } \mathbb{F}_2$$ one pt in $h_y(\mathbb{F}_2)$

$$= \text{irr} \quad \text{in } \mathbb{F}_3$$ $\longrightarrow$ defined pt $h_y(\mathbb{F}_{3^2})$

$$= (x+2)(x+3) \quad \text{in } \mathbb{F}_5$$ two pts in $h_y(\mathbb{F}_5)$

$\vdots$

III $\boxed{\text{affine } \mathbb{F}_1\text{-schemes}}$

Chr. Soulé (MPI 1999), Connes-Consani (arXiv: 0809.2926)

3 ingredients

SIMILAR TO AFF GEO

"gadget"
- (A) functor of "pts" $(\mathbb{F}_1\text{-alg}) \xrightarrow{Z} (\text{Sets})$
- (B) evaluation map, natural transf
  $Z \xrightarrow{ev} h_X$ some complex variety X
- (C) realization as subfunctor of "best" integral scheme Y

VISIBILITY MANTRA

$$Z \hookrightarrow h_Y$$

If such a best integral scheme Y exist, we say:
- Y is defined over $\mathbb{F}_1$
- or, gadget (Z, ev) is an affine $\mathbb{F}_1$ - scheme

(A) What are $\mathbb{F}_1$-algs?

only multiplicative structure $\Rightarrow$ group of units of field extns

$\exists!\ \mathbb{F}_{p^n} \,|\, \mathbb{F}_p$ dim n $\Rightarrow$ expect $\exists!\ \mathbb{F}_{1^n} \,|\, \mathbb{F}_1$ dim n = "set of n elements" + multipl. str = abelian group

gp of units cyclic $\Rightarrow$ expect: $\mathbb{F}_{1^n} = C_n$

visibility mantra: $\mathbb{F}_{1^n}$ must become visible $-\otimes \mathbb{Z}$ or $-\otimes \mathbb{C}$

SOULE - proposal $\quad \mathbb{F}_{1^n} \otimes \mathbb{Z} = \mathbb{Z} C_n \quad \mathbb{F}_{1^n} \otimes \mathbb{C} = \mathbb{C}\, C_n$

integral and complex group algebras

($\mathbb{F}_1$-algs) as $\otimes$-products of $\mathbb{F}_{1^n}$
$\parallel$
(abelian) : finite Abelian groups (CC-proposal)

so our functor of points are functors :

$$Z : (\text{abelian}) \longrightarrow (\text{Sets})$$
$$G \mapsto Z(G)$$

Example) easiest example of such a functor is forgetfull functor

$$F : (\text{abelian}) \to (\text{sets})$$
$$G \mapsto F(G) = G$$

(B) Evaluation map

want to give the sets $Z(G)$ visibility as $\mathbb{C}G$-points of a complex affine variety X

$$ev : Z \longrightarrow h_X = h_A \qquad (A = \mathbb{C}[X])$$

$$Z(G) \longrightarrow h_X(\mathbb{C}G) = Alg_{\mathbb{C}}(A, \mathbb{C}G)$$

Historic motivation comes from arithmetic geometry

$$\text{spec}(\mathbb{Z}) = \underbrace{\text{p-adic valuation}}_{\text{non arch}} + \text{real valuation} \leftarrow \text{archim}$$

these are included if we view spec($\mathbb{Z}$) as $\mathbb{F}_1$-space

this not but evaluation map substitutes for this

buzz word = "Arakelov"

Example Take $X = \mathbb{C}^*$ $\qquad A = \mathbb{C}[x, x^{-1}]$

coordinate ring of multiplicative group scheme

what is functor of pts?

$$h_X : (\mathbb{C}\text{-alg}) \to (\text{Sets})$$

$$B \mapsto \mathrm{Alg}_{\mathbb{C}}(\mathbb{C}[x,x^{-1}], B) = B^*$$

$$x \mapsto b$$

evaluation map:

$$ev : F \longrightarrow h_X$$

$$F(G) = G \longrightarrow h_X(\mathbb{C}G) = \mathrm{Alg}_{\mathbb{C}}(\mathbb{C}[x,x^{-1}], \mathbb{C}G)$$

$$g \mapsto \quad \mathbb{C}[x,x^{-1}] \to \mathbb{C}G = \sum_{h \in G} \mathbb{C} e_h$$

$$x \mapsto e_g$$

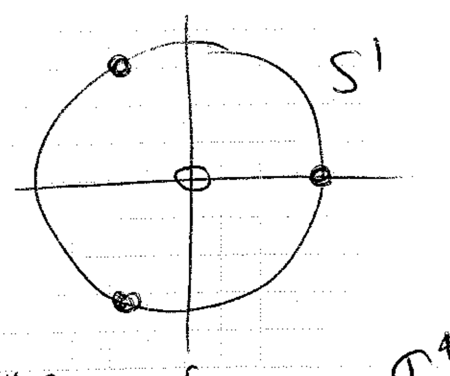
geometric picture

$$\text{affine scheme of } \mathbb{C}G = \{\text{maximal ideals of } \mathbb{C}G\}$$

$$\{\chi_1, \dots, \chi_{o(G)}\}$$

$$\{\text{characters of } G\} = \{\text{algebra maps } \mathbb{C}G \to \mathbb{C}\}$$

$$\begin{array}{ccc} \mathbb{C}[x,x^{-1}] & & \mathbb{C}^* \supset \{\chi_1(g), \dots, \chi_{o(G)}(g)\} \\ \downarrow \quad x \mapsto e_g & \sim & \uparrow \\ \mathbb{C}G & & \{\chi_1, \dots, \chi_{o(G)}\} \end{array}$$



column in char table = character values = #G roots of unity

(c) "best" integral scheme : Y

$$\begin{tikzcd}
Z \arrow[rr, "ev"] \arrow[dr] \arrow[dd, "i"'] & & h_X = h_A \arrow[d, "① = - \circ \psi"] \\
\text{subfunctor} & h_\Gamma = h_{Y'} \arrow[r, "\text{extn}", "\text{scalar}"'] & h_{Y' \otimes \mathbb{C}} = h_{\Gamma \otimes \mathbb{C}} \\
h_\Lambda = h_Y \arrow[ur, dashed, "\exists\ ② = - \circ ?"] \arrow[rr, "\text{extn}", "\text{scalar}"'] & & h_{Y \otimes \mathbb{C}} \arrow[u, dashed]
\end{tikzcd}$$

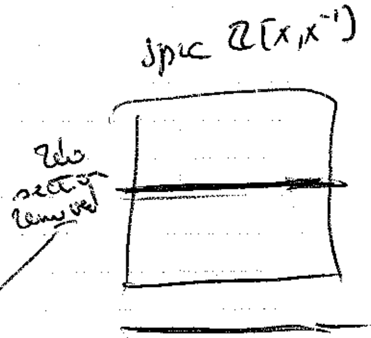
property that any other compatible nat transfos factor through i

① $\longleftrightarrow$ $\mathbb{C}$-algebra map

$$\Gamma \hookrightarrow \Gamma \otimes \mathbb{C} \xrightarrow{\psi} A$$

if exists ② $\longleftrightarrow$ $\mathbb{Z}$-algebra map

$$\Gamma \xrightarrow{?} \Lambda$$



Example Take as $\Lambda = \mathbb{Z}[x,x^{-1}]$ with $\int$ tor of path

$$h_\Lambda : (\mathbb{Z}\text{-rings}) \to (\text{Sets})$$
$$\Gamma \mapsto Alg_{\mathbb{Z}}(\mathbb{Z}[x,x^{-1}], \Gamma) = \Gamma^*$$

subfunctor

is proper subfunctor because $(\mathbb{Z}G)^* = \pm G \times G'$

can be infinite abelian

$$i : F \hookrightarrow h_\Lambda$$

$$F(G) = G \hookrightarrow h_\Lambda(\mathbb{Z}G) = (\mathbb{Z}G)^*$$
$$g \longmapsto e_g$$

Why is i "best" i.e. perhaps there is a more complicated $\mathbb{Z}$-ring realizing the subfunctor better.

Suppose there is one: Γ

$$\begin{tikzcd}
F \arrow[rr, "ev"] \arrow[dr, "j"] \arrow[dd, "i"'] & & h_{\mathbb{C}[x,x^{-1}]} \arrow[d] \\
 & h_{\Gamma} \arrow[r] \arrow[ur, phantom, "\circledast"] & h_{\Gamma\otimes\mathbb{C}} \\
h_{\mathbb{Z}[x,x^{-1}]} \arrow[ur, dashed, "?"] & &
\end{tikzcd}$$

have $\mathbb{C}$-alg map

$$\Gamma \to \Gamma\otimes\mathbb{C} \xrightarrow{\psi} \mathbb{C}[x,x^{-1}]$$

want $\mathbb{Z}$-alg map $\Gamma \to \mathbb{Z}[x,x^{-1}]$ $\Big\}$ suffices to prove $\operatorname{Im}(\Gamma) \subset \mathbb{Z}[x,x^{-1}]$

Take $\mathbb{Z}$-generator γ of Γ $\quad \psi(\gamma) = a_{-d}x^{-d} + \dots + a_d x^d$

Take $N > d$ and consider abelian group $C_N = \langle c \rangle$

Diagram chase $\circledast$

$$\begin{tikzcd}
F(C_N) \arrow[r, "ev"] \arrow[d, "\xi j"'] & \mathrm{Alg}_{\mathbb{C}}\left(\mathbb{C}[x,x^{-1}],\ \mathbb{C}C_N = \frac{\mathbb{C}[x,x^{-1}]}{(x^N-1)}\right) \arrow[d, "-\circ\psi"] \\
\mathrm{Alg}_{\mathbb{Z}}\left(\Gamma, \frac{\mathbb{Z}[x,x^{-1}]}{(x^N-1)}\right) \arrow[r, "ext"] & \mathrm{Alg}_{\mathbb{C}}\left(\Gamma\otimes\mathbb{C}, \frac{\mathbb{C}[x,x^{-1}]}{(x^N-1)}\right)
\end{tikzcd}$$

$c \longmapsto \pi : \mathbb{C}[x,x^{-1}] \to \frac{\mathbb{C}[x,x^{-1}]}{(x^N-1)}$

$\varphi \longleftarrow \pi\circ\psi$

gives commuting diagram

$$\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[d, "\varphi"'] & \Gamma \otimes \mathbb{C} \arrow[r, "\psi"] & \mathbb{C}[x, x^{-1}] \arrow[d, "\pi"] \\
\dfrac{\mathbb{Z}[x, x^{-1}]}{(x^N - 1)} \arrow[rr, hook] & & \dfrac{\mathbb{C}[x, x^{-1}]}{(x^N - 1)}
\end{tikzcd}$$

Because φ π is injective on Laurent polys of degree $\leq d$

have $\psi(\gamma) \in \mathbb{Z}[x, x^{-1}]$ Same for all $\mathbb{Z}$-generators

so have map ?

<u>That is</u> multiplicative group scheme

$$\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$$

is defined over $\mathbb{F}_1$

and gadget

$$\left(\left(\text{abelian} \xrightarrow{F} \text{sets} \right), \ \text{ev by characters} \right)$$
$$G \mapsto G$$

is an affine $\mathbb{F}_1$-scheme.

what makes trick work is that $\mathbb{Z}$ has enough cofinal subgroups and so $\mathbb{F}_1$ detects only $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}$ and observe that $\mathrm{char}(\hat{\mathbb{Z}}) = \mu_\infty$

§ alternative approach: coalgebras

Grothendieck's affine schemes are too fat for today's use, need "light" version

X $\mathbb{C}$-variety $\quad h_X : (\mathbb{C}\text{-alg}) \to (\text{Sets})$ but only use (need) this for f.d. $\mathbb{C}$-algs $\mathbb{C}G$

That is: use only points + infinitesimal info. All this is contained in

<u>dual coalgebra</u> $\quad A^\circ = \{ \lambda : A \to \mathbb{C} \text{ linear} \; \exists \underset{\text{ideal}}{I} \subset \mathrm{Ker}(\lambda) : \dim_{\mathbb{C}} A/I < \infty \}$

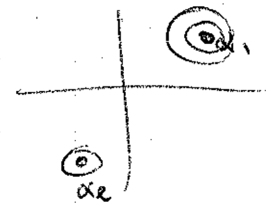
$A \otimes A \xrightarrow{m} A$ gives $A^\circ \xrightarrow[\Delta]{m^*} A^\circ \otimes A^\circ \subset (A \otimes A)^*$ and $\mathbb{C} \xrightarrow{i} A$ $\quad \varepsilon \; A^\circ \xrightarrow[\mathrm{ev}(1)]{i^*} \mathbb{C}$

KOSTANT DUALITY $\qquad$ ALG $\underset{\circ}{\overset{*}{\rightleftarrows}}$ COALG adj functors

$$\underset{\substack{\shortparallel \\ h_X(C^*)}}{\mathrm{Alg}_{\mathbb{C}}(A, C^*)} \longleftrightarrow \mathrm{Coalg}_{\mathbb{C}}(C, A^\circ)$$

so if C is f.d. (such as $\mathbb{C}G$) $\quad h_X(C^*) = \mathrm{Coalg}_{\mathbb{C}}(C, A^\circ)$ and $A^\circ = \lim$ f.d. coalgs

$\boxed{\text{Example}}$ $\; A = \mathbb{C}[x, x^{-1}]$ or $\mathbb{C}[x]$ $\quad$ what is A° ?



any ideal $I = (x-\alpha_1)^{n_1} \cdots (x-\alpha_e)^{n_e}$ is cofinite $\quad \dfrac{\mathbb{C}[x]}{I} = \dfrac{\mathbb{C}[x]}{(x-\alpha_1)^{n_1}} \oplus \cdots \oplus \dfrac{\mathbb{C}[x]}{(x-\alpha_e)^{n_e}}$

$$\left(\frac{\mathbb{C}[x]}{I}\right)^* = \left(\frac{\mathbb{C}[x]}{(x-\alpha_1)^{n_1}}\right)^* \oplus \cdots \oplus \left(\frac{\mathbb{C}[x]}{(x-\alpha_e)^{n_e}}\right)^*$$

need only compute

$$\left(\frac{\mathbb{C}[y]}{y^n}\right)^* = \mathbb{C}1 + \mathbb{C}z + \mathbb{C}z^2 + \cdots + \mathbb{C}z^{n-1}$$

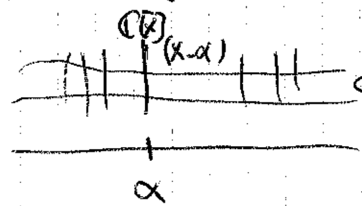
$$\Delta z^k = \sum_{l+m=k} z^l \otimes z^m \qquad \text{so } \varinjlim \left(\frac{\mathbb{C}[y]}{y^n}\right)^* = U(\mathbb{C}.1) = U(T_{\mathbb{A}^1, y}) \leftarrow \text{coalg str of env. alg.}$$

$$\mathbb{C}[x]^\circ = \bigoplus_{\alpha \in \mathbb{A}^1} U(T_{\mathbb{A}^1,\alpha}) \quad \text{and dualizing gives algebra}$$

get variety back as Coradical(A°)

$$\mathbb{C}[x] \xhookrightarrow{\text{not}} (\mathbb{C}[x]^\circ)^* = \prod_{\alpha \in \mathbb{A}^1} \mathbb{C}[[x-\alpha]]$$

$$f \longmapsto \prod_\alpha (\text{Taylor series expansion of } f \text{ in } \alpha)$$



$\leftarrow \mathbb{C}[x]$

get algebra back as global sections

In our example of forgetful functor, know that only need points in μ_∞

so relevant subcoalgebra is $\bigoplus_{\alpha \in \mu_\infty} U(T_{\mathbb{A}^1,\alpha})$

so complex algebra corresponding to "coordinate ring of $\mathbb{F}_1$-variety $\mathbb{G}_m$" is

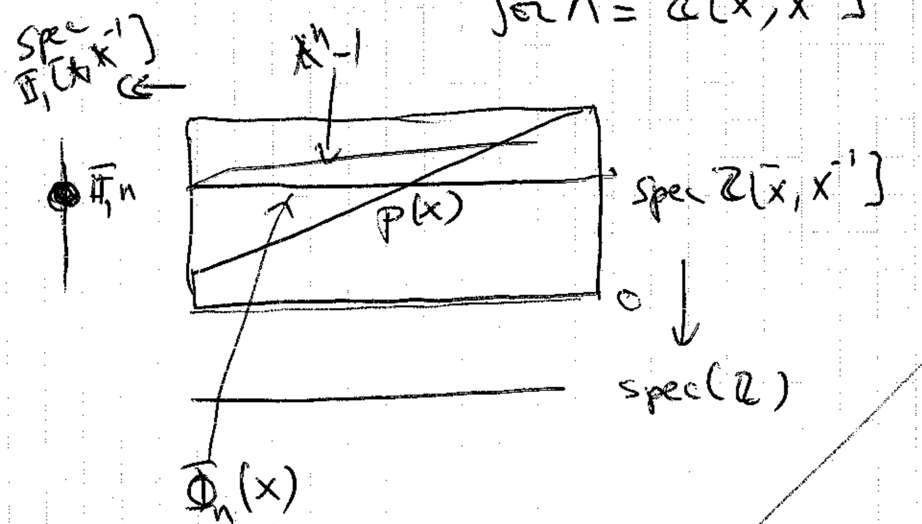
$$\mathbb{F}_1[t,t^{-1}] \otimes \mathbb{C} = \underbrace{\bigcap_{\alpha \in \mu_\infty} \mathbb{C}[[x-\alpha]]}_{\text{analytic functions at roots of unity.}} \underset{\neq}{\supset} \mathbb{C}[x] \quad \text{so obtain new interesting ring}$$

see later

But, also need this for $\mathbb{Z}$-algebras. Fortunately, for $\mathbb{Z}$ (or even Dedekind domains) can define dual coalg + prove Kostant duality when one uses

$$\Lambda^{\circ} = \{ f : \Lambda \longrightarrow \mathbb{Z} \ \ \mathbb{Z}\text{-linear} \quad \exists\, I \triangleleft \Lambda \text{ contained in } \mathrm{Ker}(\lambda) \text{ s.t. } \Lambda/I \text{ is } \underline{\text{f.g. + torsion free}} \text{ (and hence proj) } \}$$

So which ideals satisfy this property? only the "horizontal ones" of height one (for $\Lambda = \mathbb{Z}[x, x^{-1}]$)



$$\frac{\mathbb{Z}[x,x^{-1}]}{(p(x))} \overset{\text{almost}}{\underset{\hookrightarrow}{\simeq}} \frac{\mathbb{Z}[x,x^{-1}]}{p_1(x)^{e_1}} \oplus \cdots \oplus \frac{\mathbb{Z}[x,x^{-1}]}{p_k(x)^{e_k}}$$

$$\left(p(x) = p_1(x)^{e_1} \cdots p_k(x)^{e_k} \right)$$

$$\mathbb{Z}[x,x^{-1}]^{\circ} \simeq \bigoplus_{\substack{q(x) \\ \text{irred.}}} B_{q(x)} \longleftarrow \mathbb{Z}\text{-coalgebra} \quad \varinjlim \left(\frac{\mathbb{Z}[x,x^{-1}]}{q(x)^n} \right)^{*}$$

$$\mathbb{Z}[x,x^{-1}]^{\circ *} \simeq \prod_{\substack{q(x) \\ \text{irred}}} \widehat{\mathbb{Z}[x,x^{-1}]}_{q(x)}$$

$$\varprojlim_n \left(\mathbb{Z}[x,x^{-1}] / q(x)^n \right)$$

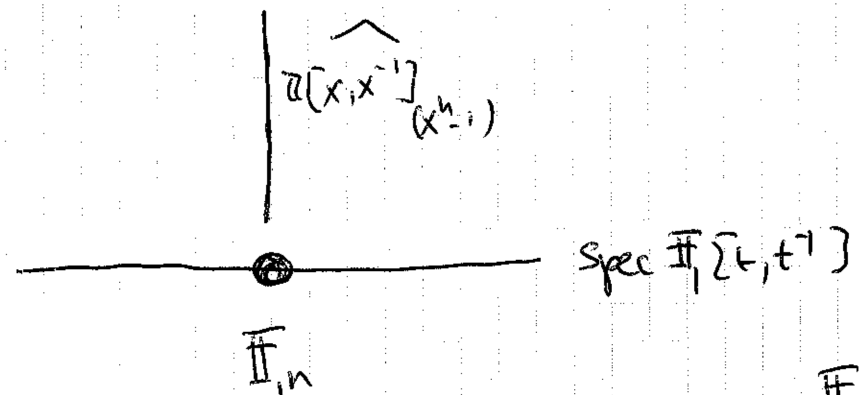
$\uparrow$ $q(x)$-adic completion of $\mathbb{Z}[x,x^{-1}]$

in general "coCauchy"

In our example only need small subset namely

$$\prod_{x^n-1} \widehat{\mathbb{Z}[x,x^{-1}]}_{(x^n-1)} \quad \text{or} \quad \prod_{\Phi_n(x)} \widehat{\mathbb{Z}[x,x^{-1}]}_{\Phi_n(x)}$$

$\Phi_n(x)$ cyclotomic polyn.



Habiro math/0209324

Manin arXiv: 0809.1564

so integral algebra corresponding to $\mathbb{F}_1$-variety $\mathbb{F}_1[x,x^{-1}]$ is

$$\mathbb{F}_1[t,t^{-1}] \otimes \mathbb{Z} = \bigcap_n \widehat{\mathbb{Z}[x,x^{-1}]}_{(x^n-1)}$$

$$\widehat{\mathbb{Z}}_{Hab} = \varprojlim_N \mathbb{Z}[x] \Big/ \big(\underbrace{(x^N-1)(x^{N-1}-1)\cdots(x-1)}_{N_x!}\big) = \text{Habiro ring}$$

$\widehat{\mathbb{Z}}_{Hab}$ consists of formal power series in x of the form

$$\sum_N a_N(x)\,(x^N-1)(x^{N-1}-1)\cdots(x-1) \quad \text{with } a_N(x) \in \mathbb{Z}[x] \text{ of degree} < N$$

Habiro found ug in unifying Witten-Reshetikhin-Turaev knot invariants'

Important fact of "Habiro functions" for every $\alpha \in \mu_\infty$ $\exists N$ s.t. $\forall M \geq N$: $(\alpha^M - 1)(\alpha^{M-1} - 1) \dots (\alpha - 1) = 0$ so Habiro ftions are well-defined in all roots of unity (and possibly nowhere else). Also, have Taylor series expansion around every root of unity

remember last time ($\mathbb{F}_1$-geometry $\Leftrightarrow$ Lichtenbaum numbers).

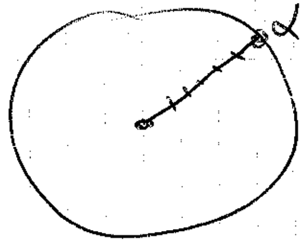
Habiro math 0605314

$$\widehat{\mathbb{Z}}_{Hab} \subset \bigcap_{\alpha \in \mu_\infty} \mathbb{C}[[x-\alpha]]$$

Some of these functions appeared earlier

$$1 + \sum_{n=1}^{\infty} (-1)^n \{n\}_x ! = 1 + \sum_{n=1}^{\infty} (1-x)(1-x^2) \dots (1-x^n)$$ in Kontsevich's work on Feynman integrals

Zagier proved that its values at $\alpha \in \mu_\infty$ are radial limits of a function which is holomorphic in unit circle

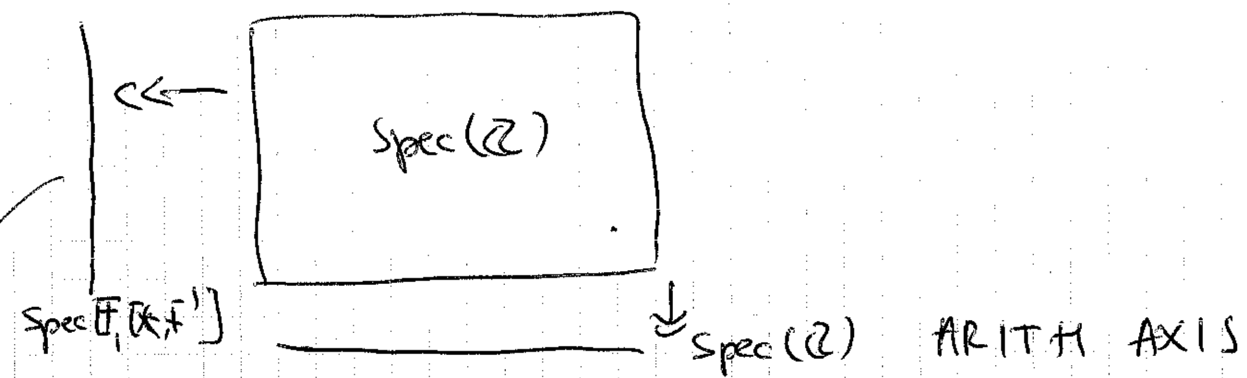


$$\frac{1}{2} \sum_{n=1}^{\infty} n \, \chi(n) \, x^{\frac{n^2-1}{24}}$$

χ: quadratic character of conductor 12

"functions leaking out of roots of unity"

GEOMETRIC AXIS



leads to $\hat{\mathbb{Z}}_{\text{Hab}} = \varprojlim \mathbb{Z}[x] / \{N_x!\}$ which has analogon along ARITH axis

$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/(N!)$ profinite integers, that is all sequences

$(\cdots c_3\, c_2\, c_1)_!$ with digits $0 \leq c_i \leq i$

every pos integer $n = c_k k! + c_{k-1}(k-1)! + \cdots + c_1 \cdot 1!$ uniquely determ

for example $-1 = (\cdots 5\,4\,3\,2\,1)_!$

and $\hat{\mathbb{Z}}$ appears also because finite dml simple complex $\hat{\mathbb{Z}}$-representation are all $\mathbb{M}_\infty$