# HALLOWEEN : $\mathbb{F}_1$ and other ghost stories

<u>idea</u>: spec($\mathbb{Z}$) is too big to be terminal object in schemes.
should exist "absolute point"

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{Z}) & & \mathbb{Z} \\ \downarrow & & \uparrow \\ \mathrm{Spec}(\mathbb{F}_1) & & \mathbb{F}_1 \end{array}$$

$\leftarrow$ field with one element.

First give brief explanation why Jacques Tits invented it
Then consider uses of $\mathbb{F}_1$ - approach to other fields.

BASIC QUESTION :
what is best way to view spec($\mathbb{Z}$) geometrically? What is its "dimension" and related to this: what geometric objects corresponds to prime numbers.

3 possible answers:

| | spec($\mathbb{Z}$) | p | uses of $\mathbb{F}_1$ |
|---|---|---|---|
| dim = 1 | classical approach global fields Spec($\mathbb{Z}$) $\leftrightarrow$ Spec $\mathbb{F}_p[t]$ | primes $\leftrightarrow$ points on curves | zeta function absolute motives |
| dim = 3 | Artin-Verdier-Mazur étale top $\Leftrightarrow$ Poincaré duality of dim 3 manif. Spec($\mathbb{Z}$) $\leftrightarrow$ $S^3$ | primes $\leftrightarrow$ knots in $S^3$ | linking numbers $\Updownarrow$ power residue symbols |
| dim = $\infty$ | spec($\mathbb{Z}$) not affine object in $\mathbb{F}_1$ - geometry, so must be $\infty$. Can approximate it formally $\widehat{\mathrm{Spec}(\mathbb{Z})} \leftrightarrow \mathrm{Spec}\,\hat{\mathbb{Z}}$ $\leftarrow$ profinite integers | primes $\leftrightarrow$ factors in $\prod \hat{\mathbb{Z}}_p = \hat{\mathbb{Z}}$ | Witt schemes in $\mathbb{F}_1$ - geometry |

# $\boxed{\text{I}}$ HISTORY OF $\mathbb{F}_1$ (Jacques Tits 1956)

SIMPLE LIE GROUPS / $\mathbb{C}$ $\xrightarrow{\text{Chevalley}}$ SIMPLE FINITE GRPS $\xrightarrow{\text{Tits}}$ GEOMETRY survives

integral form Cartan Matrix — $\mathbb{F}_{p^n}$ — study the finite geometry — $\lim p \to 1$

as if version exists over $\mathbb{F}_1$

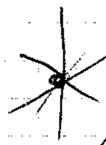
WEYL GROUP

## EXAMPLE

$PGL_3(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$; $[x:y:z]A \longleftarrow [x:y:z]$

$\mathbb{C}$ $\quad = GL_3(\mathbb{C})/\mathbb{C}^* \mathbb{1}_3$

$\mathbb{P}^2_{\mathbb{C}} = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0$



lines through pt form $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^0$

$\mathbb{F}_p$ $\quad PGL_3(\mathbb{F}_p)$ associated geometry: incidence geometry of pts / lines in $\mathbb{P}^2_{\mathbb{F}_p}$

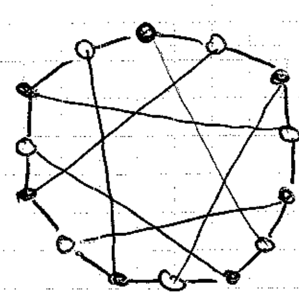
have: $p^2+p+1$ points and $p^2+p+1$ lines (dual)
every pt lies on $p+1$ lines, every line has $p+1$ pts

$p=3$ $\longrightarrow$ $p=2$ $\longrightarrow$ $p=1$ $\qquad p=1$

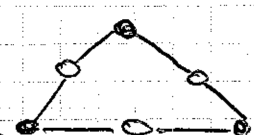
| $p=3$ | $p=2$ | $p=1$ |
|---|---|---|
| 13 pts | 7 pts | 3 pts, 3 lines, pt on 2 lines |
| 13 lines | 7 line | line contain 2 pts |
| line : 4 pts | line : 3 pts | |
| point : 4 line | point : 3 line | |

M13 - picture

Conway game



Weyl gp $PGL_3(\mathbb{C})$



Symmetry $= S_3 = D_3$

$N\begin{bmatrix}\cdot & \circ & \circ\\ \circ & \cdot & \circ\\ \circ & \circ & \cdot\end{bmatrix} = \text{Perm.}\begin{bmatrix}\cdot & \circ & \circ\\ \circ & \cdot & \circ\\ \circ & \circ & \cdot\end{bmatrix}$

does Chevalley gp exists over $\mathbb{F}_1 = \bullet$ ?

need: develop "algebraic geometry over $\mathbb{F}_1$"

next time full details, here basic idea:

$\mathbb{Z}$

$\vdots$

$\mathbb{F}_1$

"absolute pt"

$\mathbb{F}_1$-schemes $\longleftarrow$ $\mathbb{Z}$-schemes

"freeze essential content"

$\rightarrow$ combinatorial gadget.

$\mathbb{F}_1$-schemes $\longrightarrow$ $\mathbb{Z}$-schemes

"extension of scalars"

? $\rightsquigarrow$ "visible" after $- \otimes_{\mathbb{F}_1} \mathbb{Z}$

"descent" problem: what $\mathbb{Z}$-schemes have $\mathbb{F}_1$-form?

<u>Connes-Consani</u> (sept '08)

Chevalley groups have $\mathbb{F}_{1^2}$-form

$\mathbb{F}_1 = \bullet$

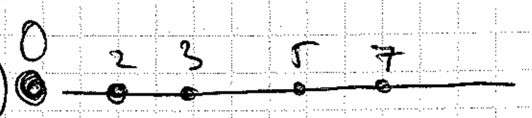
$\mathbb{F}_1 \xrightarrow{\text{quadratic extn}} \mathbb{F}_{1^2} = \text{"}\mu_2\text{"}$

$\mathbb{F}_1 \xrightarrow{\text{alg. closure}} \overline{\mathbb{F}}_1 = \text{"}\mu_\infty\text{"}$

$\mathbb{F}_1 \xrightarrow{\text{degree } n \text{ extn}} \mathbb{F}_{1^n} = \text{"}\mu_n\text{"}$

# $\boxed{\text{II}}$ PRIMES $\leftrightarrow$ POINTS ON CURVE
## (GLOBAL FIELDS DICTIONARY)

idea




spec($\mathbb{Z}$) Dedekind 1857 spec $\mathbb{F}_p[t]$

ring of integers $D \subset \mathbb{Q}(\alpha)$

$\mathrm{Spec}(D) \xrightarrow{\text{cover}} \mathrm{spec}(\mathbb{Z})$

$\mathbb{F}_p[t] \subset \mathbb{F}_p[C]$ $\quad C \to \mathbb{A}^1$

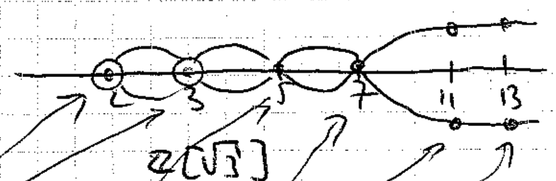
Example:

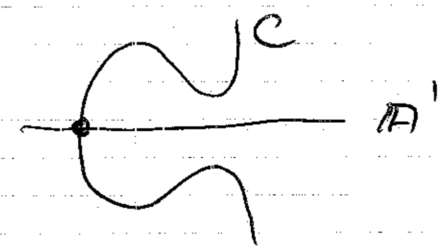
$D = \mathbb{Z}[\sqrt{d}\,]$

$C$ elliptic curve $/\mathbb{F}_p$

$y^2 + y = x^3$ over $\mathbb{F}_2$

Spec(D)




① $p \mid \mathrm{disc} \Rightarrow$ ramified prime $\mathfrak{p}$

non-reduced ~~pattern~~ coord. ring

$D/\mathfrak{p}^2$ residue $D/\mathfrak{p} = \mathbb{F}_p$

only 3 $\mathbb{F}_2$-points

$[0:0:1]$

$[0:1:0] = \infty$

$[0:1:1]$

② $\left(\frac{d}{p}\right) = 1$ ($\bar{d}$ is square in $\mathbb{F}_p$)

Legendre symbol

disj union of 2 pts $\mathfrak{p}, \mathfrak{p}'$

residue $D/\mathfrak{p} \cong D/\mathfrak{p}' = \mathbb{F}_p$

$(p) = \mathfrak{p} \cdot \mathfrak{p}'$

other points defined over field ext

$\mathbb{F}_2 \subset \mathbb{F}_{2^n}$ ← rh of pol

③ $\left(\frac{d}{p}\right) = -1$ one reduced prime $\mathfrak{p} = (p)$

with residue $D/\mathfrak{p} = \mathbb{F}_{p^2}$

$\mathbb{F}_p[C] = \dfrac{\mathbb{F}_p[x,y]}{(y^2+y-x^3)}$ is Dedekind domain

primes $\leftrightarrow$ points

max ideals $\leftrightarrow$ point

$\mathbb{F}_p[C]/\mathfrak{m} = \mathbb{F}_{2^2}$

ideals

$\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}$

(pos) divisors on $C$

~~div~~ $= \sum n_i P_i$

zeta_functions : archetype example

Riemann zeta

$s \in \mathbb{C}$ $\zeta(s) = \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$

$$= \sum_{(n) \triangleleft \mathbb{Z}} \frac{1}{n^s} \overset{\text{Euler}}{=} \prod_{p \text{ prime}} \left( \frac{1}{1 - \frac{1}{p^s}} \right)$$

over ideals — over prime ideals

every ideal = prod prime ideals

$D \subset \mathbb{Q}(\alpha)$ is Dedekind domain

1920-30 Artin-Hasse with etc

Curve / $\mathbb{F}_p$

$$\zeta_D(s) = \sum_{a \triangleleft D} \frac{1}{N(a)^s} = \prod_{p \triangleleft D} \frac{1}{(1 - N(p)^{-s})}$$

$N(a) = \#\left(\frac{D}{a}\right)$

$N_r = \# C(\mathbb{F}_{p^r})$

congruence zeta function

$$\zeta(T, C) = \exp\left( \sum_{r=1}^{\infty} \frac{N_r}{r} T^r \right)$$

(generalized) Riemann Hypo

non-trivial roots lie on $Re = 1/2$

OPEN

Riemann Hypo for curve

$$\zeta(T, C) = \frac{L(T)}{(1-T)(1-pT)}$$

$L(T)$ poly of deg $2g(C)$

$$\zeta(\mathbb{P}^1, T) = \frac{1}{(1-T)(1-pT)}$$ all reciprocal roots $|\alpha_i| = \sqrt{p}$

(Artin)-Hasse elliptic curve

$$\zeta(C, T) = \frac{1 - aT + pT^2}{(1-T)(1-pT)} = \frac{(\alpha T - 1)(\beta T - 1)}{(1-T)(1-pT)}$$

$|\alpha| = |\beta| = \sqrt{p}$

$$\begin{cases} N_1 = p + 1 - a \\ N_r = p^r + 1 - \alpha^r - \beta^r \end{cases}$$

example $\zeta = \dfrac{1+2T^2}{(1-T)(1-2T)}$

$$= \frac{(1+i\sqrt{2}T)(1-i\sqrt{2}T)}{(1-T)(1-2T)}$$

$$|i\sqrt{2}| = |-i\sqrt{2}| = \sqrt{2}$$

$$N_r = \begin{cases} 2^r + 1 & r \text{ odd} \\ 2^r + 1 - 2(-2)^{r/2} & r \text{ even.} \end{cases}$$

**Weil** (1940) : RH true for all $C/\mathbb{F}_q$

"HOPE": use tricks of Weil to get RH for #-fields

TRICKS

① 
$$\begin{tikzcd} C \arrow[d, dash] \\ \mathbb{F}_p \end{tikzcd} \qquad \text{Want} \qquad \begin{tikzcd} \mathrm{Spec}(D) \subset \mathbb{Q}(\alpha) \arrow[d, dash, "\leftarrow \text{"curve"}"] \\ \mathbb{F}_1 \end{tikzcd}$$

$\mathbb{F}_1 \longleftarrow$ absolute pt : field with 1 elem.

at moment: not in framework but can express BC-endomotive in $\mathbb{F}_1$-geometry

"best hope" to prove RH.

② 
$$\begin{tikzcd} \overline{\mathbb{F}}_p \arrow[d, dash, "\mathrm{Gal} = \hat{\mathbb{Z}}"] \\ \mathbb{F}_p \end{tikzcd} \qquad\qquad \begin{tikzcd} \overline{\mathbb{F}}_1 = \mu_\infty \arrow[d, dash] \\ \mathbb{F}_1 = \bullet \end{tikzcd}$$

generated by Frobenius $x \to x^p$

has power maps $x \mapsto x^n$ for all $n$

acts on cohomology gps.

explain emphasis on cyclotomic stuff when developing $\mathbb{F}_1$-geometry.

Manin: We will imagine that "natural factors" of zeta-function of $\mathbb{Z}$ scheme of finite type correspond to absolute motives $M$ which can be reconstructed from zeroes of $Z(M,s)$ up to (unspecified) isomorphism relation.

$C$ curve $/ \mathbb{F}_q$ Weil – Deligne

$$Z(C,s) = \sum_{\substack{a \\ \text{pos div}}} \frac{1}{N(a)^s} = \prod_{\substack{x \\ \text{points}}} \frac{1}{1 - N(x)^{-s}} = \frac{\prod_{j=1}^{2g} (1 - \varphi_j q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

$$= \prod_{w=0}^{2} \det\left(\mathrm{Id} - \mathrm{Frob}\, q^{-s} \mid H^w(C)\right)^{(-1)^{w-1}}$$

Frobenius

étale cohomology of weight $w$

$$= \prod_{w=0}^{2} Z(\underbrace{h^w(C)}, s)^{(-1)^{w-1}}$$

"motivic ~~part~~ piece of $C$" which is kind of "universal cohomology"

Chr Deninger: $\overline{\mathrm{Spec}(\mathbb{Z})} = \mathrm{Spec}(\mathbb{Z}) \cup \mathbb{R}$ real val.

$$Z(\overline{\mathrm{spec}(\mathbb{Z})}, s) \overset{\text{def}}{:=} 2^{-1/2}\, \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) \zeta(s) \overset{!}{=} \frac{\prod_\rho \frac{s-\rho}{2\pi}}{\frac{s}{2\pi}\, \frac{s-1}{2\pi}}$$

$$\overset{?}{=} \prod_{w=0}^{2} \mathrm{DET}\left(\frac{s\, \mathrm{Id} - \Phi}{2\pi} \,\Big|\, H^w_{?}(\overline{\mathrm{spec}(\mathbb{Z})})\right)^{(-1)^{w-1}}$$

(logarithm) of absolute Frobenius

conjectural infinite product

conjectural cohomology

denominators of

$$\frac{1}{(1-q^{-s})(1-q^{1-s})} = Z(\mathbb{P}^1_{\mathbb{F}_q})$$

$$\frac{1}{\frac{s}{2\pi}\, \frac{s-1}{2\pi}}$$

? zeta-function of "absolute motive" $Z(\mathbb{P}^1_{\mathbb{F}_1})$ ?

Soulé e.a. developed algebraic geometry over $\mathbb{F}_1$ (see next time)

idea $\quad \mathbb{F}_1 \quad$ ———— $\quad \mathbb{Z}$

$X \quad \rightsquigarrow \quad X_{\mathbb{Z}}$ gives $\mathbb{Z}$-scheme

but not all $\mathbb{Z}$-schemes are defined over $\mathbb{F}_1$ (sort of ideal mode)

"conjecturally" $X_{\mathbb{Z}}$ $\mathbb{Z}$-scheme ~~also~~ is defin$^X$ over $\mathbb{F}_1$

(*) if $\exists$ polynomial $N_X(t) = \sum_{k=0}^{n} a_k t^k \quad (a_i \in \mathbb{Z})$

s.t. $\# X_{\mathbb{Z}}(\mathbb{F}_{p^m}) = N_X(p^m)$ so number of pts have uniform behaviour for all $p$

so expect $\mathbb{P}^1_{\mathbb{F}_1}$ to exist but $C$ elliptic curve NOT.

TRUE

Connection between $N_X(t) = \sum_{k=0}^{n} a_k t^k$ and zeta-function

$$\zeta(s, X/\mathbb{F}_p) = \exp\left( \sum_{m=1}^{\infty} \frac{\# X(\mathbb{F}_{p^m})}{m} p^{-ms} \right) = \prod_{k=0}^{n} (1-p^{k-s})^{-a_k}$$

$$\zeta(s, X/\mathbb{Z}) \overset{\text{def}}{=} \prod_{p} \zeta(s, X/\mathbb{F}_p) = \prod_{k=0}^{n} \zeta(s-k)^{a_k}$$

KUROKAWA

For those such that are defined over $\mathbb{F}_1$ can make sense of zeta function over $\mathbb{F}_1$ which then becomes

$$\zeta(s, X/\mathbb{F}_1) = \prod_{k=0}^{n} (s-k)^{-a_k}$$

(compare last page)

in particular

$$\zeta(s, \mathbb{A}^1/\mathbb{F}_1) = \frac{1}{s-1} \qquad \zeta(s, \mathbb{P}^1/\mathbb{F}_1) = \frac{1}{s(s-1)}$$

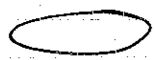
# $\boxed{\text{III}}$ PRIMES $\longleftrightarrow$ KNOTS

(KAPRANOV - REZNIKOV - MAZUR DICTIONARY)

idea 1 $\quad \mathrm{spec}(\mathbb{F}_p) \longleftrightarrow S^1$

$\pi_1(S^1) \cong \mathbb{Z}$

$$\begin{array}{c} \overline{\mathbb{F}}_p \\ \langle \mathrm{Frob} \rangle \Big| \quad \mathrm{Gal} \cong \hat{\mathbb{Z}} \\ \mathbb{F}_p \end{array}$$


$S^1$

Galois grps (covers) of geom obj $\longleftrightarrow$ profinite completion of fundamental grps of object

idea 2

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{F}_p) & \searrow & \\ & & \mathrm{Spec}(\mathbb{Z}) \\ \mathrm{Spec}(\mathbb{F}_q) & \nearrow & \end{array}$$

distincts knots in spec($\mathbb{Z}$) but can be linked

$\Downarrow$

Spec($\mathbb{Z}$) at least dim 3

Barry Mazur "Notes on étale cohomology of number fields" (1973)

Artin - Verdier duality theorem

$D$ ring of integers in number field $K$

$X = \mathrm{spec}(D)$, $F$ constructible abelian sheaf.

$$H^r_{ét}(X, F) \times \mathrm{Ext}^{3-r}_X(F, \mathbb{G}_m) \to H^3_{ét}(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

are finite and pairing non-degenerate

SORT of Poincaré duality

(is not really Poincaré duality as $\mathrm{Ext}^m_X(F, \mathbb{G}_m) \neq H^m(X, \hat{F})$)

(but only upto p-torsion for certain p)

Closed oriented connected 10

so in étale topology "Spec($\mathbb{Z}$)" 3-dim'l manifold

$\mathbb{Z} \hookrightarrow D$ corresponds to cover

$$\begin{array}{c} \mathrm{Spec}(D) \\ \downarrow \\ \mathrm{Spec}(\mathbb{Z}) \cong S^3 \end{array}$$

because $\nexists$ unramified covers, so simply connected

$S^3$ has no unbranched cover $\Rightarrow$ Poincaré conjecture should be

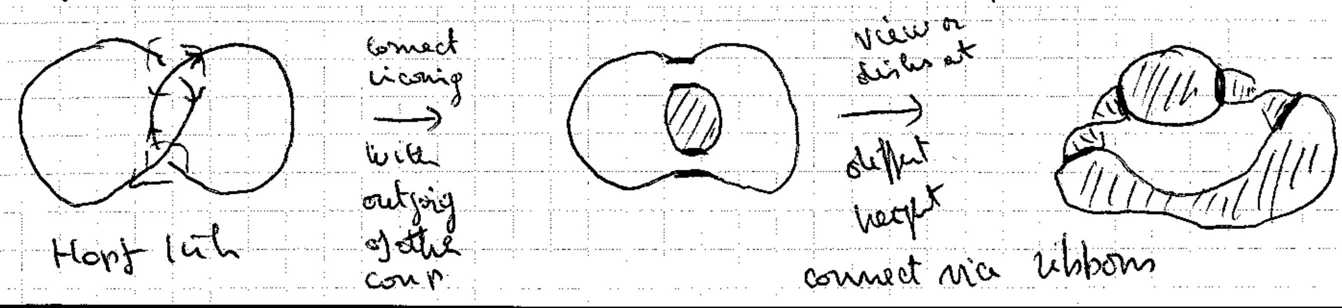
| | |
|---|---|
| Spec$(D)$ | $M$ closed oriented connected 3-manifold |
| prime ideal $\mathfrak{p} \triangleleft D$ | knot in $M$ |
| $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_e^{n_e} = \mathfrak{a} \triangleleft D$ | Link in $M$ with components corresponding to primes |
| $w \in D$ | embedded surface $S \hookrightarrow M$ possibly with boundary |
| $\downarrow$ | $\uparrow$ |
| $\mathfrak{a} = \langle w \rangle$ | Link $= \partial S$ boundary of $S$ |

[illegible]

$\boxed{\text{Seifert's algorithm to associate surface with boundary to any link}}$

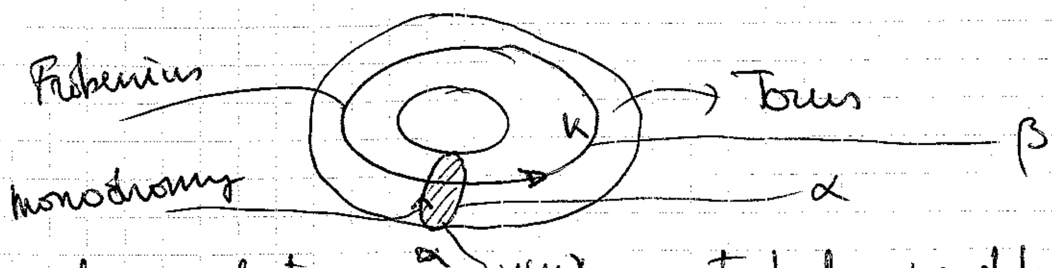
| | |
|---|---|
| $w \in D^*$ | closed surface $S \hookrightarrow M$ |

Surface with $\partial =$ link



## local picture for prime $p \triangleleft D$

$p$ — Knot $K$




p-adic completion $\mathrm{Spec}(\hat{D}_p)$ — tubular neighborhood $V(K)$

p-adic field $\mathrm{Spec}(K_p)$ — torus $= \partial V(K)$

motivation

$\pi_1(\mathrm{spec}\,\hat{D}_p) = \langle \sigma \rangle$, $\sigma$ = Frob — $\pi_1(V(K)) = \langle \beta \rangle$

$\pi_1^{\mathrm{tame}}(\mathrm{spec}\,K_p) = \langle \tau, \sigma \mid \tau^{p-1}[\tau,\sigma] = 1 \rangle$ — $\pi_1(\partial V(K)) = \langle \alpha, \beta \mid [\alpha,\beta] = 1 \rangle$

## connection with classgroups, units etc

$Cl(D)$ — $H_1(M,\mathbb{Z})_{\mathrm{tor}}$

principal ideals $\langle w \rangle$ triv in classgr — $\partial S$ represents identity in $H_1(M,\mathbb{Z})$

$D^*$ — $H_2(M,\mathbb{Z})$

spec ($\mathbb{Z}$) | $S^3$

prime number $p$ | $K \hookrightarrow S^3$

ramified only at $p$ $\longrightarrow$

$$\begin{array}{ccc} \mathbb{Z}[\varepsilon_{p^n}] & \subset & \mathbb{Q}(\varepsilon_{p^n}) \\ | & & | \\ (p) \triangleleft \mathbb{Z} & \subset & \mathbb{Q} \end{array}$$

$$\begin{array}{c} M_{p^n} \\ \downarrow \\ S^3 \end{array}$$

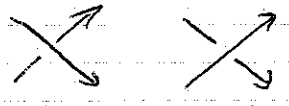
branched cyclic cover over knot

tower $\bigcup_n \mathbb{Q}(\varepsilon_{p^n})$

infinite cyclic cover of knot complement $\widetilde{S^3 \setminus K}$

~~$\boxed{\text{link with } \mathbb{F}_1}$ (1) if $\dim(\mathbb{Z}) = 3 \Rightarrow$ should be something smaller (2) invariants description in $\mathbb{F}_1$ - linear algebra~~

power residue symbols $\longleftrightarrow$ linking numbers



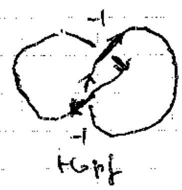


$$lk(K_1, K_2) = \frac{1}{2}\Big(\sum_{\substack{\text{crossings} \\ \text{betw} \\ K_1 \& K_2}}\Big)$$

$q \equi 1 \bmod n$

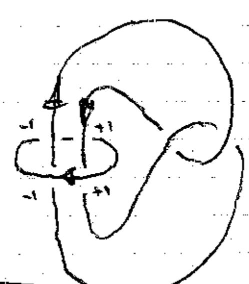
little Fermat $\mu_n \hookrightarrow \mathbb{F}_q^*$

$$\boxed{\left(\frac{p}{q}\right)_n = p^{\frac{q-1}{n}} \in \mu_n}$$





(-1)





Whitehead link

(0)

"linking number mod n"

RECIPROCITY $\Leftrightarrow$ $\boxed{\text{SYMM OF LK}}$

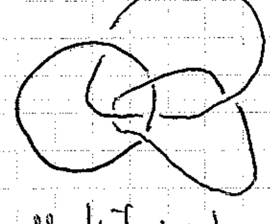
Hilbert-symbols
Milnor symbols

$\longleftrightarrow$

higher linking number

quadratic form assoc to link, Jones polys etc.

Borromean rings



all link number 0

but higher lk #'s
Milnor #'s
non-vanishing

get prime numbers relationships similar to links:

MORISHITA

$\left(\frac{p_1}{p_2}\right)_4 = \left(\frac{p_2}{p_1}\right)_4 = 1$

$p_1, p_2 \equiv 1 \bmod 4$ $\quad \mathbb{Q}\sqrt{p_1 p_2}$

$(13, 61, 937)_2$

analogy is strong enough to obtain results on 2-Sylow part of class gp of quadratic fields.

analogy not perfect match: $\exists$ imaginary quadratic UFD $\mathbb{Q}\sqrt{-n}$ for $n = 1, 2, 3, 7, 11, 19, 43, 67, 163$ and corresp 3-manifolds should be simply connected ($\Rightarrow$ "counterexample" to Poincaré conjecture)

$\downarrow$

$S^3$ is only closed 3-manifold with no unbranched cover

$\boxed{\text{connection with } \mathbb{F}_1}$, (Kapranov - Smirnov)

$\mu_n \subset \mathbb{Q}(\alpha)$ $\quad$ $\mathcal{D}$ = ring of integers in $\mathbb{Q}(\alpha)$

spec $(\mathcal{D})$ is algebraic scheme over $\mathbb{F}_{1^n}$

level 1) get dfn of power-residue symbol via linear algebra in $\mathbb{F}_{1^n}$

vector space / $\mathbb{F}_1$ $\rightsquigarrow$ set

dimension $\rightsquigarrow$ #

$GL_d$ $\rightsquigarrow$ $S_d$

det $\rightsquigarrow$ sgn

vector space / $\mathbb{F}_{1^n}$ $\rightsquigarrow$ set with free $\mu_n$-action

dimension $\rightsquigarrow$ $\#/n$ = # orbits

linear auto $\rightsquigarrow$ $e_1$ ◯ $e_2$ ◯ $e_3$ ◯ ...

$A(e_i) = \omega_{ij}\, e_j$ $\quad$ $\omega_{ij} \in \mu_n$

$$\det \rightsquigarrow \prod_{i=1}^{\#/n} \omega_{ij}$$

Take $q$ prime (or prime power) $q \equiv 1 \mod n$

$\Rightarrow \mathbb{F}_q^*$ is Vectorspace / $\mathbb{F}_{1^n}$

any element $a \in \mathbb{F}_q^*$ determines linear map

$$A_a \quad x \longrightarrow a.x$$

power residue symbol

$$\mu_n \ni a^{\frac{q-1}{n}} \overset{\text{Klassical}}{=} \left(\frac{a}{q}\right)_n \overset{!}{=} \det(A_a)$$

(level 2) $\quad X = \mathrm{Spec}(D)$ as scheme / $\mathbb{F}_{1^n}$

K-S associate to any pair of "line bundles" $L, M$ on $X$

a $\mu_n$-torsor $\langle L, M \rangle$

and to any sections $l \in \Gamma(L)$, $m \in \Gamma(M)$ which are relative prime and congruent to 1 mod $\mathfrak{N}$ an element

$(l, m)_n \in \mu_n$ ideal lying over $(n)$

satisfying natural isomorphism w.r.t tensor product of line bundles and then section and s.t.

$$(l, m)_n = (m, l)_n \qquad \text{(reciprocity law)}$$

$f \in D$ congruent to 1 mod $N$, for every $m \in \Gamma(M)$

$m \equiv 1 \mod N$ and relative prime to $f$

the element

$$(f,m)_n \in \langle \mathcal{O}_D, M \rangle \cong \mu_n$$

is equal to product of power residue symbols

$$\prod_{v \in X} \left( \frac{f}{\mathbb{F}_v} \right)_n^{\operatorname{ord}_v(m)}$$

and $\forall f, g \in D$ relative prime and $\equiv 1 \bmod N$ recover reciprocity law of class field theory

$$\prod_v \left( \frac{f}{\mathbb{F}_v} \right)_n^{\operatorname{ord}_v(g)} = \prod_v \left( \frac{g}{\mathbb{F}_v} \right)_n^{\operatorname{ord}_v(f)}$$

# $\boxed{\text{IV}}$ Primes $\longleftrightarrow$ infinite dml varieties

<u>idea</u> $\hat{\mathbb{Z}} = Gal(\overline{\mathbb{F}}_p / \mathbb{F}_p) = Gal(\overline{\mathbb{F}}_1 / \mathbb{F}_1) = \dots$

is not only group but also ring:

<u>Profinite integers</u> $\hat{\mathbb{Z}} = \prod_p \hat{\mathbb{Z}}_p$ ← p-adic integers

prime number $p \longleftrightarrow$ "factor" in this decomp.

$+$ and $\times$ on $\hat{\mathbb{Z}}$: any $n \in \mathbb{N} = \mathbb{Z}_{\geq 0}$ has $\exists !$

$$n = c_k . k! + c_{k-1} (k-1)! + \dots + c_2 . 2! + c_1 . 1!$$

with "digits" $c_k \neq 0$ & $0 \leq c_i \leq i \quad \forall i$

$n = (c_k \, c_{k-1} \dots c_2 c_1)_!$ <u>ex</u>: $25 = (1001)_!$

profinite integers are possibly infinite sequences of these

$\alpha = (\dots c_5 \, c_4 \, c_3 c_2 c_1)_!$ with $\forall i$: $c_i \leq i$

example: negative integers are of the form

$-1 = (\dots\dots\dots 6 \; 5 \; 4 \; 3 \; 2 \; 1)_!$

We like to compute digits of $\alpha + \beta$ and $\alpha . \beta$ from the digits of $\alpha$ and $\beta$.

addition: normal way by carrying over 1 and substracting $i+1$ if sum $\alpha_i + \beta_i > i$

multiplication: more difficult: use first $k$ digits of $\alpha . \beta$ depend only on first $k$ dig of $\alpha$ and $\beta$ (so reduce to $\mathbb{Z}, \times$)

MANIN sept 2008

"in a certain sense primes can be considered as cyclotomic points of spec($\mathbb{Z}$), at which the "cyclotomic coordinates", all integers, take values that are roots of unity or zero"

can have better "digits": with rigs. $\boxed{\text{First Case } \mathbb{Z}_p}$

Teichmüller digits $\quad S = \{0, \omega_p(\bar{1}), \omega_p(\bar{2}), \dots, \omega_p(\overline{p-1})\}$

$\omega_p(x)$ is unique $(p-1)$-th root of unity in $\mathbb{Q}_p$ which is congruent to $x$ modulo $p$, and therefore

$$\mathbb{F}_p^* \xrightarrow{\omega_p} S^* = S - \{0\}$$

is an isomorphism of multiplicative groups.

$$\mathbb{U}_p = \mathbb{Z}_p^* = \{x \in \mathbb{Z}_p : |x| = 1\}$$

$$\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{F}_p$$

$$\mathbb{U}_p \twoheadrightarrow \mathbb{F}_p^*$$

$$\omega_p(x) = \lim_{n \to \infty} u^{p^n} \qquad \text{when } x = \bar{u}, \quad u \in \mathbb{U}_p$$

(Fermat little thm) $\forall n$: $\quad x^{p^n} = x \quad$ so $\quad u^p = u + p a_1$ $\quad a_1 \in \mathbb{Z}_p$

by induction $\quad u^{p^n} = u + p^n a_n \qquad a_n \in \mathbb{Z}_p$

so $u^{p^n}$ is Cauchy sequence and its limit

$$\omega_p(x)^p = \omega_p(x).$$

Thm Every element $x \in \mathbb{Q}_p$ can be written uniquely

$$x = \sum_{n \gg -\infty} a_n p^n \qquad \text{with } a_n \in S$$

Every element $x \in \mathbb{Z}_p$ can be written uniquely as

$$x = \sum_{i=0}^{\infty} a_i p^i$$

so can represent

$$x = (a_0, a_1, a_2, a_3, \ldots) \qquad a_n = \omega_p(\bar{i}_n)$$

$x + y$ and $x \cdot y$ series are given by

see below $\boxed{\text{Witt's universal polynomials}}$ (defined over $\mathbb{Z}$)

procedure $\begin{cases} \text{- reduce mod } p \text{ to get } \mathbb{F}_p\text{-digits} \\ \text{- apply Witt polynomials to get digit} \\ \text{- lift to roots of unity in } \mathbb{Z}_p. \end{cases}$

$\boxed{\text{Witt functor}}$: comm $\longrightarrow$ com

$$R \longmapsto W(R) = \prod_{i=1}^{\infty} R$$

represented by spec $\mathbb{Z}[u_1, u_2, \ldots]$ image of $u_k$ is $k$th coordinate

$+$ and $\cdot$ rules of Witt give commutative ring structure on $W(R)$ functorial in $R$

making $\mathbb{Z}[u_1, u_2, \ldots]$ Hopf algebra $(+)$ coring object $(\times)$

"ghost variables" $$q_n = \sum_{d \mid n} d\, u_d^{n/d}$$

$\boxed{\text{Hazewinkel 14.3}}$

$$x = (x_1, x_2, x_3, \ldots) \in W(R)$$
$$y = (y_1, y_2, y_3, \ldots)$$

$$X +_w Y = (\mu_{S,1}(X,Y), \mu_{S,2}(X,Y), \mu_{S,3}(X,Y), \ldots)$$

$$X \cdot_w Y = (\mu_{P,1}(X,Y), \mu_{P,2}(X,Y), \mu_{P,3}(X,Y), \ldots)$$

polynomials $\mu_{S,j}$ and $\mu_{P,j}$ are <u>recursively</u> given by

$$q_n(\mu_{S,1}(X,Y), \mu_{S,2}(X,Y), \ldots) = q_n(X) + q_n(Y)$$

$$q_n(\mu_{P,1}(X,Y), \mu_{P,2}(X,Y), \ldots) = q_n(X) \cdot q_n(Y)$$

Hence $q_n(X)$ depends only on $X_d$ for $d | n$ and so $\mu_{S,n}$ and $\mu_{P,n}$ are polys only involving $X_d, Y_d$ for $d | n$.

So, for $p$ prime $\mathbb{Z}[\alpha_1, \alpha_p, \alpha_{p^2}, \ldots]$ is sub Hopf algebra and sub comp object of $\mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ (*) → next page

<u>Truncated</u> Witt functor $W^{(N)}$ represented by $\mathbb{Z}[u_1, \ldots, u_N]$

<u>$p$-Witt functor</u> $W_p$ represented by $\mathbb{Z}[u_1, u_p, u_{p^2}, \ldots]$

<u>Truncated $p$-Witt</u> $W_p^{(N)}$ rep by $\mathbb{Z}[u_1, u_p, u_{p^2} : p^k \leq N]$

$\boxed{\text{MANIN'S WITT GADGET}}$ Truncated Witt gadget $W^{(N)}$ defined over $\mathbb{F}_1$

① $V_{\mathbb{Z}}^{(N)} = W^{(N)}$

② abelian → sets $A \to \{ \alpha \in W_{\mathbb{Z}}^{(N)}(\mathbb{Z}A) :$ all ghost coordinates are $0$ or roots of unity $\}$

ESSENTIAL IDEA OF $\mathbb{F}_1$-GEO
CAN REPRESENT SUBFUNCTOR OF POINT-FTR OF $\mathbb{Z}$-SCHEME!

(*) instead of looking at inclusion (and hence quotient of functors)

$$\mathbb{Z}[u_1, u_p, u_{p^2}, \ldots] \hookrightarrow \mathbb{Z}[u_1, u_2, u_3, \ldots]$$

can also consider quotient by dividing out all other variables

$$\mathbb{Z}[u_1, u_2, u_3, \ldots] \twoheadrightarrow \mathbb{Z}[u_1, u_p, u_{p^2}, \ldots] \cong \mathbb{Z}[u_1,u_2,\ldots] / (u_j : j \neq p^n)$$

corresponds to inclusion on level of Spec's

$$\underbrace{\operatorname{spec}(\mathbb{Z}[u_1, u_p, u_{p^2}, \ldots])}_{\text{prime}} \hookrightarrow \underbrace{\operatorname{spec} \mathbb{Z}[u_1, u_2, \ldots]}_{\mathbb{Z}}$$

All this can be defined via geometry over $\mathbb{F}_1$

(back to (*) ⟸ last page)

because we define subfunctor having only 0 or roots of unity "digits", over $\mathbb{F}_1$ have description of $\hat{\mathbb{Z}}_p$ in terms of Teichmüller digits and projections (inclusion)

$$X(p) \hookrightarrow X$$

corresponds to decomposition $\hat{\mathbb{Z}} = \prod \hat{\mathbb{Z}}_p$ in suitable digits for profinite numbers (all digits being roots of unity)