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NONCOMMUTATIVE GEOMETRY AND DUAL COALGEBRAS

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ABSTRACT. In arXiv:math/0606241v2 M. Kontsevich and Y. Soibelman argue that the category of noncommutative (thin) schemes is equivalent to the category of coalgebras. We propose that under this correspondence the affine scheme $\mathtt{rep}(A)$ of a k-algebra A is the dual coalgebra A^o and draw some consequences. In particular, we describe the dual coalgebra A^o of A in terms of the A_∞-structure on the Yoneda-space of all the simple finite dimensional A-representations.

CONTENTS

1. $\mathtt{rep}(A) = A^o$

Throughout, k will be a (commutative) field with separable closure $\mathbb{k}$. In [3, §I.2] Maxim Kontsevich and Yan Soibelman define a *noncommutative thin scheme* to be a covariant functor commuting with finite projective limits

$$\mathtt{X} \,:\, \mathtt{alg}_k^{fd} \longrightarrow \mathtt{sets}$$

from the category $\mathtt{alg}_k^{fd}$ of all *finite dimensional* k-algebras (associative with unit) to the category $\mathtt{sets}$ of all sets. They prove [3, Thm. 2.1.1] that every noncommutative thin scheme is represented by a k-coalgebra.

Recall that a k-coalgebra is a k-vectorspace C equipped with linear structural morphisms : a comultiplication $\Delta \,:\, C \longrightarrow C \otimes C$ and a counit $\epsilon \,:\, C \longrightarrow k$ satisfying the coassociativity $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$ and counitary property $(id \otimes \epsilon)\Delta = (\epsilon \otimes id)\Delta = id$.

By being representable they mean that every noncommutative thin scheme $\mathtt{X}$ has associated to it a k-coalgebra $C_{\mathtt{X}}$ with the property that for any finite dimensional k-algebra B there is a natural one-to-one correspondence

$$\mathtt{X}(B) = \mathtt{alg}_k(B, C_{\mathtt{X}}^*)$$

Here, for a k-coalgebra C we denote by C^* the space of linear functionals $Hom_k(C, k)$ which acquires a k-algebra structure by dualizing the structural coalgebra morphisms.

They call $C_{\mathtt{X}}$ the *coalgebra of distributions* on $\mathtt{X}$ and define the *noncommutative algebra of functions* on $\mathtt{X}$ to be the dual k-algebra $k[\mathtt{X}] = C_{\mathtt{X}}^*$.

Whereas the dual C^* of a k-coalgebra is always a k-algebra, for a k-algebra A it is not true in general that the dual vectorspace A^* is a coalgebra, because $(A \otimes A)^* \not\simeq A^* \otimes A^*$. Still, one can define the subspace

$$A^o = \{ f \in A^* = Hom_k(A, k) \mid ker(f) \text{ contains a twosided ideal of finite codimension } \}$$

and show that the duals of the structural morphisms on A determine a k-coalgebra structure on this dual coalgebra A^o, see for example [5, Prop. 6.0.2].

With these definitions, *Kostant duality* asserts that the functors

$$\begin{tikzcd} \mathtt{alg}_k \arrow[r, bend left, "o"] & \mathtt{coalg}_k \arrow[l, bend left, "*"] \end{tikzcd}$$

are adjoint, [5, Thm. 6.0.5]. That is, for any k-algebra A and any k-coalgebra C, there is a natural one-to-one correspondence between the homomorphisms

$$\mathtt{alg}_k(A, C^*) = \mathtt{coalg}_k(C, A^o)$$

Moreover, we have [5, Lemma 6.0.1] that for $f \in \mathtt{alg}_k(A, B)$, the dual map f^* determines a k-coalgebra morphism $f^* \in \mathtt{coalg}_k(B^o, A^o)$.

For a k-algebra A one can define the contravariant functor $\mathtt{rep}(A)$ describing its finite dimensional representations [3, Example 2.1.9]

$$\mathtt{rep}(A) \ : \ \mathtt{coalg}_k^{fd} \longrightarrow \mathtt{sets} \qquad C \mapsto \mathtt{alg}_k(A, C^*)$$

from *finite dimensional* k-coalgebras $\mathtt{coalg}_k^{fd}$ to $\mathtt{sets}$, which commutes with finite direct limits. As on finite dimensional k-(co)algebras Kostant duality is an anti-equivalence of categories

$$\begin{tikzcd} \mathtt{alg}_k^{fd} \arrow[r, bend left, "*"] & \mathtt{coalg}_k^{fd} \arrow[l, bend left, "*"] \end{tikzcd}$$

we might as well describe $\mathtt{rep}(A)$ as the noncommutative thin scheme represented by A^o

$$\mathtt{rep}(A) \ : \ \mathtt{alg}_k^{fd} \longrightarrow \mathtt{sets} \qquad B = C^* \mapsto \mathtt{alg}_k(A, B = C^*) = \mathtt{coalg}_k(C = B^*, A^o)$$

the latter equality follows again from Kostant duality. Therefore, we propose

Definition 1. *The* noncommutative affine scheme $\mathtt{rep}(A)$ *is the noncommutative (thin) scheme corresponding to the dual k-coalgebra A^o of A.*

2. $\mathtt{simp}(A) = corad(A^o)$

The dual k-coalgebra A^o is usually a *huge* object and hence contains a lot of information about the k-algebra A. Let us begin by recalling how the geometry of a commutative affine k-scheme X is contained in the dual coalgebra A^o of its coordinate ring $A = \Bbbk[X]$.

Recall that a coalgebra D is said to be *simple* if it has no proper nontrivial subcoalgebras. In particular, a simple coalgebra D is finite dimensional over k and by duality is such that D^* is a simple k-algebra, that is, D^* is a central simple L-algebra where L is a finite separable extension of k.

Hence, in case $A = \Bbbk[X]$ (and $\Bbbk$ is separably closed) we have that all simple subcoalgebras of A^o are one-dimensional (and hence are spanned by a group-like element), because they correspond to simple representations of A.

That is, A^o is *pointed* and by [5, Prop. 8.0.7] we know that any cocommutative pointed coalgebra is the direct sum of its *pointed irreducible components* (at the algebra level, this says that a semi-local commutative algebra is the direct sum of locals). Therefore,

$$A^o = \oplus_{x \in X} C_x$$

where each C_x is pointed irreducible and cocommutative. As such, each C_x is a subcoalgebra of the enveloping coalgebra of the abelian Lie algebra on the tangent space $T_x(X)$. That is, we recover the points of X as well as tangent information from the dual coalgebra A^o.

But then, the dual algebra of A^o, that is the 'noncommutative' algebra of functions A^{o*} decomposes as

$$A^{o*} = \oplus_{x \in X} \hat{\mathcal{O}}_{x,X}$$

the direct sum of the completions of the local algebras at points. The diagonal embedding $A = \Bbbk[X] \hookrightarrow A^{o*}$ inevitably leads to the structure scheaf $\mathcal{O}_X$.

We will now associate a topological space associated to any k-algebra A, generalizing the space of points equipped with the Zariski topology when A is a commutative affine

k-algebra. In the next section we will describe the dual coalgebra A^o when A is a noncommutative affine $\mathbb{k}$-algebra.

The *coradical* $corad(C)$ of a k-coalgebra C is the (direct) sum of all simple subcoalgebras of C. It is also the direct sum of all simple subcomodules of C, when C is viewed as a left (or right) C-comodule.

In the example above, when $A = \mathbb{k}[X]$, we have that $corad(A^o) = \oplus_{x \in X} \mathbb{k}\; ev_x$ where the group-like element ev_x is evaluation in the point x. This motivates :

Definition 2. *For a k-algebra A we define the* space of points $\mathtt{simp}(A)$ *to be the set of direct summands of* $corad(\mathtt{rep}(A)) = corad(A^o)$. *That is,* $\mathtt{simp}(A)$ *is the set of simple subcoalgebras of* $\mathtt{rep}(A)$.

By Kostant duality it follows that $\mathtt{simp}(A)$ is the set of all finite dimensional simple algebra quotients of the k-algebra A, or equivalently, the set of all isomorphism classes of finite dimensional simple A-representations, explaining the notation.

We can equip this set with a *Zariski topology* in the usual way, using the *evaluation map*

$$A^o \times A \xrightarrow{ev} k \qquad (f, a) \mapsto f(a)$$

when restricted to the subcoalgebra $corad(A^o)$. Note that the evaluation map actually defines a *measuring* of A to k [5, Prop. 7.0.3], that is, $A^o \otimes A \xrightarrow{ev} k$ satisfies

$$ev(f \otimes aa') = \sum_{(f)} f_{(1)}(a) f_{(2)}(a') \qquad \text{and} \qquad ev(f \otimes 1) = \epsilon(f) 1_k$$

Definition 3. *The* Zariski topology *of a k-algebra A is the set* $\mathtt{simp}(A)$ *equipped with the topology generated by the basic closed sets*

$$\mathbb{V}(a) = \{ S \in \mathtt{simp}(A) \mid ev(S \otimes a) = 0, \text{ that is } f(a) = 0, \ \forall f \in S \}$$

Having associated a topological space to a k-algebra, one might ask when this is a functor. Functoriality has always been a problem in noncommutative geometry. Indeed, a simple B-representation does not have to remain a simple A-representation under restriction of scalars via $\phi \ : \ A \longrightarrow B$.

Still, if we define $\mathtt{rep}(A) = A^o$, we get functionality for free. If $A \xrightarrow{\phi} B$ is an algebra morphism, we have seen that the dual map maps B^o to A^o, so we have a morphism

$$B^o = \mathtt{rep}(B) \xrightarrow{\phi^*} \mathtt{rep}(A) = A^o$$

A coalgebra is the direct limit of its finite dimensional coalgebras, and they correspond under duality to finite dimensional algebras. Hence, ϕ^* is the natural map on finite dimensional representations by restriction of scalars.

The observed failure of functoriality on the level of points translates on the coalgebra-level to the fact that for a coalgebra map $B^o \longrightarrow A^o$ the coradical $corad(B^o)$ does not have to be mapped to $corad(A^o)$, in general.

However, when $corad(B^o)$ is cocommutative, we *do have* that $\phi^*(corad(B^o)) \subset corad(A^o)$ by [5, Thm. 9.1.4]. In particular, we recover the functor of points in *commutative* algebraic geometry.

Clearly, we still have $corad(B^o) \longrightarrow A^o$ in general. This corresponds to the fact that there is always a map $\mathtt{simp}(B) \longrightarrow \mathtt{rep}(A)$.

Next, let us turn to the algebra of functions on $\mathtt{rep}(A)$. By definition we have

$$k[\mathtt{rep}(A)] = A^{o*}$$

and we can ask how this algebra relates to the algebra A.

In general, it is *not* true that $A \hookrightarrow A^{o*}$. This only holds when A^o is dense in A^* in which case the k-algebra is said to be *proper*, see [5, §6.1].

In the commutative case, when A is a finitely generated k-algebra, then A is indeed proper and this is a consequence of the Hilbert Nullstellensatz and the Krull intersection theorem.

When A is noncommutative, this is no longer the case. For example, if $A = A_n(k)$ the *Weyl algebra* over a field of characteristic zero k, then A is simple whence has no twosided ideals of finite codimension. As a result $A^o = 0$! As our proposal for the noncommutative affine scheme $\mathtt{rep}(A)$ is based on finite dimensional representations of A, it will not be suitable for k-algebras having few such representations.

3. THE DUAL COALGEBRA A^o

In general though, A^o is a huge object, so it is very difficult to describe explicitly. In this section, we will begin to tame A^o even when A is noncommutative.

In order not to add extra problems, we will assume that $\mathbb{k}$ is separably closed in this section. The general case can be recovered by taking $Gal(\mathbb{k}/k)$-invariants (replacing quivers by *species* in the sequel).

Over a separably closed field $\mathbb{k}$ all simple subcoalgebras are full matrix coalgebras $M_n(\mathbb{k})^*$, that is, $M_n(\mathbb{k})^* = \oplus_{i,j}\mathbb{k}e_{ij}$ with $\Delta(e_{ij}) = \sum_{k=1}^n e_{ik} \otimes e_{kj}$ and $\epsilon(e_{ij}) = \delta_{ij}$.

Hence, $corad(A^o) = \oplus_S M_{n_S}(\mathbb{k})^*$ where the sum is taken over all finite dimensional simple A-representations S, each having dimension n_S.

In algebra, one can resize idempotents by Morita-theory and hence replace full matrices by the basefield. In coalgebra-theory there is an analogous duality known as *Takeuchi equivalence*, see [6].

The isotypical decomposition of $corad(A^o)$ as an A^o-comodule is of the form $\oplus_S C_S^{\oplus n_S}$, the sum again taken over all simple A-representations. Take the A^o-comodule $E = \oplus_S C_S$ and its *coendomorphism coalgebra*

$$A^\dagger = coend^{A^o}(E)$$

then Takeuchi-equivalence (see for example [1, §4, §5] and the references contained in this paper for more details) asserts that A^o is Takeuchi-equivalent to the coalgebra $A^\dagger$ which is *pointed*, that is, $corad(A^\dagger) = \mathbb{k}\ \mathtt{simp}(A) = \oplus_S \mathbb{k} g_S$ with one *group-like* element g_S for every simple A-representation. Remains to describe the structure of the full basic coalgebra $A^\dagger$.

For a (possibly infinite) quiver $\vec{Q}$ we define the *path coalgebra* $\mathbb{k}\vec{Q}$ to be the vectorspace $\oplus_p \mathbb{k}p$ with basis all oriented paths p in the quiver $\vec{Q}$ (including those of length zero, corresponding to the vertices) and with structural maps induced by

$$\Delta(p) = \sum_{p=p'p"} p' \otimes p" \qquad \text{and} \qquad \epsilon(p) = \delta_{0,l(p)}$$

where $p'p$" denotes the concatenation of the oriented paths p' and p" and where $l(p)$ denotes the length of the path p. Hence, every vertex v is a group-like element and for an arrow $v \xrightarrow{a} w$ we have $\Delta(a) = v \otimes a + a \otimes w$ and $\epsilon(a) = 0$, that is, arrows are skew-primitive elements.

For every natural number i, we define the ext^i-*quiver* $\overrightarrow{\mathtt{ext}}_A^i$ to have one vertex v_S for every $S \in \mathtt{simp}(A)$ and such that the number of arrows from v_S to v_T is equal to the dimension of the space $Ext_A^i(S,T)$. With $\mathtt{ext}_A^i$ we denote the $\mathbb{k}$-vectorspace on the arrows of $\overrightarrow{\mathtt{ext}}_A^i$.

The *Yoneda-space* $\mathtt{ext}_A^\bullet = \oplus \mathtt{ext}_A^i$ is endowed with a natural A_∞-structure [2], defining a linear map (the *homotopy Maurer-Cartan map*, [4])

$$\mu = \oplus_i m_i \ : \ \mathbb{k}\overrightarrow{\mathtt{ext}}_A^1 \longrightarrow \mathtt{ext}_A^2$$

from the path coalgebra $\mathbb{k}\overrightarrow{\mathtt{ext}}_A^1$ of the ext^1-quiver to the vectorspace $\mathtt{ext}_A^2$, see [2, §2.2] and [4].

Theorem 1. *The dual coalgebra A^o is Takeuchi-equivalent to the pointed coalgebra $A^\dagger$ which is the sum of all subcoalgebras contained in the kernel of the linear map*

$$\mu = \oplus_i m_i \ : \ \mathbb{k}\overrightarrow{\mathtt{ext}}_A^1 \longrightarrow \mathtt{ext}_A^2$$

determined by the A_∞-structure on the Yoneda-space $\mathtt{ext}_A^\bullet$.

We can reduce to finite subquivers as any subcoalgebra is the limit of finite dimensional subcoalgebras and because any finite dimensional A-representation involves only finitely many simples. Hence, the statement is a global version of the result on finite dimensional algebras due to B. Keller [2, §2.2].

Alternatively, we can use the results of E. Segal [4]. Let $S_1, \ldots, S_r$ be distinct simple finite dimensional A-representations and consider the semi-simple module $M = S_1 \oplus \ldots \oplus S_r$ which determines an algebra epimorphism

$$\pi_M \; : \; A \twoheadrightarrow M_{n_1}(\Bbbk) \oplus \ldots \oplus M_{n_r}(\Bbbk) = B$$

If $\mathfrak{m} = Ker(\pi_M)$, then the $\mathfrak{m}$-adic completion $\hat{A}_{\mathfrak{m}} = \underset{\leftarrow}{lim} \; A/\mathfrak{m}^n$ is an augmented B-algebra and we are done if we can describe its finite dimensional (nilpotent) representations. Again, consider the A_∞-structure on the Yoneda-algebra $Ext^\bullet_A(M,M)$ and the quiver on r-vertices $\overrightarrow{\mathtt{ext}}^1_A(M,M)$ and the homotopy Mauer-Cartan map

$$\mu_M = \oplus_i m_i \; : \; \Bbbk\overrightarrow{\mathtt{ext}}^1_A(M,M) \longrightarrow Ext^2_A(M,M)$$

Dualizing we get a subspace $Im(\mu_M^*)$ in the path-*algebra* $\Bbbk\overrightarrow{\mathtt{ext}}^1_A(M,M)^*$ of the dual quiver. Ed Segal's main result [4, Thm 2.12] now asserts that $\hat{A}_{\mathfrak{m}}$ is Morita-equivalent to

$$\hat{A}_{\mathfrak{m}} \underset{M}{\sim} \frac{(\Bbbk\overrightarrow{\mathtt{ext}}^1_A(M,M)^*)\hat{}}{(Im(\mu_M^*))}$$

where $(\Bbbk\overrightarrow{\mathtt{ext}}^1_A(M,M)^*)\hat{}$ is the completion of the path-algebra at the ideals generated by the paths of positive length. The statement above is the dual coalgebra version of this.

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