BRAID GROUP B_3 IRREDUCIBLES
- A DIY GUIDE -

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ABSTRACT. This note tells you how to construct a $k(n)$-dimensional family of (isomorphism classes of) irreducible representations of dimension n for the three string braid group B_3, where $k(n)$ is an admissible function of your choosing; for example take $k(n) = [\frac{n}{2}] + 1$ as in [2] and [3].

(step 1) Learn the basics. The three string braid group B_3 is the group $\langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ and its center is cyclic with generator $c = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$. The quotient group

$$B_3/\langle c \rangle = \langle u, v | u^2 = v^3 = e \rangle \simeq C_2 * C_3 \simeq \Gamma_0$$

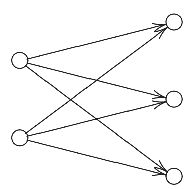
is the modular group $PSL_2(\mathbb{Z})$ where u and v are the images of $\sigma_1\sigma_2$ resp. $\sigma_1\sigma_2\sigma_1$.

By Schur's lemma, the central element c acts as λI_n (where $\lambda \in \mathbb{C}^*$) on any n-dimensional irreducible B_3-representation. Hence, it is enough to construct a $k(n) - 1$-dimensional family of n-dimensional irreducible representations of the modular group Γ_0.

If V is an n-dimensional Γ_0 representation, we can decompose it into eigenspaces for the action of $C_2 = \langle u \rangle$ and $C_3 = \langle v \rangle$:

$$V_1 \oplus V_2 = V \downarrow_{C_2} = V = V \downarrow_{C_3} = W_1 \oplus W_2 \oplus W_3$$

If the dimension of V_i is a_i and that of W_j is b_j, we say that V is a Γ_0-representation of *dimension vector* $\alpha = (a_1, a_2; b_1, b_2, b_3)$. Choosing a basis B_1 of V wrt. the decomposition $V_1 \oplus V_2$ and a basis B_2 wrt. $W_1 \oplus W_2 \oplus W_3$, we can view the basechange matrix $B_1 \longrightarrow B_2$ as an α-dimensional representation V_Q of the quiver Q

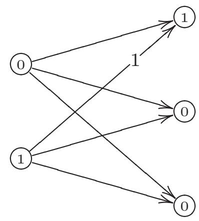


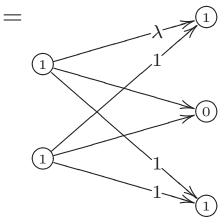
Bruce Westbury [6] has shown that V is an irreducible Γ_0-representation if and only if V_Q is a θ-*stable* Q-representation where $\theta = (-1, -1; 1, 1, 1)$ and that the two notions of isomorphism coincide. The *Euler-form* χ_Q of the quiver Q is the bilinear form on $\mathbb{Z}^{\oplus 5}$ determined by the matrix

$$\begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Westbury also showed that if there exists a θ-stable α-dimensional Q-representation, then there is an $1 - \chi_Q(\alpha, \alpha)$ dimensional family of isomorphism classes of such representations (and a Zariski open subset of them will correspond to isomorphism classes of irreducible Γ_0-representations). Hence, an *admissible* function $k(n)$ is one such that for all n we have $k(n) \leq 2 - \chi_Q(\alpha_n, \alpha_n)$ for a dimension vector $\alpha_n = (a_1, a_2; b_1, b_2, b_3)$ such that $n = a_1 + a_2$ and there exists a θ-stable α_n-dimensional Q-representation. Note that Aidan Schofield [5] gave an inductive procedure to determine the dimension vectors of stable representations.

(step 2) Choose known non-isomorphic Γ_0-irreducibles and their corresponding θ-stable Q-representations $\{V_i \ : \ i \in I\}$. Here are some obvious choices : using the foregoing and standard quiverology, there are 6 irreducible 1-dimensional Γ_0-representations S_{ij} and there are 3 one-parameter families of 2-dimensional simple Γ_0-representations $T_i(\lambda)$. Below the corresponding Q-representations for S_{21} and $T_2(\lambda)$ (the other cases are similar)

$S_{21} =$ 

$T_2(\lambda) =$ 

More interesting choices are the Q-representations corresponding to irreducible continuous representations of $\hat{\Gamma}_0$, the profinite completion of the modular group. For example, a simple factor of the monodromy representation associated to a dessin d'enfant or an irreducible representation of a finite group generated by an order two and an order three element, for example the monster group $\mathbb{M}$. Pick your favourite collection of non-isomorphic $\{V_i\}$.

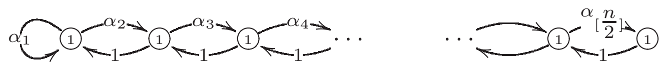
(step 3) Compute the local quiver of the collection $\{V_i \ : \ i \in I\}$ as in e.g. [1]. That is, we make a new quiver Δ having one vertex v_i for every V_i. If α_i is the dimension vector of the θ-stable Q-representation determined by V_i, then there are $1 - \chi_Q(\alpha_i, \alpha_i)$ loops in vertex v_i in Δ and there are exactly $-\chi_Q(\alpha_i, \alpha_j)$ oriented arrows starting in vertex v_i and ending in vertex v_j in Δ.

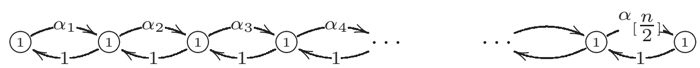
For each $n \in \mathbb{N}$ take a finite full subquiver Δ_n of Δ (say, on the vertices $\{v_{n,1}, \ldots, v_{n,k}\}$) then [1] asserts that there is an étale map between a Zariski open subset of the moduli space $M^{ss}_\alpha(Q,\theta)$ of θ-semi-stable Q-representations of dimension vector $\alpha = \alpha_{n,1} + \alpha_{n,2} + \ldots + \alpha_{n,k}$ around the Q-representation $V_{n,1} \oplus V_{n,2} \oplus \ldots \oplus V_{n,k}$ *and* the moduli space of *semi-simple* Δ_n-representations of dimension vector $\mathbf{1} = (1,1,\ldots,1)$ around the zero-representation. Moreover, in this étale correspondence, (isomorphism classes of) simple Δ_n-representations correspond to (isomorphism classes) of θ-stable representations.

By the results from [4] we have accomplished our objective, provided we can find for each n a *subquiver* Σ_n of Δ_n satisfying the following conditions

- Σ_n is strongly connected, meaning that any two vertices are connected via an oriented circuit in Σ_n, and
- $1 - \chi_{\Sigma_n}(\mathbf{1},\mathbf{1}) = k(n) - 1$ where χ_{Σ_n} is the Euler-form (as above) of the quiver Σ_n.

An example : consider the set $\{V_0 = S_{11}, V_1 = T_1(\lambda_1), V_2 = T_2(\lambda_2), V_3 = T_1(\lambda_3), V_4 = T_2(\lambda_4), V_5 = T_1(\lambda_5), \ldots\}$ with $\lambda_i \neq \lambda_j$ if $i \neq j$. Then, the quiver Δ has exactly one loop in each vertex v_i (except in v_0) and exactly one arrow $v_i \longrightarrow v_j$ whenever $i \neq j \ mod\ 2$. Let Δ_n be the full subquiver on the first $[\frac{n}{2}]$ vertices and Σ_n the subquiver below (on vertices $\{v_1, \ldots, v_{[\frac{n}{2}]}\}$ if n is even and on $\{v_0, v_1, \ldots, v_{[\frac{n}{2}]}\}$ if n is odd). Then, the indicated representations give an $[\frac{n}{2}]$-parameter family of simple Σ_n (and hence also Δ_n)-representations

(n even) : 

(n odd) : 

Using the étale map these representations give an $[\frac{n}{2}]$-parameter family of θ-stable Q-representations and hence of irreducible n-dimensional Γ_0-representations, and hence by Schur an $[\frac{n}{2}] + 1$-parameter family of isomorphism classes of irreducible B_3-representations.

(step 4) Reverse-engineer the above general argument to fit your specific example.

REFERENCES

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