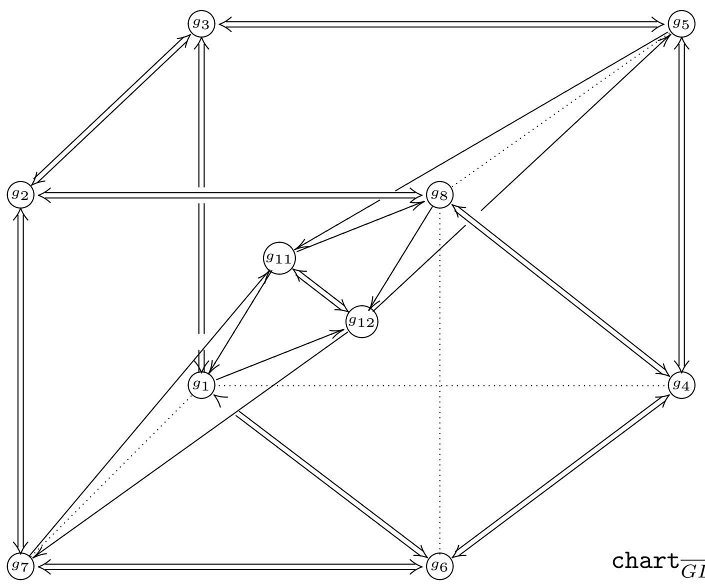
GRANADA - NAG
version 0.3

lieven le bruyn



$\mathtt{chart}_{\overline{GL_2(\mathbb{Z})}}$

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CONTENTS

INTRODUCTION

Let us take a hopeless problem, motivate why something like non-commutative algebraic geometry might help to solve it, and verify whether this promise is kept.

Suppose we want to know all solutions in invertible matrices to the braid relation (or Yang-Baxter equation)

$$XYX = YXY$$

All such solutions (for varying size of matrices) form an additive Abelian category $\mathtt{rep}\ B_3$, so a big step forward would be to know all its simple solutions (that is, those whose matrices cannot be brought in upper triangular block form). A literature check shows that even this task is far too ambitious. The best result to date is the classification due to Imre Tuba and Hans Wenzl of simple solutions of which the matrix size is at most 5.

For fixed matrix size n, finding solutions in $\mathtt{rep}\ B_3$ is the same as solving a system of n^2 cubic polynomial relations in $2n^2$ unknowns, which quickly becomes a daunting task. Algebraic geometry tells us that all solutions, say $\mathtt{rep}_n\ B_3$ form an affine closed subvariety of n^2-dimensional affine space. If we assume that $\mathtt{rep}_n\ B_3$ is a smooth variety (that is, a manifold) and if we know one solution explicitly, then we can use the tangent space in this point to linearize the problem and to get at all solutions in a neighborhood.

So, here is an idea : assume that $\mathtt{rep}\ B_3$ itself would be a non-commutative manifold, then we might linearize our problem by considering tangent spaces and obtain new solutions out of already known ones. But, what is a non-commutative manifold? Well, by the above we at least require that for all integers n the commutative variety $\mathtt{rep}_n\ B_3$ is a commutative manifold.

But, there is still some redundancy in our problem : if (X, Y) is a solution, then so is any conjugated pair $(g^{-1}Xg, g^{-1}Yg)$ where $g \in GL_n$ is a basechange matrix. In categorical terms, we are only interested in isomorphism classes of solutions. Again, if we fix the size n of matrix-solutions, we consider the affine variety $\mathtt{rep}_n\ B_3$ as a variety with a GL_n-action and we like to classify the orbits of simple solutions. If $\mathtt{rep}_n\ B_3$ is a manifold then the theory of Luna slices provides a method, both to linearize the problem as well as to reduce its complexity. Instead of the tangent space we consider the normal space N to the GL_n-orbit (in a suitable solution). On this affine space, the stabilizer subgroup $GL(\alpha)$ acts and there is a natural one-to-one correspondence between GL_n-orbits in $\mathtt{rep}_n\ B_3$ and $GL(\alpha)$-orbits in the normal space N (at least in a neighborhood of the solution).

So, here is a refinement of the idea : we would like to view $\mathtt{rep}\ B_3$ as a non-commutative manifold with a group action given by the notion of isomorphism. Then, in order to get new isoclasses of solutions from a constructed one we want to reduce the size of our problem by considering a linearization (the normal space to the orbit) and on it an easier isomorphism problem.

However, we immediately encounter a problem : calculating ranks of Jacobians we discover that already $\mathtt{rep}_2\ B_3$ is not a smooth variety so there is not a chance in the world that $\mathtt{rep}\ B_3$ might be a useful non-commutative manifold. Still, if (X, Y) is a solution to the braid relation, then the matrix $(XYX)^2$ commutes with both X and Y, for

$$\begin{aligned} (XYX)^2X &= (XYX)(XYX)X = (XYX)(YXY)X \\ &= X(YXY)(XYX) = X(XYX)(XYX) \\ &= X(XYX)^2 \end{aligned}$$

If (X, Y) is a simple solution, this means that after performing a basechange, $C = (XYX)^2$ becomes a scalar matrix, say $\lambda^6 1_n$. But then, $(X', Y') = (\lambda^{-1}X, \lambda^{-1}Y)$ is a solution to

$$X'Y'X' = Y'X'Y' \qquad \text{and} \qquad (X'Y'X')^2 = 1$$

and all such solutions form a non-commutative closed subvariety, say $\mathtt{rep}\ \Gamma$ of $\mathtt{rep}\ B_3$ and if we know all (isomorphism classes of) simple solutions in $\mathtt{rep}\ \Gamma$ we have solved our problem as we just have to bring in the additional scalar $\lambda \in \mathbb{C}^*$.

Here we strike gold : $\mathtt{rep}\ \Gamma$ is indeed a non-commutative manifold. This can be seen by identifying Γ with one of the most famous discrete infinite groups in mathematics : the modular group $PSL_2(\mathbb{Z})$. The modular group acts by Möbius transformations on the upper half plane and this action can be used to write $PSL_2(\mathbb{Z})$ as the free group product $\mathbb{Z}_2 * \mathbb{Z}_3$. Finally, using classical representation theory of finite groups it follows that indeed all $\mathtt{rep}_n\ \Gamma$ are commutative manifolds (possibly having many connected components)! So, let us try to linearize this problem by looking at its non-commutative tangent space, if we can figure out what this might be.

Here is another idea (or rather a dogma) : in the world of non-commutative manifolds, the role of affine spaces is played by $\mathtt{rep}\ Q$ the representations of finite quivers Q. A quiver is just on oriented graph and a representation of it assigns to each vertex a finite dimensional vector space and to each arrow a linear map between the vertex-vector spaces. The notion of isomorphism in $\mathtt{rep}\ Q$ is of course induced by base change actions in all of these vertex-vector spaces.

Now, can we assign such an non-commutative tangent space, that is a $\mathtt{rep}\ Q$ for some quiver Q, to $\mathtt{rep}\ \Gamma$? As $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$ we may restrict any solution $V = (X, Y)$ in $\mathtt{rep}\ \Gamma$ to the finite subgroups $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Now, representations of finite cyclic groups are decomposed into eigen-spaces. For example

$$V \downarrow_{\mathbb{Z}_2} = V_+ \oplus V_-$$

where $V_\pm = \{v \in V \mid g.v = \pm v\}$ with g the generator of $\mathbb{Z}_2$. Similarly,

$$V \downarrow_{\mathbb{Z}_3} = V_1 \oplus V_\rho \oplus V_{\rho^2}$$

where ρ is a primitive 3-rd root of unity. That is, to any solution $V \in \mathtt{rep}\ \Gamma$ we have found 5 vector spaces V_+, V_-, V_1, V_ρ and V_{ρ^2} so we would like them to correspond to the vertices of our conjectured quiver Q.

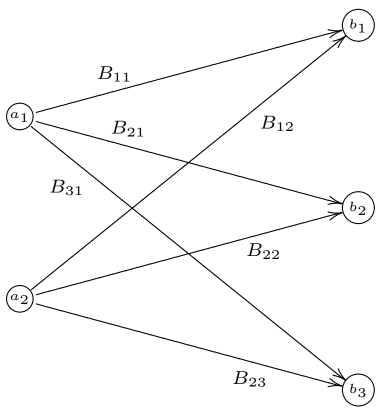
What are the arrows of Q, or equivalently, is there a natural linear map between the vertex-vector spaces? Clearly, as

$$V_+ \oplus V_- = V = V_1 \oplus V_\rho \oplus V_{\rho^2}$$

any choice of two bases of V (one compatible with the left-side decomposition, the other with the right-side decomposition) are related by a basechange matrix B which we can decompose into six blocks (corresponding to the two decompositions in 2 resp. 3 subspaces

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

which gives us 6 linear maps between the vertex-vector spaces. Hence, to $V \in$ `rep` Γ does correspond in a natural way a representation of dimension vector $\alpha = (a_1, a_2, b_1, b_2, b_3)$ (where $dim(V_+) = a_1, \ldots, dim(V_{\rho^2}) = b_3$) of the quiver Q which is of the form



Clearly, not every representation of `rep` Q is obtained in this way. For starters, the eigen-space decompositions force the numerical restriction

$$a_1 + a_2 = dim(V) = b_1 + b_2 + b_3$$

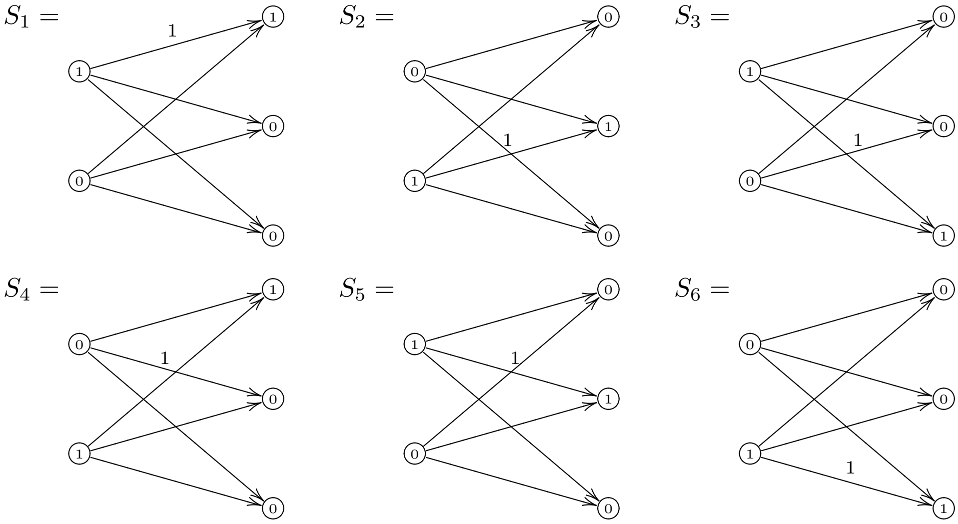
on the dimension vector and the square matrix constructed from the arrow-linear maps must be invertible. However, if both these conditions are satisfied, we can reconstruct the (isomorphism class) of the solution in `rep` Γ from this quiver representation by taking

$$X = B^{-1} \begin{bmatrix} 1_{b_1} & 0 & 0 \\ 0 & \rho^2 1_{b_2} & 0 \\ 0 & 0 & \rho 1_{b_3} \end{bmatrix} B \begin{bmatrix} 1_{a_1} & 0 \\ 0 & -1_{a_2} \end{bmatrix}$$

$$Y = \begin{bmatrix} 1_{a_1} & 0 \\ 0 & -1_{a_2} \end{bmatrix} B^{-1} \begin{bmatrix} 1_{b_1} & 0 & 0 \\ 0 & \rho^2 1_{b_2} & 0 \\ 0 & 0 & \rho 1_{b_3} \end{bmatrix} B$$

Hence, it makes sense to view `rep` Q as a linearization of, or as a tangent space to, `rep` Γ. However, though we reduced the study of solutions of the polynomial system of equations to linear algebra, we have not reduced the isomorphism problem in size. In fact, if we start of with a matrix-solution $V = (X, Y)$ of size n we end up with a quiver-representation of total dimension $2n$. So, can we construct some sort of non-commutative normal space to the isomorphism classes? That is, is there another quiver Q' whose representations can be interpreted as normal-spaces to orbits in certain points?

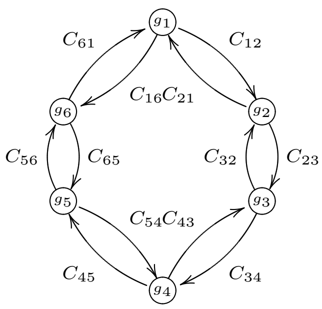
Here is the construction of this normal space or chart $\mathtt{chart}_\Gamma$. The sub-semigroup of $\mathbb{Z}^5$ (all dimension vectors of Q) consisting of those vectors $\alpha = (a_1, a_2, b_1, b_2, b_3)$ satisfying the numerical condition $a_1 + a_2 = n = b_1 + b_2 + b_3$ is generated by six dimension vectors, namely those of the 6 non-isomorphic one-dimensional solutions in $\mathtt{rep}\ \Gamma$



In particular, in any component $\mathtt{rep}_\alpha\ Q$ containing an open subset of representations corresponding to solutions in $\mathtt{rep}\ \Gamma$ we have a particular semi-simple solution

$$M = S_1^{\oplus g_1} \oplus S_2^{\oplus g_2} \oplus S_3^{\oplus g_3} \oplus S_4^{\oplus g_4} \oplus S_5^{\oplus g_5} \oplus S_6^{\oplus g_6}$$

and in particular $\alpha = (g_1 + g_3 + g_5, g_2 + g_4 + g_6, g_1 + g_4, g_2 + g_5, g_3 + g_6)$. The normal space to the $GL(\alpha)$-orbit of M in $\mathtt{rep}_\alpha\ Q$ can be identified with the representation space $\mathtt{rep}_\beta\ Q'$ where $\beta = (g_1, \ldots, g_6)$ and Q' is the quiver of the following form



and we can even identify how the small matrices C_{ij} fit into the 3×2 block-decomposition of the base-change matrix B

$$B = \left[\begin{array}{ccc|ccc} 1_{a_1} & 0 & 0 & C_{21} & 0 & C_{61} \\ 0 & C_{34} & C_{54} & 0 & 1_{a_4} & 0 \\ \hline C_{12} & C_{32} & 0 & 1_{a_2} & 0 & 0 \\ 0 & 0 & 1_{a_5} & 0 & C_{45} & C_{65} \\ \hline 0 & 1_{a_3} & 0 & C_{23} & C_{43} & 0 \\ C_{16} & 0 & C_{56} & 0 & 0 & 1_{a_6} \end{array}\right]$$

Hence, it makes sense to call Q' the non-commutative normal space to the isomorphism problem in $\mathtt{rep}\ \Gamma$. Moreover, under this correspondence simple representations of Q' (for which both the dimension vectors and distinguishing characters are known explicitly) correspond to simple solutions in $\mathtt{rep}\ \Gamma$.

Having completed our promised approach via non-commutative geometry to the classification problem of solutions to the braid relation, it is time to collect what we have learned. Let $\beta = (g_1, \ldots, g_6)$ with $n = \gamma_1 + \ldots + \gamma_6$, then for every non-zero scalar $\lambda \in \mathbb{C}^*$ the matrices

$$X = \lambda B^{-1} \begin{bmatrix} 1_{g_1+g_4} & 0 & 0 \\ 0 & \rho^2 1_{g_2+g_5} & 0 \\ 0 & 0 & \rho 1_{g_3+g_6} \end{bmatrix} B \begin{bmatrix} 1_{g_1+g_3+g_5} & 0 \\ 0 & -1_{g_2+g_4+g_6} \end{bmatrix}$$

$$Y = \lambda \begin{bmatrix} 1_{g_1+g_3+g_5} & 0 \\ 0 & -1_{g_2+g_4+g_6} \end{bmatrix} B^{-1} \begin{bmatrix} 1_{g_1+g_4} & 0 & 0 \\ 0 & \rho^2 1_{g_2+g_5} & 0 \\ 0 & 0 & \rho 1_{g_3+g_6} \end{bmatrix} B$$

give a solution of size n to the braid relation. Moreover, such a solution can be simple only if the following numerical relations are satisfied

$$g_i \leq g_{i-1} + g_{i+1}$$

where indices are viewed modulo 6. In fact, if these conditions are satisfied then a sufficiently general representation of Q' does determine a simple solution in $\mathtt{rep}\ B_3$ and conversely, any sufficiently general simple n size solution of the braid relation can be conjugated to one of the above form. Here, by sufficiently general we mean a Zariski open (hence dense) subset.

That is, for all integers n we have constructed nearly all (meaning a dense subset) simple solutions to the braid relation. As to the classification problem, if we have representants of simple β-dimensional representations of the quiver Q', then the corresponding solutions (X, Y) of the braid relation represent different orbits (up to finite overlap coming from the fact that our linearizations only give an analytic isomorphism, or in algebraic terms, an étale map). Such representants can be constructed for low dimensional β. Finally, our approach also indicates why the classification of braid-relation solutions of size ≤ 5 is easier : from size 6 on there are new classes of simple Q'-representations given by going round the whole six-cycle!

day 1

GROUPS & CHARACTERS

Today, we will introduce some interesting arithmetical groups and the third braid group which will be our principal examples. Virtually nothing is known about the finite dimensional representations of these groups. For example, the best result on the third braid group is the classification of all simple representations of dimension ≤ 5. We will see that the arithmetical groups can be constructed from finite groups and recall the representation theory of finite groups. However, this character theory does not extend immediately to discrete infinite groups as the easy example of the group of integers already clarifies. Non-commutative algebraic geometry will provide a handle to study the finite dimensional representations of such groups.

1.1 Arithmetical groups

We will focus attention to the following four groups of interest

$$\begin{tikzcd} SL_2(\mathbb{Z}) \arrow[r, hook] \arrow[d, two heads] & GL_2(\mathbb{Z}) \\ PSL_2(\mathbb{Z}) & B_3 \arrow[l, two heads] \end{tikzcd}$$

$GL_2(\mathbb{Z})$ is the *general linear group* over the integers $\mathbb{Z}$, that is, it consists of all invertible 2×2 matrices with integer coefficients. As ± 1 are the only units in $\mathbb{Z}$ we have

$$GL_2(\mathbb{Z}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc = \pm 1 \}$$

matrix-multiplication turns $GL_2(\mathbb{Z})$ into a non-Abelian infinite group.

$SL_2(\mathbb{Z})$ is the *special linear group* over the integers $\mathbb{Z}$. That is, it is the subgroup of $GL_2(\mathbb{Z})$ consisting of those invertible 2×2 matrices with determinant equal to 1

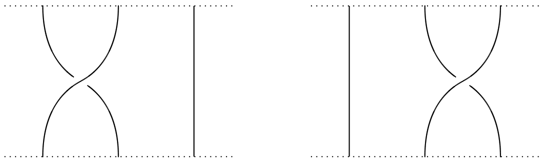
$$SL_2(\mathbb{Z}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc = 1 \}$$

This group has a finite central (hence normal) subgroup of order 2 namely

$$\mathbb{Z}_2 = \langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rangle$$

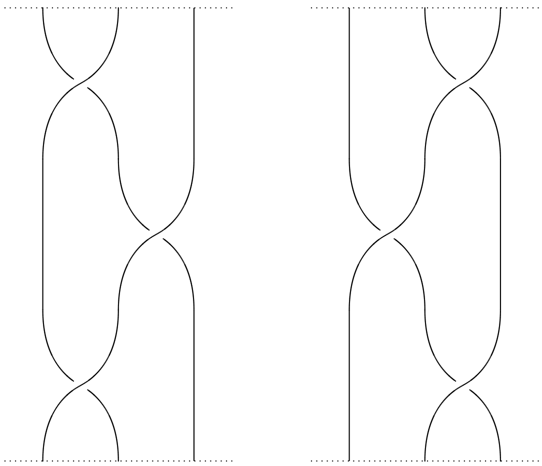
and the corresponding *quotient group* $SL_2(\mathbb{Z})/\mathbb{Z}_2$ is called the *modular group* and is denoted $PSL_2(\mathbb{Z})$.

B_3 is the *third braid group*. That is, B_3 is the group of all 3-string braids up to topological equivalence. It is generated by the two *elementary braids*



σ_1 σ_2

Multiplication is induced by concatenating braids, that is placing them on top of another. Hence, any 3-braid can be written as a noncommutative word in σ_1 and σ_2 but some of these words represent topologically equivalent braids. For example, the braids



$\sigma_1\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_2$

can be transformed into each other by pulling so we have an identity (the *Yang-Baxter equation*)

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

in B_3. In fact, Emil Artin proved that this is the only non-trivial relation among σ_1 and σ_2 so B_3 has a presentation

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Before we can explain the epimorphism $B_3 \twoheadrightarrow PSL_2(\mathbb{Z})$ we need to find presentations of the arithmetic groups $GL_2(\mathbb{Z}), SL_2(\mathbb{Z})$ and $PSL_2(\mathbb{Z})$.

Consider the following three matrices in $GL_2(\mathbb{Z})$

$$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then we claim that $GL_2(\mathbb{Z})$ is generated by U, V and R. Consider the products

$$C = UV = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad D = VU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

then by multiplying an arbitrary element of $GL_2(\mathbb{Z})$ with powers of C and D we obtain the following matrices

$$C^n \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-nc & b-nd \\ c & d \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} C^n = \begin{bmatrix} a & b-na \\ c & d-nc \end{bmatrix}$$

$$D^n \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c+na & d+nb \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} D^n = \begin{bmatrix} a+nb & b \\ c+nd & d \end{bmatrix}$$

As the determinant of an element in $GL_2(\mathbb{Z})$ is ± 1 it follows that the entries in each column (resp. row) are coprime integers. By multiplying with powers of C and D on the right we can reduce a modulo c as well as c modulo a and this procedure will finish if one of them is equal to 0 and the other is equal to ± 1 (use coprimeness of a and c). We may assume $c = 0$ (otherwise, multiply by U on the left) and hence the matrix is of the form

$$\begin{bmatrix} \pm 1 & b' \\ 0 & d' \end{bmatrix} \qquad \text{whence} \qquad d' = \pm 1$$

By multiplying this matrix on the right by a power of C we can get rid of b' and obtain the matrix

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \in \{\, id, UR, RU, U^2 \,\}$$

so working backwards we have shown that an arbitrary element of $GL_2(\mathbb{Z})$ can be written as a word in U, V and R. Clearly, there are relations between these generators and we aim to prove that a presentation of $GL_2(\mathbb{Z})$ is given as

$$GL_2(\mathbb{Z}) = \langle U, V, R \mid U^2 = V^3, U^4 = R^2 = (RU)^2 = (RV)^2 = id \,\rangle$$

Similarly, one proves that $SL_2(\mathbb{Z})$ is generated by the matrices U and V. Indeed, we used only multiplications by U or powers of C and D to reduce the matrix to the form

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

But as the determinant has to be equal to 1 only the following cases are possible

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = U^2 \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = id$$

and below we will prove that in fact $SL_2(\mathbb{Z})$ has a presentation

$$SL_2(\mathbb{Z}) = \langle\, U, V \mid U^2 = V^3, U^4 = id \,\rangle$$

To prove that the obvious relations among the generators are the only ones, we need to study the action of $SL_2(\mathbb{Z})$ and of the modular group $PSL_2(\mathbb{Z})$ on the upper-half plane $\mathcal{H}$ which also clarifies the interest of these groups for number theory as well as the study of Riemann surfaces.

It is a classical fact that the group $SL_2(\mathbb{Z})$ acts on the upper half of the complex numbers

$$\mathcal{H} = \{\, z \in \mathbb{C} \mid Im\, z > 0 \,\}$$

by *Möbius transformations*

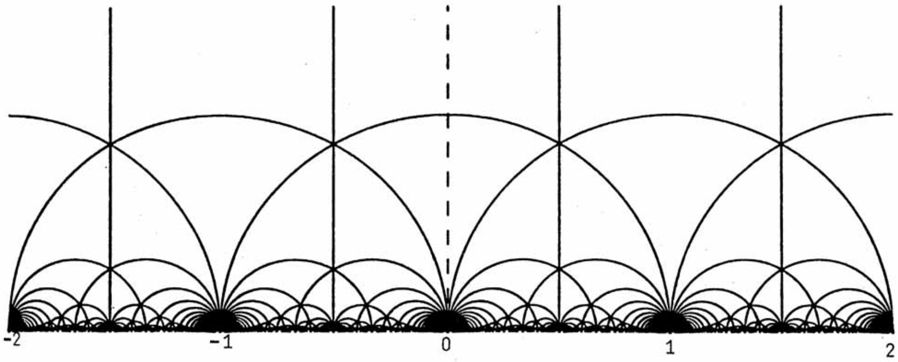
$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad : \quad \mathcal{H} \xrightarrow{g} \mathcal{H} \qquad \text{by} \qquad z \mapsto \frac{az+b}{cz+d}$$

One can compute that g maps $\mathcal{H}$ to $\mathcal{H}$ by verifying that

$$Im\ g(z) = \frac{Im\ z}{|cz+d|^2}$$

Consider the *unit-circle* $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ then one can calculate that the element $g \in SL_2(\mathbb{Z})$ carries $S^1 \cap \mathcal{H}$ to the set

$$(*) = \begin{cases} \{ z \in \mathcal{H} \mid \quad |z - \frac{ac-bd}{c^2-d^2}| = |\frac{1}{c^2-d^2}| \ \} & \text{if } c^2 \neq d^2 \\ \{ z \in \mathcal{H} \mid \quad Re\ z = ac - \frac{1}{2} \ \} & \text{if } c^2 = d^2 = 1 \end{cases}$$



Consider the arc

$$L = \{ e^{i\theta} \mid \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \}$$

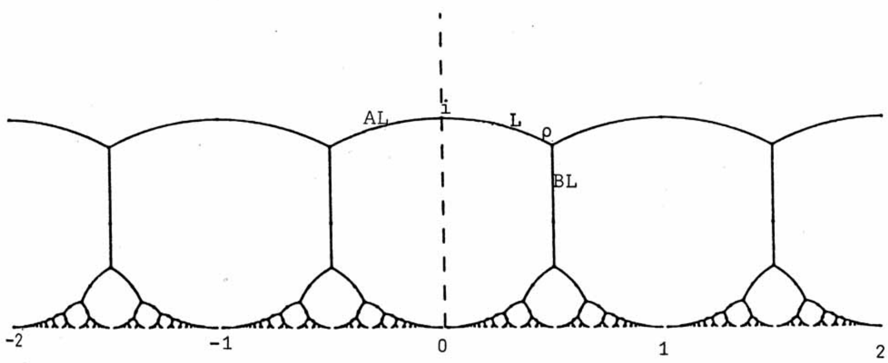
as an oriented edge

$$i \longrightarrow \rho$$

where $\rho = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Define the set $T = SL_2(\mathbb{Z})L$ then we claim that T is a tree. From $(*)$ it follows that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} L \cap L \subset \{i, \rho\}$$

whence T is a graph. Moreover, it follows from $(*)$ that the only point of T on the imaginary axis is the point i (observe that $0 \notin \mathcal{H}$). The only translates gL with $g \in SL_2(\mathbb{Z})$ containing i are L and UL therefore there are no closed circuits in T passing through L just once. If T would have a closed circuit then one can translate it by a suitable element of $SL_2(\mathbb{Z})$ so that it includes L and therefore there is no such circuit.



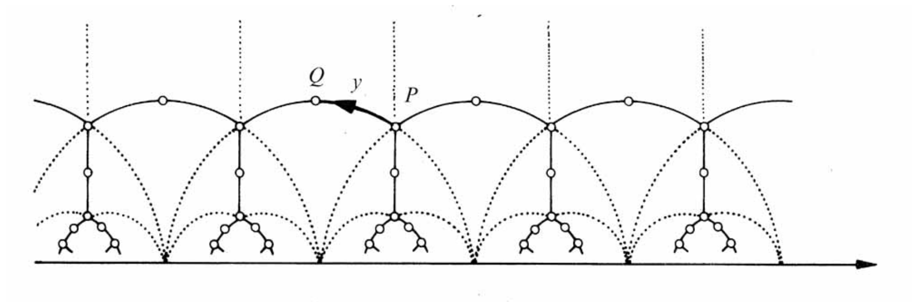
To prove that T is a tree it only remains to show that T is connected. As $SL_2(\mathbb{Z})$ is generated by U and V it is also generated by U and

$$V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

which fixes ρ. Hence,

$$L \cup UL \qquad \text{and} \qquad L \cup V^{-1}L$$

are connected and hence so is $SL_2(\mathbb{Z})L$.



L is a *fundamental domain* for the action of $SL_2(\mathbb{Z})$ on the tree T as by definition of T, L contains one point in each orbit and we have seen that it does not contain two points in the same orbit. Let us compute the *stabilizer subgroups*

$$G_i = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \frac{ai+b}{ci+d} = i \}$$

which gives the condition $ai + b = di - c$ whence

$$G_i = \{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a^2 + b^2 = 1 \} = \langle\, U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mid U^4 = id \rangle$$

In a similar way we find that the stabilizer subgroup of $SL_2(\mathbb{Z})$ at ρ is

$$G_\rho = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \frac{a\rho+b}{c\rho+d} = \rho \}$$

giving after some calculation that

$$G_\rho = \{ \begin{bmatrix} b+d & b \\ -b & d \end{bmatrix} \mid b^2 + bd + d^2 = 1 \} = \langle\, V = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \mid V^6 = id \,\rangle$$

It follows that the stabilizer subgroup G_L of L, that is those elements $g \in SL_2(\mathbb{Z})$ fixing L is the intersection

$$G_L = G_i \cap G_\rho = \langle\, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = U^2 = V^3 \,\rangle$$

Definition 1.1 If G_1 and G_2 are two finite groups having a common subgroup H, then the *amalgamated free product* $G_1 *_H G_2$ is the group having group morphisms $G_i \xrightarrow{c_i} G_1 *_H G_2$ satisfying the universal property : for any pair of group morphisms

$G_1 \xrightarrow{f} G$ and $G_2 \xrightarrow{g} G$ such that $f|H = g|H$ there is a uniquely determined group morphism h making the diagram commute

$$\begin{tikzcd} & G & \\ G_1 \arrow[ur, "f"] \arrow[r, "c_1"] & G_1 *_H G_2 \arrow[u, "\exists ! h"] & G_2 \arrow[ul, "g"'] \arrow[l, "c_2"'] \\ & H \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

The amalgamated product $G_1 *_H G_2$ is constructed to be the set of all words

$$h.s_{i_1}^{(a)} s_{i_2}^{(a+1)} s_{i_3}^{(a+2)} \ldots s_{i_k}^{(a+k)} \qquad h \in H,\ k \in \mathbb{N}$$

and where $\{1, s_i^{(c)}\}$ is a set of right coset representatives a G_c modulo H and $(a + j) = (a + j\ mod 2)$. There is a natural group structure on this set making it into the amalgamated free product.

If $H = \{id\}$ is the trivial subgroup then $G_1 *_H G_2$ is the *free product* and will be denoted by $G_1 * G_2$.

The upshot of all our calculations above is that we can prove :

Theorem 1.2 *With notations as before we have :*

1. *Let* $\mathbb{Z}_2 \simeq \langle U^2 \rangle = \langle V^3 \rangle$*, then*

$$SL_2(\mathbb{Z}) \simeq \langle U \mid U^4 = id \rangle *_{\mathbb{Z}_2} \langle V \mid V^6 = id \rangle \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$$

whence

$$SL_2(\mathbb{Z}) \simeq \langle U, V \mid U^2 = V^3, U^4 = 1 \rangle$$

2. $PSL_2(\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$

Proof. As $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/(U^2 = V^3)$ the second statement follows from the first. As for the first, by the universal property there is a uniquely determined group morphism

$$\langle U \mid U^4 = id \rangle *_{\mathbb{Z}_2} \langle V \mid V^6 = id \rangle \longrightarrow SL_2(\mathbb{Z})$$

which is surjective as the images of U and V generate $SL_2(\mathbb{Z})$. Any element in the kernel would give a relation in $SL_2(\mathbb{Z})$ of the form

$$U^2 = V^3 = U^j.V^{i_1}.U.V^{i_2}.U \ldots U.V^{i_l}.U^k \qquad j, k \in \{0, 1\},\ i_u \in \{1, 2\}$$

which would produce a nontrivial circuit in the tree, a contradiction. $\square$

Recall that Artin's theorem asserted that $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. Consider the braids

$$S = \sigma_1\sigma_2\sigma_1 \qquad \text{and} \qquad T = \sigma_1\sigma_2$$

From the Yang-Baxter equation we obtain the relations

$$T^{-1}S = \sigma_1 \qquad \text{and} \qquad ST^{-1} = \sigma_2$$

whence the braid group B_3 is also generated by S and T. Moreover,

$$S^2 = (\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_2) = (\sigma_1\sigma_2)(\sigma_1\sigma_2)(\sigma_1\sigma_2) = T^3$$

is a *central* element C in $B_3 = \langle S, T \mid S^2 = T^3 \rangle$. Dividing out the normal subgroup generated by C we obtain the group

$$B_3/\langle C \rangle = \langle \overline{S}, \overline{T} \mid \overline{S}^2 = \overline{T}^3 = id \rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_3 \simeq PSL_2(\mathbb{Z})$$

giving us the claimed epimorphism $B_3 \twoheadrightarrow PSL_2(\mathbb{Z})$.

Theorem 1.3 *With notations as before we have*

$$GL_2(\mathbb{Z}) = \langle\, U, V, R \mid U^2 = V^3, U^4 = R^2 = (RU)^2 = (RV)^2 = id \,\rangle$$

Or, alternatively, if $D_2 = \langle R, U^2 \rangle = \langle R, V^3 \rangle$, then

$$\begin{aligned} GL_2(\mathbb{Z}) = \; & \langle U, R \mid U^4 = R^2 = (RU)^2 = id \rangle *_{D_2} \langle V, R \mid V^6 = R^2 = (RV)^2 = id \rangle \\ \simeq \; & D_4 *_{D_2} D_6 \end{aligned}$$

where D_n is the dihedral group of order $2n$.

Proof. In order to get the defining relations of $GL_2(\mathbb{Z})$ from those of $SL_2(\mathbb{Z})$ we only need to know how the extra generator R operates on the generators of $SL_2(\mathbb{Z})$ and the lowest power of R belonging to $SL_2(\mathbb{Z})$. As $R \notin SL_2(\mathbb{Z})$ we have to add the relations

$$R^2 = id \qquad RUR^{-1} = U^{-1} \qquad RVR^{-1} = V^{-1}$$

to those of $SL_2(\mathbb{Z})$ to complete the set. □

Recall that the *dihedral group* D_n is the symmetry group of a regular n-gon, so D_4 is the symmetry group of the square and D_6 that of a hexagon.

1.2 Representation theory

If G is a group, an *n-dimensional representation* of G is a group morphism

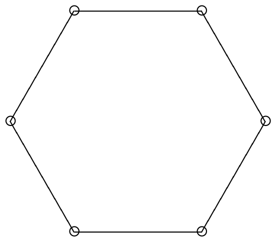
$$G \xrightarrow{\phi} GL_n(\mathbb{C})$$

and two n-dimensional representations are said to be *isomorphic* if they are conjugate, that is if the diagram below commutes

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\psi"'] & GL_n(\mathbb{C}) \arrow[dl, "m.m^{-1}"] \\ GL_n(\mathbb{C}) & \end{tikzcd}$$

that is, there is an invertible matrix $m \in GL_n(\mathbb{C})$ such that for all $g \in G$ we have $\psi(g) = m\phi(g)m^{-1}$. In case G is a *finite* group, all relevant information about representations is contained in the *character table* of G.

For example, let $G = D_6$ be the symmetry group of the hexagon and order the vertices clockwise 1 to 6



Let $V = (1,2,3,4,5,6)$ be the rotation over 60^o and $R = (2,6)(3,5)$ is flipping over the line through the vertices 1 and 4. The character table is

D_6	$1a$	$2a$	$2b$	$6a$	$3a$	$2c$
$\#$	1	3	3	2	2	1
	id	R	VR	V	V^2	V^3
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	1
χ_3	1	-1	1	-1	1	-1
χ_4	1	1	-1	-1	1	-1
χ_5	2	0	0	1	-1	-2
χ_6	2	0	0	-1	-1	2

In a character table of a finite group G, the columns correspond to the different *conjugacy classes* in G. Recall that two elements $g, h \in G$ are said to be conjugated if there is an $x \in G$ such that $xgx^{-1} = h$. Observe that the number of elements in the conjugacy class C_g of g

$$\# C_g = \frac{\# G}{\# Z_g(G)}$$

where $Z_g(G) = \{h \in G \mid gh = hg\}$.

In the example, one verifies that there are 6 conjugacy classes. One of elements of order 6 containing 2 elements $\{V, V^{-1}\}$, one of two elements of order 3 namely $\{V^2, V^{-2}\}$ and three conjugacy classes of order two elements : one containing the single (central) element $C = V^3$, the two other classes contain each 3 elements : the three flips over lines through midpoints of edges (type $2b$) resp. flips over lines through antipodal points (type $2a$). The rows of a character table correspond to the non-isomorphic *simple representations* of G. Observe that a representation $\phi \ : \ G \longrightarrow GL_n(\mathbb{C})$ defines a G-action on the column vectors $V_\phi = \mathbb{C}^n$ by the rule $g.v = \phi(g)v$. A G-action on a finite dimensional vector space V is a map $G \times V \longrightarrow V$ satisfying for all $v, v' \in V$, all $g, h \in G$ and all $\lambda \in \mathbb{C}$

$$id.v = v \ , \ g.(h.v) = (gh).v \ , \ g.(v + v') = g.v + g.v' \ , \ g.(\lambda v) = \lambda g.v$$

Observe that ϕ and ψ are isomorphic n-dimensional representations of G if and only if V_ϕ and V_ψ only differ by a basechange. A representation ϕ is said to be *simple* if and only if V_ϕ does not have a proper linear subspace $W \subset V_\phi$ such that $g.w \in W$ for all $g \in G$ and all $w \in W$. It is a fact that the character table is a square matrix, that is, the number of non-isomorphic simple representations of a finite group G is equal to the number of its conjugacy classes.

In the example, we have $D_6 = \langle V, R \mid V^6 = R^2 = (RV)^2 = id \rangle$ and so in every 1-dimensional representation $D_6 \longrightarrow \mathbb{C}^*$ (which is necessarily simple) we must have $R \mapsto \pm 1$ and then the last identity also forces $V \mapsto \pm 1$ whence there are precisely 4 one-dimensional (simple) representations of D_6 : χ_1, χ_2, χ_3 and χ_4.

There are two non-isomorphic simple 2-dimensional representations of D_6 defined by

$$\chi_5 \;:\; V \mapsto \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \quad R \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\chi_6 \;:\; V \mapsto \begin{bmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-2} \end{bmatrix} \quad R \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $\zeta = e^{2\pi i/6}$. As there must be exactly 6 non-isomorphic simple D_6-representations, we have described them all!

The (i,j)-th entry of the character table of a finite group is the *character* of the j-th conjugacy class $g_j \in C_j$ of G on the i-th simple representation $\phi_i \;:\; G \longrightarrow GL_n(\mathbb{C})$, that is

$$\chi_i(g_j) = Tr(\phi_i(g_j))$$

the trace of the matrix giving the action of g_j on V_{ϕ_i}. In particular, as the identity element of G acts trivially on each representation we have that $\chi_i(id)$ is the dimension of the simple representation. The characters of the identity are classically written in the first column of the character table. Observe that since $Tr(X) = Tr(mXm^{-1})$ for any $m \in GL_n(\mathbb{C})$ we have that the character is a *class function*, that is, is the same for all group elements in the same conjugacy class. Moreover, the class functions over all simple representations are known to be linearly independent, that is, the square matrix determined by the character table is invertible!

If $\phi \;:\; G \longrightarrow GL_n(\mathbb{C})$ and $\psi \;:\; G \longrightarrow GL_m(\mathbb{C})$ are n- resp. m-dimensional representations, then there is an $n+m$-dimensional representation called *the direct sum*

$$\phi \oplus \psi \;:\; G \longrightarrow GL_{n+m}(\mathbb{C}) \qquad g \mapsto \begin{bmatrix} \phi(g) & 0 \\ 0 & \psi(g) \end{bmatrix}$$

The fundamental theorem on representations of finite groups is that every representation is *completely reducible*, that is, every representation $\phi \;:\; G \longrightarrow GL_n(\mathbb{C})$ is isomorphic

$$\phi \simeq \chi_1^{\oplus e_1} \oplus \chi_2^{\oplus e_2} \oplus \ldots \oplus \chi_k^{\oplus e_k}$$

where the $\{\chi_1, \ldots, \chi_k\}$ are the distinct simple representations of G (in particular, k is the number of conjugacy classes of G) and where the $e_i \in \mathbb{N}$ are the *multiplicities* of the simple representations in V_ϕ. This decomposition (that is, the integers e_i) can be easily computed using the character-table. Indeed, given $\phi \;:\; G \longrightarrow GL_n(\mathbb{C})$ we can compute the character of ϕ for any element $g \in G$ (it suffices to take one representant in each conjugacy class)

$$\chi_\phi = (\chi_\phi(g_1) = Tr(\phi(g_1)) \;, \;\ldots\;, \;\chi_\phi(g_k) = Tr(\phi(g_k))) \in \mathbb{C}^k$$

which, given the decomposition above must be of the form

$$\chi_\phi = (e_1, \ldots, e_k).\texttt{CharacterMatrix(G)}$$

and therefore we obtain the multiplicities and hence the decomposition by

$$(e_1, \ldots, e_k) = \chi_\phi.\texttt{CharacterMatrix(G)}^{-1}$$

For example, there is an 8-dimensional D_6-representation V_ϕ with character

$\chi_\phi = (8, 0, 0, -1, -1, 8)$ and as $\chi_V.\texttt{CharacterMatrix}(D_6)^{-1} = (1, 1, 0, 0, 0, 3)$

this gives us the decomposition into simple representations

$$\phi \simeq \chi_1 \oplus \chi_2 \oplus \chi_6^{\oplus 3}$$

If H is a subgroup of the group G, we can *restrict* G-representations to H-representations. So, let $\phi : G \longrightarrow GL_n(\mathbb{C})$ be an n-dimensional representation of G, then the restriction is the composition

$$\phi \downarrow_H \; : H \hookrightarrow G \xrightarrow{\phi} GL_n(\mathbb{C})$$

and hence is an n-dimensional representation of G. In particular, if H is a finite group we have that $\phi \downarrow_H$ is uniquely a direct sum of simple H-representations.

For example, consider the subgroup $H = D_2 = \langle C = V^3, R \rangle$ of order 4 of D_6 which is an Abelian group isomorphic to the *Klein Vierergruppe* $\mathbb{Z}_2 \times \mathbb{Z}_2$. Consequently, all conjugacy classes consist of just one element and hence there must be 4 simple H-representations, each of dimension one. In fact, the character table of D_2 is

D_2	$1a$	$2a$	$2b$	$2c$
$\#$	1	1	1	1
	id	C	R	CR
ψ_1	1	1	1	1
ψ_2	1	-1	-1	1
ψ_3	1	-1	1	-1
ψ_4	1	1	-1	-1

$C = V^3$ defines conjugacy class $2c$ in D_6 and R conjugacy class $2a$, but in which conjugacy class lies CR? Well, as a symmetry of the hexagon, C is point-symmetry over the center and R is a flip over a line through two anti-podal vertices. But then, CR is a flip over a line through the midpoints of edges, so CR belongs to conjugacy class $2b$ of D_6. Now, all we have to do to compute the restrictions $\chi_i \downarrow_H$ is to take the columns $[id, 2c, 2a, 2b]$ of the character table of $G = D_6$ and interpret them as characters of $H = D_2$-representations. So,

	$1a$	$2c$	$2a$	$2b$
	id	C	R	CR
$\chi_1 \downarrow_H$	1	1	1	1
$\chi_2 \downarrow_H$	1	1	-1	-1
$\chi_3 \downarrow_H$	1	-1	-1	1
$\chi_4 \downarrow_H$	1	-1	1	-1
$\chi_5 \downarrow_H$	2	-2	0	0
$\chi_6 \downarrow_H$	2	2	0	0

giving us the restriction data

$$\begin{cases} \chi_1 \downarrow_H & \simeq \psi_1 \\ \chi_2 \downarrow_H & \simeq \psi_4 \\ \chi_3 \downarrow_H & \simeq \psi_2 \\ \chi_4 \downarrow_H & \simeq \psi_3 \\ \chi_5 \downarrow_H & \simeq \psi_2 \oplus \psi_3 \\ \chi_6 \downarrow_H & \simeq \psi_1 \oplus \psi_4 \end{cases}$$

However, in case the group G is infinite (as is the case for the four groups we promise to study in more detail) it is no longer true that every representation is the direct sum of simple representations nor that characters determine the representation up to isomorphism.

Example 1.4 Let $G = \mathbb{Z} \simeq \langle x, x^{-1} \rangle$ then as G is Abelian every simple representation must be one-dimensional and clearly sending

$$x \mapsto \lambda \in \mathbb{C}^*$$

defines a one-dimensional simple representation.$\phi_\lambda \; : \; \mathbb{Z} \longrightarrow \mathbb{C}^*$ and as conjugation in $\mathbb{C}^*$ is trivial they are non-isomorphic for different λ, that is

$$\texttt{simples}(\mathbb{Z}) \leftrightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$$

An n-dimensional representation $\phi \; : \; \mathbb{Z} \longrightarrow GL_n(\mathbb{C})$ is fully determined by the image $\phi(x) \in GL_n(\mathbb{C})$ and if such a representation is isomorphic to a direct sum of simples, say

$$\phi \simeq \phi_{\lambda_1} \oplus \ldots \oplus \phi_{\lambda_n}$$

(some possibly occurring more than once) this would mean that there is an invertible matrix $m \in GL_n(\mathbb{C})$ such that

$$m\phi(x)m^{-1} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

that is, ϕ is a direct sum of simple representations if and only if $\phi(x)$ is a *diagonalizable* matrix. But, we know from the *Jordan normal form* theorem that not every invertible $n \times n$ matrix is diagonalizable. For example, the 2-dimensional representation

$$\phi \; : \; \mathbb{Z} \longrightarrow GL_2(\mathbb{C}) \qquad x \mapsto \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

is not the direct sum of two simple representations. In particular, it is *not* isomorphic to the *semi-simple representation*

$$\phi_{ss} \; : \; \mathbb{Z} \longrightarrow GL_2(\mathbb{C}) \qquad x \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

As $\mathbb{Z}$ is Abelian all its conjugacy classes consist of a single element x^i and the corresponding characters are

$$\chi_\phi(x^i) = 2\lambda^i = \chi_{\phi_{ss}}(x^i)$$

whence character cannot distinguish between the two non-isomorphic 2-dimensional representations ϕ and ϕ_{ss}!

As the arithmetical groups and B_3 are more complicated, we expect similar phenomena. Hence, we have to find another approach to study their finite dimensional representations. Here, non-commutative algebraic geometry enters the picture.

day 2

ALGEBRAS & REPRESENTATIONS

If G is a group, its *group algebra* $\mathbb{C}G$ is the $\mathbb{C}$-vector space $\sum_{g \in G} \mathbb{C}e_g$ with a basis corresponding to the elements of G and with multiplication linearly induced by the rule

$$e_g.e_h = e_{gh}$$

It is easy to verify that this is an associative $\mathbb{C}$-algebra having a unit element $1 = 1.e_{id}$. Moreover, $\mathbb{C}G$ is commutative if and only if G is an Abelian group.

If G is a finite group, then the group algebra $\mathbb{C}G$ is a *semi-simple algebra*, that is a finite direct sum of full matrix algebras over $\mathbb{C}$. In fact, the character table of G indicates which matrix-algebras occur. If the complete set of simple G-representations is $\chi_1, \ldots, \chi_k$ having dimensions $n_1, \ldots, n_k$, then

$$\mathbb{C}G \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

so, in particular $\#G = n_1^2 + \ldots + n_k^2$ is the sum of the squares of the dimensions of the simple representations. For example,

$$\mathbb{C}D_6 \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$$

As $\mathbb{C}G$ is a finite dimensional vector space with a G-action by

$$G \times \mathbb{C}G \longrightarrow \mathbb{C}G \qquad (g, e_h) \mapsto e_{gh}$$

we know that $\mathbb{C}G$ must decompose as G-representation into a direct sum of simple representations. In fact, as G-representations,

$$\mathbb{C}G \simeq \chi_1^{\oplus n_1} \oplus \ldots \oplus \chi_k^{\oplus n_k}$$

the distinct χ_i-components corresponding to the columns of the matrix-component $M_{n_i}(\mathbb{C})$.

For a group G, we have already seen that an n-dimensional representation $G \xrightarrow{\phi} GL_n(\mathbb{C})$ corresponds to a G-action on an n-dimensional space $V_\phi = \mathbb{C}^n$.

The latter is the same thing as defining a *left* $\mathbb{C}G$*-module* structure on $\mathbb{C}^n$ which in turn is the same thing as defining an algebra map $\mathbb{C}G \longrightarrow M_n(\mathbb{C})$ which we can still denote by ϕ.

In analogy with the group-case we call an algebra map $\mathbb{C}G \longrightarrow M_n(\mathbb{C})$ an n-dimensional *representation* of the group algebra $\mathbb{C}G$. Hence, there are natural one-to-one correspondences between

- n-dimensional representations of G
- G-actions on $\mathbb{C}^n$
- left $\mathbb{C}G$-module structures on $\mathbb{C}^n$
- n-dimensional representations of $\mathbb{C}G$

Moreover, these correspondences preserve the natural notion of isomorphisms in each of the four settings. This allows us to extend the concept of a finite dimensional representation to an arbitrary $\mathbb{C}$-algebra.

2.1 Representation schemes

Commutative affine $\mathbb{C}$-algebras, that is the objects of the category `commalg`, are precisely the *coordinate rings* of *affine schemes*. Recall that an affine scheme V is determined by a system of polynomial equations

$$\begin{cases} g_1(x_1,\ldots,x_n) & = 0 \\ \vdots & \vdots \\ g_r(x_1,\ldots,x_n) & = 0 \end{cases}$$

with all $g_i(x_1,\ldots,x_n) \in \mathbb{C}[x_1,\ldots,x_n]$. The coordinate ring of the affine scheme V is the quotient algebra

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_1,\ldots,x_n]}{(g_1,\ldots,g_r)}$$

and as any affine commutative $\mathbb{C}$-algebra can be expressed in this way, they are precisely the coordinate rings of affine schemes.

The set of points $\mathtt{points}V$ of an affine scheme V is the set of points $c = (c_1,\ldots,c_n) \in \mathbb{C}^n$ which are solutions to the system of equations, that is, such that

$$\begin{cases} g_1(c_1,\ldots,c_n) & = 0 \\ \vdots & \vdots \\ g_r(c_1,\ldots,c_n) & = 0 \end{cases}$$

However, it is not true in general that the point set $\mathtt{points}V$ determines the affine scheme V or the ideal $I = (g_1,\ldots,g_r)$ in $\mathbb{C}[x_1,\ldots,x_n]$!

In fact, the *Hilbert Nullstellensatz* asserts that if $J = (h_1,\ldots,h_s)$ is another ideal in $\mathbb{C}[x_1,\ldots,x_n]$, with associated affine scheme W, then

$$\mathtt{points}W = \mathtt{points}V \qquad \text{iff} \qquad rad(I) = rad(J)$$

where $rad(I) = \{g \in \mathbb{C}[x_1,\ldots,x_n] \mid \exists k \in \mathbb{N} \ : \ g^k \in I\}$ is the *radical* of the ideal I. Ideals that coincide with their radical are radical (or semi-prime) ideals and the corresponding affine schemes are called *reduced* or *affine varieties*.

Affine schemes are generalizations of affine varieties so that we have an (anti)-equivalence between the category `commalg` of all commutative affine $\mathbb{C}$-algebras and

$\texttt{affine}$ the category of affine schemes. So, what is the *affine scheme* corresponding to C? Formally, it is the scheme *representing the functor*

$$\texttt{rep}_1\, C \;:\; \texttt{commalg} \longrightarrow \texttt{sets} \qquad \text{defined by} \qquad D \mapsto Hom_{\texttt{commalg}}(C, D)$$

In general, we say that an affine scheme X represents a functor

$$F \;:\; \texttt{commalg} \longrightarrow \texttt{sets} \qquad D \mapsto F(D)$$

if and only if there is a natural one-to-one correspondence for every $D \in \texttt{commalg}$

$$F(D) \leftrightarrow Hom_{\texttt{commalg}}(\mathbb{C}[X], D)$$

which in the case of $\texttt{rep}_1$ is just a tautology, so by definition, the affine scheme corresponding to C is the geometric object $\texttt{rep}_1\, C$ with coordinate ring $\mathbb{C}[\texttt{rep}_1\, C] = C$.

For those who had a course in commutative algebraic geometry, $\texttt{rep}_1\, C$ is what is usually denoted $\texttt{spec}\, C$ the *prime spectrum of* C, that is, the set of all prime ideals of C which becomes a topological space after endowing it with the *Zariski topology*, that is, a typical closed set is of the form

$$\mathbb{V}(I) = \{p \in \texttt{spec}\, C \mid I \subset p\}$$

for I an ideal of C. Again, this topological space is not sufficient to reconstruct C from it but if we equip it with a *structure sheaf* $\mathcal{O}_C$ we can recover C by taking its global sections.

We will denote the category of all $\mathbb{C}$-algebras by $\texttt{alg}$. A $\mathbb{C}$-algebra A is said to be *affine* if it is generated as a $\mathbb{C}$-algebra by finitely many elements. For example, if the group G is generated by finitely many elements (as is for instance the case for $GL_2(\mathbb{Z}), (P)SL_2(\mathbb{Z})$ and B_3) then the group algebra $\mathbb{C}G$ is an affine $\mathbb{C}$-algebra.

If A is a non-commutative affine $\mathbb{C}$-algebra, what is the geometric object associated to A? A first idea might be to take the same functor

$$\texttt{rep}_1\, A \;:\; \texttt{commalg} \longrightarrow \texttt{sets} \qquad \text{defined by} \qquad D \mapsto Hom_{\texttt{alg}}(A, D)$$

but as any algebra map from A to a *commutative* algebra D factorizes over the *Abelianization*

$$A_{ab} = \frac{A}{[A, A]}$$

we see that $\texttt{rep}_1\, A = \texttt{rep}_1\, A_{ab}$ and as A_{ab} is a commutative affine $\mathbb{C}$-algebra, the corresponding affine scheme represents the functor and we have

$$\mathbb{C}[\texttt{rep}_1\, A] = \mathbb{C}[\texttt{rep}_1\, A_{ab}] = A_{ab}$$

Example 2.1 Take the third braid group $B_3 = \langle S, T \mid S^2 = T^3 \rangle$ then we have (as the Abelianization of a group algebra is the group algebra of the Abelianized group)

$$(\mathbb{C}B_3)_{ab} = \frac{\mathbb{C}[s, t, s^{-1}, t^{-1}]}{(s^2 - t^3)}$$

and hence $\texttt{rep}_1\, \mathbb{C}B_3$ is represented by the affine (smooth) curve

$$\mathbb{V}(x^2 - y^3) - \{(0, 0)\} \subset \mathbb{C}^2$$

which is the cusp minus the singular top.

However, in this approach we have lost all non-commutative information. So, a second idea might be to try to represent the functor

$$\mathtt{rep}_n\ A\ :\ \mathtt{commalg} \longrightarrow \mathtt{sets} \qquad \text{defined by} \qquad D \mapsto Hom_{\mathtt{alg}}(A, M_n(D))$$

for any natural number $n \in \mathbb{N}$. In fact, we will show that the functor from *all* $\mathbb{C}$-algebras to sets

$$\mathtt{alg} \longrightarrow \mathtt{sets} \qquad \text{defined by} \qquad B \mapsto Hom_{\mathtt{alg}}(A, M_n(B))$$

is *representable* by the *anti-matrix algebra* $\sqrt[n]{A}$ which means that there is a natural one-to-one correspondence

$$Hom_{\mathtt{alg}}(\sqrt[n]{A}, B) \leftrightarrow Hom_{\mathtt{alg}}(A, M_n(B))$$

As a consequence, the functor $\mathtt{rep}_n\ A$ will be representable by the Abelianization of the anti-matrix algebra $\sqrt[n]{A}_{ab}$. Clearly, we have to prove all these claims

Definition 2.2 The *anti-matrix algebra* of a $\mathbb{C}$-algebra A is the subalgebra

$$\sqrt[n]{A} = \{x \in A * M_n(\mathbb{C}) \mid aE_{ij} = E_{ij}a\ \forall 1 \leq i, j \leq n\}$$

where E_{ij} are the standard matrices $E_{ij} = (\delta_{ki}\delta lj)_{k,l} \in M_n(\mathbb{C})$.

Theorem 2.3 *For any $\mathbb{C}$-algebras A and B there is a natural one-to-one correspondence*

$$Hom_{\mathtt{alg}}(A, M_n(B)) \leftrightarrow Hom_{\mathtt{alg}}(\sqrt[n]{A}, B)$$

and if A is affine, so is $\sqrt[n]{A}$. As a consequence, the functor $\mathtt{rep}_n\ A$ is represented by the affine commutative scheme $\mathtt{rep}_1\ \sqrt[n]{A}_{ab}$ with coordinate ring $\sqrt[n]{A}$.

Proof. We start with a classical result : let $M_n(\mathbb{C}) \xrightarrow{\phi} R$ be an algebra map and denote $\phi(E_{ij}) = e_{ij}$. In R we consider the subalgebra

$$S = \{r \in R \mid e_{ij}r = re_{ij}\ \forall i, j\}$$

then we claim that $R \simeq M_n(S)$. To begin,we construct an algebra map

$$R \xrightarrow{\alpha} M_n(S) \quad \text{defined by} \quad \alpha(r) = \begin{bmatrix} r_{11} & \dots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \dots & r_{nn} \end{bmatrix} \quad \text{where } r_{ij} = \sum_{k=1}^n e_{ki}rr_{jk}$$

To begin, $r_{ij} \in S$ as $r_{ij}e_{uv} = e_{uv}r_{ij}$ because

$$\begin{aligned} r_{ij}e_{uv} &= \sum_{k=1}^n e_{ki}re_{jk}e_{uv} \\ &= \sum_{k=1}^n e_{ki}ee_{jv}\delta_{uk} = e_{ui}rr_{jv} \\ e_{uv}r_{ij} &= \sum_{k=1}^n e_{uv}e_{ki}re_{jk} \\ &= \sum_{k=1}^n \delta_{vk}e_{ui}re_{jk} = e_{ui}rr_{jv} \end{aligned}$$

Moreover, α is indeed an algebra map, for if $\alpha(r)\alpha(s) = (t_{ij})_{i,j}$, then

$$\begin{aligned} t_{ij} &= \sum_{k=1}^{n} r_{ik}s_{kj} \\ &= \sum_{k=1}^{n}(\sum_{l=1}^{n} e_{li}re_{kl})(\sum_{m=1}^{n} e_{mk}se_{jm}) \\ &= \sum_{k,l=1}^{n} e_{li}re_{kk}se_{jl} = \sum_{l=1}^{n} e_{li}rse_{jl} = \alpha(rs)_{i,j} \end{aligned}$$

We also have an algebra map in the other direction

$$M_n(S) \xrightarrow{\beta} R \qquad \text{defined by} \qquad \begin{bmatrix} r_{11} & \dots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \dots & r_{nn} \end{bmatrix} \mapsto \sum_{i,j=1}^{n} r_{ij}e_{ij}$$

This is an algebra map as

$$t_{ij} = \sum_{k=1}^{n} r_{ik}s_{kj} \mapsto \sum_{i,j=1}^{n}\sum_{k=1}^{n} r_{ik}s_{kj}e_{ij}$$

$$(\sum_{i,j=1}^{n} r_{ij}e_{ij})(\sum_{k,l=1}^{n} s_{kl}e_{kl}) = \sum_{i,j=k,l} r_{ik}s_{kl}e_{il}$$

and one verifies that α and β are each other inverses, proving the claim. Now take

$$\sqrt[n]{A} = \{x \in A * M_n(\mathbb{C}) \mid xE_{ij} = E_{ij}x,\ \forall i,j\}$$

For an algebra map $A \xrightarrow{\phi} M_n(B)$ take the unique map $M_n(\mathbb{C}) \xrightarrow{i} M_n(B)$ sending E_{ij} to the standard matrix-elements $e_{ij} \in M_n(B)$ then we have a uniquely determined algebra map

$$A * M_n(\mathbb{C}) \xrightarrow{\phi * i} M_n(B)$$

which sends the centralizer $\sqrt[n]{A}$ to the subring $\{m \in M_n(B) \ :\ me_{ij} = e_{ij}m\, \forall i,j\} = B$ giving us the desired map $\sqrt[n]{A} \longrightarrow B$. Conversely, for an algebra map $\sqrt[n]{A} \xrightarrow{\psi} B$ we have the induced algebra map

$$A * M_n(\mathbb{C}) = M_n(\sqrt[n]{A}) \xrightarrow{M_n(\psi)} M_n(B)$$

and composing this with the natural inclusion $A \xrightarrow{j} A * M_n(\mathbb{C})$ we get a map $A \longrightarrow M_n(B)$. These two constructions are each other inverses and finish the proof. □

Example 2.4 For the group algebra $\mathbb{C}B_3$ of the third braid group and $n = 2$ we have

$$\sqrt[2]{\mathbb{C}B_3} = \frac{\mathbb{C}\langle s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4\rangle}{(\begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}^2 = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}^3)}$$

and hence $\mathtt{rep}_2\ \mathbb{C}B_3$ is the affine scheme corresponding to the Abelianization

$$\sqrt[2]{\mathbb{C}B_3}_{ab} = \frac{\mathbb{C}[x_1,x_2,x_3,x_4,y_1,y_2,y_{3,4}]}{(f_{11},f_{12},f_{21},f_{22})}$$

where f_{ij} is the (i,j)-entry of the matrix (with commuting entries)

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^2 - \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}^3$$

That is,

$$\begin{cases} f_{11} & = x_1^2 + x_2x_3 - y_1^3 - 2y_1y_2y_3 - y_2y_3y_4 \\ f_{12} & = x_1x_2 + x_2x_4 - y_1^2y_2 - y_2^2y_3 - y_1y_2y_4 - y_2y_4^2 \\ f_{21} & = x_1x_3 + x_3x_4 - y_1^2y_3 - y_2y_3^2 - y_1y_3y_4 - y_3y_4^2 \\ f_{22} & = x_2x_3 + x_4^2 - y_1y_2y_3 - 2y_2y_3y_4 - y_4^3 \end{cases}$$

For a general noncommutative $\mathbb{C}$-algebra A we have natural maps i_A and $j_A$where j_A satisfies the universal property

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "\phi"'] \arrow[rd, "j_A"] & A * M_n(\mathbb{C}) = M_n(\sqrt[n]{A}) \arrow[d, "\pi"] \\ M_n(\mathbb{C}) & M_n(\sqrt[n]{A}_{ab}) \arrow[l, dotted, "\exists! M_n(\psi)"] \end{tikzcd}$$

that for any $\mathbb{C}$-algebra morphism $\phi : A \longrightarrow M_n(C)$ where C is a commutative algebra, there is a unique algebra morphism $\psi : \sqrt[n]{A}_{ab} \longrightarrow C$ making the diagram commute. We will give a few applications of these universal maps.

Theorem 2.5 *There is an action of $GL_n(\mathbb{C})$ by automorphisms on $\sqrt[n]{A}$ and hence there is a GL_n-action on the affine scheme $\mathtt{rep}_n\ A$.*

Proof. For any $g \in GL_n(\mathbb{C})$ there is an algebra map $c_g : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ by conjugation and therefore also an algebra map (using the universal property of algebra free products)

$$M_n(\sqrt[n]{A}) = A * M_n(\mathbb{C}) \xrightarrow{id * c_g} A * M_n(\mathbb{C}) = M_n(\sqrt[n]{A})$$

whence by the universal property of $\sqrt[n]{A}$ an algebra map

$$\sqrt[n]{A} \xrightarrow{a_g} \sqrt[n]{A}$$

Abelianizing this action induces a GL_n-action by automorphisms on $\sqrt[n]{A}_{ab}$ and as this is the coordinate ring of the scheme $\mathtt{rep}_n\ A$, this affine scheme is a GL_n-scheme. $\square$

The *orbits* of the GL_n-action on $\mathtt{rep}_n\ A$ are precisely the isomorphism classes of n-dimensional representations.

As there is nothing special about a particular n, we argue that the *noncommutative affine scheme* corresponding to a noncommutative affine $\mathbb{C}$-algebra A is the disjoint union

$$\mathtt{rep}\ A = \bigsqcup_n \mathtt{rep}_n\ A$$

where $\mathtt{rep}\ A$ is the *category* of all finite dimensional A-modules. Observe that $\mathtt{rep}\ A$ is even an *Abelian category* meaning that A-module morphisms have kernels and cokernels.

2.2 Smooth algebras

Now that we agreed to associate to an affine non-commutative $\mathbb{C}$-algebra A as *non-commutative affine scheme* the Abelian category $\mathtt{rep}\ A$ (later we will put extra structure such as a topology on it) we want to know which of these are non-commutative manifolds, that is which algebras A deserve to be called non-commutative smooth algebras. Again, let us look at the commutative case for inspiration.

An important class of reduced schemes are the *smooth affine varieties*, that is, those affine schemes V such that $\mathtt{points}V$ is a (complex) manifold. These can be defined by requiring that the rank of the *Jacobian matrix* of the system of equations

$$\mathtt{Jac} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \end{bmatrix}$$

is locally constant on $\mathtt{points}V$. By requiring the rank to be only locally constant we allow smooth affine varieties to have several disjoint connected components, possibly of different dimensions.

The coordinate ring of a smooth affine variety V is called a *smooth commutative algebra* (sometimes also called a *regular algebra*). Alexander Grothendieck found a categorical characterization of smooth affine algebras.

Theorem 2.6 (Grothendieck) *An affine scheme V is a manifold, or equivalently, its coordinate ring $\mathbb{C}[V]$ is an affine smooth commutative algebra if and only if $\mathbb{C}[V]$ satisfies the following lifting property in* $\mathtt{commalg}$. *For any commutative algebra B and any nilpotent ideal $I \triangleleft B$ (that is, such that there is a power $k \in \mathbb{N}$ such that $I^k = (0)$) and any $\mathbb{C}$-algebra morphism $f : \mathbb{C}[V] \longrightarrow B/I$*

$$\begin{tikzcd} \mathbb{C}[V] \arrow[r, dotted, "\tilde{f}"] \arrow[dr, "f"'] & B \arrow[d, two heads, "\pi"] \\ & B/I \end{tikzcd}$$

there is an algebra morphism $A \xrightarrow{\tilde{f}} B$ making the diagram commute.

These facts motivate the approach to *noncommutative algebraic geometry* as proposed by Daniel Quillen and Maxim Kontsevich : affine $\mathbb{C}$-algebras should be thought of as coordinate rings of noncommutative affine schemes and noncommutative affine manifolds correspond to *smooth algebras*.

Definition 2.7 An affine $\mathbb{C}$-algebra A is called *smooth* if it has the following lifting property in $\mathtt{alg}$. For any $\mathbb{C}$-algebra B and any nilpotent ideal $I \triangleleft B$ (that is, such that there is a power $k \in \mathbb{N}$ such that $I^k = (0)$) and any $\mathbb{C}$-algebra morphism $f : A \longrightarrow B/I$

$$\begin{tikzcd} A \arrow[r, dotted, "\tilde{f}"] \arrow[dr, "f"'] & B \arrow[d, "\pi"] \\ & B/I \end{tikzcd}$$

there is an algebra morphism $A \xrightarrow{\tilde{f}} B$ making the diagram commute.

Before we give an alternative description and classes of examples, let us deduce an important consequence of smoothness for finite dimensional representations.

Theorem 2.8 *If A is a smooth noncommutative $\mathbb{C}$-algebra, then for all $n \in \mathbb{N}$ the affine commutative scheme $\mathtt{rep}_n\ A$ is smooth, that is an affine manifold.*

Proof. By Grothendieck's characterization we have to show that every $\mathbb{C}$-algebra morphism

$$\sqrt[n]{A}_{ab} = \mathbb{C}[\mathtt{rep}_n\ A] \xrightarrow{\phi} D/I$$

can be lifted through the nilpotent ideal $I \triangleleft D$ of the commutative algebra D. Consider the following diagram of $\mathbb{C}$-algebra maps

$$\begin{tikzcd} A \arrow[r, dotted, "\exists\tilde{\phi}"] \arrow[d, "j_A"'] & M_n(D) \arrow[d] \\ M_n(\mathbb{C}[\mathtt{rep}_n\ A]) \arrow[ur, dotted] \arrow[r, "M_n(\phi)"'] & M_n(D/I) \end{tikzcd}$$

As $M_n(I)$ is a nilpotent ideal of the $\mathbb{C}$-algebra $M_n(D)$ we can use smoothness of A to have a lifted morphism $\tilde{\phi} : A \longrightarrow M_n(D)$. Then, we use the universal property of j_A to see that the diagonal map of the form $M_n(\psi)$ exists and ψ is the required algebra lift. $\square$

Hence, as M. Kontsevich argues, noncommutative smooth algebras A can be seen as *machines* to produce an infinite family $\{\mathtt{rep}_n\ A : n \in \mathbb{N}\}$ of manifolds. We will give a few equivalent definitions of smooth algebras. To begin, it is not necessary to check all nilpotent lifts, it suffices to check those for so called *square zero extensions*.

Recall that M is called an A-bimodule if M is both a left- and a right module and satisfies

$$(a_1 m)a_2 = a_1(m a_2) \qquad \forall m \in M, \forall a_i \in A$$

It is well known that there is an equivalence of categories between A-bimod, the category of all A-bimodules and A^e-mod, the category of left A^e-modules where $A^e = A \otimes A^{opp}$ is the *enveloping algebra* of A. Indeed, the left A^e-module structure corresponding to an A-bimodule M is given by

$$(a \otimes a')m = ama'$$

Using this equivalence of categories one can extend homological properties (such as projective, free, resolutions etc.) from one-sided to bimodules.

Let $\overline{A}$ be the $\mathbb{C}$-vector space $A/\mathbb{C}.1_A$, and consider the free A-bimodules

$$A \otimes \overline{A}^{\otimes n} \otimes A = \Omega^n\ A \otimes A$$

where $\Omega^n\ A = A \otimes \overline{A}^{\otimes n}$ are the *non-commutative differential forms* using the dictionary

$$(a_0, a_1, \ldots, a_n) = a_0 da_1 \ldots da_n = \omega$$

We put a graded algebra structure on $\Omega A = \oplus_{i=0}^{\infty} \Omega^n\ A$ by

$$(a_0, a_1, \ldots, a_n)(a_{n+1}, \ldots, a_{n+k}) = \sum_{i=0}^{n} (-1)^{n-i}(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+k})$$

which determines maps $\Omega^n A \otimes \Omega^{k-1} A \longrightarrow \Omega^{n+k-1}$ and as $\Omega^0 A = A$ this makes all $\Omega^n A$ into A-bimodules. We have exact sequences of A-bimodules

$$0 \longrightarrow \Omega^{n+1}\ A \xrightarrow{j} \Omega^n\ A \otimes A \xrightarrow{m} \Omega^n\ A \longrightarrow 0$$

where the maps are defined by

$$\begin{cases} j(\omega da) & = \omega a \otimes 1 - \omega \otimes a \\ m(\omega \otimes a) & = \omega a \end{cases}$$

in particular, we have the exact sequence of A-bimodules

$$0 \longrightarrow \Omega^1\ A \xrightarrow{j} A \otimes A \xrightarrow{m} A \longrightarrow 0$$

Differential 1-forms $\Omega^1 A$ has the following universal property. A *derivation* for an A-bimodule M is a linear map $D : A \longrightarrow M$ such that

$$D(\mathbb{C}) = 0 \qquad \text{and} \qquad D(ab) = D(a)b + aD(b)$$

For example, $d : A \longrightarrow \Omega^1 A = A \otimes \overline{A}$ such that $d(a) = (1, a)$ is a derivation and any derivation D for a bimodule M has a unique factorization through d

$$\begin{tikzcd} A \arrow[r, "D"] \arrow[d, "d"'] & M \\ \Omega^1 A \arrow[ur, dotted] & \end{tikzcd}$$

For an A-bimodule M the *Hochschild cohomology* spaces $H^i(M)$ are defined by

$$H^i(M) = Ext^i_{A^e}(A, M)$$

and the first of those have the following interpretations. $H^0(M) = M^A = \{m \in M \mid am = ma \forall a \in A\}$.Moreover,

$$H^1(M) = \frac{\text{Derivations on } M}{\text{inner derivations}}$$

where an inner derivation is one of the form $D_m(a) = am - ma$ for $m \in M$.

Also, $H^2(M)$ has a concrete interpretation. A *square zero extension* of A is a $\mathbb{C}$-algebra B having an ideal M satisfying $M^2 = 0$ such that $B/M \simeq A$. The kernel M of the quotient map $B \xrightarrow{\pi} A$ can be given a natural A-bimodule structure via $a.m = bm$ whenever $\pi(b) = a$ (because $M^2 = 0$ this does not depend on the choice of b). Two square zero extensions (B_1, M) and (B_2, M) for a given A-bimdule M are sais to be equivalent if there is an algebra map $\phi : B_1 \longrightarrow B_2$ making the diagram below commute

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, "id_M"'] & B_1 \arrow[r, two heads] \arrow[d, "\phi"'] & A \arrow[d, "id_A"'] \\
M \arrow[r, hook] & B_2 \arrow[r, two heads] & A
\end{tikzcd}$$

A square-free extension is said to be *trivial* if it is of the form $B = A \oplus M$ with multiplication rule

$$(a, m)(a', m') = (aa', am' + ma')$$

For a fixed A-bimodule M, the second Hochschild space $H^2(M)$ classifies equivalence classes of square-zero extensions of A with kernel M and the zero vector corresponds to the trivial square-zero extension, that is the one where A has a lift through π.

General arguments (such as induction on nilpotency of the nilpotent ideal) assert that A is a smooth algebra if and only if A lifts through all square-zero extensions, that is that

$$0 = H^2(M) = Ext^2_{A^e}(A, M) = Ext^1_{A^e}(\Omega^1 A, M)$$

for all A-bimodules M (that is all left A^e-modules M). But this is equivalent to $\Omega^1 A$ being a projective A^e-module, that is a projective A-bimodule. So, we have the following alternative characterizations of smooth $\mathbb{C}$-algebras.

Theorem 2.9 *For a $\mathbb{C}$-algebra A, the following statements are equivalent*

1. *A is a smooth algebra*
2. *A lifts through every square-zero extension*
3. *Ω^1 A is a projective A-bimodule*

But let us return to the examples of interest.

Theorem 2.10 *If G is a finite group, then $A = \mathbb{C}G$ is a smooth algebra. In fact, any two algebra lifts through a nilpotent ideal*

$$\begin{tikzcd} \mathbb{C}G \arrow[r, "\phi,\psi"] \arrow[dr, "f"'] & B \arrow[d] \\ & B/I \end{tikzcd}$$

are conjugated, that is, there is a unit $b \in B^*$ such that $\psi(a) = b^{-1}\phi(a)b$ for all $a \in \mathbb{C}G$.

Proof. Consider the exact sequence of $\mathbb{C}G$-bimodules

$$0 \longrightarrow \Omega^1\mathbb{C}G \xrightarrow{j} \mathbb{C}G \otimes \mathbb{C}G \xrightarrow{m} \mathbb{C}G \longrightarrow 0$$

which splits as we can send 1 to the *separability idempotent*

$$\frac{1}{\#G}\sum_{g \in G} g \otimes g^{-1}$$

Hence, $\mathbb{C}G$ and $\Omega^1\ \mathbb{C}G$ are direct summands of the free $\mathbb{C}G$-bimodule $\mathbb{C}G \otimes \mathbb{C}G$. As $\Omega^1\ \mathbb{C}G$ is projective, $\mathbb{C}G$ is a smooth algebra. Moreover, A is a projective A-bimodule, whence

$$H^1(M) = Ext^1_{A^e}(A, M) = 0$$

for every A-bimodule M. As every lift through the trivial square-zero extension $A \oplus M$ defines (and is defined by) a derivation $A \longrightarrow M$ we know that these two differ by an inner derivation, which can be translated into the conjugation property. Again, standard arguments allow to extend this from square-zero extensions to arbitrary nilpotent lifts. $\square$

In fact, as the proof works for all $\mathbb{C}$-algebras S having a separability idempotent, that is when S is a semi-simple algebra.

Let $S \in$ `alg`, the category of all $\mathbb{C}$-algebras, and consider the category $S -$ `alg` of all S-algebras. That is, objects in $S -$ `alg` are pairs (A, i) where A is a $\mathbb{C}$-algebra and $S \overset{i}{\hookrightarrow} A$ is an inclusion. Morphisms in $S -$ `alg` are $\mathbb{C}$-algebra morphisms compatible with the inclusions, that is $f : (A, i) \longrightarrow (A', i')$ if and only if $f : A \longrightarrow A'$ is a $\mathbb{C}$-algebra morphism such that

$$\begin{tikzcd} A \arrow[r, "f"] & A' \\ S \arrow[u, hook, "i"] \arrow[ur, hook, "i'"'] & \end{tikzcd}$$

is a commutative diagram. If (A_1, i_1) and (A_2, i_2) are two S-algebras we define the *amalgamated free algebra product*

$$(A_1 *_S A_2, u)$$

as the S-algebra (if it exists) with the universal property that there are S-algebra embeddings $A_i \overset{j_i}{\hookrightarrow} A_1 *_S A_2$ and for any S-algebra (B, j) and S-algebra morphisms $f_1 : A_1 \longrightarrow B$ and $f_2 : A_2 \longrightarrow B$ (which in particular implies that $f_1 \circ i_1 = j = f_2 \circ i_2$) there is a unique S-algebra map g such that the diagram of $\mathbb{C}$-algebra maps is commutative

$$\begin{tikzcd}
 & A_1 \arrow[d, hook, "j_1"] \arrow[dr, "f_1"] & \\
S \arrow[ur, hook, "i_1"] \arrow[r, hook, "u"] \arrow[dr, hook, "i_2"'] & A_1 *_S A_2 \arrow[r, "g"] & B \\
 & A_2 \arrow[u, hook, "j_2"'] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

For general S there is no reason why such an algebra should exist, but one can prove (essentially by a similar method as we constructed amalgamated free products of groups) that for S a semi-simple algebra such a universal algebra always exists. When $S = \mathbb{C}$ the construction reduces to the *algebra free product* $A_1 * A_2$.

We will need this only in the following case : let H be a finite subgroup of two groups G_1 and G_2, then the amalgamated free algebra product of the group algebras $\mathbb{C}G_1$ and $\mathbb{C}G_2$ exists and is isomorphic to

$$\mathbb{C}G_1 *_{\mathbb{C}H} \mathbb{C}G_2 \simeq \mathbb{C}G_1 *_H G_2$$

the group algebra $\mathbb{C}G_1 *_H G_2$ of the amalgamated group product $G_1 *_H G_2$. Indeed, let $f_1 : \mathbb{C}G_1 \longrightarrow B$ and $f_2 : \mathbb{C}G_2 \longrightarrow B$ be two $\mathbb{C}H$-algebra morphisms. Restricting to the group-elements gives us a commutative diagram of group morphisms

$$\begin{tikzcd}
 & G_1 \arrow[d, hook, "j_1"] \arrow[dr, "f_1"] & \\
H \arrow[ur, hook, "i_1"] \arrow[r, hook, "u"] \arrow[dr, hook, "i_2"'] & G_1 *_H G_2 \arrow[r, dotted, "g"] & B^* \\
 & G_2 \arrow[u, hook, "j_2"'] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

where the (uniquely determined) group morphism g exists by the universal property of $G_1 *_H G_2$. Linearly extending this group morphism gives a $\mathbb{C}H$-algebra morphism $\mathbb{C}G_1 *_H G_2 \longrightarrow B$ whence the group-algebra has the required universal property.

Theorem 2.11 *Let S be a semi-simple algebra and let (A_1, i_1) and (A_2, i_2) be two S-algebras which are smooth as $\mathbb{C}$-algebra. Then, the amalgamated free algebra product*

$$A_1 *_S A_2$$

(which exists!) is a smooth $\mathbb{C}$-algebra. In particular, the group-algebras of the arithmetic groups

$$\mathbb{C}PSL_2(\mathbb{Z}) \qquad \mathbb{C}SL_2(\mathbb{Z}) \qquad \textit{and} \qquad \mathbb{C}GL_2(\mathbb{Z})$$

are all smooth algebras, hence are the coordinate rings of noncommutative affine manifolds.

Proof. Let $I \triangleleft B$ be a nilpotent ideal and take an algebra map $A_1 *_S A_2 \xrightarrow{f} B/I$. Composing with the universal inclusions and using smoothness of the A_k we obtain lifted algebra maps

$$\begin{tikzcd} & & & B \arrow[d, "\pi"] \\ S \arrow[r, "i_k"] & A_k \arrow[urr, dotted, "g_k"] \arrow[r, "j_k"] & A_1 *_S A_2 \arrow[r, "f"] & B/I \end{tikzcd}$$

Hence, we have two $\mathbb{C}$-algebra lifts $g_1 \circ i_1$ and $g_2 \circ i_2$ from the semi-simple algebra $S \longrightarrow B$ lifting the morphism $f \circ j_1 \circ i_1 = f \circ j_2 \circ i_2$. Therefore, these two lifts are conjugated by a unit $b = 1 + i \in B^*$. But then we have a commutative diagram

$$\begin{tikzcd} & A_1 \arrow[d, hook, "j_1"] \arrow[dr, "g_1 \circ i_1"] & \\ S \arrow[ur, hook, "i_1"] \arrow[r, hook, "u"] \arrow[dr, hook, "i_2"'] & A_1 *_S A_2 \arrow[r, dotted] & B \\ & A_2 \arrow[u, hook', "j_2"] \arrow[ur, "b g_2 \circ i_2 b^{-1}"'] & \end{tikzcd}$$

whence the universal property of $A_1 *_S A_2$ provides us with the required lifted algebra map. The second statement follows from this using the fact that

$$\mathbb{C}PSL_2(\mathbb{Z}) \simeq \mathbb{C}\mathbb{Z}_2 * \mathbb{C}\mathbb{Z}_3 \qquad \mathbb{C}SL_2(\mathbb{Z}) \simeq \mathbb{C}\mathbb{Z}_4 *_{\mathbb{C}\mathbb{Z}_2} \mathbb{C}\mathbb{Z}_6 \qquad \mathbb{C}GL_2(\mathbb{Z}) \simeq \mathbb{C}D_4 *_{\mathbb{C}D_2} \mathbb{C}D_6$$

and the fact that semi-simple algebras are smooth algebras. $\square$

In particular, $\mathtt{rep}_n\ GL_2(\mathbb{Z})$ (and similarly for the other arithmetical groups) are all smooth varieties (which would be pretty hard to prove by hand). On the other hand, the group algebra $\mathbb{C}B_3$ is *not* a smooth algebra as one can verify by proving that $\mathtt{rep}_2\ B_3$ is not a smooth variety.

day 3

QUIVERS & EXAMPLES

As we have described the arithmetical groups $GL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})$ and $PSL_2(\mathbb{Z})$ as (amalgamated) free products of finite subgroups, restricting representations to these subgroups and applying the representation theory of finite groups gives us a handle on the representation theory of these infinite groups.

We will consider the easiest case, that of $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$, or more explicitly

$$PSL_2(\mathbb{Z}) = \langle\ \sigma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\ ,\ \tau = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\ |\ \sigma^2 = id = \tau^3\ \rangle$$

The character tables of the Abelian cyclic groups are easy to work out. In our case we have

$\mathbb{Z}_2$	$1a$	$1b$
S_1	1	1
S_2	1	-1

and

$\mathbb{Z}_3$	$1a$	$1b$	$1c$
T_1	1	1	1
T_2	1	ρ	ρ^2
T_3	1	ρ^2	ρ

where $\rho = e^{2\pi i/3}$. The S_i and T_j are all one-dimensional simple representations and we will use the same notation for the one-dimensional space having a $\mathbb{Z}_k$-action. If $\phi : PSL_2(\mathbb{Z}) \longrightarrow GL_n(\mathbb{C})$ is an n-dimensional representation of $PSL_2(\mathbb{Z})$ then the restrictions must be isomorphic to

$$V_\phi \downarrow_{\mathbb{Z}_2} = S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \qquad \text{and} \qquad V_\phi \downarrow_{\mathbb{Z}_3} = T_1^{\oplus b_1} \oplus T_2^{\oplus b_2} \oplus T_3^{\oplus b_3}$$

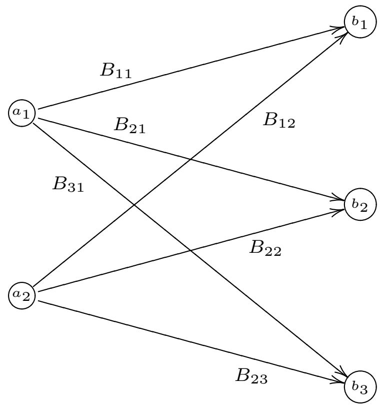
with a_i and b_j integers and clearly they have to satisfy

$$a_1 + a_2 = n = b_1 + b_2 + b_3$$

All this does is to divide the n-dimensional space $V_\phi = \mathbb{C}^n$ in two different ways : one time with respect to the eigenspaces of the order two operator σ and another time with respect to the eigenspaces of the rank three operator τ. If we take a basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of V_ϕ compatible with the first decomposition and a basis $\mathcal{F} = \{f_1, \ldots, f_n\}$ compatible with the second, then the base-change matrix can be decomposed into block matrices

$$\mathcal{E} \xrightarrow{B} \mathcal{F} \qquad \text{where} \qquad B = \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \\ \hline \end{array} \in GL_n(\mathbb{C})$$

where the block B_{ij} has sizes $b_i \times a_j$. This information can be encoded into the *quiver-representation* of *dimension-vector* $\alpha = (a_1, a_2; b_1, b_2, b_3)$ depicted by

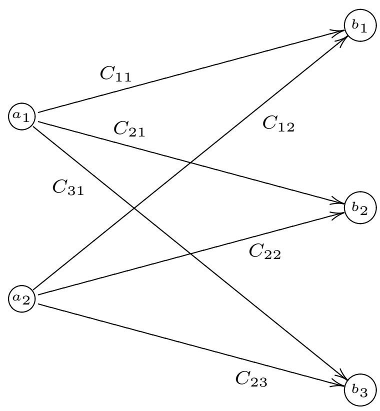


By this we mean that to each *vertex* of the quiver corresponds a vector space of dimension the indicated component of the dimension vector (in our case, these vector spaces are the eigenspaces, those of σ to the left, those of τ to the right) and to each *arrow* of the quiver corresponds a linear map from the starting-vertex space to the end-vertex space (in our case these are the different blocks in the base-change matrix B).

Conversely, to a representation of this quiver of dimension vector $\alpha = (a_1, a_2; b_1, b_2, b_3)$ such that $a_1 + a_2 = n = b_1 + b_2 + b_3$ such that the matrix B constructed from the arrow maps in an invertible $n \times n$ matrix, we can associate the n-dimensional representation

$$PSL_2(\mathbb{Z}) \xrightarrow{\phi} GL_n(\mathbb{C}) \qquad \sigma \mapsto \begin{bmatrix} 1_{a_1} & 0 \\ 0 & -1_{a_2} \end{bmatrix} \qquad \tau \mapsto B^{-1} \begin{bmatrix} 1_{b_1} & 0 & 0 \\ 0 & \rho 1_{b_2} & 0 \\ 0 & 0 & \rho^2 1_{b_3} \end{bmatrix} B$$

If $PSL_2(\mathbb{Z}) \xrightarrow{\psi} GL_n(\mathbb{C})$ is a representation isomorphic to ϕ then clearly the two eigen-*space* decompositions of $\mathbb{C}^n$ are the same and hence the numbers a_i and b_j are the same (isomorphic just means that the images of σ and τ are computed with respect to a different basis of $V_\phi = \mathbb{C}^n = V_\psi$) but possibly we have to choose a different basis in each of the eigen-spaces to get $\psi(\sigma)$ and $\psi(\tau)$ into the matrix-form corresponding to a quiver representation



This means that we have 'little' base change matrices in the eigenspaces

$$X_i \in GL_{a_i}(\mathbb{C}) \qquad \text{and} \qquad Y_j \in GL_{b_j}(\mathbb{C})$$

such that the 'big' base change matrices associated to ϕ and ψ are related by

$$\begin{array}{|c|c|} \hline C_{11} & C_{12} \\ \hline C_{21} & C_{22} \\ \hline C_{31} & C_{32} \\ \hline \end{array} = \begin{bmatrix} Y_1^{-1} & 0 & 0 \\ 0 & Y_2^{-1} & 0 \\ 0 & 0 & Y_3^{-1} \end{bmatrix} \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \\ \hline \end{array} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

We will see today that this is exactly the base change action on quiver representations.

3.1 Quiver representations

A finite *quiver* Q is a directed graph having

- k vertices $\{v_1, \ldots, v_k\}$
- l directed arrows $v_i \longrightarrow v_j$ having a starting vertex v_i and ending vertex v_j where we allow loops (that is, v_i and v_j may be the same vertex).

This directed graph can be encoded by a matrix in $\chi_Q \in M_k(\mathbb{Z})$ or by the *Euler* bilinear form it defines where

$$\chi_Q = \begin{bmatrix} \chi_{11} & \ldots & \chi_{1k} \\ \vdots & & \vdots \\ \chi_{k1} & \ldots & \chi_{kk} \end{bmatrix} \qquad \mathbb{Z}^k \times \mathbb{Z}^k \xrightarrow{\chi_Q} \mathbb{Z} \quad \chi_Q(v, w) = v \chi_Q w^{tr}$$

and with $\chi_{ij} = \delta_{ij} - \#\{a : v_i \xrightarrow{a} v_j \}$. A path of length z is an orientation preserving walk along z arrows

$$v_{i_z} \longleftarrow \qquad \cdots \longleftarrow v_{i_2} \longleftarrow v_{i_1} \longleftarrow v_{i_0}$$

and we include k paths of length zero which correspond to the vertices.

To such a quiver Q we associate its *path algebra* $\mathbb{C}Q$ which is a vector space having as basis all paths in the quiver Q and where multiplication is induced by concatenation of paths. That is, $\mathbb{C}Q$ is an affine algebra generated by $k + l$-elements : $\{e_1, \ldots, e_k\}$ corresponding to the paths of length zero (the vertices) and which satisfy the relations

$$e_i e_j = \delta_{ij} e_i \qquad e_1 + \ldots + e_k = 1$$

so they form a complete set of orthogonal idempotents in $\mathbb{C}Q$, and with $\{a_1, \ldots, a_l\}$generators corresponding to the paths of length one (the arrows). If a and a' are the arrows

$$v_a \xleftarrow[a]{} v_b \qquad \text{and} \qquad v_c \xleftarrow[a']{} d$$

then we have the following relations

$$v_i a = \delta_{ia} a \quad a v_j = \delta_{jb} a \quad v_i a' = \delta_{ic} a' \quad a' v_j = \delta_{jd} a' \quad a a' = \delta_{cb} p \quad a' a = \delta_{ad} q$$

where p (resp. q) is the path of length two, which exists only if $v_c = v_b$ (resp. if $v_a = v_d$)

$$\textcircled{v_a} \xleftarrow[a]{} \textcircled{v_b} \xleftarrow[a']{} \textcircled{v_d} \qquad \text{resp.} \qquad \textcircled{v_c} \xleftarrow[a']{} \textcircled{v_d} \xleftarrow[a]{} \textcircled{v_b}$$

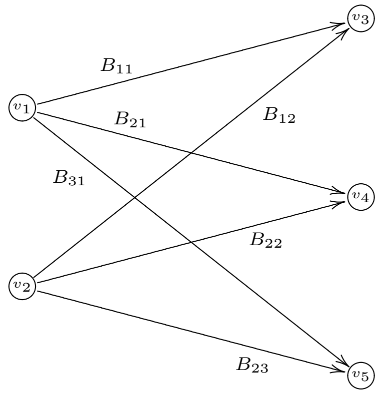
If Q has no oriented cycles (that is a path having the same beginning and ending vertex) then the path algebra $\mathbb{C}Q$ is finite dimensional. For example,

$$Q = \textcircled{v_3} \xleftarrow[b]{} \textcircled{v_2} \xleftarrow[a]{} \textcircled{v_1} \qquad \text{then} \qquad \mathbb{C}Q \simeq \begin{bmatrix} \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \end{bmatrix}$$

where the correspondence between paths and basis vectors is indicated by

$$\begin{bmatrix} e_1 & 0 & 0 \\ a & e_2 & 0 \\ ba & b & e_3 \end{bmatrix}$$

As another example, consider the quiver we encountered in the investigation of representations of $PSL_2(\mathbb{Z})$.



then we obtain as the path algebra $\mathbb{C}Q$ the 11-dimensional algebra (with correspondence indicated)

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & \mathbb{C} \end{bmatrix} \qquad \begin{bmatrix} e_1 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 \\ B_{11} & B_{12} & e_3 & 0 & 0 \\ B_{21} & B_{22} & 0 & e_4 & 0 \\ B_{31} & B_{32} & 0 & 0 & e_5 \end{bmatrix}$$

Theorem 3.1 *If $\mathbb{C}Q$ is a finite quiver, then its path algebra $\mathbb{C}Q$ is a smooth algebra.*

Proof. We have to lift an algebra morphism $\mathbb{C}Q \xrightarrow{f} B/I$ through the nilpotent ideal I. Consider the subalgebra $\mathbb{C}V = \mathbb{C} \times \ldots \times \mathbb{C}$ (k copies) generated by the vertex idempotent e_i. As this is a semi-simple algebra the map $f \circ i$ lifts to an algebra map F

$$\begin{tikzcd} & & B \arrow[d, "\pi"] \\ \mathbb{C}V \arrow[r, "i"] \arrow[urr, "F"] & \mathbb{C}Q \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & B/I \end{tikzcd}$$

Let b_a be any element of B mapping onto $f(a)$ where a is an arrow $\textcircled{v_i} \longrightarrow \textcircled{v_j}$ then we can define a map $\tilde{f}$ mapping a to $F(v_j)b_aF(v_i)$ and one verifies that these images satisfy all defining equations in $\mathbb{C}Q$ whence $\tilde{f}$ is the required algebra lift. $\square$

Next, let us study the representation theory of $\mathbb{C}Q$. If $V = \mathbb{C}^n$ is an n-dimensional left $\mathbb{C}Q$-module, we can use the vertex-idempotents e_i to decompose V into subspaces

$$V = e_1V \oplus e_2V \oplus \ldots \oplus e_kV$$

and if we denote $dim_{\mathbb{C}} e_iV = a_i$ we see that every n-dimensional representation of $\mathbb{C}Q$ determines a ***dimension vector*** $\alpha = (a_1, \ldots, a_k)$ such that the total dimension $|\alpha| = a_1 + \ldots + a_k = n$.

As for the action of an arrow $\textcircled{v_i} \xrightarrow{a} \textcircled{v_j}$ on V we use the fact that $a = e_jae_i$ to see that the action is the zero map on all components e_xV with $x \neq i$ and that the ae_iV is contained in the component e_jV. That is, the action of a on V is given by a $a_j \times a_i$-matrix representing a linear map $e_iV \longrightarrow e_jV$.

That is, to any n-dimensional representation of dimension vector α we can associate an α-dimensional quiver representation by assigning to the vertex v_i the vector space e_iV and to each arrow a the matrix representing the linear map $e_iV \longrightarrow e_jV$ describing the action of a on V. Conversely, a quiver-representation of dimension vector α determines an $n = |\alpha|$-dimensional representation of $\mathbb{C}Q$ by taking as the images of the vertex-idempotents e_i and the arrows a the $n \times n$-matrices

$$e_i \mapsto \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 1_{a_i} & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} \qquad a \mapsto \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & M_a & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

where the block matrix at block-position (j, i) has sizes $a_j \times a_i$. Fixing basis vectors in each of the vertex spaces e_iV we can conjugate any representation of $\mathbb{C}Q$ into such a standard quiver-representation form and two such quiver-representations determine isomorphic $\mathbb{C}Q$-representations if they can be conjugated by an element of the *vertex base change group*

$$GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k} = \{ \begin{bmatrix} g_1 & & & & & & \\ & \ddots & & & & & \\ & & g_j & & & & \\ & & & \ddots & & & \\ & & & & g_i & & \\ & & & & & \ddots & \\ & & & & & & g_k \end{bmatrix} \mid g_i \in GL_{a_i} \ \}$$

Under this basechange, the e_i are mapped to the same matrix-idempotents of rank a_i and the arrow a is mapped to

$$a \mapsto \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & g_j^{-1} M_a g_i & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

Two quiver-representations which transform into each other in this way under the action of the base-change group $GL(\alpha)$ are said to be *isomorphic*. Hence, every n-dimensional representation of the path algebra $\mathbb{C}Q$ determines an α-dimensional quiver-representation of some dimension vector α with total dimension $|\alpha| = n$ and the two notions of isomorphisms are compatible.

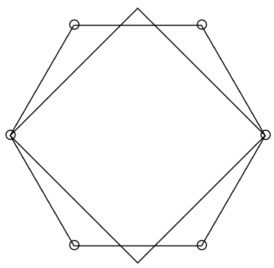
3.2 Quiver examples

Recall that we encountered the vertex-basechange group action on quiver-representations already before in the investigation of representations of $PSL_2(\mathbb{Z})$ in terms of the quiver with the 11-dimensional path algebra. Reinterpreting this we have :

Theorem 3.2 *The study of the isomorphism problem of finite dimensional representations of the modular group $PSL_2(\mathbb{Z})$ can be reduced to that of* certain *finite dimensional representations of the* 11*-dimensional algebra*

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & \mathbb{C} \end{bmatrix}$$

Do we have a similar result for the more complicated arithmetical groups $SL_2(\mathbb{Z})$ and $GL_2(\mathbb{Z})$? Take $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. As all occurring groups are Abelian, all their simple representations are one-dimensional and the character tables are easy to work out. We will need the restrictions of $\mathbb{Z}_4$ resp. $\mathbb{Z}_6$-representations to the common subgroup $\mathbb{Z}_2$, so it is best to consider a hexagon and a square lined up so that they share two vertices :



Let $\mathbb{Z}_4 = \langle u | u^4 = 1 \rangle$ be rotation over 90^o, $Z_6 = \langle v | v^6 = 1 \rangle$ rotation over 60^o and let the common subgroup $\mathbb{Z}^2 = \langle c | c^2 = 1 \rangle$ where c is reflection over the central point. With these conventions we have the following character-tables and restriction data

$\mathbb{Z}_2$	1	c
Z_1	1	1
Z_2	1	-1

$\mathbb{Z}_4$	$\boxed{1}$	v	$\boxed{c}$	v^3
X_1	1	1	1	1
X_2	1	i	-1	$-i$
X_3	1	-1	1	-1
X_4	1	$-i$	-1	i

$\mathbb{Z}_6$	$\boxed{1}$	v	v^2	$\boxed{c}$	v^4	v^5
Y_1	1	1	1	1	1	1
Y_2	1	ρ	ρ^2	-1	ρ^4	ρ^5
Y_3	1	ρ^2	ρ^4	1	ρ^2	ρ^4
Y_4	1	-1	1	-1	1	-1
Y_5	1	ρ^4	ρ^2	1	ρ^4	ρ^2
Y_6	1	ρ^5	ρ^4	-1	ρ^2	ρ

To determine the restrictions $X_i \downarrow_{\mathbb{Z}_2}$ and $Y_j \downarrow_{\mathbb{Z}_2}$ we only have to consider the boxed columns. We obtain

$$\begin{cases} X_1 \downarrow_{\mathbb{Z}_2} & = Z_1 \\ X_2 \downarrow_{\mathbb{Z}_2} & = Z_2 \\ X_3 \downarrow_{\mathbb{Z}_2} & = Z_1 \\ X_4 \downarrow_{\mathbb{Z}_2} & = Z_2 \end{cases} \qquad \begin{cases} Y_1 \downarrow_{\mathbb{Z}_2} & = Z_1 \\ Y_2 \downarrow_{\mathbb{Z}_2} & = Z_2 \\ Y_3 \downarrow_{\mathbb{Z}_2} & = Z_1 \\ Y_4 \downarrow_{\mathbb{Z}_2} & = Z_2 \\ Y_5 \downarrow_{\mathbb{Z}_2} & = Z_1 \\ Y_6 \downarrow_{\mathbb{Z}_2} & = Z_2 \end{cases}$$

If V is an n-dimensional $SL_2(\mathbb{Z})$-representation, the restrictions to the subgroups $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are of the form

$$V \downarrow_{\mathbb{Z}_4} = X_1^{\oplus a_1} \oplus X_2^{\oplus a_2} \oplus X_3^{\oplus a_3} \oplus X_4^{\oplus a_4} \quad \text{and} \quad V \downarrow_{\mathbb{Z}_6} = Y_1^{\oplus b_1} \oplus \ldots \oplus Y_6^{\oplus b_6}$$

giving a dimension vector $\alpha = (a_1, \ldots, a_4, b_1, \ldots, b_6)$ satisfying $\sum_i a_i = n = \sum_j b_j$. However, this time not all of these dimension vectors can occur as we must have that

$$(V \downarrow_{\mathbb{Z}_4}) \downarrow_{\mathbb{Z}_2} = V \downarrow_{\mathbb{Z}_2} = (V \downarrow_{\mathbb{Z}_6}) \downarrow_{\mathbb{Z}_2}$$

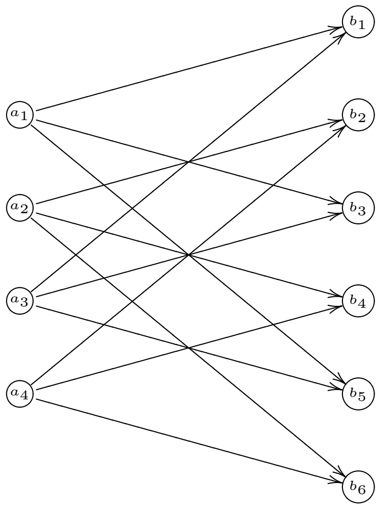
whence we must have that $a_1 + a_3 = p = b_1 + b_3 + b_5$, $a_2 + a_4 = q = b_2 + b_4 + b_6$ and $p + q = n$. Moreover, the base-change between a basis compatible with the decomposition in the $\mathbb{Z}_4$-restriction and a basis compatible with the $\mathbb{Z}_6$-restriction

$$\mathcal{E} = \{e_1, \ldots, e_{a_1}, \ldots, e_n\} \xrightarrow{B} \mathcal{F} = \{f_1, \ldots, f_{b_1}, \ldots, b_n\}$$

must be an isomorphism of $\mathbb{Z}_2$-representations, so B can only have non-zero entries at places where the corresponding left and right factors are the same $\mathbb{Z}_2$-representation. That is, B is an invertible $n \times n$ matrix with the following (checker-board) block-decomposition

$$B = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 \\ 0 & B_{21} & 0 & B_{24} \\ B_{31} & 00 & B_{33} & 0 \\ 0 & B_{42} & 0 & B_{44} \\ B_{51} & 0 & B_{53} & 0 \\ 0 & B_{62} & 0 & B_{64} \end{bmatrix}$$

That is, we have associated to an n-dimensional $SL_2(\mathbb{Z})$-representation an α-dimensional representation of the quiver



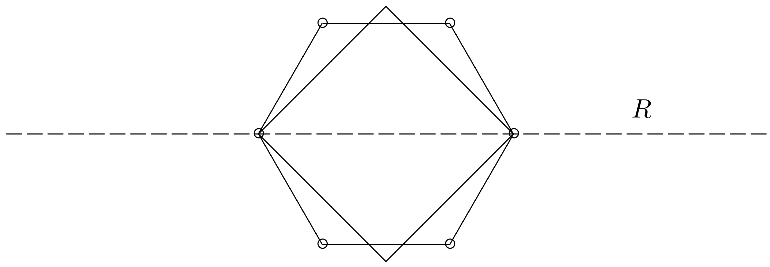
which is the disjoint union of two copies of the quiver constructed for $PSL_2(\mathbb{Z})$. Conversely, to any α-dimensional representation of this quiver (such that the $n \times n$ matrix B constructed as before from the arrow-matrix blocks is invertible) one associates an n-dimensional representation of $SL_2(\mathbb{Z}) = \langle U, V \mid U^2 = V^3, U^4 = 1 \rangle$

$$\begin{cases} U \mapsto \begin{bmatrix} 1_{a_1} & 0 & 0 & 0 \\ 0 & i1_{a_2} & 0 & 0 \\ 0 & 0 & -1_{a_3} & 0 \\ 0 & 0 & 0 & -i1_{a_4} \end{bmatrix} \\ V \mapsto B^{-1} \begin{bmatrix} 1_{b_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho 1_{b_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho^2 1_{b_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_{b_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^4 1_{b_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho^5 1_{b_6} \end{bmatrix} B \end{cases}$$

Theorem 3.3 *The study of the isomorphism problem of finite dimensional representations of $SL_2(\mathbb{Z})$ can be reduced to that of* certain *finite dimensional representations of the 22-dimensional algebra*

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & \mathbb{C} \end{bmatrix} \oplus \begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & \mathbb{C} \end{bmatrix}$$

For $GL_2(\mathbb{Z}) = D_6 *_{D_2} D_4$ we line up as before two vertices of a hexagon (having symmetry group D_6) and of a square (having symmetry group D_4).



Observe that D_4 is generated by U (rotation by 90^o) and R (symmetry along the line) and D_4 has 5 conjugacy classes : $\{1, R, R', U, C\}$ where C is the central point reflection (which is also U^2) and R' is reflection along a line through two midpoints of edges of the square. D_6 is generated by V (rotation over 60^o) and R and has 6 conjugacy classes " $\{1, R, R", V, V^2, C\}$ where R" is a reflection along a line through two midpoints of the hexagon.

The common subgroup D_2 is generated by C and R and is Klein's Vierergruppe having elements (=conjugacy classes) $\{1, C, R, CR\}$. CR viewed as a symmetry of the square is reflection along a line through two vertices, so belongs to the conjugacy class of R and therefore we have the following character tables and restriction data.

D_2	1	C	R	CR
Z_1	1	1	1	1
Z_2	1	-1	-1	1
Z_3	1	-1	1	-1
Z_4	1	1	-1	-1

D_4	1	R	R'	U	C
X_1	1	1	1	1	1
X_2	1	-1	-1	1	1
X_3	1	-1	1	-1	1
X_4	1	1	-1	-1	1
X_5	2	0	0	0	-2

$D_4 \downarrow_{D_2}$	1	C	R	R
$X_1 \downarrow_{D_2}$	1	1	1	1
$X_2 \downarrow_{D_2}$	1	1	-1	-1
$X_3 \downarrow_{D_2}$	1	1	1	1
$X_4 \downarrow_{D_2}$	1	1	-1	-1
$X_5 \downarrow_{D_2}$	2	-2	0	0

whence

$$X_1 \downarrow_{D_2} = Z_1 \quad X_2 \downarrow_{D_2} = Z_4 \quad X_3 \downarrow_{D_2} = Z_1 \quad X_4 \downarrow_{D_2} = Z_4 \quad X_5 \downarrow_{D_2} = Z_2 \oplus Z_3$$

CR when viewed as a symmetry of the hexagon is a reflection along a line through two midpoints of edges and hence belongs to the conjugacy class R". As we have given the character table of D_6 before we give only the restriction data

$D_6 \downarrow_{D_2}$	1	C	R	R"
$Y_1 \downarrow_{D_2}$	1	1	1	1
$Y_2 \downarrow_{D_2}$	1	1	-1	-1
$Y_3 \downarrow_{D_2}$	1	-1	-1	1
$Y_4 \downarrow_{D_2}$	1	-1	1	-1
$Y_5 \downarrow_{D_2}$	2	-2	0	0
$Y_6 \downarrow_{D_2}$	2	2	0	0

whence

$$Y_1 \downarrow_{D_2} = Z_1 \quad Y_2 \downarrow_{D_2} = Z_4 \quad Y_3 \downarrow_{D_2} = Z_2$$
$$Y_4 \downarrow_{D_2} = Z_3 \quad Y_5 \downarrow_{D_2} = Z_2 \oplus Z_3 \quad Y_6 \downarrow_{D_2} = Z_1 \oplus Z_4$$

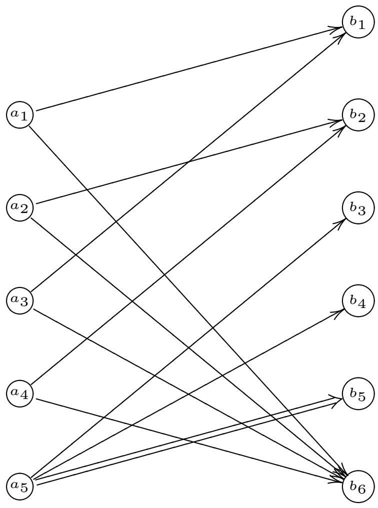
An n-dimensional representation V of $GL_2(\mathbb{Z})$ can be restricted to the finite subgroups D_4 and D_6 giving decompositions

$$V \downarrow_{D_4} = X_1^{\oplus a_1} \oplus \ldots \oplus X_5^{\oplus a_5} \quad \text{and} \quad V \downarrow_{D_6} = Y_1^{\oplus b_1} \oplus \ldots \oplus Y_6^{\oplus b_6}$$

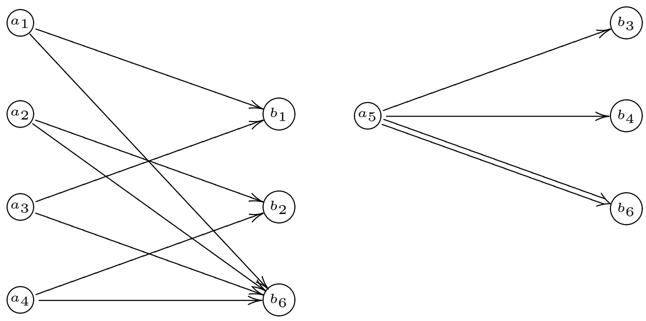
giving a dimension vector $\alpha = (a_1, \ldots, a_5, b_1, \ldots, b_6)$ this time satisfying (using the facts that the dimensions of the simples X_5, Y_5 and Y_6 is two)

$$a_1 + a_2 + a_3 + a_4 + 2a_5 = n = b_1 + b_2 + b_3 + b_4 + 2b_5 + 2b_6$$

Again, the basechange matrix between D_2-bases of $V \downarrow_{D_4}$ and $V \downarrow_{D_6}$ must be a D_2-isomorphism, leading to an α-dimensional representation of the quiver



This quiver is the disjoint union of two connected components



Theorem 3.4 *The study of the isomorphism problem of finite dimensional representations of $GL_2(\mathbb{Z})$ can be reduced to that of* certain *finite dimensional representations of*

the 23*-dimensional algebra*

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{C} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 & \mathbb{C} & 0 & 0 \\ 0 & \mathbb{C} & 0 & \mathbb{C} & 0 & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 & \mathbb{C} \end{bmatrix} \oplus \begin{bmatrix} \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 \\ \mathbb{C}x + \mathbb{C}y & 0 & 0 & \mathbb{C} \end{bmatrix}$$

day 4

MANIFOLDS & COMPACTIFICATIONS

So far, we have viewed the non-commutative affine scheme $\mathtt{rep}\ A = \bigsqcup_n \mathtt{rep}_n\ A$ only as an Abelian category or, at best, as the disjoint union of the family of commutative GL_n-schemes $\mathtt{rep}_n\ A$. Today, we will define a topology (actually two topologies) on $\mathtt{rep}\ A$. This will then allow us to define more general non-commutative varieties and manifolds by *gluing* affine pieces together. We will apply this idea to get a natural compactification of $\mathtt{rep}\ PSL_2(\mathbb{Z})$ (and of the other arithmetical groups) using their associated finite dimensional path algebras we constructed last time. Then, we will outline the construction of a truly non-commutative topology on $\mathtt{rep}\ A$.

4.1 Commutative topologies

Let A be a $\mathbb{C}$-algebra and take a finite set $\Delta = \{\delta_1, \ldots, \delta_k\}$ of A-module morphisms between projective left A-modules

$$P_i \xrightarrow{\delta_i} Q_i \qquad 1 \leq i \leq k$$

The *universal localization* A_Δ is the algebra $A \xrightarrow{j_\Delta} A_\Delta$ which has the following universal property. All extended morphisms $A_\Delta \otimes \delta_i$ are isomorphisms of left projective A_Δ-modules and if there is an A-algebra $A \longrightarrow B$ such that all the extended morphisms $B \otimes \delta_i$ are isomorphisms, then there is an algebra map completing the commutative diagram

$$\begin{tikzcd} A \arrow[d, "j_\Delta"'] \arrow[dr] & \\ A_\Delta \arrow[r, dotted] & B \end{tikzcd}$$

Restricting finite dimensional representations gives natural maps

$$i_\Delta\ :\ \mathtt{rep}\ A_\Delta \longrightarrow \mathtt{rep}\ A$$

and we denote the image of i_Δ by $\mathbb{X}(\Delta)$. From the universal property of universal localizations, it follows that

$$A_{\Delta_1 \cup \Delta_2} = (A_{\Delta_i})_{\Delta_j} \qquad \text{and} \qquad \mathbb{X}(\Delta_1) \cap \mathbb{X}(\Delta_2) = \mathbb{X}(\Delta_1 \cup \Delta_2)$$

whence we can view the sets $\mathbb{X}(\Delta)$ as the basic open sets of a topology on $\mathtt{rep}\ A$.

Definition 4.1 The topology on $\texttt{rep}\ A$ generated by the basic open sets $\mathbb{X}(\Delta)$ will be called the *non-commutative Zariski topology* on the non-commutative affine scheme $\texttt{rep}\ A$.

Having associated to an affine noncommutative $\mathbb{C}$-algebra its *noncommutative affine scheme* $\texttt{rep}\ A$ which is an Abelian category equipped with a topology, we can 'glue' these schemes together to construct more general noncommutative schemes (and non-commutative manifolds).

Definition 4.2 An *aggregate* $\texttt{agg}$ is an Abelian $\mathbb{C}$-category (meaning that all objects are $\mathbb{C}$-vector spaces and all morphisms are $\mathbb{C}$-linear maps) having the following properties :

1. $\texttt{agg}$ is additive, that is for any two objects the direct sum exists in $\texttt{agg}$.
2. $\texttt{agg}$ is Krull-Schmidt, that is, for any *indecomposable* object V (that is, one which cannot be written as a direct sum of proper subobjects) the endomorphism algebra $End(V) = Hom_{\texttt{agg}}(V,V)$ is a local $\mathbb{C}$-algebra (that is, the non-units form a twosided ideal).
3. $\texttt{agg}$ is hom-finite, that is all homomorphism spaces $Hom_{\texttt{alg}}(V,W)$ are finite dimensional $\mathbb{C}$-vector spaces.

Observe that for any noncommutative algebra A the category $\texttt{rep}\ A$ is an aggregate. The Krull-Schmidt property implies that any object V of an aggregate $\texttt{agg}$ can be written as a direct sum of indecomposable objects

$$V \simeq W_1^{\oplus e_1} \oplus \ldots \oplus W_k^{\oplus e_k} \qquad \text{with } W_i \text{ indecomposable}$$

and that this decomposition is unique up to isomorphism.

Definition 4.3 A *noncommutative scheme* is an aggregate $\texttt{agg}$ equipped with a topology such that there are open subsets $\{U_i\}$ satisfying the following properties

1. U_i is an Abelian subcategory of $\texttt{agg}$
2. U_i is equivalent and homeomorphic to $\texttt{rep}\ A_i$ for some affine non-commutative algebra A_i.
3. For all i,j the intersection $U_i \cap U_j$ is equivalent and homeomorphic to $\texttt{rep}\ A_{ij}$ and the inclusion maps

$$\begin{tikzcd}
U_{ij} = \texttt{rep}\ A_{ij} \arrow[r, hook] \arrow[d, hook] & U_i = \texttt{rep}\ A_i \\
U_j = \texttt{rep}\ A_j &
\end{tikzcd}$$

are induced by algebra morphisms

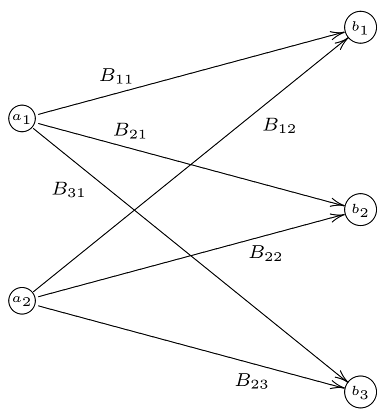
$$\begin{tikzcd} A_{ij} & A_i \arrow[l] \\ A_j \arrow[u] & \end{tikzcd}$$

If all A_i and A_{ij} are noncommutative smooth $\mathbb{C}$-algebras, then the noncommutative scheme `agg` is called a noncommutative manifold.

In order to give an example of a non-commutative manifold, let us construct a natural compactification $\overline{PSL_2(\mathbb{Z})}$ of `rep` $PSL_2(\mathbb{Z})$. We will see that similar constructions also work for the other arithmetical groups.

With Q we will denote the quiver we have associated to $PSL_2(\mathbb{Z})$, that is

Q :



Clearly, `rep` $\mathbb{C}Q$ is an affine noncommutative manifold with `comp` $\mathbb{C}Q = \mathbb{Z}^5$ and every dimension vector $\beta \in \mathbb{Z}^5$ determines a component isomorphic to $GL_{|\beta|} \times^{GL(\beta)}$ $\mathtt{rep}_\beta\, Q$. In the study of $PSL_2(\mathbb{Z})$-representations we were interested in the subset of dimension vectors $\alpha = (a_1, a_2, b_1, b_2, b_3)$ satisfying the numerical restriction

$$a_1 + a_2 = b_1 + b_2 + b_3$$

An equivalent way to describe this is as follows : let $\theta = (-1, -1, 1, 1, 1) \in \mathbb{Z}^5$ then α satisfies the restriction if and only if $\theta.\alpha = 0$. θ corresponds to a *character* of $GL(\alpha)$ namely

$$\chi_\theta \;:\; GL(\alpha) = GL_{a_1} \times \ldots \times GL_{b_3} \longrightarrow \mathbb{C}^* \qquad (g_1, g_2, g_3, g_4, g_5) \mapsto det(g_1 g_2)^{-1} det(g_3 g_4 g_5)$$

Such characters allow us to define a *stability structure* on `rep` Q. If $V \in$ `rep` Q is a representation with dimension vector β we will denote $dim(V) = \beta$.

Definition 4.4 Let $\alpha \in \mathbb{Z}^5$ such that $\theta.\alpha = 0$. An α-dimensional representation $V \in$ $\mathtt{rep}_\alpha\, Q$ is said to be

1. θ-semistable if and only if for all subrepresentations $W \subset V$ we have $\theta.dim(W) \geq 0$.

2. θ-stable if and only if for all *proper* subrepresentations $0 \neq W \subsetneq V$ we have $\theta.dim(W) > 0$.

We will denote the subset of all θ-semistable representations of the quiver Q by $\mathtt{rep}^\theta\ Q$ or of the path algebra by $\mathtt{rep}^\theta\ \mathbb{C}Q$.

It is an easy verification that $\mathtt{rep}^\theta\ Q$ (using the fact that the dimension vector is additive on short exact sequences) is an Abelian subcategory of $\mathtt{rep}\ Q$, that is, the kernel and cokernel of maps between two θ-semistable representations is again θ-semistable. In particular, $\mathtt{rep}^\theta\ Q$ and hence also $\mathtt{rep}^\theta\ \mathbb{C}Q$ is an aggregate. In fact, we claim that

$$\mathtt{rep}^\theta\ \mathbb{C}Q = \overline{PSL_2(\mathbb{Z})}$$

a natural compactification of $\mathtt{rep}\ PSL_2(\mathbb{Z})$.

So, let us begin to relate the representation theory of $PSL_2(\mathbb{Z})$ with this particular stability structure.

Theorem 4.5 *For $\theta = (-1,-1,1,1,1)$ and $V \in \mathtt{rep}_\alpha\ Q$ a representation corresponding to a $PSL_2(\mathbb{Z})$-representation, then*

1. *V is θ-semistable, and*

2. *V is θ stable if V determines a simple $PSL_2(\mathbb{Z})$-representation.*

Proof. As $V \in \mathtt{rep}\ PSL_2(\mathbb{Z})$ we know already that $\theta.dim(V) = 0$. Let the matrices of V be denoted by $(B_{11}, \ldots, B_{32})$ and the vertex-spaces by $V_1, \ldots, V_5$. If W is a subrepresentation of V of dimension vector $\beta = (c_1, c_2, d_2, d_2, d_3)$ and associated matrices $(C_{11}, \ldots, C_{32})$ then W being a subrepresentation means that the diagram below is commutative

$$\begin{tikzcd}
V_1 \oplus V_2 \arrow[rr, "\phi = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}"] & & V_3 \oplus V_4 \oplus V_5 \\
W_1 \oplus W_2 \arrow[u, hook] \arrow[rr, "\phi|_W = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}"'] & & W_3 \oplus W_4 \oplus W_5 \arrow[u, hook]
\end{tikzcd}$$

Now, assume $\theta.dim(W) < 0$ this means that $dim_{\mathbb{C}} W_1 \oplus W_2 > dim_{\mathbb{C}} W_3 \oplus W_4 \oplus W_5$ whence $\phi|_W$ must have a kernel. But this is impossible as ϕ is a linear isomorphism because the matrix is invertible (because $V \in \mathtt{rep}\ PSL_2(\mathbb{Z})$. This proves (1). As for (2) it follows from the above argument that every quiver-subrepresentation $W \subset V$ must satisfy $\theta.dim(W) \geq 0$ and W represents a $PSL_2(\mathbb{Z})$-subrepresentation of V if and only if $\theta.dim(W) = 0$. Hence, V is θ-stable iff V is a simple $PSL_2(\mathbb{Z})$-representation. $\square$

An element $(g_1, \ldots, g_5) \in GL(\alpha)$ acts on V and hence on the matrix-components B_{ij} so that the composite matrix is mapped to

$$B' = \begin{bmatrix} g_3^{-1}B_{11}g_1 & g_3^{-1}B_{12}g_2 \\ g_4^{-1}B_{21}g_1 & g_4^{-1}B_{22}g_2 \\ g_5^{-1}B_{31}g_1 & g_5^{-1}B_{32}g_2 \end{bmatrix} = \begin{bmatrix} g_3^{-1} & 0 & 0 \\ 0 & g_4^{-1} & 0 \\ 0 & 0 & g_5^{-1} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

whence

$$det(B') = det(g_1g_2)det(g_3^{-1}g_4^{-1}g_5^{-1})det(B) = \chi_\theta^{-1}(g_1, \ldots, g_5)det(B)$$

We say that $det(B)$ is a polynomial *θ-semi invariant* of weight -1. More generally,

Definition 4.6 A polynomial function f on $\mathtt{rep}_\alpha\ Q$ is said to be a θ-semi invariant of weight $-l$ if and only if

$$g.f = \chi_\theta^{-l} f \qquad \forall g \in GL(\alpha)$$

The ring a θ-semi invariants is the positively graded subalgebra of $\mathbb{C}[\mathtt{rep}_\alpha\ Q]$

$$R_\alpha^\theta = \oplus_{l \leq 0} R_{-l} = \oplus_{l \leq 0} \{ f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q \mid g.f = \chi_\theta^{-l} f\ \forall g \in GL(\alpha)\ \}$$

Observe that R_0 is the ring of polynomial invariants which is known (in general) to be generated by traces along oriented in the quiver, but as there are no such cycles in Q we have that $R_0 = \mathbb{C}$ and hence

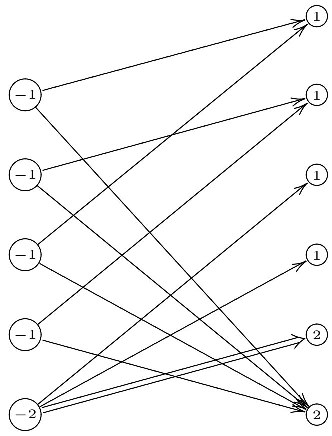
$$\mathtt{proj}\ R_\alpha^\theta$$

is a projective variety. In fact, one can show that the points of $\mathtt{proj}\ R_\alpha^\theta$ correspond to isoclasses of direct sums of θ-stable representations of total dimension α.

The last fact clarifies why we say that $\mathtt{rep}^\theta\ Q$ is a noncommutative projective variety because the best algebraic solution to classifying its isomorphism classes is given by the family of projective varieties $\mathtt{proj}\ R_\alpha^\theta$. In fact we have a characterization of the representation theoretic notion of θ-semistability in terms of these semi-invariants : the following are equivalent

- $V \in \mathtt{rep}_\alpha\ Q$ is θ-semistable
- there is a θ-semi invariant $f \in R_\alpha^\theta$ such that $f(V) \neq 0$.

So, in order to study $\mathtt{rep}^\theta\ Q$ we have to know a generating set for θ-semi invariants. Such a set is given by *determinantal semi invariants*. We will define them here in the special case of a *bipartite quiver* Q and a stability structure θ having all its left components strict negative and all its right components strict positive. From the discussion above we see that $PSL_2(\mathbb{Z})$ corresponds to such a setting, hence also $SL_2(\mathbb{Z})$ (as the quiver is just two copies of Q) but also $GL_2(\mathbb{Z})$ is such a setting with quiver and stability structure θ depicted by



For such a setting (Q, θ) with Q having l left vertices with θ-components $(-t_1, \ldots, -t_l)$ and r right vertices with θ-components $(s_1, \ldots, s_r)$ we construct a θ-semi invariant of weight $-a$ by taking a matrix with block decomposition

$$\Delta = \begin{bmatrix} \boxed{A_{11}} & \ldots & \boxed{A_{l1}} \\ \vdots & & \vdots \\ \boxed{A_{r1}} & \ldots & \boxed{A_{rl}} \end{bmatrix}$$

where the block A_{ij} has sizes $at_j \times as_i$ and all its entries are linear combinations of arrows in the quiver Q from the i-th vertex on the left to the j-th vertex on the right. If α is a dimension vector of Q such that $\theta.\alpha = 0$ we can evaluate Δ in every representation $V \in \mathtt{rep}_\alpha\ Q$ and the matrix obtained $\Delta(V)$ becomes a square matrix whence the determinant $det(\Delta(V))$ is a polynomial function on $\mathtt{rep}_\alpha\ Q$ which is verified to be a θ-semi invariant of weight $-a$. One can prove that these determinantal semi-invariants generate all!

To a block-matrix Δ as above we will associate a noncommutative smooth algebra $A_\Delta = \mathbb{C}Q_\Delta/(R_\Delta)$. To begin we construct an extended quiver Q_Δ which is Q together with a bunch of $a^2t_js_i$ extra arrows $d^{(ji)}_{u,v}$ from the j-th right vertex to the i-th left vertex and let these extra arrows be the components of a matrix D_{ij} of sizes $as_i \times at_j$

$$D_{ij} = \begin{bmatrix} d^{(ij)}_{1,1} & \ldots & d^{(ij)}_{1,at_j} \\ \vdots & & \vdots \\ d^{(ij)}_{as_i,1} & \ldots & d^{(ij)}_{as_i,at_j} \end{bmatrix}$$

and make the bigger block-matrix

$$D = \begin{bmatrix} \boxed{D_{11}} & \ldots & \boxed{D_{1r}} \\ \vdots & & \vdots \\ \boxed{D_{l1}} & \ldots & \boxed{D_{lr}} \end{bmatrix}$$

Now, the algebra A_Δ is the quotient of the path algebra $\mathbb{C}Q_\Delta$ of the extended quiver modulo the ideal of relations coming from the following two matrix-identities in $\mathbb{C}Q_\Delta$

$$D.\Delta = \begin{bmatrix} e_1 1_{as_1} & & 0 \\ & \ddots & \\ 0 & & e_l 1_{as_l} \end{bmatrix} \qquad \text{and} \qquad \Delta.D = \begin{bmatrix} f_1 1_{at_1} & & 0 \\ & \ddots & \\ 0 & & f_r 1_{at_r} \end{bmatrix}$$

where e_i (resp. f_j) is the vertex idempotent of the i-th left vertex in Q (resp. of the j-th right vertex in Q). After all these definitions it is about time to illustrate its use

Theorem 4.7 *With notations as before*

1. A_Δ *is an affine smooth* $\mathbb{C}$*-algebra.*
2. $\mathtt{rep}\ A_\Delta = \{V \in \mathtt{rep}^\theta\ \mathbb{C}Q \mid det\ \Delta(V) \neq 0\}$

and as all θ*-semi invariants are generated by those coming from* Δ*'s we have*

$$\mathtt{rep}^\theta\ Q = \bigcup_D \mathtt{rep}\ A_\Delta$$

is a noncommutative manifold.

Proof. Another description of the algebra A_Δ is as a *universal localization* of the smooth algebra $\mathbb{C}Q$. Let $P_i = e_i\mathbb{C}Q$ be the *projective* right ideal generated by the i-th left vertex idempotent of Q and $Q_j = f_j\mathbb{C}Q$ that generated by the j-th right vertex idempotent. The matrix Δ describes a $\mathbb{C}Q$-module morphism

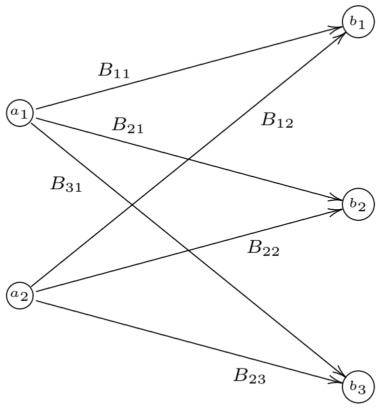
$$P_1^{\oplus as_1} \oplus \ldots \oplus P_l^{\oplus as_l} \xrightarrow{\phi} Q_1^{\oplus at_1} \oplus \ldots \oplus Q_r^{\oplus at_r}$$

As universal localizations of smooth algebras are again smooth (use the universal property to lift modulo nilpotents) the first statement follows.

As for the second, let M be an α-dimensional representation of A_Δ determined by an α-dimensional Q_Δ quiver representation satisfying the required identities R_Δ and let $V = M|_Q$ be the restriction of M to the arrows of Q. Then, by the very definition of R_Δ it follows that $det\ \Delta(V) \neq 0$ and therefore V is a θ-semistable representation. Conversely, any θ-stable representation such that $det\ \Delta(V) \neq 0$ can be extended to a representation of A_Δ by assigning to the additional arrows the block-matrices occurring in the description of the inverse of $\Delta(V)$. In fact, by an argument as before for $PSL_2(\mathbb{Z})$ we also have that there is a natural one-to-one correspondence between simple α-dimensional A_Δ-representations and θ-stable representations in $\mathtt{rep}_\alpha\ Q$ such that $det\ \Delta(V) \neq 0$. $\square$

We havent brought in the topology yet, but we can give $\mathtt{rep}^\theta\ \mathbb{C}Q$ the *induced* non-commutative Zariski topology of $\mathtt{rep}\ \mathbb{C}Q$ and then use properties of universal localizations that all fits well together as demanded by the definition of a non-commutative manifold.

In the $PSL_2(\mathbb{Z})$-example with the usual notation for arrows of the corresponding quiver



we have seen that $\mathtt{rep}\ PSL_2(\mathbb{Z}) = \mathtt{rep}\ A_\Delta$ for

$$\Delta = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

but we might have considered other affine smooth pieces of $\overline{PSL_2(\mathbb{Z})}$ such as those determined by the matrices

$$\Delta_0 = \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \\ B_{31} & 0 \end{bmatrix} \qquad \Delta_1 = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \\ B_{31} & 0 \end{bmatrix} \qquad \Delta_2 = \begin{bmatrix} B_{11} & 0 \\ B_{21} & 0 \\ 0 & B_{32} \end{bmatrix}$$

For example, if $\alpha = (2, 1, 1, 1, 1)$ one can show that the determinants of these three generate the whole ring of semi-invariants, that is

$$R^\theta_\alpha = \mathbb{C}[det\ \Delta_0, det\ \Delta_1, det\ \Delta_2] \qquad \text{whence} \qquad \mathtt{proj}\ R^\theta_\alpha = \mathbb{P}^2 = \overline{PSL_2(\mathbb{Z})_\alpha}$$

4.2 Non-commutative topologies

If A is smooth, we have seen that all representation schemes are smooth (hence in particular reduced) but they may have several connected (which in this case is the same as irreducible) components and we give each of these components a label

$$\mathtt{rep}_n\ A = \bigsqcup_{|\alpha|=n} \mathtt{rep}_\alpha\ A$$

and we say that α is a dimension vector of total dimension $|\alpha| = n$.

Let $\mathtt{comp}\ A$ be the set of all labels α for all natural numbers $n \in \mathbb{N}$, that is the set of all (non-empty) connected components in $\mathtt{rep}\ A$. We define an Abelian semigroup structure on $\mathtt{comp}\ A$ by bringing in the sum-maps

$$\mathtt{rep}_n\ A = \bigsqcup_{|\alpha|=n} \mathtt{rep}_\alpha\ A \times \mathtt{rep}_m\ A = \bigsqcup_{|\beta|=m} \mathtt{rep}_\beta\ A \xrightarrow{\oplus} \mathtt{rep}_{m+n}\ A = \bigsqcup_{|\gamma|=m+n} \mathtt{rep}_\gamma\ A$$

We define $\alpha + \beta = \gamma$ if $\texttt{rep}_\gamma\ A$ is the connected component of $\texttt{rep}_{m+n}\ A$ containing the image of the connected and irreducible variety $\texttt{rep}_\alpha\ A \times \texttt{rep}_\beta\ A$. With $\texttt{gen}\ A$ we will denote the set of semigroup generators for the *component-semigroup* $\texttt{comp}\ A$. It is unknown whether this set is always finite for a smooth affine algebra A. We have a representation theoretic description for the set $\texttt{gen}\ A$.

Theorem 4.8 *The generator set* $\texttt{gen}\ A$ *are precisely those components* $\alpha \in \texttt{comp}\ A$ *for which* $\texttt{rep}_\alpha\ A$ *consisting entirely of simple representations of* A*.*

Proof. If $V \in \texttt{rep}_\alpha\ A$ is not simple, then it has a *Jordan-Hölder filtration*

$$0 \subset V_l \subset V_{l-1} \subset \ldots \subset V_1 \subset V_0 = V \qquad \text{with all factors } V_i/V_{i+1} = S_i \text{ simples}$$

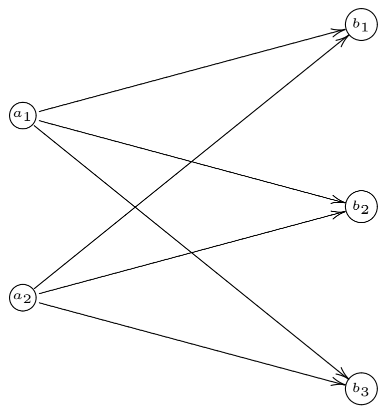
One can show that the *semi-simplification* of V

$$V^{ss} = S_0 \oplus \ldots \oplus S_l$$

lies in the closure of the orbit $\mathcal{O}(V) = GL_n.V$ wrt. the Zariski topology on $\texttt{rep}_\alpha\ A$ (and in particular is contained in the same connected component). But then, if $S_i \in \texttt{rep}_{\beta_i}\ A$ we have that $\alpha = \beta_0 + \ldots + \beta_l$ whence α is not a generator for $\texttt{comp}\ A$. □

Example 4.9 (path algebras) If Q is a quiver on k vertices, then $\texttt{comp}\ \mathbb{C}Q \simeq \mathbb{N}^k$ and is generated by the vertex-dimension vectors $\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. The addition on $\texttt{comp}\ \mathbb{C}Q$ is the ordinary addition on $\mathbb{N}^k$.

Example 4.10 ($PSL_2(\mathbb{Z})$) The corresponding quiver is the full bipartite quiver



For a dimension vector $\alpha = (a_1, a_2, b_1, b_2, b_3)$ let us denote $|\alpha| = n$ if $a_1 + a_2 = n = b_1 + b_2 + b_3$. For any $|\alpha| = n$ there is a non-empty open subset U_α of $\texttt{rep}_\alpha\ Q$ defining a component $GL_n.U_\alpha$ of $\texttt{rep}_n\ PSL_2(\mathbb{Z})$. As a consequence the component semigroup $\texttt{comp}\ PSL_2(\mathbb{Z})$ is generated by the six dimension vectors (for $n = 1$)

$$\begin{cases} g_1 & = (1,0,1,0,0) \\ g_2 & = (0,1,0,1,0) \\ g_3 & = (1,0,0,0,1) \\ g_4 & = (0,1,1,0,0) \\ g_5 & = (1,0,0,1,0) \\ g_6 & = (0,1,0,0,1) \end{cases}$$

and as the addition on $\mathtt{com}\ PSL_2(\mathbb{Z})$ is as a sub-semigroup of $\mathtt{comp}\ \mathbb{C}Q = \mathbb{N}^5$ there must be relations among these generators. In fact, we have

$$g_1 + g_2 = g_4 + g_5 \qquad g_6 + g_1 = g_3 + g_4 \qquad \text{and} \qquad g_2 + g_3 = g_5 + g_6$$

Example 4.11 ($SL_2(\mathbb{Z})$) As the quiver for $SL_2(\mathbb{Z})$ (see before) is the disjoint union of two copies of that of $PSL_2(\mathbb{Z})$ we have that $\mathtt{comp}\ SL_2(\mathbb{Z})$ has exactly 12 generators (all dimension vectors for $n = 1$ mentioned before)

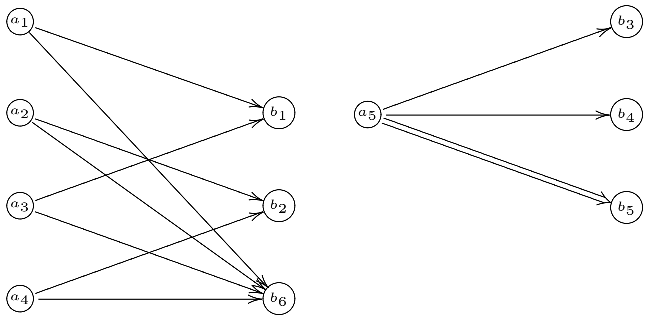
$$\begin{cases} g_1 & = (1,0,0,0,1,0,0,0,0,0) \\ g_2 & = (0,0,1,0,0,0,1,0,0,0) \\ g_3 & = (1,0,0,0,0,0,0,0,1,0) \\ g_4 & = (0,0,1,0,1,0,0,0,0,0) \\ g_5 & = (1,0,0,0,0,0,1,0,0,0) \\ g_6 & = (0,0,1,0,0,0,0,0,1,0) \\ g_7 & = (0,1,0,0,0,1,0,0,0,0) \\ g_8 & = (0,0,0,1,0,0,0,1,0,0) \\ g_9 & = (0,1,0,0,0,0,0,1,0,0) \\ g_{10} & = (0,0,0,1,0,1,0,0,0,0) \\ g_{11} & = (0,1,0,0,0,0,0,1,0,0) \\ g_{12} & = (0,0,0,1,0,0,0,0,0,1) \end{cases}$$

and there are the following relations among these generators in $\mathtt{comp}\ SL_2(\mathbb{Z})$

$$g_1 + g_2 = g_4 + g_5 \qquad g_6 + g_1 = g_3 + g_4 \qquad \text{and} \qquad g_2 + g_3 = g_5 + g_6$$

$$g_7 + g_8 = g_{10} + g_{11} \qquad g_{12} + g_7 = g_9 + g_{10} \qquad \text{and} \qquad g_8 + g_9 = g_{11} + g_{12}$$

Example 4.12 ($GL_2(\mathbb{Z})$) The corresponding quiver has two components



and the dimension vector $\alpha = (a_1, \ldots, a_5, b_1, \ldots, b_6)$ must certainly satisfy the condition denoted by $|\alpha| = n$

$$a_1 + a_2 + a_3 + a_4 + 2a_5 = n = b_1 + b_2 + b_3 + b_4 + 2b_5 + 2b_6$$

However, not every $|\alpha| = n$ has a non-empty subset $U_\alpha \hookrightarrow \mathtt{rep}_\alpha\ Q$ consisting of $GL_2(\mathbb{Z})$ representations. The baechange matrix must be an isomorphism of D_2-representations, hence in each of the two components every irreducible D_2-representation Z_i gives an additional linear condition on the components of α expressing the fact that the total number of Z_i-components in the left-hand vertices is equal to that in the right-hand vertices. Recalling the restriction data, this gives the additional conditions on α to determine a component in $\texttt{rep}_n \; GL_2(\mathbb{Z})$.

$$\begin{cases} (Z_1) \; : & a_1 + a_3 = b_1 + b_6 \\ (Z_2) \; : & a_5 = b_3 + b_5 \\ (Z_3) \; : & a_5 = b_4 + b_5 \\ (Z_4) \; : & a_2 + a_4 = b_2 + b_6 \end{cases}$$

The first component gives us 8 generators (4 for $n = 1$ and 4 for $n = 2$)

	n	a_1	a_2	a_3	a_4	b_1	b_2	b_6
g_1	1	1	0	0	0	1	0	0
g_2	2	1	1	0	0	0	0	1
g_3	2	0	0	1	1	0	0	1
g_4	2	1	0	0	1	0	0	1
g_5	1	0	1	0	0	0	1	0
g_6	2	0	1	1	0	0	0	1
g_7	1	0	0	0	1	0	1	0
g_8	1	0	0	1	0	1	0	0

The second component gives us an additional 2 generators (for $n = 2$).

	n	a_5	b_3	b_4	b_5
g_9	2	1	1	1	0
g_{10}	2	1	0	0	1

and again there are plenty of obvious relations between these generators in $\texttt{comp} \; GL_2(\mathbb{Z})$. In terms of universal localizations (or open subsets of $\texttt{rep}^\theta \; Q$ we can identify $\texttt{rep} \; GL_2(\mathbb{Z})$ with $\texttt{rep} \; \mathbb{C}Q_\Delta$ for Δ the matrix (with natural notation in terms of the arrows of Q)

$$\Delta = \begin{bmatrix} B_{11} & 0 & B_{31} & 0 & 0 & 0 \\ 0 & B_{22} & 0 & B_{42} & 0 & 0 \\ B_{16} & 0 & B_{36} & 0 & 0 & 0 \\ 0 & B_{26} & 0 & B_{46} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{53} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{54} \\ 0 & 0 & 0 & 0 & B_{55}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{55}^{(2)} \end{bmatrix}$$

We will give the construction of all blocks for a smooth algebra A. Consider a connected component $\texttt{rep}_\alpha \; A$ of $\texttt{rep}_n \; A$ such that there is a non-empty Zariski open subset $\texttt{bricks}_\alpha \; A$ of bricks in $\texttt{rep}_\alpha \; A$. As the endomorphism ring of a brick is the field $\mathbb{C}$ geometric invariant theory implies that if there is a brick in $\texttt{rep}_\alpha \; A$ there is a Zariski open subset of bricks.

Consider an irreducible closed GL_n-stable subvariety $X \subset \texttt{rep}_\alpha \; A$ such that $X \cap \texttt{bricks}_\alpha \; A \neq \emptyset$ then one can associate to X an epimorphism

$$A \xrightarrow{f_X} M_d(D)$$

with D a division algebra and $d|n$.

For those who know some GIT here is the construction : the GL_n action on $Y = X \cap \texttt{bricks}_\alpha\ A$ is really a free PGL_n-action whence one has an orbits space Y/PGL_n and the quotient map $Y \twoheadrightarrow Y/PGL_n$ is a principal PGL_n-fibration. Now, principal PGL_n-fibrations correspond to Azumaya algebras and as Y/PGL_n is an irreducible variety, its classical ring of fractions is a central simple algebra Σ_X of dimension n^2 over the function field $K = \mathbb{C}(Y/PGL_n)$. By the structure theory of central simple algebras we have

$$\Sigma_X = M_d(D) \qquad \text{with } D \text{ a central } K\text{-division algebra of dimension } (n/d)^2.$$

Two irreducible GL_n-stable subvarieties X and X' define the same block if and only if X and X' have a common Zariski open subset (that is, are birational).

This construction shows that the topology induced on $\texttt{bricks}_\alpha\ A$ by the closed subsets $\mathbb{V}(X)$ of $\texttt{blocks}\ A$ is *roughly* as fine as the Zariski topology on $\texttt{bricks}_\alpha\ A$.

The underlying idea to construct a *non-commutative topology* on $\texttt{rep}\ A$ is first to define a (commutative) topology on a certain subset of all finite dimensional representations including all simples and then use finite filtrations á la Jordan-Hölder sequences to extend this topology to all of $\texttt{rep}\ A$.

Definition 4.13 For $A \in \texttt{alg}$ a *block* is a left A-module (possibly infinite dimensional) X such that its endomorphism ring $D = End_A(X)$ is a division ring and X considered as a right D-module is finite dimensional.

A *brick* S for A is a block which is finite dimensional, in particular it follows that $D = End_A(S) = \mathbb{C}$.

Observe that all simple finite dimensional representations of A are bricks. We will relate blocks to *epimorphisms* of algebras. Recall that a $\mathbb{C}$-algebra morphism $A \xrightarrow{f} B$ is said too be an epimorphism in $\texttt{alg}$ if for all algebra maps

$$A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C \qquad \text{satisfying } g_1 \circ f = g_2 \circ f \text{ we have that } g_1 = g_2$$

Common examples of epimorphisms are quotients as well as localizations. Blocks are defined by certain special epimorphisms

Theorem 4.14 (Ringel) *There is a natural one-to-one correspondence between*

1. *blocks of A, and*
2. *epimorphisms $A \longrightarrow M_n(D)$ with D a division algebra.*

If $A \xrightarrow{f} M_n(D)$ is an epimorphism one considers the block $D^{\oplus n}$ (viewed as an n-dimensional column-space which becomes a left A-module via left multiplication via f. Conversely, if X is a block with endomorphism division algebra $D = End_A(X)$ and if $dim_D\ X = n$ one constructs an algebra map

$$A \xrightarrow{f} M_n(D) = End_D(X_D)$$

where $f(a)$ is left multiplication by a on X. One verifies that this map is an epimorphism and that the two constructions are each other inverses.

We will equip the set $\mathtt{blocks}\ A$ of all A-blocks with a partial order coming from the notion of *specialization*. If D and E are division algebras we say that the epimorphism $A \xrightarrow{g} M_m(E)$ is a specialization of the epimorphism $A \xrightarrow{f} M_n(D)$ if there is an epimorphism $A \xrightarrow{h} B$ such that the diagram

$$\begin{tikzcd}
 & M_n(D) \\
A \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & B \arrow[u, hook, "i"'] \arrow[d, two heads, "\pi"'] \\
 & M_m(E)
\end{tikzcd}$$

is commutative where i is an embedding and π is a quotient map. If X resp. Y are the blocks corresponding to f resp. g we will denote the specialization by $X \leq Y$. It is easy to verify that this notion turns $(\mathtt{blocks}\ A, \leq)$ into a partially ordered set. This allows us to define for each block X the *closed subset* on the set of all bricks $\mathtt{bricks}\ A$

$$\mathbb{V}(X) = \{Y \in \mathtt{bricks}\ A \mid X \leq Y\}$$

We will give the construction of all blocks for a smooth algebra A. Consider a connected component $\mathtt{rep}_\alpha\ A$ of $\mathtt{rep}_n\ A$ such that there is a non-empty Zariski open subset $\mathtt{bricks}_\alpha\ A$ of bricks in $\mathtt{rep}_\alpha\ A$. As the endomorphism ring of a brick is the field $\mathbb{C}$ geometric invariant theory implies that if there is a brick in $\mathtt{rep}_\alpha\ A$ there is a Zariski open subset of bricks.

Consider an irreducible closed GL_n-stable subvariety $X \subset \mathtt{rep}_\alpha\ A$ such that $X \cap \mathtt{bricks}_\alpha\ A \neq \emptyset$ then one can associate to X an epimorphism

$$A \xrightarrow{f_X} M_d(D)$$

with D a division algebra and $d|n$.

For those who know some GIT here is the construction : the GL_n action on $Y = X \cap \mathtt{bricks}_\alpha\ A$ is really a free PGL_n-action whence one has an orbits space Y/PGL_n and the quotient map $Y \twoheadrightarrow Y/PGL_n$ is a principal PGL_n-fibration. Now, principal PGL_n-fibrations correspond to Azumaya algebras and as Y/PGL_n is an irreducible variety, its classical ring of fractions is a central simple algebra Σ_X of dimension n^2 over the function field $K = \mathbb{C}(Y/PGL_n)$. By the structure theory of central simple algebras we have

$$\Sigma_X = M_d(D) \qquad \text{with } D \text{ a central } K\text{-division algebra of dimension } (n/d)^2.$$

Two irreducible GL_n-stable subvarieties X and X' define the same block if and only if X and X' have a common Zariski open subset (that is, are birational).

This construction shows that the topology induced on $\mathtt{bricks}_\alpha\ A$ by the closed subsets $\mathbb{V}(X)$ of $\mathtt{blocks}\ A$ is *roughly* as fine as the Zariski topology on $\mathtt{bricks}_\alpha\ A$.

Definition 4.15 The *block-topology* on the set $\texttt{bricks}\ A = \bigsqcup_\alpha\ \texttt{bricks}_\alpha\ A$ is the topology generated by taking as a subset of its closed sets the sets

$$\mathbb{V}(X) = \{S \in \texttt{bricks}\ A \mid X \leq S\}$$

where X runs over $\texttt{blocks}\ A$. With $\mathcal{L}_A$ we denote the set of all closed subsets of $\texttt{bricks}\ A$ in the block-topology. $\mathcal{L}_A$ will be the set of *letters* on which to base our non-commutative topology.

If M is an n-dimensional representation of A we call a finite filtration of length u

$$\mathcal{F}^u\ :\ 0 = M_0 \subset M_1 \subset \ldots \subset M_u = M$$

of A-representations a *brick filtration* if the successive quotients

$$\mathcal{F}_i = \frac{M_i}{M_{i-1}}$$

are bricks. As simple representations are bricks, any Jordan-Hölder filtration of M is a brick filtration, but there may be others.

Definition 4.16 With $\mathbb{W}_A$ we denote the non-commutative words in the letters $\mathcal{L}_A$.

$$\mathbb{W}_A = \{V_1 \ldots V_k \mid V_i \in \mathcal{L}_A, k \in \mathbb{N}\}$$

For a given word $w = V_1V_2 \ldots V_k \in \mathbb{W}_A$ we define the *left basic open set*

$$\mathcal{O}^l_w = \{M \in \texttt{rep}\ A \mid\ \nexists \mathcal{F}^u \text{ brick filtration on } M \text{ such that } \mathcal{F}_i \in V_i\}$$

and the *right basic open set*

$$\mathcal{O}^r_w = \{M \in \texttt{rep}\ A \mid\ \nexists \mathcal{F}^u \text{ brick filtration on } M \text{ such that } \mathcal{F}_{u-i} \in V_{k-i}\}$$

Finally, to make these definitions symmetric we define the *basic open set*

$$\mathcal{O}_w = \{M \in \texttt{rep}\ A \mid\ \nexists \mathcal{F}^u \text{ brick filtration on } M \text{ such that } \mathcal{F}_{i_j} \in V_j$$
$$\text{for some } 1 \leq i_1 < i_2 < \ldots < i_k \leq u\ \}$$

Clearly, $\mathcal{O}^l_w$ consists of those representations having restricted bottom structure, whereas $\mathcal{O}^r_w$ consists of those with restricted top structure. In order to avoid three sets of definitions we will denote from now on $\mathcal{O}^\bullet_w$ whenever we mean $\bullet \in \{l, r, \emptyset\}$.

If $w = L_1 \ldots L_k$ and $w' = M_1 \ldots M_l$, we will denote with $w \cup w'$ the *multi-set* $\{N_1, \ldots, N_m\}$ where each N_i is one of L_j, M_j and N_i occurs in $w \cup w'$ as many times as its maximum number of factors in w or w'. With $\texttt{rep}(w \cup w')$ we denote the subset of $\texttt{rep}\ A$ consisting of the representations of M having a Jordan-Hölder filtration having factor-multi-set containing $w \cup w'$. For any triple of words w, w' and w" we denote $\mathcal{O}^\bullet_{w"}(w \cup w') = \mathcal{O}^\bullet_{w"} \cap \texttt{rep}(w \cup w')$.

We define an equivalence relation on the basic open sets by

$$\mathcal{O}^\bullet_w \approx \mathcal{O}^\bullet_{w'} \qquad \Leftrightarrow \qquad \mathcal{O}^\bullet_w(w \cup w') = \mathcal{O}^\bullet_{w'}(w \cup w')$$

The reason for this definition is that the condition of $M \in \mathtt{rep}\ A - \mathcal{O}^{\bullet}_w$ is void if M does not have enough brick components to get all factors of w which makes it impossible to define equality of basic open sets defined by different words.

We can now define the partially ordered sets $\Lambda^{\bullet}_A$ as consisting of all basic open subsets $\mathcal{O}^{\bullet}_w$ of $\mathtt{rep}\ A$. The partial ordering $\leq$ is induced by set-theoretic inclusion modulo equivalence, that is,

$$\mathcal{O}^{\bullet}_w \leq \mathcal{O}^{\bullet}_{w'} \qquad \Leftrightarrow \qquad \mathcal{O}^{\bullet}_w(w \cup w') \subseteq \mathcal{O}^{\bullet}_{w'}(w \cup w')$$

As a consequence, equality $=$ in the set $\Lambda^{\bullet}_A$ coincides with equivalence $\approx$. Observe that these partially ordered sets have a unique minimal and a unique maximal element (up to equivalence)

$$0 = \emptyset = \mathcal{O}^{\bullet}_{\mathtt{bricks}\ A} \qquad \text{and} \qquad 1 = \mathtt{rep}\ A = \mathcal{O}^{\bullet}_{\emptyset}$$

The operations $\vee$ and $\wedge$ are defined as follows : $\wedge$ is induced by ordinary set-theoretic intersection and $\vee$ is induced by concatenation of words, that is

$$\mathcal{O}^{\bullet}_w \vee \mathcal{O}^{\bullet}_{w'} \approx \mathcal{O}^{\bullet}_{ww'}$$

This will turn out to be an example of a *non-commutative topology* of which we recall the definition. We fix a partially ordered set $(\Lambda, \leq)$ with a unique minimal element 0 and a unique maximal element 1, equipped with two operations $\wedge$ and $\vee$. With i_{Λ} we will denote the set of all *idempotent elements* of Λ, that is, those $x \in \Lambda$ such that $x \wedge x = x$. A *finite global cover* is a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ such that $1 = \lambda_1 \vee \ldots \vee \lambda_n$. In the table below we have listed the conditions for a (one-sided) non-commutative topology.

$(A1)$	$x \wedge y \leq x$	$x \wedge y \leq y$
$(A2)$	$x \wedge 1 = x$ $x \wedge 0 = 0$	$1 \wedge x = x$ $0 \wedge x = 0$
$(A3)$	$(x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y \wedge z$	
$(A4)$	$x \leq y \Rightarrow z \wedge x \leq z \wedge y$	$x \leq y \Rightarrow x \wedge z \leq y \wedge z$
$(A5)$	$x \leq x \vee y$	$y \leq x \vee y$
$(A6)$	$x \vee 1 = 1$ $x \vee 0 = x$	$1 \vee x = 1$ $0 \vee x = x$
$(A7)$	$(x \vee y) \vee z = x \vee (y \vee z) = x \vee y \vee z$	
$(A8)$	$x \leq y \Rightarrow x \vee z \leq y \vee z$	$x \leq y \Rightarrow z \vee x \leq z \vee y$
$(A9)$	$a \vee (a \wedge b) \leq (a \vee a) \wedge b$	$a \vee (b \wedge a) \leq (a \vee b) \wedge a$
$(A10)$	$x = (x \wedge \lambda_1) \vee \ldots \vee (x \wedge \lambda_n)$	$x = (\lambda_1 \wedge x) \vee \ldots \vee (\lambda_n \wedge x)$

where $(A3)$ and $(A7)$ are symmetric conditions.

Definition 4.17 Let $(\Lambda, \leq)$ be a partially ordered set with minimal and maximal element 0 and 1 and operations $\wedge$ and $\vee$. Then,

Λ is said to be a *left non-commutative topology* if and only if the left column conditions of (A1)-(A10) are valid for all $x, y, z \in \Lambda$, all $a, b \in i_\Lambda$ with $a \leq b$ and all finite global covers $\{\lambda_1, \ldots, \lambda_n\}$.

Λ is said to be a *right non-commutative topology* if and only if the right column (together with $(A3)$ and $(A7)$) conditions of (A1)-(A10) are valid for all $x, y, z \in \Lambda$, all $a, b \in i_\Lambda$ with $a \leq b$ and all finite global covers $\{\lambda_1, \ldots, \lambda_n\}$.

Λ is said to be a *non-commutative topology* if and only if the conditions (A1)-(A10) are valid for all $x, y, z \in \Lambda$, all $a, b \in i_\Lambda$ with $a \leq b$ and all finite global covers $\{\lambda_1, \ldots, \lambda_n\}$.

There are at least two ways of building a genuine non-commutative topology out of these sets of basic opens.

Let $T(\Lambda)$ be the set of all finite $(\wedge, \vee)$-words in the *contractible* idempotent elements i_Λ (that is, $\lambda \in i_\Lambda$ such that for all λ_1, λ_2 with $\lambda \leq \lambda_1 \vee \lambda_2$ we have that $\lambda = (\lambda \wedge \lambda_1) \vee (\lambda \wedge \lambda_2)$). If Λ is a (left,right) non-commutative topology, then so is $T(\Lambda)$. The $\vee$-*complete topology of virtual opens* $T'(\Lambda)$ is then the set of all $(\wedge, \vee)$-words in the contractible idempotents of finite length in $\wedge$ (but not necessarily of finite length in $\vee$). This non-commutative topology has properties very similar to that of an ordinary topology and, in fact, has associated to it a *commutative shadow*.

The second construction, leading to the *pattern topology*, starts with the equivalence classes of *directed systems* $S \subset \Lambda$ (that is, if for all $x, y \in S$ there is a $z \in S$ such that $z \leq x$ and $z \leq y$) and where the equivalence relation $S \sim S'$ is defined by

$$\begin{cases} \forall a \in S, \exists a' \in S, a' \leq a \text{ and } b \leq a' \leq b' \text{ for some } b, b' \in S' \\ \forall b \in S', \exists b' \in S', b' \leq b \text{ and } a \leq b' \leq a' \text{ for some } a, a' \in S \end{cases}$$

One can extend the $\wedge, \vee$ operations on Λ to the equivalence classes $C(\Lambda) = \{[S] \mid S$ directed $\}$ in the obvious way such that also $C(\Lambda)$ is a (left,right) non-commutative topology. A directed set $S \subset \Lambda$ is said to be *idempotent* if for all $a \in S$, there is an $a' \in S \cap i_\Lambda$ such that $a' \leq a$. If S is idempotent then $[S] \in i_{C(\Lambda)}$ and those idempotents will be called *strong idempotents*. The pattern topology $\Pi(\Lambda)$ is the (left,right) non-commutative topology of finite $(\wedge, \vee)$-words in the strong idempotents of $C(\Lambda)$. A directed system $[S]$ is called a *point* iff $[S] \leq \vee [S_\alpha]$ implies that $[S] \leq [S_\alpha]$ for some α.

Theorem 4.18 *With notations as before,*

1. $(\Lambda^l_A, \leq, =, 0, 1, \vee, \wedge)$ *is a left non-commutative topology on* `rep` A.
2. $(\Lambda^r_A, \leq, =, 0, 1, \vee, \wedge)$ *is a right non-commutative topology on* `rep` A.

day 5

CHARTS & SIMPLES

Today we finally come to applications of non-commutative algebraic geometry to the representation theory of arithmetical groups (and the third braid group B_3). The crucial ingredient is the Euler-form which exists on $\mathtt{rep}\ A$ whenever A is a smooth algebra as smooth algebras are *hereditary*. This form then allows us to define the *chart* of A which is a quiver $\mathtt{chart}_A$ containing enough information to reduce all questions on $\mathtt{rep}\ A$ to quiver-problems. One might view $\mathtt{chart}_A$ as a sort of *tangent space* to the non-commutative manifold $\mathtt{rep}\ A$. We will state just one application of it : to construct nearly all simple representations of A. We will work through the details in the case of $PSL_2(\mathbb{Z})$ and finish by giving nearly all simple representations of B_3.

5.1 Euler forms

Definition 5.1 Let M and N be two representations of dimensions m and n of $A \in \mathtt{alg}$. A representation P of dimension $m+n$ is said to be an *extension of* N *by* M if there exists a short exact sequence of left A-modules

$$e : \qquad 0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

Define an equivalence relation on extensions (P,e) of N by M : $(P,e) \cong (P',e')$ if and only if there is an isomorphism $P \xrightarrow{\phi} P'$ of left A-modules such that the diagram below is commutative

$$\begin{tikzcd}
e : & 0 \arrow[r] & M \arrow[r] \arrow[d, "id_M"'] & P \arrow[r] \arrow[d, "\phi"] & N \arrow[r] \arrow[d, "id_N"] & 0 \\
e' : & 0 \arrow[r] & M \arrow[r] & P' \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

The set of equivalence classes of extensions of N by M will be denoted by $Ext^1_A(N,M)$.

An alternative description of $Ext^1_A(N,M)$ is as follows. Let $\rho : A \longrightarrow M_m(\mathbb{C})$ and $\sigma : A \longrightarrow M_n(\mathbb{C})$ be the representations defining M and N. For an extension (P,e) we identify the $\mathbb{C}$-vector space with $M \oplus N$ and the A-module structure on P

gives a algebra map $\mu : A \longrightarrow M_{m+n}(\mathbb{C})$. We represent the action of a on P by left multiplication of the block-matrix

$$\mu(a) = \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix},$$

where $\lambda(a)$ is an $m \times n$ matrix and hence defines a linear map

$$\lambda : A \longrightarrow Hom_{\mathbb{C}}(N, M).$$

The condition that μ is an algebra morphism is equivalent to the condition

$$\lambda(aa') = \rho(a)\lambda(a') + \lambda(a)\sigma(a')$$

and we denote the set of all liner maps $\lambda : A \longrightarrow Hom_{\mathbb{C}}(N, M)$ by $Z(N, M)$ and call it the space of *cycle* .

The extensions of N by M corresponding to two cycles λ and λ' from $Z(N, M)$ are equivalent if and only if there is an A-module isomorphism in block form

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{with } \beta \in Hom_{\mathbb{C}}(N, M)$$

between them. A-linearity of this map translates to the matrix relation

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} . \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix} = \begin{bmatrix} \rho(a) & \lambda'(a) \\ 0 & \sigma(a) \end{bmatrix} . \begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{for all } a \in A$$

or equivalently, that $\lambda(a) - \lambda'(a) = \rho(a)\beta - \beta\sigma(a)$ for all $a \in A$. We will define the subspace of $Z(N, M)$ of *boundaries* $B(N, M)$

$$\{\delta \in Hom_{\mathbb{C}}(N, M) \mid \exists \beta \in Hom_{\mathbb{C}}(N, M) : \forall a \in A : \delta(a) = \rho(a)\beta - \beta\sigma(a)\}.$$

Therefore, $Ext^1_A(N, M) = \frac{Z(N,M)}{B(N,M)}$.

Recall that the *Euler form* of a quiver Q on k vertices is the bilinear form on $\mathbb{Z}^k$

$$\chi_Q(.,.) : \mathbb{Z}^k \times \mathbb{Z}^k \longrightarrow \mathbb{Z} \quad \text{defined by} \quad \chi_Q(\alpha, \beta) = \alpha.\chi_Q.\beta^\tau$$

for all row vectors $\alpha, \beta \in \mathbb{Z}^k$.

Theorem 5.2 *Let V resp. W be representations of the quiver Q of dimension vector α resp. β, then*

$$dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, W) - dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V, W) = \chi_Q(\alpha, \beta)$$

In particular, the right-hand side does not depend on the particular representations but only on the dimension vector.

Proof. There is an exact sequence of $\mathbb{C}$-vector spaces

$$0 \longrightarrow Hom_{\mathbb{C}Q}(V, W) \xrightarrow{\gamma} \oplus_{v_i \in Q_v} Hom_{\mathbb{C}}(V_i, W_i) \xrightarrow{d^V_W}$$

$$\xrightarrow{d^V_W} \oplus_{a \in Q_a} Hom_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) \xrightarrow{\epsilon} Ext^1_{\mathbb{C}Q}(V, W) \longrightarrow 0$$

Here, $\gamma(\phi) = (\phi_1, \ldots, \phi_k)$ and d^V_W maps a family of linear maps $(f_1, \ldots, f_k)$ to the linear maps $\mu_a = f_j V_a - W_a f_i$ for any arrow $j \xleftarrow{a} i$ in Q, that is, to the obstruction of the following diagram to be commutative

$$\begin{tikzcd} V_i \arrow[r, "V_a"] \arrow[d, "f_i"'] \arrow[dr, dotted, "\mu_a"] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_a"] & W_j \end{tikzcd}$$

By the definition of morphisms between representations of Q it is clear that the kernel of d^V_W coincides with $Hom_{\mathbb{C}Q}(V, W)$.

The map ϵ is defined by sending a family of maps $(g_1, \ldots, g_s) = (g_a)_{a \in Q_a}$ to the equivalence class of the exact sequence

$$0 \longrightarrow W \xrightarrow{i} E \xrightarrow{p} V \longrightarrow 0$$

where for all $v_i \in Q_v$ we have $E_i = W_i \oplus V_i$ and the inclusion i and projection map p are the obvious ones and for each arrow $a \in Q_a$ the action of a on E is defined by the matrix

$$E_a = \begin{bmatrix} W_a & g_a \\ 0 & V_a \end{bmatrix} : E_i = W_i \oplus V_i \longrightarrow W_j \oplus V_j = E_j$$

This makes E into a $\mathbb{C}Q$-representation and one verifies that the above short exact sequence is one of $\mathbb{C}Q$-representations. Remains to prove that the cokernel of d^V_W can be identified with $Ext^1_{\mathbb{C}Q}(V, W)$.

A set of algebra generators of $\mathbb{C}Q$ is given by $\{v_1, \ldots, v_k, a_1, \ldots, a_l\}$. A cycle is given by a linear map $\lambda : \mathbb{C}Q \longrightarrow Hom_{\mathbb{C}}(V, W)$ such that for all $f, f' \in \mathbb{C}Q$ we have the condition

$$\lambda(ff') = \rho(f)\lambda(f') + \lambda(f)\sigma(f')$$

where ρ determines the action on W and σ that on V. For any v_i the condition is $\lambda(v_i^2) = \lambda(v_i) = p_i^W \lambda(v_i) + \lambda(v_i) p_i^V$ whence $\lambda(v_i) : V_i \longrightarrow W_i$ but then applying again the condition we see that $\lambda(v_i) = 2\lambda(v_i)$ so $\lambda(v_i) = 0$. Similarly, for the arrow $j \xleftarrow{a} i$ the condition on $a = v_j a = a v_i$ implies that $\lambda(a) : V_i \longrightarrow W_j$. That is, we can identify $\oplus_{a \in Q_a} Hom_{\mathbb{C}}(V_i, W_j)$ with $Z(V, W)$ under the map ϵ. Moreover, the image of δ gives rise to a family of morphisms $\lambda(a) = f_j V_a - W_a f_i$ for a linear map $f = (f_i) : V \longrightarrow W$ so this image coincides precisely to the subspace of boundaries $B(V, W)$ proving that indeed the cokernel of d^V_W is $Ext^1_{\mathbb{C}Q}(V, W)$.

If $dim(V) = \alpha = (r_1, \ldots, r_k)$ and $dim(W) = \beta = (s_1, \ldots, s_k)$, then $dim\ Hom(V, W) - dim\ Ext^1(V, W)$ is equal to

$$\sum_{v_i \in Q_v} dim\ Hom_{\mathbb{C}}(V_i, W_i) - \sum_{j \xleftarrow{a} i} dim\ Hom_{\mathbb{C}}(V_i, W_j)$$

$$= \sum_{v_i \in Q_v} r_i s_i - \sum_{j \xleftarrow{a} i} r_i s_j$$

$$= (r_1, \ldots, r_k) \chi_Q (s_1, \ldots, s_k)^\tau = \chi_Q(\alpha, \beta)$$

□

For any algebra A and finite dimensional representations V and W we can define

$$\chi_A(V,W) = dim_{\mathbb{C}}\ Hom_A(V,W) - dim_{\mathbb{C}}\ Ext^1_A(V,W)$$

but this has no good properties in general. However, if A is smooth, then the functions $\chi_A(V,-)$ and $\chi_A(-,W)$ are *additive* on short exact sequences! This follows from standard homological algebra and the fact that smooth algebras are *hereditary*. Recall that A is hereditary if every left A-module Mhas a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

has length ≤ 1 which implies that $Ext^i_A(M,N) = 0$ whenever $i \geq 2$ and then the theory of derived functors implies additivity.

Theorem 5.3 *A smooth algebra A is hereditary.*

Proof. Because $A \otimes A$ is a free one-sided A-module, the sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

splits as a sequence of right A-modules. Therefore, tensoring this sequence with a left A-module M we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 A \otimes_A M \arrow[r] \arrow[d, "="'] & A \otimes A \otimes_A M \arrow[r] \arrow[d, "="'] & A \otimes_A M \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & \Omega^1 \otimes_A A \arrow[r] & A \otimes M \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The middle term is a free left A-module and as $\Omega^1 A$ is a projective A-bimodule it is a direct summand of some $A \otimes V \otimes A$. But then,

$$\Omega^1 A \otimes_A A \triangleleft A \otimes V \otimes A \otimes_A M = A \otimes V \otimes M$$

is also a projective left A-module. $\square$

Using the additivity and some heavy geometric invariant theory we were able to prove

Theorem 5.4 *Let A be a smooth algebra and $V \in \mathtt{rep}_\alpha\ A$, $W \in \mathtt{rep}_\beta\ A$. Then,*

$$\chi_A(V,W) = dim_{\mathbb{C}} Hom_A(V,W) - dim_{\mathbb{C}} Ext^1_A(V,W)$$

does depend only on the components α and β and not on the particular choice of representations. Therefore, we have a bilinear form

$$\chi_A\ :\ \mathtt{comp}\ A \times \mathtt{comp}\ A \longrightarrow \mathbb{Z}$$

on the component semigroup $\mathtt{comp}\ A$ *which we call the* Euler form *of teh smooth algebra A.*

Example 5.5 Take $A = \mathbb{C}B_3$ the group algebra of the third braid group. We have seen that

$$\mathtt{rep}_1\ B_3 = \mathbb{V}(x^3 - y^2) - \{(0,0)\}$$

and consists entirely of simple representations. Let S and T be two such simples corresponding to distinct points (x,y) and (x',y') on the cusp. The space a cycles are those $(a,b) \in \mathbb{C}^2$ such that

$$T \mapsto \begin{bmatrix} x' & a \\ 0 & x \end{bmatrix} \qquad \text{and} \qquad S \mapsto \begin{bmatrix} y' & b \\ 0 & y \end{bmatrix}$$

is a 2-dimensional representation giving the condition that

$$a(x^2 + xx' + x'^2) = b(y + y')$$

which for general S,T gives one relation between a and b so the cycle-space is 1-dimensional generically. Two such extensions (a,b) and (a',b') are equivalent iff

$$a - a' = \lambda(x' - x) \qquad \text{and} \qquad b - b' = \lambda(y' - y)$$

giving a one-dimensional subspace of boundaries. So, for general S,T we have that $Ext^1_A(S,T) = 0$. However, for $(S,T) \in \Delta \cup \Delta_1 \cup \Delta_2$ where

$$\begin{cases} \Delta & = \{((x,y),(x,y)) \ : \ x^3 = y^2 \ \} \\ \Delta_1 & = \{((x,y),(\rho x,-y)) \ : \ x^3 = y^2 \ \} \\ \Delta_2 & = \{((x,y),(\rho^2 x,-y) \ : \ x^3 = y^2 \ \} \end{cases}$$

the cycle space is two-dimensional whence $Ext^1_A(S,T) \simeq \mathbb{C}$. So, in this case the Euler-form does depend on the choice of representations and hence $\mathbb{C}B_3$ is *not* a smooth algebra. In fact, the calculations above can be used to find singular points in $\mathtt{rep}_2\ \mathbb{C}B_3$.

Definition 5.6 If A is a smooth algebra, we define its *chart* $\mathtt{chart}_A$ to be the quiver with vertices corresponding to the generating set $\mathtt{gen}\ A$ of the component semigroup $\mathtt{comp}\ A$ and if $\alpha,\beta \in \mathtt{gen}\ A$, the number of directed arrows between the corresponding vertices v_α and v_β in $\mathtt{chart}_A$ is given by

$$\#\{ \ v_\alpha \longrightarrow v_\beta \ \} = \delta_{\alpha\beta} - \chi_A(\alpha,\beta)$$

Example 5.7 If Q is a quiver, then the chart $\mathtt{chart}_{\mathbb{C}Q} = Q$. Indeed, we have seen that the generators of $\mathtt{comp}\ \mathbb{C}Q$ correspond to the vertex-simples S_v (with dimension vector δ_v) and we have seen that

$$\chi_{\mathbb{C}Q}(S_v,S_w) = \chi_Q(\delta_v,\delta_w) = \delta_{vw} - \#\{ v \longrightarrow w \ \}$$

For the arithmetical groups $(P)SL_2(\mathbb{Z}), GL_2(\mathbb{Z})$ we can use the Euler form of the corresponding quiver to calculate the dimension of the ext-spaces.

Example 5.8 The Euler-form of the quiver associated to $PSL_2(\mathbb{Z})$ is

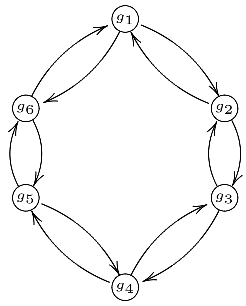
$$\begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So, for $g_1 = (1,0,1,0,0)$ we have that $g_1.\chi_Q = (1,0,0,-1,-1)$ implying that

$$\chi_Q(g_1,g_1) = 1 \quad \chi_Q(g_1,g_2) = -1 = \chi_Q(g_1,g_6) \quad \text{and} \quad \chi_Q(g_1,g_i) = 0$$

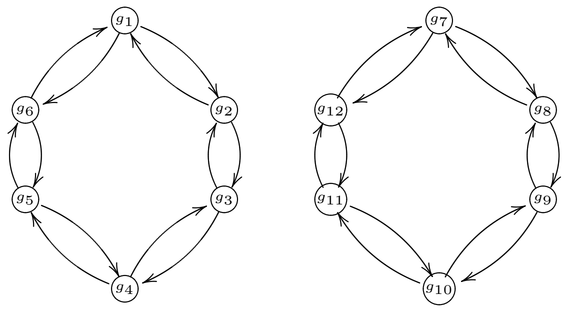
for $i \neq 1,2,6$. Performing similar computations for the other generators we see that the chart of $PSL_2(\mathbb{Z})$ has the following form

$\texttt{chart}_{PSL_2(\mathbb{Z})} =$



As the quiver of $SL_2(\mathbb{Z})$ is the disjoint union of two copies of that of $PSL_2(\mathbb{Z})$ we immediately obtain that the chart of $SL_2(\mathbb{Z})$ has the following form

$\texttt{chart}_{SL_2(\mathbb{Z})} =$



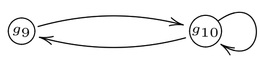
Example 5.9 The Euler-form matrices for the two components of the quiver associated to $GL_2(\mathbb{Z})$ are respectively

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

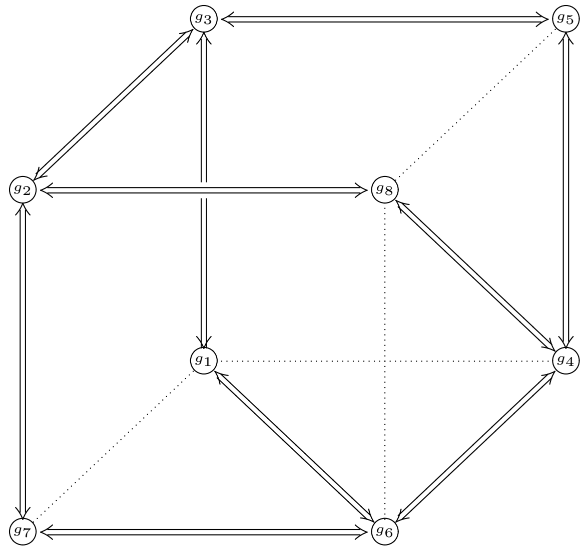
From this is it easy to work out the inproducts. For example, the inproduct matrix of the second component is given by

$\chi(g_i,g_j)$	g_9	g_{10}
g_9	1	-1
g_{10}	-1	0

whence the corresponding component of $\texttt{chart}_{GL_2(\mathbb{Z})}$ has the following form

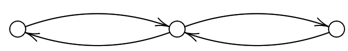


Calculating all inproduct $\chi_Q(g_i, g_j)$ for $1 \leq i, j \leq 8$ we find that the other component of $\mathtt{chart}_{GL_2(\mathbb{Z})}$ has the following form

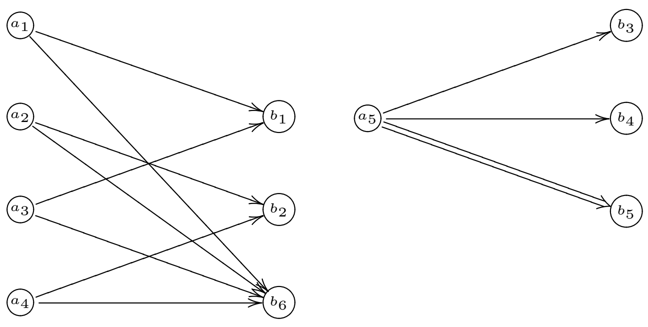


where every double arrow means 'one arrow in each direction'.

It is an interesting exercise to compute the charts of other smooth algebras, such as those coming from universal localizations of path algebras (as in the compactification of $PSL_2(\mathbb{Z})$). In the $PSL_2(\mathbb{Z})$-case one can show that every chart of such a universal localization is a subquiver of $\mathtt{chart}_{PSL_2(\mathbb{Z})}$, mainly because in the bipartite quiver associated to $PSL_2(\mathbb{Z})$ there is only one arrow between a left and a right vertex. For example, for each of the three universal localizations at Δ_i the chart is of the form



A more interesting example is for the components of the bipartite quiver Q associated to $GL_2(\mathbb{Z})$ which is



For the second component, let A be the universal localization of the path algebra at the matrix with natural notation

$$\Delta = \begin{bmatrix} B_{53} & 0 \\ 0 & B_{54} \\ B_{55}^{(1)} & B_{55}^{(2)} \\ B_{55}^{(2)} & B_{55}^{(1)} \end{bmatrix}$$

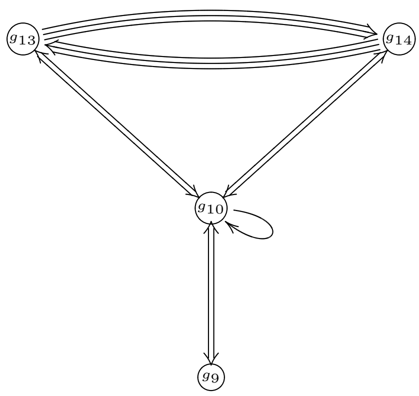
then we only have to satisfy the numerical condition

$$2a_5 = b_3 + b_4 + 2b_5$$

and not the two extra conditions coming from the requirement that on the left and right hand side there must be the same D_2-representation. Hence, in addition to the generators g_9 and g_{10} the component semigroup `comp` A has the two additional generators (for dimension $n = 4$) g_{13} and g_{14}

	n	a_5	b_3	b_4	b_5
g_{13}	4	2	2	0	1
g_9	2	1	1	1	0
g_{14}	4	2	0	2	1
g_{10}	2	1	0	0	1

and computing the inproducts one finds that $\texttt{chart}_A$ has the following shape



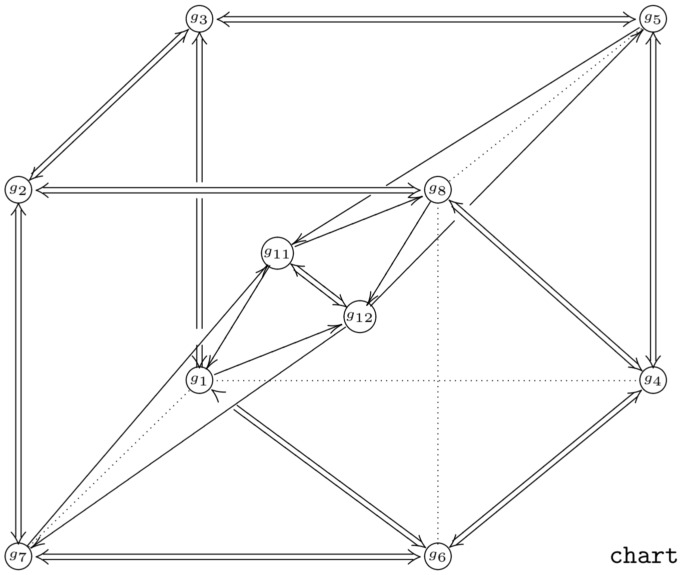
If B is the universal localization of the path algebra of the first component at the matrix

$$\begin{bmatrix} B_{11} & 0 & B_{31} & 0 \\ 0 & B_{22} & 0 & B_{42} \\ B_{16} & B_{26} & B_{36} & 0 \\ B_{16} & B_{26} & 0 & B_{46} \end{bmatrix}$$

then `comp` B has, in addition to the 8 generators $g_1, \ldots, g_8$ of `comp` $GL_2(\mathbb{Z})$ the additional two generators g_{11} and g_{12}

	n	a_1	a_2	a_3	a_4	b_1	b_2	b_6
g_{11}	2	1	0	1	0	0	0	1
g_{12}	2	0	1	0	1	0	0	1

and computing all inproducts gives us that $\texttt{chart}_B$ has the form, also depicted in the fronti-piece



$\texttt{chart}_B$

which is a non-symmetric quiver having the interesting property that every vertex has 3 incoming and 3 outgoing arrows.

5.2 Constructing simples

What can we do with the chart of a smooth algebra? To start it gives us a way to construct nearly all representations of the smooth algebra. Let us sketch the general procedure and then work it out in the special case of $PSL_2(\mathbb{Z})$-representations.

Let $\gamma \in \texttt{comp}\ A$, then α can be written as an integral combination of the generators $\{\gamma_1, \ldots, \gamma_k\} = \texttt{gen}\ A$

$$\gamma = a_1\gamma_1 + \ldots + a_k\gamma_k$$

giving us a dimension vector $\alpha = (a_1, \ldots, a_k)$ of the chart $\texttt{chart}_A$. Let S_i be simple A-representations in $\texttt{rep}_{\gamma_1}\ A$, then by construction of the chart, we see that we can identify

$$\texttt{rep}_\alpha\ \texttt{chart}_A = Ext^1_A(M, M) \qquad \text{with} \qquad M = S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k}$$

the space of self-extensions of a point $M \in \texttt{rep}_\gamma\ A$ and this identification is one as $GL(\alpha) = Stab(M)$-modules.

The space of self-extensions has another interpretation in terms of the GL_n-structure on $\texttt{rep}_\gamma\ A$ where $n = |\gamma|$. Let $\mathcal{O}(M)$ be the GL_n-orbit of M in $\texttt{rep}_\gamma\ A$, then because $Stab(M) = GL(\alpha)$ we have the following $GL(\alpha)$-modules

$$\begin{cases} T_M\ \texttt{rep}_\gamma\ A, & \text{the tangent space in } M \text{ to the component } \texttt{rep}_\gamma\ A \\ T_M\mathcal{O}(M), & \text{the tangent space in } M \text{ to the orbit } \mathcal{O}(M) \end{cases}$$

and as $GL(\alpha)$ is a reductive group the subspace $T_M\ \mathcal{O}(M)$ is a direct $GL(\alpha)$-summand of $T_M\ \mathtt{rep}_\gamma\ A$ and the quotient, that is the *normal space to the orbit*

$$N_M = \frac{T_M\ \mathtt{rep}_\gamma\ A}{T_M\ \mathcal{O}(M)} \simeq Ext^1_A(M,M) = \mathtt{rep}_\alpha\ \mathtt{chart}_A$$

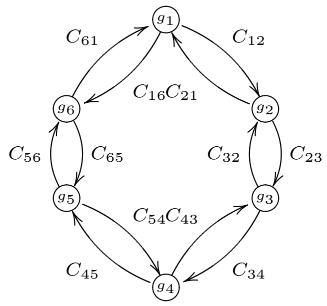
where the isomorphism is one of $Stab(M) = GL(\alpha)$-modules.

Because A is a smooth algebra we know that the component $\mathtt{rep}_\gamma\ A$ is smooth in M and therefore we can apply the *Luna slice theorem* which asserts that locally around the orbit $\mathcal{O}(M)$ the GL_n-structure of $\mathtt{rep}_\gamma\ A$ looks like that of the fiber bundle

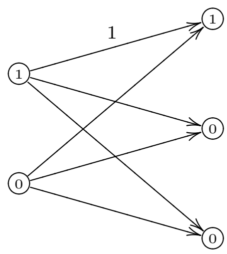
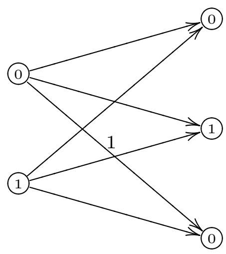
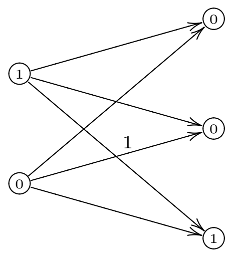
$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ \mathtt{chart}_A$$

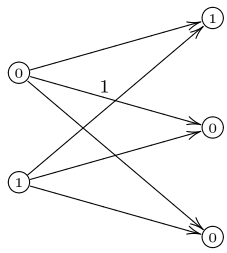
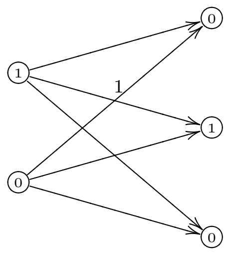
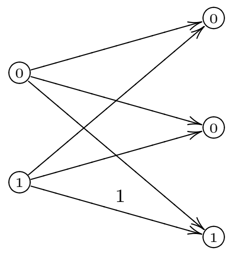
where 'locally' means in the étale (or if you want the analytic) topology. All this sounds pretty scary so let us work it out in the case of $PSL_2(\mathbb{Z})$ of which we calculated the chart to be of the form

$\mathtt{chart}_{PSL_2(\mathbb{Z})} =$



where the g_i are the generators corresponding to the six one-dimensional simple $PSL_2(\mathbb{Z})$-representations S_i, corresponding to the quiver representations

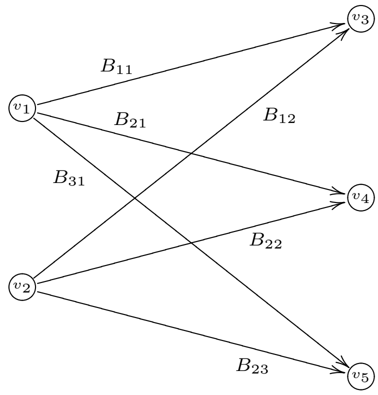
$S_1 =$ 
$S_2 =$ 
$S_3 =$ 

$S_4 =$ 
$S_5 =$ 
$S_6 =$ 

Take the component $\mathtt{rep}_\gamma\ PSL_2(\mathbb{Z})$ containing the semi-simple representation

$$M = S_1^{\oplus a_1} \oplus S_2^{\oplus a_2} \oplus S_3^{\oplus a_3} \oplus S_4^{\oplus a_4} \oplus S_5^{\oplus a_5} \oplus S_6^{\oplus a_6}$$

that is $\gamma = (a_1 + a_3 + a_5, a_2 + a_4 + a_6, a_1 + a_4, a_2 + a_5, a_3 + a_6)$ and M is the PSL_2-representation corresponding to the quiver representation



with the B_{ij} block-matrices, each block of size $a_u \times a_v$ for the appropriate u, v

$$B_{11} = \begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{a_5} \end{bmatrix} \qquad B_{31} = \begin{bmatrix} 0 & 1_{a_3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1_{a_4} & 0 \end{bmatrix} \qquad B_{22} = \begin{bmatrix} 1_{a_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{a_6} \end{bmatrix}$$

Clearly, as $\mathtt{rep}_\gamma\ Q$ is an affine space, the tangent space $T_M\ \mathtt{rep}_\gamma\ PSL_2(\mathbb{Z}) = T_M\ \mathtt{rep}_\gamma\ Q$ can be identified with $\mathtt{rep}_\gamma\ Q$. The stabilizer subgroup of M is $GL(\alpha) = GL_{a_1} \times \ldots \times GL(a_6)$. To compute the components of the tangent space in M to the orbit, take $Lie(GL(\alpha))$ as the set of matrices

$$\begin{bmatrix} A_1 & A_{13} & A_{15} \\ A_{31} & A_3 & A_{35} \\ A_{51} & A_{53} & A_5 \end{bmatrix} \oplus \begin{bmatrix} A_2 & A_{24} & A_{26} \\ A_{42} & A_4 & A_{46} \\ A_{62} & A_{64} & A_6 \end{bmatrix} \oplus \begin{bmatrix} A'_1 & A_{14} \\ A_{41} & A'_4 \end{bmatrix} \oplus \begin{bmatrix} A'_2 & A_{25} \\ A_{52} & A'_5 \end{bmatrix} \oplus \begin{bmatrix} A'_3 & A_{36} \\ A_{63} & A'_6 \end{bmatrix}$$

and hence the tangent space to the orbit is computed using the action of $GL(\gamma)$ on the quiver-representations, giving for example for the B_{11}-arrow

$$(\begin{bmatrix} 1_{a_1} & 0 \\ 0 & 1_{a_4} \end{bmatrix} + \epsilon \begin{bmatrix} A'_1 & A_{14} \\ A_{41} & A'_4 \end{bmatrix}). \begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .(\begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 1_{a_3} & 0 \\ 0 & 0 & 1_{a_5} \end{bmatrix} - \epsilon \begin{bmatrix} A_1 & A_{13} & A_{15} \\ A_{31} & A_3 & A_{35} \\ A_{51} & A_{53} & A_5 \end{bmatrix})$$

which is equal to

$$\begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} A_1 - A'_1 & -A_{13} & -A_{15} \\ A_{41} & 0 & 0 \end{bmatrix}$$

and the ϵ-components of $B_{21}, B_{31}, B_{12}, B_{22}$ resp. B_{23}.

$$B_{21} : \begin{bmatrix} 0 & 0 & A_{25} \\ -A_{51} & -A_{53} & A_5 - A'_5 \end{bmatrix} \qquad B_{31} : \begin{bmatrix} -A_{31} & A_3 - A'_3 & -A_{35} \\ 0 & A_{63} & 0 \end{bmatrix}$$

$$B_{12} : \begin{bmatrix} 0 & A_{41} & 0 \\ -A_{42} & A_4 - A'_4 & -A_{46} \end{bmatrix} \qquad B_{22} : \begin{bmatrix} A_2 - A'_2 & -A_{24} & -A_{26} \\ A_{52} & 0 & 0 \end{bmatrix}$$

$$B_{32}: \begin{bmatrix} 0 & 0 & A_{36} \\ -A_{62} & -A_{64} & A_6 - A_6' \end{bmatrix}$$

Hence, the only non-zero blocks correspond precisely to the matrices in $\mathtt{rep}_\alpha\ \mathtt{ch}_{PSL_2(\mathbb{Z})}$ which can therefore be identified with the normal space to the orbit. So, representations of the form $M + N_M$ are determined by the matrices

$$B_{11} = \begin{bmatrix} 1_{a_1} & 0 & 0 \\ 0 & C_{34} & C_{54} \end{bmatrix} \qquad B_{21} = \begin{bmatrix} C_{12} & C_{32} & 0 \\ 0 & 0 & 1_{a_5} \end{bmatrix}$$

$$B_{31} = \begin{bmatrix} 0 & 1_{a_3} & 0 \\ C_{16} & 0 & C_{56} \end{bmatrix} \qquad B_{12} = \begin{bmatrix} C_{21} & 0 & C_{61} \\ 0 & 1_{a_4} & 0 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} 1_{a_2} & 0 & 0 \\ 0 & C_{45} & C_{65} \end{bmatrix} \qquad B_{32} = \begin{bmatrix} C_{23} & C_{43} & 0 \\ 0 & 0 & 1_{a_6} \end{bmatrix}$$

and the Luna slice theorem asserts that every representation in $\mathtt{rep}_\gamma\ PSL_2(\mathbb{Z})$ near M can be brought in this form. In fact, there is a lot more to be said about the connection between representations of a smooth algebra and representations of its chart. We state these facts here in the special case of $PGL_2(\mathbb{Z})$ but they hold in general.

Theorem 5.10 *Let $\gamma = a_1 g_1 + \ldots + a_6 g_6$ then there is a GL_n ($n = \sum_i a_i$)-equivariant étale isomorphism on Zariski open subsets between*

$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ \mathtt{ch}_{PSL_2(\mathbb{Z})} \qquad \textit{and} \qquad \mathtt{rep}_\gamma PSL_2(\mathbb{Z})$$

where $\alpha = (a_1, \ldots, a_6)$ and the correspondence is given by

$$\overline{(g, V)} \leftrightarrow g.(M + V) \qquad \textit{where} \quad M = S_1^{\oplus a_1} \oplus \ldots \oplus S_6^{\oplus a_6}$$

and $\mathtt{rep}_\alpha\ \mathtt{ch}_{PSL_2(\mathbb{Z})}$ is identified with the normal space to the orbit in M. Explicitly, to a representation $(C_{12}, \ldots, C_{61})$ in $\mathtt{rep}_\alpha\ \mathtt{ch}_{PSL_2(\mathbb{Z})}$ corresponds the n-dimensional representation $PSL_2(\mathbb{Z}) \longrightarrow GL_n(\mathbb{C})$

$$\sigma \mapsto \begin{bmatrix} 1_{a_1+a_3+a_5} & 0 \\ 0 & -1_{a_2+a_4+a_6} \end{bmatrix} \qquad \textit{and} \qquad \tau \mapsto B^{-1} \begin{bmatrix} 1_{a_1+a_4} & 0 & 0 \\ 0 & \rho 1_{a_2+a_5} & 0 \\ 0 & 0 & \rho^2 1_{a_3+a_6} \end{bmatrix} B$$

whenever the following matrix is invertible

$$B = \begin{bmatrix} 1_{a_1} & 0 & 0 & C_{21} & 0 & C_{61} \\ 0 & C_{34} & C_{54} & 0 & 1_{a_4} & 0 \\ C_{12} & C_{32} & 0 & 1_{a_2} & 0 & 0 \\ 0 & 0 & 1_{a_5} & 0 & C_{45} & C_{65} \\ 0 & 1_{a_3} & 0 & C_{23} & C_{43} & 0 \\ C_{16} & 0 & C_{56} & 0 & 0 & 1_{a_6} \end{bmatrix}$$

Under this correspondence, a simple chart representation corresponds to a simple $PSL_2(\mathbb{Z})$ representation.

Moreover, two simple chart representations determine isomorphic $PSL_2(\mathbb{Z})$ representations if and only if all their traces along oriented cycles in the chart are the same (hence these can be viewed as generalized characters) and if α is the dimension vector *of a simple chart representation, then the dimension of the quotient variety parametrizing isomorphism classes of γ-dimensional $PSL_2(\mathbb{Z})$-representations is equal to*

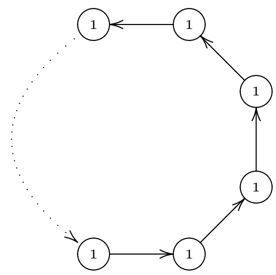
$$dim\ \mathtt{iss}_\gamma\ PSL_2(\mathbb{Z}) = 1 - \chi_{\mathtt{ch}}(\alpha, \alpha)$$

In particular, if we have representants of isoclasses of α-dimensional simple chart representations, then we have representants of isoclasses of simple $PSL_2(\mathbb{Z})$-representations near M.

In order for this result to be useful, we need a classification of all dimension vectors of simple quiver representations. Such a classification is known

Theorem 5.11 $\alpha = (d_1, \ldots, d_k) \in \mathtt{simp}\mathbb{C}Q$ *if and only if one of the following two cases holds*

1. $\mathtt{supp}\alpha = \tilde{A}_k$*, the extended Dynkin quiver on k vertices with cyclic orientation and $d_i = 1$ for all $1 \leq i \leq k$*



2. $\mathtt{supp}\alpha \neq \tilde{A}_k$*. Then, $\mathtt{supp}\alpha$ is strongly connected (meaning that any pair of vertices belongs to an oriented cycle) and for all $1 \leq i \leq k$ we have*

$$\begin{cases} \chi_Q(\alpha, \epsilon_i) & \leq 0 \\ \chi_Q(\epsilon_i, \alpha) & \leq 0 \end{cases}$$

In either case, $\mathtt{simp}\mathbb{C}Q$ is a cone in $\mathtt{comp}\mathbb{C}Q = \mathbb{N}^k$.

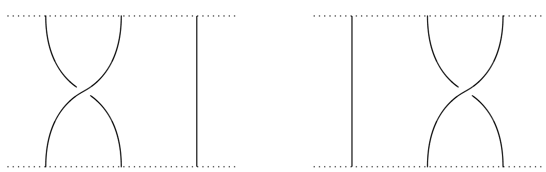
Applying this in the case of $PSL_2(\mathbb{Z})$ we deduce

Theorem 5.12 *If $\gamma = a_1g_1 + \ldots + a_6g_6$, then $\mathtt{rep}_\gamma\ PSL_2(\mathbb{Z})$ contains a simple representation if and only if*

$$a_i \leq a_{i-1} + a_{i+1}$$

where subscripts are taken modulo 6. The only exceptional case is when $\mathtt{supp}(\alpha) = \{v_i, v_{i+1}\}$ in which case the two non-zero components of α must be equal to 1.

So how can we use this to get at the promised simple representations of the third braid group B_3. Recall that B_3 was generated by the two elementary braids



σ_1 σ_2

and the defining relation of B_3 is the Yang-Baxter equation

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

For the braids $S = \sigma_1\sigma_2\sigma_1$ and $T = \sigma_1\sigma_2$ we see that $C = S^2 = T^3$ is a central element of B_3 and the quotient

$$B_3/\langle C\rangle = \langle \sigma = \overline{S}, \tau = \overline{T} \mid \sigma^2 = 1 = \tau^3\rangle = \mathbb{Z}_2 * \mathbb{Z}_3 = PSL_2(\mathbb{Z})$$

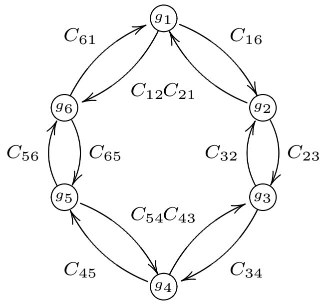
Because the central element C acts by a non-zero scalar on every simple B_3-representation we see that every simple $PSL_2(\mathbb{Z})$-representation determines (and is determined by) a one-parameter family of simple B_3-representations. Finally, to get at matrices satisfying the Yang-Baxter equation we have to recall that

$$\sigma_1 = T^{-1}S \qquad \text{and} \qquad \sigma_2 = ST^{-1}$$

These facts allow us to determine nearly all simple B_3-representations in any dimension!

Theorem 5.13 *Consider the quiver (which is the chart of $PSL_2(\mathbb{Z})$)*

$$\mathtt{chart}_{PSL_2(\mathbb{Z})} =$$



and construct from a representation $V = (C_{ij}) \in \mathtt{rep}_\alpha\ \mathtt{chart}_{PSL_2(\mathbb{Z})}$ with $\alpha = (a_1, \ldots, a_6)$ the $n \times n$ matrix (where $n = \sum_i a_i$)

$$B_V = \begin{bmatrix} 1_{a_1} & 0 & 0 & C_{21} & 0 & C_{61} \\ 0 & C_{34} & C_{54} & 0 & 1_{a_4} & 0 \\ C_{12} & C_{32} & 0 & 1_{a_2} & 0 & 0 \\ 0 & 0 & 1_{a_5} & 0 & C_{45} & C_{65} \\ 0 & 1_{a_3} & 0 & C_{23} & C_{43} & 0 \\ C_{16} & 0 & C_{56} & 0 & 0 & 1_{a_6} \end{bmatrix}$$

Then, for any non-zero scalar $\lambda \in \mathbb{C}^*$ we have that

$$\sigma_1 \mapsto \lambda B^{-1} \begin{bmatrix} 1_{a_1+a_4} & 0 & 0 \\ 0 & \rho^2 1_{a_2+a_5} & 0 \\ 0 & 0 & \rho 1_{a_3+a_6} \end{bmatrix} B \begin{bmatrix} 1_{a_1+a_3+a_5} & 0 \\ 0 & -1_{a_2+a_4+a_6} \end{bmatrix}$$

$$\sigma_2 \mapsto \lambda \begin{bmatrix} 1_{a_1+a_3+a_5} & 0 \\ 0 & -1_{a_2+a_4+a_6} \end{bmatrix} B^{-1} \begin{bmatrix} 1_{a_1+a_4} & 0 & 0 \\ 0 & \rho^2 1_{a_2+a_5} & 0 \\ 0 & 0 & \rho 1_{a_3+a_6} \end{bmatrix} B$$

is an n-dimensional representation of the third braid group B_3.

If for all i modulo 6 we have that $a_i \leq a_{i-1}+a_{i+1}$ then this representation is simple for sufficiently general V and any sufficiently general simple n-dimensional representation of B_3 (meaning a Zariski open subset of simples) can be conjugated to one of this form.

Finally, if we have representants of isomorphism classes of simple α-dimensional representations of $\mathtt{chart}_{PSL_2(\mathbb{Z})}$, then the corresponding B_3-representations classify the isomorphism classes of simple B_3-representations, finite to one.

The methods we used to construct these simple representations are general, hence also for $GL_2(\mathbb{Z})$ we can use its chart to construct nearly all simple representations (at least in principle).