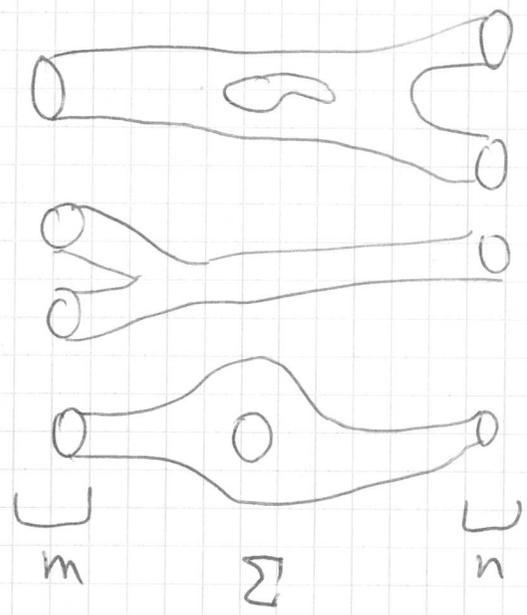


Topological Fields Theories & Algebra  
 (hep-th/9401023)      Voronov

$V$  vector space (state space). Every "string"  
 Riemann surface with  $m$  input and  $n$  output  
 cutles



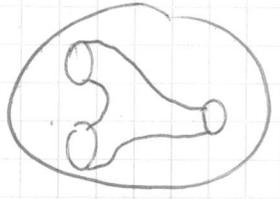
gives linear operator

$$|\Sigma\rangle: V^{\otimes m} \rightarrow V^{\otimes n}$$

satisfying certain conditions such as  
 associativity, multiplicativity, topological invariance, factorization

property (sum of surfaces = composition of maps)

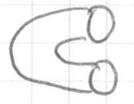
Can split any of these steps in smaller pieces out  
 hence corresponding maps.



$$V \otimes V \rightarrow V$$



$$V \otimes V \rightarrow \mathbb{C}$$
  
 (multiplication)



$$\mathbb{C} \rightarrow V \otimes V$$

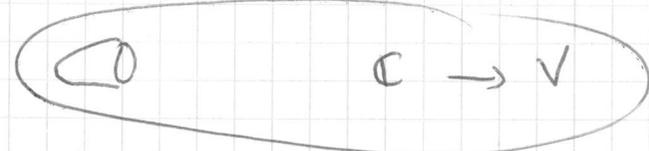
(adjoint to multiplication)



$$V \rightarrow V$$



$$V \rightarrow \mathbb{C}$$



$$\mathbb{C} \rightarrow V$$

(unit)

and physical properties can be translated into algebraic conditions.

Then A topological field theory is equivalent to a Frobenius algebra, a commutative algebra with unit with a nondegenerate symmetric bilinear form  $\langle -, - \rangle: V \otimes V \rightarrow \mathbb{C}$  invariant under multiplication.

$$\langle ab, c \rangle = \langle a, bc \rangle$$

which has an adjoint  $\mathbb{C} \rightarrow V \otimes V$

D-branes are higher dim analogs of strings and their properties can be summarized by more involved algebra:

(hep-th/0507222, hep-th/0305095)

Thm: topological D-branes  $\Leftrightarrow$  cyclic & unital weak  $A_\infty$ -category.

Finite collection of branes  $u_1, \dots, u_k$

Finite dim super-vectorspaces  $\text{Hom}(u, v)$

$\text{Hom}(u, v)$   all worldsheets from  $u$  to  $v$

super-vector space  $H = H_0 \oplus H_1$  is  $\mathbb{Z}_2$ -graded  

 $\underbrace{\quad}_{\text{even elements}}$ 
 $\underbrace{\quad}_{\text{odd elements}}$

$H[1]$  is  $H$  with degree shift so is

$$H[1] = (H_0)_1 \oplus (H_1)_0$$

suspension map  $\Sigma: H \rightarrow H[1]$  is identity  
 only "phase" of element changes  $\Sigma^2 = id$ .

maps between super-spaces can be homogeneous and be either odd or even

$$u \xrightarrow{x} v$$

$x \in \text{Hom}(u, v)$  then define  $\begin{cases} t(x) = u & \text{tail} \\ h(x) = v & \text{head} \end{cases}$

ordered collection of maps  $(x_1, \dots, x_n)$  is compatible iff  $t(x_{2+i}) = h(x_i)$



$$[x_1 \dots x_n] = \underbrace{t(x_1)t(x_2) \dots t(x_n)h(x_n)}_{\text{nc. word in the } \langle u_1, \dots, u_n \rangle}$$

if  $x$  is homogeneous element of  $\text{Hom}(u, v)$

$$\begin{cases} |x| = \deg x \text{ in } \text{Hom}(u, v) \\ \tilde{x} = \deg x \text{ in } \text{Hom}(u, v)[1] \end{cases}$$

For all  $v_1, \dots, v_{n+1}$  among  $\langle u_1, \dots, u_n \rangle$  have odd multilinear maps (rule:  $\deg 0 \rightarrow \deg 1$ ,  $\deg 1 \rightarrow \deg 0$ )

$$\tau_{v_1, \dots, v_{n+1}} : \quad (\text{for all } n \geq 0)$$

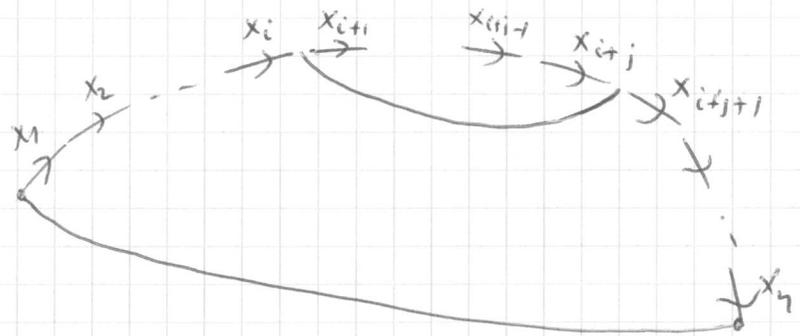
$$\text{Hom}(u_1, v_2)[1] \times \dots \times \text{Hom}(v_n, v_{n+1})[1] \rightarrow \text{Hom}(u_1, v_{n+1})[1]$$

which again describe physical content of composition of world sheets

For  $n=0$  have odd map

$$\tau_u : \mathbb{C} \rightarrow \text{Hom}(u, u)[1]$$

$1 \mapsto \sigma_u$  over element  $i \in \text{Hom}(u, u)$



Def: weak  $A_\infty$ -category iff for all homogeneous elements and composable  $(x_1, \dots, x_n)$  must satisfy:

$$\sum_{0 \leq (i,j) \leq n} (-1)^{\tilde{x}_1 + \dots + \tilde{x}_i} \tau_{[x_1 \dots x_i][x_{i+j+1} \dots x_n]}(x_1, \dots, x_i, \tau_{[x_{i+1} \dots x_{i+j}]}(x_{i+1}, \dots, x_{i+j}), x_{i+j+1}, \dots, x_n) \stackrel{!}{=} 0$$

Def: such an  $A_\infty$ -category is called

(1) STRONG iff all  $\tau_u = 0 \quad \forall u \in \text{Ob}$

(2) MINIMAL iff STRONG and  $\tau_{uv} = 0 \quad \forall u, v \in \text{Ob}$   
"units"                      "odd units"

(3) UNITAL iff  $\forall u \in \text{Ob} \exists (\mathbb{1}_u \in \text{Hom}(u, u)_0)$  with  $\lambda_u = \sum \mathbb{1}_u \in \text{Hom}(u, u)[1]$  satisfying following.

•  $\forall n \neq 2, \forall j : \tau_{[x_1 \dots x_{j-1}][x_{j+1} \dots x_n]}(x_1, \dots, x_{j-1}, \lambda_u, x_{j+1}, \dots, x_n) = 0$

•  $\tau_{[\lambda_u, x]}(\lambda_u, x) = -x$  and  $\tau_{[y, \lambda_u]}(y, \lambda_u) = (-1)^{\tilde{y}} y$

whenever morphisms are composable.



(4) CYCLIC: if there are non-degenerate bilinear forms.

$$P_{uv}: \text{Hom}(u, v) \times \text{Hom}(v, u) \rightarrow \mathbb{C}$$

all of the same  $\mathbb{Z}_2$ -degree  $\tilde{\omega}$  satisfy

$$P_{uv}(x, y) = (-1)^{|x||y|} P_{vu}(y, x)$$

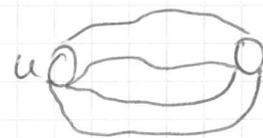
and whenever  $[x_0 \dots x_n]$  is a cyclic word in  $\text{Hom}(u_1, u_2)$ , then

$$P_{h(x_0)h(x_0)}(x_0, \tau_{[x_1 \dots x_n]}(x_1, \dots, x_n)) = (-1)^{\tilde{\omega}(x_1 + \dots + x_n)}$$

$$\tilde{\omega}(x_1 + \dots + x_n)$$

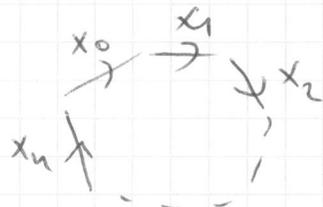
$$P_{t(x_1)h(x_1)}(x_1, \tau_{[x_2 \dots x_0]}(x_2, \dots, x_n, x_0))$$

$$(h(x_n) = t(x_0))$$



closed worldline  
 $\Rightarrow$  metric or expectation value of process

even or odd system  $\tilde{\omega}=0$   
 $\tilde{\omega}=1$



next time

nc-symplectic geometry of super-quivvers will give method to produce examples of such weak unital cyclic  $A_\infty$  algebras

MORE ON  $\left\{ \begin{array}{l} A_\infty\text{-algebras: math.RA/9910179} \\ A_\infty\text{-categories: math.CT/0310337} \end{array} \right.$

explain connection with the above.

Ob =  $\{u_1, \dots, u_n\}$  take  $R = \mathbb{C}x \dots x\mathbb{C}$  semisimple comm with idempotents  $e_i$

Take  $U = \bigoplus_{u, v} \text{Hom}(u, v)$  as super  $R$ -bimodule where bimodule structure is given via  $e_i \text{Hom}(u, v) e_j = \delta_{i, \text{source}(u)} \delta_{j, \text{target}(v)} \text{Hom}(u, v)$

Dfn  $U$  super  $R$ -bimodule  $\sigma: U \times U \rightarrow R$  homogeneous  $R$ -bilinear form of degree  $\tilde{\sigma}$   
 iff it defines an  $R$ -superbimodule map

$$U^{\text{opp}} \longrightarrow U[\tilde{\sigma}]^* \quad x \mapsto f_x(\cdot) = \sigma(\cdot, x)$$

where  $U^{\text{opp}}$  is opposite  $R$ -bimodule i.e.  $\pi \cdot u \cdot \pi' = \pi' u \pi$ . Form is non-degenerate  
 iff  $x \mapsto f_x$  is vector space iso.

Dfn non-degenerate  $R$ -bilinear form on  $U$  is said to be a  
 • metric iff graded-symmetric i.e.  $\sigma(x, y) = (-1)^{\deg x \deg y} \sigma(y, x)$   
 • symplectic iff graded-antisymmetric  $\sigma(x, y) = (-1)^{1 + \deg x \deg y} \sigma(y, x)$

For our  $U = \bigoplus_{u,v} \text{Hom}(u, v)$  coming from cyclic  $A_\infty$ -category have induced metric  
 coming from  $\rho_{uv}: \text{Hom}(u, v) \times \text{Hom}(v, u) \rightarrow \mathbb{C}$  satisfying  $\rho_{uv}(x, y) = (-1)^{|x||y|} \rho_{vu}(y, x)$ .

Observation:  $U$  has metric  $\Leftrightarrow U[\tilde{\sigma}]$  has symplectic form when  $\omega(x, y) = (-1)^{|x|} \rho(x, y)$   
 $\omega: U[\tilde{\sigma}] \times U[\tilde{\sigma}] \rightarrow U[\tilde{\sigma}]$

So: that will be our symplectic form. Where is the quiver?

$V$   $R$ -bimodule = quiver on  $k$  vertices with  $\#(i \rightarrow j) = \text{dim}_{\mathbb{C}}(e_i V e_j)$

In our case take  $V = U[1]$  gives quiver and using the decomposition

$$U[1] = U[1]_0 \oplus U[1]_1, \text{ gives certain arrow degree 0 other degree 1.}$$

Moreover, as  $V$  is symplectic  $\Rightarrow$  iso as  $R$ -bimod  $\underbrace{U[1]^{opp}}_{\text{reversed arrow}} \rightarrow \underbrace{U[1]^*}_{\text{same arrow}}$

so get a symmetric quiver, that is,

$$\#(o_i \rightarrow o_j) = \#(o_j \rightarrow o_i) \text{ if } i \neq j$$

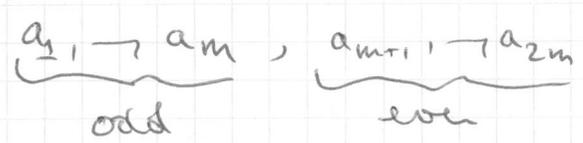
Moreover, putting (graded) symplectic form in canonical form can pick basis of  $U[1]_0$  and  $U[1]_1$  (that is: arrows in our quiver) such that symplectic form has particular form depending on whether have even or odd system:

next time

odd system ( $\tilde{w} = 1$ )

$$\omega = \sum_{i=1}^m da_i da_i^*$$

even # of arrows

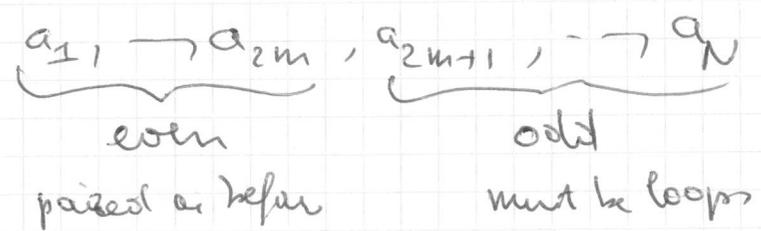


$$a_{i+m} = a_i^*$$

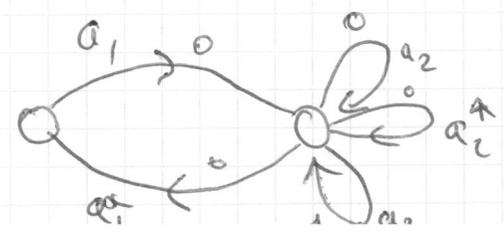
next time

even system ( $\tilde{w} = 0$ )

$$\omega = \sum_{i=1}^m da_i da_i^* + \frac{1}{2} \sum_{j=2m+1}^N da_j da_j$$



SUPER QUIVERS



|||

Q finite quiver } vertices: idempotents  
 } arrows

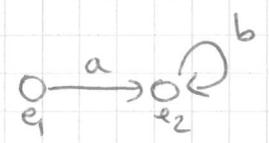
develop theory of nc differential forms on CQ

CQ path algebra: } basis oriented paths in Q,  
 } multipl. concatenation.

Recall classical case  $A = \mathcal{O}(M)$   
 $\Omega_A^1$  } generated by  $da \in A$   
 } relation  $d(ab) = (da)b + a(db)$

Example:  $e_1 \xrightarrow{a} e_2 \xrightarrow{b} e_3$

$$CQ = \begin{pmatrix} e & e & e \\ 0 & e & e \\ 0 & 0 & e \end{pmatrix} = \begin{pmatrix} ce_1 & ca & cab \\ 0 & ce_2 & cb \\ 0 & 0 & ce_3 \end{pmatrix}$$



$$CQ = \begin{pmatrix} e & a \cdot C[b] \\ 0 & C[b] \end{pmatrix}$$



$$CQ = \mathbb{C}\langle x_1, \dots, x_k \rangle$$

CQ is formally smooth algebra, so should be viewed as coordinatizing of nc manifold i fact of nc affine space.

$A \xrightarrow{d} \Omega_A^1$  universal derivation.

$$\Omega_A^n = \bigwedge_A^n \Omega_A^1 \quad \Omega_A^0 = A$$

elements  $a_0 da_1 \dots da_n$

$$\rightarrow \Omega_A^n \xrightarrow{d} \Omega_A^{n+1} \rightarrow \dots \text{ complex}$$

$a_0 da_1 \dots da_n \sim da_0 da_1 \dots da_n$

homology of complex;  $H_n$  is homology of  $A$

$$H_n^{\text{DR}} A = \frac{\text{Ker}(d^n \rightarrow \Omega^{n+1})}{\text{Im}(\Omega^{n-1} \rightarrow \Omega^n)} \quad \text{invariants of } A \text{ of } M$$

$(CQ_1) = U =$  vector space spanned by arrows

$$U = \mathbb{C}\langle x_1, \dots, x_k \rangle \quad k = \# \text{ vertices} \quad \text{bimodule}$$

$$\text{and } T_R(U) \cong CQ$$

Math. AG / 0010030 + Beckwith  
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$R = \mathbb{C}x \dots x \mathbb{C} \subset \mathbb{C}Q$  consider  $V = \mathbb{C}Q/R$  as  $R$ -bimodule

dfn: nc differential  $n$ -forms  $= \Omega_R^n \mathbb{C}Q = \mathbb{C}Q \otimes_R V^{\otimes n}$  has basis  $p_0 \otimes \bar{p}_1 \otimes \dots \otimes \bar{p}_n = p_0 dp_1 \dots dp_n$   
 whenever  $o_{i_1} p_0 \dots p_{i_1} \dots o_{i_n} p_n$  is oriented path in  $Q$ .

nc substitute for  $\wedge$  exterior product on commut. case

dfn:  $\Omega_R^n \mathbb{C}Q = \bigoplus_{n=0}^{\infty} \Omega_R^n \mathbb{C}Q$  is d-algebra with differential  $\Omega_R^n \mathbb{C}Q \xrightarrow{d} \Omega_R^{n+1} \mathbb{C}Q$   
 and multiplication

$$(a_0, a_1, \dots, a_n)(a_{n+1}, a_{n+2}, \dots, a_k) = \sum_{i=0}^n (-1)^{n-i} (a_0, \dots, a_i a_{i+1}, \dots, a_k)$$

examples:  $\begin{cases} a_0 \cdot da_1 = (a_0, a_1) = (a_0, a_1) \\ da_1 \cdot a_0 = (1, a_1)(a_0) = -(a_1, a_0) + (1, a_1 a_0) = -a_1 da_0 + d(a_1 a_0) \end{cases}$  etc.

dfn:  $d^2 = 0$  so complex so can define BGG cohomology  $H_{BGG}^n \mathbb{C}Q = \frac{\text{Ker}(\Omega^n \xrightarrow{d} \Omega^{n+1})}{\text{Im}(\Omega^{n-1} \xrightarrow{d} \Omega^n)}$

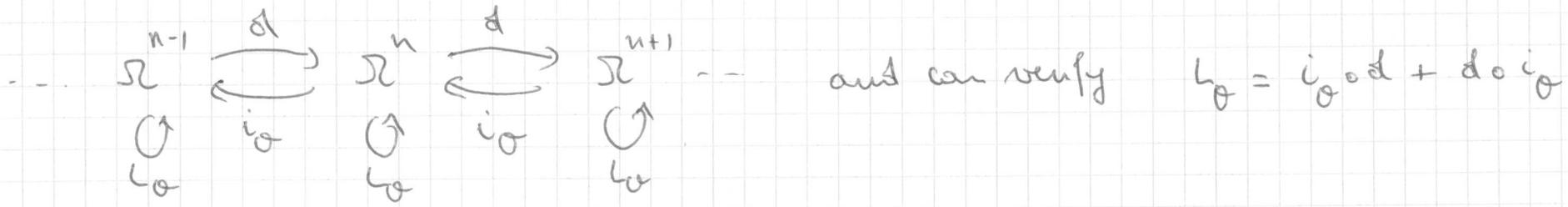
Thm: (path algebras resemble affine spaces so are contractible)

(ACYCLICITY)  $n > 1$ :  $H_{BGG}^n \mathbb{C}Q = 0$  and  $H_{BGG}^0 \mathbb{C}Q = \mathbb{C}x \dots x \mathbb{C} = R$

For the most need action of Euler derivation or complex. This is more general:

$\sigma$   $R$ -derivation on  $\mathbb{C}Q$  (i.e.  $\sigma(R) = 0$ ), then we define

- degree preserving derivation  $L_\sigma$  on  $\Omega_R^0 \mathbb{C}Q$  by the rule  $L_\sigma(a) = \sigma(a)$ ,  $L_\sigma(da) = d\sigma(a)$
- degree -1 super-derivation  $i_\sigma$  on  $\Omega_R^0 \mathbb{C}Q$  by  $i_\sigma(a) = 0$ ;  $i_\sigma(da) = \sigma(a)$   
 $i_\sigma(\omega\omega') = i_\sigma(\omega)\omega' + (-1)^{\deg \omega} \omega i_\sigma(\omega')$



proof of thm:  $E$  Euler derivation on  $\mathbb{C}Q$ :  $E(\text{vertex}) = 0$   $E(\text{arrow}) = \text{arrow}$

so if  $p$  is oriented path of length  $k$  then  $E(p) = k \cdot p$ . But also:

Assume  $\omega \in \text{Ker}(\Omega^n \rightarrow \Omega^{n+1})$   $L_E(p_0 dp_1 \dots dp_n) = (k_0 + k_1 + \dots + k_n) p_0 dp_1 \dots dp_n$

then  $L_E(\omega) = i_E d\omega + d i_E \omega = d i_E \omega$  so  $\omega = \frac{1}{k} d i_E \omega \in \text{Im}(\Omega^{n-1} \rightarrow \Omega^n)$

$k \cdot \omega$   $\downarrow$   $\square$

and for  $n=0$   $\text{Ker}(\Omega^0 \xrightarrow{d} \Omega^1) = \text{Ker } d = R \times \dots \times R$   $a + da \in \frac{\mathbb{C}Q}{R}$

~~thm~~  $\Omega_R^n \mathbb{C}Q$  is too big, so alt shown to "Koszul complex"

$\omega \in \Omega^i, \omega' \in \Omega^j$  super-commutator  $[\omega, \omega'] = \omega \cdot \omega' - (-1)^{ij} \omega' \cdot \omega$

Because  $d$  is superderivation have  $d[\omega, \omega'] = [d\omega, \omega'] + (-1)^i [\omega, d\omega']$  so  $d$  induces complex

$$\rightarrow DR_R^{n-1} \mathbb{C}Q \rightarrow DR_R^n \mathbb{C}Q \rightarrow DR_R^{n+1} \mathbb{C}Q \quad \text{with} \quad DR_R^n \mathbb{C}Q = \frac{\text{ker } d: DR^n \mathbb{C}Q}{\sum_{i=0}^n [\Omega_R^i \mathbb{C}Q, \Omega_R^{n-i} \mathbb{C}Q]}$$

and small cohomology  $H_{\text{Small}}^n \mathbb{C}Q = \frac{\text{ker } DR^n \rightarrow DR^{n+1}}{\text{im } DR^{n-1} \rightarrow DR^n}$

Thm (acyclicity)  $H_{\text{Small}}^n \mathbb{C}Q = 0$  for  $n \geq 1$  and  $H_{\text{Small}}^0 \mathbb{C}Q = \mathbb{C}x \dots x \mathbb{C} = \mathbb{R}$ .

*pf*: (same use that  $i_E$  and  $L_E$  induce maps on smaller complex.)

Interpretation of the lowest small nc differential forms

①  $DR_R^0 \mathbb{C}Q = \frac{\mathbb{C}Q}{[\mathbb{C}Q, \mathbb{C}Q]}$  is vector space with basis necklaces in  $Q$   
↳ oriented circuits in  $Q$  upto cyclic permutation  
are "nc-functions" on nc-manifold associated to  $Q$ .

*pf*:  $\mathbb{C}Q \xrightarrow{n} \text{neck}$  defined by  $p \mapsto \omega_p$  if  $p$  is cycle  $p \neq \emptyset$  otherwise. Because  $\omega_{p_1 p_2} = \omega_{p_2 p_1}$   
we have  $[\mathbb{C}Q, \mathbb{C}Q] \subset \text{ker } n$

$x = x_0 + x_1 + \dots + x_m$  where  $x_0$  is lin combi of non circuits

non-any circuit  $p = a \cdot p' \omega$   
be with  $a$   $p = a \cdot p' = [a, p'] \subset \text{ker } n$

$x_i = a_1 p_1 + a_2 p_2 + \dots$   
 $n = a \cdot a' + \dots + a \cdot b \cdot b'$   
 $x_i$  lin combi of circuits map to same necklace

+ and d.o. is commutative (e.g.)

so  $x_i = \text{conn} + \text{lin combi of fewer oriented circuits} \Rightarrow$  induction and  $x_i \in [\mathbb{CQ}, \mathbb{CQ}] \boxtimes$

$$\square \quad DR_R^1 \mathbb{CQ} \cong \bigoplus_{\textcircled{0} \xrightarrow{a} \textcircled{1}} \textcircled{0} \rightsquigarrow \textcircled{1} \quad d(\textcircled{0} \xrightarrow{a} \textcircled{1}) \cong \sum_a (e_j \otimes \mathbb{CQ} e_i) da$$

Pr: If  $p, q$  not circuit  $\Rightarrow pdq = [p, dq]$  is commutator so vanishes in  $DR^1$   
 Have only to consider  $pdq$  with  $p, q$  circuit. If  $p, q, r$  paths in  $Q$  then have

$[p, qdr] = pqdr - qd(rp) + qrdp$  and so in  $DR^1$ ;  $qd(rp) = pqdr + qrdp$   
 so can reduce  $d$  part to combination of arrows. This defines maps

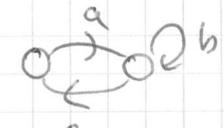
$$DR_R^1 \mathbb{CQ} \rightarrow \sum_a (e_j \otimes \mathbb{CQ} e_i) da$$

and check that  $[DR^0, DR^1]$  is in kernel and similar argument as before that this is iso  $\boxtimes$ .

Immediate use: can define "partial arrow derivatives" for necklaces.

$n = \text{necklace in } Q \quad DR^0 \xrightarrow{d} DR^1 \quad \text{with } dn = \sum_a \underbrace{p_a}_{\text{paths}} da$

then  $\boxed{\frac{\partial n}{\partial a} \stackrel{\text{def}}{=} p_a}$

Example:   $n = ab^k cae \quad \frac{\partial n}{\partial a} = b^k cae + cae b^k$

Symplectic structure : classical case  $\omega$  non degenerate closed 2-form on  $M$

induces isomorphism  $T_P M \cong T_P^* M$   
to space cotangent space

so  $\text{Vector fields} \leftrightarrow \text{diff 1-forms}$   
derivation on  $\mathcal{O}(M)$

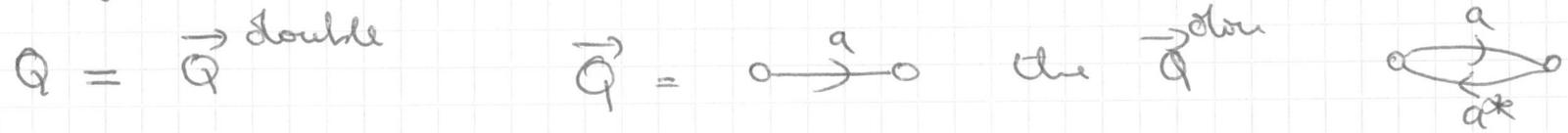
"symplectic vectorfield" if  $L_X(\omega) = 0$   
 $\text{Vect}_\omega M$  Lie algebra of symplectic vectorfields.

$\mathcal{O}(M)$  is also Lie algebra (Poisson algebra)  
via  $\{f, g\} = \omega(X_f, X_g)$

and have classical sequence Lie map

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O}(M) \xrightarrow{L} \text{Vect}_\omega M \rightarrow H^1_{dR} M \rightarrow 0$$

Will now generalize this to case  $\mathbb{C}Q$  where  $Q$  is symplectic quiver that is



and we have canonical 2-form  $\omega = \sum_{a \in \vec{Q}} da da^* \in DR^2_{\mathbb{R}} \mathbb{C}Q$

Noncommutative vectorfields  $\leftrightarrow$  Noncomm 1-forms  
"  $\xrightarrow{\tau}$   $DR^1_{\mathbb{R}} \mathbb{C}Q$   
R-derivations of  $\mathbb{C}Q$

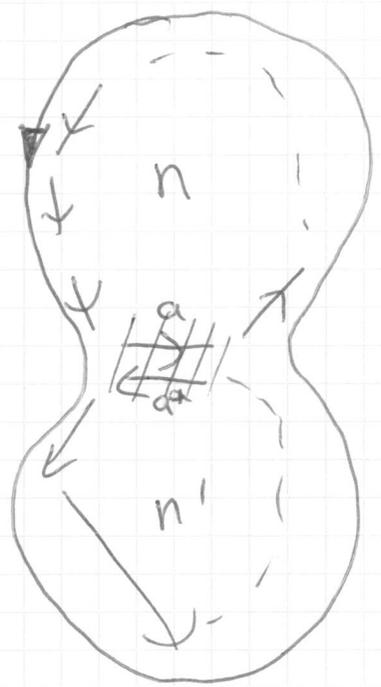
then  $\theta \mapsto i_\theta(\omega) \in DR^1_{\mathbb{R}} \mathbb{C}Q$

is isomorphism and  $i_\theta(\omega) = \sum_{a \in \vec{Q}} i_\theta(da) da^* - i_\theta(da^*) da = \sum \theta(a) da^* - \theta(a^*) da$   
as  $\theta$  is determined by  $\theta(a)$



graphical

$$\frac{\partial n}{\partial a} \frac{\partial n'}{\partial a^*} =$$



one verifies that is indeed lie algebra (satisfies Jacobi) and that square  $\otimes$  is lie algebra map.

next time: give super-version of this and relate back to weak cyclic  $A_{\infty}$ -categories, unital

Recall: top D-branes  $\Leftrightarrow$  weak cyclic unital  $A_\infty$ -category  
 $\{u_1, \dots, u_n\}$

$$U = \bigoplus_{\substack{u, v \\ \in \text{Ob}}} \text{Hom}(u, v) \quad R = \mathbb{C}x \dots x^{\ell}$$

$R$  super bimodule

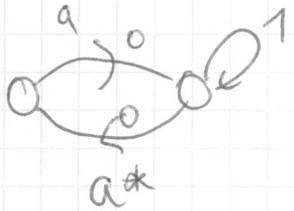
$U[\tilde{\omega}]$  can be represented by symmetric quiver

$U$  metric  $\Leftrightarrow U[\tilde{\omega}]$  symplectic.

even sytle

$$\tilde{\omega} = 0$$

$$\omega = \sum_{i=1}^m \underbrace{d a d a^*}_{\text{odd even}} + \frac{1}{2} \sum_{\text{odd}} d a d a$$

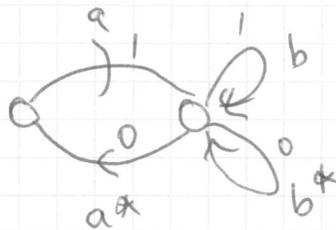


super quiver.

odd sytle

$$\tilde{\omega} = 1$$

$$\omega = \sum_{\text{odd even}} d a d a^*$$



super quiver.

$(U[\tilde{\omega}], \omega) =$  space on arrows as  $R$ -superbimodule

What are additional requirements (cyclic unital  $A_\infty$ -cat) in term of  $R$ -superbimodule?

$\exists$  odd  $R$ -linear maps

(local  $A_\infty$ -cat)

$\tau_n: U[\tilde{\omega}]^{\otimes n} \rightarrow U[\tilde{\omega}]$  satisfying for all homop. elements  $x_i$  ( $R$ -lin forces the to be compatible)

category of  $R$ -bimodule

$$\sum_{0 \leq i+j \leq n} (-1)^{\tilde{x}_1 + \dots + \tilde{x}_i} \tau_{n-j+1}(x_1, \dots, x_i, \tau_j(x_{i+1}, \dots, x_{i+j}), x_{i+j+1}, \dots, x_n) = 0$$

UNITAL:  $\exists$  even element  $1 \in U^R$

$\exists$  odd el  $\lambda = \sum 1 \in U[\tilde{\omega}]^R$

$$-\tau_2(\lambda, x) = (-1)^{\tilde{x}} \tau_2(x, \lambda) = x$$

$$\tau_n(x_1, \dots, x_i, \lambda, x_{i+1}, \dots, x_n) = 0 \quad \forall n \geq 2 \quad \forall i$$

CYCLIC :  $U[1]$  has symplectic form  $\omega$  st.

$$\omega(x_0, \mathbb{Z}_n(x_1, \dots, x_n)) = (-1)^{\tilde{x}_0 + \tilde{x}_1 + \tilde{x}_0(\tilde{x}_1 + \dots + \tilde{x}_n)} \omega(x_1, \mathbb{Z}_n(x_2, \dots, x_n, x_0))$$

superstructure on  $\mathbb{Q}$  makes  $\mathbb{C}\mathbb{Q}$  into an  $R$ -superalgebra ( $\mathbb{Z}_2$ -gradation coming from gradation on arrows) and hence the  $R$ -bimodule

$$V = \mathbb{C}\mathbb{Q}/R \text{ is super } R\text{-bimodule so also } \Omega_R^0 \mathbb{C}\mathbb{Q} = \bigoplus \Omega_R^i \mathbb{C}\mathbb{Q} = \bigoplus (\mathbb{C}\mathbb{Q} \otimes V^{\otimes i})$$

become super bimodules.

become  $\mathbb{N} \times \mathbb{Z}_2$  graded  
degree of form  $\uparrow$   $\uparrow$  sym

Can repeat all nc-symplectic geometry as before, but have to take care of superstructure in defining commutators.

$$[\Omega^i, \Omega^j]_{\text{new}} = \left\{ \begin{array}{l} [\omega_1, \omega_2] = \omega_1 \omega_2 - (-1)^{\deg \omega_1 \deg \omega_2} \omega_2 \omega_1 \end{array} \right.$$

with  $\deg \omega_1 = (n, \alpha)$   
 $\deg \omega_2 = (m, \beta)$

the  $\deg \omega_1 \cdot \deg \omega_2 = \underbrace{[nm]}_{nm \pmod 2} + \alpha\beta$

so may get other (super)  $R$ -bimodules

$$DR_R^n \mathbb{C}\mathbb{Q} = \frac{\Omega_R^n \mathbb{C}\mathbb{Q}}{\sum_i [\Omega_R^i, \Omega_R^{n-i}]_{\text{new}}}$$



$A = \mathbb{C}Q$  superalgebra

$\text{Der}_R(A)$

are (super)  $R$ -derivations of  $A$ , that is

spanned by homogeneous derivs  $D$  satyf.

$$D(ab) = D(a)b + (-1)^{\tilde{a}\tilde{D}} a D(b) \quad \text{where } \tilde{D} \text{ is degree of } D$$

becomes Lie superalgebra via  $[D_1, D_2] = D_1 \circ D_2 - (-1)^{\tilde{D}_1 \tilde{D}_2} D_2 \circ D_1$

$\theta \in \text{Der}_R(A)$  like as before have  $\bar{L}_\theta \in \text{Der}_R^{-1, \tilde{\theta}}(R^*A)$ ,  $L_\theta \in \text{Der}_R^{0, \tilde{\theta}}(R^*A)$

$$\Rightarrow \bar{L}_\theta, L_\theta \text{ on } DR^*A$$

### Symplectic derivations

$$\text{Der}_R^\omega A = \{ \theta \in \text{Der}_R A \mid \bar{L}_\theta \omega = 0 \} \subset \text{Der}_R(A) \quad \text{is Lie superalgebra}$$

Can repeat cyclicity result in super-xttyf, only have to worry about  $\mathbb{Z}_2$ -degree of  $\omega$  (i.e. even or odd system) and get situation.

$$\begin{array}{ccccccc} 0 \rightarrow R & \rightarrow & DR^0 \mathbb{C}Q & \rightarrow & (DR^1 \mathbb{C}Q)_{\text{exact}} & \rightarrow & 0 \\ & & \parallel & & \downarrow \tau^{-1} & & \\ 0 \rightarrow R & \rightarrow & \text{superneck} & \rightarrow & \text{Der}_\omega \mathbb{C}Q[\tilde{\omega}] & \rightarrow & 0 \end{array}$$



Super

Analogue of Lie algebra structure as before is that the time Kontsevich bracket satisfies relation

$$\{g, f\} = (-1)^{1 + (\tilde{f} + \tilde{w})(\tilde{g} + \tilde{w})} \{f, g\}$$

and so with this bracket

superneck  $[\tilde{w}]$  becomes super Lie algebra and map

$$\text{superneck}[\tilde{w}] \rightarrow \text{Der}_{\omega} \mathbb{C}Q \text{ is super Lie algebra map.}$$

epi

After this: RETURN TO  $A_{\infty}$ -categories and make CONNECTION.

$U(1)$  has  $\mathbb{C}$ -basis around  $Q$  as  $\{a_1, \dots, a_n\}$  fix order.

Crucial observation: take  $Q$  (odd)  $\mathbb{R}$ -derivation on  $\mathbb{C}Q$ . Fully determined by images  $Q(a)$  where  $a$  is arrow

Write.

$$Q(a) = \sum_{n \geq 0} Q_{a_1 \dots a_n}^a \overbrace{a_1 \dots a_n}^{\text{paths} = Q}$$

$\in \mathbb{C}$

Because  $Q$  odd derivation  $\Rightarrow Q^2 \stackrel{!}{=} \frac{1}{2}\{Q, Q\}$  is derivation and hence  $\textcircled{6}$

$$Q^2 = 0 \Leftrightarrow Q^2(a) = 0 \text{ for all arrows } a.$$

Call this det.  $Q$

OBSERVATION  $\textcircled{4}$   $Q$  odd derivation  $(\text{Der}_{\mathbb{R}}(Q))_1$ , define odd linear maps  
 main content of hep-th/0507222  $\otimes n$

$$\tau_n : U[1] \rightarrow U[1] \text{ via } \tau_n(a_1, \dots, a_n) = \sum_a Q_{a_1 \dots a_n}^a a$$

$\textcircled{A}$  Then these maps define weak  $A_\infty$ -structure  $\Leftrightarrow Q^2 = 0$

$\textcircled{B}$  This structure is cyclic  $\Leftrightarrow Q$  is supersymplectic derivation  $\in (\text{Der}^\omega(Q))_1$

Alternatively, using exact sequence

$$0 \rightarrow \mathbb{R}[\tilde{\omega}] \rightarrow \text{Supermod}[\tilde{\omega}] \rightarrow \text{Der}_\omega(Q) \rightarrow 0$$

topological D-branes = weak cyclic  $A_\infty$ -structure

= giving ~~even~~ supermodular  $W$  of degree  $\tilde{\omega} + 1$

$$\text{s.t. } \{W, W\} = 0$$

Moreover, then  $W$  is uniquely determined upto constant  $W_{min} \in \mathbb{R}$  (7)  
 so if choose  $W$  s.t. vanishing at  $W_{min}$  unique: SUPERPOTENTIAL of system.

Remain UNITAL-condition.

lin comb  
of loops in vector  $v$ .

$$\lambda_v = \sum_i a_i^v \text{ loop}_i$$

$$\boxed{10} \text{ UNITAL} \Leftrightarrow \exists \lambda = \sum_{\text{vectors}} \lambda_v \text{ s.t.}$$

~~Vertices~~  
 $\int$  ~~vertices~~

$$\Leftrightarrow \frac{\partial W}{\partial \lambda} = -\frac{1}{2} \sum_{a \in \vec{Q}} [a, a^*]$$

//

right hand side is called moment map for  $g$ -curve representation.

$$\sum_{\text{vert}} \sum_i a_i^v \frac{\partial W}{\partial l_i}$$

Variation  $A = \frac{\mathbb{C}Q}{\left( \frac{\partial W}{\partial a} = a \text{ arrow} \right)}$