# noncommutative geometry@n

volume 1 : the tools

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for my mother
simonne stevens (1926-2004)

# Contents

# Introduction

A crucial result in (commutative) algebraic geometry is the anti-equivalence of categories

$$\begin{tikzcd} \mathtt{commalg} \arrow[r, bend left, "\mathtt{spec}"] & \mathtt{affine} \arrow[l, bend left, "\mathbb{C}[-]"] \end{tikzcd}$$

between the category $\mathtt{commalg}$ of all affine commutative $\mathbb{C}$-algebras and the category $\mathtt{affine}$ of all affine schemes, determined by associating to an affine commutative $\mathbb{C}$-algebra $C$ its *affine scheme* $\mathtt{spec}\ C$ and to an affine scheme $\mathtt{X}$ its coordinate ring $\mathbb{C}[\mathtt{X}]$.

The *points* of $\mathtt{spec}\ C$ correspond to the maximal ideals $\mathfrak{m}$ of $C$, or equivalently, to the *one-dimensional representations* of $C$ (that is, to the algebra morphisms $C \longrightarrow \mathbb{C}$). We will see that the set of all one-dimensional representations of $C$ can be given the structure of an affine scheme, $\mathtt{rep}_1\ C$, such that there is an isomorphism of affine schemes $\mathtt{spec}\ C \simeq \mathtt{rep}_1\ C$. Hence, the above anti-equivalence can be rephrased as

$$\begin{tikzcd} \mathtt{commalg} \arrow[r, bend left, "\mathtt{rep}_1"] & \mathtt{affine} \arrow[l, bend left, "\mathbb{C}[-]"] \end{tikzcd}$$

In this book we will prove a natural extension of this anti-equivalence to the category $\mathtt{alg}$ of *all* affine $\mathbb{C}$-algebras. For a non-commutative algebra $A$, it is not natural to restrict to one-dimensional representations so we will define an affine scheme $\mathtt{rep}_n\ A$ whose points are precisely the *$n$-dimensional representations* of $A$, that is, the $\mathbb{C}$-algebra morphisms $A \longrightarrow M_n(\mathbb{C})$. We will view $\mathtt{rep}_n\ A$ as a *level $n$ approximation* of a non-commutative affine scheme associated to $A$. Hence, we can define a functor

$$\mathtt{alg} \xrightarrow{\mathtt{rep}_n} \mathtt{affine}$$

but this can never be close to an anti-equivalence.

To begin, the map is not surjective as the affine scheme $\mathtt{rep}_n\ A$ has some additional structure.

For example, we can *conjugate* an algebra morphism $\phi$

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\phi_g"'] & M_n(\mathbb{C}) \arrow[d, "g.g^{-1}"] \\ & M_n(\mathbb{C}) \end{tikzcd}$$

by any invertible $n \times n$ matrix $g \in GL_n$ to obtain another algebra morphism $\phi_g$. This defines an *action* of the linear reductive group $GL_n$ on the affine scheme $\mathtt{rep}_n\ A$. Therefore, the image of the above functor must be contained in `GL(n)-affine`, the category of all affine schemes with a $GL_n$-action. Remark that in the special case of one-dimensional representations (that is $n = 1$) we considered before, we didn't spot this extra structure as the natural $\mathbb{C}^*$-action on $\mathtt{rep}_1\ A$ is trivial. So, for fixed $n$, we'd better consider the functor

$$\mathtt{alg} \xrightarrow{\mathtt{rep}_n} \mathtt{GL(n)\text{-}affine}$$

Still, this cannot be an anti-equivalence because the map is not injective. There may be non-isomorphic affine $\mathbb{C}$-algebras $A$ and $B$ with $\mathtt{rep}_n\ A \simeq \mathtt{rep}_n\ B$. For example, assume that $A$ does *not* satisfy all the polynomial identities of $n \times n$ matrices and let $I_n$ be the twosided ideal of $A$ generated by all evaluations $p_n(a_1, \ldots, a_k)$ of polynomial identities $p_n$of $M_n(\mathbb{C})$ in elements $a_i \in A$, then it follows that every $\mathbb{C}$-algebra morphism $A \longrightarrow M_n(\mathbb{C})$ factors through $\overline{A} = A/I_n$ whence $\mathtt{rep}_n\ A \simeq \mathtt{rep}_n\ \overline{A}$. So, we better restrict to algebras satisfying all polynomial identities of $n \times n$ matrices.

In fact, we will consider a slightly different category, `alg@n`, the category of all affine *Cayley-Hamilton algebras of degree n*. Consider the category `alg@` of affine algebras $A$ with a *trace map* $tr_A \ : \ A \longrightarrow A$ and with *trace preserving algebra maps* as morphisms. There is a functor $\int \ : \ \mathtt{alg} \longrightarrow \mathtt{alg@}$ which assigns to an affine $\mathbb{C}$-algebra $A$ the algebra $\int A$ obtained by tensoring $A$ with the symmetric algebra on the vector-space quotient $A/[A,A]_v$ and equipped with the trace map which sends $a \in A$ to its image $\overline{a}$ in the space $A/[A,A]_v$. Factor $\int A$ by the two-sided ideal of all *trace identities* holding in $n \times n$ matrices to obtain an (affine) algebra $\int_n A$. We have a commuting triangle of functors

$$\begin{tikzcd} & \mathtt{alg@n} \arrow[dr, "\mathtt{trep}_n"] & \\ \mathtt{alg} \arrow[ur, "\int_n"] \arrow[rr, "\mathtt{rep}_n"] & & \mathtt{GL(n)\text{-}affine} \end{tikzcd}$$

The functor $\mathtt{trep}_n$ (which assigns to $A \in \mathtt{alg@n}$ the affine scheme $\mathtt{trep}_n\ A$ of *trace preserving* $n$-dimensional representations of $A$) is our best hope to extend the classical anti-equivalence between commutative affine algebras and affine schemes to level $n$, that is to `noncommutative geometry@n`.

For $\mathtt{X} \in \mathtt{GL(n)\text{-}affine}$, a natural substitute for the coordinate ring $\mathbb{C}[\mathtt{X}]$ of polynomial functions is the algebra $\Uparrow^n [\mathtt{X}]$ of all $GL_n$*-equivariant polynomial maps* $\mathtt{X} \longrightarrow M_n(\mathbb{C})$. It turns out that this *witness algebra* $\Uparrow^n [\mathtt{X}]$ is indeed a Cayley-Hamilton algebra of degree $n$ and so we do have functors

$$\begin{tikzcd} \mathtt{alg@n} \arrow[r, bend left, "\mathtt{trep}_n"] & \mathtt{GL(n)\text{-}affine} \arrow[l, bend left, "\Uparrow^n"] \end{tikzcd}$$

The desired extension of the anti-equivalence to level $n$ noncommutative geometry is the following result, due to Claudio Procesi [68]

**Theorem 0.1 (Procesi)** *The witness functor* $\Uparrow^n$ *is a* left inverse *to the functor* $\mathtt{trep}_n$ *associating to a Cayley-Hamilton algebra* $A \in \mathtt{alg@n}$ *the affine* $GL_n$*-scheme of trace preserving* $n$*-dimensional representations.*

Hence, we can recover the Cayley-Hamilton algebra $A \in \mathtt{alg@n}$ from the $GL_n$-geometry of the affine scheme $\mathtt{trep}_n\ A$. However, we will give examples that these functors do *not* determine an anti-equivalence of categories. In fact, it is a major open problem to identify among all $GL_n$-affine varieties the representation schemes of algebras.

We can connect this *near miss* anti-equivalence at level $n$ to the anti-equivalence of commutative algebraic geometry. We associate to an $A \in \mathtt{alg@n}$ the commutative subalgebra $\oint_n A = tr_A(A)$. Conversely, geometric invariant theory associates to an affine $GL_n$-scheme $\mathtt{trep}_n\ A$ the *quotient scheme*

$$\mathtt{trep}_n\ A/GL_n \simeq \mathtt{triss}_n\ A$$

whose points classify the *closed* orbits. We will see that $GL_n$-closed orbits correspond to the isomorphism classes of $n$-dimensional *semi-simple* representations of $A$. We obtain a commuting diagram of functors

$$\begin{tikzcd} \mathtt{alg@n} \arrow[r, bend left, "\mathtt{trep}_n"] \arrow[d, "\oint_n"] & \mathtt{GL(n)\text{-}affine} \arrow[l, bend left, "\Uparrow^n"] \arrow[d, "/GL_n"] \\ \mathtt{commalg} \arrow[r, bend left, "\mathtt{spec}"] & \mathtt{affine} \arrow[l, bend left, "\mathbb{C}[-]"] \end{tikzcd}$$

Hence, in particular, we recover the central subalgebra $\oint_n A$ as the coordinate algebra of the scheme $\mathtt{triss}_n\ A$ classifying isomorphism classes of $n$-dimensional semi-simple representations.

Having generalized the classical anti-equivalence of categories $\mathtt{commalg} \simeq \mathtt{affine}^o$ to level $n$, we turn to defining and classifying smooth objects in $\mathtt{alg@n}$. These *Cayley-smooth* algebras are defined in terms of a lifting property with respect to nilpotent ideals, motivated by Grothendieck's characterization of commutative regular algebras. We will prove Procesi's result that $A \in \mathtt{alg@n}$ is Cayley-smooth if and only if the corresponding representation scheme $\mathtt{trep}_n\ A$ is a smooth affine variety. An important source of examples of Cayley-smooth algebras is the level $n$ approximations $\int_n\ A$ of *Quillen-smooth* algebras $A$, that is, quasi-free algebras in the terminology of J. Cuntz and D. Quillen [23] or formally smooth algebras in the terminology of M. Kontsevich [46].

A commutative smooth variety is locally diffeomorphic to affine space. Rephrased in algebraic terms, for every maximal ideal $\mathfrak{m}$ of $C$, the coordinate ring of an affine variety $\mathtt{X}$ of dimension $d$, we have that the $\mathfrak{m}$-adic completion

$$\hat{C}_{\mathfrak{m}} \simeq \mathbb{C}[[x_1, \ldots, x_d]]$$

is isomorphic to the algebra of formal power series in $d$ variables. In this book we will be able to extend this *étale local* classification to Cayley-smooth algebras. It is no longer true that there is just one local type for every central dimension $d$, but the different types can be classified, up to Morita equivalence, by a combinatorial gadget : a *marked quiver* $Q^\bullet$ and a *dimension vector* $\alpha$.

Let $A \in \mathtt{alg@n}$ and consider a maximal ideal $\mathfrak{m}$ of the central subalgebra $tr_A(A)$. As this is the coordinate ring of the quotient variety $\mathtt{triss}_n\ A$, the ideal $\mathfrak{m}$ determines the isomorphism class of an $n$-dimensional semi-simple representation

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the $S_i$ are simple representations of $A$ of dimension $d_i$ and occurring in $M$ with multiplicity $e_i$ (so $n = \sum d_i e_i$). We associate to $M$ a quiver $Q$ on $k$ vertices (where vertex $i$ corresponds to the simple factor $S_i$) and where the number of arrows in $Q$ between vertices is given by the formula

$$\#\ \{\ \textcircled{i} \longrightarrow \textcircled{j}\ \} = dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j)$$

Remark that taking the multiplicities $e_i$ to be the components of the dimension vector $\alpha = (e_1, \ldots, e_k)$, then the affine space of $\alpha$-dimensional *quiver representations* $rep_\alpha\ Q$ can be identified with the space

$$rep_\alpha\ Q \simeq Ext^1_A(M, M)$$

of *self-extensions* of $M$. A self-extension $e \in Ext^1_A(M, M)$ defines an algebra morphism

$$\phi_e\ :\ A \longrightarrow M_n(\mathbb{C}[\epsilon])$$

to $n \times n$-matrices over the dual numbers $\mathbb{C}[\epsilon] = \mathbb{C}[x]/(x^2)$, so we can look at the subspace $Ext^t_A(M, M)$ of trace-preserving self-extensions. We will see that this subspace can be identified with the representation space $rep_\alpha\ Q^\bullet$, this time of a marked quiver $Q^\bullet$ which is obtained from $Q$ by removing certain loops and possibly marking others. We call the pair $(Q^\bullet, \alpha)$ the *local quiver setting* of the Cayley-Hamilton algebra $A$ in $\mathfrak{m}$. The desired étale local characterization was proved in [58].

**Theorem 0.2** *If $A \in \mathtt{alg@n}$ is Cayley-smooth and $\mathfrak{m}$ is a maximal ideal of the central subalgebra $tr_A(A)$, then the $\mathfrak{m}$-adic completion $\hat{A}_{\mathfrak{m}}$ can be reconstructed from the local quiver setting of $A$ in $\mathfrak{m}$ together with knowledge of the dimensions of the simple components of the semi-simple representation $M$ determined by $\mathfrak{m}$.*

# 1 — Cayley-Hamilton Algebras

In this chapter we will define the category $\texttt{alg@n}$ of Cayley-Hamilton algebras of degree $n$. These are affine $\mathbb{C}$-algebras $A$ equipped with a trace map $tr_A$ such that all trace identities holding in $n \times n$ matrices also hold in $A$. Hence, we have to study trace identities and, closely related to them, necklace relations.This requires the description of the *generic algebras*

$$\int_n \mathbb{C}\langle x_1, \ldots, x_m \rangle = \mathbb{T}_n^m \qquad \text{and} \qquad \oint_n \mathbb{C}\langle x_1, \ldots, x_m \rangle = \mathbb{N}_n^m$$

called the *trace algebra of $m$ generic $n \times n$ matrices*, respectively the *necklace algebra of $m$ generic $n \times n$ matrices*. For every $A \in \texttt{alg@n}$ there are epimorphisms $\mathbb{T}_n^m \twoheadrightarrow A$ and $\mathbb{N}_n^m \twoheadrightarrow tr_A(A)$ for some $m$.

In chapter 2 we will reconstruct the Cayley-Hamilton algebra $A$ (and its central subalgebra $tr_A(A)$) as the ring of $GL_n$-equivariant polynomial functions (resp. invariant polynomials) on the representation scheme $\texttt{rep}_n\, A$. Using the Reynolds operator in geometric invariant theory, it suffices to prove these results for the generic algebras mentioned above. An $n$-dimensional representation of the free algebra $\mathbb{C}\langle x_1, \ldots, x_m \rangle$ is determined by the images of the generators $x_i$ in $M_n(\mathbb{C})$ whence

$$\texttt{rep}_n\, \mathbb{C}\langle x_1, \ldots, x_m \rangle \simeq \underbrace{M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})}_{m}$$

and the $GL_n$-action on it is that of *simultaneous conjugation*. For this reason we have to understand the fundamental results on the invariant theory of $m$-tuples on $n \times n$ matrices, due to Claudio Procesi [67].

## 1.1 Conjugacy classes of matrices

In this section we recall the standard results in the case when $m = 1$, that is, the study of conjugacy classes of $n \times n$ matrices. Clearly, the conjugacy classes are determined by matrices in Jordan normal form. Though this gives a complete set-theoretic solution to the orbit problem in this case, there cannot be an orbit variety due to the existence of non-closed orbits. Hence, the geometric study of the conjugacy classes splits up into a *quotient problem* (the polynomial invariants determine an affine variety whose points correspond to the closed orbits) and a *nullcone problem* (the study of the orbits having a given closed orbit in their closures). In this section we will solve the first part in full detail, the second part will be solved in section 2.7. A recurrent theme of this book will be to generalize this two part approach to the orbit-space problem to other representation varieties.

We denote by $M_n$ the space of all $n \times n$ matrices $M_n(\mathbb{C})$ and by $GL_n$ the general linear group $GL_n(\mathbb{C})$. A matrix $A \in M_n$ determines by left multiplication a linear operator on the $n$-dimensional vectorspace $V_n = \mathbb{C}^n$ of *column vectors* . If $g \in GL_n$ is the matrix describing the *base change* from the canonical basis of $V_n$ to a new basis, then the linear operator expressed in this new basis is represented by the matrix $gAg^{-1}$. For a given matrix $A$ we want to find a suitable basis such that the *conjugated matrix* $gAg^{-1}$ has a simple form.

Consider the linear action of $GL_n$ on the $n^2$-dimensional vectorspace $M_n$

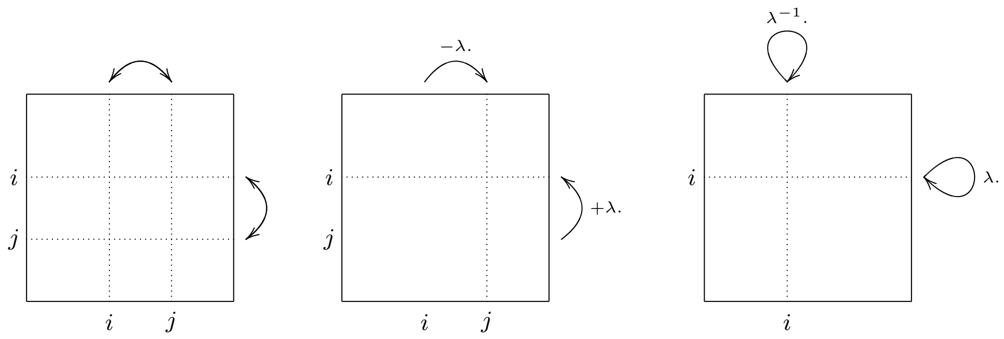
$$GL_n \times M_n \longrightarrow M_n \qquad (g, A) \mapsto g.A = gAg^{-1}.$$

The *orbit* $\mathcal{O}(A) = \{ gAg^{-1} \mid g \in GL_n \}$ of $A$ under this action is called the *conjugacy class* of $A$. We look for a particularly nice representative in a given conjugacy class. The answer to this problem is, of course, given by the *Jordan normal form* of the matrix.

With $e_{ij}$ we denote the matrix whose unique non-zero entry is 1 at entry $(i, j)$. Recall that the group $GL_n$ is generated by the following three classes of matrices :

- the *permutation* matrices $p_{ij} = \mathbb{1}_n + e_{ij} + e_{ji} - e_{ii} - e_{jj}$ for all $i \neq j$,
- the *addition* matrices $a_{ij}(\lambda) = \mathbb{1}_n + \lambda e_{ij}$ for all $i \neq j$ and $0 \neq \lambda$, and
- the *multiplication* matrices $m_i(\lambda) = \mathbb{1}_n + (\lambda - 1)e_{ii}$ for all $i$ and $0 \neq \lambda$.

Conjugation by these matrices determine the three types of *Jordan moves* on $n \times n$ matrices, as depicted below, where the altered rows and columns are indicated.




type p      type a      type m

Therefore, it suffices to consider sequences of these moves on a given $n \times n$ matrix $A \in M_n$. The *characteristic polynomial* of $A$ is defined to be the polynomial of degree $n$ in the variable $t$

$$\chi_A(t) = det(t\mathbb{1}_n - A) \in \mathbb{C}[t].$$

As $\mathbb{C}$ is algebraically closed, $\chi_A(t)$ decomposes as a product of linear terms

$$\prod_{i=1}^{e}(t-\lambda_i)^{d_i}$$

Here, the $\{\lambda_1,\ldots,\lambda_e\}$ are called the *eigenvalues* of the matrix $A$. Observe that $\lambda_i$ is an eigenvalue of $A$ if and only if there is a non-zero *eigenvector* $v \in V_n = \mathbb{C}^n$ with eigenvalue $\lambda_i$, that is, $A.v = \lambda_i v$. In particular, the *rank* $r_i$ of the matrix $A_i = \lambda_i \mathbb{1}_n - A$ satisfies $n - d_i \leq r_i < n$. A nice inductive procedure using only Jordan moves is given in [28] and proves the *Jordan-Weierstrass theorem* .

**Theorem 1.1 (Jordan-Weierstrass)** *Let $A \in M_n$ with characteristic polynomial $\chi_A(t) = \prod_{i=1}^{e}(t-\lambda_i)^{d_i}$. Then, $A$ determines unique partitions*

$$p_i = (a_{i1}, a_{i2}, \ldots, a_{im_i}) \quad \textit{of} \quad d_i$$

*associated to the eigenvalues $\lambda_i$ of $A$ such that $A$ is conjugated to a unique (up to permutation of the blocks) block-diagonal matrix*

$$J_{(p_1,\ldots,p_e)} = \left[\begin{array}{c|c|c|c} B_1 & 0 & \ldots & 0 \\ \hline 0 & B_2 & & 0 \\ \hline \vdots & & \ddots & \vdots \\ \hline 0 & 0 & \ldots & B_m \end{array}\right]$$

*with $m = m_1 + \ldots + m_e$ and exactly one block $B_l$ of the form $J_{a_{ij}}(\lambda_i)$ for all $1 \leq i \leq e$ and $1 \leq j \leq m_i$ where*

$$J_{a_{ij}}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in M_{a_{ij}}(\mathbb{C})$$

Let us prove uniqueness of the partitions $p_i$ of $d_i$ corresponding to the eigenvalue $\lambda_i$ of $A$. Assume $A$ is conjugated to another Jordan block matrix $J_{(q_1,\ldots,q_e)}$, necessarily with partitions $q_i = (b_{i1},\ldots,b_{im'_i})$ of $d_i$. To begin, observe that for a Jordan block of size $k$ we have that

$$rk\ J_k(0)^l = k - l \quad \text{for all } l \leq k \text{ and if } \mu \neq 0 \text{ then} \quad rk\ J_k(\mu)^l = k$$

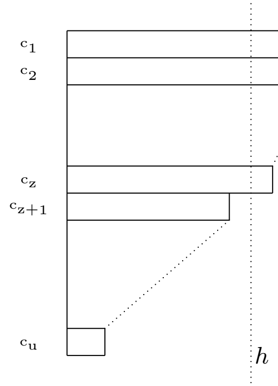
for all $l$. As $J_{(p_1,\ldots,p_e)}$ is conjugated to $J_{(q_1,\ldots,q_e)}$ we have for all $\lambda \in \mathbb{C}$ and all $l$

$$rk\ (\lambda \mathbb{1}_n - J_{(p_1,\ldots,p_e)})^l = rk\ (\lambda \mathbb{1}_n - J_{(q_1,\ldots,q_e)})^l$$

Now, take $\lambda = \lambda_i$ then only the Jordan blocks with eigenvalue $\lambda_i$ are important in the calculation and one obtains for the ranks

$$n - \sum_{h=1}^{l} \#\{j \mid a_{ij} \geq h\} \quad \text{respectively} \quad n - \sum_{h=1}^{l} \#\{j \mid b_{ij} \geq h\}.$$

Now, for any partition $p = (c_1, \ldots, c_u)$ and any natural number $h$ we see that the number $z = \#\{j \mid c_j \geq h\}$




is the number of blocks in the $h$-th row of the dual partition $p^*$ which is defined to be the partition obtained by interchanging rows and columns in the *Young diagram* of $p$ (see section 1.5 for the definition). Therefore, the above rank equality implies that $p_i^* = q_i^*$ and hence that $p_i = q_i$. As we can repeat this argument for the other eigenvalues we have the required uniqueness.

Hence, the Jordan normal form shows that the classification of $GL_n$-orbits in $M_n$ consists of two parts : a discrete part choosing

- a partition $p = (d_1, d_2, \ldots, d_e)$ of $n$, and for each $d_i$,
- a partition $p_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$ of $d_i$,

determining the sizes of the Jordan blocks and a continuous part choosing

- an $e$-tuple of distinct complex numbers $(\lambda_1, \lambda_2, \ldots, \lambda_e)$.

fixing the eigenvalues. Moreover, this $e$-tuple $(\lambda_1, \ldots, \lambda_e)$ is determined only up to permutations of the subgroup of all permutations $\pi$ in the symmetric group $S_e$ such that $p_i = p_{\pi(i)}$ for all $1 \leq i \leq e$.

Whereas this gives a satisfactory set-theoretical description of the orbits we cannot put an Hausdorff topology on this set due to the existence of non-closed orbits in $M_n$. For example, if $n = 2$, consider the matrices

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

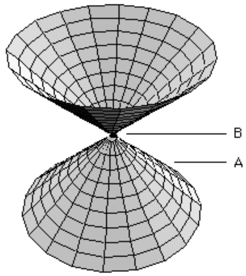



Figure 1.1: Orbit closure for $2 \times 2$ matrices

which are in different normal form so correspond to distinct orbits. For any $\epsilon \neq 0$ we have that

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & \epsilon \\ 0 & \lambda \end{bmatrix}$$

belongs to the orbit of $A$. Hence if $\epsilon \longrightarrow 0$, we see that $B$ lies in the closure of $\mathcal{O}(A)$. As any matrix in $\mathcal{O}(A)$ has trace $2\lambda$, the orbit is contained in the 3-dimensional subspace

$$\begin{bmatrix} \lambda + x & y \\ z & \lambda - x \end{bmatrix} \hookrightarrow M_2$$

In this space, the orbit-closure $\overline{\mathcal{O}(A)}$ is the set of points satisfying $x^2 + yz = 0$ (the determinant has to be $\lambda^2$), which is a cone having the origin as its top : The orbit $\mathcal{O}(B)$ is the top of the cone and the orbit $\mathcal{O}(A)$ is the complement, see figure 1.1.

Still, for general $n$ we can try to find the best separated topological quotient space for the action of $GL_n$ on $M_n$. We will prove that this space coincide with the quotient variety determined by the invariant polynomial functions.

If two matrices are conjugated $A \sim B$, then $A$ and $B$ have the same *unordered* $n$-tuple of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ (occurring with multiplicities). Hence any symmetric function in the $\lambda_i$ will have the same values in $A$ as in $B$. In particular this is the case for the *elementary symmetric functions* $\sigma_l$

$$\sigma_l(\lambda_1, \ldots, \lambda_l) = \sum_{i_1 < i_2 < \ldots < i_l} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_l}.$$

Observe that for every $A \in M_n$ with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ we have

$$\prod_{j=1}^{n} (t - \lambda_j) = \chi_A(t) = det(t\mathbb{1}_n - A) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(A) t^{n-i}$$

Developing the determinant $det(t\mathbb{1}_n - A)$ we see that each of the coefficients $\sigma_i(A)$ is in fact a *polynomial function* in the entries of $A$. A fortiori, $\sigma_i(A)$ is a complex valued continuous function on $M_n$. The above equality also implies that the functions $\sigma_i : M_n \longrightarrow \mathbb{C}$ are constant along orbits. We now construct the continuous map

$$M_n \xrightarrow{\pi} \mathbb{C}^n$$

sending a matrix $A \in M_n$ to the point $(\sigma_1(A), \ldots, \sigma_n(A))$ in $\mathbb{C}^n$. Clearly, if $A \sim B$ then they map to the same point in $\mathbb{C}^n$. We claim that $\pi$ is surjective. Take any point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ and consider the matrix $A \in M_n$

$$A = \begin{bmatrix} 0 & & & & a_n \\ -1 & 0 & & & a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & a_2 \\ & & & -1 & a_1 \end{bmatrix} \tag{1.1}$$

then we will show that $\pi(A) = (a_1, \ldots, a_n)$, that is,

$$det(t\mathbb{1}_n - A) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \ldots + (-1)^n a_n.$$

Indeed, developing the determinant of $t\mathbb{1}_n - A$ along the first column we obtain

$$\begin{vmatrix} t & 0 & 0 & \cdots & 0 & -a_n \\ 1 & t & 0 & & 0 & -a_{n-1} \\ 0 & 1 & t & & 0 & -a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & t & -a_2 \\ 0 & 0 & & & 1 & t-a_1 \end{vmatrix} - \begin{vmatrix} t & 0 & 0 & \cdots & 0 & -a_n \\ 1 & t & 0 & & 0 & -a_{n-1} \\ 0 & 1 & t & & 0 & -a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & t & -a_2 \\ 0 & 0 & & & 1 & t-a_1 \end{vmatrix}$$

Here, the second determinant is equal to $(-1)^{n-1}a_n$ and by induction on $n$ the first determinant is equal to $t.(t^{n-1} - a_1 t^{n-2} + \ldots + (-1)^{n-1}a_{n-1})$, proving the claim.

Next, we will determine which $n \times n$ matrices can be conjugated to a matrix in the canonical form $A$ as above. We call a matrix $B \in M_n$ *cyclic* if there is a (column) vector $v \in \mathbb{C}^n$ such that $\mathbb{C}^n$ is spanned by the vectors $\{v, B.v, B^2.v, \ldots, B^{n-1}.v\}$. Let $g \in GL_n$ be the basechange transforming the standard basis to the ordered basis

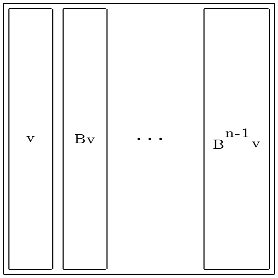
$$(v, -B.v, B^2.v, -B^3.v, \ldots, (-1)^{n-1}B^{n-1}.v).$$

In this new basis, the linear map determined by $B$ (or equivalently, $g.B.g^{-1}$) is equal to the matrix in canonical form

$$\begin{bmatrix} 0 & & & & b_n \\ -1 & 0 & & & b_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & b_2 \\ & & & -1 & b_1 \end{bmatrix}$$

where $B^n.v$ has coordinates $(b_n, \ldots, b_2, b_1)$ in the new basis. Conversely, any matrix in this form is a cyclic matrix.

We claim that the set of all cyclic matrices in $M_n$ is a *dense* open subset. To see this take $v = (x_1, \ldots, x_n)^\tau \in \mathbb{C}^n$ and compute the determinant of the $n \times n$ matrix




This gives a polynomial of total degree $n$ in the $x_i$ with all its coefficients polynomial functions $c_j$ in the entries $b_{kl}$ of $B$. Now, $B$ is a cyclic matrix if and only if at least one of these coefficients is non-zero. That is, the set of non-cyclic matrices is exactly the intersection of the finitely many *hypersurfaces*

$$V_j = \{B = (b_{kl})_{k,l} \in M_n \mid c_j(b_{11}, b_{12}, \ldots, b_{nn}) = 0\}$$

in the vectorspace $M_n$.

**Theorem 1.2** *The best continuous approximation to the orbit space is given by the surjection*

$$M_n \xrightarrow{\pi} \mathbb{C}^n$$

*mapping a matrix* $A \in M_n(\mathbb{C})$ *to the* $n$*-tuple* $(\sigma_1(A), \ldots, \sigma_n(A))$.

Let $f : M_n \longrightarrow \mathbb{C}$ be a continuous function which is constant along conjugacy classes. We will show that $f$ factors through $\pi$, that is, $f$ is really a continuous function in the $\sigma_i(A)$. Consider the diagram

$$\begin{tikzcd} M_n \arrow[r, "f"] \arrow[d, shift left, "\pi"] & \mathbb{C} \\ \mathbb{C}^n \arrow[u, shift left, "s"] \arrow[ur, dotted, "f'=f\circ s"'] & \end{tikzcd}$$

where $s$ is the *section* of $\pi$ (that is, $\pi \circ s = id_{\mathbb{C}^n}$) determined by sending a point $(a_1, \ldots, a_n)$ to the cyclic matrix in canonical form $A$ as in equation (1.1). Clearly, $s$ is continuous, hence so is $f' = f \circ s$. The approximation property follows if we prove that $f = f' \circ \pi$. By continuity, it suffices to check equality on the dense open set of cyclic matrices in $M_n$.

There it is a consequence of the following three facts we have proved before : (1) : any cyclic matrix lies in the same orbit as one in standard form, (2) : $s$ is a section of $\pi$ and (3) : $f$ is constant along orbits.

**Example 1.1 (Orbits in $M_2$)** A $2 \times 2$ matrix $A$ can be conjugated to an upper triangular matrix with diagonal entries the eigenvalues $\lambda_1, \lambda_2$ of $A$. As the trace and determinant of both matrices are equal we have

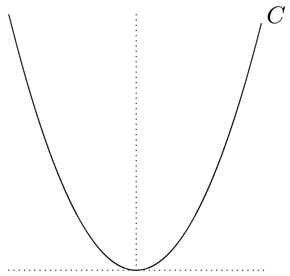
$$\sigma_1(A) = tr(A) \text{ and } \sigma_2(A) = det(A).$$

The best approximation to the orbitspace is therefore given by the surjective map

$$M_2 \overset{\pi}{\twoheadrightarrow} \mathbb{C}^2 \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a+d, ad-bc)$$

The matrix $A$ has two equal eigenvalues if and only if the discriminant of the characteristic polynomial $t^2 - \sigma_1(A)t + \sigma_2(A)$ is zero, that is when $\sigma_1(A)^2 - 4\sigma_2(A) = 0$. This condition determines a closed curve $C$ in $\mathbb{C}^2$ where

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 - 4y = 0\}.$$



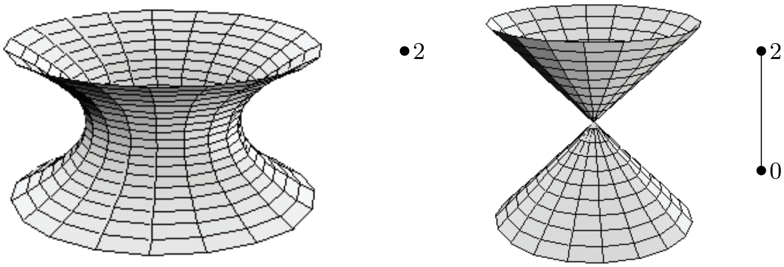



Figure 1.2: Orbit closures of $2 \times 2$ matrices

Observe that $C$ is a smooth 1-dimensional submanifold of $\mathbb{C}^2$. We will describe the *fibers* (that is, the inverse images of points) of the surjective map $\pi$.

If $p = (x, y) \in \mathbb{C}^2 - C$, then $\pi^{-1}(p)$ consists of precisely one orbit (which is then necessarily closed in $M_2$) namely that of the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{where} \quad \lambda_{1,2} = \frac{-x \pm \sqrt{x^2 - 4y}}{2}$$

If $p = (x, y) \in C$ then $\pi^{-1}(p)$ consists of two orbits,

$$\mathcal{O}_{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}} \quad \text{and} \quad \mathcal{O}_{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}$$

where $\lambda = \frac{1}{2}x$. We have seen that the second orbit lies in the closure of the first. Observe that the second orbit reduces to one point in $M_2$ and hence is closed. Hence, also $\pi^{-1}(p)$ contains a unique closed orbit.

To describe the fibers of $\pi$ as closed subsets of $M_2$ it is convenient to write any matrix $A$ as a linear combination

$$A = u(A) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + v(A) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + w(A) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z(A) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Expressed in the coordinate functions $u, v, w$ and $z$ the fibers $\pi^{-1}(p)$ of a point $p = (x, y) \in \mathbb{C}^2$ are the common zeroes of

$$\begin{cases} u & = x \\ v^2 + 4wz & = x^2 - 4y \end{cases}$$

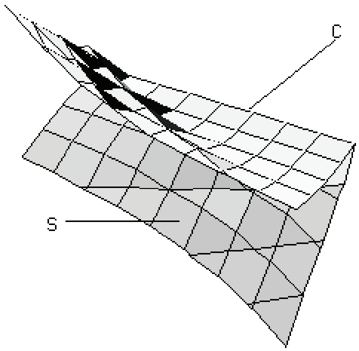



Figure 1.3: Representation strata for $3 \times 3$ matrices.

The first equation determines a three dimensional affine subspace of $M_2$ in which the second equation determines a *quadric*. If $p \notin C$ this quadric is non-degenerate and thus $\pi^{-1}(p)$ is a smooth 2-dimensional submanifold of $M_2$. If $p \in C$, the quadric is a cone with top lying in the point $\frac{x}{2}\mathbb{1}_2$. Under the $GL_2$-action, the unique singular point of the cone must be clearly fixed giving us the closed orbit of dimension 0 corresponding to the diagonal matrix. The other orbit is the complement of the top and hence is a smooth 2-dimensional (non-closed) submanifold of $M_2$. The graphs in figure 1.2 represent the orbit-closures and the dimensions of the orbits.

**Example 1.2 (Orbits in $M_3$)** We will describe the fibers of the surjective map $M_3 \xrightarrow{\pi} \mathbb{C}^3$. If a $3 \times 3$ matrix has multiple eigenvalues then the *discriminant* $d = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2$ is zero. Clearly, $d$ is a symmetric polynomial and hence can be expressed in terms of $\sigma_1, \sigma_2$ and $\sigma_3$. More precisely,

$$d = 4\sigma_1^3\sigma_3 + 4\sigma_2^3 + 27\sigma_3^2 - \sigma_1^2\sigma_2^2 - 18\sigma_1\sigma_2\sigma_3$$

The set of points in $\mathbb{C}^3$ where $d$ vanishes is a surface $S$ with *singularities*. These singularities are the common zeroes of the $\frac{\partial d}{\partial \sigma_i}$ for $1 \leq i \leq 3$. One computes that these singularities form a *twisted cubic* curve $C$ in $\mathbb{C}^3$, that is,

$$C = \{(3c, 3c^2, c^3) \mid c \in \mathbb{C}\}.$$

The description of the fibers $\pi^{-1}(p)$ for $p = (x, y, z) \in \mathbb{C}^3$ is as follows. When $p \notin S$, then $\pi^{-1}(p)$ consists of a unique orbit (which is therefore closed in $M_3$), the conjugacy class of a matrix with paired distinct eigenvalues. If $p \in S - C$, then $\pi^{-1}(p)$ consists of the orbits of

$$A_1 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

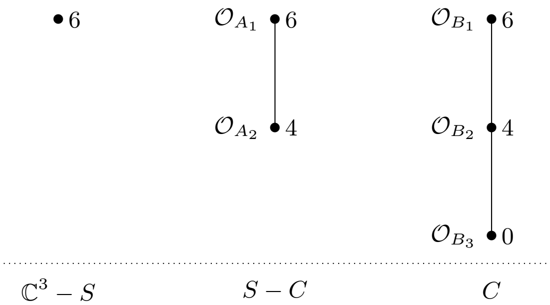
Finally, if $p \in C$, then the matrices in the fiber $\pi^{-1}(p)$ have a single eigenvalue $\lambda = \frac{1}{3}x$ and the fiber consists of the orbits of the matrices

$$B_1 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad B_2 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad B_3 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

We observe that the *strata* with distinct fiber behavior (that is, $\mathbb{C}^3 - S$, $S - C$ and $C$) are all submanifolds of $\mathbb{C}^3$, see figure 1.3.

The dimension of an orbit $\mathcal{O}(A)$ in $M_n$ is computed as follows. Let $C_A$ be the subspace of all matrices in $M_n$ commuting with $A$. Then, the *stabilizer* subgroup of $A$ is a dense open subset of $C_A$ whence the dimension of $\mathcal{O}(A)$ is equal to $n^2 - \dim C_A$.

Performing these calculations for the matrices given above, we obtain the following graphs representing orbit-closures and the dimensions of orbits




Returning to $M_n$, the set of cyclic matrices is a Zariski open subset of $M_n$. For, consider the generic matrix of coordinate functions and generic column vector

$$X = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

and form the square matrix

$$\begin{bmatrix} v & X.v & X^2.v & \dots & X^{n-1}.v \end{bmatrix} \in M_n(\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, v_1, \dots, v_n])$$

Then its determinant can be written as $\sum_{l=1}^{z} p_l(x_{ij})q_l(v_k)$ where the $q_l$ are polynomials in the $v_k$ and the $p_l$ polynomials in the $x_{ij}$. Let $A \in M_n$ be such that at least one of the $p_l(A) \neq 0$, then the polynomial $d = \sum_l p_l(A)q_l(v_k) \in \mathbb{C}[v_1, \ldots, v_k]$ is non-zero. But then there is a $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ such that $d(c) \neq 0$ and hence $c^\tau$ is a cyclic vector for $A$. The converse implication is obvious.

**Theorem 1.3** *Let $f : M_n \longrightarrow \mathbb{C}$ is a regular (that is, polynomial) function on $M_n$ which is constant along conjugacy classes, then*

$$f \in \mathbb{C}[\sigma_1(X), \ldots, \sigma_n(X)]$$

*Proof.* Consider again the diagram

$$\begin{tikzcd} M_n \arrow[r, "f"] \arrow[d, shift left, "\pi"] & \mathbb{C} \\ \mathbb{C}^n \arrow[u, shift left, "s"] \arrow[ur, dotted, "f'=f \circ s"'] & \end{tikzcd}$$

The function $f' = f \circ s$ is a regular function on $\mathbb{C}^n$ whence is a polynomial in the coordinate functions of $\mathbb{C}^m$ (which are the $\sigma_i(X)$), so

$$f' \in \mathbb{C}[\sigma_1(X), \ldots, \sigma_n(X)] \hookrightarrow \mathbb{C}[M_n].$$

Moreover, $f$ and $f'$ are equal on a Zariski open (dense) subset of $M_n$ whence they are equal as polynomials in $\mathbb{C}[M_n]$. $\square$

The ring of polynomial functions on $M_n$ which are constant along conjugacy classes can also be viewed as a ring of invariants. The group $GL_n$ acts as algebra automorphisms on the polynomial ring $\mathbb{C}[M_n]$. The automorphism $\phi_g$ determined by $g \in GL_n$ sends the variable $x_{ij}$ to the $(i,j)$-entry of the matrix $g^{-1}.X.g$ which is a linear form in $\mathbb{C}[M_n]$. This action is determined by the property that for all $g \in GL_n$, $A \in A$ and $f \in \mathbb{C}[M_n]$ we have that

$$\phi_g(f)(A) = f(g.A.g^{-1})$$

The *ring of polynomial invariants* is the algebra of polynomials left invariant under this action

$$\mathbb{C}[M_n]^{GL_n} = \{f \in \mathbb{C}[M_n] \mid \phi_g(f) = f \text{ for all } \ g \in GL_n\}$$

and hence is the ring of polynomial functions on $M_n$ which are constant along orbits. The foregoing theorem determines the ring of polynomials invariants

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[\sigma_1(X), \ldots, \sigma_n(X)]$$

We will give an equivalent description of this ring below.

Consider the variables $\lambda_1, \ldots, \lambda_n$ and consider the polynomial

$$f_n(t) = \prod_{i=1}^{n}(t-\lambda_i) = t^n + \sum_{i=1}^{n}(-1)^i\sigma_i t^{n-i}$$

then $\sigma_i$ is the $i$-th elementary symmetric polynomial in the $\lambda_j$. We know that these polynomials are algebraically independent and generate the *ring of symmetric polynomials* in the $\lambda_j$, that is,

$$\mathbb{C}[\sigma_1, \ldots, \sigma_n] = \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{S_n}$$

where $S_n$ is the *symmetric group* on $n$ letters acting by automorphisms on the polynomial ring $\mathbb{C}[\lambda_1, \ldots, \lambda_n]$ via $\pi(\lambda_i) = \lambda_{\pi(i)}$ and the algebra of polynomials which are fixed under these automorphisms are precisely the symmetric polynomials in the $\lambda_j$.

Consider the symmetric Newton functions $s_i = \lambda_1^i + \ldots + \lambda_n^i$, then we claim that this is another generating set of symmetric polynomials, that is,

$$\mathbb{C}[\sigma_1, \ldots, \sigma_n] = \mathbb{C}[s_1, \ldots, s_n].$$

To prove this it suffices to express each $\sigma_i$ as a polynomial in the $s_j$. More precisely, we claim that the following identities hold for all $1 \leq j \leq n$

$$s_j - \sigma_1 s_{j-1} + \sigma_2 s_{j-2} - \ldots + (-1)^{j-1}\sigma_{j-1}s_1 + (-1)^j\sigma_j.j = 0 \tag{1.2}$$

For $j = n$ this identity holds because we have

$$0 = \sum_{i=1}^{n} f_n(\lambda_i) = s_n + \sum_{i=1}^{n}(-1)^i\sigma_i s_{n-i}$$

if we take $s_0 = n$. Assume now $j < n$ then the left hand side of equation 1.2 is a symmetric function in the $\lambda_i$ of degree $\leq j$ and is therefore a polynomial $p(\sigma_1, \ldots, \sigma_j)$ in the first $j$ elementary symmetric polynomials. Let $\phi$ be the algebra epimorphism

$$\mathbb{C}[\lambda_1, \ldots, \lambda_n] \xrightarrow{\phi} \mathbb{C}[\lambda_1, \ldots, \lambda_j]$$

defined by mapping $\lambda_{j+1}, \ldots, \lambda_j$ to zero. Clearly, $\phi(\sigma_i)$ is the $i$-th elementary symmetric polynomial in $\{\lambda_1, \ldots, \lambda_j\}$ and $\phi(s_i) = \lambda_1^i + \ldots + \lambda_j^i$. Repeating the above $j = n$ argument (replacing $n$ by $j$) we have

$$0 = \sum_{i=1}^{j} f_j(\lambda_i) = \phi(s_j) + \sum_{i=1}^{j}(-1)^i\phi(\sigma_i)\phi(s_{n-i})$$

(this time with $s_0 = j$). But then, $p(\phi(\sigma_1), \ldots, \phi(\sigma_j)) = 0$ and as the $\phi(\sigma_k)$ for $1 \leq k \leq j$ are algebraically independent we must have that $p$ is the zero polynomial finishing the proof of the claimed identity.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix $A$, then $A$ can be conjugated to an upper triangular matrix $B$ with diagonal entries $(\lambda_1, \ldots, \lambda_1)$. Hence, the *trace* $tr(A) = tr(B) = \lambda_1 + \ldots + \lambda_n = s_1$. In general, $A^i$ can be conjugated to $B^i$ which is an upper triangular matrix with diagonal entries $(\lambda_1^i, \ldots, \lambda_n^i)$ and hence the traces of $A^i$ and $B^i$ are equal to $\lambda_1^i + \ldots + \lambda_n^i = s_i$. Concluding, we have

**Theorem 1.4** *Consider the action of conjugation by $GL_n$ on $M_n$. Let $X$ be the generic matrix of coordinate functions on $M_n$*

$$X = \begin{bmatrix} x_{11} & \ldots & x_{nn} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix}$$

*Then, the ring of polynomial invariants is generated by the traces of powers of $X$, that is,*

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[tr(X), tr(X^2), \ldots, tr(X^n)]$$

*Proof.* The result follows from theorem 1.3 and the fact that

$$\mathbb{C}[\sigma_1(X), \ldots, \sigma_n(X)] = \mathbb{C}[tr(X), \ldots, tr(X^n)]$$

□

## 1.2 Simultaneous conjugacy classes

As mentioned in the introduction, we need to extend what we have done for conjugacy classes of matrices to simultaneous conjugacy classes of *m-tuples of matrices* . Consider the $mn^2$-dimensional complex vectorspace

$$M_n^m = \underbrace{M_n \oplus \ldots \oplus M_n}_{m}$$

of $m$-tuples $(A_1, \ldots, A_m)$ of $n \times n$-matrices $A_i \in M_n$. On this space we let the group $GL_n$ act by simultaneous conjugation, that is

$$g.(A_1, \ldots, A_m) = (g.A_1.g^{-1}, \ldots, g.A_m.g^{-1})$$

for all $g \in GL_n$ and all $m$-tuples $(A_1, \ldots, A_m)$. Unfortunately, there is no substitute for the Jordan normalform result in this more general setting.

Still, for small $m$ and $n$ one can work out the $GL_n$-orbits by brute force methods. In this section we will give the details for the first non-trivial case, that of couples of $2 \times 2$ matrices. These explicit calculations will already exhibit some of the general features we will prove later. For example, that all subvarieties of the quotient variety determined by points of the same representation type are smooth and that the fiber structure depends only on the representation type.

**Example 1.3 (Orbits in $M_2^2 = M_2 \oplus M_2$)** We can try to mimic the geometric approach to the conjugacy class problem, that is, we will try to approximate the orbitspace via polynomial functions on $M_2^2$ which are constant along orbits. For $(A,B) \in M_2^2 = M_2 \oplus M_2$ clearly the polynomial functions we have encountered before $tr(A), det(A)$ and $tr(B), det(B)$ are constant along orbits. However, there are more : for example $tr(AB)$. In the next section, we will show that these five functions generate all polynomials functions which are constant along orbits. Here, we will show that the map $M_2^2 = M_2 \oplus M_2 \xrightarrow{\pi} \mathbb{C}^5$ defined by

$$(A,B) \mapsto (tr(A), det(A), tr(B), det(B), tr(AB))$$

is surjective such that each fiber contains precisely one closed orbit. In the next chapter, we will see that this property characterizes the best polynomial approximation to the (non-existent) orbit space.

First, we will show surjectivity of $\pi$, that is, for every $(x_1, \ldots, x_5) \in \mathbb{C}^5$ we will construct a couple of $2 \times 2$ matrices $(A,B)$ (or rather its orbit) such that $\pi(A,B) = (x_1, \ldots, x_5)$. Consider the open set where $x_1^2 \neq 4x_2$. We have seen that this property characterizes those $A \in M_2$ such that $A$ has distinct eigenvalues and hence diagonalizable. Hence, we can take a representative of the orbit $\mathcal{O}(A,B)$ to be a couple

$$( \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} , \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} )$$

with $\lambda \neq \mu$. We need a solution to the set of equations

$$\begin{cases} x_3 = & c_1 + c_4 \\ x_4 = & c_1c_4 - c_2c_3 \\ x_5 = & \lambda c_1 + \mu c_4 \end{cases}$$

Because $\lambda \neq \mu$ the first and last equation uniquely determine $c_1, c_4$ and substitution in the second gives us $c_2c_3$. Analogously, points of $\mathbb{C}^5$ lying in the open set $x_3^2 \neq x_4$ lie in the image of $\pi$. Finally, for a point in the complement of these open sets, that is when $x_1^2 = x_2$ and $x_3^2 = 4x_4$ we can consider a couple $(A,B)$

$$( \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} , \begin{bmatrix} \mu & 0 \\ c & \mu \end{bmatrix} )$$

where $\lambda = \frac{1}{2}x_1$ and $\mu = \frac{1}{2}x_3$. Observe that the remaining equation $x_5 = tr(AB) = 2\lambda\mu + c$ has a solution in $c$.

Now, we will describe the fibers of $\pi$. Assume $(A,B)$ is such that $A$ and $B$ have a common eigenvector $v$. Simultaneous conjugation with a $g \in GL_n$ expressing a basechange from the standard basis to $\{v,w\}$ for some $w$ shows that the orbit $\mathcal{O}(A,B)$ contains a couple of upper-triangular matrices. We want to describe the image of these matrices under $\pi$. Take an upper triangular representative in $\mathcal{O}(A,B)$

$$( \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} , \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} ).$$

with $\pi$-image $(x_1, \ldots, x_5)$. The coordinates $x_1, x_2$ determine the eigenvalues $a_1, a_3$ of $A$ only as an unordered set (similarly, $x_3, x_4$ only determine the set of eigenvalues $\{b_1, b_3\}$ of $B$). Hence, $tr(AB)$ is one of the following two expressions

$$a_1b_1 + a_3b_3 \quad \text{or} \quad a_1b_3 + a_3b_1$$

and therefore satisfies the equation

$$(tr(AB) - a_1b_1 - a_3b_3)(tr(AB) - a_1b_3 - a_3b_1) = 0.$$

Recall that $x_1 = a_1 + a_3, x_2 = a_1a_3,\ x_3 = b_1 + b_3, x_4 = b_1b_3$ and $x_5 = tr(AB)$ we can express this equation as

$$x_5^2 - x_1x_3x_5 + x_1^2x_4 + x_3^2x_2 - 4x_2x_4 = 0.$$

This determines an hypersurface $H \hookrightarrow \mathbb{C}^5$. If we view the left-hand side as a polynomial $f$ in the coordinate functions of $\mathbb{C}^5$ we see that $H$ is a four dimensional subset of $\mathbb{C}^5$ with singularities the common zeroes of the partial derivatives

$$\frac{\partial f}{\partial x_i} \text{ for } 1 \leq i \leq 5$$

These singularities for the 2-dimensional submanifold $S$ of points of the form $(2a, a^2, 2b, b^2, 2ab)$. We now claim that the smooth submanifolds $\mathbb{C}^5 - H$, $H - S$ and $S$ of $\mathbb{C}^5$ describe the different types of fiber behavior. In chapter 6 we will see that the subsets of points with different fiber behavior (actually, of different representation type) are manifolds for $m$-tuples of $n \times n$ matrices.

If $p \notin H$ we claim that $\pi^{-1}(p)$ is a unique orbit, which is therefore closed in $M_2^2$. Let $(A, B) \in \pi^{-1}$ and assume first that $x_1^2 \neq 4x_2$ then there is a representative in $\mathcal{O}(A, B)$ of the form

$$\left( \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} , \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right)$$

with $\lambda \neq \mu$. Moreover, $c_2c_3 \neq 0$ (for otherwise $A$ and $B$ would have a common eigenvector whence $p \in H$) hence we may assume that $c_2 = 1$ (eventually after simultaneous conjugation with a suitable diagonal matrix $diag(t, t^{-1})$). The value of $\lambda, \mu$ is determined by $x_1, x_2$. Moreover, $c_1, c_3, c_4$ are also completely determined by the system of equations

$$\begin{cases} x_3 &= c_1 + c_4 \\ x_4 &= c_1c_4 - c_3 \\ x_5 &= \lambda c_1 + \mu c_4 \end{cases}$$

and hence the point $p = (x_1, \ldots, x_5)$ completely determines the orbit $\mathcal{O}(A, B)$. Remains to consider the case when $x_1^2 = 4x_2$ (that is, when $A$ has a single eigenvalue). Consider the couple $(uA+vB, B)$

for $u, v \in \mathbb{C}^*$. To begin, $uA + vB$ and $B$ do not have a common eigenvalue. Moreover, $p = \pi(A, B)$ determines $\pi(uA + vB, B)$ as

$$\begin{cases} tr(uA + vB) & = utr(A) + vtr(B) \\ det(uA + vB) & = u^2 det(A) + v^2 det(B) + uv(tr(A)tr(B) - tr(AB)) \\ tr((uA + vB)B) & = utr(AB) + v(tr(B)^2 - 2det(B)) \end{cases}$$

Assume that for all $u, v \in \mathbb{C}^*$ we have the equality $tr(uA + vB)^2 = 4det(uA + vB)$ then comparing coefficients of this equation expressed as a polynomial in $u$ and $v$ we obtain the conditions $x_1^2 = 4x_2$, $x_3^2 = 4x_4$ and $2x_5 = x_1x_3$ whence $p \in S \hookrightarrow H$, a contradiction. So, fix $u, v$ such that $uA + vB$ has distinct eigenvalues. By the above argument $\mathcal{O}(uA + vB, B)$ is the unique orbit lying over $\pi(uA + vB, B)$, but then $\mathcal{O}(A, B)$ must be the unique orbit lying over $p$.

Let $p \in H - S$ and $(A, B) \in \pi^{-1}(p)$, then $A$ and $B$ are simultaneous upper triangularizable, with eigenvalues $a_1, a_2$ respectively $b_1, b_2$. Either $a_1 \neq a_2$ or $b_1 \neq b_2$ for otherwise $p \in S$. Assume $a_1 \neq a_2$, then there is a representative in the orbit $\mathcal{O}(A, B)$ of the form

$$( \begin{bmatrix} a_i & 0 \\ 0 & a_j \end{bmatrix} , \begin{bmatrix} b_k & b \\ 0 & b_l \end{bmatrix})$$

for $\{i, j\} = \{1, 2\} = \{k, l\}$. If $b \neq 0$ we can conjugate with a suitable diagonal matrix to get $b = 1$ hence we get at most 9 possible orbits. Checking all possibilities we see that only three of them are distinct, those corresponding to the couples

$$(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}) \quad (\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}) \quad (\begin{bmatrix} a_2 & 0 \\ 0 & a_1 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix})$$

Clearly, the first and last orbit have the middle one lying in its closure. Observe that the case assuming that $b_1 \neq b_2$ is handled similarly. Hence, if $p \in H - S$ then $\pi^{-1}(p)$ consists of three orbits, two of dimension three whose closures intersect in a (closed) orbit of dimension two.

Finally, consider the case when $p \in S$ and $(A, B) \in \pi^{-1}(p)$. Then, both $A$ and $B$ have a single eigenvalue and the orbit $\mathcal{O}(A, B)$ has a representative of the form

$$(\begin{bmatrix} a & x \\ 0 & a \end{bmatrix}, \begin{bmatrix} b & y \\ 0 & b \end{bmatrix})$$

for certain $x, y \in \mathbb{C}$. If either $x$ or $y$ are non-zero, then the subgroup of $GL_2$ fixing this matrix consists of the matrices of the form

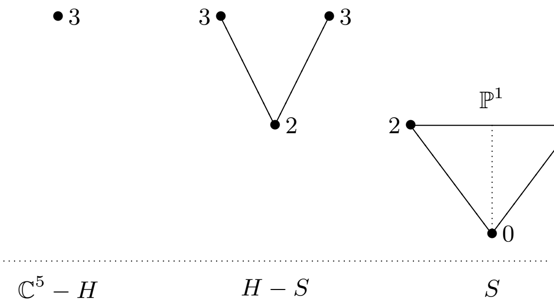
$$Stab \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} = \{\begin{bmatrix} u & v \\ 0 & u \end{bmatrix} \mid u \in \mathbb{C}^*, v \in \mathbb{C}\}$$

but these matrices also fix the second component. Therefore, if either $x$ or $y$ is nonzero, the orbit is fully determined by $[x : y] \in \mathbb{P}^1$. That is, for $p \in S$, the fiber $\pi^{-1}(p)$ consists of an infinite family of orbits of dimension 2 parameterized by the points of the projective line $\mathbb{P}^1$ together with the orbit of

$$
( \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} , \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} )
$$

which consists of one point (hence is closed in $M_2^2$) and lies in the closure of each of the 2-dimensional orbits.

Concluding, we see that each fiber $\pi^{-1}(p)$ contains a unique closed orbit (that of minimal dimension). The orbitclosure and dimension diagrams have the following shapes




The reader is invited to try to extend this to the case of three $2 \times 2$ matrices (relatively easy) or to two $3 \times 3$ matrices (substantially harder). By the end of this book you will have learned enough techniques to solve the general case, *at least in principle*. As this problem is the archetypical example of a *wild representation problem* it is customary to view it as 'hopeless'. Hence, sooner or later we will hit the wall, but what this book will show you is that you can push the wall a bit further than was generally expected.

## 1.3 Matrix invariants and necklaces

In this section we will determine the ring of all polynomial maps

$$
M_n^m = \underbrace{M_n \oplus \ldots \oplus M_n}_{m} \xrightarrow{f} \mathbb{C}
$$

which are constant along orbits under the action of $GL_n$ on $M_n^m$ by simultaneous conjugation. The strategy we will use is classical in invariant theory.

- First, we will determine the *multilinear* maps which are constant along orbits, equivalently, the *linear* maps

$$
M_n^{\otimes m} = \underbrace{M_n \otimes \ldots \otimes M_n}_{m} \longrightarrow \mathbb{C}
$$

which are constant along $GL_n$-orbits where $GL_n$ acts by the diagonal action, that is,

$$g.(A_1 \otimes \ldots \otimes A_m) = gA_1g^{-1} \otimes \ldots \otimes gA_mg^{-1}.$$

- Afterwards, we will be able to obtain from them all polynomial invariant maps by using *polarization* and *restitution* operations.

First, we will translate our problem into one studied in classical invariant theory of $GL_n$.

Let $V_n \simeq \mathbb{C}^n$ be the $n$-dimensional vectorspace of column vectors on which $GL_n$ acts naturally by left multiplication

$$V_n = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \\ \vdots \\ \mathbb{C} \end{bmatrix} \qquad \text{with action} \qquad g.\begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix}$$

In order to define an action on the dual space $V_n^* = Hom(V_n, \mathbb{C}) \simeq \mathbb{C}^n$ of *covectors* (or, row vectors) we have to use the *contragradient* action

$$V_n^* = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \ldots & \mathbb{C} \end{bmatrix} \qquad \text{with action} \qquad \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \end{bmatrix}.g^{-1}$$

Observe, that we have an *evaluation* map $V_n^* \times V_n \longrightarrow \mathbb{C}$ which is given by the scalar product $f(v)$ for all $f \in V_n^*$ and $v \in V_n$

$$\begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \end{bmatrix}.\begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix} = \phi_1\nu_1 + \phi_2\nu_2 + \ldots + \phi_n\nu_n$$

which is invariant under the diagonal action of $GL_n$ on $V_n^* \times V_n$. Further, we have the natural identification

$$M_n = V_n \otimes V_n^* = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \\ \vdots \\ \mathbb{C} \end{bmatrix} \otimes \begin{bmatrix} \mathbb{C} & \mathbb{C} & \ldots & \mathbb{C} \end{bmatrix}.$$

Under this identification, a *pure tensor* $v \otimes f$ corresponds to the rank one matrix (or rank one endomorphism of $V_n$) defined by

$$v \otimes f : V_n \longrightarrow V_n \quad \text{with} \quad w \mapsto f(w)v$$

and we observe that the rank one matrices span $M_n$. The diagonal action of $GL_n$ on $V_n \otimes V_n^*$ is then determined by its action on the pure tensors where it is equal to

$$g.\begin{bmatrix} \nu_1 \\ \nu_2 \\ \ldots \\ \nu_n \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \end{bmatrix}.g^{-1}$$

and therefore coincides with the action of conjugation on $M_n$. Now, let us consider the identification

$$(V_n^{*\otimes m} \otimes V_n^{\otimes m})^* \simeq End(V_n^{\otimes m})$$

obtained from the nondegenerate pairing

$$End(V_n^{\otimes m}) \times (V_n^{*\otimes m} \otimes V_n^{\otimes m}) \longrightarrow \mathbb{C}$$

given by the formula

$$\langle \lambda, f_1 \otimes \ldots \otimes f_m \otimes v_1 \otimes \ldots \otimes v_m \rangle = f_1 \otimes \ldots \otimes f_m(\lambda(v_1 \otimes \ldots \otimes v_m))$$

$GL_n$ acts diagonally on $V_n^{\otimes m}$ and hence again by conjugation on $End(V_n^{\otimes m})$ after embedding $GL_n \hookrightarrow GL(V_n^{\otimes m}) = GL_{mn}$. Thus, the above identifications are isomorphism as vectorspaces with $GL_n$-action. But then, the space of $GL_n$-invariant linear maps

$$V_n^{*\otimes m} \otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

can be identified with the space $End_{GL_n}(V_n^{\otimes m})$ of $GL_n$-linear endomorphisms of $V_n^{\otimes m}$. We will now give a different presentation of this vectorspace relating it to the symmetric group.

Apart from the diagonal action of $GL_n$ on $V_n^{\otimes m}$ given by

$$g.(v_1 \otimes \ldots \otimes v_m) = g.v_1 \otimes \ldots \otimes g.v_m$$

we have an action of the symmetric group $S_m$ on $m$ letters on $V_n^{\otimes m}$ given by

$$\sigma.(v_1 \otimes \ldots \otimes v_m) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)}$$

These two actions commute with each other and give embeddings of $GL_n$ and $S_m$ in $End(V_n^{\otimes m})$. The subspace of $V_n^{\otimes m}$ spanned by the image of $GL_n$ will be denoted by $\langle GL_n \rangle$. Similarly, with $\langle S_m \rangle$ we denote the subspace spanned by the image of $S_m$.

**Theorem 1.5** *With notations as above we have :*

1. $\langle GL_n \rangle = End_{S_m}(V_n^{\otimes m})$
2. $\langle S_m \rangle = End_{GL_n}(V_n^{\otimes m})$

*Proof.* (1) : Under the identification $End(V_n^{\otimes m}) = End(V_n)^{\otimes m}$ an element $g \in GL_n$ is mapped to the symmetric tensor $g \otimes \ldots \otimes g$. On the other hand, the image of $End_{S_m}(V_n^{\otimes m})$ in $End(V_n)^{\otimes m}$ is the subspace of all symmetric tensors in $End(V)^{\otimes m}$. We can give a basis of this subspace as follows. Let $\{e_1, \ldots, e_{n^2}\}$ be a basis of $End(V_n)$, then the vectors $e_{i_1} \otimes \ldots \otimes e_{i_m}$ form a basis of $End(V_n)^{\otimes m}$ which is stable under the $S_m$-action. Further, any $S_m$-orbit contains a unique representative of the form

$$e_1^{\otimes h_1} \otimes \ldots \otimes e_{n^2}^{\otimes h_{n^2}}$$

with $h_1+\ldots+h_{n^2}=m$. If we denote by $r(h_1,\ldots,h_{n^2})$ the sum of all elements in the corresponding $S_m$-orbit then these vectors are a basis of the symmetric tensors in $End(V_n)^{\otimes m}$.

The claim follows if we can show that every linear map $\lambda$ on the symmetric tensors which is zero on all $g\otimes\ldots\otimes g$ with $g\in GL_n$ is the zero map. Write $e=\sum x_i e_i$, then

$$\lambda(e\otimes\ldots\otimes e)=\sum x_1^{h_1}\ldots x_{n^2}^{h_{n^2}}\lambda(r(h_1,\ldots,h_{n^2}))$$

is a polynomial function on $End(V_n)$. As $GL_n$ is a Zariski open subset of $End(V)$ on which by assumption this polynomial vanishes, it must be the zero polynomial. Therefore, $\lambda(r(h_1,\ldots,h_{n^2}))=0$ for all $(h_1,\ldots,h_{n^2})$ finishing the proof.

(2) : Recall that the groupalgebra $\mathbb{C}S_m$ of $S_m$ is a *semisimple algebra* . Any epimorphic image of a semisimple algebra is semisimple. Therefore, $\langle S_m\rangle$ is a semisimple subalgebra of the matrixalgebra $End(V_n^{\otimes m})\simeq M_{nm}$. By the *double centralizer theorem* (see for example [66]), it is therefore equal to the centralizer of $End_{S_m}(V_m^{\otimes m})$. By the first part, it is the centralizer of $\langle GL_n\rangle$ in $End(V_n^{\otimes m})$ and therefore equal to $End_{GL_n}(V_n^{\otimes m})$. □

Because $End_{GL_n}(V_n^{\otimes m})=\langle S_m\rangle$, every $GL_n$-endomorphism of $V_n^{\otimes m}$ can be written as a linear combination of the morphisms $\lambda_\sigma$ describing the action of $\sigma\in S_m$ on $V_n^{\otimes m}$. Our next job is to trace back these morphisms $\lambda_\sigma$ through the canonical identifications until we can express them in terms of matrices.

To start let us compute the linear invariant

$$\mu_\sigma : V_n^{*\otimes m}\otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

corresponding to $\lambda_\sigma$ under the identification $(V_n^{*\otimes m}\otimes V_n^{\otimes m})^*\simeq End(V_n^{\otimes m})$. By the identification we know that $\mu_\sigma(f_1\otimes\ldots f_m\otimes v_1\otimes\ldots\otimes v_m)$ is equal to

$$\begin{aligned}\langle\lambda_\sigma, f_1\otimes\ldots f_m\otimes v_1\otimes\ldots\otimes v_m\rangle &= f_1\otimes\ldots\otimes f_m(v_{\sigma(1)}\otimes\ldots v_{\sigma(m)})\\ &= \prod_i f_i(v_{\sigma(i)})\end{aligned}$$

That is, we have proved

**Proposition 1.1** *Any multilinear $GL_n$-invariant map*

$$\gamma : V_n^{*\otimes m}\otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

*is a linear combination of the invariants*

$$\mu_\sigma(f_1\otimes\ldots f_m\otimes v_1\otimes\ldots\otimes v_m)=\prod_i f_i(v_{\sigma(i)})$$

*for $\sigma\in S_m$.*

Using the identification $M_n(\mathbb{C}) = V_n \otimes V_n^*$ a multilinear $GL_n$-invariant map

$$(V_n^* \otimes V_n)^{\otimes m} = V_n^{*\otimes m} \otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

corresponds to a multilinear $GL_n$-invariant map

$$M_n(\mathbb{C}) \otimes \ldots \otimes M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

We will now give a description of the generating maps $\mu_\sigma$ in terms of matrices. Under the identification, matrix multiplication is induced by composition on rank one endomorphisms and here the rule is given by

$$v \otimes f.v' \otimes f' = f(v')v \otimes f'$$

$$\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \ldots & \phi_n \end{bmatrix} . \begin{bmatrix} \nu_1' \\ \vdots \\ \nu_n' \end{bmatrix} \otimes \begin{bmatrix} \phi_1' & \ldots & \phi_n' \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} f(v') \otimes \begin{bmatrix} \phi_1' & \ldots & \phi_n' \end{bmatrix} .$$

Moreover, the trace map on $M_n$ is induced by that on rank one endomorphisms where it is given by the rule

$$tr(v \otimes f) = f(v)$$

$$tr(\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \ldots & \phi_n \end{bmatrix}) = tr(\begin{bmatrix} \nu_1\phi_1 & \ldots & \nu_1\phi_n \\ \vdots & \ddots & \vdots \\ \nu_n\phi_1 & \ldots & \nu_n\phi_n \end{bmatrix}) = \sum_i \nu_i\phi_i = f(v)$$

With these rules we can now give a matrix-interpretation of the $GL_n$-invariant maps $\mu_\sigma$.

**Proposition 1.2** *Let $\sigma = (i_1 i_2 \ldots i_\alpha)(j_1 j_2 \ldots j_\beta) \ldots (z_1 z_2 \ldots z_\zeta)$ be a decomposition of $\sigma \in S_m$ into cycles (including those of length one). Then, under the above identification we have*

$$\mu_\sigma(A_1 \otimes \ldots \otimes A_m) = tr(A_{i_1} A_{i_2} \ldots A_{i_\alpha}) tr(A_{j_1} A_{j_2} \ldots A_{j_\beta}) \ldots tr(A_{z_1} A_{z_2} \ldots A_{z_\zeta})$$

*Proof.* Both sides are multilinear hence it suffices to verify the equality for rank one matrices. Write $A_i = v_i \otimes f_i$, then we have that

$$\begin{aligned} \mu_\sigma(A_1 \otimes \ldots \otimes A_m) = & \quad \mu_\sigma(v_1 \otimes \ldots v_m \otimes f_1 \otimes \ldots \otimes f_m) \\ = & \quad \prod_i f_i(v_{\sigma(i)}) \end{aligned}$$

Consider the subproduct

$$f_{i_1}(v_{i_2}) f_{i_2}(v_{i_3}) \ldots f_{i_{\alpha-1}}(v_{i_\alpha}) = S$$

Now, look at the matrixproduct

$$v_{i_1} \otimes f_{i_1} . v_{i_2} \otimes f_{i_2} . \ \ldots \ . v_{i_\alpha} \otimes f_{i_\alpha}$$

which is by the product rule equal to

$$f_{i_1}(v_{i_2})f_{i_2}(v_{i_3})\ldots f_{i_{\alpha-1}}(v_{i_\alpha})v_{i_1}\otimes f_{i_\alpha}$$

Hence, by the trace rule we have that

$$tr(A_{i_1}A_{i_2}\ldots A_{i_\alpha}) = \prod_{j=1}^{\alpha} f_{i_j}(v_{\sigma(i_j)}) = S$$

$\square$

Having found a description of the multilinear invariant polynomial maps

$$M_n^m = \underbrace{M_n \oplus \ldots \oplus M_n}_{m} \longrightarrow \mathbb{C}$$

we will now describe all polynomial maps which are constant along orbits by polarization. The coordinate algebra $\mathbb{C}[M_n^m]$ is the polynomial ring in $mn^2$ variables $x_{ij}(k)$ where $1 \leq k \leq m$ and $1 \leq i,j \leq n$. Consider the $m$ generic $n \times n$ matrices

$$\boxed{k} = X_k = \begin{bmatrix} x_{11}(k) & \ldots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \ldots & x_{nn}(k) \end{bmatrix} \in M_n(\mathbb{C}[M_n^m]).$$

The action of $GL_n$ on polynomial maps $f \in \mathbb{C}[M_n^m]$ is fully determined by the action on the coordinate functions $x_{ij}(k)$. As in the case of one $n \times n$ matrix we see that this action is given by

$$g.x_{ij}(k) = (g^{-1}.X_k.g)_{ij}.$$

We see that this action preserves the subspaces spanned by the entries of any of the generic matrices. Hence, we can define a gradation on $\mathbb{C}[M_n^m]$ by $deg(x_{ij}(k)) = (0,\ldots,0,1,0,\ldots,0)$ (with 1 at place $k$) and decompose

$$\mathbb{C}[M_n^m] = \bigoplus_{(d_1,\ldots,d_m)\in\mathbb{N}^m} \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)}$$

where $\mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)}$ is the subspace of all multihomogeneous forms $f$ in the $x_{ij}(k)$ of degree $(d_1,\ldots,d_m)$, that is, in each monomial term of $f$ there are exactly $d_k$ factors coming from the entries of the generic matrix $X_k$ for all $1 \leq k \leq m$. The action of $GL_n$ stabilizes each of these subspaces, that is,

$$\text{if} \quad f \in \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)} \quad \text{then} \quad g.f \in \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)} \quad \text{for all } g \in GL_n.$$

In particular, if $f$ determines a polynomial map on $M_n^m$ which is constant along orbits, that is, if $f$ belongs to the ring of invariants $\mathbb{C}[M_n^m]^{GL_n}$ then each of its multihomogeneous components is also an invariant and therefore it suffices to determine all multihomogeneous invariants.

Let $f \in \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)}$ and take for each $1 \leq k \leq m$ $d_k$ new variables $t_1(k),\ldots,t_{d_k}(k)$. Expand

$$f(t_1(1)A_1(1)+\ldots+t_{d_1}A_{d_1}(1),\ldots,t_1(m)A_1(m)+\ldots+t_{d_m}(m)A_{d_m}(m))$$

as a polynomial in the variables $t_i(k)$, then we get an expression

$$\sum t_1(1)^{s_1(1)}\ldots t_{d_1}^{s_{d_1}(1)}\ldots t_1(m)^{s_1(m)}\ldots t_{d_m}(m)^{s_{d_m}(m)}. \\ f_{(s_1(1),\ldots,s_{d_1}(1),\ldots,s_1(m),\ldots,s_{d_m}(m))}(A_1(1),\ldots,A_{d_1}(1),\ldots,A_1(m),\ldots,A_{d_m}(m))$$

such that for all $1 \leq k \leq m$ we have $\sum_{i=1}^{d_k} s_i(k) = d_k$. Moreover, each of the $f_{(s_1(1),\ldots,s_{d_1}(1),\ldots,s_1(m),\ldots,s_{d_m}(m))}$ is a multi-homogeneous polynomial function on

$$\underbrace{M_n \oplus \ldots \oplus M_n}_{d_1} \oplus \underbrace{M_n \oplus \ldots \oplus M_n}_{d_2} \oplus \ldots \oplus \underbrace{M_n \oplus \ldots \oplus M_n}_{d_m}$$

of multi-degree $(s_1(1),\ldots,s_{d_1}(1),\ldots,s_1(m),\ldots,s_{d_m}(m))$. Observe that if $f$ is an invariant polynomial function on $M_n^m$, then each of these multi homogeneous functions is an invariant polynomial function on $M_n^D$ where $D = d_1+\ldots+d_m$.

In particular, we consider the multi-*linear* function

$$f_{1,\ldots,1} : M_n^D = M_n^{d_1} \oplus \ldots \oplus M_n^{d_m} \longrightarrow \mathbb{C}$$

which we call the *polarization* of the polynomial $f$ and denote with $Pol(f)$. Observe that $Pol(f)$ in symmetric in each of the entries belonging to a block $M_n^{d_k}$ for every $1 \leq k \leq m$. If $f$ is invariant under $GL_n$, then so is the multilinear function $Pol(f)$ and we know the form of all such functions by the results given before (replacing $M_n^m$ by $M_n^D$).

Finally, we want to recover $f$ back from its polarization. We claim to have the equality

$$Pol(f)(\underbrace{A_1,\ldots,A_1}_{d_1},\ldots,\underbrace{A_m,\ldots,A_m}_{d_m}) = d_1!\ldots d_m! f(A_1,\ldots,A_m)$$

and hence we recover $f$. This process is called *restitution* . The claim follows from the observation that

$$f(t_1(1)A_1+\ldots+t_{d_1}(1)A_1,\ldots,t_1(m)A_m+\ldots+t_{d_m}(m)A_m) = \\ f((t_1(1)+\ldots+t_{d_1}(1))A_1,\ldots,(t_1(m)+\ldots+t_{d_m}(m))A_m) = \\ (t_1(1)+\ldots+t_{d_1}(1))^{d_1}\ldots(t_1(m)+\ldots+t_{d_m}(m))^{d_m} f(A_1,\ldots,A_m)$$

and the definition of $Pol(f)$. Hence we have proved that any multi-homogeneous invariant polynomial function $f$ on $M_n^m$ of multidegree $(d_1, \ldots, d_m)$ can be obtained by restitution of a multilinear invariant function

$$Pol(f) : M_n^D = M_n^{d_1} \oplus \ldots \oplus M_n^{d_m} \longrightarrow \mathbb{C}$$

If we combine this fact with our description of all multilinear invariant functions on $M_n \oplus \ldots \oplus M_n$ we finally obtain :

**Theorem 1.6 (First fundamental theorem of matrix invariants)** *Any polynomial function $M_n^m \xrightarrow{f} \mathbb{C}$ which is constant along orbits under the action of $GL_n$ by simultaneous conjugation is a polynomial in the invariants*

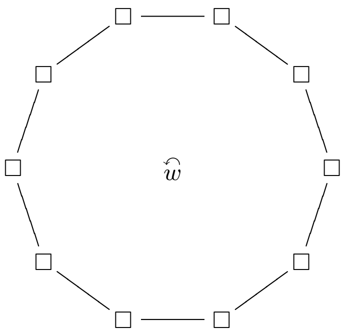
$$tr(X_{i_1} \ldots X_{i_l})$$

*where $X_{i_1} \ldots X_{i_l}$ run over all possible noncommutative polynomials in the generic matrices $\{X_1, \ldots, X_m\}$.*

We will call the algebra $\mathbb{C}[M_n^m]$ generated by these invariants the *necklace algebra* $\mathbb{N}_n^m = \mathbb{C}[M_n^m]^{GL_n}$. The terminology is justified by the observation that the generators

$$tr(X_{i_1} X_{i_2} \ldots X_{i_l})$$

are only determined up to cyclic permutation of the factors $X_j$. They correspond to a *necklace word* $w$




where each $i$-colored bead $\boxed{i}$ corresponds to a generic matrix $X_i$. To obtain an invariant, these bead-matrices are cyclically multiplied to obtain an $n \times n$ matrix with coefficients in $M_n(\mathbb{C}[M_n^m])$. The trace of this matrix is called $tr(w)$ and theorem 1.6 asserts that these elements generate the ring of polynomial invariants.

## 1.4 The trace algebra

In this section we will prove that there is a bound on the length of the necklace words $w$ necessary for the $tr(w)$ to generate $\mathbb{N}_n^m$. Later, after we have determined the relations between these necklaces $tr(w)$, we will be able to improve this bound.

First, we will characterize all $GL_n$*-equivariant maps* from $M_n^m$ to $M_n$, that is all polynomial maps $M_n^m \xrightarrow{f} M_n$ such that for all $g \in GL_n$ the diagram below is commutative

$$\begin{tikzcd} M_n^m \arrow[r, "f"] \arrow[d, "g.g^{-1}"'] & M_n \arrow[d, "g.g^{-1}"] \\ M_n^m \arrow[r, "f"] & M_n \end{tikzcd}$$

With pointwise addition and multiplication in the target algebra $M_n$, these polynomial maps form a noncommutative algebra $\mathbb{T}_n^m$ called the *trace algebra*. Obviously, the trace algebra is a subalgebra of the algebra of *all* polynomial maps from $M_n^m$ to $M_n$, that is,

$$\mathbb{T}_n^m \hookrightarrow M_n(\mathbb{C}[M_n^m])$$

Clearly, using the diagonal embedding of $\mathbb{C}$ in $M_n$ any invariant polynomial on $M_n^m$ determines a $GL_n$-equivariant map. Equivalently, using the diagonal embedding of $\mathbb{C}[M_n^m]$ in $M_n(\mathbb{C}[M_n^m])$ we can embed the necklace algebra

$$\mathbb{N}_n^m = \mathbb{C}[M_n^m]^{GL_n} \hookrightarrow \mathbb{T}_n^m$$

Another source of $GL_n$-equivariant maps are the *coordinate maps*

$$X_i : M_n^m = M_n \oplus \ldots \oplus M_n^m \longrightarrow M_n \qquad (A_1, \ldots, A_m) \mapsto A_i$$

Observe that the coordinate map $X_i$ is represented by the generic matrix $\boxed{i} = X_i$ in $M_n(\mathbb{C}[M_n^m])$.

**Proposition 1.3** *As an algebra over the necklace algebra $\mathbb{N}_n^m$, the trace algebra $\mathbb{T}_n^m$ is generated by the elements $\{X_1, \ldots, X_m\}$.*

*Proof.* Consider a $GL_n$-equivariant map $M_n^m \xrightarrow{f} M_n$ and associate to it the polynomial map

$$M_n^{m+1} = M_n^m \oplus M_n \xrightarrow{tr(fX_{m+1})} \mathbb{C}$$

defined by sending $(A_1, \ldots, A_m, A_{m+1})$ to $tr(f(A_1, \ldots, A_m).A_{m+1})$. For all $g \in GL_n$ we have that $f(g.A_1.g^{-1}, \ldots, g.A_m.g^{-1})$ is equal to $g.f(A_1, \ldots, A_m).g^{-1}$ and hence

$$\begin{aligned} tr(f(g.A_1.g^{-1}, \ldots, g.A_m.g^{-1}).g.A_{m+1}.g^{-1}) &= tr(g.f(A_1, \ldots, A_m).g^{-1}.g.A_{m+1}.g^{-1}) \\ &= tr(g.f(A_1, \ldots, A_m).A_{m+1}.g^{-1}) \\ &= tr(f(A_1, \ldots, A_m).A_{m+1}) \end{aligned}$$

so $tr(fX_{m+1})$ is an invariant polynomial function on $M_n^{m+1}$ which is *linear* in $X_{m+1}$. By theorem 1.6 we can therefore write

$$tr(fX_{m+1}) = \sum \underbrace{g_{i_1 \ldots i_l}}_{\in \mathbb{N}_n^m} tr(X_{i_1} \ldots X_{i_l} X_{m+1})$$

Here, we used the necklace property allowing to permute cyclically the trace terms in which $X_{m+1}$ occurs such that $X_{m+1}$ occurs as the last factor. But then, $tr(fX_{m+1}) = tr(gX_{m+1})$ where

$$g = \sum g_{i_1 \ldots i_l} X_{i_1} \ldots X_{i_l}.$$

Finally, using the *nondegeneracy* of the trace map on $M_n$ (that is, if $A, B \in M_n$ such that $tr(AC) = tr(BC)$ for all $C \in M_n$, then $A = B$) it follows that $f = g$. $\square$

If we give each of the generic matrices $X_i$ degree one, we see that the trace algebra $\mathbb{T}_n^m$ is a *connected positively graded algebra*

$$\mathbb{T}_n^m = T_0 \oplus T_1 \oplus T_2 \oplus \ldots \qquad \text{with} \quad T_0 = \mathbb{C}.$$

Our aim is to bound the length of the monomials in the $X_i$ necessary to generate $\mathbb{T}_n^m$ as a module over the necklace algebra $\mathbb{N}_n^m$. Before we can do this we need to make a small detour in one of the more exotic realms of noncommutative algebra : the *Nagata-Higman problem* .

**Theorem 1.7 (Nagata-Higman)** *Let $R$ be an associative algebra* without a unit element. *Assume there is a fixed natural number $n$ such that $x^n = 0$ for all $x \in R$. Then, $R^{2^n-1} = 0$, that is*

$$x_1.x_2.\ldots.x_{2^n-1} = 0$$

*for all $x_j \in R$.*

*Proof.* We use induction on $n$, the case $n = 1$ being obvious. Consider for all $x, y \in R$

$$f(x, y) = yx^{n-1} + xyx^{n-2} + x^2yx^{n-3} + \ldots + x^{n-2}yx + x^{n-1}y.$$

Because for all $c \in \mathbb{C}$ we must have that

$$0 = (y + cx)^n = x^n c^n + f(x, y)c^{n-1} + \ldots + y^n$$

it follows that all the coefficients of the $c^i$ with $1 \leq i < n$ must be zero, in particular $f(x, y) = 0$. But then we have for all $x, y, z \in R$ that

$$\begin{aligned} 0 &= f(x, z)y^{n-1} + f(x, zy)y^{n-2} + f(x, zy^2)y^{n-3} + \ldots + f(x, zy^{n-1}) \\ &= nx^{n-1}zy^{n-1} + zf(y, x^{n-1}) + xzf(y, x^{n-2}) + x^2zf(y, x^{n-3}) + \ldots + x^{n-2}zf(y, x) \end{aligned}$$

and therefore $x^{n-1}zy^{n-1} = 0$. Let $I \triangleleft R$ be the twosided ideal of $R$ generated by all elements $x^{n-1}$, then we have that $I.R.I = 0$. In the quotient algebra $\overline{R} = R/I$ every element $\overline{x}$ satisfies $\overline{x}^{n-1} = 0$.

By induction we may assume that $\overline{R}^{2^{n-1}-1} = 0$, or equivalently that $R^{2^{n-1}-1}$ is contained in $I$. But then,

$$R^{2^n-1} = R^{2(2^{n-1}-1)+1} = R^{2^{n-1}-1}.R.R^{2^{n-1}-1} \hookrightarrow I.R.I = 0$$

finishing the proof. $\square$

**Proposition 1.4** *The trace algebra $\mathbb{T}_n^m$ is spanned as a module over the necklace algebra $\mathbb{N}_n^m$ by all monomials in the generic matrices*

$$X_{i_1}X_{i_2}\ldots X_{i_l}$$

*of degree $l \leq 2^n - 1$.*

*Proof.* By the diagonal embedding of $\mathbb{N}_n^m$ in $M_n(\mathbb{C}[M_n^m])$ it is clear that $\mathbb{N}_n^m$ commutes with any of the $X_i$. Let $\mathbb{T}_+$ and $\mathbb{N}_+$ be the strict positive degrees of $\mathbb{T}_n^m$ and $\mathbb{N}_n^m$ and form the graded associative algebra (without unit element)

$$R = \mathbb{T}_+/\mathbb{N}_+.\mathbb{T}_+$$

Observe that any element $t \in \mathbb{T}_+$ satisfies an equation of the form

$$t^n + c_1t^{n-1} + c_2t^{n-2} + \ldots + c_n = 0$$

with all of the $c_i \in \mathbb{N}_+$. Indeed we have seen that all the coefficients of the characteristic polynomial of a matrix can be expressed as polynomials in the traces of powers of the matrix. But then, for any $x \in R$ we have that $x^n = 0$.

By the Nagata-Higman theorem we know that $R^{2^n-1} = (R_1)^{2^n-1} = 0$. Let $\mathbb{T}'$ be the graded $\mathbb{N}_n^m$-submodule of $\mathbb{T}_n^m$ spanned by all monomials in the generic matrices $X_i$ of degree at most $2^n - 1$, then the above can be reformulated as

$$\mathbb{T}_n^m = \mathbb{T}' + \mathbb{N}_+.\mathbb{T}_n^m.$$

We claim that $\mathbb{T}_m^n = \mathbb{T}'$. Assume not, then there is a homogeneous $t \in \mathbb{T}_n^m$ *of minimal degree* $d$ not contained in $\mathbb{T}'$ but still we have a description

$$t = t' + c_1.t_1 + \ldots + c_s.t_s$$

with $t'$ and all $c_i, t_i$ homogeneous elements. As $deg(t_i) < d$, $t_i \in \mathbb{T}'$ for all $i$ but then is $t \in \mathbb{T}'$ a contradiction. $\square$

Finally we are in a position to bound the length of the necklaces generating $\mathbb{N}_n^m$ as an algebra.

**Theorem 1.8** *The necklace algebra $\mathbb{N}_n^m$ is generated by all necklaces $tr(w)$ where $w$ is a necklace word in the bead-matrices $\{X_1, \ldots, X_m\}$ of length $l \leq 2^n$.*

*Proof.* Let $\mathbb{T}'$ be the $\mathbb{C}$-subalgebra of $\mathbb{T}_n^m$ generated by the generic matrices $X_i$. Then, $tr(\mathbb{T}'_+)$ generates the ideal $\mathbb{N}_+$. Let $\mathbb{S}$ be the set of all monomials in the $X_i$ of degree at most $2^n - 1$. By the foregoing proposition we know that $\mathbb{T}' \hookrightarrow \mathbb{N}_n^m.\mathbb{S}$. The trace map

$$tr : \mathbb{T}_n^m \longrightarrow \mathbb{N}_n^m$$

is $\mathbb{N}_n^m$-linear and therefore, because $\mathbb{T}'_+ \subset \mathbb{T}'.(\mathbb{C}X_1 + \ldots + \mathbb{C}X_m)$ we have

$$tr(\mathbb{T}'_+) \subset tr(\mathbb{N}_n^m.\mathbb{S}.(\mathbb{C}X_1 + \ldots + \mathbb{C}X_m)) \subset \mathbb{N}_n^m.tr(\mathbb{S}')$$

where $\mathbb{S}'$ is the set of monomials in the $X_i$ of degree at most $2^n$. If $\mathbb{N}'$ is the $\mathbb{C}$-subalgebra of $\mathbb{N}_n^m$ generated by all $tr(S')$, then we have $tr(\mathbb{T}'_+) \subset \mathbb{N}_n^m.\mathbb{N}'_+$. But then, we have

$$\mathbb{N}_+ = \mathbb{N}_n^m tr(\mathbb{T}_+) \subset \mathbb{N}_n^m \mathbb{N}'_+ \qquad \text{and thus} \qquad \mathbb{N}_n^m = \mathbb{N}' + \mathbb{N}_n^m \mathbb{N}'_+$$

from which it follows that $\mathbb{N}_n^m = \mathbb{N}'$ by a similar argument as in the foregoing proof. □

**Example 1.4 (The algebras $\mathbb{N}_2^2$ and $\mathbb{T}_2^2$)** When working with $2 \times 2$ matrices, the following identities are often helpful

$$0 = A^2 - tr(A)A + det(A)$$
$$A.B + B.A = tr(AB) - tr(A)tr(B) + tr(A)B + tr(B)A$$

for all $A, B \in M_2$. Let $\mathbb{N}'$ be the subalgebra of $\mathbb{N}_2^2$ generated by $tr(X_1), tr(X_2)$, $det(X_1), det(X_2)$ and $tr(X_1X_2)$. Using the two formulas above and $\mathbb{N}_2^2$-linearity of the trace on $\mathbb{T}_2^2$ we see that the trace of any monomial in $X_1$ and $X_2$ of degree $d \geq 3$ can be expressed in elements of $\mathbb{N}'$ and traces of monomials of degree $\leq d-1$. Hence, we have

$$\mathbb{N}_2^2 = \mathbb{C}[tr(X_1), tr(X_2), det(X_1), det(X_2), tr(X_1X_2)].$$

Observe that there can be no algebraic relations between these generators as we have seen that the induced map $\pi : M_2^2 \longrightarrow \mathbb{C}^5$ is surjective. Another consequence of the above identities is that over $\mathbb{N}_2^2$ any monomial in the $X_1, X_2$ of degree $d \geq 3$ can be expressed as a linear combination of $1, X_1, X_2$ and $X_1X_2$ and so these elements generate $\mathbb{T}_2^2$ as a $\mathbb{N}_2^2$-module. In fact, they are a basis of $\mathbb{T}_2^2$ over $\mathbb{N}_2^2$. Assume otherwise, there would be a relation say

$$X_1X_2 = \alpha I_2 + \beta X_1 + \gamma X_2$$

with $\alpha, \beta, \gamma \in \mathbb{C}(tr(X_1), tr(X_2), det(X_1), det(X_2), tr(X_1X_2))$. Then this relation has to hold for all matrix couples $(A, B) \in M_2^2$ and we obtain a contradiction if we take the couple

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{whence} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Concluding, we have the following description of $\mathbb{N}_2^2$ and $\mathbb{T}_2^2$ as a subalgebra of $\mathbb{C}[M_2^2]$ respectively $M_2(\mathbb{C}[M_2^2])$

$$\begin{cases} \mathbb{N}_2^2 = & \mathbb{C}[tr(X_1), tr(X_2), det(X_1), det(X_2), tr(X_1X_2)] \\ \mathbb{T}_2^2 = & \mathbb{N}_2^2.I_2 \oplus \mathbb{N}_2^2.X_1 \oplus \mathbb{N}_2^2.X_2 \oplus \mathbb{N}_2^2.X_1X_2 \end{cases}$$

Observe that we might have taken the generators $tr(X_i^2)$ rather than $det(X_i)$ because $det(X_i) = \frac{1}{2}(tr(X_i)^2 - tr(X_i)^2)$ as follows from taking the trace of characteristic polynomial of $X_i$.

## 1.5 The symmetric group

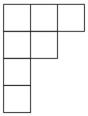
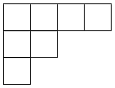
Let $S_d$ be the symmetric group of all permutations on $d$ letters. The group algebra $\mathbb{C}\ S_d$ is a semisimple algebra. In particular, any simple $S_d$-representation is isomorphic to a minimal left ideal of $\mathbb{C}\ S_d$ which is generated by an *idempotent* . We will now determine these idempotents.

To start, conjugacy classes in $S_d$ correspond naturally to *partitions* $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $d$, that is, decompositions in natural numbers

$$d = \lambda_1 + \ldots + \lambda_k \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$$

The correspondence associates to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ the conjugacy class of a permutation consisting of disjoint cycles of lengths $\lambda_1, \ldots, \lambda_k$. It is traditional to assign to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ a *Young diagram* with $\lambda_i$ boxes in the $i$-th row, the rows of boxes lined up to the left. The *dual partition* $\lambda^* = (\lambda_1^*, \ldots, \lambda_r^*)$ to $\lambda$ is defined by interchanging rows and columns in the Young diagram of $\lambda$ .

For example, to the partition $\lambda = (3, 2, 1, 1)$ of 7 we assign the Young diagram

$\lambda =$  $\lambda^* =$ 

with dual partition $\lambda^* = (4, 2, 1)$. A *Young tableau* is a numbering of the boxes of a Young diagram by the integers $\{1, 2, \ldots, d\}$. For example, two distinct Young tableaux of type $\lambda$ are

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

Now, fix a Young tableau $T$ of type $\lambda$ and define subgroups of $S_d$ by

$$P_\lambda = \{\sigma \in S_d \mid \sigma \text{ preserves each row } \}$$

$$Q_\lambda = \{\sigma \in S_d \mid \sigma \text{ preserves each column } \}$$

For example, for the second Young tableaux given above we have that

$$\begin{cases} P_\lambda & = S_{\{1,3,5\}} \times S_{\{2,4\}} \times \{(6)\} \times \{(7)\} \\ Q_\lambda & = S_{\{1,2,6,7\}} \times S_{\{3,4\}} \times \{(5)\} \end{cases}$$

Observe that different Young tableaux for the same $\lambda$ define different subgroups and different elements to be defined below. Still, the simple representations we will construct from them turn out to be isomorphic.

Using these subgroups, we define the following elements in the groupalgebra $\mathbb{C}S_d$

$$a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma \quad , \quad b_\lambda = \sum_{\sigma \in Q_\lambda} sgn(\sigma) e_\sigma \quad \text{and} \quad c_\lambda = a_\lambda . b_\sigma$$

The element $c_\lambda$ is called a *Young symmetrizer* . The next result gives an explicit one-to-one correspondence between the simple representations of $\mathbb{C}S_d$ and the conjugacy classes in $S_d$ (or, equivalently, Young diagrams).

**Theorem 1.9** *For every partition $\lambda$ of $d$ the left ideal $\mathbb{C}S_d.c_\lambda = V_\lambda$ is a simple $S_d$-representations and, conversely, any simple $S_d$-representation is isomorphic to $V_\lambda$ for a unique partition $\lambda$.*

*Proof.* (sketch) Observe that $P_\lambda \cap Q_\lambda = \{e\}$ (any permutation preserving rows as well as columns preserves all boxes) and so any element of $S_d$ can be written in at most one way as a product $p.q$ with $p \in P_\lambda$ and $q \in Q_\lambda$. In particular, the Young symmetrizer can be written as $c_\lambda = \sum \pm e_\sigma$ with $\sigma = p.q$ for unique $p$ and $q$ and the coefficient $\pm 1 = sgn(q)$. From this it follows that for all $p \in P_\lambda$ and $q \in Q_\lambda$ we have

$$p.a_\lambda = a_\lambda.p = a_\lambda \quad , \quad sgn(q)q.b_\lambda = b_\lambda.sgn(q)q = b_\lambda \quad , \quad p.c_\lambda.sgn(q)q = c_\lambda$$

Moreover, we claim that $c_\lambda$ is the unique element in $\mathbb{C}S_d$ (up to a scalar factor) satisfying the last property. This requires a few preparations.

Assume $\sigma \notin P_\lambda.Q_\lambda$ and consider the tableaux $T' = \sigma T$, that is, replacing the label $i$ of each box in $T$ by $\sigma(i)$. We claim that there are two distinct numbers which belong to the same row in $T$ and to the same column in $T'$. If this were not the case, then all the distinct numbers in the first row of $T$ appear in different columns of $T'$. But then we can find an element $q_1'$ in the subgroup $\sigma.Q_\lambda.\sigma^{-1}$ preserving the columns of $T'$ to take all these elements to the first row of $T'$. But then, there is an element $p_1 \in T_\lambda$ such that $p_1 T$ and $q_1' T'$ have the same first row. We can proceed to the second row and so on and obtain elements $p \in P_\lambda$ and $q' \in \sigma.Q_\lambda, \sigma^{-1}$ such that the tableaux $pT$ and $q'T'$ are equal. Hence, $pT = q'\sigma T$ entailing that $p = q'\sigma$. Further, $q' = \sigma.q.\sigma^{-1}$ but then $p = q'\sigma = \sigma q$ whence $\sigma = p.q^{-1} \in P_\lambda.Q_\lambda$, a contradiction. Therefore, to $\sigma \notin P_\lambda.Q_\lambda$ we can assign a *transposition* $\tau = (ij)$ (replacing the two distinct numbers belonging to the same row in $T$ and to the same column in $T'$) for which $p = \tau \in P_\lambda$ and $q = \sigma^{-1}.\tau.\sigma \in Q_\lambda$.

After these preliminaries, assume that $c' = \sum a_\sigma e_\sigma$ is an element such that

$$p.c'.sgn(q)q = c' \quad \text{for all} \quad p \in P_\lambda, q \in Q_\lambda$$

We claim that $a_\sigma = 0$ whenever $\sigma \notin P_\lambda.Q_\lambda$. For take the transposition $\tau$ found above and $p = \tau$, $q = \sigma^{-1}.\tau.\sigma$, then $p.\sigma.q = \tau.\sigma.\sigma^{-1}.\tau.\sigma = \sigma$. However, the coefficient of $\sigma$ in $c'$ is $a_\sigma$ and that of $p.c'.q$ is $-a_\sigma$ proving the claim. That is,

$$c' = \sum_{p,q} a_{pq} e_{p.q}$$

but then by the property of $c'$ we must have that $a_{pq} = sgn(q)a_e$ whence $c' = a_e c_\lambda$ finishing the proof of the claimed uniqueness of the element $c_\lambda$.

As a consequence we have for all elements $x \in \mathbb{C}S_d$ that $c_\lambda.x, c_\lambda = \alpha_x c_\lambda$ for some scalar $\alpha_x \in \mathbb{C}$ and in particular that $c_\lambda^2 = n_\lambda c_\lambda$, for,

$$\begin{aligned} p.(c_\lambda.x.c_\lambda).sgn(q)q &= p.a_\lambda.b_\lambda.x.a_\lambda.b_\lambda.sgn(q)q \\ &= a_\lambda.b_\lambda.x.a_\lambda.b_\lambda = c_\lambda.x.c_\lambda \end{aligned}$$

and the statement follows from the uniqueness result for $c_\lambda$.

Define $V_\lambda = \mathbb{C}S_d.c_\lambda$ then we have $c_\lambda.V_\lambda \subset \mathbb{C}c_\lambda$. We claim that $V_\lambda$ is a simple $S_d$-representation. Let $W \subset V_\lambda$ be a simple subrepresentation, then being a left ideal of $\mathbb{C}S_d$ we can write $W = \mathbb{C}S_d.x$ with $x^2 = x$ (note that $W$ is a direct summand). Assume that $c_\lambda.W = 0$, then $W.W \subset \mathbb{C}S_d.c_\lambda.W = 0$ implying that $x = 0$ whence $W = 0$, a contradiction. Hence, $c_\lambda.W = \mathbb{C}c_\lambda \subset W$, but then

$$V_\lambda = \mathbb{C}S_d.c_\lambda \subset W \quad \text{whence} V_\lambda = W$$

is simple. Remains to show that for different partitions, the corresponding simple representations cannot be isomorphic.

We put a *lexicographic* ordering on the partitions by the rule that

$$\lambda > \mu \quad \text{if the first nonvanishing } \lambda_i - \mu_i \text{ is positive}$$

We claim that if $\lambda > \mu$ then $a_\lambda.\mathbb{C}S_d.b_\mu = 0$. It suffices to check that $a_\lambda.\sigma.b_\mu = 0$ for $\sigma \in S_d$. As $\sigma.b_\mu.\sigma^{-1}$ is the "$b$-element" constructed from the tableau $b.T'$ where $T'$ is the tableaux fixed for $\mu$, it is sufficient to check that $a_\lambda.b_\mu = 0$. As $\lambda > \mu$ there are distinct numbers $i$ and $j$ belonging to the same row in $T$ and to the same column in $T'$. If not, the distinct numbers in any fixed row of $T$ must belong to different columns of $T'$, but this can only happen for all rows if $\mu \geq \lambda$. So consider $\tau = (ij)$ which belongs to $P_\lambda$ and to $Q_\mu$, whence $a_\lambda.\tau = a_\lambda$ and $\tau.b_\mu = -b_\mu$. But then,

$$a_\lambda.b_\mu = a_\lambda.\tau, \tau, b_\mu = -a_\lambda.b_\mu$$

proving the claim.

If $\lambda \neq \mu$ we claim that $V_\lambda$ is not isomorphic to $V_\mu$. Assume that $\lambda > \mu$ and $\phi$ a $\mathbb{C}S_d$-isomorphism with $\phi(V_\lambda) = V_\mu$, then

$$\phi(c_\lambda V_\lambda) = c_\lambda \phi(V_\lambda) = c_\lambda V_\mu = c_\lambda \mathbb{C}S_d c_\mu = 0$$

Hence, $c_\lambda V_\lambda = \mathbb{C}c_\lambda \neq 0$ lies in the kernel of an isomorphism which is clearly absurd.

Summarizing, we have constructed to distinct partitions of $d$, $\lambda$ and $\mu$ nonisomorphic simple $\mathbb{C}S_d$-representations $V_\lambda$ and $V_\mu$. As we know that there are as many isomorphism classes of simples as there are conjugacy classes in $S_d$ (or partitions), the $V_\lambda$ form a complete set of isomorphism classes of simple $S_d$-representations, finishing the proof of the theorem. $\square$

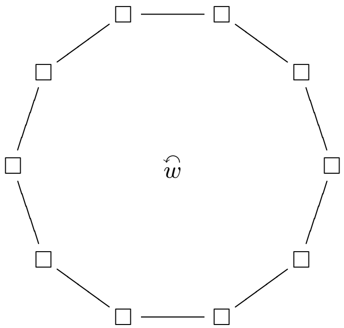
## 1.6 Necklace relations

In this section we will prove that all the relations holding among the elements of the necklace algebra $\mathbb{N}_n^m$ are formal consequences of the Cayley-Hamilton theorem. First, we will have to set up some notation to clarify what we mean by this.

For technical reasons it is sometimes convenient to have an infinite supply of noncommutative variables $\{x_1, x_2, \ldots, x_i, \ldots\}$. Two monomials of the same degree $d$ in these variables

$$M = x_{i_1} x_{i_2} \ldots x_{i_d} \quad \text{and} \quad M' = x_{j_1} x_{j_2} \ldots x_{j_d}$$

are said to be *equivalent* if $M'$ is obtained from $M$ by a cyclic permutation, that is, there is a $k$ such that $i_1 = j_k$ and all $i_a = j_b$ with $b = k + a - 1$ *mod* $d$. That is, if they determine the same necklace word




with each of the beads one of the noncommuting variables $\boxed{i} = x_i$. To each equivalence class we assign a formal variable that we denote by

$$t(x_{i_1} x_{i_2} \ldots x_{i_d}).$$

The *formal necklace algebra* $\mathbb{N}^\infty$ is then the polynomial algebra on all these (infinitely many) letters. Similarly, we define the *formal trace algebra* $\mathbb{T}^\infty$ to be the algebra

$$\mathbb{T}^\infty = \mathbb{N}^\infty \otimes_{\mathbb{C}} \mathbb{C}\langle x_1, x_2, \ldots, x_i, \ldots \rangle$$

that is, the free associative algebra on the noncommuting variables $x_i$ with coefficients in the polynomial algebra $\mathbb{N}^\infty$.

Crucial for our purposes is the existence of an $\mathbb{N}^\infty$-linear *formal trace map*

$$t : \mathbb{T}^\infty \twoheadrightarrow \mathbb{N}^\infty$$

defined by the formula

$$t(\sum a_{i_1 \ldots i_k} x_{i_1} \ldots x_{i_k}) = \sum a_{i_1 \ldots i_k} t(x_{i_1} \ldots x_{i_k})$$

where $a_{i_1 \ldots i_k} \in \mathbb{N}^\infty$.

In an analogous manner we will define infinite versions of the necklace and trace algebras. Let $M_n^\infty$ be the space of all ordered sequences $(A_1, A_2, \ldots, A_i, \ldots)$ with $A_i \in M_n$ and all but finitely many of the $A_i$ are the zero matrix. Again, $GL_n$ acts on $M_n^\infty$ by simultaneous conjugation and we denote the *infinite necklace algebra* $\mathbb{N}_n^\infty$ to be the algebra of polynomial functions $f$

$$M_n^\infty \xrightarrow{f} \mathbb{C}$$

which are constant along orbits. Clearly, $\mathbb{N}_n^\infty$ is generated as $\mathbb{C}$-algebra by the invariants $tr(M)$ where $M$ runs over all monomials in the coordinate generic matrices $X_k = (x_{ij}(k))_{i,j}$ belonging to the $k$-th factor of $M_n^\infty$. Similarly, the *infinite trace algebra* $\mathbb{T}_n^\infty$ is the algebra of $GL_n$-equivariant polynomial maps

$$M_n^\infty \longrightarrow M_n.$$

Clearly, $\mathbb{T}_n^\infty$ is the $\mathbb{C}$-algebra generated by $\mathbb{N}_n^\infty$ and the generic matrices $X_k$ for $1 \leq k < \infty$. Observe that $\mathbb{T}_n^\infty$ is a subalgebra of the matrixring

$$\mathbb{T}_n^\infty \hookrightarrow M_n(\mathbb{C}[M_n^\infty])$$

and as such has a trace map $tr$ defined on it and from our knowledge of the generators of $\mathbb{N}_n^\infty$ we know that $tr(\mathbb{T}_n^\infty) = \mathbb{N}_n^\infty$.

Now, there are natural algebra epimorphisms

$$\mathbb{T}^\infty \overset{\tau}{\twoheadrightarrow} \mathbb{T}_n^\infty \quad \text{and} \quad \mathbb{N}^\infty \overset{\nu}{\twoheadrightarrow} \mathbb{N}_n^\infty$$

defined by $\tau(t(x_{i_1} \ldots x_{i_k})) = \nu(t(x_{i_1} \ldots x_{i_k})) = tr(X_{i_1} \ldots X_{i_k})$ and $\tau(x_i) = X_i$. That is, $\nu$ and $\tau$ are compatible with the trace maps

$$\begin{tikzcd}
\mathbb{T}^\infty \arrow[r, two heads, "\tau"] \arrow[d, "t"'] & \mathbb{T}_n^\infty \arrow[d, "tr"] \\
\mathbb{N}^\infty \arrow[r, two heads, "\nu"] & \mathbb{N}_n^\infty
\end{tikzcd}$$

We are interested in describing the *necklace relations* , that is, the kernel of $\nu$. In the next section we will describe the *trace relations* which is the kernel of $\tau$. Note that we obtain the relations holding among the necklaces in $\mathbb{N}_n^m$ by setting all $x_i = 0$ with $i > m$ and all $t(x_{i_1} \ldots x_{i_k}) = 0$ containing a variable with $i_j > m$.

In the description a map $T : \mathbb{C}S_d \longrightarrow \mathbb{N}^\infty$ will be important. Let $S_d$ be the symmetric group of permutations on $\{1, \ldots, d\}$ and let

$$\sigma = (i_1 i_1 \ldots i_\alpha)(j_1 j_2 \ldots j_\beta) \ldots (z_1 z_2 \ldots z_\zeta)$$

be a decomposition of $\sigma \in S_d$ into cycles including those of length one. The map $T$ assigns to $\sigma$ a formal necklace $T_\sigma(x_1, \ldots, x_d)$ defined by

$$T_\sigma(x_1, \ldots, x_d) = t(x_{i_1} x_{i_2} \ldots x_{i_\alpha}) t(x_{j_1} x_{j_2} \ldots x_{j_\beta}) \ldots t(x_{z_1} x_{z_2} \ldots x_{z_\zeta})$$

Let $V = V_n$ be again the $n$-dimensional vectorspace of column vectors, then $S_d$ acts naturally on $V^{\otimes d}$ via

$$\sigma.(v_1 \otimes \ldots \otimes v_d) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$$

hence determines a linear map $\lambda_\sigma \in End(V^{\otimes d})$. Recall from section 3 that under the natural identifications

$$(M_n^{\otimes d})^* \simeq (V^{*\otimes d} \otimes V^{\otimes d})^* \simeq End(V^{\otimes d})$$

the map $\lambda_\sigma$ defines the multilinear map

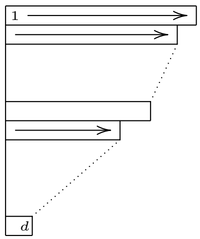
$$\mu_\sigma : \underbrace{M_n \otimes \ldots \otimes M_n}_{d} \longrightarrow \mathbb{C}$$

defined by (using the cycle decomposition of $\sigma$ as before)

$$\mu_\sigma(A_1 \otimes \ldots \otimes A_d) = tr(A_{i_1} A_{i_2} \ldots A_{i_\alpha}) tr(A_{j_1} A_{j_2} \ldots A_{j_\beta}) \ldots tr(A_{z_1} A_{z_2} \ldots A_{z_\zeta}) \quad .$$

Therefore, a linear combination $\sum a_\sigma T_\sigma(x_1, \ldots, x_d)$ is a necklace relation (that is, belongs to $Ker\ \nu$) if and only if the multilinear map $\sum a_\sigma \mu_\sigma : M_n^{\otimes d} \longrightarrow \mathbb{C}$ is zero. This, in turn, is equivalent to the endomorphism $\sum a_\sigma \lambda_\sigma \in End(V^{\otimes m})$, induced by the action of the element $\sum a_\sigma e_\sigma \in \mathbb{C}S_d$ on $V^{\otimes d}$, being zero. In order to answer the latter problem we have to understand the action of a Young symmetrizer $c_\lambda \in \mathbb{C}S_d$ on $V^{\otimes d}$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $d$ and equip the corresponding Young diagram with the standard tableau (that is, order first the boxes in the first row from left to right, then the second row from left to right and so on).




The subgroup $P_\lambda$ of $S_d$ which preserves each row then becomes

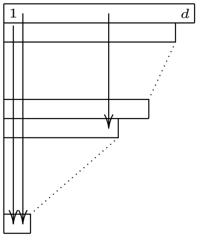
$$P_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_k} \hookrightarrow S_d.$$

As $a_\lambda = \sum_{p \in P_\lambda} e_p$ we see that the image of the action of $a_\lambda$ on $V^{\otimes d}$ is the subspace

$$Im(a_\lambda) = Sym^{\lambda_1}\ V \otimes Sym^{\lambda_2}\ V \otimes \ldots \otimes Sym^{\lambda_k}\ V \hookrightarrow V^{\otimes d} \quad .$$

Here, $Sym^i\ V$ denotes the subspace of symmetric tensors in $V^{\otimes i}$.

Similarly, equip the Young diagram of $\lambda$ with the tableau by ordering first the boxes in the first column from top to bottom, then those of the second column from top to bottom and so on.




Equivalently, give the Young diagram corresponding to the dual partition of $\lambda$

$$\lambda^* = (\mu_1, \mu_2, \ldots, \mu_l)$$

the standard tableau. Then, the subgroup $Q_\lambda$ of $S_d$ which preserves each row of $\lambda$ (or equivalently, each column of $\lambda^*$) is

$$Q_\lambda = S_{\mu_1} \times S_{\mu_2} \times \ldots \times S_{\mu_l} \hookrightarrow S_d$$

As $b_\lambda = \sum_{q \in Q_\lambda} sgn(q) e_q$ we see that the image of $b_\lambda$ on $V^{\otimes d}$ is the subspace

$$Im(b_\lambda) = \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \ldots \otimes \bigwedge^{\mu_l} V \hookrightarrow V^{\otimes d} \quad .$$

Here, $\bigwedge^i\ V$ is the subspace of all anti-symmetric tensors in $V^{\otimes i}$. Note that $\bigwedge^i\ V = 0$ whenever $i$ is greater than the dimension $dim\ V = n$. That is, the image of the action of $b_\lambda$ on $V^{\otimes d}$ is zero whenever the dual partition $\lambda^*$ contains a row of length $\geq n+1$, or equivalently, whenever $\lambda$ has $\geq n+1$ rows. Because the Young symmetrizer $c_\lambda = a_\lambda . b_\lambda \in \mathbb{C}\ S_d$ we have proved the first result on necklace relations.

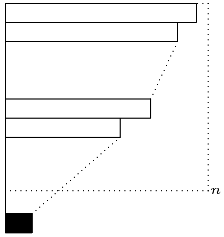
**Theorem 1.10 (Second fundamental theorem of matrix invariants)** *A formal necklace*

$$\sum_{\sigma \in S_d} a_\sigma T_\sigma(x_1, \ldots, x_d)$$

*is a necklace relation (for $n \times n$ matrices) if and only if the element*

$$\sum a_\sigma e_\sigma \in \mathbb{C} S_d$$

*belongs to the ideal of $\mathbb{C}S_d$ spanned by the Young symmetrizers $c_\lambda$ relative to partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$*




*with a least $n+1$ rows, that is, $k \geq n+1$.*

**Example 1.5** (Fundamental necklace and trace relation.) Consider the partition $\lambda = (1, 1, \ldots, 1)$ of $n+1$, with corresponding Young tableau

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline \text{n+1} \\ \hline \end{array}$$

Then, $P_\lambda = \{e\}$, $Q_\lambda = S_{n+1}$ and we have the Young symmetrizer

$$a_\lambda = 1 \qquad b_\lambda = c_\lambda = \sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma.$$

The corresponding element is called the *fundamental necklace relation*

$$\mathtt{fund}_n(x_1, \ldots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} sgn(\sigma) T_\sigma(x_1, \ldots, x_{n+1}).$$

Clearly, $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$ is multilinear of degree $n+1$ in the variables $\{x_1, \ldots, x_{n+1}\}$. Conversely, any multilinear necklace relation of degree $n+1$ must be a scalar multiple of $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$. This follows from the proposition as the ideal described there is for $d = n+1$ just the scalar multiples of $\sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma$.

Because $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$ is multilinear in the variables $x_i$ we can use the cyclic permutation property of the formal trace $t$ to write it in the form

$$\mathtt{fund}_n(x_1, \ldots, x_{n+1}) = t(\mathtt{cha}_n(x_1, \ldots, x_n) x_{n+1}) \quad \text{with} \quad \mathtt{cha}_n(x_1, \ldots, x_n) \in \mathbb{T}^\infty$$

Observe that $\mathtt{cha}_n(x_1, \ldots, x_n)$ is multilinear in the variables $x_i$. Moreover, by the nondegeneracy of the trace map $tr$ and the fact that $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$ is a necklace relation, it follows that

$\mathtt{cha}_n(x_1, \dots, x_n)$ is a trace relation. Again, any multilinear trace relation of degree $n$ in the variables $\{x_1, \dots, x_n\}$ is a scalar multiple of $\mathtt{cha}_n(x_1, \dots, x_n)$. This follows from the corresponding uniqueness result for $\mathtt{fund}_n(x_1, \dots, x_{n+1})$.

We can give an explicit expression of this *fundamental trace relation*

$$\mathtt{cha}_n(x_1, \dots, x_n) = \sum_{k=0}^{n} (-1)^k \sum_{i_1 \neq i_2 \neq \dots \neq i_k} x_{i_1} x_{i_2} \dots x_{i_k} \sum_{\sigma \in S_J} sgn(\sigma) T_\sigma(x_{j_1}, \dots, x_{j_{n-k}})$$

where $J = \{1, \dots, n\} - \{i_1, \dots, i_k\}$. In a moment we will see that $\mathtt{cha}_n(x_1, \dots, x_n)$ and hence also $\mathtt{fund}_n(x_1, \dots, x_{n+1})$ is obtained by polarization of the Cayley-Hamilton identity for $n \times n$ matrices.

We will explain what we mean by the Cayley-Hamilton polynomial for an element of $\mathbb{T}^\infty$. Recall that when $X \in M_n(A)$ is a matrix with coefficients in a commutative $\mathbb{C}$-algebra $A$ its *characteristic polynomial* is defined to be

$$\chi_X(t) = det(t \mathbb{1}_n - X) \in A[t]$$

and by the *Cayley-Hamilton theorem* we have the basic relation that $\chi_X(X) = 0$. We have seen that the coefficients of the characteristic polynomial can be expressed as polynomial functions in the $tr(X^i)$ for $1 \leq i \leq n$.

For example if $n = 2$, then the characteristic polynomial can we written as

$$\chi_X(t) = t^2 - tr(X)t + \frac{1}{2}(tr(X)^2 - tr(X^2)).$$

For general $n$ the method for finding these polynomial functions is based on the formal recursive algorithm expressing elementary symmetric functions in term of *Newton functions* , usually expressed by the formulae

$$f(t) = \prod_{i=1}^{n} (t - \lambda_i),$$

$$\frac{f'(t)}{f(t)} = \frac{d \; log \; f(t)}{dt} = \sum_{i=1}^{n} \frac{1}{t - \lambda_i} = \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} (\sum_{i=1}^{n} \lambda_i^k)$$

Note, if $\lambda_i$ are the eigenvalues of $X \in M_n$, then $f(t) = \chi_X(t)$ and $\sum_{i=1}^n \lambda_i^k = tr(X^k)$. Therefore, one can use the formulae to express $f(t)$ in terms of the elements $\sum_{i=1}^n \lambda_i^k$. To get the required expression for the characteristic polynomial of $X$ one only has to substitute $\sum_{i=1}^n \lambda_i^k$ with $tr(X^k)$.

This allows us to construct a *formal Cayley-Hamilton polynomial* $\chi_x(x) \in \mathbb{T}^\infty$ of an element $x \in \mathbb{T}^\infty$ by replacing in the above characteristic polynomial the term $tr(X^k)$ with $t(x^k)$ and $t^l$ with $x^l$. If $x$ is one of the variables $x_i$ then $\chi_x(x)$ is an element of $\mathbb{T}^\infty$ homogeneous of degree $n$. Moreover, by the Cayley-Hamilton theorem it follows immediately that $\chi_x(x)$ is a trace relation. Hence, if we fully polarize $\chi_x(x)$ (say, using the variables $\{x_1, \dots, x_n\}$) we obtain a multilinear trace relation of degree $n$. By the argument given in the example above we know that this element must be a scalar multiple of $\mathtt{cha}_n(x_1,\ldots,x_n)$. In fact, one can see that this scale factor must be $(-1)^n$ as the leading term of the multilinearization is $\sum_{\sigma\in S_n} x_{\sigma(1)}\ldots x_{\sigma(n)}$ and compare this with the explicit form of $\mathtt{cha}_n(x_1,\ldots,x_n)$.

**Example 1.6** Consider the case $n=2$. The formal Cayley-Hamilton polynomial of an element $x\in\mathbb{T}^\infty$ is

$$\chi_x(x) = x^2 - t(x)x + \frac{1}{2}(t(x)^2 - t(x^2)) \quad .$$

Polarization with respect to the variables $x_1$ and $x_2$ gives the expression

$$x_1x_2 + x_2x_1 - t(x_1)x_2 - t(x_2)x_1 + t(x_1)t(x_2) - t(x_1x_2)$$

which is $\mathtt{cha}_2(x_1,x_2)$. Indeed, multiplying it on the right with $x_3$ and applying the formal trace $t$ to it we obtain

$$\begin{aligned} & t(x_1x_2x_3) + t(x_2x_1x_3) - t(x_1)t(x_2x_3) - t(x_2)t(x_1x_3) \\ & \quad + t(x_1)t(x_2)t(x_3) - t(x_1x_2)t(x_3) \\ = \; & T_{(123)}(x_1,x_2,x_3) + T_{(213)}(x_1,x_2,x_3) - T_{(1)(23)}(x_1,x_2,x_3) - T_{(2)(13)}(x_1,x_2,x_3) \\ & \quad + T_{(1)(2)(3)}(x_1,x_2,x_3) - T_{(12)(3)}(x_1,x_2,x_3) \\ & \quad = \textstyle\sum_{\sigma\in S_3} T_\sigma(x_1,x_2,x_3) = \mathtt{fund}_2(x_1,x_2,x_3) \end{aligned}$$

as required.

**Theorem 1.11** *The necklace relations $Ker\ \nu$ is the ideal of $\mathbb{N}^\infty$ generated by all the elements*

$$\mathtt{fund}_n(m_1,\ldots,m_{n+1})$$

*where the $m_i$ run over all monomials in the variables $\{x_1,x_2,\ldots,x_i,\ldots\}$.*

*Proof.* Take a homogeneous necklace relation $f\in Ker\ \nu$ of degree $d$ and polarize it to get a multilinear element $f'\in\mathbb{N}^\infty$. Clearly, $f'$ is also a necklace relation and if we can show that $f'$ belongs to the described ideal, then so does $f$ as the process of restitution maps this ideal into itself.

Therefore, we may assume that $f$ is multilinear of degree $d$. A priori $f$ may depend on more than $d$ variables $x_k$, but we can separate $f$ as a sum of multilinear polynomials $f_i$ each depending on precisely $d$ variables such that for $i\neq j$ $f_i$ and $f_j$ do not depend on the same variables. Setting some of the variables equal to zero, we see that each of the $f_i$ is again a necklace relation.

Thus, we may assume that $f$ is a multilinear necklace identity of degree $d$ depending on the variables $\{x_1,\ldots,x_d\}$. But then we know from theorem 1.10 that we can write

$$f = \sum_{\tau\in S_d} a_\tau T_\tau(x_1,\ldots,x_d)$$

where $\sum a_\tau e_\tau \in \mathbb{C}S_d$ belongs to the ideal spanned by the Young symmetrizers of Young diagrams $\lambda$ having at least $n+1$ rows.

We claim that this ideal is generated by the Young symmetrizer of the partition $(1,\ldots,1)$ of $n+1$ under the natural embedding of $S_{n+1}$ into $S_d$. Let $\lambda$ be a Young diagram having $k \geq n+1$ boxes and let $c_\lambda$ be a Young symmetrizer with respect to a tableau where the boxes in the first column are labeled by the numbers $I = \{i_1,\ldots,i_k\}$ and let $S_I$ be the obvious subgroup of $S_d$. As $Q_\lambda = S_I \times Q'$ we see that $b_\lambda = (\sum_{\sigma \in S_I} sgn(\sigma)e_\sigma).b'$ with $b' \in \mathbb{C}Q'$. Hence, $c_\lambda$ belongs to the twosided ideal generated by $c_I = \sum_{\sigma \in S_I} sgn(\sigma)e_\sigma$ but this is also the twosided ideal generated by $c_k = \sum_{\sigma \in S_k} sgn(\sigma)e_\sigma$ as one verifies by conjugation with a partition sending $I$ to $\{1,\ldots,k\}$. Moreover, by induction one shows that the twosided ideal generated by $c_k$ belongs to the twosided ideal generated by $c_d = \sum_{\sigma \in S_d} sgn(\sigma)e_\sigma$, finishing the proof of the claim.

From this claim, we can write

$$\sum_{\tau \in S_d} a_\tau e_\tau = \sum_{\tau_i,\tau_j \in S_d} a_{ij} e_{\tau_i}.(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_{\tau_j}$$

and therefore it suffices to analyze the form of the necklace identity associated to an element of the form

$$e_\tau.(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_{\tau'} \quad \text{with} \quad \tau,\tau' \in S_d$$

Now, if a groupelement $\sum_{\mu \in S_d} b_\mu e_\mu$ corresponds to the formal necklace polynomial $\mathtt{g}(x_1,\ldots,x_d)$, then the element $e_\tau.(\sum_{\mu \in S_d} b_\mu e_\mu).e_{\tau^{-1}}$ corresponds to the formal necklace polynomial $\mathtt{g}(x_{\tau(1)},\ldots,x_{\tau(d)})$.

Therefore, we may replace the element $e_\tau.(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_{\tau'}$ by the element

$$(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\eta \quad \text{with} \quad \eta = \tau'.\tau \in S_d$$

We claim that we can write $\eta = \sigma'.\theta$ with $\sigma' \in S_{n+1}$ and $\theta \in S_d$ such that each cycle of $\theta$ contains at most one of the elements from $\{1,2,\ldots,n+1\}$. Indeed assume that $\eta$ contains a cycle containing more than one element from $\{1,\ldots,n+1\}$, say 1 and 2, that is

$$\eta = (1 i_1 i_2 \ldots i_r 2 j_1 j_2 \ldots j_s)(k_1 \ldots k_\alpha)\ldots(z_1 \ldots z_\zeta)$$

then we can express the product $(12).\eta$ in cycles as

$$(1 i_1 i_2 \ldots i_r)(2 j_1 j_2 \ldots j_s)(k_1 \ldots k_\alpha)\ldots(z_1 \ldots z_\zeta)$$

Continuing in this manner we reduce the number of elements from $\{1.\ldots,n+1\}$ in every cycle to at most one.

But then as $\sigma' \in S_{n+1}$ we have seen that $(\sum sgn(\sigma)e_\sigma).e_{\sigma'} = sgn(\sigma')(\sum sgn(\sigma)e_\sigma)$ and consequently

$$(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\eta = \pm(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\theta$$

where each cycle of $\theta$ contains at most one of $\{1,\ldots,n+1\}$. Let us write

$$\theta = (1i_1 \ldots i_\alpha)(2j_1 \ldots j_\beta)\ldots(n+1s_1 \ldots s_\kappa)(t_1 \ldots t_\lambda)\ldots(z_1 \ldots z_\zeta)$$

Now, let $\sigma \in S_{n+1}$ then the cycle decomposition of $\sigma.\theta$ is obtained as follows : substitute in each cycle of $\sigma$ the element 1 formally by the string $1i_1 \ldots i_\alpha$, the element 2 by the string $2j_1 \ldots j_\beta$, and so on until the element $n+1$ by the string $n+1s_1 \ldots s_\kappa$ and finally adjoin the cycles of $\theta$ in which no elements from $\{1,\ldots,n+1\}$ appear.

Finally, we can write out the formal necklace element corresponding to the element $(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\theta$ as

$$\mathtt{fund}_n(x_1x_{i_1}\ldots x_{i_\alpha}, x_2x_{j_1}\ldots x_{j_\beta},\ldots,x_{n+1}x_{s_1}\ldots x_{s_\kappa})t(x_{t_1}\ldots x_{t_\lambda})\ldots t(x_{z_1}\ldots x_{z_\zeta})$$

finishing the proof of the theorem. □

## 1.7 Trace relations

We will again use the non-degeneracy of the trace map to deduce the trace relations. That is, we will describe the kernel of the epimorphism

$$\tau \; : \; \int \mathbb{C}\langle x_1,x_2,\ldots\rangle = \mathbb{T}^\infty \twoheadrightarrow \mathbb{T}^\infty_n = \int_n \mathbb{C}\langle x_1,x_2,\ldots\rangle$$

from the description of the necklace relations.

**Theorem 1.12** *The trace relations $Ker\ \tau$ is the twosided ideal of the formal trace algebra $\mathbb{T}^\infty$ generated by all elements*

$$\mathtt{fund}_n(m_1,\ldots,m_{n+1}) \quad \textit{and} \quad \mathtt{cha}_n(m_1,\ldots,m_n)$$

*where the $m_i$ run over all monomials in the variables $\{x_1,x_2,\ldots,x_i,\ldots\}$.*

*Proof.* Consider a trace relation $\mathtt{h}(x_1,\ldots,x_d) \in Ker\ \tau$. Then, we have a necklace relation of the form

$$t(\mathtt{h}(x_1,\ldots,x_d)x_{d+1}) \in Ker\ \nu$$

By theorem 1.11 we know that this element must be of the form

$$\sum n_{i_1\ldots i_{n+1}}\mathtt{fund}_n(m_{i_1},\ldots,m_{i_{n+1}})$$

where the $m_i$ are monomials, the $n_{i_1\ldots i_{n+1}} \in \mathbb{N}^\infty$ and the expression must be linear in the variable $x_{d+1}$. That is, $x_{d+1}$ appears linearly in each of the terms

$$n\mathtt{fund}_n(m_1,\ldots,m_{n+1})$$

so appears linearly in $n$ or in precisely one of the monomials $m_i$. If $x_{d+1}$ appears linearly in $n$ we can write

$$n = t(n'.x_{d+1}) \quad \text{where} \quad n' \in \mathbb{T}^\infty.$$

If $x_{d+1}$ appears linearly in one of the monomials $m_i$ we may assume that it does so in $m_{n+1}$, permuting the monomials if necessary. That is, we may assume $m_{n+1} = m'_{n+1}.x_{d+1}.m"_{n+1}$ with $m, m'$ monomials. But then, we can write

$$\begin{aligned} n\mathtt{fund}_n(m_1,\ldots,m_{n+1}) = &\quad nt(\mathtt{cha}_n(m_1,\ldots,m_n).m'_{n+1}.x_{d+1}.m"_{n+1}) \\ = &\quad t(n.m"_{n+1}.\mathtt{cha}_n(m_1,\ldots,m_n).m'_{n+1}.x_{d+1}) \end{aligned}$$

using $\mathbb{N}^\infty$-linearity and the cyclic permutation property of the formal trace $t$. But then, separating the two cases, one can write the total expression

$$\begin{aligned} t(\mathtt{h}(x_1,\ldots,x_d)x_{d+1}) = t([&\sum_{\underline{i}} n'_{i_1\ldots i_{n+1}}\mathtt{fund}_n(m_{i_1},\ldots,m_{i_{n+1}}) \\ &+ \sum_{\underline{j}} n_{j_1\ldots j_{n+1}}.m"_{j_{n+1}}.\mathtt{cha}_n(m_{j_1},\ldots,m_{j_n}).m'_{j_{n+1}}] \quad x_{d+1}) \end{aligned}$$

Finally, observe that two formal trace elements $\mathtt{h}(x_1,\ldots,x_d)$ and $\mathtt{k}(x_1,\ldots,x_d)$ are equal if the formal necklaces

$$t(\mathtt{h}(x_1,\ldots,x_d)x_{d+1}) = t(\mathtt{k}(x_1,\ldots,x_d)x_{d+1})$$

are equal, finishing the proof. $\square$

We will give another description of the necklace relations $Ker\ \tau$ which is better suited for the categorical interpretation of $\mathbb{T}^\infty_n$ to be given in the next chapter. Consider formal trace elements $m_1, m_2, \ldots, m_i, \ldots$ with $m_j \in \mathbb{T}^\infty$. The *formal substitution*

$$f \mapsto f(m_1, m_2, \ldots, m_i, \ldots)$$

is the uniquely determined algebra endomorphism of $\mathbb{T}^\infty$ which maps the variable $x_i$ to $m_i$ and is compatible with the formal trace $t$. That is, the substitution sends a monomial $x_{i_1}x_{i_1}\ldots x_{i_k}$ to the element $g_{i_1}g_{i_2}\ldots g_{i_k}$ and an element $t(x_{i_1}x_{i_2}\ldots x_{i_k})$ to the element $t(g_{i_1}g_{i_2}\ldots g_{i_k})$. A *substitution invariant ideal* of $\mathbb{T}^\infty$ is a twosided ideal of $\mathbb{T}^\infty$ that is closed under all possible substitutions as well as under the formal trace $t$. For any subset of elements $E \subset \mathbb{T}^\infty$ there is a minimal substitution invariant ideal containing $E$. This is the ideal generated by all elements obtained from $E$ by making all possible substitutions and taking all their formal traces. We will refer to this ideal as the *substitution invariant ideal generated by* $E$.

Recall the definition of the formal Cayley-Hamilton polynomial $\chi_x(x)$ of an element $x \in \mathbb{T}^\infty$ given in the previous section.

**Theorem 1.13** *The trace relations* $Ker\ \tau$ *is the substitution invariant ideal of* $\mathbb{T}^\infty$ *generated by the formal Cayley-Hamilton polynomials*

$$\chi_x(x) \quad \textit{for all} \quad x \in \mathbb{T}^\infty$$

*Proof.* The result follows from theorem 1.12 and the definition of a substitution invariant ideal once we can show that the full polarization of $\chi_x(x)$, which we have seen is $\mathtt{cha}_n(x_1, \ldots, x_n)$, lies in the substitution invariant ideal generated by the $\chi_x(x)$.

This is true since we may replace the process of polarization with the process of multilinearization, whose first step is to replace, for instance

$$\chi_x(x) \quad \text{by} \quad \chi_{x+y}(x+y) - \chi_x(x) - \chi_y(y).$$

The final result of multilinearization is the same as of full polarization and the claim follows as multilinearizing a polynomial in a substitution invariant ideal, we remain in the same ideal. □

We will use our knowledge on the necklace and trace relations to improve the bound of $2^n - 1$ in the *Nagata-Higman problem* to $n^2$. Recall that this problem asks for a number $N(n)$ with the property that if $R$ is an associative $\mathbb{C}$-algebra without unit such that $r^n = 0$ for all $r \in R$, then we must have for all $r_i \in R$ the identity

$$r_1 r_2 \ldots r_{N(n)} = 0 \quad \text{in} \quad R.$$

We start by reformulating the problem. Consider the positive part $\mathbb{F}_+$ of the free $\mathbb{C}$-algebra generated by the variables $\{x_1, x_2, \ldots, x_i, \ldots\}$

$$\mathbb{F}_+ = \mathbb{C}\langle x_1, x_2, \ldots, x_i, \ldots\rangle_+$$

which is an associative $\mathbb{C}$-algebra without unit. Let $T(n)$ be the twosided ideal of $\mathbb{F}_+$ generated by all $n$-powers $f^n$ with $f \in \mathbb{F}_+$. Note that the ideal $T(n)$ is invariant under all substitutions of $\mathbb{F}_+$. The Nagata-Higman problem then asks for a number $N(n)$ such that the product

$$x_1 x_2 \ldots x_{N(n)} \in T(n).$$

We will now give an alternative description of the quotient algebra $\mathbb{F}_+/T(n)$. Let $\mathbb{N}_+$ be the positive part of the infinite necklace algebra $\mathbb{N}_n^\infty$ and $\mathbb{T}_+$ the positive part of the infinite trace algebra $\mathbb{T}_n^\infty$. Consider the quotient associative $\mathbb{C}$-algebra without unit

$$\overline{\mathbb{T}_+} = \mathbb{T}_+/(\mathbb{N}_+\mathbb{T}_n^\infty).$$

Observe the following facts about $\overline{\mathbb{T}_+}$ : as a $\mathbb{C}$-algebra it is generated by the variables $X_1, X_2, \ldots$ as all the other algebra generators of the form $t(x_{i_1} \ldots x_{i_r})$ of $\mathbb{T}^\infty$ are mapped to zero in $\overline{\mathbb{T}_+}$. Further, from the form of the Cayley-Hamilton polynomial it follows that every $t \in \overline{\mathbb{T}_+}$ satisfies $t^n = 0$. That is, we have an algebra epimorphism

$$\mathbb{F}_+/T(n) \twoheadrightarrow \overline{\mathbb{T}_+}$$

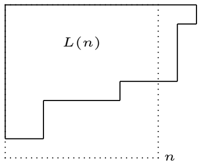
and we claim that it is also injective. To see this, observe that the quotient $\mathbb{T}^\infty/\mathbb{N}_+^\infty\mathbb{T}^\infty$ is just the free $\mathbb{C}$-algebra on the variables $\{x_1, x_2, \ldots\}$. To obtain $\overline{\mathbb{T}_+}$ we have to factor out the ideal of trace relations. Now, a formal Cayley-Hamilton polynomial $\chi_x(x)$ is mapped to $x^n$ in $\mathbb{T}^\infty/\mathbb{N}_+^\infty\mathbb{T}^\infty$. That is, to obtain $\overline{\mathbb{T}_+}$ we factor out the substitution invariant ideal (observe that $t$ is zero here) generated by the elements $x^n$, but this is just the definition of $\mathbb{F}_+/T(n)$.

Therefore, a reformulation of the Nagata-Higman problem is to find a number $N = N(n)$ such that the product of the first $N$ generic matrices

$$X_1X_2\ldots X_N \in \mathbb{N}_+^\infty\mathbb{T}_n^\infty \qquad \text{or, equivalently that} \qquad tr(X_1X_2\ldots X_NX_{N+1})$$

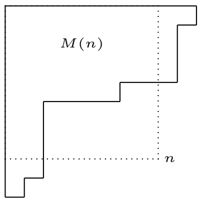
can be expressed as a linear combination of products of traces of lower degree. Using the description of the necklace relations given in theorem 1.10 we can reformulate this conditions in terms of the group algebra $\mathbb{C}S_{N+1}$. Let us introduce the following subspaces of the groupalgebra :

- $A$ will be the subspace spanned by all $N+1$ cycles in $S_{N+1}$,
- $B$ will be the subspace spanned by all elements except $N+1$ cycles,
- $L(n)$ will be the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions




  with $\leq n$ rows, and

- $M(n)$ will be the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions




  having more than $n$ rows.

With these notations, we can reformulate the above condition as

$$(12\ldots NN+1) \in B + M(n) \quad \text{and consequently} \quad \mathbb{C}S_{N+1} = B + M(n)$$

Define an inner product on the groupalgebra $\mathbb{C}S_{N+1}$ such that the groupelements form an orthonormal basis, then $A$ and $B$ are orthogonal complements and also $L(n)$ and $M(n)$ are orthogonal complements. But then, taking orthogonal complements the condition can be rephrased as

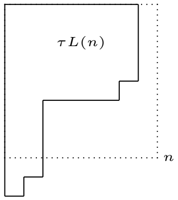
$$(B + M(n))^{\perp} = A \cap L(n) = 0.$$

Finally, let us define an automorphism $\tau$ on $\mathbb{C}S_{N+1}$ induced by sending $e_\sigma$ to $sgn(\sigma)e_\sigma$. Clearly, $\tau$ is just multiplication by $(-1)^N$ on $A$ and therefore the above condition is equivalent to

$$A \cap L(n) \cap \tau L(n) = 0.$$

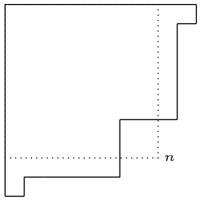
Moreover, for any Young tableau $\lambda$ we have that $\tau(a_\lambda) = b_{\lambda^*}$ and $\tau(b_\lambda) = a_{\lambda^*}$. Hence, the automorphism $\tau$ sends the Young symmetrizer associated to a partition to the Young symmetrizer of the dual partition. This gives the following characterization

- $\tau L(n)$ is the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions

  with $\leq n$ columns.

Now, specialize to the case $N = n^2$. Clearly, any Young diagram having $n^2 + 1$ boxes must have either more than $n$ columns or more than $n$ rows




and consequently we indeed have for $N = n^2$ that

$$A \cap L(n) \cap \tau L(n) = 0$$

finishing the proof of the promised refinement of the Nagata-Higman bound

**Theorem 1.14** *Let $R$ be an associative $\mathbb{C}$-algebra without unit element. Assume that $r^n = 0$ for all $r \in R$. Then, for all $r_i \in R$ we have*

$$r_1 r_2 \ldots r_{n^2} = 0$$

**Theorem 1.15** *The necklace algebra $\mathbb{N}_n^m$ is generated as a $\mathbb{C}$-algebra by all elements of the form*

$$tr(X_{i_1} X_{i_2} \ldots X_{i_l})$$

*with $l \leq n^2 + 1$. The trace algebra $\mathbb{T}_n^m$ is spanned as a module over the necklace algebra $\mathbb{N}_n^m$ by all monomials in the generic matrices*

$$X_{i_1} X_{i_2} \ldots X_{i_l}$$

*of degree $l \leq n^2$.*

## 1.8 Cayley-Hamilton algebras

In this section we define the category `alg@n` of Cayley-Hamilton algebras of degree $n$.

**Definition 1.1** *A* trace map *on an (affine) $\mathbb{C}$-algebra $A$ is a $\mathbb{C}$-linear map*

$$tr : A \longrightarrow A$$

*satisfying the following three properties for all $a, b \in A$ :*

1. $tr(a)b = btr(a)$,
2. $tr(ab) = tr(ba)$ *and*
3. $tr(tr(a)b) = tr(a)tr(b)$.

Note that it follows from the first property that the image $tr(A)$ of the trace map is contained in the *center* of $A$. Consider two algebras $A$ and $B$ equipped with a trace map which we will denote by $tr_A$ respectively $tr_B$. A *trace morphism* $\phi : A \longrightarrow B$ will be a $\mathbb{C}$-algebra morphism which is compatible with the trace maps, that is, the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "tr_A"'] & B \arrow[d, "tr_B"] \\ A \arrow[r, "\phi"] & B \end{tikzcd}$$

This definition turns algebras with a trace map into a category, denoted by `alg@`. We will say that an algebra $A$ with trace map $tr$ is *trace generated* by a subset of elements $I \subset A$ if the $\mathbb{C}$-algebra generated by $B$ and $tr(B)$ is equal to $A$ where $B$ is the $\mathbb{C}$-subalgebra generated by the elements of $I$. Note that $A$ does not have to be generated as a $\mathbb{C}$-algebra by the elements from $I$.

Observe that for $\mathbb{T}^\infty$ the formal trace $t : \mathbb{T}^\infty \twoheadrightarrow \mathbb{N}^\infty \hookrightarrow \mathbb{T}^\infty$ is a trace map. Property (1) follows because $\mathbb{N}^\infty$ commutes with all elements of $\mathbb{T}^\infty$, property (2) is the cyclic permutation property for $t$ and property (3) is the fact that $t$ is a $\mathbb{N}^\infty$-linear map. The formal trace algebra $\mathbb{T}^\infty$ is trace generated by the variables $\{x_1, x_2, \ldots, x_i, \ldots\}$ but *not* as a $\mathbb{C}$-algebra.

Actually, $\mathbb{T}^\infty$ is the *free algebra* in the generators $\{x_1, x_2, \ldots, x_i, \ldots\}$ in the category of algebras with a trace map, `alg@`. That is, if $A$ is an algebra with trace $tr$ which is trace generated by $\{a_1, a_2, \ldots\}$, then there is a trace preserving algebra epimorphism

$$\mathbb{T}^\infty \overset{\pi}{\twoheadrightarrow} A \quad .$$

For example, define $\pi(x_i) = a_i$ and $\pi(t(x_{i_1} \ldots x_{i_l})) = tr(\pi(x_{i_1}) \ldots \pi(x_{i_l}))$. Also, the formal trace algebra $\mathbb{T}^m$, that is the subalgebra of $\mathbb{T}^\infty$ trace generated by $\{x_1, \ldots, x_m\}$, is the free algebra in the category of algebras with trace that are trace generated by at most $m$ elements.

Given a trace map $tr$ on $A$, we can define for any $a \in A$ a *formal Cayley-Hamilton polynomial of degree n* . Indeed, express

$$f(t) = \prod_{i=1}^{n} (t - \lambda_i)$$

as a polynomial in $t$ with coefficients polynomial functions in the Newton functions $\sum_{i=1}^n \lambda_i^k$. Replacing the Newton function $\sum \lambda_i^k$ by $tr(a^k)$ we obtain the Cayley-Hamilton polynomial of degree $n$

$$\chi_a^{(n)}(t) \in A[t] \quad .$$

**Definition 1.2** *An (affine) $\mathbb{C}$-algebra $A$ with trace map $tr : A \longrightarrow A$ is said to be a* Cayley-Hamilton algebra of degree n *if the following two properties are satisfied :*

1. *$tr(1) = n$, and*
2. *For all $a \in A$ we have $\chi_a^{(n)}(a) = 0$ in $A$.*

`alg@n` *is the category of Cayley-Hamilton algebras of degree n with trace preserving morphisms.*

Observe that if $R$ is a commutative $\mathbb{C}$-algebra, then $M_n(R)$ is a Cayley-Hamilton algebra of degree $n$. The corresponding trace map is the composition of the usual trace with the inclusion of $R \twoheadrightarrow M_n(R)$ via scalar matrices. As a consequence, the infinite trace algebra $\mathbb{T}_n^\infty$ has a trace map induced by the natural inclusion

$$\begin{tikzcd}
\mathbb{T}_n^\infty \arrow[r, hook] \arrow[d, dotted, "tr"'] & M_n(\mathbb{C}[M_n^\infty]) \arrow[d, "tr"] \\
\mathbb{N}_n^\infty \arrow[r, hook] & \mathbb{C}[M_n^\infty]
\end{tikzcd}$$

which has image $tr(\mathbb{T}_n^\infty)$ the infinite necklace algebra $\mathbb{N}_n^\infty$. Clearly, being a trace preserving inclusion, $\mathbb{T}_n^\infty$ is a Cayley-Hamilton algebra of degree $n$. With this definition, we have the following categorical description of the trace algebra $\mathbb{T}_n^\infty$.

**Theorem 1.16** *The trace algebra $\mathbb{T}_n^\infty$ is the free algebra in the generic matrix generators $\{X_1, X_2, \ldots, X_i, \ldots\}$ in the category of Cayley-Hamilton algebras of degree $n$.*

*For any $m$, the trace algebra $\mathbb{T}_n^m$ is the free algebra in the generic matrix generators $\{X_1, \ldots, X_m\}$ in the category* `alg@n` *of Cayley-Hamilton algebras of degree $n$ which are trace generated by at most $m$ elements.*

*Proof.* Let $F_n$ be the free algebra in the generators $\{y_1, y_2, \ldots\}$ in the category `alg@n`, then by freeness of $\mathbb{T}^\infty$ there is a trace preserving algebra epimorphism

$$\mathbb{T}^\infty \xrightarrow{\pi} F_n \quad \text{with} \quad \pi(x_i) = y_i.$$

By the universal property of $F_n$, the ideal $Ker\ \pi$ is the minimal ideal $I$ of $\mathbb{T}^\infty$ such that $\mathbb{T}^\infty/I$ is Cayley-Hamilton of degree $n$.

We claim that $Ker\ \pi$ is substitution invariant. Consider a substitution endomorphism $\phi$ of $\mathbb{T}^\infty$ and consider the diagram

$$\begin{tikzcd}
\mathbb{T}^\infty \arrow[r, "\phi"] \arrow[d, dotted] \arrow[dr, dotted, "\chi"] & \mathbb{T}^\infty \arrow[d, "\pi"] \\
\mathbb{T}^\infty/Ker\ \chi \arrow[r, hook] & F_n
\end{tikzcd}$$

then $Ker\ \chi$ is an ideal closed under traces such that $\mathbb{T}^\infty/Ker\ \chi$ is a Cayley-Hamilton algebra of degree $n$ (being a subalgebra of $F_n$). But then $Ker\ \pi \subset Ker\ \chi$ (by minimality of $Ker\ \pi$) and therefore $\chi$ factors over $F_n$, that is, the substitution endomorphism $\phi$ descends to an endomorphism $\overline{\phi} : F_n \longrightarrow F_n$ meaning that $Ker\ \pi$ is left invariant under $\phi$, proving the claim. Further, any formal Cayley-Hamilton polynomial $\chi_x^{(n)}(x)$ of degree $n$ of $x \in \mathbb{T}^\infty$ maps to zero under $\pi$. By substitution invariance it follows that the ideal of trace relations $Ker\ \tau \subset Ker\ \pi$. We have seen that $\mathbb{T}^\infty/Ker\ \tau = \mathbb{T}_n^\infty$ is the infinite trace algebra which is a Cayley-Hamilton algebra of degree $n$. Thus, by minimality of $Ker\ \pi$ we must have $Ker\ \tau = Ker\ \pi$ and hence $F_n \simeq \mathbb{T}_n^\infty$. The second assertion follows immediately. □

Let $A$ be a Cayley-Hamilton algebra of degree $n$ which is trace generated by the elements $\{a_1, \ldots, a_m\}$. We have a trace preserving algebra epimorphism $p_A$ defined by $p(X_i) = a_i$

$$\begin{tikzcd} \mathbb{T}_n^m \arrow[r, two heads, "p_a"] \arrow[d, "tr"'] & A \arrow[d, "tr_A"] \\ \mathbb{T}_n^m \arrow[r, two heads, "p_a"] & A \end{tikzcd}$$

and hence a *presentation* $A \simeq \mathbb{T}_n^m / T_A$ where $T_A \triangleleft \mathbb{T}_n^m$ is the *ideal of trace relations* holding among the generators $a_i$. We recall that $\mathbb{T}_n^m$ is the ring of $GL_n$-equivariant polynomial maps $M_n^m \xrightarrow{f} M_n$, that is,

$$M_n(\mathbb{C}[M_n^m])^{GL_n} = \mathbb{T}_n^m$$

where the action of $GL_n$ is the diagonal action on $M_n(\mathbb{C}[M_n^m]) = M_n \otimes \mathbb{C}[M_n^m]$.

Observe that if $R$ is a commutative algebra, then any twosided ideal $I \triangleleft M_n(R)$ is of the form $M_n(J)$ for an ideal $J \triangleleft R$. Indeed, the subsets $J_{ij}$ of $(i,j)$ entries of elements of $I$ is an ideal of $R$ as can be seen by multiplication with scalar matrices. Moreover, by multiplying on both sides with *permutation matrices* one verifies that $J_{ij} = J_{kl}$ for all $i, j, k, l$ proving the claim.

Applying this to the induced ideal $M_n(\mathbb{C}[M_n^m])\ T_A\ M_n(\mathbb{C}[M_n^m]) \triangleleft M_n(\mathbb{C}[M_n^m])$ we find an ideal $N_A \triangleleft \mathbb{C}[M_n^m]$ such that

$$M_n(\mathbb{C}[M_n^m])\ T_A\ M_n(\mathbb{C}[M_n^m]) = M_n(N_A)$$

Observe that both the induced ideal and $N_A$ are stable under the respective $GL_n$-actions.

Assume that $V$ and $W$ are two (not necessarily finite dimensional) $\mathbb{C}$-vectorspaces with a locally finite $GL_n$-action (that is, every finite dimensional subspace is contained in a finite dimensional $GL_n$-stable subspace) and that $V \xrightarrow{f} W$ is a linear map commuting with the $GL_n$-action. In section 2.5 we will see that we can decompose $V$ and $W$ uniquely in direct sums of simple representations and in their isotypical components (that is, collecting all factors isomorphic to a given simple $GL_n$-representation) and prove that $V_{(0)} = V^{GL_n}$ respectively $W_{(0)} = W^{GL_n}$ where $(0)$ denotes the trivial $GL_n$-representation. We obtain a commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, two heads, "R"'] & W \arrow[d, two heads, "R"] \\ V^{GL_n} \arrow[r, "f_0"] & W^G \end{tikzcd}$$

where $R$ is the *Reynolds operator* , that is, the canonical projection to the isotypical component of the trivial representation. Clearly, the Reynolds operator commutes with the $GL_n$-action. Moreover, using complete decomposability we see that $f_0$ is surjective (resp. injective) if $f$ is surjective (resp. injective).

Because $N_A$ is a $GL_n$-stable ideal of $\mathbb{C}[M_n^m]$ we can apply the above in the situation

$$\begin{tikzcd} M_n(\mathbb{C}[M_n^m]) \arrow[r, two heads, "\pi"] \arrow[d, "R"] & M_n(\mathbb{C}[M_n^m]/N_A) \arrow[d, "R"] \\ \mathbb{T}_n^m \arrow[r, two heads, "\pi_0"] & M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n} \end{tikzcd}$$

and the bottom map factorizes through $A = \mathbb{T}_n^m/T_A$ giving a surjection

$$A \twoheadrightarrow M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n}.$$

In order to verify that this map is injective (and hence an isomorphism) it suffices to check that

$$M_n(\mathbb{C}[M_n^m])\ T_A\ M_n(\mathbb{C}[M_n^m]) \cap \mathbb{T}_n^m = T_A.$$

Using the functoriality of the Reynolds operator with respect to multiplication in $M_n(\mathbb{C}[M_n^\infty])$ with an element $x \in \mathbb{T}_n^m$ or with respect to the trace map (both commuting with the $GL_n$-action) we deduce the following relations :

- For all $x \in \mathbb{T}_n^m$ and all $z \in M_n(\mathbb{C}[M_n^\infty])$ we have $R(xz) = xR(z)$ and $R(zx) = R(z)x$.
- For all $z \in M_n(\mathbb{C}[M_n^\infty])$ we have $R(tr(z)) = tr(R(z))$.

Assume that $z = \sum_i t_i x_i n_i \in M_n(\mathbb{C}[M_n^m])\ T_A\ M_n(\mathbb{C}[M_n^m]) \cap \mathbb{T}_n^m$ with $m_i, n_i \in M_n(\mathbb{C}[M_n^m])$ and $t_i \in T_A$. Now, consider $X_{m+1} \in \mathbb{T}_n^\infty$. Using the cyclic property of traces we have

$$tr(zX_{m+1}) = \sum_i tr(m_i t_i n_i X_{m+1}) = \sum_i tr(n_i X_{m+1} m_i t_i)$$

and if we apply the Reynolds operator to it we obtain the equality

$$tr(zX_{m+1}) = tr(\sum_i R(n_i X_{m+1} m_i) t_i)$$

For any $i$, the term $R(n_i X_{m+1} m_i)$ is invariant so belongs to $\mathbb{T}_n^{m+1}$ and is linear in $X_{m+1}$. Knowing the generating elements of $\mathbb{T}_n^{m+1}$ we can write

$$R(n_i X_{m+1} m_i) = \sum_j s_{ij} X_{m+1} t_{ij} + \sum_k tr(u_{ik} X_{m+1}) v_{ik}$$

with all of the elements $s_{ij}, t_{ij}, u_{ik}$ and $v_{ik}$ in $\mathbb{T}_n^m$. Substituting this information and again using the cyclic property of traces we obtain

$$tr(zX_{m+1}) = tr((\sum_{i,j,k} s_{ij} t_{ij} t_i + tr(v_{ik} t_i)) X_{m+1})$$

and by the nondegeneracy of the trace map we again deduce from this the equality

$$z = \sum_{i,j,k} s_{ij} t_{ij} t_i + tr(v_{ik} t_i)$$

Because $t_i \in T_A$ and $T_A$ is stable under taking traces we deduce from this that $z \in T_A$ as required.

Because $A = M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n}$ we can apply functoriality of the Reynolds operator to the setting

$$\begin{tikzcd}
M_n(\mathbb{C}[M_n^m]/N_A) \arrow[r, two heads, shift left, "tr"] \arrow[d, "R"'] & \mathbb{C}[M_n]/N_A \arrow[l, hook', shift left] \arrow[d, "R"] \\
A \arrow[r, two heads, shift left, "tr_A"] & (\mathbb{C}[M_n]/N_A)^{GL_n} \arrow[l, hook', shift left]
\end{tikzcd}$$

Concluding we also have the equality

$$tr_A(A) = (\mathbb{C}[M_n^m]/J_A)^{GL_n}.$$

Summarizing, we have proved the following invariant theoretic reconstruction result for Cayley-Hamilton algebras.

**Theorem 1.17** *Let $A$ be a Cayley-Hamilton algebra of degree $n$, with trace map $tr_A$, which is trace generated by at most $m$ elements. Then , there is a canonical ideal $N_A \triangleleft \mathbb{C}[M_n^m]$ from which we can reconstruct the algebras $A$ and $tr_A(A)$ as invariant algebras*

$$A = M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n} \quad \textit{and} \quad tr_A(A) = (\mathbb{C}[M_n^m]/N_A)^{GL_n}$$

A direct consequence of the above proof is the following *universal property* of the embedding

$$A \overset{i_A}{\hookrightarrow} M_n(\mathbb{C}[M_n^m]/N_A).$$

Let $R$ be a commutative $\mathbb{C}$-algebra, then $M_n(R)$ with the usual trace is a Cayley-Hamilton algebra of degree $n$. If $f : A \longrightarrow M_n(R)$ is a trace preserving morphism, we claim that there exists a natural algebra morphism $\overline{f} : \mathbb{C}[M_n^m]/N_A \longrightarrow R$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i_A"'] & M_n(R) \\
M_n(\mathbb{C}[M_n^m]/N_A) \arrow[ur, dotted, "M_n(\overline{f})"'] &
\end{tikzcd}$$

where $M_n(\overline{f})$ is the algebra morphism defined entrywise. To see this, consider the composed trace preserving morphism $\phi : \mathbb{T}_n^m \twoheadrightarrow A \xrightarrow{f} M_n(R)$. Its image is fully determined by the images of the trace generators $X_k$ of $\mathbb{T}_n^m$ which are say $m_k = (m_{ij}(k))_{i,j}$. But then we have an algebra morphism $\mathbb{C}[M_n^m] \xrightarrow{g} R$ defined by sending the variable $x_{ij}(k)$ to $m_{ij}(k)$. Clearly, $T_A \subset Ker\ \phi$ and after inducing to $M_n(\mathbb{C}[M_n^m])$ it follows that $N_A \subset Ker\ g$ proving that $g$ factors through $\mathbb{C}[M_n^m]/J_A \longrightarrow R$. This morphism has the required universal property.

## References

The first fundamental theorem of matrix invariants, theorem 1.6, is due independently to G. B. Gurevich [31], C. Procesi [67] and K. S. Siberskii [78]. The second fundamental theorem of matrix invariants, theorem 1.10 is due independently to C. Procesi [67] and Y. P. Razmyslov [69]. Our treatment follows the paper [67] of C. Procesi, supplemented with material taken from the lecture notes of H-P. Kraft [52] and E. Formanek [26]. The invariant theoretic reconstruction result, theorem 1.17, is due to C. Procesi [68].

# 2 — Reconstructing Algebras

We will associate to an affine $\mathbb{C}$-algebra $A$ its affine scheme of $n$-dimensional representations $\mathtt{rep}_n\ A$. There is a basechange action by $GL_n$ on this scheme and its orbits are exactly the isomorphism classes of $n$-dimensional representations. We will prove the *Hilbert criterium* which describes the nullcone via one-parameter subgroups and apply it to prove Michael Artin's result that the closed orbits in $\mathtt{rep}_n\ A$ correspond to *semi-simple* representations.

We recall the basic results on algebraic quotient varieties in geometric invariant theory and apply them to prove Procesi's reconstruction result. If $A \in \mathtt{alg@n}$, then we can recover $A$ as

$$A \simeq \Uparrow^n [\mathtt{trep}_n\ A]$$

the ring of $GL_n$-equivariant polynomial maps from the trace preserving representation scheme $\mathtt{trep}_n\ A$ to $M_n(\mathbb{C})$. However, the functors

$$\begin{tikzcd} \mathtt{alg@n} \arrow[r, bend left, "\mathtt{trep}_n"] & \mathtt{GL(n)\text{-}affine} \arrow[l, bend left, "\Uparrow^n"] \end{tikzcd}$$

do not determine an anti-equivalence of categories (as they do in commutative algebraic geometry, which is the special case $n = 1$). We will illustrate this by calculating the rings of equivariant maps of orbit-closures of nilpotent matrices. These orbit-closures are described by the *Gerstenhaber-Hesselink* theorem. Later, we will be able to extend this result and study the nullcones of more general representation varieties.

## 2.1 Representation schemes

For a noncommutative affine algebra $A$ with generating set $\{a_1, \ldots, a_m\}$, there is an epimorphism

$$\mathbb{C}\langle x_1, \ldots, x_m\rangle \xrightarrow{\phi} A$$

defined by $\phi(x_i) = a_i$. That is, a presentation of $A$ as

$$A \simeq \mathbb{C}\langle x_1, \ldots, x_m\rangle / I_A$$

where $I_A$ is the *twosided ideal of relations* holding among the $a_i$. For example, if $A = \mathbb{C}[x_1, \ldots, x_m]$, then $I_A$ is the twosided ideal of $\mathbb{C}\langle x_1, \ldots, x_m\rangle$ generated by the elements $x_i x_j - x_j x_i$ for all $1 \leq i, j \leq m$.

An *$n$-dimensional representation* of $A$ is an algebra morphism

$$A \xrightarrow{\psi} M_n$$

from $A$ to the algebra of $n \times n$ matrices over $\mathbb{C}$. If $A$ is generated by $\{a_1, \ldots, a_m\}$, then $\psi$ is fully determined by the point

$$(\psi(a_1), \psi(a_2), \ldots, \psi(a_m)) \in M_n^m = \underbrace{M_n \oplus \ldots \oplus M_n}_{m}.$$

We claim that $rep_n(A)$, the set of all $n$-dimensional representations of $A$, forms a Zariski closed subset of $M_n^m$. To begin, observe that

$$rep_n(\mathbb{C}\langle x_1, \ldots, x_m\rangle) = M_n^m$$

as any $m$-tuple of $n \times n$ matrices $(A_1, \ldots, A_m) \in M_n^m$ determines an algebra morphism $\mathbb{C}\langle x_1, \ldots, x_m\rangle \xrightarrow{\psi} M_n$ by taking $\psi(x_i) = A_i$.

Given a presentation $A = \mathbb{C}\langle x_1, \ldots, x_m\rangle / I_A$ an $m$-tuple $(A_1, \ldots, A_m) \in M_n^m$ determines an $n$-dimensional representation of $A$ if (and only if) for every noncommutative polynomial $r(x_1, \ldots, x_m) \in I_A \triangleleft \mathbb{C}\langle x_1, \ldots, x_m\rangle$ we have that

$$r(A_1, \ldots, A_m) = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in M_n.$$

Hence, consider the ideal $I_A(n)$ of $\mathbb{C}[M_n^m] = \mathbb{C}[x_{ij}(k) \mid 1 \leq i, j \leq n, 1 \leq k \leq m]$ generated by all the entries of the matrices in $M_n(\mathbb{C}[M_n^m])$ of the form

$$r(X_1, \ldots, X_m) \quad \text{for all} \quad r(x_1, \ldots, x_m) \in I_A.$$

We see that the *reduced representation variety* $rep_n\ A$ is the set of simultaneous zeroes of the ideal $I_A(n)$, that is,

$$rep_n\ A = \mathbb{V}(I_A(n)) \hookrightarrow M_n^m$$

proving the claim. Here, $\mathbb{V}$ denotes the *closed* set in the *Zariski topology* determined by an ideal. The complement of $\mathbb{V}(I)$ we will denote with $\mathbb{X}(I)$). Observe that, even when $A$ is not finitely presented, the ideal $I_A(n)$ is finitely generated as an ideal of the commutative (Noetherian) polynomial algebra $\mathbb{C}[M_n^m]$.

**Example 2.1** It may happen that $rep_n\ A = \emptyset$. For example, consider the *Weyl algebra*

$$A_1(\mathbb{C}) = \mathbb{C}\langle x, y\rangle / (xy - yx - 1)$$

If a couple of $n \times n$-matrices $(A, B) \in rep_n\ A_1(\mathbb{C})$ then we must have

$$A.B - B.A = \mathbb{1}_n \in M_n$$

However, taking traces on both sides gives a contradiction as $tr(AB) = tr(BA)$ and $tr(\mathbb{1}_n) = n \neq 0$.

Often, the ideal $I_A(n)$ contains more information than the closed subset $rep_n(A) = \mathbb{V}(I_A(n))$ which, using the *Hilbert Nullstellensatz*, only determines the radical ideal of $I_A(n)$. This fact forces us to consider the *representation variety* (or scheme) $\mathtt{rep}_n\ A$.

In the foregoing chapter we studied the action of $GL_n$ by simultaneous conjugation on $M_n^m$. We claim that $rep_n\ A \hookrightarrow M_n^m$ is *stable* under this action, that is, if $(A_1, \ldots, A_m) \in rep_n\ A$, then also $(gA_1g^{-1}, \ldots, gA_mg^{-1}) \in rep_n\ A$. This is clear by composing the $n$-dimensional representation $\psi$ of $A$ determined by $(A_1, \ldots, A_m)$ with the algebra automorphism of $M_n$ given by conjugation with $g \in GL_n$,

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[dr, dotted, "g.\psi"'] & M_n \arrow[d, "g.g^{-1}"] \\
& M_n
\end{tikzcd}$$

Therefore, $rep_n\ A$ is a *$GL_n$-variety* . We will give an interpretation of the orbits under this action.

Recall that a *left $A$-module* $M$ is a vectorspace on which elements of $A$ act on the left as linear operators satisfying the conditions

$$1.m = m \quad \text{and} \quad a.(b.m) = (ab).m$$

for all $a, b \in A$ and all $m \in M$. An $A$-module *morphism* $M \xrightarrow{f} N$ between two left $A$-modules is a linear map such that $f(a.m) = a.f(m)$ for all $a \in A$ and all $m \in M$. An $A$-module *automorphism* is an $A$-module morphism $M \xrightarrow{f} N$ such that there is an $A$-module morphism $N \xrightarrow{g} M$ such that $f \circ g = id_M$ and $g \circ f = id_N$.

Assume the $A$-module $M$ has dimension $n$, then after fixing a basis we can identify $M$ with $\mathbb{C}^n$ (column vectors). For any $a \in A$ we can represent the linear action of $a$ on $M$ by an $n \times n$ matrix $\psi(a) \in M_n$. The condition that $a.(b.m) = (ab).m$ for all $m \in M$ asserts that $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in A$, that is, $\psi$ is an algebra morphism $A \xrightarrow{\psi} M_n$ and hence $M$ determines an $n$-dimensional representation of $A$. Conversely, an $n$-dimensional representation $A \xrightarrow{\psi} M_n$ determines an $A$-module structure on $\mathbb{C}^n$ by the rule

$$a.v = \psi(a)v \quad \text{for all} \quad v \in \mathbb{C}^n.$$

Hence, there is a one-to-one correspondence between the $n$-dimensional representations of $A$ and the $A$-module structures on $\mathbb{C}^n$. If two $n$-dimensional $A$-module structures $M$ and $N$ on $\mathbb{C}^n$ are isomorphic (determined by a linear invertible map $g \in GL_n$) then for all $a \in A$ we have the commutative diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "a."'] & N \arrow[d, "a."] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

Hence, if the action of $a$ on $M$ is represented by the matrix $A$, then the action of $a$ on $M$ is represented by the matrix $g.A.g^{-1}$. Therefore, two $A$-module structures on $\mathbb{C}^n$ are isomorphic if and only if the points of $rep_n\ A$ corresponding to them lie in the same $GL_n$-orbit. Concluding, studying $n$-dimensional $A$-modules up to isomorphism is the same as studying the $GL_n$-orbits in the reduced representation variety $rep_n\ A$.

If the defining ideal $I_A(n)$ is a radical ideal, the above suffices. In general, the *scheme* structure of the representation variety $\mathtt{rep}_n\ A$ will be important. By definition, the scheme $\mathtt{rep}_n\ A$ is the functor assigning to any (affine) commutative $\mathbb{C}$-algebra $R$, the set

$$\mathtt{rep}_n\ A(R) = Alg_{\mathbb{C}}(\mathbb{C}[M_n^m]/I_A(n), R)$$

of $\mathbb{C}$-algebra morphisms $\frac{\mathbb{C}[M_n^m]}{I_A(n)} \xrightarrow{\psi} R$. Such a map $\psi$ is determined by the image $\psi(x_{ij}(k)) = r_{ij}(k) \in R$. That is, $\psi \in \mathtt{rep}_n\ A(R)$ determines an $m$-tuple of $n \times n$ matrices with coefficients in $R$

$$(r_1, \ldots, r_m) \in \underbrace{M_n(R) \oplus \ldots \oplus M_n(R)}_{m} \quad \text{where} \quad r_k = \begin{bmatrix} r_{11}(k) & \ldots & r_{1n}(k) \\ \vdots & & \vdots \\ r_{n1}(k) & \ldots & r_{nn}(k) \end{bmatrix}.$$

Clearly, for any $r(x_1, \ldots, x_m) \in I_A$ we must have that $r(r_1, \ldots, r_m)$ is the zero matrix in $M_n(R)$. That is, $\psi$ determines uniquely an $R$-algebra morphism

$$\psi : R \otimes_{\mathbb{C}} A \longrightarrow M_n(R) \quad \text{by mapping} \quad x_k \mapsto r_k.$$

Alternatively, we can identify the set $\mathtt{rep}_n(R)$ with the set of left $R \otimes A$-module structures on the free $R$-module $R^{\oplus n}$ of rank $n$.

## 2.2 Some algebraic geometry

Throughout this book we assume that the reader has some familiarity with algebraic geometry, as contained in the first two chapters of the textbook [33]. In this section we restrict to the dimension formulas and the relation between Zariski and analytic closures. We will illustrate these results by examples from representation varieties. We will consider only the reduced varieties in this section.

A morphism $X \xrightarrow{\phi} Y$ between two affine *irreducible varieties* (that is, the coordinate rings $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are domains) is said to be *dominant* if the image $\phi(X)$ is Zariski dense in $Y$. On the level of the coordinate algebras dominance is equivalent to $\phi^* : \mathbb{C}[Y] \longrightarrow \mathbb{C}[X]$ being injective and hence inducing a fieldextension $\phi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ between the functionfields. Indeed, for $f \in \mathbb{C}[Y]$ the function $\phi^*(f)$ is by definition the composition

$$X \xrightarrow{\phi} Y \xrightarrow{f} \mathbb{C}$$

and therefore $\phi^*(f) = 0$ iff $f(\phi(X)) = 0$ iff $f(\overline{\phi(X)}) = 0$.

A morphism $X \xrightarrow{\phi} Y$ between two affine varieties is said to be *finite* if under the algebra morphism $\phi^*$ the coordinate algebra $\mathbb{C}[X]$ is a finite $\mathbb{C}[Y]$-module. An important property of finite morphisms is that they are *closed* , that is the image of a closed subset is closed. Indeed, we can replace without loss of generality $Y$ by the closed subset $\overline{\phi(X)} = \mathbb{V}_Y(Ker\ \phi^*)$ and hence assume that $\phi^*$ is an inclusion $\mathbb{C}[Y] \hookrightarrow \mathbb{C}[X]$. The claim then follows from the fact that in a finite extension there exists for any maximal ideal $N \triangleleft \mathbb{C}[Y]$ a maximal ideal $M \triangleleft \mathbb{C}[X]$ such that $M \cap \mathbb{C}[Y] = \mathbb{C}[X]$.

**Example 2.2** Let $X$ be an irreducible affine variety of dimension $d$. By the *Noether normalization* result $\mathbb{C}[X]$ is a finite module over a polynomial subalgebra $\mathbb{C}[f_1, \ldots, f_d]$. But then, the finite inclusion $\mathbb{C}[f_1, \ldots, f_d] \hookrightarrow \mathbb{C}[X]$ determines a finite surjective morphism

$$X \overset{\phi}{\twoheadrightarrow} \mathbb{C}^d$$

An important source of finite morphisms is given by integral extensions. Recall that, if $R \hookrightarrow S$ is an inclusion of domains we call $S$ *integral* over $R$ if every $s \in S$ satisfies an equation

$$s^n = \sum_{i=0}^{n-1} r_i s^i \quad \text{with} \quad r_i \in R.$$

A *normal* domain $R$ has the property that any element of its field of fractions which is integral over $R$ belongs already to $R$. If $X \xrightarrow{\phi} Y$ is a dominant morphism between two irreducible affine varieties, then $\phi$ is finite if and only if $\mathbb{C}[X]$ in integral over $\mathbb{C}[Y]$ for the embedding coming from $\phi^*$.

**Proposition 2.1** *Let $X \xrightarrow{\phi} Y$ be a dominant morphism between irreducible affine varieties. Then, for any $x \in X$ and any irreducible component $C$ of the fiber $\phi^{-1}(\phi(z))$ we have*

$$dim\ C \geq dim\ X - dim\ Y.$$

*Moreover, there is a nonempty open subset $U$ of $Y$ contained in the image $\phi(X)$ such that for all $u \in U$ we have*

$$dim\ \phi^{-1}(u) = dim\ X - dim\ Y.$$

*Proof.* Let $d = dim\ X - dim\ Y$ and apply the Noether normalization result to the affine $\mathbb{C}(Y)$-algebra $\mathbb{C}(Y)\mathbb{C}[X]$. Then, we can find a function $g \in \mathbb{C}[Y]$ and algebraic independent functions $f_1, \ldots, f_d \in \mathbb{C}[X]_g$ ($g$ clears away any denominators that occur after applying the normalization result) such that $\mathbb{C}[X]_g$ is *integral* over $\mathbb{C}[Y]_g[f_1, \ldots, f_d]$. That is, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{X}_X(g) \arrow[rr, two heads, "\rho"] \arrow[d, hook] & & \mathbb{X}_Y(g) \times \mathbb{C}^d \arrow[d, "pr_1"'] \\
X \arrow[r, "\phi"] & Y & \mathbb{X}_Y(g) \arrow[l, hook']
\end{tikzcd}$$

where we know that $\rho$ is finite and surjective. But then we have that the open subset $\mathbb{X}_Y(g)$ lies in the image of $\phi$ and in $\mathbb{X}_Y(g)$ all fibers of $\phi$ have dimension $d$. For the first part of the statement we have to recall the statement of *Krull's Hauptideal result* : if $X$ is an irreducible affine variety and $g_1, \ldots, g_r \in \mathbb{C}[X]$ with $(g_1, \ldots, g_r) \neq \mathbb{C}[X]$, then any component $C$ of $\mathbb{V}_X(g_1, \ldots, g_r)$ satisfies the inequality

$$dim\ C \geq dim\ X - r.$$

If $dim\ Y = r$ apply this result to the $g_i$ determining the morphism

$$X \xrightarrow{\phi} Y \twoheadrightarrow \mathbb{C}^r$$

where the latter morphism is the one from example 2.2. □

In fact, a stronger result holds. *Chevalley's theorem* asserts the following.

**Theorem 2.1** *Let $X \xrightarrow{\phi} Y$ be a morphism between affine varieties, the function*

$$X \longrightarrow \mathbb{N} \quad \textit{defined by} \quad x \mapsto dim_x\ \phi^{-1}(\phi(x))$$

*is upper-semicontinuous. That is, for all $n \in \mathbb{N}$, the set*

$$\{x \in X \mid dim_x\ \phi^{-1}(\phi(x)) \leq n\}$$

*is Zariski open in $X$.*

*Proof.* Let $Z(\phi, n)$ be the set $\{x \in X \mid dim_x\ \phi^{-1}(\phi(x)) \geq n\}$. We will prove that $Z(\phi, n)$ is closed by induction on the dimension of $X$. We first make some reductions. We may assume that $X$ is irreducible. For, let $X = \cup_i X_i$ be the decomposition of $X$ into irreducible components, then $Z(\phi, n) = \cup Z(\phi \mid X_i, n)$. Next, we may assume that $Y = \overline{\phi(X)}$ whence $Y$ is also irreducible and $\phi$ is a dominant map. Now, we are in the setting of proposition 2.1. Therefore, if $n \leq dim\ X - dim\ Y$

we have $Z(\phi, n) = X$ by that proposition, so it is closed. If $n > dim\ X - dim\ Y$ consider the open set $U$ in $Y$ of proposition 2.1. Then, $Z(\phi, n) = Z(\phi \mid (X - \phi^{-1}(U)), n)$. the dimension of the closed subvariety $X - \phi^{-1}(U)$ is strictly smaller that $dim\ X$ hence by induction we may assume that $Z(\phi \mid (X - \phi^{-1}(U)), n)$ is closed in $X - \phi^{-1}(U)$ whence closed in $X$. □

An immediate consequence of the foregoing proposition is that for any morphism $X \xrightarrow{\phi} Y$ between affine varieties, the image $\phi(X)$ contains an open dense subset of $\overline{\phi(Z)}$ (reduce to irreducible components and apply the proposition).

**Example 2.3** Let $A$ be an affine $\mathbb{C}$-algebra and $M \in rep_n\ A$. We claim that the orbit

$$\mathcal{O}(M) = GL_n.M \qquad \text{is Zariski open in its closure} \qquad \overline{\mathcal{O}(M)}.$$

Consider the *'orbit-map'* $GL_n \xrightarrow{\phi} rep_n\ A$ defined by $g \mapsto g.M$. Then, by the above remark $\overline{\mathcal{O}(M)} = \overline{\phi(GL_n)}$ contains a Zariski open subset $U$ of $\overline{\mathcal{O}(M)}$ contained in the image of $\phi$ which is $\mathcal{O}(M)$. But then,

$$\mathcal{O}(M) = GL_n.M = \cup_{g \in GL_n} g.U$$

is also open in $\overline{\mathcal{O}(M)}$. Next, we claim that $\overline{\mathcal{O}(M)}$ contains a closed orbit. Indeed, assume $\mathcal{O}(M)$ is not closed, then the complement $C_M = \overline{\mathcal{O}(M)} - \mathcal{O}(M)$ is a proper Zariski closed subset whence $dim\ C < dim\ \overline{\mathcal{O}(M)}$. But, $C$ is the union of $GL_n$-orbits $\mathcal{O}(M_i)$ with $dim\ \overline{\mathcal{O}(M_i)} < dim\ \overline{\mathcal{O}(M)}$. Repeating the argument with the $M_i$ and induction on the dimension we will obtain a closed orbit in $\overline{\mathcal{O}(M)}$.

Next, we want to relate the Zariski closure with the $\mathbb{C}$-closure (that is, closure in the usual complex or analytic topology). Whereas they are usually not equal (for example, the unit circle in $\mathbb{C}^1$), we will show that they coincide for the important class of *constructible* subsets. A subset $Z$ of an affine variety $X$ is said to be *locally closed* if $Z$ is open in its Zariski closure $\overline{Z}$. A subset $Z$ is said to be *constructible* if $Z$ is the union of finitely many locally closed subsets. Clearly, finite unions, finite intersections and complements of constructible subsets are again constructible. The importance of constructible sets for algebraic geometry is clear from the following result.

**Proposition 2.2** *Let $X \xrightarrow{\phi} Y$ be a morphism between affine varieties. If $Z$ is a constructible subset of $X$, then $\phi(Z)$ is a constructible subset of $Y$.*

*Proof.* Because every open subset of $X$ is a finite union of special open sets which are themselves affine varieties, it suffices to show that $\phi(X)$ is constructible. We will use induction on $dim\ \overline{\phi(X)}$. There exists an open subset $U \subset \overline{\phi(X)}$ which is contained in $\phi(X)$. Consider the closed complement $W = \overline{\phi(X)} - U$ and its inverse image $X' = \phi^{-1}(W)$. Then, $X'$ is an affine variety and by induction we may assume that $\phi(X')$ is constructible. But then, $\phi(X) = U \cup \phi(X')$ is also constructible. □

**Example 2.4** Let $A$ be an affine $\mathbb{C}$-algebra. The subset $ind_n\ A \hookrightarrow rep_n\ A$ of the *indecomposable* $n$-dimensional $A$-modules is constructible. Indeed, define for any pair $k, l$ such that $k + l = n$ the morphism

$$GL_n \times rep_k\ A \times rep_l\ A \longrightarrow rep_n\ A$$

by sending a triple $(g, M, N)$ to $g.(M \oplus N)$. By the foregoing result the image of this map is constructible. The decomposable $n$-dimensional $A$-modules belong to one of these finitely many sets whence are constructible, but then so is its complement which in $ind_n\ A$.

Apart from being closed, finite morphisms often satisfy the *going-down property* . That is, consider a finite and surjective morphism

$$X \xrightarrow{\phi} Y$$

where $X$ is irreducible and $Y$ is normal (that is, $\mathbb{C}[Y]$ is a normal domain). Let $Y' \hookrightarrow Y$ an irreducible Zariski closed subvariety and $x \in X$ with image $\phi(x) = y' \in Y'$. Then, the going-down property asserts the existence of an irreducible Zariski closed subvariety $X' \hookrightarrow X$ such that $x \in X'$ and $\phi(X') = Y'$. In particular, the morphism $X' \xrightarrow{\phi} Y'$ is again finite and surjective and in particular $dim\ X' = dim\ Y'$.

**Lemma 2.1** *Let $x \in X$ an irreducible affine variety and $U$ a Zariski open subset. Then, there is an irreducible curve $C \hookrightarrow X$ through $x$ and intersecting $U$.*

*Proof.* If $d = dim\ X$ consider the finite surjective morphism $X \xrightarrow{\phi} \mathbb{C}^d$ of example 2.2. Let $y \in \mathbb{C}^d - \phi(X - U)$ and consider the line $L$ through $y$ and $\phi(x)$. By the going-down property there is an irreducible curve $C \hookrightarrow X$ containing $x$ such that $\phi(C) = L$ and by construction $C \cap U \neq \emptyset$. □

**Proposition 2.3** *Let $X \xrightarrow{\phi} Y$ be a dominant morphism between irreducible affine varieties any $y \in Y$. Then, there is an irreducible curve $C \hookrightarrow X$ such that $y \in \overline{\phi(C)}$.*

*Proof.* Consider an open dense subset $U \hookrightarrow Y$ contained in the image $\phi(X)$. By the lemma there is a curve $C' \hookrightarrow Y$ containing $y$ and such that $C' \cap U \neq \emptyset$. Then, again applying the lemma to an irreducible component of $\phi^{-1}(C')$ not contained in a fiber, we obtain an irreducible curve $C \hookrightarrow X$ with $\overline{\phi(C)} = \overline{C'}$. □

Any affine variety $X \hookrightarrow \mathbb{C}^k$ can also be equipped with the induced $\mathbb{C}$-topology (or *analytic topology*) from $\mathbb{C}^k$ which is much finer than the *Zariski topology* . Usually there is no relation between the closure $\overline{Z}^{\mathbb{C}}$ of a subset $Z \hookrightarrow X$ in the $\mathbb{C}$-topology and the Zariski closure $\overline{Z}$.

**Lemma 2.2** *Let $U \subset \mathbb{C}^k$ containing a subset $V$ which is Zariski open and dense in $\overline{U}$. Then,*

$$\overline{U}^{\mathbb{C}} = \overline{U}$$

*Proof.* By reducing to irreducible components, we may assume that $\overline{U}$ is irreducible. Assume first that $dim\ \overline{U} = 1$, that is, $\overline{U}$ is an irreducible curve in $\mathbb{C}^k$. Let $U_s$ be the subset of points where $\overline{U}$ is a complex manifold, then $\overline{U} - U_s$ is finite and by the *implicit function theorem* in analysis every $u \in U_s$ has a $\mathbb{C}$-open neighborhood which is $\mathbb{C}$-homeomorphic to the complex line $\mathbb{C}^1$, whence the result holds in this case.

If $\overline{U}$ is general and $x \in \overline{U}$ we can take by the lemma above an irreducible curve $C \hookrightarrow \overline{U}$ containing $z$ and such that $C \cap V \neq \emptyset$. Then, $C \cap V$ is Zariski open and dense in $C$ and by the curve argument above $x \in \overline{(C \cap V)}^{\mathbb{C}} \subset \overline{U}^{\mathbb{C}}$. We can do this for any $x \in \overline{U}$ finishing the proof. □

Consider the embedding of an affine variety $X \hookrightarrow \mathbb{C}^k$, proposition 2.2 and the fact that any constructible set $Z$ contains a subset $U$ which is open and dense in $\overline{Z}$ we deduce from the lemma at once the next result.

**Proposition 2.4** *If $Z$ is a constructible subset of an affine variety $X$, then*

$$\overline{Z}^{\mathbb{C}} = \overline{Z}$$

**Example 2.5** Let $A$ be an affine $\mathbb{C}$-algebra and $M \in rep_n\ A$. We have proved in example 2.3 that the orbit $\mathcal{O}(M) = GL_n.M$ is Zariski open in its closure $\overline{\mathcal{O}(M)}$. Therefore, the orbit $\mathcal{O}(M)$ is a constructible subset of $rep_n\ A$. By the proposition above, the Zariski closure $\overline{\mathcal{O}(M)}$ of the orbit coincides with the closure of $\mathcal{O}(M)$ in the $\mathbb{C}$-topology.

## 2.3 The Hilbert criterium

A *one parameter subgroup* of a linear algebraic group $G$ is a morphism

$$\lambda : \mathbb{C}^* \longrightarrow G$$

of affine algebraic groups. That is, $\lambda$ is both a groupmorphism and a morphism of affine varieties. The set of all one parameter subgroup of $G$ will be denoted by $Y(G)$.

If $G$ is commutative algebraic group, then $Y(G)$ is an Abelian group with additive notation

$$\lambda_1 + \lambda_2 : \mathbb{C}^* \longrightarrow G \quad \text{with } (\lambda_1 + \lambda_2)(t) = \lambda_1(t).\lambda_2(t)$$

Recall that an $n$-dimensional torus is an affine algebraic group isomorphic to

$$\underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{n} = T_n$$

the closed subgroup of invertible diagonal matrices in $GL_n$.

**Lemma 2.3** *$Y(T_n) \simeq \mathbb{Z}^n$. The correspondence is given by assigning to $(r_1,\ldots,r_n) \in \mathbb{Z}^n$ the one-parameter subgroup*

$$\lambda : \mathbb{C}^* \longrightarrow T_n \quad \textit{given by} \quad t \mapsto (t^{r_1},\ldots,t^{r_n})$$

*Proof.* For any two affine algebraic groups $G$ and $H$ there is a canonical bijection $Y(G \times H) = Y(G) \times Y(H)$ so it suffices to verify that $Y(\mathbb{C}^*) \simeq \mathbb{Z}$ with any $\lambda : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ given by $t \mapsto t^r$ for some $r \in \mathbb{Z}$. This is obvious as $\lambda$ induces the algebra morphism

$$\mathbb{C}[\mathbb{C}^*] = \mathbb{C}[x,x^{-1}] \xrightarrow{\lambda^*} \mathbb{C}[x,x^{-1}] = \mathbb{C}[\mathbb{C}^*]$$

which is fully determined by the image of $x$ which must be an invertible element. Now, any invertible element in $\mathbb{C}[x,x^{-1}]$ is homogeneous of the form $cx^r$ for some $r \in \mathbb{Z}$ and $c \in \mathbb{C}^*$. The corresponding morphism maps $t$ to $ct^r$ which is only a groupmorphism if it maps the identity element 1 to 1 so $c = 1$, finishing the proof. $\square$

**Proposition 2.5** *Any one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ is of the form*

$$t \mapsto g^{-1}.\begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}.g$$

*for some $g \in GL_n$ and some $n$-tuple $(r_1,\ldots,r_n) \in \mathbb{Z}^n$.*

*Proof.* Let $H$ be the image under $\lambda$ of the subgroup $\mu_\infty$ of roots of unity in $\mathbb{C}^*$. We claim that there is a basechange matrix $g \in GL_n$ such that

$$g.H.g^{-1} \hookrightarrow \begin{bmatrix} \mathbb{C}^* & & 0 \\ & \ddots & \\ 0 & & \mathbb{C}^* \end{bmatrix}$$

Assume $h \in H$ not a scalar matrix, then $h$ has a proper eigenspace decomposition $V \oplus W = \mathbb{C}^n$. We use that $h^l = \mathbb{1}_n$ and hence all its Jordan blocks must have size one as for any $\lambda \neq 0$ we have

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}^l = \begin{bmatrix} \lambda^l & l\lambda^{l-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & l\lambda^{l-1} \\ & & & \lambda^l \end{bmatrix} \neq \mathbb{1}_n$$

Because $H$ is commutative, both $V$ and $W$ are stable under $H$. By induction on $n$ we may assume that the images of $H$ in $GL(V)$ and $GL(W)$ are diagonalizable, but then the same holds in $GL_n$.

As $\mu_\infty$ is infinite, it is Zariski dense in $\mathbb{C}^*$ and because the diagonal matrices are Zariski closed in $GL_n$ we have

$$g.\lambda(\mathbb{C}^*).g^{-1} = g.\overline{H}.g^{-1} \hookrightarrow T_n$$

and the result follows from the lemma above □

Let $V$ be a general $GL_n$-representation considered as an affine space with $GL_n$-action, let $X$ be a $GL_n$-stable closed subvariety and consider a point $x \in X$. A one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ determines a morphism

$$\mathbb{C}^* \xrightarrow{\lambda_x} X \quad \text{defined by} \quad t \mapsto \lambda(t).x$$

Observe that the image $\lambda_x(\mathbb{C}^*)$ lies in the orbit $GL_n.x$ of $x$. Assume there is a continuous extension of this map to the whole of $\mathbb{C}$. We claim that this extension must then be a morphism. If not, the induced algebra morphism

$$\mathbb{C}[X] \xrightarrow{\lambda_x^*} \mathbb{C}[t, t^{-1}]$$

does not have its image in $\mathbb{C}[t]$, so for some $f \in \mathbb{C}[Z]$ we have that

$$\lambda_x^*(f) = \frac{a_0 + a_1 t + \ldots + a_z t^z}{t^s} \quad \text{with} \quad a_0 \neq 0 \text{ and } \; s > 0$$

But then $\lambda_x^*(f)(t) \longrightarrow \pm\infty$ when $t$ goes to zero, that is, $\lambda_x^*$ cannot have a continuous extension, a contradiction.

So, *if* a continuous extension exists there is morphism $\lambda_x : \mathbb{C} \longrightarrow X$. Then, $\lambda_x(0) = y$ and we denote this by

$$\lim_{t \to 0} \lambda(t).x = y$$

Clearly, the point $y \in X$ must belong to the orbitclosure $\overline{GL_n.x}$ in the Zariski topology (or in the $\mathbb{C}$-topology as orbits are constructible). Conversely, one might ask whether if $y \in \overline{GL_n.x}$ we can always approach $y$ via a one-parameter subgroup. The *Hilbert criterium* gives situations when this is indeed possible.

The only ideals of the formal power series $\mathbb{C}[[t]]$ are principal and generated by $t^r$ for some $r \in \mathbb{N}_+$. With $\mathbb{C}((t))$ we will denote the field of fractions of the domain $\mathbb{C}((t))$.

**Lemma 2.4** *Let $V$ be a $GL_n$-representation, $v \in V$ and a point $w \in V$ lying in the orbitclosure $\overline{GL_n.v}$. Then, there exists a matrix $g$ with coefficients in the field $\mathbb{C}((t))$ and $det(g) \neq 0$ such that*

$$(g.v)_{t=0} \quad \textit{is well defined and is equal to} \quad w$$

*Proof.* Note that $g.v$ is a vector with coordinates in the field $\mathbb{C}((t))$. If all coordinates belong to $\mathbb{C}[[t]]$ we can set $t = 0$ in this vector and obtain a vector in $V$. It is this vector that we denote with $(g.v)_{t=0}$.

Consider the orbit map $\mu : GL_n \longrightarrow V$ defined by $g' \mapsto g'.v$. As $w \in \overline{GL_n.v}$ we have seen that there is an irreducible curve $C \hookrightarrow GL_n$ such that $w \in \overline{\mu(C)}$. We obtain a diagram of $\mathbb{C}$-algebras

$$\begin{tikzcd}
\mathbb{C}[GL_n] \arrow[r] & \mathbb{C}[C] \arrow[r, hook] & \mathbb{C}(C) \\
\mathbb{C}[V] \arrow[r] \arrow[u, "\mu^*"'] & \mathbb{C}[\overline{\mu(C)}] \arrow[r, hook] \arrow[u, hook, "\mu^*"] & \mathbb{C}[C'] \arrow[u, hook]
\end{tikzcd}$$

Here, $\mathbb{C}[C]$ is defined to be the integral closure of $\mathbb{C}[\overline{\mu(C)}]$ in the functionfield $\mathbb{C}(C)$ of $C$. Two things are important to note here : $C' \longrightarrow \overline{\mu(C)}$ is finite, so surjective and take $c \in C'$ be a point lying over $w \in \overline{\mu(C)}$. Further, $C'$ having an integrally closed coordinate ring is a complex manifold. Hence, by the implicit function theorem polynomial functions on $C$ can be expressed in a neighborhood of $c$ as power series in one variable, giving an embedding $\mathbb{C}[C'] \hookrightarrow \mathbb{C}[[t]]$ with $(t) \cap \mathbb{C}[C'] = M_c$. This inclusion extends to one on the level of their fields of fractions. That is, we have a diagram of $\mathbb{C}$-algebra morphisms

$$\begin{tikzcd}
\mathbb{C}[GL_n] \arrow[r] & \mathbb{C}(C) \arrow[r, phantom, "="] & \mathbb{C}(C') \arrow[r, hook] & \mathbb{C}((t)) \\
\mathbb{C}[V] \arrow[r] \arrow[u, "\mu^*"'] & \mathbb{C}[\overline{\mu(C)}] \arrow[r, hook] \arrow[u, hook] & \mathbb{C}[C'] \arrow[r, hook] \arrow[u, hook] & \mathbb{C}[[t]] \arrow[u, hook]
\end{tikzcd}$$

The upper composition defines an invertible matrix $g(t)$ with coefficients in $\mathbb{C}((t))$, its $(i,j)$-entry being the image of the coordinate function $x_{ij} \in \mathbb{C}[GL_n]$. Moreover, the inverse image of the maximal ideal $(t) \triangleleft \mathbb{C}[[t]]$ under the lower composition gives the maximal ideal $M_w \triangleleft \mathbb{C}[V]$. This proves the claim. $\square$

**Lemma 2.5** *Let $g$ be an $n \times n$ matrix with coefficients in $\mathbb{C}((t))$ and $\det g \neq 0$. Then there exist $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ such that*

$$g = u_1 . \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} .u_2$$

*with $r_i \in \mathbb{Z}$ and $r_1 \leq r_2 \leq \ldots \leq r_n$.*

*Proof.* By multiplying $g$ with a suitable power of $t$ we may assume that $g = (g_{ij}(t))_{i,j} \in M_n(\mathbb{C}[[t]])$. If $f(t) = \sum_{i=0}^{\infty} f_i t^i \in \mathbb{C}[[t]]$ define $v(f(t))$ to be the minimal $i$ such that $a_i \neq 0$. Let $(i_0, j_0)$ be an entry where $v(g_{ij}(t))$ attains a minimum, say $r_1$. That is, for all $(i,j)$ we have $g_{ij}(t) = t^{r_1} t^r f(t)$ with $r \geq 0$ and $f(t)$ an invertible element of $\mathbb{C}[[t]]$.

By suitable row and column interchanges we can take the entry $(i_0, j_0)$ to the $(1,1)$-position. Then, multiplying with a unit we can replace it by $t^{r_1}$ and by elementary row and column operations all the remaining entries in the first row and column can be made zero. That is, we have invertible matrices $a_1, a_2 \in GL_n(\mathbb{C}[[t]])$ such that

$$g = a_1 . \left[ \begin{array}{c|ccc} t^{r_1} & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & g_1 & \\ 0 & & & \end{array} \right] . a_2$$

Repeating the same idea on the submatrix $g_1$ and continuing gives the result. □

We can now state and prove the *Hilbert criterium* which allows us to study orbit-closures by one parameter subgroups.

**Theorem 2.2** *Let $V$ be a $GL_n$-representation and $X \hookrightarrow V$ a closed $GL_n$-stable subvariety. Let $\mathcal{O}(x) = GL_n.x$ be the orbit of a point $x \in X$. Let $Y \hookrightarrow \overline{\mathcal{O}(x)}$ be a closed $GL_n$-stable subset. Then, there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ such that*

$$\lim_{t \rightarrow 0} \lambda(t).x \in Y$$

*Proof.* It suffices to prove the result for $X = V$. By lemma 2.4 there is an invertible matrix $g \in M_n(\mathbb{C}((t)))$ such that

$$(g.x)_{t=0} = y \in Y$$

By lemma 2.5 we can find $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ such that

$$g = u_1.\lambda'(t).u_2 \quad \text{with} \quad \lambda'(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}$$

a one-parameter subgroup. There exist $x_i \in V$ such that $u_2.x = \sum_{i=0}^{\infty} z_i t^i$ in particular $u_2(0).x = x_0$. But then,

$$\begin{aligned} (\lambda'(t).u_2.x)_{t=0} &= \sum_{i=0}^{\infty} (\lambda'(t).x_i t^i)_{t=0} \\ &= (\lambda'(t).x_0)_{t=0} + (\lambda'(t).x_1 t)_{t=0} + \dots \end{aligned}$$

But one easily verifies (using a basis of eigenvectors of $\lambda'(t)$) that

$$\underset{s\rightarrow 0}{lim}\lambda'^{-1}(s).(\lambda'(t)x_it^i)_{t=0} = \begin{cases} (\lambda'(t).x_0)_{t=0} & \text{if } i=0, \\ 0 & \text{if } i \neq 0 \end{cases}$$

As $(\lambda'(t).u_2.x)_{t=0} \in Y$ and $Y$ is a closed $GL_n$-stable subset, we also have that

$$\underset{s\rightarrow 0}{lim}\lambda'^{-1}(s).(\lambda'(t).u_2.x)_{t=0} \in Y \quad \text{that is,} \quad (\lambda'(t).x_0)_{t=0} \in Y$$

But then, we have for the one-parameter subgroup $\lambda(t) = u_2(0)^{-1}.\lambda'(t).u_2(0)$ that

$$\underset{t\rightarrow 0}{lim}\lambda(t).x \in Y$$

finishing the proof. □

An important special case occurs when $x \in V$ belongs to the *nullcone* , that is, when the orbit closure $\overline{\mathcal{O}(x)}$ contains the fixed point $0 \in V$. The original Hilbert criterium asserts the following.

**Proposition 2.6** *Let $V$ be a $GL_n$-representation and $x \in V$ in the nullcone. Then, there is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that*

$$\underset{t\rightarrow 0}{lim}\ \lambda(t).x = 0$$

In the statement of theorem 2.2 it is important that $Y$ is closed. In particular, it does *not* follow that any orbit $\mathcal{O}(y) \hookrightarrow \overline{\mathcal{O}(x)}$ can be reached via one-parameter subgroups, see example 2.7 below.

## 2.4 Semisimple modules

In this section we will characterize the closed $GL_n$-orbits in the representation variety $rep_n\ A$ for an affine $\mathbb{C}$-algebra $A$. We have seen that any point $\psi \in rep_n\ A$ (that is any $n$-dimensional representation $A \xrightarrow{\psi} M_n$) determines an $n$-dimensional $A$-module which we will denote with $M_\psi$.

A *finite filtration* $F$ on an $n$-dimensional module $M$ is a sequence of $A$-submodules

$$F \quad : \quad 0 = M_{t+1} \subset M_t \subset \ldots \subset M_1 \subset M_0 = M.$$

The *associated graded* $A$-module is the $n$-dimensional module

$$gr_F\ M = \oplus_{i=0}^t M_i/M_{i+1}.$$

We have the following ringtheoretical interpretation of the action of one-parameter subgroups of $GL_n$ on the representation variety $rep_n\ A$.

**Lemma 2.6** *Let $\psi, \rho \in rep_n\ A$. Equivalent are,*

1. *There is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that*

$$\lim_{t \to 0} \lambda(t).\psi = \rho$$

2. *There is a finite filtration $F$ on the $A$-module $M_\psi$ such that*

$$gr_F\ M_\psi \simeq M_\rho$$

*as $A$-modules.*

*Proof.* $(1) \Rightarrow (2)$ : If $V$ is any $GL_n$-representation and $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ a one-parameter subgroup, we have an induced *weight space decomposition* of $V$

$$V = \oplus_i V_{\lambda,i} \quad \text{where} \quad V_{\lambda,i} = \{v \in V \mid \lambda(t).v = t^i v, \forall t \in \mathbb{C}^*\}.$$

In particular, we apply this to the underlying vectorspace of $M_\psi$ which is $V = \mathbb{C}^n$ (column vectors) on which $GL_n$ acts by left multiplication. We define

$$M_j = \oplus_{i > j} V_{\lambda,i}$$

and claim that this defines a finite filtration on $M_\psi$ with associated graded $A$-module $M_\rho$. For any $a \in A$ (it suffices to vary $a$ over the generators of $A$) we can consider the linear maps

$$\phi_{ij}(a) : V_{\lambda,i} \hookrightarrow V = M_\psi \xrightarrow{a.} M_\psi = V \twoheadrightarrow V_{\lambda,j}$$

(that is, we express the action of $a$ in a blockmatrix $\Phi_a$ with respect to the decomposition of $V$). Then, the action of $a$ on the module corresponding to $\lambda(t).\psi$ is given by the matrix $\Phi'_a = \lambda(t).\Phi_a.\lambda(t)^{-1}$ with corresponding blocks

$$\begin{tikzcd}
V_{\lambda,i} \arrow[r, "\phi_{ij}(a)"] & V_{\lambda,j} \arrow[d, "\lambda(t)"] \\
V_{\lambda,i} \arrow[u, "\lambda(t)^{-1}"] \arrow[r, "\phi'_{ij}(a)"'] & V_{\lambda,j}
\end{tikzcd}$$

that is $\phi'_{ij}(a) = t^{j-i}\phi_{ij}(a)$. Therefore, if $lim_{t \to 0}\lambda(t).\psi$ exists we must have that

$$\phi_{ij}(a) = 0 \quad \text{for all} \quad j < i.$$

But then, the action by $a$ sends any $M_k = \oplus_{i>k} V_{\lambda,i}$ to itself, that is, the $M_k$ are $A$-submodules of $M_\psi$. Moreover, for $j > i$ we have

$$\lim_{t\longrightarrow 0} \phi'_{ij}(a) = \lim_{t\longrightarrow 0} t^{j-i}\phi_{ij}(a) = 0$$

Consequently, the action of $a$ on $\rho$ is given by the diagonal blockmatrix with blocks $\phi_{ii}(a)$, but this is precisely the action of $a$ on $V_i = M_{i-1}/M_i$, that is, $\rho$ corresponds to the associated graded module.

(2) $\Rightarrow$ (1) : Given a finite filtration on $M_\psi$

$$F \quad : \quad 0 = M_{t+1} \subset \ldots \subset M_1 \subset M_0 = M_\psi$$

we have to find a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ which induces the filtration $F$ as in the first part of the proof. Clearly, there exist subspaces $V_i$ for $0 \leq i \leq t$ such that

$$V = \oplus_{i=0}^t V_i \quad \text{and} \quad M_j = \oplus_{j=i}^t V_i.$$

Then we take $\lambda$ to be defined by $\lambda(t) = t^i Id_{V_i}$ for all $i$ and it verifies the claims. $\square$

**Example 2.6** Let $M_\psi$ we the 2-dimensional $\mathbb{C}[x]$-module determined by the Jordan block and consider the canonical basevectors

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then, $\mathbb{C}e_1$ is a $\mathbb{C}[x]$-submodule of $M_\psi$ and we have a filtration

$$0 = M_2 \subset \mathbb{C}e_1 = M_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 = M_0 = M_\psi$$

Using the conventions of the second part of the above proof we then have

$$V_1 = \mathbb{C}e_1 \qquad V_2 = \mathbb{C}e_2 \quad \text{hence} \quad \lambda(t) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

Indeed, we then obtain that

$$\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix}$$

and the limit $t \longrightarrow 0$ exists and is the associated graded module $gr_F\ M_\psi = M_\rho$ determined by the diagonal matrix.

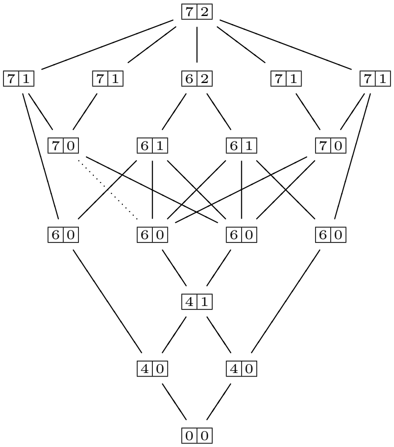



Figure 2.1: Kraft's diamond describing the nullcone of $M_3^2$.

Consider two modules $M_\psi, M_\psi \in rep_n\ A$. Assume that $\mathcal{O}(M_\rho) \hookrightarrow \overline{\mathcal{O}(M_\psi)}$ and that we can reach the orbit of $M_\rho$ via a one-parameter subgroup. Then, lemma 2.6 asserts that $M_\rho$ must be *decomposable* as it is the associated graded of a nontrivial filtration on $M_\psi$. This gives us a criterium to construct examples showing that the closedness assumption in the formulation of Hilbert's criterium is essential.

**Example 2.7 (Nullcone of $M_3^2 = M_3 \oplus M_3$)** In chapter 6 we will describe a method to determine the nullcones of $m$-tuples of $n \times n$ matrices. The special case of two $3 \times 3$ matrices has been worked out by H.P. Kraft in [50, p.202]. The orbits are depicted in figure 2.1 In this picture, each node corresponds to a torus. The right hand number is the dimension of the torus and the left hand number is the dimension of the orbit represented by a point in the torus. The solid or dashed lines indicate orbitclosures. For example, the dashed line corresponds to the following two points in $M_3^2 = M_3 \oplus M_3$

$$\psi = (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) \qquad \rho = (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

We claim that $M_\rho$ is an indecomposable 3-dimensional module of $\mathbb{C}\langle x, y\rangle$. Indeed, the only subspace of the column vectors $\mathbb{C}^3$ left invariant under both $x$ and $y$ is equal to

$$\begin{bmatrix} \mathbb{C} \\ 0 \\ 0 \end{bmatrix}$$

hence $M_\rho$ cannot have a direct sum decomposition of two or more modules. Next, we claim that $\mathcal{O}(M_\rho) \hookrightarrow \overline{\mathcal{O}(M_\psi)}$. Indeed, simultaneous conjugating $\psi$ with the invertible matrix

$$\begin{bmatrix} 1 & \epsilon^{-1}-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon^{-1} \end{bmatrix} \quad \text{we obtain the couple} \quad ( \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \epsilon \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} )$$

and letting $\epsilon \longrightarrow 0$ we see that the limiting point is $\rho$.

The *Jordan-Hölder theorem* , see for example [66, 2.6] asserts that any finite dimensional $A$-module $M$ has a *composition series* , that is, $M$ has a finite filtration

$$F \quad : \quad 0 = M_{t+1} \subset M_t \subset \ldots \subset M_1 \subset M_0 = M$$

such that the successive quotients $S_i = M_i / M_{i+1}$ are all *simple* $A$-modules for $0 \leq i \leq t$. Moreover, these *composition factors* $S$ and their *multiplicities* are independent of the chosen composition series, that is, the set $\{S_0, \ldots, S_t\}$ is the same for every composition series. In particular, the associated graded module for a composition series is determined only up to isomorphism and is the semisimple $n$-dimensional module

$$gr \ M = \oplus_{i=0}^t S_i$$

**Theorem 2.3** *Let $A$ be an affine $\mathbb{C}$-algebra and $M \in rep_n \ A$.*

1. *The orbit $\mathcal{O}(M)$ is closed in $rep_n \ A$ if and only if $M$ is an $n$-dimensional semisimple $A$-module.*
2. *The orbitclosure $\overline{\mathcal{O}(M)}$ contains exactly one closed orbit, corresponding to the direct sum of the composition factors of $M$.*
3. *The points of the quotient variety of $rep_n \ A$ under $GL_n$ classify the isomorphism classes of $n$-dimensional semisimple $A$-modules. We will denote the quotient variety by $iss_n \ A$.*

*Proof.* (1) : Assume that the orbit $\mathcal{O}(M)$ is Zariski closed. Let $gr \ M$ be the associated graded module for a composition series of $M$. From lemma 2.6 we know that $\mathcal{O}(gr \ M)$ is contained in $\overline{\mathcal{O}(M)} = \mathcal{O}(M)$. But then $gr \ M \simeq M$ whence $M$ is semisimple.

Conversely, assume $M$ is semisimple. We know that the orbitclosure $\overline{\mathcal{O}(M)}$ contains a closed orbit, say $\mathcal{O}(N)$. By the Hilbert criterium we have a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that

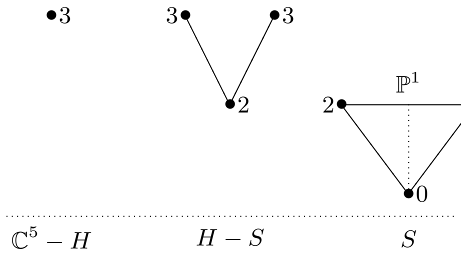
$$\lim_{t \rightarrow 0} \lambda(t).M = N' \simeq N.$$

By lemma 2.6 this means that there is a finite filtration $F$ on $M$ with associated graded module $gr_F\ M \simeq N$. For the semisimple module $M$ the only possible finite filtrations are such that each of the submodules is a direct sum of simple components, so $gr_F\ M \simeq M$, whence $M \simeq N$ and hence the orbit $\mathcal{O}(M)$ is closed.

(2) : Remains only to prove uniqueness of the closed orbit in $\overline{\mathcal{O}(M)}$. This either follows from the Jordan-Hölder theorem or, alternatively, from the separation property of the quotient map to be proved in the next section.

(3) : We will prove in the next section that the points of a quotient variety parameterize the closed orbits. □

**Example 2.8** Recall the description of the orbits in $M_2^2 = M_2 \oplus M_2$ given in the previous chapter.




and each fiber contains a unique closed orbit. The one over a point in $H - S$ corresponding to the matrix couple

$$(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix})$$

which is indeed a semi-simple module of $\mathbb{C}\langle x, y\rangle$ (the direct sum of the two 1-dimensional simple representations determined by $x \mapsto a_i$ and $y \mapsto b_i$). In case $a_1 = a_2$ and $b_1 = b_2$ these two simples coincide and the semi-simple module having this factor with multiplicity two is the unique closed orbit in the fiber of a point in $S$.

**Example 2.9** Assume $A$ is a finite dimensional $\mathbb{C}$-algebra. Then, there are only a finite number, say $k$, of nonisomorphic $n$-dimensional semisimple $A$-modules. Hence $iss_n\ A$ is a finite number of $k$ points, whence $rep_n\ A$ is the disjoint union of $k$ connected components, each consisting of all $n$-dimensional $A$-modules with the same composition factors. Connectivity follows from the fact that the orbit of the sum of the composition factors lies in the closure of each orbit.

**Example 2.10** Let $A$ be an affine commutative algebra with presentation $A = \mathbb{C}[x_1, \ldots, x_m]/I_A$ and let $X$ be the affine variety $\mathbb{V}(I_A)$. Observe that any simple $A$-module is one-dimensional hence corresponds to a point in $X$. (Indeed, for any algebra $A$ a simple $k$-dimensional module determines an epimorphism $A \longrightarrow\!\!\!\!\!\rightarrow M_k$ and $M_k$ is only commutative if $k = 1$). Applying the Jordan-Hölder theorem we see that

$$iss_n\ A \simeq X^{(n)} = \underbrace{X \times \ldots \times X}_{n}/S_n$$

the $n$-th symmetric product of $X$.

## 2.5 Some invariant theory

The results in this section hold for arbitrary reductive algebraic groups. Because we will only work with $GL_n$ (or later with products $GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k}$) we include a proof in this case. Our first aim is to prove that $GL_n$ is a *reductive group* , that is, all $GL_n$-representations are completely reducible. Consider the *unitary group*

$$U_n = \{A \in GL_n \mid A.A^* = \mathbb{1}_n\}$$

where $A^*$ is the *Hermitian transpose* of $A$. Clearly, $U_n$ is a compact Lie group. Any compact Lie group has a so called *Haar measure* which allows one to integrate continuous complex valued functions over the group in an invariant way. That is, there is a linear function assigning to every continuous function $f : U_n \longrightarrow \mathbb{C}$ its integral

$$f \mapsto \int_{U_n} f(g)dg \in \mathbb{C}$$

which is normalized such that $\int_{U_n} dg = 1$ and is left and right invariant, which means that for all $u \in U_n$ we have the equalities

$$\int_{U_n} f(gu)dg = \int_{U_n} f(g)dg = \int_{U_n} f(ug)dg.$$

This integral replaces the classical idea in representation theory of averaging functions over a finite group.

**Proposition 2.7** *Every $U_n$-representation is completely reducible.*

*Proof.* Take a finite dimensional complex vectorspace $V$ with a $U_n$-action and assume that $W$ is a subspace of $V$ left invariant under this action. Extending a basis of $W$ to $V$ we get a linear map $V \xrightarrow{\phi} W$ which is the identity on $W$. For any $v \in V$ we have a continuous map

$$U_n \longrightarrow W \qquad g \mapsto g.\phi(g^{-1}.v)$$

(use that $W$ is left invariant) and hence we can integrate it over $U_n$ (integrate the coordinate functions). Hence we can define a map $\phi_0 : V \longrightarrow W$ by

$$\phi_0(v) = \int_{U_n} g.\phi(g^{-1}.v)dg.$$

Clearly, $\phi_0$ is linear and is the identity on $W$. Moreover,

$$\begin{aligned}\phi_0(u.v) = \int_{U_n} g.\phi(g^{-1}u.v)dg = u. \int_{U_n} u^{-1}g.\phi(g^{-1}u.v)dg \\ \stackrel{*}{=} u. \int_{U_n} g\phi(g^{-1}.v)dg = u.\phi_0(v)\end{aligned}$$

where the starred equality uses the invariance of the Haar measure. Hence, $V = W \oplus Ker\ \phi_0$ is a decomposition as $U_n$-representations. Continuing whenever one of the components has a nontrivial subrepresentation we arrive at a decomposition of $V$ into simple $U_n$-representations. $\square$

We claim that for any $n$, $U_n$ is Zariski dense in $GL_n$. Let $D_n$ be the group of all diagonal matrices in $GL_n$. The *Cartan decomposition* for $GL_n$ asserts that

$$GL_n = U_n.D_n.U_n$$

For, take $g \in GL_n$ then $g.g^*$ is an Hermitian matrix and hence diagonalizable by unitary matrices. So, there is a $u \in U_n$ such that

$$u^{-1}.g.g^*.u = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix} = \underbrace{s^{-1}.g.s}_{p} . \underbrace{s^{-1}.g^*.s}_{p^*}$$

Then, each $\alpha_i > 0 \in \mathbb{R}$ as $\alpha_i = \sum_{j=1}^n \parallel p_{ij} \parallel^2$. Let $\beta_i = \sqrt{\alpha_i}$ and let $d$ the diagonal matrix $diag(\beta_1, \ldots, \beta_n)$. Clearly,

$$g = u.d.(d^{-1}.u^{-1}.g) \quad \text{and we claim} \quad v = d^{-1}.u^{-1}.g \in U_n.$$

Indeed, we have

$$\begin{aligned} v.v^* &= (d^{-1}.u^{-1}.g).(g^*.u.d^{-1}) = d^{-1}.(u^{-1}.g.g^*.u).d^{-1} \\ &= d^{-1}.d^2.d^{-1} = \mathbb{1}_n \end{aligned}$$

proving the Cartan decomposition. Now, $D_n = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ and $D_n \cap U_n = U_1 \times \ldots \times U_1$ and because $U_1 = \mu$ is Zariski dense (being infinite) in $D_1 = \mathbb{C}^*$, we have that $D_n$ is contained in the Zariski closure of $U_n$. By the Cartan decomposition we then have that the Zariski closure of $U_n$ is $GL_n$.

**Theorem 2.4** *$GL_n$ is a reductive group. That is, all $GL_n$-representations are completely reducible.*

*Proof.* Let $V$ be a $GL_n$-representation having a subrepresentation $W$. In particular, $V$ and $W$ are $U_n$-representations, so by the foregoing proposition we have a decomposition $V = W \oplus W'$ as $U_n$-representations. Consider the subgroup

$$N = N_{GL_n}(W') = \{g \in GL_n \mid g.W' \subset W'\}$$

then $N$ is a Zariski closed subgroup of $GL_n$ containing $U_n$. As the Zariski closure of $U_n$ is $GL_n$ we have $N = GL_n$ and hence that $W'$ is a representation of $GL_n$. Continuing gives a decomposition of $V$ in simple $GL_n$-representations. $\square$

Let $S = S_{GL_n}$ be the set of isomorphism classes of simple $GL_n$-representations. If $W$ is a simple $GL_n$-representation belonging to the isomorphism class $s \in S$, we say that $W$ is *of type* $s$ and denote this by $W \in s$. Let $X$ be a complex vectorspace (not necessarily finite dimensional) with a linear action of $GL_n$. We say that the action is *locally finite* on $X$ if, for any finite dimensional subspace $Y$ of $X$, there exists a finite dimensional subspace $Y \subset Y' \subset X$ which is a $GL_n$-representation. The *isotypical component* of $X$ of type $s \in S$ is defined to be the subspace

$$X_{(s)} = \sum \{W \mid W \subset X, W \in s\}.$$

If $V$ is a $GL_n$-representation, we have seen that $V$ is completely reducible. Then, $V = \oplus V_{(s)}$ and every isotypical component $V_{(s)} \simeq W^{\oplus e_s}$ for $W \in s$ and some number $e_s$. Clearly, $e_s \neq 0$ for only finitely many classes $s \in S$. We call the decomposition $V = \oplus_{s \in S} V_{(s)}$ the *isotypical decomposition* of $V$ and we say that the simple representation $W \in s$ occurs *with multiplicity* $e_s$ in $V$.

If $V'$ is another $GL_n$-representation and if $V \xrightarrow{\phi} V'$ is a morphism of $GL_n$-representations (that is, a linear map commuting with the action), then for any $s \in S$ we have that $\phi(V_{(s)}) \subset V'_{(s)}$. If the action of $GL_n$ on $X$ is locally finite, we can reduce to finite dimensional $GL_n$-subrepresentation and obtain a decomposition

$$X = \oplus_{s \in S} X_{(s)},$$

which is again called the *isotypical decomposition* of $X$.

Let $V$ be a $GL_n$-representation of some dimension $m$. Then, we can view $V$ as an affine space $\mathbb{C}^m$ and we have an induced action of $GL_n$ on the polynomial functions $f \in \mathbb{C}[V]$ by the rule

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g."'] & \mathbb{C} \\ V \arrow[ur, dotted, "g.f"'] & \end{tikzcd}$$

that is $(g.f)(v) = f(g^{-1}.v)$ for all $g \in GL_n$ and all $v \in V$. If $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_m]$ is graded by giving all the $x_i$ degree one, then each of the homogeneous components of $\mathbb{C}[V]$ is a finite dimensional $GL_n$-representation. Hence, the action of $GL_n$ on $\mathbb{C}[V]$ is locally finite. Indeed, let $\{y_1, \ldots, y_l\}$ be a basis of a finite dimensional subspace $Y \subset \mathbb{C}[V]$ and let $d$ be the maximum of the $deg(y_i)$. Then $Y' = \oplus_{i=0}^{d}\mathbb{C}[V]_i$ is a $GL_n$-representation containing $Y$.

Therefore, we have an isotypical decomposition $\mathbb{C}[V] = \oplus_{s\in S}\mathbb{C}[V]_{(s)}$. In particular, if $0 \in S$ denotes the isomorphism class of the trivial $GL_n$-representation ($\mathbb{C}_{triv} = \mathbb{C}x$ with $g.x = x$ for every $g \in GL_n$) then we have

$$\mathbb{C}[V]_{(0)} = \{f \in \mathbb{C}[V] \mid g.f = f, \forall g \in GL_n\} = \mathbb{C}[V]^{GL_n}$$

the ring of *polynomial invariants* , that is, of polynomial functions which are constant along orbits in $V$.

**Lemma 2.7** *Let $V$ be a $GL_n$-representation.*

1. *Let $I \lhd \mathbb{C}[V]$ be a $GL_n$-stable ideal, that is, $g.I \subset I$ for all $g \in GL_n$, then*

$$(\mathbb{C}[V]/I)^{GL_n} \simeq \mathbb{C}[V]^{GL_n}/(I \cap \mathbb{C}[V]^{GL_n}).$$

2. *Let $J \lhd \mathbb{C}[V]^{GL_n}$ be an ideal, then we have a lying-over property*

$$J = J\mathbb{C}[V] \cap \mathbb{C}[V]^{GL_n}.$$

   *Hence, $\mathbb{C}[V]^{GL_n}$ is Noetherian, that is, every increasing chain of ideals stabilizes.*

3. *Let $I_j$ be a family of $GL_n$-stable ideals of $\mathbb{C}[V]$, then*

$$(\sum_j I_j) \cap \mathbb{C}[V]^{GL_n} = \sum_j (I_j \cap \mathbb{C}[V]^{GL_n}).$$

*Proof.* (1) : As $I$ has the induced $GL_n$-action which is locally finite we have the isotypical decomposition $I = \oplus I_{(s)}$ and clearly $I_{(s)} = \mathbb{C}[V]_{(s)} \cap I$. But then also, taking quotients we have

$$\oplus_s(\mathbb{C}[V]/I)_{(s)} = \mathbb{C}[V]/I = \oplus_s\mathbb{C}[V]_{(s)}/I_{(s)}.$$

Therefore, $(\mathbb{C}[V]/I)_{(s)} = \mathbb{C}[V]_{(s)}/I_{(s)}$ and taking the special case $s = 0$ is the statement.

(2) : For any $f \in \mathbb{C}[V]^{GL_n}$ left-multiplication by $f$ in $\mathbb{C}[V]$ commutes with the $GL_n$-action, whence $f.\mathbb{C}[V]_{(s)} \subset \mathbb{C}[V]_{(s)}$. That is, $\mathbb{C}[V]_{(s)}$ is a $\mathbb{C}[V]^{GL_n}$-module. But then, as $J \subset \mathbb{C}[V]^{GL_n}$ we have

$$\oplus_s(J\mathbb{C}[V])_{(s)} = J\mathbb{C}[V] = \oplus_s J\mathbb{C}[V]_{(s)}.$$

Therefore, $(J\mathbb{C}[V])_{(s)} = J\mathbb{C}[V]_{(s)}$ and again taking the special value $s = 0$ we obtain $J\mathbb{C}[V] \cap \mathbb{C}[V]^{GL_n} = (J\mathbb{C}[V])_{(0)} = J$. The Noetherian statement follows from the fact that $\mathbb{C}[V]$ is Noetherian (the Hilbert basis theorem).

(3) : For any $j$ we have the decomposition $I_j = \oplus_s (I_j)_{(s)}$. But then, we have

$$\oplus_s (\sum_j I_j)_{(s)} = \sum_j I_j = \sum_j \oplus_s (I_j)_{(s)} = \oplus_s \sum_j (I_j)_{(s)}.$$

Therefore, $(\sum_j I_j)_{(s)} = \sum_j (I_j)_{(s)}$ and taking $s = 0$ gives the required statement. $\square$

**Theorem 2.5** *Let $V$ be a $GL_n$-representation. Then, the ring of polynomial invariants $\mathbb{C}[V]^{GL_n}$ is an affine $\mathbb{C}$-algebra.*

*Proof.* Because the action of $GL_n$ on $\mathbb{C}[V]$ preserves the gradation, the ring of invariants is also graded

$$\mathbb{C}[V]^{GL_n} = R = \mathbb{C} \oplus R_1 \oplus R_2 \oplus \dots \quad .$$

From lemma 2.7(2) we know that $\mathbb{C}[V]^{GL_n}$ is Noetherian and hence the ideal $R_+ = R_1 \oplus R_2 \oplus \dots$ is finitely generated $R_+ = Rf_1 + \dots + Rf_l$ by homogeneous elements $f_1, \dots, f_l$. We claim that as a $\mathbb{C}$-algebra $\mathbb{C}[V]^{GL_n}$ is generated by the $f_i$. Indeed, we have $R_+ = \sum_{i=1}^{l} \mathbb{C} f_i + R_+^2$ and then also

$$R_+^2 = \sum_{i,j=1}^{l} \mathbb{C} f_i f_j + R_+^3$$

and iterating this procedure we obtain for all powers $m$ that

$$R_+^m = \sum_{\sum m_i = m} \mathbb{C} f_1^{m_1} \dots f_l^{m_l} + R_+^{m+1}.$$

Now, consider the subalgebra $\mathbb{C}[f_1, \dots, f_l]$ of $R = \mathbb{C}[V]^{GL_n}$, then we obtain for any power $d > 0$ that

$$\mathbb{C}[V]^{GL_n} = \mathbb{C}[f_1, \dots, f_l] + R_+^d.$$

For any $i$ we then have for the homogeneous components of degree $i$

$$R_i = \mathbb{C}[f_1, \dots, f_l]_i + (R_+^d)_i.$$

Now, if $d > i$ we have that $(R_+^d)_i = 0$ and hence that $R_i = \mathbb{C}[f_1, \dots, f_l]_i$. As this holds for all $i$ we proved the claim. $\square$

Choose generating invariants $f_1, \dots, f_l$ of $\mathbb{C}[V]^{GL_n}$, consider the morphism

$$V \xrightarrow{\phi} \mathbb{C}^l \quad \text{defined by} \quad v \mapsto (f_1(v), \dots, f_l(v))$$

and define $W$ to be the Zariski closure $\overline{\phi(V)}$ in $\mathbb{C}^l$. Then, we have a diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \mathbb{C}^l \\ & W \arrow[u, hook] \end{tikzcd}$$

and an isomorphism $\mathbb{C}[W] \xrightarrow{\pi^*} \mathbb{C}[V]^{GL_n}$. More general, let $X$ be a closed $GL_n$-stable subvariety of $V$, then $X = \mathbb{V}_V(I)$ for some $GL_n$-stable ideal $I$ of $\mathbb{C}[V]$. From lemma 2.7(1) we obtain

$$\mathbb{C}[X]^{GL_n} = (\mathbb{C}[V]/I)^{GL_n} = \mathbb{C}[V]^{GL_n}/(I \cap \mathbb{C}[V]^{GL_n})$$

whence $\mathbb{C}[X]^{GL_n}$ is also an affine algebra (and generated by the images of the $f_i$). Define $Y$ to be the Zariski closure of $\phi(X)$ in $\mathbb{C}^l$, then we have a diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \mathbb{C}^l \\ & Y \arrow[u, hook] \end{tikzcd}$$

and an isomorphism $\mathbb{C}[Y] \xrightarrow{\pi} \mathbb{C}[X]^{GL_n}$. We call the morphism $X \xrightarrow{\pi} Y$ an *algebraic quotient* of $X$ under $GL_n$. We will now prove some important properties of this quotient.

**Proposition 2.8 (universal property)** *If $X \xrightarrow{\mu} Z$ is a morphism which is constant along $GL_n$-orbits in $X$, then there exists a unique factoring morphism $\overline{\mu}$*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\mu"'] & & Y \arrow[dl, dotted, "\overline{\mu}"] \\ & Z & \end{tikzcd}$$

*Proof.* As $\mu$ is constant along $GL_n$-orbits in $X$, we have an inclusion $\mu^*(\mathbb{C}[Z]) \subset \mathbb{C}[X]^{GL_n}$. We have the commutative diagram

$$\begin{tikzcd}
 & \mathbb{C}[X] & \\
 & \mathbb{C}[X]^{GL_n} \arrow[u, hook] & \\
\mathbb{C}[Z] \arrow[uur, "\mu^*"] \arrow[ur] \arrow[rr, dotted, "\overline{\mu}^*"'] & & \mathbb{C}[Y] \arrow[uul, "\pi^*"'] \arrow[ul, "\simeq"]
\end{tikzcd}$$

from which the existence and uniqueness of $\overline{\mu}$ follows. $\square$

As a consequence, an algebraic quotient is uniquely determined up to isomorphism (that is, we might have started from other generating invariants and still obtain the same quotient variety up to isomorphism).

**Proposition 2.9 (onto property)** *The algebraic quotient* $X \xrightarrow{\pi} Y$ *is surjective. Moreover, if* $Z \hookrightarrow X$ *is a closed* $GL_n$*-stable subset, then* $\pi(Z)$ *is closed in* $Y$ *and the morphism*

$$\pi_X \mid Z : Z \longrightarrow \pi(Z)$$

*is an algebraic quotient, that is,* $\mathbb{C}[\pi(Z)] \simeq \mathbb{C}[Z]^{GL_n}$.

*Proof.* Let $y \in Y$ with maximal ideal $M_y \triangleleft \mathbb{C}[Y]$. By lemma 2.7(2) we have $M_y\mathbb{C}[X] \neq \mathbb{C}[X]$ and hence there is a maximal ideal $M_x$ of $\mathbb{C}[X]$ containing $M_y\mathbb{C}[X]$, but then $\pi(x) = y$. Let $Z = \mathbb{V}_X(I)$ for a $G$-stable ideal $I$ of $\mathbb{C}[X]$, then $\overline{\pi(Z)} = \mathbb{V}_Y(I \cap \mathbb{C}[Y])$. That is, $\mathbb{C}[\overline{\pi(Z)}] = \mathbb{C}[Y]/(I \cap \mathbb{C}[Y])$. However, we have from lemma 2.7(1) that

$$\mathbb{C}[Y]/(\mathbb{C}[Y] \cap I) \simeq (\mathbb{C}[X]/I)^{GL_n} = \mathbb{C}[Z]^{GL_n}$$

and hence $\mathbb{C}[\overline{\pi(Z)}] = \mathbb{C}[Z]^{GL_n}$. Finally, surjectivity of $\pi \mid Z$ is proved as above. $\square$

An immediate consequence is that the Zariski topology on $Y$ is the quotient topology of that on $X$. For, take $U \subset Y$ with $\pi^{-1}(U)$ Zariski open in $X$. Then, $X - \pi^{-1}(U)$ is a $GL_n$-stable closed subset of $X$. Then, $\pi(X - \pi^{-1}(U)) = Y - U$ is Zariski closed in $Y$.

**Proposition 2.10 (separation property)** *The quotient $X \xrightarrow{\pi} Y$ separates disjoint closed $GL_n$-stable subvarieties of $X$.*

*Proof.* Let $Z_j$ be closed $GL_n$-stable subvarieties of $X$ with defining ideals $Z_j = \mathbb{V}_X(I_j)$. Then, $\cap_j Z_j = \mathbb{V}_X(\sum_j I_j)$. Applying lemma 2.7(3) we obtain

$$\overline{\pi(\cap_j Z_j)} = \mathbb{V}_Y((\sum_j I_j) \cap \mathbb{C}[Y]) = \mathbb{V}_Y(\sum_j (I_j \cap \mathbb{C}[Y]))$$
$$= \cap_j \mathbb{V}_Y(I_j \cap \mathbb{C}[Y]) = \cap_j \overline{\pi(Z_j)}$$

The onto property implies that $\overline{\pi(Z_j)} = \pi(Z_j)$ from which the statement follows. □

It follows from the universal property that the quotient variety $Y$ determined by the ring of polynomial invariants $\mathbb{C}[Y]^{GL_n}$ is the best algebraic approximation to the orbit space problem. From the separation property a stronger fact follows.

**Proposition 2.11** *The algebraic quotient $X \xrightarrow{\pi} Y$ is the best continuous approximation to the orbit space. That is, points of $Y$ parameterize the closed $GL_n$-orbits in $X$. In fact, every fiber $\pi^{-1}(y)$ contains exactly one closed orbit $C$ and we have*

$$\pi^{-1}(y) = \{x \in X \mid C \subset \overline{GL_n.x}\}$$

*Proof.* The fiber $F = \pi^{-1}(y)$ is a $GL_n$-stable closed subvariety of $X$. Take any orbit $GL_n.x \subset F$ then either it is closed or contains in its closure an orbit of strictly smaller dimension. Induction on the dimension then shows that $\overline{G.x}$ contains a closed orbit $C$. On the other hand, assume that $F$ contains two closed orbits, then they have to be disjoint contradicting the separation property. □

## 2.6 Geometric reconstruction

In this section we will give a geometric interpretation of the reconstruction result of theorem 1.17. Let $A$ be a Cayley-Hamilton algebra of degree $n$, with trace map $tr_A$, which is generated by at most $m$ elements $a_1, \ldots, a_m$. We will give a functorial interpretation to the affine scheme determined by the canonical ideal $N_A \triangleleft \mathbb{C}[M_n^m]$ in the formulation of theorem 1.17. First, let us identify the reduced affine variety $\mathbb{V}(N_A)$. A point $m = (m_1, \ldots, m_m) \in \mathbb{V}(N_A)$ determines an algebra map

$f_m : \mathbb{C}[M_n^m]/N_A \longrightarrow \mathbb{C}$ and hence an algebra map $\phi_m$

$$\begin{tikzcd} A \arrow[r, dotted, "\phi_m"] \arrow[d, hook] & M_n(\mathbb{C}) \\ M_n(\mathbb{C}[M_n^m]/N_A) \arrow[ur, "M_n(f_m)"'] & \end{tikzcd}$$

which is trace preserving. Conversely, from the universal property it follows that any trace preserving algebra morphism $A \longrightarrow M_n(\mathbb{C})$ is of this form by considering the images of the trace generators $a_1, \ldots, a_m$ of $A$. Alternatively, the points of $\mathbb{V}(N_A)$ classify $n$-dimensional *trace preserving* representations of $A$. That is, $n$-dimensional representations for which the morphism $A \longrightarrow M_n(\mathbb{C})$ describing the action is trace preserving. For this reason we will denote the variety $\mathbb{V}(N_A)$ by $trep_n\ A$ and call it the *trace preserving reduced representation variety* of $A$.

Assume that $A$ is generated as a $\mathbb{C}$-algebra by $a_1, \ldots, a_m$ (observe that this is no restriction as trace affine algebras are affine) then clearly $I_A(n) \subset N_A$. That is,

**Lemma 2.8** *For $A$ a Cayley-Hamilton algebra of degree $n$ generated by $\{a_1, \ldots, a_m\}$, the reduced trace preserving representation variety*

$$trep_n\ A \hookrightarrow rep_n\ A$$

*is a closed subvariety of the reduced representation variety.*

It is easy to determine the additional defining equations. Write any trace monomial out in the generators

$$tr_A(a_{i_1} \ldots a_{i_k}) = \sum \alpha_{j_1 \ldots j_l} a_{j_1} \ldots a_{j_l}$$

then for a point $m = (m_1, \ldots, m_m) \in rep_n\ A$ to belong to $trep_n\ A$, it must satisfy all the relations of the form

$$tr(m_{i_1} \ldots m_{i_k}) = \sum \alpha_{j_1 \ldots j_l} m_{j_1} \ldots m_{j_l}$$

with $tr$ the usual trace on $M_n(\mathbb{C})$. These relations define the closed subvariety $trep_n(A)$. Usually, this is a proper subvariety.

**Example 2.11** Let $A$ be a finite dimensional semi-simple algebra $A = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C})$, then $A$ has precisely $k$ distinct simple modules $\{M_1, \ldots, M_k\}$ of dimensions $\{d_1, \ldots, d_k\}$. Here, $M_i$ can be viewed as column vectors of size $d_i$ on which the component $M_{d_i}(\mathbb{C})$ acts by left multiplication and the other factors act as zero. Because $A$ is semi-simple every $n$-dimensional $A$-representation $M$ is isomorphic to

$$M = M_1^{\oplus e_1} \oplus \ldots \oplus M_k^{\oplus e_k}$$

for certain multiplicities $e_i$ satisfying the numerical condition

$$n = e_1 d_1 + \ldots + e_k d_k$$

That is, $rep_n\ A$ is the disjoint union of a finite number of (closed) orbits each determined by an integral vector $(e_1, \ldots, e_k)$ satisfying the condition called the *dimension vector* of $M$.

$$rep_n\ A \simeq \bigsqcup_{(e_1,\ldots,e_k)} GL_n/(GL_{e_1} \times \ldots GL_{e_k})$$

Let $f_i \geq 1$ be natural numbers such that $n = f_1 d_1 + \ldots + f_k d_k$ and consider the embedding of $A$ into $M_n(\mathbb{C})$ defined by

$$(a_1, \ldots, a_k) \in A \longrightarrow \begin{bmatrix} \underbrace{\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_1 \end{bmatrix}}_{f_1} & & \\ & \ddots & \\ & & \overbrace{\begin{bmatrix} a_k & & 0 \\ & \ddots & \\ 0 & & a_k \end{bmatrix}}^{f_k} \end{bmatrix} \in M_n(\mathbb{C})$$

Via this embedding, $A$ becomes a Cayley-Hamilton algebra of degree $n$ when equipped with the induced trace $tr$ from $M_n(\mathbb{C})$.

Let $M$ be the $n$-dimensional $A$-representation with dimension vector $(e_1, \ldots, e_k)$ and choose a basis compatible with this decomposition. Let $E_i$ be the idempotent of $A$ corresponding to the identity matrix $I_{d_i}$ of the $i$-th factor. Then, the trace of the matrix defining the action of $E_i$ on $M$ is clearly $e_i d_i . I_n$. On the other hand, $tr(E_i) = f_i d_i . I_n$, hence the only trace preserving $n$-dimensional $A$-representation is that of dimension vector $(f_1, \ldots, f_k)$. Therefore, $trep_n\ A$ consists of the single closed orbit determined by the integral vector $(f_1, \ldots, f_k)$.

$$trep_n\ A \simeq GL_n/(GL_{f_1} \times \ldots \times GL_{f_k})$$

Consider the scheme structure of the trace preserving representation variety $\mathtt{trep}_n\ A$. The corresponding functor assigns to a commutative affine $\mathbb{C}$-algebra $R$

$$\mathtt{trep}_n(R) = Alg_{\mathbb{C}}(\mathbb{C}[M_n^m]/N_A, R).$$

An algebra morphism $\psi : \mathbb{C}[M_n^m]/N_A \longrightarrow R$ determines uniquely an $m$-tuple of $n \times n$ matrices with coefficients in $R$ by

$$r_k = \begin{bmatrix} \psi(x_{11}(k)) & \ldots & \psi(x_{1n}(k)) \\ \vdots & & \vdots \\ \psi(x_{n1}(k)) & \ldots & \psi(x_{nn}(k)) \end{bmatrix}$$

Composing with the canonical embedding

$$\begin{tikzcd} A \arrow[r, dotted, "\phi"] \arrow[d, hook] & M_n(R) \\ M_n(\mathbb{C}[M_n^m]/N_A) \arrow[ur, "M_n(\psi)"'] & \end{tikzcd}$$

determines the trace preserving algebra morphism $\phi : A \longrightarrow M_n(R)$ where the trace map on $M_n(R)$ is the usual trace. By the universal property any trace preserving map $A \longrightarrow M_n(R)$ is also of this form.

**Lemma 2.9** *Let $A$ be a Cayley-Hamilton algebra of degree $n$ which is generated by $\{a_1, \ldots, a_m\}$. The trace preserving representation variety $\mathtt{trep}_n\ A$ represents the functor*

$$\mathtt{trep}_n\ A(R) = \{A \xrightarrow{\phi} M_n(R) \ \mid\ \phi \text{ is trace preserving } \}$$

*Moreover, $\mathtt{trep}_n\ A$ is a closed subscheme of $\mathtt{rep}_n\ A$.*

Recall that there is an action of $GL_n$ on $\mathbb{C}[M_n^m]$ and from the definition of the ideals $I_A(n)$ and $N_A$ it is clear that they are stable under the $GL_n$-action. That is, there is an action by automorphisms on the quotient algebras $\mathbb{C}[M_n^m]/I_A(n)$ and $\mathbb{C}[M_n^m]/N_A$. But then, their algebras of invariants are equal to

$$\begin{cases} \mathbb{C}[\mathtt{rep}_n\ A]^{GL_n} & = (\mathbb{C}[M_n^m]/I_A(n))^{GL_n} = \mathbb{N}_n^m/(I_A(n) \cap \mathbb{N}_n^m) \\ \mathbb{C}[\mathtt{trep}_n\ A]^{GL_n} & = (\mathbb{C}[M_n^m]/N_A)^{GL_n} = \mathbb{N}_n^m/(N_A \cap \mathbb{N}_n^m) \end{cases}$$

That is, these rings of invariants define closed subschemes of the affine (reduced) variety associated to the necklace algebra $\mathbb{N}_n^m$. We will call these schemes the *quotient schemes* for the action of $GL_n$ and denote them respectively by

$$\mathtt{iss}_n\ A = \mathtt{rep}_n\ A/GL_n \quad \text{and} \quad \mathtt{triss}_n\ A = \mathtt{trep}_n\ A/GL_n.$$

We have seen that the geometric points of the reduced variety $iss_n\ A$ of the affine quotient scheme $\mathtt{iss}_n\ A$ parameterize the isomorphism classes of $n$-dimensional semisimple $A$-representations. Similarly, the geometric points of the reduced variety $triss_n\ A$ of the quotient scheme $\mathtt{triss}_n\ A$ parameterize isomorphism classes of *trace preserving* $n$-dimensional semisimple $A$-representations.

**Proposition 2.12** *Let $A$ be a Cayley-Hamilton algebra of degree $n$ with trace map $tr_A$. Then, we have that*

$$tr_A(A) = \mathbb{C}[\mathtt{triss}_n\ A],$$

*the coordinate ring of the quotient scheme $\mathtt{triss}_n\ A$. In particular, maximal ideals of $tr_A(A)$ parameterize the isomorphism classes of trace preserving $n$-dimensional semi-simple $A$-representations.*

By definition, a $GL_n$-equivariant map between the affine $GL_n$-schemes

$$\mathtt{trep}_n\ A \xrightarrow{f} M_n = \mathtt{M}_n$$

means that for any commutative affine $\mathbb{C}$-algebra $R$ the corresponding map

$$\mathtt{trep}_n\ A(R) \xrightarrow{f(R)} M_n(R)$$

commutes with the action of $GL_n(R)$. Alternatively, the ring of all morphisms $\mathtt{trep}_n\ A \longrightarrow M_n$ is the matrixalgebra $M_n(\mathbb{C}[M_n^m]/N_A)$ and those that commute with the $GL_n$ action are precisely the invariants. That is, we have the following description of $A$.

**Theorem 2.6** *Let $A$ be a Cayley-Hamilton algebra of degree $n$ with trace map $tr_A$. Then, we can recover $A$ as the ring of $GL_n$-equivariant maps*

$$A = \{ f : \mathtt{trep}_n\ A \longrightarrow M_n \quad GL_n\text{-equivariant} \ \}$$

Summarizing the results of this and the previous section we have

**Theorem 2.7** *The functor*

$$\mathtt{alg@n} \xrightarrow{\mathtt{trep}_n} \mathtt{GL(n)\text{-}affine}$$

*which assigns to a Cayley-Hamilton algebra $A$ of degree $n$ the $GL_n$-affine scheme $\mathtt{trep}_n\ A$ of trace preserving $n$-dimensional representations has a left inverse. This left inverse functor*

$$\mathtt{GL(n)\text{-}affine} \xrightarrow{\Uparrow^n} \mathtt{alg@n}$$

*assigns to a $GL_n$-affine scheme $\mathtt{X}$ its* witness algebra $\Uparrow^n [X] = M_n(\mathbb{C}[\mathtt{X}])^{GL_n}$ *which is a Cayley-Hamilton algebra of degree $n$.*

Note however that this functor is *not* an equivalence of categories. For, there are many affine $GL_n$-schemes having the same witness algebra as we will see in the next section.

We will give an application of the algebraic reconstruction result, theorem 1.17, to finite dimensional algebras.

Let $A$ be a Cayley-Hamilton algebra of degree $n$ wit trace map $tr$, then we can define a *norm map* on $A$ by

$$N(a) = \sigma_n(a) \qquad \text{for all } a \in A.$$

Recall that the elementary symmetric function $\sigma_n$ is a polynomial function $f(t_1, t_2, \ldots, t_n)$ in the Newton functions $t_i = \sum_{j=1}^n x_j^i$. Then, $\sigma(a) = f(tr(a), tr(a^2), \ldots, tr(a^n))$. Because, we have a trace preserving embedding $A \hookrightarrow M_n(\mathbb{C}[\mathtt{trep}_n\ A])$ and the norm map $N$ coincides with the determinant in this matrix-algebra, we have that

$$N(1) = 1 \quad \text{ and } \quad \forall a, b \in A \quad N(ab) = N(a)N(b).$$

Furthermore, the norm map extends to a polynomial map on $A[t]$ and we have that $\chi_a^{(n)}(t) = N(t-a)$. In particular we can obtain the trace by polarization of the norm map. Consider a finite dimensional semi-simple $\mathbb{C}$-algebra

$$A = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C}),$$

then all the Cayley-Hamilton structures of degree $n$ on $A$ with trace values in $\mathbb{C}$ are given by the following result.

**Lemma 2.10** *Let $A$ be a semi-simple algebra as above and $tr$ a trace map on $A$ making it into a Cayley-Hamilton algebra of degree $n$ with $tr(A) = \mathbb{C}$. Then, there exist a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}_+^k$ such that $n = \sum_{i=1}^k m_i d_i$ and for any $a = (A_1, \ldots, A_k) \in A$ with $A_i \in M_{d_i}(\mathbb{C})$ we have that*

$$tr(a) = m_1 Tr(A_1) + \ldots + m_k Tr(A_k)$$

*where $Tr$ are the usual trace maps on matrices.*

*Proof.* The norm-map $N$ on $A$ defined by the trace map $tr$ induces a group morphism on the invertible elements of $A$

$$N : A^* = GL_{d_1}(\mathbb{C}) \times \ldots \times GL_{d_k}(\mathbb{C}) \longrightarrow \mathbb{C}^*$$

that is, a *character*. Now, any character is of the following form, let $A_i \in GL_{d_i}(\mathbb{C})$, then for $a = (A_1, \ldots, A_k)$ we must have

$$N(a) = det(A_1)^{m_1} det(A_2)^{m_2} \ldots det(A_k)^{m_k}$$

for certain integers $m_i \in \mathbb{Z}$. Since $N$ extends to a polynomial map on the whole of $A$ we must have that all $m_i \geq 0$. By polarization it then follows that

$$tr(a) = m_1 Tr(A_1) + \ldots + m_k Tr(A_k)$$

and it remains to show that no $m_i = 0$. Indeed, if $m_i = 0$ then $tr$ would be the zero map on $M_{d_i}(\mathbb{C})$, but then we would have for any $a = (0, \ldots, 0, A, 0, \ldots, 0)$ with $A \in M_{d_i}(\mathbb{C})$ that

$$\chi_a^{(n)}(t) = t^n$$

whence $\chi_a^{(n)}(a) \neq 0$ whenever $A$ is not nilpotent. This contradiction finishes the proof. $\square$

We can extend this to all finite dimensional $\mathbb{C}$-algebras. Let $A$ be a finite dimensional algebra with radical $J$ and assume there is a trace map $tr$ on $A$ making $A$ into a Cayley-Hamilton algebra of degree $n$ and such that $tr(A) = \mathbb{C}$. We claim that the norm map $N : A \longrightarrow \mathbb{C}$ is zero on $J$.

Indeed, any $j \in J$ satisfies $j^l = 0$ for some $l$ whence $N(j)^l = 0$. But then, polarization gives that $tr(J) = 0$ and we have that the semisimple algebra

$$A^{ss} = A/J = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C})$$

is a semisimple Cayley-hamilton algebra of degree $n$ on which we can apply the foregoing lemma. Finally, note that $A \simeq A^{ss} \oplus J$ as $\mathbb{C}$-vectorspaces. This concludes the proof of

**Proposition 2.13** *Let $A$ be a finite dimensional $\mathbb{C}$-algebra with radical $J$ and semisimple part*

$$A^{ss} = A/J = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C}).$$

*Let $tr : A \longrightarrow \mathbb{C} \hookrightarrow A$ be a trace map such that $A$ is a Cayley-Hamilton algebra of degree $n$. Then, there exists a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}^k_+$ such that for all $a = (A_1, \ldots, A_k, j)$ with $A_i \in M_{d_i}(\mathbb{C})$ and $j \in J$ we have*

$$tr(a) = m_1 Tr(A_1) + \ldots + m_k Tr(A_k)$$

*with $Tr$ the usual traces on $M_{d_i}(\mathbb{C})$ and $\sum_i m_i d_i = n$.*

Fix a trace map $tr$ on $A$ determined by a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}^k$. Then, the trace preserving variety $\mathtt{trep}_n\ A$ is the scheme of $A$-modules of *dimension vector* $\alpha$, that is, those $A$-modules $M$ such that

$$M^{ss} = S_1^{\oplus m_1} \oplus \ldots \oplus S_k^{\oplus m_k}$$

where $S_i$ is the simple $A$-module of dimension $d_i$ determined by the $i$-th factor in $A^{ss}$. An immediate consequence of the reconstruction theorem 2.6 is

**Proposition 2.14** *Let $A$ be a finite dimensional algebra with trace map $tr : A \longrightarrow \mathbb{C}$ determined by a dimension vector $\alpha = (m_1, \ldots, m_k)$ as before with all $m_i > 0$. Then, $A$ can be recovered from the $GL_n$-structure of the affine scheme $\mathtt{trep}_n\ A$ of all $A$-modules of dimension vector $\alpha$.*

Still, there can be other trace maps on $A$ making $A$ into a Cayley-Hamilton algebra of degree $n$. For example let $C$ be a finite dimensional commutative $\mathbb{C}$-algebra with radical $N$, then $A = M_n(C)$ is finite dimensional with radical $J = M_n(N)$ and the usual trace map $tr : M_n(C) \longrightarrow C$ makes $A$ into a Cayley-Hamilton algebra of degree $n$ such that $tr(J) = N \neq 0$. Still, if $A$ is semi-simple, the center $Z(A) = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$ (as many terms as there are matrix components in $A$) and any subring of $Z(A)$ is of the form $\mathbb{C} \oplus \ldots \oplus \mathbb{C}$. In particular, $tr(A)$ has this form and composing the trace map with projection on the $j$-th component we have a trace map $tr_j$ on which we can apply lemma 2.10.

## 2.7 The Gerstenhaber-Hesselink theorem

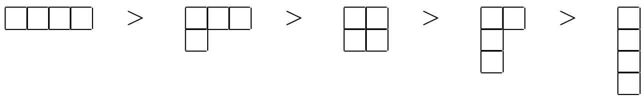
In this section we will give examples of distinct $GL_n$-affine schemes having the same witness algebra, proving that the left inverse of theorem 2.7 is *not* an equivalence of categories. We will study the orbits in $rep_n\ \mathbb{C}[x]$ or, equivalent, conjugacy classes of $n \times n$ matrices.

It is sometimes convenient to relax our definition of partitions to include zeroes at its tail. That is, a partition $p$ of $n$ is an integral $n$-tuple $(a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ with $\sum_{i=1}^n a_i = n$. As before, we represent a partition by a Young diagram by omitting rows corresponding to zeroes.

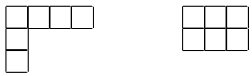
If $q = (b_1, \ldots, b_n)$ is another partition of $n$ we say that $p$ *dominates* $q$ and write

$$p > q \quad \text{if and only if} \quad \sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i \quad \text{for all } 1 \leq r \leq n.$$

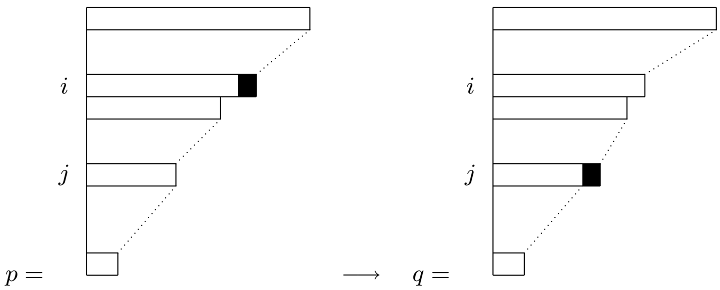
For example, the partitions of 4 are ordered as indicated below



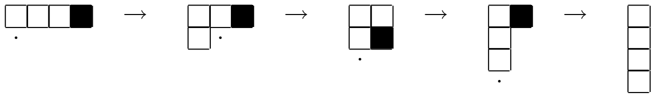

Note however that the dominance relation is *not* a total ordering whenever $n \geq 6$. For example, the following two partition of 6



are not comparable. The *dominance order* is induced by the *Young move* of throwing a row-ending box down the diagram. Indeed, let $p$ and $q$ be partitions of $n$ such that $p > q$ and assume there is no partition $r$ such that $p > r$ and $r > q$. Let $i$ be the minimal number such that $a_i > b_i$. Then by the assumption $a_i = b_i + 1$. Let $j > i$ be minimal such that $a_j \neq b_j$, then we have $b_j = a_j + 1$ because $p$ dominates $q$. But then, the remaining rows of $p$ and $q$ must be equal. That is, a Young move can be depicted as



For example, the Young moves between the partitions of 4 given above are as indicated



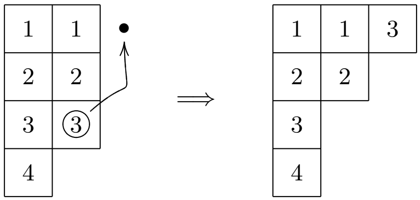
A *Young p-tableau* is the Young diagram of $p$ with the boxes labeled by integers from $\{1, 2, \ldots, s\}$ for some $s$ such that each label appears at least ones. A Young $p$-tableau is said to be *of type* $q$ for some partition $q = (b_1, \ldots, b_n)$ of $n$ if the following conditions are met :

- the labels are non-decreasing along rows,
- the labels are strictly increasing along columns, and
- the label $i$ appears exactly $b_i$ times.

For example, if $p = (3, 2, 1, 1)$ and $q = (2, 2, 2, 1)$ then the $p$-tableau below

| | | |
|---|---|---|
| 1 | 1 | 3 |
| 2 | 2 | |
| 3 | | |
| 4 | | |

is of type $q$ (observe that $p > q$ and even $p \to q$). In general, let $p = (a_1, \ldots, a_n)$ and $q = (b_1, \ldots, b_n)$ be partitions of $n$ and assume that $p \to q$. Then, there is a Young $p$-tableau of type $q$. For, fill the Young diagram of $q$ by putting 1's in the first row, 2's in the second and so on. Then, upgrade the fallen box together with its label to get a Young $p$-tableau of type $q$. In the example above




Conversely, assume there is a Young $p$-tableau of type $q$. The number of boxes labeled with a number $\leq i$ is equal to $b_1 + \ldots + b_i$. Further, any box with label $\leq i$ must lie in the first $i$ rows (because the labels strictly increase along a column). There are $a_1 + \ldots + a_i$ boxes available in the first $i$ rows whence

$$\sum_{j=1}^{i} b_i \leq \sum_{j=1}^{i} a_i \quad \text{for all} \quad 1 \leq i \leq n$$

and therefore $p > q$. After these preliminaries on partitions, let us return to nilpotent matrices.

Let $A$ be a nilpotent matrix of type $p = (a_1, \ldots, a_n)$, that is, conjugated to a matrix with Jordan blocks (all with eigenvalue zero) of sizes $a_i$. We have seen before that the subspace $V_l$ of column vectors $v \in \mathbb{C}^n$ such that $A^l.v = 0$ has dimension

$$\sum_{h=1}^{l} \#\{j \mid a_j \geq h\} = \sum_{h=1}^{l} a_h^*$$

where $p^* = (a_1^*, \ldots, a_n^*)$ is the dual partition of $p$. Choose a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$ such that for all $l$ the first $a_1^* + \ldots + a_l^*$ base vectors span the subspace $V_l$. For example, if $A$ is in Jordan normal form of type $p = (3, 2, 1, 1)$

$$\begin{bmatrix} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & & & & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

then $p^* = (4, 2, 1)$ and we can choose the standard base vectors ordered as follows

$$\{\underbrace{e_1, e_4, e_6, e_7}_{4}, \underbrace{e_2, e_5}_{2}, \underbrace{e_3}_{1}\}.$$

Take a partition $q = (b_1, \ldots, b_n)$ with $p \rightarrow q$ (in particular, $p > q$), then for the dual partitions we have $q^* \rightarrow p^*$ (and thus $q^* > p^*$). But then there is a Young $q^*$-tableau of type $p^*$. In the example with $q = (2, 2, 2, 1)$ we have $q^* = (4, 3)$ and $p^* = (4, 2, 1)$ and we can take the Young $q^*$-tableau of type $p^*$

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 2 | 3 | |

Now label the boxes of this tableau by the base vectors $\{v_1, \ldots, v_n\}$ such that the boxes labeled $i$ in the Young $q^*$-tableau of type $p^*$ are filled with the base vectors from $V_i - V_{i-1}$. Call this tableau

$T$. In the example, we can take

$$T = \begin{array}{|c|c|c|c|} \hline e_1 & e_4 & e_6 & e_7 \\ \hline e_2 & e_5 & e_3 & \\ \cline{1-3} \end{array}$$

Define a linear operator $F$ on $\mathbb{C}^n$ by the rule that $F(v_i) = v_j$ if $v_j$ is the label of the box in $T$ just above the box labeled $v_i$. In case $v_i$ is a label of a box in the first row of $T$ we take $F(v_i) = 0$. Obviously, $F$ is a nilpotent $n \times n$ matrix and by construction we have that

$$rk\ F^l = n - (b_1^* + \ldots + b_l^*)$$

That is, $F$ is nilpotent of type $q = (b_1, \ldots, b_n)$. Moreover, $F$ satisfies $F(V_i) \subset V_{i-1}$ for all $i$ by the way we have labeled the tableau $T$ and defined $F$.

In the example above, we have $F(e_2) = e_1$, $F(e_5) = e_4$, $F(e_3) = e_6$ and all other $F(e_i) = 0$. That is, $F$ is the matrix

$$\begin{bmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & & & 0 & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & 1 & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

which is seen to be of type $(2, 2, 2, 1)$ after performing a few Jordan moves.

Returning to the general case, consider for all $\epsilon \in \mathbb{C}$ the $n \times n$ matrix :

$$F_\epsilon = (1 - \epsilon)F + \epsilon A.$$

We claim that for all but finitely many values of $\epsilon$ we have $F_\epsilon \in \mathcal{O}(A)$. Indeed, we have seen that $F(V_i) \subset V_{i-1}$ where $V_i$ is defined as the subspace such that $A^i(V_i) = 0$. Hence, $F(V_1) = 0$ and therefore

$$F_\epsilon(V_1) = (1 - \epsilon)F + \epsilon A(V_1) = 0.$$

Assume by induction that $F_\epsilon^i(V_i) = 0$ holds for all $i < l$, then we have that

$$\begin{aligned} F_\epsilon^l(V_l) &= F_\epsilon^{l-1}((1 - \epsilon)F + \epsilon A)(V_l) \\ &\subset F_\epsilon^{l-1}(V_{l-1}) = 0 \end{aligned}$$

because $A(V_l) \subset V_{l-1}$ and $F(V_l) \subset V_{l-1}$. But then we have for all $l$ that

$$rk\ F_\epsilon^l \leq dim\ V_l = n - (a_1^* + \ldots + a_l^*) = rk\ A^l \overset{def}{=} r_l.$$

Then for at least one $r_l \times r_l$ submatrix of $F^l_\epsilon$ its determinant considered it as a polynomial of degree $r_l$ in $\epsilon$ is not identically zero (as it is nonzero for $\epsilon = 1$). But then this determinant is non-zero for all but finitely many $\epsilon$. Hence, $rk\ F^l_\epsilon = rk\ A^l$ for all $l$ for all but finitely many $\epsilon$. As these numbers determine the dual partition $p^*$ of the type of $A$, $F_\epsilon$ is a nilpotent $n \times n$ matrix of type $p$ for all but finitely many values of $\epsilon$, proving the claim. But then, $F_0 = F$ which we have proved to be a nilpotent matrix of type $q$ belongs to the closure of the orbit $\mathcal{O}(A)$. That is, we have proved the difficult part of the *Gerstenhaber-Hesselink theorem* .

**Theorem 2.8** *Let $A$ be a nilpotent $n \times n$ matrix of type $p = (a_1, \ldots, a_n)$ and $B$ nilpotent of type $q = (b_1, \ldots, b_n)$. Then, $B$ belongs to the closure of the orbit $\mathcal{O}(A)$, that is,*

$$B \in \overline{\mathcal{O}(A)} \quad \text{if and only if} \quad p > q$$

*in the domination order on partitions of $n$.*

To prove the theorem we only have to observe that if $B$ is contained in the closure of $A$, then $B^l$ is contained in the closure of $A^l$ and hence $rk\ A^l \geq rk\ B^l$ (because $rk\ A^l < k$ is equivalent to vanishing of all determinants of $k \times k$ minors which is a closed condition). But then,

$$n - \sum_{i=1}^{l} a_i^* \geq n - \sum_{i=1}^{l} b_i^*$$

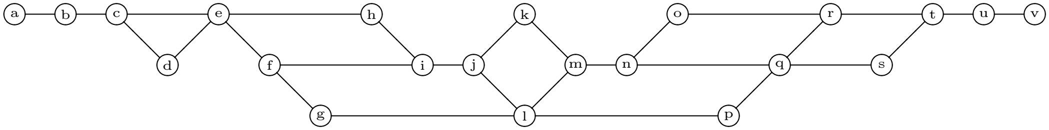
for all $l$, that is, $q^* > p^*$ and hence $p > q$. The other implication was proved above if we remember that the domination order was induced by the Young moves and clearly we have that if $B \in \overline{\mathcal{O}(C)}$ and $C \in \overline{\mathcal{O}(A)}$ then also $B \in \overline{\mathcal{O}(A)}$.

**Example 2.12 (Nilpotent matrices for small $n$)** We will apply theorem 2.8 to describe the orbit-closures of nilpotent matrices of $8 \times 8$ matrices. The following table lists all partitions (and their dual in the other column)

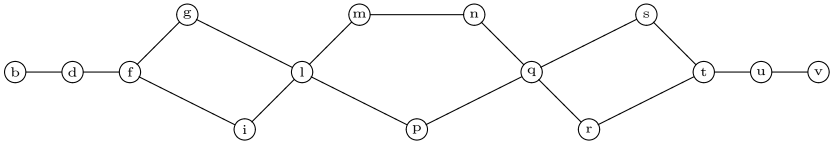
The partitions of 8.

| | | | |
|---|---|---|---|
| a | (8) | v | (1,1,1,1,1,1,1,1) |
| b | (7,1) | u | (2,1,1,1,1,1,1) |
| c | (6,2) | t | (2,2,1,1,1,1) |
| d | (6,1,1) | s | (3,1,1,1,1,1) |
| e | (5,3) | r | (2,2,2,1,1) |
| f | (5,2,1) | q | (3,2,1,1,1) |
| g | (5,1,1,1) | p | (4,1,1,1,1) |
| h | (4,4) | o | (2,2,2,2) |
| i | (4,3,1) | n | (3,2,2,1) |
| j | (4,2,2) | m | (3,3,1,1) |
| k | (3,3,2) | k | (3,3,2) |
| l | (4,2,1,1) | l | (4,2,1,1) |

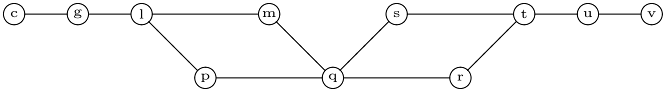
The domination order between these partitions can be depicted as follows where all the Young moves are from left to right




Of course, from this graph we can read off the dominance order graphs for partitions of $n \leq 8$. The trick is to identify a partition of $n$ with that of 8 by throwing in a tail of ones and to look at the relative position of both partitions in the above picture. Using these conventions we get the following graph for partitions of 7




and for partitions of 6 the dominance order is depicted as follows




The dominance order on partitions of $n \leq 5$ is a total ordering.

The Gerstenhaber-Hesselink theorem can be applied to describe the module varieties of the algebras $\mathbb{C}[x]/(x^r)$.

**Example 2.13 (The representation variety $rep_n \frac{\mathbb{C}[x]}{(x^r)}$)** Any algebra morphism from $\mathbb{C}[x]$ to $M_n$ is determined by the image of $x$, whence $rep_n(\mathbb{C}[x]) = M_n$. We have seen that conjugacy classes in $M_n$ are classified by the Jordan normalform. Let $A$ is conjugated to a matrix in normalform

$$\begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_s} \end{bmatrix}$$

where $J_i$ is a Jordan block of size $d_i$, hence $n = d_1 + d_2 + \ldots + d_s$. Then, the $n$-dimensional $\mathbb{C}[x]$-module $M$ determined by $A$ can be decomposed uniquely as

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_s$$

where $M_i$ is a $\mathbb{C}[x]$-module of dimension $d_i$ which is *indecomposable* , that is, cannot be decomposed as a direct sum of proper submodules.

Now, consider the quotient algebra $R = \mathbb{C}[x]/(x^r)$, then the ideal $I_R(n)$ of $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$ is generated by the $n^2$ entries of the matrix

$$\begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix}^r .$$

For example if $r = m = 2$, then the ideal is generated by the entries of the matrix

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^2 = \begin{bmatrix} x_1^2 + x_2x_3 & x_2(x_1 + x_4) \\ x_3(x_1 + x_4) & x_4^2 + x_2x_3 \end{bmatrix}$$

That is, the ideal with generators

$$I_R = (x_1^2 + x_2x_3, x_2(x_1 + x_4), x_3(x_1 + x_4), (x_1 - x_4)(x_1 + x_4))$$

The variety $\mathbb{V}(I_R) \hookrightarrow M_2$ consists of all matrices $A$ such that $A^2 = 0$. Conjugating $A$ to an upper triangular form we see that the eigenvalues of $A$ must be zero, hence

$$rep_2\ \mathbb{C}[x]/(x^2) = \mathcal{O}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \cup \mathcal{O}(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$$

and we have seen that this variety is a cone with top the zero matrix and defining equations

$$\mathbb{V}(x_1 + x_4, x_1^2 + x_2x_3)$$

and we see that $I_R$ is properly contained in this ideal. Still, we have that

$$rad(I_R) = (x_1 + x_4, x_1^2 + x_3x_4)$$

for an easy computation shows that $\overline{x_1 + x_4}^3 = 0 \in \mathbb{C}[x_1, x_2, x_3, x_4]/I_R$. Therefore, even in the easiest of examples, the representation variety does not have to be reduced.

For the general case, observe that when $J$ is a Jordan block of size $d$ with eigenvalue zero an easy calculation shows that

$$J^{d-1} = \begin{bmatrix} 0 & \ldots & 0 & d-1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix} \quad \text{and} \quad J^d = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}$$

Therefore, we see that the representation variety $rep_n\ \mathbb{C}[x]/(x^r)$ is the union of all conjugacy classes of matrices having 0 as only eigenvalue and all of which Jordan blocks have size $\leq r$. Expressed in module theoretic terms, any $n$-dimensional $R = \mathbb{C}[x]/(x^r)$-module $M$ is isomorphic to a direct sum of indecomposables

$$M = I_1^{\oplus e_1} \oplus I_2^{\oplus e_2} \oplus \ldots \oplus I_r^{\oplus e_r}$$

where $I_j$ is the unique indecomposable $j$-dimensional $R$-module (corresponding to the Jordan block of size $j$). Of course, the multiplicities $e_i$ of the factors must satisfy the equation

$$e_1 + 2e_2 + 3e_3 + \ldots + re_r = n$$

In $M$ we can consider the subspaces for all $1 \leq i \leq r-1$

$$M_i = \{m \in M \mid x^i.m = 0\}$$

the dimension of which can be computed knowing the powers of Jordan blocks (observe that the dimension of $M_i$ is equal to $n - \mathrm{rank}(A^i)$)

$$t_i = dim_{\mathbb{C}}\ M_i = e_1 + 2e_2 + \ldots (i-1)e_i + i(e_i + e_{i+1} + \ldots + e_r)$$

Observe that giving $n$ and the $r-1$-tuple $(t_1, t_2, \ldots, t_{n-1})$ is the same as giving the multiplicities $e_i$ because

$$\begin{cases} 2t_1 & = t_2 + e_1 \\ 2t_2 & = t_3 + t_1 + e_2 \\ 2t_3 & = t_4 + t_2 + e_3 \\ & \vdots \\ 2t_{n-2} & = t_{n-1} + t_{n-3} + e_{n-2} \\ 2t_{n-1} & = n + t_{n-2} + e_{n-1} \\ n & = t_{n-1} + e_n \end{cases}$$

Let $n$-dimensional $\mathbb{C}[x]/(x^r)$-modules $M$ and $M'$ (or associated matrices $A$ and $A'$) be determined by the $r-1$-tuples $(t_1, \ldots, t_{r-1})$ respectively $(t'_1, \ldots, t'_{r-1})$ then we have that

$$\mathcal{O}(A') \hookrightarrow \overline{\mathcal{O}(A)} \quad \text{if and only if} \quad t_1 \leq t'_1, t_2 \leq t'_2, \ldots, t_{r-1} \leq t'_{r-1}$$

Therefore, we have an inverse order isomorphism between the orbits in $rep_n(\mathbb{C}[x]/(x^r))$ and the $r-1$-tuples of natural numbers $(t_1, \ldots, t_{r-1})$ satisfying the following linear inequalities (which follow from the above system)

$$2t_1 \geq t_2, 2t_2 \geq t_3 + t_1, 2t_3 \geq t_4 + t_2, \ldots, 2t_{n-1} \geq n + t_{n-2}, n \geq t_{n-2}.$$

Let us apply this general result in a few easy cases. First, consider $r = 2$, then the orbits in $rep_n\ \mathbb{C}[x]/(x^2)$ are parameterized by a natural number $t_1$ satisfying the inequalities $n \geq t_1$ and

$2t_1 \geq n$, the multiplicities are given by $e_1 = 2t_1 - n$ and $e_2 = n - t_1$. Moreover, the orbit of the module $M(t_1')$ lies in the closure of the orbit of $M(t_1)$ whenever $t_1 \leq t_1'$.

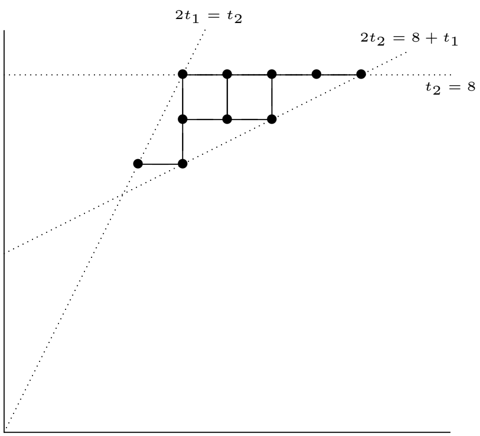
That is, if $n = 2k + \delta$ with $\delta = 0$ or 1, then $rep_n\ \mathbb{C}[x]/(x^2)$ is the union of $k+1$ orbits and the orbitclosures form a linear order as follows (from big to small)

$$I_1^{\delta} \oplus I_2^{\oplus k} \;—\!—\; I_1^{\oplus \delta+2} \oplus I_2^{\oplus k-1} \;—\!—\; \ldots \;—\!—\; I_1^{\oplus n}$$

If $r = 3$, orbits in $rep_n\ \mathbb{C}[x]/(x^3)$ are determined by couples of natural numbers $(t_1, t_2)$ satisfying the following three linear inequalities

$$\begin{cases} 2t_1 & \geq t_2 \\ 2t_2 & \geq n + t_1 \\ n & \geq t_2 \end{cases}$$

For example, for $n = 8$ we obtain the following situation




Therefore, $rep_8\ \mathbb{C}[x]/(x^3)$ consists of 10 orbits with orbit closure diagram as in figure 2.2 (the nodes represent the multiplicities $[e_1e_2e_3]$).

Here we used the equalities $e_1 = 2t_1 - t_2$, $e_2 = 2t_2 - n - t_1$ and $e_3 = n - t_2$. For general $n$ and $r$ this result shows that $rep_n\ \mathbb{C}[x]/(x^r)$ is the closure of the orbit of the module with decomposition

$$M_{gen} = I_r^{\oplus e} \oplus I_s \quad \text{if} \quad n = er + s$$

We are now in a position to give the promised examples of affine $GL_n$-schemes having the same witness algebra.

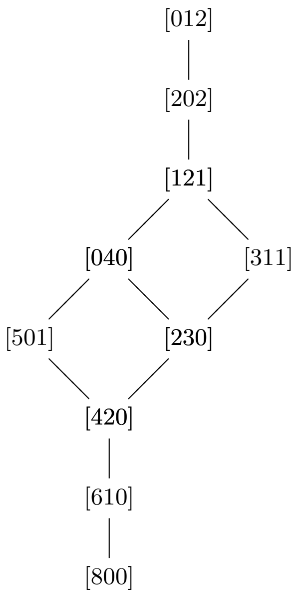



Figure 2.2: Orbit closures in $rep_8\ \mathbb{C}[x]/(x^3)$.

**Example 2.14** Consider the action of $GL_n$ on $M_n$ by conjugation and take a nilpotent matrix $A$. All eigenvalues of $A$ are zero, so the conjugacy class of $A$ is fully determined by the sizes of its Jordan blocks. These sizes determine a partition $\lambda(A) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. Moreover, we have given an algorithm to determine whether an orbit $\mathcal{O}(B)$ of another nilpotent matrix $B$ is contained in the orbit closure $\overline{\mathcal{O}(A)}$, the criterium being that

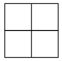
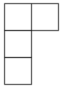
$$\mathcal{O}(B) \subset \overline{\mathcal{O}(A)} \Longleftrightarrow \lambda(B)^* \geq \lambda(A)^*.$$

where $\lambda^*$ denotes the dual partition. We see that the witness algebra of $\overline{\mathcal{O}(A)}$ is equal to

$$M_n(\mathbb{C}[\overline{\mathcal{O}(A)}])^{GL_n} = \mathbb{C}[X]/(X^k)$$

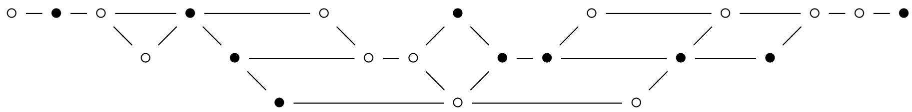
where $k$ is the number of columns of the Young diagram $\lambda(A)$.

Hence, the orbit closures of nilpotent matrices such that their associated Young diagrams have equal number of columns have the same witness algebras. For example, if $n = 4$ then the closures of the orbits corresponding to


and


have the same witness algebra, although the closure of the second is a proper closed subscheme of the closure of the first.

Recall the orbitclosure diagram of conjugacy classes of nilpotent $8 \times 8$ matrices given by the Gerstenhaber-Hesselink theorem. In the picture below, the closures of orbits corresponding to connected nodes of the same color have the same witness algebra.



## 2.8 The real moment map

In this section we will give another interpretation of the algebraic quotient variety $\mathtt{triss}_n\ A$ with methods coming from symplectic geometry. We have an involution

$$GL_n \xrightarrow{i} GL_n \qquad \text{defined by} \qquad g \longrightarrow (g^*)^{-1}$$

where $A^*$ is the *adjoint matrix* of $g$, that is, the conjugate transpose

$$M = \begin{bmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{bmatrix} \qquad M^* = \begin{bmatrix} \overline{m_{11}} & \dots & \overline{m_{n1}} \\ \vdots & & \vdots \\ \overline{m_{1n}} & \dots & \overline{m_{nn}} \end{bmatrix}$$

The *real points* of this involution, that is

$$(GL_n)^i = \{g \in GL_n \mid g = (g^*)^{-1}\} = U_n = \{u \in GL_n \mid uu^* = \mathbb{1}_n\}$$

is the *unitary group* . On the level of Lie algebras, the involution $i$ gives rise to the linear map

$$M_n \xrightarrow{di} M_n \qquad \text{defined by} \qquad M \longrightarrow -M^*$$

corresponding to the fact that the Lie algebra of the unitary group, that is, the kernel of $di$, is the space of *skew-Hermitian matrices*

$$Lie\ U_n = \{M \in M_n \mid M = -M^*\} = iHerm_n$$

Consider the standard *Hermitian inproduct* on $M_n$ defined by

$$(A,B) = tr(A^*B) \qquad \text{which satisfies} \qquad \begin{cases} (cA,B) & = \overline{c}(A,B) \\ (A,cB) & = c(A,B) \\ (B,A) & = \overline{(A,B)} \end{cases}$$

As a subgroup of $GL_n$, $U_n$ acts on $M_n$ by conjugation and because $(uAu^*, uBu^*) = tr(uA^*u^*uBu^*) = tr(A^*B)$, the inproduct is invariant under the $U_n$-action. The action of $U_n$ on $M_n$ induces an action of $Lie\ U_n$ on $M_n$ given for all $h \in Lie\ U_n$ and $M \in M_n$

$$h.M = hM + Mh^* = hM - Mh$$

Using this action, we define the *real moment map* $\mu$ for the action of $U_n$ on $M_n$ as the map from $M_n$ to the linear dual of the Lie algebra

$$M_n \xrightarrow{\mu} (iLie\ U_n)^* \qquad M \longrightarrow (h \mapsto i(h.M, M))$$

We will identify the inverse image of the zero map $\underline{0} : Lie\ U_n \longrightarrow 0$ under $\mu$. Because

$$\begin{aligned} (h.M, M) &= tr((h.M - M.h)^*M) \\ &= tr(M^*h^*M - h^*M^*M) \\ &= tr(h^*(MM^* - M^*M)) \end{aligned}$$

and using the nondegeneracy of the *Killing form* on $Lie\ U_n$ we have the identification

$$\mu^{-1}(\underline{0}) = \{ M \in M_n \mid MM^* = M^*M \} = Nor_n$$

the space of *normal matrices* . Alternatively, we can define the real moment map to be determined by

$$M_n \xrightarrow{\mu_{\mathbb{R}}} Lie\ U_n \qquad M \longrightarrow i(MM^* - M^*M) = i[M, M^*]$$

Recall that a matrix $M \in M_n(\mathbb{C})$ is said to be *normal* if its commutes with its adjoint. For example, diagonal matrices are normal as are unitary matrices. Further, it is clear that if $M$ is normal and $u$ unitary, then the conjugated matrix $uMu^{-1} = uMu^*$ is again a normal matrix, that is we have an action of the compact Lie group $U_n$ on the subset $Nor_n \hookrightarrow M_n(\mathbb{C})$ of normal matrices. We recall the proof of the following classical result

**Theorem 2.9** *Every $U_n$ orbit in $Nor_n$ contains a diagonal matrix. This gives a natural one-to-one correspondence*

$$\mu^{-1}(\underline{0})/U_n = Nor_n/U_n \longleftrightarrow M_n/GL_n$$

*between the $U_n$-orbits in $Nor_n$ and the closed $GL_n$-orbits in $M_n$.*

*Proof.* Equip $\mathbb{C}^n$ with the standard Hermitian form, that is,

$$\langle v, w \rangle = \overline{v}^\tau . w = \overline{v_1} w_1 + \ldots + \overline{v_n} w_n$$

Take a non-zero eigenvector $v$ of $M \in Nor_n$ and normalize it such that $\langle v, v \rangle = 1$. Extend $v = v_1$ to an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$ and let $u$ be the basechange matrix from the standard basis. With respect to the new basis, the linear map determined by $M$ and $M^*$ are represented by the normal matrices

$$M_1 = uMu^* = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \ldots & a_{nn} \end{bmatrix} \qquad M_1^* = uM^*u^* = \begin{bmatrix} \overline{a_{11}} & 0 & \ldots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & \ldots & \overline{a_{n2}} \\ \vdots & \vdots & & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \ldots & \overline{a_{nn}} \end{bmatrix}$$

Because $M$ is normal, so is $M_1$. The left hand corner of $M_1^* M_1$ is $a_{11}\overline{a_{11}}$ whereas that of $M_1 M_1^*$ is $a_{11}\overline{a_{11}} + a_{12}\overline{a_{12}} + \ldots + a_{1n}\overline{a_{1n}}$, whence

$$a_{12}\overline{a_{12}} + \ldots + a_{1n}\overline{a_{1n}} = 0$$

but as all $a_{1i}\overline{a_{1i}} = \| a_{1i} \| \geq 0$, this implies that all $a_{1i} = 0$, whence

$$M_1 = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

and induction finishes the claim. Because permutation matrices are unitary we see that the diagonal entries are determined up to permutation, so every $U_n$-orbit determines a unique conjugacy class of semi-simple matrices, that is, a closed $GL_n$-orbit in $M_n$. □

We will generalize this classical result to $m$-tuples of $n \times n$ matrices, $M_n^m$, and then by restriction to trace preserving representation varieties. Take $A = (A_1, \ldots, A_m)$ and $B = (B_1, \ldots, B_m)$ in $M_n^m$ and define an Hermitian inproduct on $M_n^m$ by

$$(A, B) = tr(A_1^* B_1 + \ldots + A_m^* B_m)$$

which is again invariant under the action of $U_n$ by simultaneous conjugation on $M_n^m$. The induced action of *Lie* $U_n$ on $M_n^m$ is given by

$$h.A = (hA_1 - A_1 h, \ldots, hA_m - A_m h)$$

This allows us to define the *real moment map* $\mu$ for the action of $U_n$ on $M_n^m$ to be the assignment

$$M_n^m \xrightarrow{\mu} (iLie\ U_n)^* \qquad A \longrightarrow (h \mapsto i(h.A, A))$$

and again using the nondegeneracy of the Killing form on $Lie\ U_n$ we have the identification

$$\mu^{-1}(\underline{0}) = \{A \in M_n^m \mid \sum_{i=1}^{m}(A_iA_i^* - A_i^*A_i) = 0\}$$

Again, the real moment map is determined by

$$M_n^m \xrightarrow{\mu_{\mathbb{R}}} Lie\ U_n \qquad A = (A_1, \ldots, A_m) \mapsto i[A, A^*] = i\sum_{j=1}^{m}[A_j, A_j^*]$$

We will show that there is a natural one-to-one correspondence between $U_n$-orbits in the set $\mu^{-1}(\underline{0})$ and closed $GL_n$-orbits in $M_n^m$. We first consider the properties of the real valued function $p_A$ defined as the norm on the orbit of any $A \in M_n^m$

$$GL_n \xrightarrow{p_A} \mathbb{R}_+ \qquad g \longrightarrow \|g.A\|^2$$

Because the Hermitian inproduct is invariant under $U_n$ we have $p_A(ug) = p_A(g)$ for any $u \in U_n$. If $Stab(A)$ denotes the stabilizer subgroup of $A \in GL_n$, then for any $s \in Stab(A)$ we also have $p_A(gs) = p_A(g)$ hence $p_A$ is constant along $U_n g Stab(A)$-cosets. We aim to prove that the critical points of $p_A$ are minima and that the minimum is attained if and only if $\mathcal{O}(A)$ is a closed $GL_n$-orbit.

Consider the restriction of $p_A$ to the maximal torus $T_n \hookrightarrow GL_n$ of invertible diagonal matrices. Then, $T_n \cap U_n = K = U_1 \times \ldots \times U_1$ is the subgroup

$$K = \{ \begin{bmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{bmatrix} \mid \forall i : |k_i| = 1 \}$$

The action by conjugation of $T_n$ on $M_n^m$ decomposes this space into *weight spaces*

$$M_n^m = M_n^m(0) \oplus \bigoplus_{i,j=1}^{n} M_n^m(\pi_i - \pi_j)$$

where $M_n^m(\pi_i - \pi_j) = \{A \in M_n^m \mid diag(t_1, \ldots, t_n).A = t_i t_j^{-1} A\}$. It follows from the definition of the Hermitian inproduct on $M_n^m$ that the different weightspaces are orthogonal to each other. We decompose $A \in M_n^m$ into eigenvectors for the $T_n$-action as

$$A = A(0) + \sum_{i,j=1}^{n} A(i,j) \qquad \text{with} \qquad \begin{cases} A(0) & \in M_n^m(0) \\ A(i,j) & \in M_n^m(\pi_i - \pi_j) \end{cases}$$

With this convention we have for $t = diag(t_1, \ldots, t_n) \in T_n$ that

$$
\begin{aligned}
p_A(t) &= \|A(0) + \sum_{i,j=1}^{n} t_i t_j^{-1} A(i,j)\|^2 \\
&= \|A(0)\|^2 + \sum_{i,j=1}^{n} t_i^2 t_j^{-2} \|A(i,j)\|^2
\end{aligned}
$$

where the last equality follows from the orthogonality of the different weight spaces. Further, remark that the stabilizer subgroup $Stab_T(A)$ of $A$ in $T$ can be identified with

$$
Stab_T(A) = \{t = diag(t_1, \ldots, t_n) \mid t_i = t_j \text{ if } A(i,j) \neq 0\}.
$$

As before, $p_A$ induces a function on double cosets $K\backslash T_n/Stab_T(A)$, in particular $p_M$ determines a real valued function on $K\backslash T_n \simeq \mathbb{R}^n$ (the isomorphism is given by the map $diag(t_1, \ldots, t_n) \xrightarrow{log} (log\ |t_1|, \ldots, log\ |t_n|))$. That is,

$$
\begin{tikzcd}
T_n \arrow[r, two heads, "log"] \arrow[d, "p_A"'] & K\backslash T_n \simeq \mathbb{R}^n \arrow[dl, "p'_A"] \arrow[d] \\
\mathbb{R}_+ & K\backslash T_n/Stab_T(A) \arrow[l, "p_A\text{''}"]
\end{tikzcd}
$$

where the function $p'_M$ is the *special function*

$$
p'_A(r_1, \ldots, r_n) = e^{2log\ \|A(0)\|} + \sum_{i,j:A(i,j)\neq 0} e^{2log\ \|A(i,j)\| + 2x_i - 2x_j}
$$

and where $K\backslash T_n/Stab_T(A)$ is the quotient space of $\mathbb{R}^n$ by the subspace $V_A$ which is the image of $Stab_T(A)$ under $log$

$$
V_A = \sum_{i:\nexists A(i,j)\neq 0} \mathbb{R}e_i + \sum_{i,j:A(i,j)\neq 0} \mathbb{R}(e_i - e_j)
$$

where $e_i$ are the standard basis vectors of $\mathbb{R}^n$. Let $\{i_1, \ldots, i_k\}$ be the minimal elements of the non-empty equivalence classes induced by the relation $i \sim j$ iff $A(i,j) \neq 0$, then

$$
\begin{cases}
K\backslash T_n/Stab_T(A) \simeq \sum_{j=1}^{k} \mathbb{R}e_{i_j} \\
p_A\text{''}(y_1, \ldots, y_k) = c_0 + \sum_{j=1}^{k} (\sum_{l(j)} c_{l(j)} e^{a_{l(j)} y_j})
\end{cases}
$$

for certain positive real numbers $c_0, c_{l(j)}$ and real numbers $a_{l(j)}$. But then, elementary calculus shows that the $k \times k$ matrix

$$\begin{bmatrix} \frac{\partial^2 p_A"}{\partial y_1 \partial y_1}(m) & \dots & \frac{\partial^2 p_A"}{\partial y_1 \partial y_k}(m) \\ \vdots & & \vdots \\ \frac{\partial^2 p_A"}{\partial y_k \partial y_1}(m) & \dots & \frac{\partial^2 p_A"}{\partial y_k \partial y_k}(m) \end{bmatrix}$$

is a positive definite diagonal matrix in every point $m \in \mathbb{R}^k$. That is, $p_A"$ is a strictly convex *Morse function* and if it has a critical point $m_0$ (that is, if all $\frac{\partial p_A"}{\partial y_i}(m_0) = 0$), it must be a unique minimum. Lifting this information from the double coset space $K \backslash T_n / Stab_T(A)$ to $T_n$ we have proved

**Proposition 2.15** *Let $T_n$ be the maximal torus of invertible diagonal matrices in $GL_n$ and consider the restriction of the function $GL_n \xrightarrow{p_A} \mathbb{R}_+$ to $T_n$ for $A \in M_n^m$, then*

1. *Any critical point of $p_A$ is a point where $p_A$ obtains its minimal value.*
2. *If $p_A$ obtains a minimal value, then*
   - *the set $V$ where $p_A$ obtains this minimum consists of a single $K - Stab_T(A)$ coset in $T_n$ and is connected.*
   - *the second order variation of $p_A$ at a point of $V$ in any direction not tangent to $V$ is positive.*

The same proof applies to all maximal tori $T$ of $GL_n$ which are defined over $\mathbb{R}$. Recall the *Cartan decomposition* of $GL_n$ which we proved before theorem 2.4 : any $g \in GL_n$ can be written as $g = udu'$ where $u, u' \in U_n$ and $d$ is a diagonal matrix with positive real entries. Using this fact we can now extend the above proposition to $GL_n$.

**Theorem 2.10** *Consider the function $GL_n \xrightarrow{p_A} \mathbb{R}_+$ for $A \in M_n^m$.*

1. *Any critical point of $p_A$ is a point where $p_A$ obtains its minimal value.*
2. *If $p_A$ obtains its minimal value, it does so on a single $U_n - Stab(A)$-coset.*

*Proof.* (1) : Because for any $h \in GL_n$ we have that $p_{h.A}(g) = p_A(gh)$ we may assume that $\mathbb{1}_n$ is the critical point of $p_A$. We have to prove that $p_A(g) \geq p_A(\mathbb{1}_n)$ for all $g \in GL_n$. By the Cartan decomposition $g = udu'$ whence $g = u"t$ where $u" = uu' \in U_n$ and $t = u'^{-1}du' \in T$ a maximal torus of $GL_n$ defined over $\mathbb{R}$. Because the Hermitian inproduct is invariant under $U_n$ we have that $p_A(g) = p_A(t)$. Because $\mathbb{1}_n$ is a critical point for the restriction of $p_A$ to $T$ we have by proposition 2.15 that $p_A(t) \geq p_A(\mathbb{1}_n)$, proving the claim.

(2) : Because for any $h \in GL_n$, $p_{h.A}(g) = p_A(gh)$ and $Stab(h.A) = hStab(A)h^{-1}$ we may assume that $p_A$ obtains its minimal value at $\mathbb{1}_n$. If $V$ denotes the subset of $GL_n$ where $p_A$ obtains

its minimal value we then have that $U_n Stab(A) \subset V$ and we have to prove the reverse inclusion. Assume $g \in V$ and write as before $g = u"t$ with $u" \in U_n$ and $t \in T$ a maximal torus defined over $\mathbb{R}$. Then, by unitary invariance of the inproduct, $t$ is a point of $T$ where the restriction of $p_A$ to $T$ obtains its minimal value $p_A(\mathbb{1}_n)$. By proposition 2.15 we conclude that $t \in K_T Stab_T(A)$ where $K_T = U_n \cap T$. But then,

$$V \subset U_n(\bigcup_T K_T Stab_T(A)) \subset U_n Stab(A)$$

where $T$ runs over all maximal tori of $GL_n$ which are defined over $\mathbb{R}$, finishing the proof. □

**Proposition 2.16** *The function $p_A : GL_n \longrightarrow \mathbb{R}_+$ obtains a minimal value if and only if $\mathcal{O}(A)$ is a closed orbit in $M_n^m$, that is, determines a semi-simple representation.*

*Proof.* If $\mathcal{O}(A)$ is closed then $p_A$ clearly obtains a minimal value. Conversely, assume that $\mathcal{O}(A)$ is not closed, that is, $A$ does not determine a semi-simple $n$-dimensional representation $M$ of $\mathbb{C}\langle x_1, \ldots, x_m \rangle$. By choosing a basis in $M$ (that is, possibly going to another point in the orbit $\mathcal{O}(A)$) we have a one-parameter subgroup $\mathbb{C}^* \xhookrightarrow{\lambda} T_n \hookrightarrow GL_n$ corresponding to the Jordan-Hölder filtration of $M$ with $\lim_{t \rightarrow 0} \lambda(t)A = B$ with $B$ corresponding to the semi-simplification of $M$. Now consider the restriction of $p'_A$ to $U_1 \backslash \mathbb{C}^* \simeq \mathbb{R}$, then as before we can write it uniquely in the form

$$p'_A(x) = \sum_i a_i e^{l_i x} \qquad a_i > 0, \quad l_1 < l_2 < \ldots < l_z$$

for some real numbers $l_i$ and some $z$. Because the above limit exists, the limit

$$\lim_{x \rightarrow -\infty} p'_A(x) \in \mathbb{R}$$

and hence none of the $l_i$ are negative. Further, because $\mathcal{O}(A) \neq \mathcal{O}(B)$ at least one of the $l_i$ must be positive. Therefore, $p'_A$ is a strictly increasing function on $\mathbb{R}$ whence never obtains a minimal value, whence neither does $p_A$. □

Finally, we have to clarify the connection between the function $p_A$ and the *real moment map*

$$\begin{cases} M_n^m \xrightarrow{\mu} (Lie\ U_n)^* & A \longrightarrow (h \mapsto (h.A, A)) \\ M_n^n \xrightarrow{\mu_{\mathbb{R}}} Lie\ U_n & A \longrightarrow i[A, A^*] \end{cases}$$

Assume $A \in M_n^m$ is such that $p_A$ has a critical point, which we may assume to be $\mathbb{1}_n$ by an argument as in the proof of theorem 2.10. Then, the differential in $\mathbb{1}_n$

$$(dp_A)_{\mathbb{1}_n} : M_n = T_{\mathbb{1}_n}\ GL_n \longrightarrow \mathbb{R} \qquad \text{satisfies} \qquad (dp_A)_{\mathbb{1}_n}(h) = 0 \quad \forall h \in M_n$$

Let us work out this differential

$$\begin{aligned} p_A(\mathbb{1}_n) + \epsilon (dp_A)_{\mathbb{1}_n}(h) &= tr((A^* + \epsilon(A^*h^* - h^*A^*)(A + \epsilon(hA - Ah)) \\ &= tr(A^*A) + \epsilon tr(A^*hA - A^*Ah + A^*h^*A - h^*A^*A) \\ &= tr(A^*A) + \epsilon tr((AA^* - A^*A)(h - h^*)) \end{aligned}$$

But then, vanishing of the differential for all $h \in M_n$ is equivalent by the nondegeneracy of the Killing form on $Lie\ U_n$ to

$$AA^* - A^*A = \sum_{i=1}^m A_iA_i^* - A_i^*A_i = 0$$

that is, to $A \in \mu_{\mathbb{R}}^{-1}(\underline{0})$. This concludes the proof of the main result on the real moment map for $M_n^m$.

**Theorem 2.11** *There are natural one-to-one correspondences between*

1. *isomorphism classes of semi-simple $n$-dimensional representations of $\mathbb{C}\langle x_1, \ldots, x_m\rangle$,*
2. *closed $GL_n$-orbits in $M_n^m$,*
3. *$U_n$-orbits in the subset $\mu_{\mathbb{R}}^{-1}(\underline{0}) = \{A \in M_n^m \mid \sum_{i=1}^m [A_i, A_i^*] = 0\}$.*

Let $A \in$ `alg@n` be an affine Cayley-Hamilton algebra of degree $n$, then we can embed the reduced variety of $\mathtt{trep}_n\ A$ in $M_n^m$ and obtain as a consequence :

**Theorem 2.12** *For $A \in$ `alg@n`, there are natural one-to-one correspondences between*

1. *isomorphism classes of semi-simple $n$-dimensional trace preserving representations of $A$,*
2. *closed $GL_n$-orbits in the representation variety $\mathtt{trep}_n\ A$,*
3. *$U_n$-orbits in the intersection $\mathtt{trep}_n\ A \cap \mu_{\mathbb{R}}^{-1}(\underline{0})$.*

## References.

The generalization of the Hilbert criterium, theorem 2.2 is due to D. Birkes [8]. The connection between semi-simple representations and closed orbits is due to M. Artin [2] . The geometric reconstruction result follows from theorem 1.17 and is due to C. Procesi [68] . The results on the real moment map are due to G. Kempf and L. Ness [42] . The treatment of the Hilbert criterium and invariant theory follows the textbook of H-P. Kraft [51] , that of the Gerstenhaber-Hesselink result owes to the exposition of M. Hazewinkel in [34].

# 3 — Etale Technology

Etale topology was introduced in algebraic geometry to bypass the coarseness of the Zariski topology in classification problems. Let us give an elementary example : the local classification of smooth varieties in the Zariski topology is a hopeless task, whereas in the étale topology there is just one local type of smooth variety in each dimension $d$, namely affine $d$-space $\mathbb{A}^d$. A major theme of this book is to generalize this result to `noncommutative geometry@n`.

Etale cohomology groups are used to classify *central simple algebras* over function fields of varieties. *Orders* in such central simple algebras (over the central structure sheaf) are an important class of Cayley-Hamilton algebras.

Over the years, one has tried to construct a suitable class of *smooth orders* which allows an étale local description. But, except in the case of curves and surfaces, no such classification is known say for orders of finite global dimension. In this book we introduce the class of *Cayley-smooth orders* which does allow an étale local description in arbitrary dimensions. In this chapter we will lay the foundations for this classification by investigating étale slices of representation varieties in semi-simple representations. In the next chapter we will then show that this local structure is determined by a combinatorial gadget : a *quiver setting.*

## 3.1 Etale topology

A closed subvariety $X \hookrightarrow \mathbb{C}^m$ can be equipped with the *Zariski topology* or with the much finer *analytic topology* . A major disadvantage of the coarseness of the Zariski topology is the failure to have an *implicit function* theorem in algebraic geometry. Etale morphisms are introduced to bypass this problem.

We will define étale morphisms which determine the *étale topology* . This is no longer a usual topology determined by subsets, but rather a *Grothendieck topology* determined by *covers* .

**Definition 3.1** *A finite morphism $A \xrightarrow{f} B$ of commutative $\mathbb{C}$-algebras is said to be* étale *if and only if $B = A[t_1, \ldots, t_k]/(f_1, \ldots, f_k)$ such that the* Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial t_1} & \ldots & \frac{\partial f_1}{\partial t_k} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial t_1} & \ldots & \frac{\partial f_k}{\partial t_k} \end{bmatrix}$$

*has a determinant which is a unit in $B$.*

Recall that by $\mathtt{spec}\ A$ we denote the *prime ideal spectrum* or the *affine scheme* of a commutative $\mathbb{C}$-algebra $A$ (even when $A$ is not affine as a $\mathbb{C}$-algebra). That is, $\mathtt{spec}\ A$ is the set of all *prime ideals* of $A$ equipped with the *Zariski topology* , that is the open subset are of the form

$$\mathbb{X}(I) = \{P \in \mathtt{spec}\ A \mid I \not\subset P\}$$

for some ideal $I \triangleleft A$. If $A$ is an affine $\mathbb{C}$-algebra, the points of the corresponding affine variety correspond to the *maximal ideals* of $A$ and the induced Zariski topology coincides with the one introduced before. In this chapter, however, not all $\mathbb{C}$-algebras will be affine.

**Example 3.1** Consider the morphism $\mathbb{C}[x,x^{-1}] \hookrightarrow \mathbb{C}[x,x^{-1}][\sqrt[n]{x}]$ and the induced map on the affine schemes

$$\mathtt{spec}\ \mathbb{C}[x,x^{-1}][\sqrt[n]{x}] \xrightarrow{\psi} \mathtt{spec}\ \mathbb{C}[x,x^{-1}] = \mathbb{C} - \{0\}.$$

Clearly, every point $\lambda \in \mathbb{C} - \{0\}$ has exactly $n$ preimages $\lambda_i = \zeta^i \sqrt[n]{\lambda}$. Moreover, in a neighborhood of $\lambda_i$, the map $\psi$ is a diffeomorphism. Still, we do not have an inverse map in algebraic geometry as $\sqrt[n]{x}$ is not a polynomial map. However, $\mathbb{C}[x,x^{-1}][\sqrt[n]{x}]$ is an étale extension of $\mathbb{C}[x,x^{-1}]$. In this way étale morphisms can be seen as an algebraic substitute for the failure of an inverse function theorem in algebraic geometry.

**Proposition 3.1** *Etale morphisms satisfy 'sorite', that is, they satisfy the commutative diagrams of figure 3.1. In these diagrams, et denotes an étale morphism, f.f. denotes a faithfully flat morphism and the dashed arrow is the étale morphism implied by 'sorite'.*

With these properties we can define a *Grothendieck topology* on the collection of all étale morphisms.

**Definition 3.2** *The* étale site *of $A$, which we will denote by $\mathtt{A_{et}}$ is the category with*

- *objects : the étale extensions $A \xrightarrow{f} B$ of $A$*
- *morphisms : compatible $A$-algebra morphisms*

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ B_1 \arrow[rr, "\phi"] & & B_2 \end{tikzcd}$$

*By proposition 3.1 all morphisms in $\mathtt{A_{et}}$ are étale. We can turn $\mathtt{A_{et}}$ into a Grothendieck topology by defining*

$$\begin{tikzcd} A' \arrow[r, dotted, "et"] & A' \otimes_A B \\ A \arrow[u] \arrow[r, "et"] & B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, "et"] & \\ A \arrow[ur, "et"] \arrow[rr, dotted, "et"] & & C \end{tikzcd}$$

$(basechange)$ $\qquad$ $(composition)$

$$\begin{tikzcd} A' \arrow[r, "et"] & A' \otimes_A B \\ A \arrow[u, "f.f."] \arrow[r, dotted, "et"] & B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "et"'] \arrow[dr, "et"] & \\ B \arrow[rr, dotted, "et", "A-alg"'] & & B' \end{tikzcd}$$

$(descent)$ $\qquad$ $(morphisms)$

Figure 3.1: Sorite for étale morphisms

- *cover : a collection* $\mathcal{C} = \{B \xrightarrow{f_i} B_i\}$ *in* $\mathtt{A_{et}}$ *such that*

$$\mathtt{spec}\ B = \cup_i\ Im\ (\mathtt{spec}\ B_i \xrightarrow{f} \mathtt{spec}\ B\ )$$

**Definition 3.3** *An* étale presheaf *of groups on* $\mathtt{A_{et}}$ *is a functor*

$$\mathbb{G} : \mathtt{A_{et}} \longrightarrow \mathtt{groups}$$

*In analogy with usual (pre)sheaf notation we denote for each*

- *object* $B \in \mathtt{A_{et}}$ *the* global sections $\Gamma(B, \mathbb{G}) = \mathbb{G}(B)$
- *morphism* $B \xrightarrow{\phi} C$ *in* $\mathtt{A_{et}}$ *the* restriction map $Res^B_C = \mathbb{G}(\phi) : \mathbb{G}(B) \longrightarrow \mathbb{G}(C)$ *and* $g \mid C = \mathbb{G}(\phi)(g)$.

*An étale presheaf* $\mathbb{G}$ *is an* étale sheaf *provided for every* $B \in \mathtt{A_{et}}$ *and every cover* $\{B \longrightarrow B_i\}$ *we have exactness of the* equalizer diagram

$$0 \longrightarrow \mathbb{G}(B) \longrightarrow \prod_i \mathbb{G}(B_i) \rightrightarrows \prod_{i,j} \mathbb{G}(B_i \otimes_B B_j)$$

**Example 3.2 (Constant sheaf )** If $G$ is a group, then

$$\mathbb{G} : \mathtt{A_{et}} \longrightarrow \mathtt{groups} \quad B \mapsto G^{\oplus \pi_0(B)}$$

is a sheaf where $\pi_0(B)$ is the number of connected components of $\mathtt{spec}\ B$.

**Example 3.3 (Multiplicative group $\mathbb{G}_m$ )** The functor

$$\mathbb{G}_m : \mathtt{A_{et}} \longrightarrow \mathtt{groups} \quad B \mapsto B^*$$

is a sheaf on $\mathtt{A_{et}}$.

A sequence of sheaves of Abelian groups on $\mathtt{A_{et}}$ is said to be exact

$$\mathbb{G}' \xrightarrow{f} \mathbb{G} \xrightarrow{g} \mathbb{G}"$$

if for every $B \in \mathtt{A_{et}}$ and $s \in \mathbb{G}(B)$ such that $g(s) = 0 \in \mathbb{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in $\mathtt{A_{et}}$ and sections $t_i \in \mathbb{G}'(B_i)$ such that $f(t_i) = s \mid B_i$.

**Example 3.4 (Roots of unity $\mu_n$)** We have a sheaf morphism

$$\mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m$$

and we denote the kernel with $\mu_n$. As $A$ is a $\mathbb{C}$-algebra we can identify $\mu_n$ with the constant sheaf $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ via the isomorphism $\zeta^i \mapsto i$ after choosing a primitive $n$-th root of unity $\zeta \in \mathbb{C}$.

**Lemma 3.1** *The Kummer sequence of sheaves of Abelian groups*

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 0$$

*is exact on* $\mathtt{A_{et}}$ *(but not necessarily on* $\mathtt{spec}\ A$ *with the Zariski topology).*

*Proof.* We only need to verify surjectivity. Let $B \in \mathtt{A_{et}}$ and $b \in \mathbb{G}_m(B) = B^*$. Consider the étale extension $B' = B[t]/(t^n - b)$ of $B$, then $b$ has an $n$-th root over in $\mathbb{G}_m(B')$. Observe that this $n$-th root does not have to belong to $\mathbb{G}_m(B)$. $\square$

If $\mathfrak{p}$ is a prime ideal of $A$ we will denote with $\mathbf{k}_{\mathfrak{p}}$ the algebraic closure of the field of fractions of $A/\mathfrak{p}$. An *étale neighborhood* of $\mathfrak{p}$ is an étale extension $B \in \mathtt{A_{et}}$ such that the diagram below is commutative

$$\begin{tikzcd} A \arrow[r, "nat"] \arrow[d, "et"'] & \mathbf{k}_{\mathfrak{p}} \\ B \arrow[ur] & \end{tikzcd}$$

The analogue of the localization $A_{\mathfrak{p}}$ for the étale topology is the *strict Henselization*

$$A_{\mathfrak{p}}^{sh} = \underrightarrow{lim}\ B$$

where the limit is taken over all étale neighborhoods of $\mathfrak{p}$.

Recall that a local algebra $L$ with maximal ideal $m$ and residue map $\pi : L \twoheadrightarrow L/m = k$ is said to be *Henselian* if the following condition holds. Let $f \in L[t]$ be a monic polynomial such that $\pi(f)$ factors as $g_0.h_0$ in $k[t]$, then $f$ factors as $g.h$ with $\pi(g) = g_0$ and $\pi(h) = h_0$. If $L$ is Henselian then tensoring with $k$ induces an equivalence of categories between the étale $A$-algebras and the étale $k$-algebras.

An Henselian local algebra is said to be *strict Henselian* if and only if its residue field is algebraically closed. Thus, a strict Henselian ring has no proper finite étale extensions and can be viewed as a local algebra for the étale topology.

**Example 3.5 (The algebraic functions $\mathbb{C}\{x_1,\ldots,x_d\}$)** Consider the local algebra of $\mathbb{C}[x_1,\ldots,x_d]$ in the maximal ideal $(x_1,\ldots,x_d)$, then the Henselization and strict Henselization are both equal to

$$\mathbb{C}\{x_1,\ldots,x_d\}$$

the ring of *algebraic functions* . That is, the subalgebra of $\mathbb{C}[[x_1,\ldots,x_d]]$ of formal power-series consisting of those series $\phi(x_1,\ldots,x_d)$ which are algebraically dependent on the coordinate functions $x_i$ over $\mathbb{C}$. In other words, those $\phi$ for which there exists a non-zero polynomial $f(x_i,y) \in \mathbb{C}[x_1,\ldots,x_d,y]$ with $f(x_1,\ldots,x_d,\phi(x_1,\ldots,x_d)) = 0$.

These algebraic functions may be defined implicitly by polynomial equations. Consider a system of equations

$$f_i(x_1,\ldots,x_d;y_1,\ldots,y_m) = 0 \text{ for } f_i \in \mathbb{C}[x_i,y_j] \text{ and } 1 \leq i \leq m$$

Suppose there is a solution in $\mathbb{C}$ with

$$x_i = 0 \text{ and } y_j = y_j^o$$

such that the Jacobian matrix is non-zero

$$det\ (\frac{\partial f_i}{\partial y_j}(0,\ldots,0;y_1^o,\ldots,y_m^0)) \neq 0$$

Then, the system can be solved uniquely for power series $y_j(x_1,\ldots,x_d)$ with $y_j(0,\ldots,0) = y_j^o$ by solving inductively for the coefficients of the series. One can show that such implicitly defined series $y_j(x_1,\ldots,x_d)$ are algebraic functions and that, conversely, any algebraic function can be obtained in this way.

If $\mathbb{G}$ is a sheaf on $\mathtt{A_{et}}$ and $\mathfrak{p}$ is a prime ideal of $A$, we define the *stalk* of $\mathbb{G}$ in $\mathfrak{p}$ to be

$$\mathbb{G}_{\mathfrak{p}} = \underrightarrow{lim}\ \mathbb{G}(B)$$

where the limit is taken over all étale neighborhoods of $\mathfrak{p}$. One can verify mono- epi- or isomorphisms of sheaves by checking it in all the stalks.

If $A$ is an affine algebra defined over an algebraically closed field, then it suffices to verify it in the maximal ideals of $A$.

Before we define cohomology of sheaves on $\mathtt{A_{et}}$ let us recall the definition of *derived functors* . Let $\mathcal{A}$ be an *Abelian category* . An object $I$ of $\mathcal{A}$ is said to be *injective* if the functor

$$\mathcal{A} \longrightarrow \texttt{abelian} \quad M \mapsto Hom_{\mathcal{A}}(M, I)$$

is exact. We say that $\mathcal{A}$ has enough injectives if, for every object $M$ in $\mathcal{A}$, there is a monomorphism $M \hookrightarrow I$ into an injective object.

If $\mathcal{A}$ has enough injectives and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor from $\mathcal{A}$ into a second Abelian category $\mathcal{B}$, then there is an essentially unique sequence of functors

$$R^i\ f : \mathcal{A} \longrightarrow \mathcal{B} \quad i \geq 0$$

called the *right derived functors* of $f$ satisfying the following properties

- $R^0\ f = f$
- $R^i\ I = 0$ for $I$ injective and $i > 0$
- For every short exact sequence in $\mathcal{A}$

  $$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

  there are connecting morphisms $\delta^i : R^i\ f(M") \longrightarrow R^{i+1}\ f(M')$ for $i \geq 0$ such that we have a long exact sequence

  $$\ldots \longrightarrow R^i\ f(M) \longrightarrow R^i\ f(M") \xrightarrow{\delta^i} R^{i+1}\ f(M') \longrightarrow R^{i+1}\ f(M) \longrightarrow \ldots$$

- For any morphism $M \longrightarrow N$ there are morphisms $R^i\ f(M) \longrightarrow R^i\ f(N)$ for $i \geq 0$

In order to compute the objects $R^i\ f(M)$ define an object $N$ in $\mathcal{A}$ to be $f$-*acyclic* if $R^i\ f(M) = 0$ for all $i > 0$. If we have an *acyclic resolution* of $M$

$$0 \longrightarrow M \longrightarrow N_0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \ldots$$

by $f$-acyclic object $N_i$, then the objects $R^i\ f(M)$ are canonically isomorphic to the cohomology objects of the complex

$$0 \longrightarrow f(N_0) \longrightarrow f(N_1) \longrightarrow f(N_2) \longrightarrow \ldots$$

One can show that all injectives are $f$-acyclic and hence that derived objects of $M$ can be computed from an *injective resolution* of $M$.

Now, let $\mathbf{S}^{ab}(\mathtt{A_{et}})$ be the category of all sheaves of Abelian groups on $\mathtt{A_{et}}$. This is an Abelian category having enough injectives whence we can form right derived functors of left exact functors. In particular, consider the global section functor

$$\Gamma : \mathbf{S}^{ab}(\mathtt{A_{et}}) \longrightarrow \mathtt{abelian} \quad \mathbb{G} \mapsto \mathbb{G}(A)$$

which is left exact. The right derived functors of $\Gamma$ will be called the *étale cohomology functors* and we denote

$$R^i\ \Gamma(\mathbb{G}) = H^i_{et}(A, \mathbb{G})$$

In particular, if we have an exact sequence of sheaves of Abelian groups $0 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 0$, then we have a long exact cohomology sequence

$$\ldots \longrightarrow H^i_{et}(A, \mathbb{G}) \longrightarrow H^i_{et}(A, \mathbb{G}") \longrightarrow H^{i+1}_{et}(A, \mathbb{G}') \longrightarrow \ldots$$

If $\mathbb{G}$ is a sheaf of non-Abelian groups (written multiplicatively), we cannot define cohomology groups. Still, one can define a *pointed set* $H^1_{et}(A, \mathbb{G})$ as follows. Take an étale cover $\mathcal{C} = \{A \longrightarrow A_i\}$ of $A$ and define a 1-cocycle for $\mathcal{C}$ with values in $\mathbb{G}$ to be a family

$$g_{ij} \in \mathbb{G}(A_{ij}) \text{ with } A_{ij} = A_i \otimes_A A_j$$

satisfying the cocycle condition

$$(g_{ij} \mid A_{ijk})(g_{jk} \mid A_{ijk}) = (g_{ik} \mid A_{ijk})$$

where $A_{ijk} = A_i \otimes_A A_j \otimes_A A_k$.

Two cocycles $g$ and $g'$ for $\mathcal{C}$ are said to be cohomologous if there is a family $h_i \in \mathbb{G}(A_i)$ such that for all $i, j \in I$ we have

$$g'_{ij} = (h_i \mid A_{ij}) g_{ij} (h_j \mid A_{ij})^{-1}$$

This is an equivalence relation and the set of cohomology classes is written as $H^1_{et}(\mathcal{C}, \mathbb{G})$. It is a pointed set having as its distinguished element the cohomology class of $g_{ij} = 1 \in \mathbb{G}(A_{ij})$ for all $i, j \in I$.

We then define the non-Abelian first *cohomology pointed set* as

$$H^1_{et}(A, \mathbb{G}) = \underrightarrow{lim}\ H^1_{et}(\mathcal{C}, \mathbb{G})$$

where the limit is taken over all étale coverings of $A$. It coincides with the previous definition in case $\mathbb{G}$ is Abelian.

A sequence $1 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 1$ of sheaves of groups on $\mathtt{A_{et}}$ is said to be exact if for every $B \in \mathtt{A_{et}}$ we have

- $\mathbb{G}'(B) = Ker\ \mathbb{G}(B) \longrightarrow \mathbb{G}"(B)$
- For every $g" \in \mathbb{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in $\mathtt{A_{et}}$ and sections $g_i \in \mathbb{G}(B_i)$ such that $g_i$ maps to $g" \mid B$.

**Proposition 3.2** *For an exact sequence of groups on* $\mathtt{A_{et}}$

$$1 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 1$$

*there is associated an exact sequence of pointed sets*

$$1 \longrightarrow \mathbb{G}'(A) \longrightarrow \mathbb{G}(A) \longrightarrow \mathbb{G}"(A) \xrightarrow{\delta} H^1_{et}(A,\mathbb{G}') \longrightarrow$$
$$\longrightarrow H^1_{et}(A,\mathbb{G}) \longrightarrow H^1_{et}(A,\mathbb{G}") \dashrightarrow H^2_{et}(A,\mathbb{G}')$$

*where the last map exists when* $\mathbb{G}'$ *is contained in the center of* $\mathbb{G}$ *(and therefore is Abelian whence* $H^2$ *is defined).*

*Proof.* The connecting map $\delta$ is defined as follows. Let $g" \in \mathbb{G}"(A)$ and let $\mathcal{C} = \{A \longrightarrow A_i\}$ be an étale covering of $A$ such that there are $g_i \in \mathbb{G}(A_i)$ that map to $g \mid A_i$ under the map $\mathbb{G}(A_i) \longrightarrow \mathbb{G}"(A_i)$. Then, $\delta(g)$ is the class determined by the one cocycle

$$g_{ij} = (g_i \mid A_{ij})^{-1}(g_j \mid A_{ij})$$

with values in $\mathbb{G}'$. The last map can be defined in a similar manner, the other maps are natural and one verifies exactness. $\square$

The main applications of this non-Abelian cohomology to non-commutative algebra is as follows. Let $\Lambda$ be a not necessarily commutative $A$-algebra and $M$ an $A$-module. Consider the sheaves of groups $\mathtt{Aut}(\Lambda)$ resp. $\mathtt{Aut}(M)$ on $\mathtt{A_{et}}$ associated to the presheaves

$$B \mapsto Aut_{B-alg}(\Lambda \otimes_A B) \text{ resp. } B \mapsto Aut_{B-mod}(M \otimes_A B)$$

for all $B \in \mathtt{A_{et}}$. A *twisted form* of $\Lambda$ (resp. $M$) is an $A$-algebra $\Lambda'$ (resp. an $A$-module $M'$) such that there is an étale cover $\mathcal{C} = \{A \longrightarrow A_i\}$ of $A$ such that there are isomorphisms

$$\begin{cases} \Lambda \otimes_A A_i \xrightarrow{\phi_i} \Lambda' \otimes_A A_i \\ M \otimes_A A_i \xrightarrow{\psi_i} M' \otimes_A A_i \end{cases}$$

of $A_i$-algebras (resp. $A_i$-modules). The set of $A$-algebra isomorphism classes (resp. $A$-module isomorphism classes) of twisted forms of $\Lambda$ (resp. $M$) is denoted by $Tw_A(\Lambda)$ (resp. $Tw_A(M)$). To a twisted form $\Lambda'$ one associates a cocycle on $\mathcal{C}$

$$\alpha_{\Lambda'} = \alpha_{ij} = \phi_i^{-1} \circ \phi_j$$

with values in Aut$(\Lambda)$. Moreover, one verifies that two twisted forms are isomorphic as $A$-algebras if their cocycles are cohomologous. That is, there are embeddings

$$\begin{cases} Tw_A(\Lambda) \hookrightarrow H^1_{et}(A, \mathtt{Aut}(\Lambda)) \\ Tw_A(M) \hookrightarrow H^1_{et}(A, \mathtt{Aut}(M)) \end{cases}$$

In favorable situations one can even show bijectivity. In particular, this is the case if the automorphisms group is a smooth affine algebraic group-scheme.

**Example 3.6 (Azumaya algebras)** Consider $\Lambda = M_n(A)$, then the automorphism group is $\mathtt{PGL}_n$ and twisted forms of $\Lambda$ are classified by elements of the cohomology group

$$H^1_{et}(A, \mathtt{PGL}_n)$$

These twisted forms are precisely the *Azumaya algebras* of rank $n^2$ with center $A$. When $A$ is an affine commutative $\mathbb{C}$-algebra and $\Lambda$ is an $A$-algebra with center $A$, then $\Lambda$ is an *Azumaya algebra* of rank $n^2$ if and only if

$$\frac{\Lambda}{\Lambda\mathfrak{m}\Lambda} \simeq M_n(\mathbb{C})$$

for every maximal ideal $\mathfrak{m}$ of $A$.

Azumaya algebras arise in representation theory as follows. Let $A$ be this time a *noncommutative* affine $\mathbb{C}$-algebra and assume that the following two conditions are satisfied

- $A$ has a simple representation of dimension $n$,
- $\mathtt{rep}_n\ A$ is an irreducible variety.

Then $\oint_n A = \mathbb{C}[\mathtt{rep}_n\ A]^{GL_n}$ is a domain (whence $\mathtt{iss}_n\ A$ is irreducible) and we have an onto trace preserving algebra map corresponding to the simple representation

$$\int_n A = M_n(\mathbb{C}[\mathtt{rep}_n A])^{GL_n} \overset{\phi}{\twoheadrightarrow} M_n(\mathbb{C})$$

Lift the standard basis $e_{ij}$ of $M_n(\mathbb{C})$ to elements $a_{ij} \in \int_n A$ and consider the determinant $d$ of the $n^2 \times n^2$ matrix $(tr(a_{ij}a_{kl}))_{ij,kl}$ with values in $\oint_n A$. Then $d \neq 0$ and consider the Zariski open affine subset of $\mathtt{iss}_n\ A$

$$\mathbb{X}(d) = \{ \int_n A \xrightarrow{\psi} M_n(\mathbb{C}) \ | \ \psi \text{ semisimple and } det(tr(\psi(a_{ij})\psi(a_{kl}))) \neq 0 \}$$

If $\psi \in \mathbb{X}(d)$, then $\psi : \int_n A \longrightarrow M_n(\mathbb{C})$ is onto as the $\psi(a_{ij})$ form a basis of $M_n(\mathbb{C})$ whence $\psi$ determines a simple $n$-dimensional representation.

**Proposition 3.3** *With notations as above,*

1. *The localization of $\int_n A$ at the central multiplicative set $\{1, d, d^2, \ldots\}$ is an affine Azumaya algebra with center $\mathbb{C}[\mathbb{X}(d)]$ which is the localization of $\oint_n A$ at this multiplicative set.*

2. *The restriction of the quotient map* $\mathtt{rep}_n\ A \xrightarrow{\pi} \mathtt{iss}_n\ A$ *to the open set* $\pi^{-1}(\mathbb{X}(d))$ *is a principal* $PGL_n$*-fibration and determines an element in*

$$H^1_{et}(\mathbb{C}[\mathbb{X}(d)], \mathtt{PGL}_n)$$

*giving the class of the Azumaya algebra.*

*Proof.* (1) : If $m = Ker\ \psi$ is the maximal ideal of $\mathbb{C}[\mathbb{X}(d)]$ corresponding to the semisimple representation $\psi : \int_n A \longrightarrow M_n(\mathbb{C})$, then we have seen that the quotient

$$\frac{\int_n A}{\int_n A m \int_n A} \simeq M_n(\mathbb{C})$$

whence $\int_n A \otimes_{\oint_n A} \mathbb{C}[\mathbb{X}(d)]$ is an Azumaya algebra. (2) will follow from the theory of Knop-Luna slices and will be proved in chapter 5. $\square$

An Azumaya algebra over a field is a central simple algebra. Under the above conditions we have that

$$\int_n A \otimes_{\oint_n A} \mathbb{C}(\mathtt{iss}_n\ A)$$

is a central simple algebra over the functionfield of $\mathtt{iss}_n\ A$ and hence determines a class in its Brauer group, which is an important birational invariant. In the following section we recall the cohomological description of Brauer groups of fields.

## 3.2 Central simple algebras

Let $K$ be a field of characteristic zero, choose an algebraic closure $\mathbb{K}$ with *absolute Galois group* $G_K = Gal(\mathbb{K}/K)$.

**Lemma 3.2** *The following are equivalent*

1. $K \longrightarrow A$ *is étale*
2. $A \otimes_K \mathbb{K} \simeq \mathbb{K} \times \ldots \times \mathbb{K}$
3. $A = \prod L_i$ *where* $L_i/K$ *is a finite field extension*

*Proof.* Assume (1), then $A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ where $f_i$ have invertible Jacobian matrix. Then $A \otimes \mathbb{K}$ is a smooth commutative algebra (hence reduced) of dimension 0 so (2) holds.

Assume (2), then

$$Hom_{K-alg}(A, \mathbb{K}) \simeq Hom_{\mathbb{K}-alg}(A \otimes \mathbb{K}, \mathbb{K})$$

has $dim_{\mathbb{K}}(A \otimes \mathbb{K})$ elements. On the other hand we have by the *Chinese remainder theorem* that

$$A/Jac\ A = \prod_i L_i$$

with $L_i$ a finite field extension of $K$. However,

$$dim_{\mathbb{K}}(A \otimes \mathbb{K}) = \sum_i dim_K(L_i) = dim_K(A/Jac\ A) \leq dim_K(A)$$

and as both ends are equal $A$ is reduced and hence $A = \prod_i L_i$ whence (3).

Assume (3), then each $L_i = K[x_i]/(f_i)$ with $\partial f_i/\partial x_i$ invertible in $L_i$. But then $A = \prod L_i$ is étale over $K$ whence (1). □

To every finite étale extension $A = \prod L_i$ we can associate the finite set $rts(A) = Hom_{K-alg}(A, \mathbb{K})$ on which the Galois group $G_K$ acts via a finite quotient group. If we write $A = K[t]/(f)$, then $rts(A)$ is the *set of roots* in $\mathbb{K}$ of the polynomial $f$ with obvious action by $G_K$. Galois theory, in the interpretation of Grothendieck, can now be stated as

**Proposition 3.4** *The functor*

$$\mathtt{K_{et}} \xrightarrow{rts(-)} \mathtt{finite}\ G_K - \mathtt{sets}$$

*is an anti-equivalence of categories.*

We will now give a similar interpretation of the Abelian sheaves on $\mathtt{K_{et}}$. Let $\mathbb{G}$ be a presheaf on $\mathtt{K_{et}}$. Define

$$M_{\mathbb{G}} = \underrightarrow{lim}\ \mathbb{G}(L)$$

where the limit is taken over all subfields $L \hookrightarrow \mathbb{K}$ which are finite over $K$. The Galois group $G_K$ acts on $\mathbb{G}(L)$ on the left through its action on $L$ whenever $L/K$ is Galois. Hence, $G_K$ acts an $M_{\mathbb{G}}$ and $M_{\mathbb{G}} = \cup M_{\mathbb{G}}^H$ where $H$ runs through the *open subgroups* (that is, containing a normal subgroup having a finite quotient) of $G_K$. That is, $M_{\mathbb{G}}$ is a *continuous $G_K$-module* .

Conversely, given a continuous $G_K$-module $M$ we can define a presheaf $\mathbb{G}_M$ on $\mathtt{K_{et}}$ such that

- $\mathbb{G}_M(L) = M^H$ where $H = G_L = Gal(\mathbb{K}/L)$.
- $\mathbb{G}_M(\prod L_i) = \prod \mathbb{G}_M(L_i)$.

One verifies that $\mathbb{G}_M$ is a sheaf of Abelian groups on $\mathtt{K_{et}}$.

**Theorem 3.1** *There is an equivalence of categories*

$$\mathbf{S}(\mathtt{K_{et}}) \rightleftarrows G_K - \mathtt{mod}$$

*induced by the correspondences $\mathbb{G} \mapsto M_{\mathbb{G}}$ and $M \mapsto \mathbb{G}_M$. Here, $G_K - \mathtt{mod}$ is the category of continuous $G_K$-modules.*

*Proof.* A $G_K$-morphism $M \longrightarrow M'$ induces a morphism of sheaves $\mathbb{G}_M \longrightarrow \mathbb{G}_{M'}$. Conversely, if $H$ is an open subgroup of $G_K$ with $L = \mathbb{K}^H$, then if $\mathbb{G} \xrightarrow{\phi} \mathbb{G}'$ is a sheafmorphism, $\phi(L) : \mathbb{G}(L) \longrightarrow \mathbb{G}'(L)$ commutes with the action of $G_K$ by functoriality of $\phi$. Therefore, $\underrightarrow{lim}\ \phi(L)$ is a $G_K$-morphism $M_{\mathbb{G}} \longrightarrow M_{\mathbb{G}'}$.

One verifies easily that $Hom_{G_K}(M, M') \longrightarrow Hom(\mathbb{G}_M, \mathbb{G}_{M'})$ is an isomorphism and that the canonical map $\mathbb{G} \longrightarrow \mathbb{G}_{M_{\mathbb{G}}}$ is an isomorphism. □

In particular, we have that $\mathbb{G}(K) = \mathbb{G}(\mathbb{K})^{G_K}$ for every sheaf $\mathbb{G}$ of Abelian groups on $\mathtt{K_{et}}$ and where $\mathbb{G}(\mathbb{K}) = M_{\mathbb{G}}$. Hence, the right derived functors of $\Gamma$ and $(-)^G$ coincide for Abelian sheaves.

The category $G_K - \mathtt{mod}$ of continuous $G_K$-modules is Abelian having enough injectives. Therefore, the left exact functor

$$(-)^G : G_K - \mathtt{mod} \longrightarrow \mathtt{abelian}$$

admits right derived functors. They are called the *Galois cohomology groups* and denoted

$$R^i\ M^G = H^i(G_K, M)$$

Therefore, we have.

**Proposition 3.5** *For any sheaf of Abelian groups $\mathbb{G}$ on $\mathtt{K_{et}}$ we have a group isomorphism*

$$H^i_{et}(K, \mathbb{G}) \simeq H^i(G_K, \mathbb{G}(\mathbb{K}))$$

Therefore, étale cohomology is a natural extension of Galois cohomology to arbitrary commutative algebras. The following definition-characterization of central simple algebras is classical, see for example [66].

**Proposition 3.6** *Let $A$ be a finite dimensional $K$-algebra. The following are equivalent :*

1. *$A$ has no proper twosided ideals and the center of $A$ is $K$.*
2. *$A_{\mathbb{K}} = A \otimes_K \mathbb{K} \simeq M_n(\mathbb{K})$ for some $n$.*
3. *$A_L = A \otimes_K L \simeq M_n(L)$ for some $n$ and some finite Galois extension $L/K$.*
4. *$A \simeq M_k(D)$ for some $k$ where $D$ is a division algebra of dimension $l^2$ with center $K$.*

The last part of this result suggests the following definition. Call two central simple algebras $A$ and $A'$ equivalent if and only if $A \simeq M_k(\Delta)$ and $A' \simeq M_l(\Delta)$ with $\Delta$ a division algebra. From the second characterization it follows that the tensorproduct of two central simple $K$-algebras is again central simple. Therefore, we can equip the set of equivalence classes of central simple algebras with a product induced from the tensorproduct. This product has the class $[K]$ as unit element and $[\Delta]^{-1} = [\Delta^{opp}]$, the opposite algebra as $\Delta \otimes_K \Delta^{opp} \simeq End_K(\Delta) = M_{l^2}(K)$. This group is called the *Brauer group* and is denoted $Br(K)$. We will quickly recall its cohomological description, all of which is classical.

$GL_r$ is an affine smooth algebraic group defined over $K$ and is the automorphism group of a vectorspace of dimension $r$. It defines a sheaf of groups on $\mathtt{K_{et}}$ that we will denote by $\mathtt{GL}_r$. Using the fact that the first cohomology classifies twisted forms of vectorspaces of dimension $r$ we have

**Lemma 3.3**

$$H^1_{et}(K, \mathtt{GL}_r) = H^1(G_K, GL_r(\mathbb{K})) = 0$$

*In particular, we have* 'Hilbert's theorem 90'

$$H^1_{et}(K, \mathbb{G}_m) = H^1(G_K, \mathbb{K}^*) = 0$$

*Proof.* The cohomology group classifies $K$-module isomorphism classes of twisted forms of $r$-dimensional vectorspaces over $K$. There is just one such class. □

$PGL_n$ is an affine smooth algebraic group defined over $K$ and it is the automorphism group of the $K$-algebra $M_n(K)$. It defines a sheaf of groups on $\mathtt{K_{et}}$ denoted by $\mathtt{PGL}_n$. By proposition 3.6 we know that any central simple $K$-algebra $\Delta$ of dimension $n^2$ is a twisted form of $M_n(K)$. Therefore,

**Lemma 3.4** *The pointed set of $K$-algebra isomorphism classes of central simple algebras of dimension $n^2$ over $K$ coincides with the cohomology set*

$$H^1_{et}(K, \mathtt{PGL}_n) = H^1(G_K, PGL_n(\mathbb{K}))$$

**Theorem 3.2** *There is a natural inclusion*

$$H^1_{et}(K, \mathtt{PGL}_n) \hookrightarrow H^2_{et}(K, \mu_n) = Br_n(K)$$

*where $Br_n(K)$ is the $n$-torsion part of the Brauer group of $K$. Moreover,*

$$Br(K) = H^2_{et}(K, \mathbb{G}_m)$$

*is a torsion group.*

*Proof.* Consider the exact commutative diagram of sheaves of groups on $\mathtt{K_{et}}$ of figure 3.2. Taking cohomology of the second exact sequence we obtain

$$GL_n(K) \xrightarrow{det} K^* \longrightarrow H^1_{et}(K, \mathtt{SL}_n) \longrightarrow H^1_{et}(K, \mathtt{GL}_n)$$

where the first map is surjective and the last term is zero, whence

$$H^1_{et}(K, \mathtt{SL}_n) = 0$$

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r, "(-)^n"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathtt{SL}_n \arrow[r] \arrow[d] & \mathtt{GL}_n \arrow[r, "det"] \arrow[d] & \mathbb{G}_m \arrow[r] & 1 \\
 & \mathtt{PGL}_n \arrow[r, equal] \arrow[d] & \mathtt{PGL}_n \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

Figure 3.2: Brauer group diagram.

Taking cohomology of the first vertical exact sequence we get

$$H^1_{et}(K, \mathtt{SL}_n) \longrightarrow H^1_{et}(K, \mathtt{PGL}_n) \longrightarrow H^2_{et}(K, \mu_n)$$

from which the first claim follows.

As for the second assertion, taking cohomology of the first exact sequence we get

$$H^1_{et}(K, \mathbb{G}_m) \longrightarrow H^2_{et}(K, \mu_n) \longrightarrow H^2_{et}(K, \mathbb{G}_m) \xrightarrow{n.} H^2_{et}(K, \mathbb{G}_m)$$

By Hilbert 90, the first term vanishes and hence $H^2_{et}(K, \mu_n)$ is equal to the $n$-torsion of the group

$$H^2_{et}(K, \mathbb{G}_m) = H^2(G_K, \mathbb{K}^*) = Br(K)$$

where the last equality follows from the crossed product result, see for example [66]. $\square$

$E_1^{p,q} \quad =$

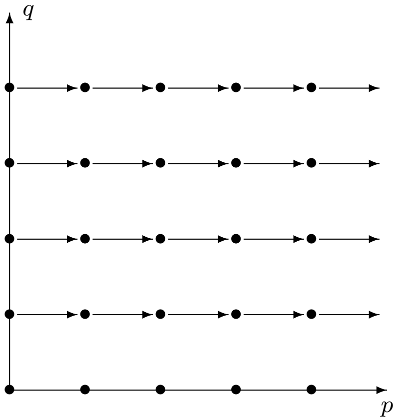



Figure 3.3: level 1

So far, the field $K$ was arbitrary. If $K$ is of transcendence degree $d$, this will put restrictions on the 'size' of the Galois group $G_K$. In particular this will enable us to show in section 3.4 that $H^i(G_K, \mu_n) = 0$ for $i > d$. But first, we need to recall the definition of spectral sequences.

## 3.3 Spectral sequences

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be Abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and consider left exact functors

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

Let the functors be such that $f$ maps injectives of $\mathcal{A}$ to $g$-acyclic objects in $\mathcal{B}$, that is $R^i\ g(f\ I) = 0$ for all $i > 0$. Then, there are connections between the objects

$$R^p\ g(R^q\ f(A)) \text{ and } R^n\ gf(A)$$

for all objects $A \in \mathcal{A}$. These connections can be summarized by giving a spectral sequence

**Theorem 3.3** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories with $\mathcal{A}, \mathcal{B}$ having enough injectives and left exact functors*

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

*such that $f$ takes injectives to $g$-acyclics.*

*Then, for any object $A \in \mathcal{A}$ there is a* spectral sequence

$$E_2^{p,q} = R^p\ g(R^q\ f(A)) \Longrightarrow R^n\ gf(A)$$

$E_2^{p,q} \quad =$

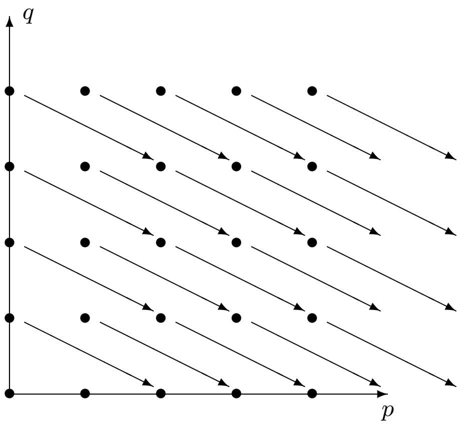



Figure 3.4: level 2

*In particular, there is an exact sequence*

$$0 \longrightarrow R^1\ g(f(A)) \longrightarrow R^1\ gf(A) \longrightarrow g(R^1\ f(A)) \longrightarrow R^2\ g(f(A)) \longrightarrow \ \ldots$$

*Moreover, if $f$ is an exact functor, then we have*

$$R^p\ gf(A) \simeq R^p\ g(f(A))$$

A spectral sequence $E_2^{p\cdot q} \Longrightarrow E^n$ (or $E_1^{p,q} \Longrightarrow E^n$) consists of the following data

1. A family of objects $E_r^{p,q}$ in an Abelian category for $p, q, r \in \mathbb{Z}$ such that $p, q \geq 0$ and $r \geq 2$ (or $r \geq 1$).

2. A family of morphisms in the Abelian category

$$d_r^{p\cdot q} : E_r^{p\cdot q} \longrightarrow E_r^{p+r,q-r+1}$$

   satisfying the complex condition

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$$

   and where we assume that $d_r^{p\cdot q} = 0$ if any of the numbers $p, q, p+r$ or $q-r+1$ is $< 1$. At level one we have the situation of figure 3.3. At level two we have the situation of figure 3.4

3. The objects $E_{r+1}^{p,q}$ on level $r+1$ are derived from those on level $r$ by taking the cohomology objects of the complexes, that is,

$$E_{r+1}^p = Ker\ d_r^{p,q}\ /\ Im\ d_r^{p-r,q+r-1}$$

At each place $(p,q)$ this process converges as there is an integer $r_0$ depending on $(p,q)$ such that for all $r \geq r_0$ we have $d_r^{p,q} = 0 = d_r^{p-r,q+r-1}$. We then define

$$E_\infty^{p,q} = E_{r_0}^{p,q} (= E_{r_0+1}^{p,q} = \ldots)$$

Observe that there are injective maps $E_\infty^{0,q} \hookrightarrow E_2^{0,q}$.

4. A family of objects $E^n$ for integers $n \geq 0$ and for each we have a filtration

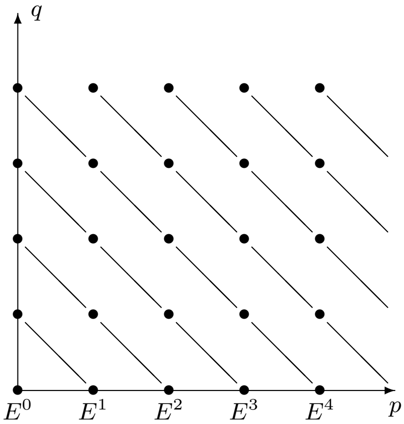
$$0 \subset E_n^n \subset E_{n-1}^n \subset \ldots \subset E_1^n \subset E_0^n = E^n$$

such that the successive quotients are given by

$$E_p^n \ / \ E_{p+1}^n = E_\infty^{p,n-p}$$

That is, the terms $E_\infty^{p,q}$ are the composition terms of the *limiting terms* $E^{p+q}$. Pictorially,

$E_\infty^{p,q} \quad =$




For small $n$ one can make the relation between $E^n$ and the terms $E_2^{p,q}$ explicit. First note that

$$E_2^{0,0} = E_\infty^{0,0} = E^0$$

Also, $E_1^1 = E_\infty^{1,0} = E_2^{1,0}$ and $E^1/E_1^1 = E_\infty^{0,1} = Ker\ d_2^{0,1}$. This gives an exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow E^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0}$$

Further, $E^2 \supset E_1^2 \supset E_2^2$ where

$$E_2^2 = E_\infty^{2,0} = E_2^{2,0} \ / \ Im\ d_2^{0,1}$$

and $E_1^2/E_2^2 = E_\infty^{1,1} = Ker\ d_2^{1,1}$ whence we can extend the above sequence to

$$\ldots \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow E_1^2 \longrightarrow E_2^{1,1} \xrightarrow{d_2^{1,1}} E_2^{3,0}$$

as $E^2/E_1^2 = E_\infty^{0,2} \hookrightarrow E_2^{0,2}$ we have that $E_1^2 = Ker\ (E^2 \longrightarrow E_2^{0,2})$. If we specialize to the spectral sequence $E_2^{p,q} = R^p\ g(R^q\ f(A)) \Longrightarrow R^n\ gf(A)$ we obtain the exact sequence

$$0 \longrightarrow R^1\ g(f(A)) \longrightarrow R^1\ gf(A) \longrightarrow g(R^1\ f(A)) \longrightarrow R^2\ g(f(A)) \longrightarrow$$

$$\longrightarrow E_1^2 \longrightarrow R^1\ g(R^1\ f(A)) \longrightarrow R^3\ g(f(A))$$

where $E_1^2 = Ker\ (R^2\ gf(A) \longrightarrow g(R^2\ f(A)))$.

An important example of a spectral sequence is the *Leray spectral sequence* . Assume we have an algebra morphism $A \xrightarrow{f} A'$ and a sheaf of groups $\mathbb{G}$ on $A'_{et}$. We define the *direct image* of $\mathbb{G}$ under $f$ to be the sheaf of groups $f_*\ \mathbb{G}$ on $\mathtt{A_{et}}$ defined by

$$f_*\ \mathbb{G}(B) = \mathbb{G}(B \otimes_A A')$$

for all $B \in \mathtt{A_{et}}$ (recall that $B \otimes_A A' \in A'_{et}$ so the right hand side is well defined).

This gives us a left exact functor

$$f_* : \mathbf{S}^{ab}(A'_{et}) \longrightarrow \mathbf{S}^{ab}(\mathtt{A_{et}})$$

and therefore we have right derived functors of it $R^i\ f_*$. If $\mathbb{G}$ is an Abelian sheaf on $A'_{et}$, then $R^i\ f_*\mathbb{G}$ is a sheaf on $\mathtt{A_{et}}$. One verifies that its stalk in a prime ideal $\mathfrak{p}$ is equal to

$$(R^i\ f_*\mathbb{G})_{\mathfrak{p}} = H^i_{et}(A^{sh}_{\mathfrak{p}} \otimes_A A', \mathbb{G})$$

where the right hand side is the direct limit of cohomology groups taken over all étale neighborhoods of $\mathfrak{p}$. We can relate cohomology of $\mathbb{G}$ and $f_*\mathbb{G}$ by the following

**Theorem 3.4** *(Leray spectral sequence) If $\mathbb{G}$ is a sheaf of Abelian groups on $A'_{et}$ and $A \xrightarrow{f} A'$ an algebra morphism, then there is a spectral sequence*

$$E_2^{p,q} = H^p_{et}(A, R^q\ f_*\mathbb{G}) \Longrightarrow H^n_{et}(A, \mathbb{G})$$

*In particular, if $R^j\ f_*\mathbb{G} = 0$ for all $j > 0$, then for all $i \geq 0$ we have isomorphisms*

$$H^i_{et}(A, f_*\mathbb{G}) \simeq H^i_{et}(A', \mathbb{G})$$

## 3.4 Tsen and Tate fields

In this section we will use spectral sequences to control the size of the Brauer group of a function field in terms of its transcendence degree.

**Definition 3.4** *A field $K$ is said to be a $Tsen^d$-field if every homogeneous form of degree deg with coefficients in $K$ and $n > deg^d$ variables has a non-trivial zero in $K$.*

For example, an algebraically closed field $\mathbb{K}$ is a $Tsen^0$-field as any form in $n$-variables defines a hypersurface in $\mathbb{P}^{n-1}_{\mathbb{K}}$. In fact, algebraic geometry tells us a stronger story

**Lemma 3.5** *Let $\mathbb{K}$ be algebraically closed. If $f_1, \ldots, f_r$ are forms in $n$ variables over $\mathbb{K}$ and $n > r$, then these forms have a common non-trivial zero in $\mathbb{K}$.*

*Proof.* Each $f_i$ defines a hypersurface $V(f_i) \hookrightarrow \mathbb{P}^{n-1}_{\mathbb{K}}$. The intersection of $r$ hypersurfaces has dimension $\geq n-1-r$ from which the claim follows. □

We want to extend this fact to higher Tsen-fields. The proof of the following result is technical unenlightening inequality manipulation, see for example [77].

**Proposition 3.7** *Let $K$ be a $Tsen^d$-field and $f_1, \ldots, f_r$ forms in $n$ variables of degree deg. If $n > rdeg^d$, then they have a non-trivial common zero in $K$.*

For our purposes the main interest in Tsen-fields comes from :

**Theorem 3.5** *Let $K$ be of transcendence degree $d$ over an algebraically closed field $\mathbb{C}$, then $K$ is a $Tsen^d$-field.*

*Proof.* First we claim that the purely transcendental field $\mathbb{C}(t_1, \ldots, t_d)$ is a $Tsen^d$-field. By induction we have to show that if $L$ is $Tsen^k$, then $L(t)$ is $Tsen^{k+1}$.

By homogeneity we may assume that $f(x_1, \ldots, x_n)$ is a form of degree $deg$ with coefficients in $L[t]$ and $n > deg^{k+1}$. For fixed $s$ we introduce new variables $y^{(s)}_{ij}$ with $i \leq n$ and $0 \leq j \leq s$ such that

$$x_i = y^{(s)}_{i0} + y^{(s)}_{i1} t + \ldots + y^{(s)}_{is} t^s$$

If $r$ is the maximal degree of the coefficients occurring in $f$, then we can write

$$f(x_i) = f_0(y^{(s)}_{ij}) + f_1(y^{(s)}_{ij})t + \ldots + f_{deg.s+r}(y^{(s)}_{ij})t^{deg.s+r}$$

where each $f_j$ is a form of degree $deg$ in $n(s+1)$-variables. By the proposition above, these forms have a common zero in $L$ provided

$$n(s+1) > deg^k(ds+r+1) \Longleftrightarrow (n - deg^{i+1})s > deg^i(r+1) - n$$

which can be satisfied by taking $s$ large enough. the common non-trivial zero in $L$ of the $f_j$, gives a non-trivial zero of $f$ in $L[t]$.

By assumption, $K$ is an algebraic extension of $\mathbb{C}(t_1, \ldots, t_d)$ which by the above argument is $Tsen^d$. As the coefficients of any form over $K$ lie in a finite extension $E$ of $\mathbb{C}(t_1, \ldots, t_d)$ it suffices to prove that $E$ is $Tsen^d$.

Let $f(x_1, \ldots, x_n)$ be a form of degree $deg$ in $E$ with $n > deg^d$. Introduce new variables $y_{ij}$ with

$$x_i = y_{i1}e_1 + \ldots y_{ik}e_k$$

where $e_i$ is a basis of $E$ over $\mathbb{C}(t_1, \ldots, t_d)$. Then,

$$f(x_i) = f_1(y_{ij})e_1 + \ldots + f_k(y_{ij})e_k$$

where the $f_i$ are forms of degree $deg$ in $k.n$ variables over $\mathbb{C}(t_1, \ldots, t_d)$. Because $\mathbb{C}(t_1, \ldots, t_d)$ is $Tsen^d$, these forms have a common zero as $k.n > k.deg^d$. Finding a non-trivial zero of $f$ in $E$ is equivalent to finding a common non-trivial zero to the $f_1, \ldots, f_k$ in $\mathbb{C}(t_1, \ldots, t_d)$, done. □

A direct application of this result is *Tsen's theorem* :

**Theorem 3.6** *Let $K$ be the functionfield of a curve $C$ defined over an algebraically closed field. Then, the only central simple $K$-algebras are $M_n(K)$. That is, $Br(K) = 1$.*

*Proof.* Assume there exists a central division algebra $\Delta$ of dimension $n^2$ over $K$. There is a finite Galois extension $L/K$ such that $\Delta \otimes L = M_n(L)$. If $x_1, \ldots, x_{n^2}$ is a $K$-basis for $\Delta$, then the reduced norm of any $x \in \Delta$,

$$N(x) = det(x \otimes 1)$$

is a form in $n^2$ variables of degree $n$. Moreover, as $x \otimes 1$ is invariant under the action of $Gal(L/K)$ the coefficients of this form actually lie in $K$.

By the main result, $K$ is a $Tsen^1$-field and $N(x)$ has a non-trivial zero whenever $n^2 > n$. As the reduced norm is multiplicative, this contradicts $N(x)N(x^{-1}) = 1$. Hence, $n = 1$ and the only central division algebra is $K$ itself. □

If $K$ is the functionfield of a surface, we also have an immediate application :

**Proposition 3.8** *Let $K$ be the functionfield of a surface defined over an algebraically closed field. If $\Delta$ is a central simple $K$-algebra of dimension $n^2$, then the reduced norm map*

$$N \; : \; \Delta \longrightarrow K$$

*is surjective.*

*Proof.* Let $e_1, \ldots, e_{n^2}$ be a $K$-basis of $\Delta$ and $k \in K$, then

$$N(\sum x_i e_i) - k x_{n^2+1}^n$$

is a form of degree $n$ in $n^2+1$ variables. Since $K$ is a $Tsen^2$ field, it has a non-trivial solution $(x_i^0)$, but then, $\delta = (\sum x_i^0 e_i) x_{n^2+1}^{-1}$ has reduced norm equal to $k$. □

From the cohomological description of the Brauer group it is clear that we need to have some control on the absolute Galois group $G_K = Gal(\mathbb{K}/K)$. We will see that finite transcendence degree forces some cohomology groups to vanish.

**Definition 3.5** *The* cohomological dimension *of a group* $G$, $cd(G) \leq d$ *if and only if* $H^r(G, A) = 0$ *for all* $r > d$ *and all torsion modules* $A \in G$*-mod.*

**Definition 3.6** *A field* $K$ *is said to be a* $Tate^d$*-field if the absolute Galois group* $G_K = Gal(\mathbb{K}/K)$ *satisfies* $cd(G) \leq d$.

First, we will reduce the condition $cd(G) \leq d$ to a more manageable one. To start, one can show that a profinite group $G$ (that is, a projective limit of finite groups, see [77] for more details) has $cd(G) \leq d$ if and only if

$$H^{d+1}(G, A) = 0 \text{ for all torsion } G\text{-modules } A$$

Further, as all Galois cohomology groups of profinite groups are torsion, we can decompose the cohomology in its $p$-primary parts and relate their vanishing to the cohomological dimension of the $p$-Sylow subgroups $G_p$ of $G$. This problem can then be verified by computing cohomology of finite simple $G_p$-modules of $p$-power order, but for a profinite $p$-group there is just one such module namely $\mathbb{Z}/p\mathbb{Z}$ with the trivial action.

Combining these facts we have the following manageable criterium on cohomological dimension.

**Proposition 3.9** $cd(G) \leq d$ *if* $H^{d+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$ *for the simple* $G$*-modules with trivial action* $\mathbb{Z}/p\mathbb{Z}$.

We will need the following spectral sequence in Galois cohomology

**Proposition 3.10** *(Hochschild-Serre spectral sequence) If* $N$ *is a closed normal subgroup of a profinite group* $G$*, then*

$$E_2^{p,q} = H^p(G/N, H^q(N, A)) \Longrightarrow H^n(G, A)$$

*holds for every continuous* $G$*-module* $A$*.*

Now, we are in a position to state and prove *Tate's theorem*

**Theorem 3.7** *Let $K$ be of transcendence degree $d$ over an algebraically closed field, then $K$ is a $Tate^d$-field.*

*Proof.* Let $\mathbb{C}$ denote the algebraically closed basefield, then $K$ is algebraic over $\mathbb{C}(t_1,\ldots,t_d)$ and therefore

$$G_K \hookrightarrow G_{\mathbb{C}(t_1,\ldots,t_d)}$$

Thus, $K$ is $Tate^d$ if $\mathbb{C}(t_1,\ldots,t_d)$ is $Tate^d$. By induction it suffices to prove

$$\text{If } cd(G_L) \leq k \text{ then } cd(G_{L(t)}) \leq k+1$$

Let $\mathbb{L}$ be the algebraic closure of $L$ and $\mathbb{M}$ the algebraic closure of $L(t)$. As $L(t)$ and $\mathbb{L}$ are linearly disjoint over $L$ we have the following diagram of extensions and Galois groups

$$\begin{tikzcd} \mathbb{L} \arrow[r, hook] & \mathbb{L}(t) \arrow[r, hook, "G_{\mathbb{L}(t)}"] & \mathbb{M} \\ L \arrow[u, hook, "G_L"] \arrow[r, hook] & L(t) \arrow[u, hook, "G_L"] \arrow[ur, hook, "G_{L(t)}"'] & \end{tikzcd}$$

where $G_{L(t)}/G_{\mathbb{L}(t)} \simeq G_L$.

We claim that $cd(G_{\mathbb{L}(t)}) \leq 1$. Consider the exact sequence of $G_{L(t)}$-modules

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{M}^* \xrightarrow{(-)^p} \mathbb{M}^* \longrightarrow 0$$

where $\mu_p$ is the subgroup (of $\mathbb{C}^*$) of $p$-roots of unity. As $G_{L(t)}$ acts trivially on $\mu_p$ it is after a choice of primitive $p$-th root of one isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Taking cohomology with respect to the subgroup $G_{\mathbb{L}(t)}$ we obtain

$$0 = H^1(G_{\mathbb{L}(t)}, \mathbb{M}^*) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{M}^*) = Br(\mathbb{L}(t))$$

But the last term vanishes by Tsen's theorem as $\mathbb{L}(t)$ is the functionfield of a curve defined over the algebraically closed field $\mathbb{L}$. Therefore, $H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all simple modules $\mathbb{Z}/p\mathbb{Z}$, whence $cd(G_{\mathbb{L}(t)}) \leq 1$.

By the inductive assumption we have $cd(G_L) \leq k$ and now we are going to use exactness of the sequence

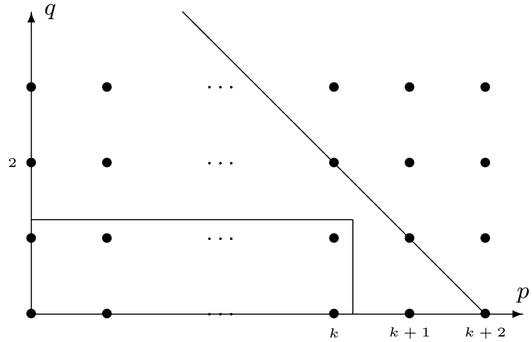
$$0 \longrightarrow G_L \longrightarrow G_{L(t)} \longrightarrow G_{\mathbb{L}(t)} \longrightarrow 0$$

to prove that $cd(G_{L(t)}) \leq k+1$. For, let $A$ be a torsion $G_{L(t)}$-module and consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_L, H^q(G_{\mathbb{L}(t)}, A)) \Longrightarrow H^n(G_{L(t)}, A)$$

By the restrictions on the cohomological dimensions of $G_L$ and $G_{\mathbb{L}(t)}$ the level two term has following shape

$E_2^{p,q} =$




where the only non-zero groups are lying in the lower rectangular region. Therefore, all $E_\infty^{p,q} = 0$ for $p + q > k + 1$. Now, all the composition factors of $H^{k+2}(G_{L(t)}, A)$ are lying on the indicated diagonal line and hence are zero. Thus, $H^{k+2}(G_{L(t)}, A) = 0$ for all torsion $G_{L(t)}$-modules $A$ and hence $cd(G_{L(t)}) \leq k + 1$. $\square$

**Theorem 3.8** *If* $\mathbf{A}$ *is a constant sheaf of an Abelian torsion group* $A$ *on* $\mathtt{K_{et}}$*, then*

$$H^i_{et}(K, \mathbf{A}) = 0$$

*whenever* $i > trdeg_{\mathbb{C}}(K)$.

## 3.5 Coniveau spectral sequence

In this section we will describe a particularly useful spectral sequence. Consider the setting $k \overset{\pi}{\twoheadleftarrow} A \overset{i}{\hookrightarrow} K$ where $A$ is a discrete valuation ring in $K$ with residue field $A/m = k$. As always, we will assume that $A$ is a $\mathbb{C}$-algebra. By now we have a grip on the Galois cohomology groups

$$H^i_{et}(K, \mu_n^{\otimes l}) \text{ and } H^i_{et}(k, \mu_n^{\otimes l})$$

and we will use this information to compute the étale cohomology groups

$$H^i_{et}(A, \mu_n^{\otimes l})$$

Here, $\mu_n^{\otimes l} = \underbrace{\mu_n \otimes \ldots \otimes \mu_n}_{l}$ where the tensorproduct is as sheafs of invertible $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$-modules.

We will consider the Leray spectral sequence for $i$ and hence have to compute the derived sheaves of the direct image

| | | | |
|---|---|---|---|
| | | | |
| $0$ | $0$ | $0$ | $\dots$ |
| $H^0(k,\mu_n^{\otimes l-1})$ | $H^1(k,\mu_n^{\otimes l-1})$ | $H^2(k,\mu_n^{\otimes l-1})$ | $\dots$ |
| $H^0(A,\mu_n^{\otimes l})$ | $H^1(A,\mu_n^{\otimes l})$ | $H^2(A,\mu_n^{\otimes l})$ | $\dots$ |

Figure 3.5: Second term of Leray sequence

**Lemma 3.6** *1.* $R^0\ i_*\mu_n^{\otimes l} \simeq \mu_n^{\otimes l}$ *on* $\mathtt{A_{et}}$*.*

*2.* $R^1\ i_*\mu_n^{\otimes l} \simeq \mu_n^{\otimes l-1}$ *concentrated in* $m$*.*

*3.* $R^j\ i_*\mu_n^{\otimes l} \simeq 0$ *whenever* $j \geq 2$*.*

*Proof.* The strict Henselizations of $A$ at the two primes $\{0, m\}$ are resp.

$$A_0^{sh} \simeq \mathbb{K} \text{ and } A_m^{sh} \simeq \mathbf{k}\{t\}$$

where $\mathbb{K}$ (resp. $\mathbf{k}$) is the algebraic closure of $K$ (resp. $k$). Therefore,

$$(R^j\ i_*\mu_n^{\otimes l})_0 = H^j_{et}(\mathbb{K},\mu_n^{\otimes l})$$

which is zero for $i \geq 1$ and $\mu_n^{\otimes l}$ for $j = 0$. Further, $A_m^{sh} \otimes_A K$ is the field of fractions of $\mathbf{k}\{t\}$ and hence is of transcendence degree one over the algebraically closed field $\mathbf{k}$, whence

$$(R^j\ i_*\mu_n^{\otimes l})_m = H^j_{et}(L,\mu_n^{\otimes l})$$

which is zero for $j \geq 2$ because $L$ is $Tate$[1].

For the field-tower $K \subset L \subset \mathbb{K}$ we have that $G_L = \hat{\mathbb{Z}} = \varprojlim \mu_m$ because the only Galois extensions of $L$ are the Kummer extensions obtained by adjoining $\sqrt[m]{t}$. But then,

$$H^1_{et}(L,\mu_n^{\otimes l}) = H^1(\hat{Z},\mu_n^{\otimes l}(\mathbb{K})) = Hom(\hat{Z},\mu_n^{\otimes l}(\mathbb{K})) = \mu_n^{\otimes l-1}$$

from which the claims follow. $\square$

**Theorem 3.9** *We have a long exact sequence*

$$0 \longrightarrow H^1(A,\mu_n^{\otimes l}) \longrightarrow H^1(K,\mu_n^{\otimes l}) \longrightarrow H^0(k,\mu_n^{\otimes l-1}) \longrightarrow$$
$$H^2(A,\mu_n^{\otimes l}) \longrightarrow H^2(K,\mu_n^{\otimes l}) \longrightarrow H^1(k,\mu_n^{\otimes l-1}) \longrightarrow \dots$$

*Proof.* By the foregoing lemma, the second term of the Leray spectral sequence for $i_*\mu_n^{\otimes l}$ is depicted in figure 3.5 with connecting morphisms

$$H_{et}^{i-1}(k,\mu_n^{\otimes l-1}) \xrightarrow{\alpha_i} H_{et}^{i+1}(A,\mu_n^{\otimes l})$$

The spectral sequences converges to its limiting term which looks like

| | | | |
|---|---|---|---|
| | | | |
| $0$ | $0$ | $0$ | $\dots$ |
| $Ker\ \alpha_1$ | $Ker\ \alpha_2$ | $Ker\ \alpha_3$ | $\dots$ |
| $H^0(A,\mu_n^{\otimes l})$ | $H^1(A,\mu_n^{\otimes l})$ | $Coker\ \alpha_1$ | $\dots$ |

and the Leray sequence gives the short exact sequences

$$0 \longrightarrow H_{et}^1(A,\mu_n^{\otimes l}) \longrightarrow H_{et}^1(K,\mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_1 \longrightarrow 0$$
$$0 \longrightarrow Coker\ \alpha_1 \longrightarrow H_{et}^2(K,\mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_2 \longrightarrow 0$$
$$0 \longrightarrow Coker\ \alpha_{i-1} \longrightarrow H_{et}^i(K,\mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_i \longrightarrow 0$$

and gluing these sequences gives us the required result. □

In particular, if $A$ is a discrete valuation ring of $K$ with residue field $k$ we have for each $i$ a connecting morphism

$$H_{et}^i(K,\mu_n^{\otimes l}) \xrightarrow{\partial_{i,A}} H_{et}^{i-1}(k,\mu_n^{\otimes l-1})$$

Like any other topology, the étale topology can be defined locally on any scheme $X$. That is, we call a morphism of schemes

$$Y \xrightarrow{f} X$$

an étale extension (resp. cover) if locally $f$ has the form

$$f^a \mid U_i : A_i = \Gamma(U_i, \mathcal{O}_X) \longrightarrow B_i = \Gamma(f^{-1}(U_i), \mathcal{O}_Y)$$

with $A_i \longrightarrow B_i$ an étale extension (resp. cover) of algebras.

Again, we can construct the étale site of $X$ locally and denote it with $X_{et}$. Presheaves and sheaves of groups on $X_{et}$ are defined similarly and the right derived functors of the left exact global sections functor

$$\Gamma : \mathbf{S}^{ab}(X_{et}) \longrightarrow \texttt{abelian}$$

will be called the cohomology functors and we denote

$$R^i\ \Gamma(\mathbb{G}) = H^i_{et}(X, \mathbb{G})$$

From now on we restrict to the case when $X$ is a smooth, irreducible projective variety of dimension $d$ over $\mathbb{C}$. In this case, we can initiate the computation of the cohomology groups $H^i_{et}(X, \mu_n^{\otimes l})$ via Galois cohomology of functionfields of subvarieties using the coniveau spectral sequence

**Theorem 3.10** *Let $X$ be a smooth irreducible variety over $\mathbb{C}$. Let $X^{(p)}$ denote the set of irreducible subvarieties $x$ of $X$ of codimension $p$ with functionfield $\mathbb{C}(x)$, then there exists a* coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H^{q-p}_{et}(\mathbb{C}(x), \mu_n^{\otimes l-p}) \Longrightarrow H^{p+q}_{et}(X, \mu_n^{\otimes l})$$

In contrast to the spectral sequences used before, the existence of the coniveau spectral sequence by no means follows from general principles. In it, a lot of heavy machinery on étale cohomology of schemes is encoded. In particular,

- cohomology groups with support of a closed subscheme, see for example [64, p. 91-94], and
- cohomological purity and duality, see [64, p. 241-252]

a detailed exposition of which would take us too far afield. For more details we refer the reader to [18].

Using the results on cohomological dimension and vanishing of Galois cohomology of $\mu_n^{\otimes k}$ when the index is larger than the transcendence degree, we see that the coniveau spectral sequence has shape as in figure 3.6 where the only non-zero terms are in the indicated region.

Let us understand the connecting morphisms at the first level, a typical instance of which is

$$\bigoplus_{x \in X^{(p)}} H^i(\mathbb{C}(x), \mu_n^{\oplus l-p}) \longrightarrow \bigoplus_{y \in X^{(p+1)}} H^{i-1}(\mathbb{C}(y), \mu_n^{\oplus l-p-1})$$

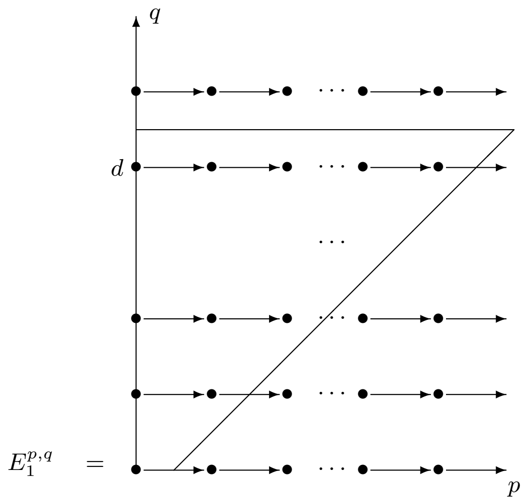



Figure 3.6: Coniveau spectral sequence

and consider one of the closed irreducible subvarieties $x$ of $X$ of codimension $p$ and one of those $y$ of codimension $p+1$. Then, either $y$ is not contained in $x$ in which case the component map

$$H^i(\mathbb{C}(x), \mu_n^{\oplus l-p}) \longrightarrow H^{i-1}(\mathbb{C}(y), \mu_n^{\oplus l-p-1})$$

is the zero map. Or, $y$ is contained in $x$ and hence defines a codimension one subvariety of $x$. That is, $y$ defines a discrete valuation on $\mathbb{C}(x)$ with residue field $\mathbb{C}(y)$. In this case, the above component map is the connecting morphism defined above.

In particular, let $K$ be the functionfield of $X$. Then we can define the unramified cohomology groups

$$F_n^{i,l}(K/\mathbb{C}) = Ker\ H^i(K, \mu_n^{\otimes l}) \xrightarrow{\oplus \partial_{i,A}} \oplus H^{i-1}(k_A, \mu_n^{\otimes l-1})$$

where the sum is taken over all discrete valuation rings $A$ of $K$ (or equivalently, the irreducible codimension one subvarieties of $X$) with residue field $k_A$. By definition, this is a (stable) birational invariant of $X$. In particular, if $X$ is (stably) rational over $\mathbb{C}$, then

$$F_n^{i,l}(K/\mathbb{C}) = 0 \text{ for all } i, l \geq 0$$

## 3.6 The Artin-Mumford exact sequence

The coniveau spectral sequence allows us to control the Brauer group of function fields of surfaces. This result, due to Michael Artin and David Mumford, was used by them to construct unirational

| | | | | |
|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 0 | 0 | 0 | 0 | $\cdots$ |
| $H^2(\mathbb{C}(S), \mu_n)$ | $\oplus_C H^1(\mathbb{C}(S), \mathbb{Z}_n)$ | $\oplus_P \mu_n^{-1}$ | 0 | $\cdots$ |
| $H^1(\mathbb{C}(S), \mu_n)$ | $\oplus_C \mathbb{Z}_n$ | 0 | 0 | $\cdots$ |
| $\mu_n$ | 0 | 0 | 0 | $\cdots$ |

Figure 3.7: First term of coniveau spectral sequence for $S$

non-rational varieties. Our main application of the description is to classify in chapter 5 the Brauer classes which do admit a Cayley-smooth noncommutative model. It will turn out that even in the case of surfaces, not every central simple algebra over the function field allows such a noncommutative model. Let $S$ be a smooth irreducible projective surface.

**Definition 3.7** *$S$ is called* simply connected *if every étale cover $Y \longrightarrow S$ is trivial, that is, $Y$ is isomorphic to a finite disjoint union of copies of $S$.*

The first term of the coniveau spectral sequence of $S$ has the shape of figure 3.7 where $C$ runs over all irreducible curves on $S$ and $P$ over all points of $S$.

**Lemma 3.7** *For any smooth $S$ we have $H^1(\mathbb{C}(S), \mu_n) \twoheadrightarrow \oplus_C \mathbb{Z}_n$. If $S$ is simply connected, $H^1_{et}(S, \mu_n) = 0$.*

*Proof.* Using the Kummer sequence $1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)} \mathbb{G}_m \longrightarrow 1$ and Hilbert 90 we obtain that

$$H^1_{et}(\mathbb{C}(S), \mu_n) = \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$$

The first claim follows from the exact diagram describing divisors of rational functions given in figure 3.8 By the coniveau spectral sequence we have that $H^1_{et}(S, \mu_n)$ is equal to the kernel of the morphism

$$H^1_{et}(\mathbb{C}(S), \mu_n) \xrightarrow{\gamma} \oplus_C \mathbb{Z}_n$$

and in particular, $H^1(S, \mu_n) \hookrightarrow H^1(\mathbb{C}(S), \mu_n)$.

$$\begin{tikzcd}
& \mu_n \arrow[r, phantom, "\simeq"] \arrow[d] & \mu_n \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d] & \mathbb{C}(S)^* \arrow[r, "div"] \arrow[d, "(-)^n"'] & \oplus_C \mathbb{Z} \arrow[r] \arrow[d, "n."'] & 0 \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d] & \mathbb{C}(S)^* \arrow[r, "div"] \arrow[d] & \oplus_C \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
& 0 & \oplus_C \mathbb{Z}_n \arrow[r, phantom, "\simeq"] & \oplus_C \mathbb{Z}_n &
\end{tikzcd}$$

Figure 3.8: Divisors of rational functions on $S$.

As for the second claim, an element in $H^1(S, \mu_n)$ determines a cyclic extension $L = \mathbb{C}(S)\sqrt[n]{f}$ with $f \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$ such that in each fieldcomponent $L_i$ of $L$ there is an étale cover $T_i \longrightarrow S$ with $\mathbb{C}(T_i) = L_i$. By assumption no non-trivial étale covers exist whence $f = 1 \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$. □

If we invoke another major tool in étale cohomology of schemes, *Poincaré duality* , see for example [64, VI,§11], we obtain the following information on the cohomology groups for $S$.

**Proposition 3.11** *(Poincaré duality for S) If S is simply connected, then*

1. $H^0_{et}(S, \mu_n) = \mu_n$
2. $H^1_{et}(S, \mu_n) = 0$
3. $H^3_{et}(S, \mu_n) = 0$
4. $H^4_{et}(S, \mu_n) = \mu_n^{-1}$

*Proof.* The third claim follows from the second as both groups are dual to each other. The last claim follows from the fact that for any smooth irreducible projective variety $X$ of dimension $d$ one has that

$$H^{2d}_{et}(X, \mu_n) \simeq \mu_n^{\otimes 1-d}$$

□

We are now in a position to state and prove the important

**Theorem 3.11** *(Artin-Mumford exact sequence) If $S$ is a simply connected smooth projective surface, then the sequence*

$$0 \longrightarrow Br_n(S) \longrightarrow Br_n(\mathbb{C}(S)) \longrightarrow \oplus_C \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow$$
$$\longrightarrow \oplus_P \mu_n^{-1} \longrightarrow \mu_n^{-1} \longrightarrow 0$$

*is exact.*

*Proof.* The top complex in the first term of the coniveau spectral sequence for $S$ was

$$H^2(\mathbb{C}(S), \mu_n) \xrightarrow{\alpha} \oplus_C H^1(\mathbb{C}(C), \mathbb{Z}_n) \xrightarrow{\beta} \oplus_P \mu_n$$

The second term of the spectral sequence (which is also the limiting term) has the following form

| | | | | |
|---|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ | |
| 0 | 0 | 0 | 0 | ... |
| $Ker\ \alpha$ | $Ker\ \beta/Im\ \alpha$ | $Coker\ \beta$ | 0 | ... |
| $Ker\ \gamma$ | $Coker\ \gamma$ | 0 | 0 | ... |
| $\mu_n$ | 0 | 0 | 0 | ... |

By the foregoing lemma we know that $Coker\ \gamma = 0$. By Poincare duality we know that $Ker\ \beta = Im\ \alpha$ and $Coker\ \beta = \mu_n^{-1}$. Hence, the top complex was exact in its middle term and can be extended to an exact sequence

$$0 \longrightarrow H^2(S, \mu_n) \longrightarrow H^2(\mathbb{C}(S), \mu_n) \longrightarrow \oplus_C H^1(\mathbb{C}(C), \mathbb{Z}_n) \longrightarrow$$
$$\oplus_P \mu_n^{-1} \longrightarrow \mu_n^{-1} \longrightarrow 0$$

As $\mathbb{Z}_n \simeq \mu_n$ the third term is equal to $\oplus_C \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$ by the argument given before and the second term we remember to be $Br_n(\mathbb{C}(S)$. The identification of $Br_n(S)$ with $H^2(S, \mu_n)$ will be explained below. □

Some immediate consequences can be drawn from this : For a smooth simply connected surface $S$, $Br_n(S)$ is a birational invariant (it is the birational invariant $F_n^{2,1}(\mathbb{C}(S)/\mathbb{C})$ of the foregoing section. In particular, if $S = \mathbb{P}^2$ we have that $Br_n(\mathbb{P}^2) = 0$ and as

$$0 \longrightarrow Br_n\ \mathbb{C}(x,y) \longrightarrow \oplus_C\ \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow \oplus_P\ \mu_n^{-1} \longrightarrow \mu_n \longrightarrow 0$$

we obtain a description of $Br_n\ \mathbb{C}(x,y)$ by a certain geo-combinatorial package which we call a $\mathbb{Z}_n$-*wrinkle* over $\mathbb{P}^2$. A $\mathbb{Z}_n$-wrinkle is determined by

- A finite collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of *irreducible curves* in $\mathbb{P}^2$, that is, $C_i = V(F_i)$ for an irreducible form in $\mathbb{C}[X,Y,Z]$ of degree $d_i$.
- A finite collection $\mathcal{P} = \{P_1, \ldots, P_l\}$ of *points* of $\mathbb{P}^2$ where each $P_i$ is either an intersection point of two or more $C_i$ or a singular point of some $C_i$.
- For each $P \in \mathcal{P}$ the *branch-data* $b_P = (b_1, \ldots, b_{i_P})$ with $b_i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\{1, \ldots, i_P\}$ the different branches of $\mathcal{C}$ in $P$. These numbers must satisfy the admissibility condition

$$\sum_i b_i = 0 \in \mathbb{Z}_n$$

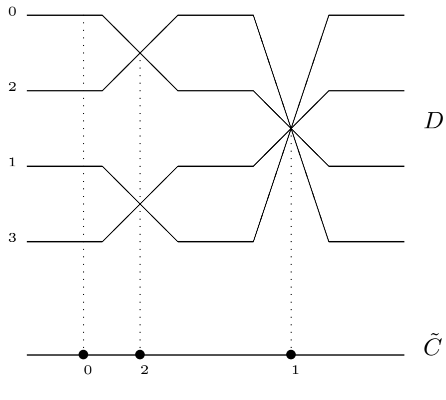
  for every $P \in \mathcal{P}$
- for each $C \in \mathcal{C}$ we fix a cyclic $\mathbb{Z}_n$-cover of smooth curves

$$D \twoheadrightarrow \tilde{C}$$

  of the desingularization $\tilde{C}$ of $C$ which is compatible with the branch-data. That is, if $Q \in \tilde{C}$ corresponds to a $\mathcal{C}$-branch $b_i$ in $P$, then $D$ is ramified in $Q$ with stabilizer subgroup

$$Stab_Q = \langle b_i \rangle \subset \mathbb{Z}_n$$

For example, a portion of a $\mathbb{Z}_4$-wrinkle can have the following picture



It is clear that the cover-data is the most intractable part of a $\mathbb{Z}_n$-wrinkle, so we want to have some control on the covers $D \twoheadrightarrow \tilde{C}$. Let $\{Q_1, \ldots, Q_z\}$ be the points of $\tilde{C}$ where the cover ramifies with branch numbers $\{b_1, \ldots, b_z\}$, then $D$ is determined by a continuous module structure (that is, a cofinite subgroup acts trivially) of

$$\pi_1(\tilde{C} - \{Q_1, \ldots, Q_z\}) \text{ on } \mathbb{Z}_n$$

where the fundamental group of the Riemann surface $\tilde{C}$ with $z$ punctures is known (topologically) to be equal to the group

$$\langle u_1, v_1, \ldots, u_g, v_g, x_1, \ldots, x_z \rangle / ([u_1, v_1] \ldots [u_g, v_g] x_1 \ldots x_z)$$

where $g$ is the genus of $\tilde{C}$. The action of $x_i$ on $\mathbb{Z}_n$ is determined by multiplication with $b_i$. In fact, we need to use the étale fundamental group, see [64], but this group has the same finite continuous modules as the topological fundamental group.

**Example 3.7 (Covers of $\mathbb{P}^1$ and elliptic curves)** 1. If $\tilde{C} = \mathbb{P}^1$ then $g = 0$ and hence $\pi_1(\mathbb{P}^1 - \{Q_1, \ldots, Q_z\}$ is zero if $z \leq 1$ (whence no covers exist) and is $\mathbb{Z}$ if $z = 2$. Hence, there exists a unique cover $D \twoheadrightarrow \mathbb{P}^1$ with branch-data $(1, -1)$ in say $(0, \infty)$ namely with $D$ the normalization of $\mathbb{P}^1$ in $\mathbb{C}(\sqrt[n]{x})$.

2. If $\tilde{C} = E$ an elliptic curve, then $g = 1$. Hence, $\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$ and there exist unramified $\mathbb{Z}_n$-covers. They are given by the isogenies

$$E' \twoheadrightarrow E$$

where $E'$ is another elliptic curve and $E = E'/\langle \tau \rangle$ where $\tau$ is an $n$-torsion point on $E'$.

Any $n$-fold cover $D \twoheadrightarrow \tilde{C}$ is determined by a function $f \in \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$. This allows us to put a group-structure on the equivalence classes of $\mathbb{Z}_n$-wrinkles. In particular, we call a wrinkle *trivial* provided all coverings $D_i \twoheadrightarrow \tilde{C}_i$ are trivial (that is, $D_i$ is the disjoint union of $n$ copies of $\tilde{C}_i$). The Artin-Mumford theorem for $\mathbb{P}^2$ can now be stated as

**Theorem 3.12** *If $\Delta$ is a central simple $\mathbb{C}(x, y)$-algebra of dimension $n^2$, then $\Delta$ determines uniquely a $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$. Conversely, any $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$ determines a unique division $\mathbb{C}(x, y)$- algebra whose class in the Brauer group has order $n$.*

**Example 3.8** If $S$ is not necessarily simply connected, any class in $Br(\mathbb{C}(S))_n$ still determines a $\mathbb{Z}_n$-wrinkle.

**Example 3.9** If $X$ is a smooth irreducible rational projective variety of dimension $d$, the obstruction to classifying $Br(\mathbb{C}(X))_n$ by $\mathbb{Z}_n$-wrinkles is given by $H^3_{et}(X, \mu_n)$.

We will give a ringtheoretical interpretation of the maps in the Artin-Mumford sequence. Observe that nearly all maps are those of the top complex of the first term in the coniveau spectral sequence for $S$. We gave an explicit description of them using discrete valuation rings. The statements below follow from this description.

Let us consider a discrete valuation ring $A$ with field of fractions $K$ and residue field $k$. Let $\Delta$ be a central simple $K$-algebra of dimension $n^2$.

**Definition 3.8** *An $A$-subalgebra $\Lambda$ of $\Delta$ will be called an* $A$-order *if it is a free $A$-module of rank $n^2$ with $\Lambda.K = \Delta$. An $A$-order is said to be* maximal *if it is not properly contained in any other order.*

In order to study maximal orders in $\Delta$ (they will turn out to be all conjugated), we consider the completion $\hat{A}$ with respect to the $m$-adic filtration where $m = At$ with $t$ a uniformizing parameter of $A$. $\hat{K}$ will denote the field of fractions of $\hat{A}$ and $\hat{\Delta} = \Delta \otimes_K \hat{K}$.

Because $\hat{\Delta}$ is a central simple $\hat{K}$-algebra of dimension $n^2$ it is of the form

$$\hat{\Delta} = M_t(D)$$

where $D$ is a division algebra with center $\hat{K}$ of dimension $s^2$ and hence $n = s.t$. We call $t$ the capacity of $\Delta$ at $A$.

In $D$ we can construct a unique maximal $\hat{A}$-order $\Gamma$, namely the integral closure of $\hat{A}$ in $D$. We can view $\Gamma$ as a discrete valuation ring extending the valuation $v$ defined by $A$ on $K$. If $v : \hat{K} \longrightarrow \mathbb{Z}$, then this extended valuation

$$w : D \longrightarrow n^{-2}\mathbb{Z} \text{ is defined as } w(a) = (\hat{K}(a) : \hat{K})^{-1} v(N_{\hat{K}(a)/\hat{K}}(a))$$

for every $a \in D$ where $\hat{K}(a)$ is the subfield generated by $a$ and $N$ is the norm map of fields.

The image of $w$ is a subgroup of the form $e^{-1}\mathbb{Z} \hookrightarrow n^{-2}.\mathbb{Z}$. The number $e = e(D/\hat{K})$ is called the *ramification index* of $D$ over $\hat{K}$. We can use it to normalize the valuation $w$ to

$$v_D : D \longrightarrow \mathbb{Z} \text{ defined by } v_D(a) = \frac{e}{n^2} v(N_{D/\hat{K}}(a))$$

With these conventions we have that $v_D(t) = e$.

The maximal order $\Gamma$ is then the subalgebra of all elements $a \in D$ with $v_D(a) \geq 0$. It has a unique maximal ideal generated by a prime element $T$ and we have that $\overline{\Gamma} = \frac{\Gamma}{T\,\Gamma}$ is a division algebra finite dimensional over $\hat{A}/t\hat{A} = k$ (but not necessarily having $k$ as its center).

The *inertial degree* of $D$ over $\hat{K}$ is defined to be the number $f = f(D/\hat{K}) = (\overline{\Gamma} : k)$ and one shows that

$$s^2 = e.f \text{ and } e \mid s \text{ whence } s \mid f$$

After this detour, we can now take $\Lambda = M_t(\Gamma)$ as a maximal $\hat{A}$-order in $\hat{\Delta}$. One shows that all other maximal $\hat{A}$-orders are conjugated to $\Lambda$. $\Lambda$ has a unique maximal ideal $M$ with $\overline{\Lambda} = M_t(\overline{\Gamma})$.

**Definition 3.9** *With notations as above, we call the numbers $e = e(D/\hat{K})$, $f = f(D/\hat{K})$ and $t$ resp. the* ramification, inertia and capacity *of the central simple algebra $\Delta$ at $A$. If $e = 1$ we call $\Lambda$ an Azumaya algebra over $A$, or equivalently, if $\Lambda/t\Lambda$ is a central simple $k$-algebra of dimension $n^2$.*

Now let us consider the case of a discrete valuation ring $A$ in $K$ such that the residue field $k$ is $Tsen^1$. The center of the division algebra $\overline{\Gamma}$ is a finite dimensional field extension of $k$ and hence is also $Tsen^1$ whence has trivial Brauer group and therefore must coincide with $\overline{\Gamma}$. Hence,

$$\overline{\Gamma} = k(\overline{a})$$

a commutative field, for some $a \in \Gamma$. But then, $f \leq s$ and we have $e = f = s$ and $k(\overline{a})$ is a cyclic degree $s$ field extension of $k$.

Because $s \mid n$, the cyclic extension $k(\overline{a})$ determines an element of $H^1_{et}(k, \mathbb{Z}_n)$.

**Definition 3.10** *Let $Z$ be a normal domain with field of fractions $K$ and let $\Delta$ be a central simple $K$-algebra of dimension $n^2$. A $Z$-order $B$ is a subalgebra which is a finitely generated $Z$-module. It is called maximal if it is not properly contained in any other order. One can show that $B$ is a maximal $Z$-order if and only if $\Lambda = B_p$ is a maximal order over the discrete valuation ring $A = Z_p$ for every height one prime ideal $p$ of $Z$.*

Return to the situation of an irreducible smooth projective surface $S$. If $\Delta$ is a central simple $\mathbb{C}(S)$-algebra of dimension $n^2$, we define a maximal order as a sheaf $\mathcal{A}$ of $\mathcal{O}_S$-orders in $\Delta$ which for an open affine cover $U_i \hookrightarrow S$ is such that

$$A_i = \Gamma(U_i, \mathcal{A}) \text{ is a maximal } Z_i = \Gamma(U_i, \mathcal{O}_S) \text{ order in } \Delta$$

Any irreducible curve $C$ on $S$ defines a discrete valuation ring on $\mathbb{C}(S)$ with residue field $\mathbb{C}(C)$ which is $Tsen^1$. Hence, the above argument can be applied to obtain from $\mathcal{A}$ a cyclic extension of $\mathbb{C}(C)$, that is, an element of $\mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$.

**Definition 3.11** *We call the union of the curves $C$ such that $\mathcal{A}$ determines a non-trivial cyclic extension of $\mathbb{C}(C)$ the* ramification divisor *of $\Delta$ (or of $\mathcal{A}$).*

The map in the Artin-Mumford exact sequence

$$Br_n(\mathbb{C}(S)) \longrightarrow \bigoplus_C H^1_{et}(\mathbb{C}(C), \mu_n)$$

assigns to the class of $\Delta$ the cyclic extensions introduced above.

**Definition 3.12** *An $S$-Azumaya algebra (of index $n$) is a sheaf of maximal orders in a central simple $\mathbb{C}(S)$-algebra $\Delta$ of dimension $n^2$ such that it is Azumaya at each curve $C$, that is, such that $[\Delta]$ lies in the kernel of the above map.*

Observe that this definition of Azumaya algebra coincides with the one given in the discussion of twisted forms of matrices. One can show that if $\mathcal{A}$ and $\mathcal{A}'$ are $S$-Azumaya algebras of index $n$ resp. $n'$, then $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}'$ is an Azumaya algebra of index $n.n'$. We call an Azumaya algebra trivial if it is of the form $End(\mathcal{P})$ where $\mathcal{P}$ is a vectorbundle over $S$. The equivalence classes of $S$-Azumaya algebras can be given a group-structure called the Brauer-group $Br(S)$ of the surface $S$.

Let us briefly sketch how Michael Artin and David Mumford used their sequence to construct unirational non-rational threefolds via *Brauer-Severi varieties*. Let $K$ be a field and $\Delta = (a,b)_K$ the quaternion algebra determined by $a, b \in K^*$. That is,

$$\Delta = K.1 \oplus K.i \oplus K.j \oplus K.ij \quad \text{with} \quad i^2 = a \quad j^2 = b \quad \text{and} \quad ji = -ij$$

The norm map on $\Delta$ defines a conic in $\mathbb{P}^2_K$ called the Brauer-Severi variety of $\Delta$

$$BS(\Delta) = \mathbb{V}(x^2 - ay^2 - bz^2) \hookrightarrow \mathbb{P}^2_K = \mathtt{proj}\ K[x,y,z].$$

Its characteristic property is that a fieldextension $L$ of $K$ admits an $L$-rational point on $BS(\Delta)$ if and only if $\Delta \otimes_K L$ admits zero-divisors and hence is isomorphic to $M_2(L)$.

In general, let $\mathbb{K}$ be the algebraic closure of $K$, then we have seen that the Galois cohomology pointed set

$$H^1(Gal(\mathbb{K}/K), PGL_n(\mathbb{K}))$$

classifies at the same time the isomorphism classes of the following geometric and algebraic objects

- Brauer-Severi $K$-varieties $BS$, which are smooth projective $K$-varieties such that $BS_{\mathbb{K}} \simeq \mathbb{P}^{n-1}_{\mathbb{K}}$.
- Central simple $K$-algebras $\Delta$, which are $K$-algebras of dimension $n^2$ such that $\Delta \otimes_K \mathbb{K} \simeq M_n(\mathbb{K})$.

The one-to-one correspondence between these two sets is given by associating to a central simple $K$-algebra $\Delta$ its Brauer-Severi variety $BS(\Delta)$ which represents the functor associating to a fieldextension $L$ of $K$ the set of left ideals of $\Delta \otimes_K L$ which have $L$-dimension equal to $n$. In particular, $BS(\Delta)$ has an $L$-rational point if and only if $\Delta \otimes_K L \simeq M_n(L)$ and hence the geometric object $BS(\Delta)$ encodes the algebraic splitting behavior of $\Delta$.

Now restrict to the case when $K$ is the functionfield $\mathbb{C}(X)$ of a projective variety $X$ and let $\Delta$ be a central simple $\mathbb{C}(X)$-algebra of dimension $n^2$. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-orders in $\Delta$ then we one can show that there is a Brauer-Severi scheme $BS(\mathcal{A})$ which is a projective space bundle over $X$ with general fiber isomorphic to $\mathbb{P}^{n-1}(\mathbb{C})$ embedded in $\mathbb{P}^N(\mathbb{C})$ where $N = \binom{n+k-1}{k} - 1$. Over an arbitrary point of $x$ the fiber may degenerate.

For example if $n = 2$ the $\mathbb{P}^1(\mathbb{C})$ embedded as a conic in $\mathbb{P}^2(\mathbb{C})$ can degenerate into a pair of $\mathbb{P}^1(\mathbb{C})$'s. Now, let us specialize further and consider the case when $X = \mathbb{P}^2$. Consider $E_1$ and $E_2$ two elliptic curves in $\mathbb{P}^2$ and take $\mathcal{C} = \{E_1, E_2\}$ and $\mathcal{P} = \{P_1, \ldots P_9\}$ the intersection points and all the branch data zero. Let $E_i'$ be a twofold unramified cover of $E_i$ , by the Artin-Mumford result there is a quaternion algebra $\Delta$ corresponding to this $\mathbb{Z}_2$-wrinkle.

Next, blow up the intersection points to get a surface $S$ with disjoint elliptic curves $C_1$ and $C_2$. Now take a maximal $\mathcal{O}_S$ order in $\Delta$ then the relevance of the curves $C_i$ is that they are the locus of the points $s \in S$ where $\overline{\mathcal{A}}_s \not\simeq M_2(\mathbb{C})$, the so called *ramification locus* of the order $\mathcal{A}$. The local structure of $\mathcal{A}$ in a point $s \in S$ is

- when $s \notin C_1 \cup C_2$, then $\mathcal{A}_s$ is an Azumaya $\mathcal{O}_{S,s}$-algebra in $\Delta$,
- when $s \in C_i$, then $\mathcal{A}_s = \mathcal{O}_{S,s}.1 \oplus \mathcal{O}_{S,s}.i \oplus \mathcal{O}_{S,s}.j \oplus \mathcal{O}_{S,s}.ij$ with

$$\begin{cases} i^2 &= a \\ j^2 &= bt \\ ji &= -ij \end{cases}$$

  where $t = 0$ is a local equation for $C_i$ and $a$ and $b$ are units in $\mathcal{O}_{S,s}$.

In chapter 5 we will see that this is the local description of a Cayley-smooth order over a smooth surface in a quaternion algebra. Artin and Mumford then define the Brauer-Severi scheme of $\mathcal{A}$ as representing the functor which assigns to an $S$-scheme $S'$ the set of left ideals of $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ which are locally free of rank 2. Using the local description of $\mathcal{A}$ they show that $BS(\mathcal{A})$ is a projective space bundle over $S$ as in figure 3.9 with the properties that $BS(\mathcal{A})$ is a smooth variety and the projection morphism $BS(\mathcal{A}) \xrightarrow{\pi} S$ is flat, all of the geometric fibers being isomorphic to $\mathbb{P}^1$ (resp. to $\mathbb{P}^1 \vee \mathbb{P}^1$) whenever $s \notin C_1 \cup C_2$ (resp. $s \in C_1 \cup C_2$).

Finally, for specific starting configurations $E_1$ and $E_2$, they prove that the obtained Brauer-Severi variety $BS(\mathcal{A})$ cannot be rational because there is torsion in $H^4(BS(\mathcal{A}), \mathbb{Z}_2)$, whereas $BS(\mathcal{A})$ can be shown to be unirational.

## 3.7 Normal spaces

In the next section we will see that in the étale topology we can describe the local structure of representation varieties in the neighborhood of a closed orbit in terms of the normal space to this orbit. In this section we will give a representation theoretic description of this normal space.

We recall some standard facts about tangent spaces first. Let $\mathtt{X}$ be a not necessarily reduced affine variety with coordinate ring $\mathbb{C}[\mathtt{X}] = \mathbb{C}[x_1, \ldots, x_n]/I$. If the origin $o = (0, \ldots, 0) \in \mathbb{V}(I)$, elements of $I$ have no constant terms and we can write any $p \in I$ as

$$p = \sum_{i=1}^{\infty} p^{(i)} \qquad \text{with } p^{(i)} \text{ homogeneous of degree } i.$$

The *order* $ord(p)$ is the least integer $r \geq 1$ such that $p^{(r)} \neq 0$. Define the following two ideals in $\mathbb{C}[x_1, \ldots, x_n]$

$$I_l = \{p^{(1)} \mid p \in I\} \quad \text{ and } I_m = \{p^{(r)} \mid p \in I \text{ and } ord(p) = r\}.$$

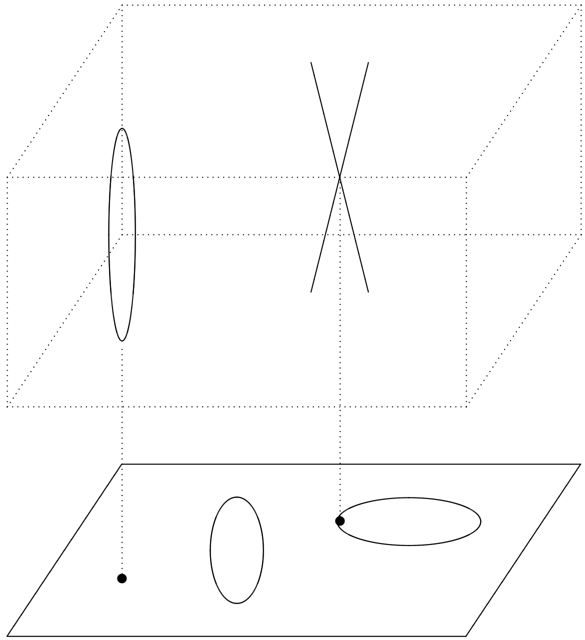


Figure 3.9: The Artin-Mumford bundle

The subscripts $l$ (respectively $m$) stand for *linear terms* (respectively, terms of *minimal* degree).

The *tangent space to* $\mathtt{X}$ *in* $o$ , $T_o(\mathtt{X})$ is by definition the subscheme of $\mathbb{C}^n$ determined by $I_l$. Observe that

$$I_l = (a_{11}x_1 + \ldots + a_{1n}x_n, \ldots, a_{l1}x_1 + \ldots + a_{ln}x_n)$$

for some $l \times n$ matrix $A = (a_{ij})_{i,j}$ of rank $l$. That is, we can express all $x_k$ as linear combinations of some $\{x_{i_1}, \ldots, x_{i_{n-l}}\}$, but then clearly

$$\mathbb{C}[T_o(\mathtt{X})] = \mathbb{C}[x_1, \ldots, x_n]/I_l = \mathbb{C}[x_{i_1}, \ldots, x_{i_{n-l}}]$$

In particular, $T_o(\mathtt{X})$ is reduced and is a linear subspace of dimension $n - l$ in $\mathbb{C}^n$ through the point $o$.

Next, consider an arbitrary geometric point $x$ of $\mathtt{X}$ with coordinates $(a_1, \ldots, a_n)$. We can translate $x$ to the origin $o$ and the translate of $\mathtt{X}$ is then the scheme defined by the ideal

$$(f_1(x_1 + a_1, \ldots, x_n + a_n), \ldots, f_k(x_1 + a_1, \ldots, x_n + a_n))$$

Now, the linear term of the translated polynomial $f_i(x_1 + a_1, \ldots, x_n + a_n)$ is equal to

$$\frac{\partial f_i}{\partial x_1}(a_1, \ldots, a_n)x_1 + \ldots + \frac{\partial f_i}{\partial x_n}(a_1, \ldots, a_n)x_n$$

and hence the *tangent space to* $\mathtt{X}$ *in* $x$ , $T_x(\mathtt{X})$ is the linear subspace of $\mathbb{C}^n$ defined by the set of zeroes of the linear terms

$$T_x(\mathtt{X}) = \mathbb{V}(\sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j}(x)x_j, \ldots, \sum_{j=1}^{n} \frac{\partial f_k}{\partial x_j}(x)x_j) \hookrightarrow \mathbb{C}^n.$$

In particular, the dimension of this linear subspace can be computed from the *Jacobian matrix* in $x$ associated with the polynomials $(f_1, \ldots, f_k)$

$$dim\ T_x(\mathtt{X}) = n - rk \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(x) & \ldots & \frac{\partial f_k}{\partial x_n}(x) \end{bmatrix}.$$

Let $\mathbb{C}[\varepsilon]$ be the *algebra of dual numbers* , that is, $\mathbb{C}[\varepsilon] \simeq \mathbb{C}[y]/(y^2)$. Consider a $\mathbb{C}$-algebra morphism

$$\mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\phi} \mathbb{C}[\varepsilon] \quad \text{defined by } x_i \mapsto a_i + c_i\varepsilon.$$

Because $\varepsilon^2 = 0$ it is easy to verify that the image of a polynomial $f(x_1, \ldots, x_n)$ under $\phi$ is of the form

$$\phi(f(x_1, \ldots, x_n)) = f(a_1, \ldots, a_n) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a_1, \ldots, a_n)c_j\varepsilon$$

Therefore, $\phi$ factors through $I$, that is $\phi(f_i) = 0$ for all $1 \leq i \leq k$, if and only if $(c_1, \ldots, c_n) \in T_x(\mathtt{X})$. Hence, we can also identify the tangent space to $\mathtt{X}$ in $x$ with the algebra morphisms $\mathbb{C}[\mathtt{X}] \xrightarrow{\phi} \mathbb{C}[\varepsilon]$ whose composition with the projection $\pi : \mathbb{C}[\varepsilon] \twoheadrightarrow \mathbb{C}$ (sending $\varepsilon$ to zero) is the evaluation in $x = (a_1, \ldots, a_n)$. That is, let $ev_x \in \mathtt{X}(\mathbb{C})$ be the point corresponding to evaluation in $x$, then

$$T_x(\mathtt{X}) = \{\phi \in \mathtt{X}(\mathbb{C}[\varepsilon]) \mid \mathtt{X}(\pi)(\phi) = ev_x\}.$$

The following two examples compute the tangent spaces to the (trace preserving) representation varieties.

**Example 3.10 (Tangent space to $\mathtt{rep}_n$)** Let $A$ be an affine $\mathbb{C}$-algebra generated by $\{a_1, \ldots a_m\}$ and $\rho : A \longrightarrow M_n(\mathbb{C})$ an algebra morphism, that is, $\rho \in rep_n\ A$. We call a linear map $A \xrightarrow{D} M_n(\mathbb{C})$ a *$\rho$-derivation* if and only if for all $a, a' \in A$ we have that

$$D(aa') = D(a).\rho(a') + \rho(a).D(a').$$

We denote the vectorspace of all $\rho$-derivations of $A$ by $Der_\rho(A)$. Observe that any $\rho$-derivation is determined by its image on the generators $a_i$, hence $Der_\rho(A) \subset M_n^m$. We claim that

$$T_\rho(\mathtt{rep}_n\ A) = Der_\rho(A).$$

Indeed, we know that $\mathtt{rep}_n\ A(\mathbb{C}[\varepsilon])$ is the set of algebra morphisms

$$A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$$

By the functorial characterization of tangentspaces we have that $T_\rho(\mathtt{rep}_n\ A)$ is equal to

$$\{D : A \longrightarrow M_n(\mathbb{C}) \text{ linear} \mid \rho + D\varepsilon : A \longrightarrow M_n(\mathbb{C}[\varepsilon]) \text{ is an algebra map}\}.$$

Because $\rho$ is an algebra morphism, the algebra map condition

$$\rho(aa') + D(aa')\varepsilon = (\rho(a) + D(a)\varepsilon).(\rho(a') + D(a')\varepsilon)$$

is equivalent to $D$ being a $\rho$-derivation.

**Example 3.11 (Tangent space to $\mathtt{trep}_n$)** Let $A$ be a Cayley-Hamilton algebra of degree $n$ with trace map $tr_A$ and trace generated by $\{a_1, \ldots, a_m\}$. Let $\rho \in trep_n\ A$, that is, $\rho : A \longrightarrow M_n(\mathbb{C})$ is a *trace preserving* algebra morphism. Because $\mathtt{trep}_n\ A(\mathbb{C}[\varepsilon])$ is the set of all trace preserving algebra morphisms $A \longrightarrow M_n(\mathbb{C}[\varepsilon])$ (with the usual trace map $tr$ on $M_n(\mathbb{C}[\varepsilon])$) and the previous example one verifies that

$$T_\rho(\mathtt{trep}_n\ A) = Der_\rho^{tr}(A) \subset Der_\rho(A)$$

the subset of *trace preserving $\rho$-derivations* $D$, that is, those satisfying

$$D \circ tr_A = tr \circ D \qquad \begin{array}{ccc} A & \xrightarrow{D} & M_n(\mathbb{C}) \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr} \\ A & \xrightarrow{D} & M_n(\mathbb{C}) \end{array}$$

Again, using this property and the fact that $A$ is *trace* generated by $\{a_1, \ldots, a_m\}$ a trace preserving $\rho$-derivation is determined by its image on the $a_i$ so is a subspace of $M_n^m$.

The *tangent cone to* $\mathtt{X}$ *in* $o$ , $TC_o(\mathtt{X})$, is by definition the subscheme of $\mathbb{C}^n$ determined by $I_m$, that is,

$$\mathbb{C}[TC_o(\mathtt{X})] = \mathbb{C}[x_1, \ldots, x_n]/I_m.$$

It is called a *cone* because if $c$ is a point of the underlying variety of $TC_o(\mathtt{X})$, then the line $l = \overrightarrow{oc}$ is contained in this variety because $I_m$ is a graded ideal. Further, observe that as $I_l \subset I_m$, the tangent cone is a closed subscheme of the tangent space at $\mathtt{X}$ in $o$. Again, if $x$ is an arbitrary geometric point of $\mathtt{X}$ we define the *tangent cone to* $\mathtt{X}$ *in* $x$ , $TC_x(\mathtt{X})$ as the tangent cone $TC_o(\underline{X'})$ where $\underline{X'}$ is the translated scheme of $\mathtt{X}$ under the translation taking $x$ to $o$. Both the tangent space and tangent cone contain *local information* of the scheme $\mathtt{X}$ in a neighborhood of $x$.

Let $m_x$ be the maximal ideal of $\mathbb{C}[\mathtt{X}]$ corresponding to $x$ (that is, the ideal of polynomial functions vanishing in $x$). Then, its complement $\mathcal{S}_x = \mathbb{C}[\mathtt{X}] - m_x$ is a multiplicatively closed subset and the *local algebra* $\mathcal{O}_x(\mathtt{X})$ is the corresponding localization $\mathbb{C}[\mathtt{X}]_{\mathcal{S}_x}$. It has a unique maximal ideal $\mathfrak{m}_x$ with residue field $\mathcal{O}_x(\mathtt{X})/\mathfrak{m}_x = \mathbb{C}$. We equip the local algebra $\mathcal{O}_x = \mathcal{O}_x(\mathtt{X})$ with the $\mathfrak{m}_x$-adic filtration that is the increasing $\mathbb{Z}$-filtration

$$\mathcal{F}_x : \qquad ... \subset \mathfrak{m}^i \subset \mathfrak{m}^{i-1} \subset \ldots \subset \mathfrak{m} \subset \mathcal{O}_x = \mathcal{O}_x = \ldots = \mathcal{O}_x = \ldots$$

with *associated graded algebra*

$$gr(\mathcal{O}_x) = \qquad \ldots \oplus \frac{\mathfrak{m}_x^i}{\mathfrak{m}_x^{i+1}} \oplus \frac{\mathfrak{m}_x^{i-1}}{\mathfrak{m}_x^i} \oplus \ldots \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \oplus \mathbb{C} \oplus 0 \oplus \ldots \oplus 0 \oplus \ldots$$

**Proposition 3.12** *If $x$ is a geometric point of the affine scheme $\mathtt{X}$, then*

1. *$\mathbb{C}[T_x(\mathtt{X})]$ is isomorphic to the polynomial algebra $\mathbb{C}[\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}]$.*
2. *$\mathbb{C}[TC_x(\mathtt{X})]$ is isomorphic to the associated graded algebra $gr(\mathcal{O}_x(\mathtt{X}))$.*

*Proof.* After translating we may assume that $x = o$ lies in $\mathbb{V}(I) \hookrightarrow \mathbb{C}^n$. That is,

$$\mathbb{C}[\mathtt{X}] = \mathbb{C}[x_1, \ldots, x_n]/I \quad \text{and} \quad m_x = (x_1, \ldots, x_n)/I.$$

(1) : Under these identifications we have

$$\begin{aligned} \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} &\simeq m_x/m_x^2 \\ &\simeq (x_1, \ldots, x_n)/((x_1, \ldots, x_n)^2 + I) \\ &\simeq (x_1, \ldots, x_n)/((x_1, \ldots, x_n)^2 + I_l) \end{aligned}$$

and as $I_l$ is generated by linear terms it follows that the polynomial algebra on $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ is isomorphic to the quotient algebra $\mathbb{C}[x_1, \ldots, x_n]/I_l$ which is by definition the coordinate ring of the tangent space.
(2) : Again using the above identifications we have

$$\begin{aligned} gr(\mathcal{O}_x) &\simeq \oplus_{i=0}^{\infty} \mathfrak{m}_x^i/\mathfrak{m}_x^{i+1} \\ &\simeq \oplus_{i=0}^{\infty} m_x^i/m_x^{i+1} \\ &\simeq \oplus_{i=0}^{\infty} (x_1, \ldots, x_n)^i/((x_1, \ldots, x_n)^{i+1} + (I \cap (x_1, \ldots, x_n)^i)) \\ &\simeq \oplus_{i=0}^{\infty} (x_1, \ldots, x_n)^i/((x_1, \ldots, x_n)^{i+1} + I_m(i)) \end{aligned}$$

where $I_m(i)$ is the homogeneous part of $I_m$ of degree $i$. On the other hand, the $i$-th homogeneous part of $\mathbb{C}[x_1,\ldots,x_n]/I_m$ is equal to

$$\frac{(x_1,\ldots,x_n)^i}{(x_1,\ldots,x_n)^{i+1}+I_m(i)}$$

we obtain the required isomorphism. □

This gives a third interpretation of the tangent space as

$$T_x(\mathtt{X}) = Hom_{\mathbb{C}}(\frac{m_x}{m_x^2},\mathbb{C}) = Hom_{\mathbb{C}}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2},\mathbb{C}).$$

Hence, we can also view the tangent space $T_x(\mathtt{X})$ as the space of *point derivations* $Der_x(\mathcal{O}_x)$ on $\mathcal{O}_x(\mathtt{X})$ (or of the point derivations $Der_x(\mathbb{C}[\mathtt{X}])$ on $\mathbb{C}[\mathtt{X}]$). That is, $\mathbb{C}$-linear maps $D : \mathcal{O}_x \longrightarrow \mathbb{C}$ (or $D : \mathbb{C}[\mathtt{X}] \longrightarrow \mathbb{C}$) such that for all functions $f, g$ we have

$$D(fg) = D(f)g(x) + f(x)D(g).$$

If we define the *local dimension of an affine scheme* $\mathtt{X}$ *in a geometric point* $x$ $dim_x\ \mathtt{X}$ to be the maximal dimension of irreducible components of the reduced variety $X$ passing through $x$, then

$$dim_x\ \mathtt{X} = dim_o\ TC_x(\mathtt{X}).$$

We say that $\mathtt{X}$ *is nonsingular at* $x$ (or equivalently, that $x$ *is a nonsingular point of* $\mathtt{X}$) if the tangent cone to $\mathtt{X}$ in $x$ coincides with the tangent space to $\mathtt{X}$ in $x$. An immediate consequence is

**Proposition 3.13** *If* $\mathtt{X}$ *is nonsingular at* $x$*, then* $\mathcal{O}_x(\mathtt{X})$ *is a domain. That is, in a Zariski neighborhood of* $x$ *,* $\mathtt{X}$ *is an irreducible variety.*

*Proof.* If $\mathtt{X}$ is nonsingular at $x$, then

$$gr(\mathcal{O}_x) \simeq \mathbb{C}[TC_x(\mathtt{X})] = \mathbb{C}[T_x(\mathtt{X})]$$

the latter one being a polynomial algebra whence a domain. Now, let $0 \neq a, b \in \mathcal{O}_x$ then there exist $k, l$ such that $a \in \mathfrak{m}^k - \mathfrak{m}^{k+1}$ and $b \in \mathfrak{m}^l - \mathfrak{m}^{l+1}$, that is $\overline{a}$ is a nonzero homogeneous element of $gr(\mathcal{O}_x)$ of degree $-k$ and $\overline{b}$ one of degree $-l$. But then, $\overline{a}.\overline{b} \in \mathfrak{m}^{k+l} - \mathfrak{m}^{k+l-1}$ hence certainly $a.b \neq 0$ in $\mathcal{O}_x$.

Now, consider the natural map $\phi : \mathbb{C}[\mathtt{X}] \longrightarrow \mathcal{O}_x$. Let $\{P_1,\ldots,P_l\}$ be the minimal prime ideals of $\mathbb{C}[\mathtt{X}]$. All but one of them, say $P_1 = \phi^{-1}(0)$, extend to the whole ring $\mathcal{O}_x$. Taking the product of $f$ functions $f_i \in P_i$ nonvanishing in $x$ for $2 \leq i \leq l$ gives the Zariski open set $\mathbb{X}(f)$ containing $x$ and whose coordinate ring is a domain, whence $\mathbb{X}(f)$ is an affine irreducible variety. □

When restricting to nonsingular points we reduce to irreducible affine varieties. From the Jacobian condition it follows that nonsingularity is a Zariski open condition on $\mathtt{X}$ and by the implicit function theorem, $\mathtt{X}$ is a complex manifold in a neighborhood of a nonsingular point.

Let $\mathtt{X} \xrightarrow{\phi} \mathtt{Y}$ be a morphism of affine varieties corresponding to the algebra morphism $\mathbb{C}[\mathtt{Y}] \xrightarrow{\phi^*} \mathbb{C}[\mathtt{X}]$. Let $x$ be a geometric point of $\mathtt{X}$ and $y = \phi(x)$. As $\phi^*(m_y) \subset m_x$, $\phi$ induces a linear map $\frac{m_y}{m_y^2} \longrightarrow \frac{m_x}{m_x^2}$ and taking the dual map gives the *differential of* $\phi$ *in* $x$ which is a linear map

$$d\phi_x : T_x(\mathtt{X}) \longrightarrow T_{\phi(x)}(\mathtt{Y}).$$

Assume $\mathtt{X}$ a closed subscheme of $\mathbb{C}^n$ and $\mathtt{Y}$ a closed subscheme of $\mathbb{C}^m$ and let $\phi$ be determined by the $m$ polynomials $\{f_1, \ldots, f_m\}$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then, the Jacobian matrix in $x$

$$J_x(\phi) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_m}{\partial x_1}(x) \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

defines a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ and the differential $d\phi_x$ is the induced linear map from $T_x(\mathtt{X}) \subset \mathbb{C}^n$ to $T_{\phi(x)}(\mathtt{Y}) \subset \mathbb{C}^m$. Let $D \in T_x(\mathtt{X}) = Der_x(\mathbb{C}[\mathtt{X}])$ and $x_D$ the corresponding element of $\mathtt{X}(\mathbb{C}[\varepsilon])$ defined by $x_D(f) = f(x) + D(f)\varepsilon$, then $x_D \circ \phi^* \in \mathtt{Y}(\mathbb{C}[\varepsilon])$ is defined by

$$x_D \circ \phi^*(g) = g(\phi(x)) + (D \circ \phi^*)\varepsilon = g(\phi(x)) + d\phi_x(D)\varepsilon$$

giving us the $\varepsilon$-interpretation of the differential

$$\phi(x + v\varepsilon) = \phi(x) + d\phi_x(v)\varepsilon$$

for all $v \in T_x(\mathtt{X})$.

**Proposition 3.14** *Let* $X \xrightarrow{\phi} Y$ *be a dominant morphism between irreducible affine varieties. There is a Zariski open dense subset* $U \hookrightarrow X$ *such that* $d\phi_x$ *is surjective for all* $x \in U$.

*Proof.* We may assume that $\phi$ factorizes into

$$\begin{tikzcd} X \arrow[r, two heads, "\rho"] \arrow[dr, "\phi"'] & Y \times \mathbb{C}^d \arrow[d, "pr_Y"] \\ & Y \end{tikzcd}$$

with $\phi$ a finite and surjective morphism. Because the tangent space of a product is the sum of the tangent spaces of the components we have that $d(pr_W)_z$ is surjective for all $z \in Y \times \mathbb{C}^d$, hence it suffices to verify the claim for a *finite* morphism $\phi$. That is, we may assume that $S = \mathbb{C}[Y]$ is a finite module over $R = \mathbb{C}[X]$ and let $L/K$ be the corresponding extension of the function fields. By the *principal element theorem* we know that $L = K[s]$ for an element $s \in L$ which is integral over $R$ with minimal polynomial

$$F = t^n + g_{n-1}t^{n-1} + \ldots + g_1 t + g_0 \quad \text{with } g_i \in R$$

Consider the ring $S' = R[t]/(F)$ then there is an element $r \in R$ such that the localizations $S'_r$ and $S_r$ are isomorphic. By restricting we may assume that $X = \mathbb{V}(F) \hookrightarrow Y \times \mathbb{C}$ and that

$$\begin{tikzcd} X = \mathbb{V}(F) \arrow[r, hook] \arrow[dr, "\phi"'] & Y \times \mathbb{C} \arrow[d, "pr_Y"] \\ & Y \end{tikzcd}$$

Let $x = (y, c) \in X$ then we have (again using the identification of the tangent space of a product with the sum of the tangent spaces of the components) that

$$T_x(X) = \{(v, a) \in T_y(Y) \oplus \mathbb{C} \mid c\frac{\partial F}{\partial t}(x) + vg_{n-1}c^{n-1} + \ldots + vg_1c + vg_0 = 0\}.$$

But then, $d\phi_x$ i surjective whenever $\frac{\partial F}{\partial t}(x) \neq 0$. This condition determines a *non-empty* open subset of $X$ as otherwise $\frac{\partial F}{\partial t}$ would belong to the defining ideal of $X$ in $\mathbb{C}[Y \times \mathbb{C}]$ (which is the principal ideal generated by $F$) which is impossible by a degree argument $\square$

**Example 3.12 (Differential of orbit map)** Let $\mathtt{X}$ be a closed $GL_n$-stable subscheme of a $GL_n$-representation $V$ and $x$ a geometric point of $\mathtt{X}$. Consider the orbitclosure $\overline{\mathcal{O}(x)}$ of $x$ in $V$. Because the orbit map

$$\mu : \; GL_n \twoheadrightarrow GL_n.x \hookrightarrow \overline{\mathcal{O}(x)}$$

is dominant we have that $\mathbb{C}[\overline{\mathcal{O}(x)}] \hookrightarrow \mathbb{C}[GL_n]$ and therefore a domain, so $\overline{\mathcal{O}(x)}$ is an irreducible affine variety. We define the *stabilizer subgroup* $Stab(x)$ to be the fiber $\mu^{-1}(x)$, then $Stab(x)$ is a closed subgroup of $GL_n$. We claim that the differential of the orbit map in the identity matrix $e = \mathbb{1}_n$

$$d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\mathtt{X})$$

satisfies the following properties

$$Ker\ d\mu_e = \mathfrak{stab}(x) \quad \text{and} \quad Im\ d\mu_e = T_x(\overline{\mathcal{O}(x)}).$$

By the proposition we know that there is a dense open subset $U$ of $GL_n$ such that $d\mu_g$ is surjective for all $g \in U$. By $GL_n$-equivariance of $\mu$ it follows that $d\mu_g$ is surjective for all $g \in GL_n$, in particular $d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\overline{\mathcal{O}(x)})$ is surjective. Further, all fibers of $\mu$ over $\mathcal{O}(x)$ have the same dimension. But then it follows from the *dimension formula* of proposition that

$$dim\ GL_n = dim\ Stab(x) + dim\ \overline{\mathcal{O}(x)}$$

(which, incidentally gives us an algorithm to compute the dimensions of orbitclosures). Combining this with the above surjectivity, a dimension count proves that $Ker\ d\mu_e = \mathfrak{stab}(x)$, the Lie algebra of $Stab(x)$.

Let $A$ be a $\mathbb{C}$-algebra and let $M$ and $N$ be two $A$-representations of dimensions say $m$ and $n$. An $A$-representation $P$ of dimension $m+n$ is said to be an *extension of $N$ by $M$* if there exists a short exact sequence of left $A$-modules

$$e : \qquad 0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

We define an equivalence relation on extensions $(P,e)$ of $N$ by $M$ : $(P,e) \cong (P',e')$ if and only if there is an isomorphism $P \xrightarrow{\phi} P'$ of left $A$-modules such that the diagram below is commutative

$$\begin{tikzcd}
e: & 0 \arrow[r] & M \arrow[r] \arrow[d, "id_M"'] & P \arrow[r] \arrow[d, "\phi"] & N \arrow[r] \arrow[d, "id_N"] & 0 \\
e': & 0 \arrow[r] & M \arrow[r] & P' \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

The set of equivalence classes of extensions of $N$ by $M$ will be denoted by $Ext^1_A(N,M)$.

An alternative description of $Ext^1_A(N,M)$ is as follows. Let $\rho : A \longrightarrow M_m$ and $\sigma : A \longrightarrow M_n$ be the representations defining $M$ and $N$. For an extension $(P,e)$ we can identify the $\mathbb{C}$-vectorspace with $M \oplus N$ and the $A$-module structure on $P$ gives a algebra map $\mu : A \longrightarrow M_{m+n}$ and we can represent the action of $a$ on $P$ by left multiplication of the block-matrix

$$\mu(a) = \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix},$$

where $\lambda(a)$ is an $m \times n$ matrix and hence defines a linear map

$$\lambda : A \longrightarrow Hom_{\mathbb{C}}(N,M).$$

The condition that $\mu$ is an algebra morphism is equivalent to the condition

$$\lambda(aa') = \rho(a)\lambda(a') + \lambda(a)\sigma(a')$$

and we denote the set of all liner maps $\lambda : A \longrightarrow Hom_{\mathbb{C}}(N,M)$ by $Z(N,M)$ and call it the space of *cycle* . The extensions of $N$ by $M$ corresponding to two cycles $\lambda$ and $\lambda'$ from $Z(N,M)$ are equivalent if and only if we have an $A$-module isomorphism in block form

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{with } \beta \in Hom_{\mathbb{C}}(N,M)$$

between them. $A$-linearity of this map translates into the matrix relation

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} . \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix} = \begin{bmatrix} \rho(a) & \lambda'(a) \\ 0 & \sigma(a) \end{bmatrix} . \begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{for all } a \in A$$

or equivalently, that $\lambda(a) - \lambda'(a) = \rho(a)\beta - \beta\sigma(a)$ for all $a \in A$. We will now define the subspace of $Z(N,M)$ of *boundaries* $B(N,M)$

$$\{\delta \in Hom_{\mathbb{C}}(N,M) \mid \exists \beta \in Hom_{\mathbb{C}}(N,M) : \forall a \in A : \delta(a) = \rho(a)\beta - \beta\sigma(a)\}.$$

We then have the description $Ext^1_A(N,M) = \frac{Z(N,M)}{B(N,M)}$.

**Example 3.13 (Normal space to $\mathtt{rep}_n$)** Let $A$ be an affine $\mathbb{C}$-algebra generated by $\{a_1,\ldots,a_m\}$ and $\rho : A \longrightarrow M_n(\mathbb{C})$ an algebra morphism, that is, $\rho \in rep_n\ A$ determines an $n$-dimensional $A$-representation $M$. We claim to have the following description of the *normal space* to the orbitclosure $C_\rho = \overline{\mathcal{O}(\rho)}$ of $\rho$

$$N_\rho(\mathtt{rep}_n\ A) \overset{def}{=} \frac{T_\rho(\mathtt{rep}_n\ A)}{T_\rho(C_\rho)} = Ext^1_A(M,M).$$

We have already seen that the space of cycles $Z(M,M)$ is the space of $\rho$-derivations of $A$ in $M_n(\mathbb{C})$, $Der_\rho(A)$, which we know to be the tangent space $T_\rho(\mathtt{rep}_n\ A)$. Moreover, we know that the differential $d\mu_e$ of the orbit map $GL_n \xrightarrow{\mu} C_\rho \hookrightarrow M_n^m$

$$d\mu_e : \quad \mathfrak{gl}_n = M_n \longrightarrow T_\rho(C_\rho)$$

is surjective. Now, $\rho = (\rho(a_1),\ldots,\rho(a_m)) \in M_n^m$ and the action of action of $GL_n$ is given by simultaneous conjugation. But then we have for any $A \in \mathfrak{gl}_n = M_n$ that

$$(I_n + A\varepsilon).\rho(a_i).(I_n - A\varepsilon) = \rho(a_i) + (A\rho(a_i) - \rho(a_i)A)\varepsilon.$$

Therefore, by definition of the differential we have that

$$d\mu_e(A)(a) = A\rho(a) - \rho(a)A \qquad \text{for all } a \in A.$$

that is, $d\mu_e(A) \in B(M,M)$ and as the differential map is surjective we have $T_\rho(C_\rho) = B(M,M)$ from which the claim follows.

**Example 3.14 (Normal space to $\mathtt{trep}_n$)** Let $A$ be a Cayley-Hamilton algebra with trace map $tr_A$ and trace generated by $\{a_1,\ldots,a_m\}$. Let $\rho \in trep_n\ A$, that is, $\rho : A \longrightarrow M_n(\mathbb{C})$ is a trace preserving algebra morphism. Any cycle $\lambda : A \longrightarrow M_n(\mathbb{C})$ in $Z(M,M) = Der_\rho(A)$ determines an algebra morphism

$$\rho + \lambda\varepsilon : A \longrightarrow M_n(\mathbb{C}[\varepsilon])$$

We know that the tangent space $T_\rho(\mathtt{trep}_n\ A)$ is the subspace $Der^{tr}_\rho(A)$ of trace preserving $\rho$-derivations, that is, those satisfying

$$\lambda(tr_A(a)) = tr(\lambda(a)) \quad \text{for all } a \in A$$

Observe that for all boundaries $\delta \in B(M,M)$, that is, such that there is an $m \in M_n(\mathbb{C})$ with $\delta(a) = \rho(a).m - m.\rho(a)$ are trace preserving as

$$\begin{aligned} \delta(tr_A(a)) = & \quad \rho(tr_A(a)).m - m.\rho(tr_A(a)) = tr(\rho(a)).m - m.tr(\rho(a)) \\ = & \quad 0 = tr(m.\rho(a) - \rho(a).m) = tr(\delta(a)) \end{aligned}$$

Hence, we can define the space of *trace preserving self-extensions*

$$Ext^{tr}_A(M,M) = \frac{Der^{tr}_\rho(A)}{B(M,M)}$$

and obtain as before that the *normal space* to the orbit closure $C_\rho = \overline{\mathcal{O}(\rho)}$ is equal to

$$N_\rho(\mathtt{trep}_n\ A) \overset{def}{=} \frac{T_\rho(\mathtt{trep}_n\ A)}{T_\rho(C_\rho)} = Ext^{tr}_A(M,M)$$

## 3.8 Knop-Luna slices

Let $A$ be an affine $\mathbb{C}$-algebra and $\xi \in \mathtt{iss}_n\ A$ a point in the quotient space corresponding to an $n$-dimensional semi-simple representation $M_\xi$ of $A$. In the next chapter we will present a method to study the étale local structure of $\mathtt{iss}_n\ A$ near $\xi$ and the étale local $GL_n$-structure of the representation variety $\mathtt{rep}_n\ A$ near the closed orbit $\mathcal{O}(M_\xi) = GL_n.M_\xi$. First, we will outline the main idea in the setting of differential geometry.

Let $M$ be a compact $\mathcal{C}^\infty$-manifold on which a compact Lie group $G$ acts differentially. By a usual averaging process we can define a $G$-invariant Riemannian metric on $M$. For a point $m \in M$ we define

- The $G$-orbit $\mathcal{O}(m) = G.m$ of $m$ in $M$,
- the stabilizer subgroup $H = Stab_G(m) = \{g \in G \mid g.m = m\}$ and
- the normal space $N_m$ defined to be the orthogonal complement to the tangent space in $m$ to the orbit in the tangent space to $M$. That is, we have a decomposition of $H$-vectorspaces

$$T_m\ M = T_m\ \mathcal{O}(m) \oplus N_m$$

The normal spaces $N_x$ when $x$ varies over the points of the orbit $\mathcal{O}(m)$ define a vectorbundle $\mathcal{N} \overset{p}{\twoheadrightarrow} \mathcal{O}(m)$ over the orbit. We can identify the bundle with the associated fiber bundle

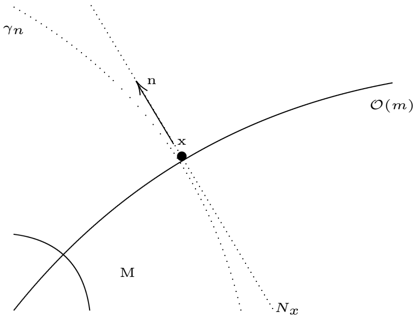
$$\mathcal{N} \simeq G \times^H N_m$$

Any point $n \in \mathcal{N}$ in the normal bundle determines a geodesic

$$\gamma_n : \mathbb{R} \longrightarrow M \quad \text{defined by} \quad \begin{cases} \gamma_n(0) & = p(n) \\ \frac{d\gamma_n}{dt}(0) & = n \end{cases}$$

Using this geodesic we can define a $G$-equivariant exponential map from the normal bundle $\mathcal{N}$ to the manifold $M$ via

$$\mathcal{N} \xrightarrow{exp} M \qquad \text{where} \qquad exp(n) = \gamma_n(1)$$




Now, take $\varepsilon > 0$ and define the $\mathcal{C}^\infty$ slice $S_\varepsilon$ to be

$$S_\varepsilon = \{ n \in N_m \quad | \quad \| n \| < \varepsilon \}$$

then $G \times^H S_\varepsilon$ is a $G$-stable neighborhood of the zero section in the normal bundle $\mathcal{N} = G \times^H N_m$. But then we have a $G$-equivariant exponential

$$G \times^H S_\varepsilon \xrightarrow{exp} M$$

which for small enough $\varepsilon$ gives a diffeomorphism with a $G$-stable tubular neighborhood $U$ of the orbit $\mathcal{O}(m)$ in $M$ as in figure 3.10 If we assume moreover that the action of $G$ on $M$ and the action of $H$ on $N_m$ are such that the orbit-spaces are manifolds $M/G$ and $N_m/H$, then we have the situation

$$\begin{tikzcd} G \times^H S_\varepsilon \arrow[r, "exp", "\simeq"'] \arrow[d] & U \arrow[r, hook] \arrow[d] & M \arrow[d] \\ S_\varepsilon/H \arrow[r, "\simeq"'] & U/G \arrow[r, hook] & M/G \end{tikzcd}$$

giving a local diffeomorphism between a neighborhood of $\overline{0}$ in $N_m/H$ and a neighborhood of the point $\overline{m}$ in $M/G$ corresponding to the orbit $\mathcal{O}(m)$.

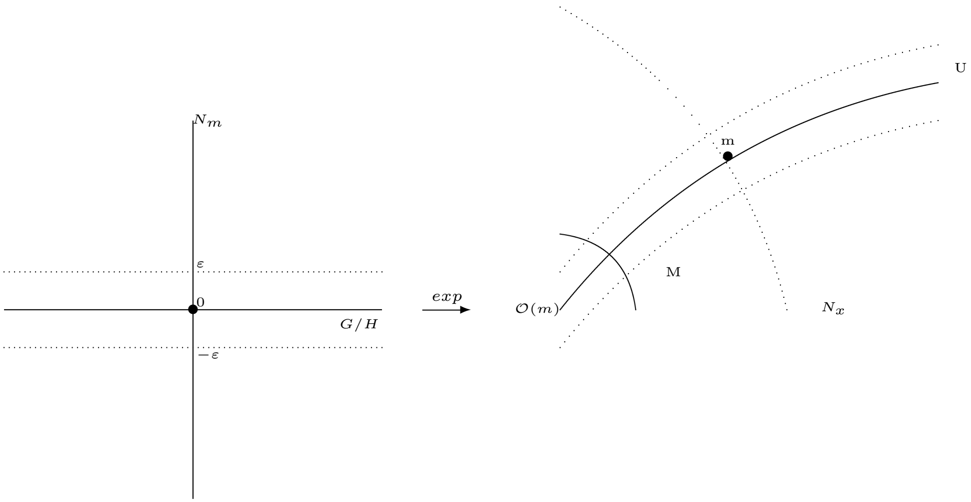



Figure 3.10: Tubular neighborhood of the orbit.

Returning to the setting of the orbit $\mathcal{O}(M_\xi)$ in $\mathtt{rep}_n\ A$ we would equally like to define a $GL_n$-equivariant morphism from an associated fiber bundle

$$GL_n \times^{GL(\alpha)} N_\xi \xrightarrow{e} \mathtt{rep}_n\ A$$

where $GL(\xi)$ is the stabilizer subgroup of $M_\xi$ and $N_\xi$ is a normal space to the orbit $\mathcal{O}(M_\xi)$. Because we do not have an exponential-map in the setting of algebraic geometry, the map $e$ will have to be an étale map. Such a map does exist and is usually called a *Luna slice* in case of a smooth point on $\mathtt{rep}_n\ A$. Later, F. Knop extended this result to allow singular points, or even points in which the scheme is not reduced.

Although the result holds for any reductive algebraic group $G$, we will apply them only in the case $G = GL_n$ or $GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k}$, so restrict to the case of $GL_n$. We fix the setting :

$\mathtt{X}$ and $\mathtt{Y}$ are (not necessarily reduced) affine $GL_n$-varieties, $\psi$ is a $GL_n$-equivariant map

$$\begin{tikzcd} x = \psi(y) \arrow[r, phantom, "\vdash"] & \mathtt{X} \arrow[d, "\pi_X"'] & \mathtt{Y} \arrow[l, "\psi"'] \arrow[d, "\pi_Y"] \arrow[r, phantom, "\dashv"] & y \\ & \mathtt{X}/GL_n & \mathtt{Y}/GL_n & \end{tikzcd}$$

and we assume the following restrictions :

- $\psi$ is étale in $y$,
- the $GL_n$-orbits $\mathcal{O}(y)$ in $\mathtt{Y}$ and $\mathcal{O}(x)$ in $\mathtt{X}$ are closed. For example, in representation varieties, we restrict to semi-simple representations,
- the stabilizer subgroups are equal $Stab(x) = Stab(y)$. In the case of representation varieties, for a semi-simple $n$-dimensional representation with decomposition

  $$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

  into distinct simple components, this stabilizer subgroup is

  $$GL(\alpha) = \begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{d_1}) & & \\ & \ddots & \\ & & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{d_k}) \end{bmatrix} \hookrightarrow GL_n$$

  where $d_i = dim\ S_i$. In particular, the stabilizer subgroup is again reductive.

In algebraic terms : consider the coordinate rings $R = \mathbb{C}[\mathtt{X}]$ and $S = \mathbb{C}[\mathtt{Y}]$ and the dual morphism $R \xrightarrow{\psi^*} S$. Let $I \triangleleft R$ be the ideal describing $\mathcal{O}(x)$ and $J \triangleleft S$ the ideal describing $\mathcal{O}(y)$. With $\widehat{R}$ we will denote the $I$-adic completion $\underset{\leftarrow}{lim}\ \frac{R}{I^n}$ of $R$ and with $\widehat{S}$ the $J$-adic completion of $S$.

**Lemma 3.8** *The morphism $\psi^*$ induces for all $n$ an isomorphism*

$$\frac{R}{I^n} \xrightarrow{\psi^*} \frac{S}{J^n}$$

*In particular, $\widehat{R} \simeq \widehat{S}$.*

*Proof.* Let $\underline{Z}$ be the closed $GL_n$-stable subvariety of $\mathtt{Y}$ where $\psi$ is not étale. By the separation property, there is an invariant function $f \in S^{GL_n}$ vanishing on $\underline{Z}$ such that $f(y) = 1$ because the two closed $GL_n$-subschemes $\underline{Z}$ and $\mathcal{O}(y)$ are disjoint. Replacing $S$ by $S_f$ we may assume that $\psi^*$ is an étale morphism. Because $\mathcal{O}(x)$ is smooth, $\psi^{-1}\ \mathcal{O}(x)$ is the disjoint union of its irreducible components and restricting $\mathtt{Y}$ if necessary we may assume that $\psi^{-1}\ \mathcal{O}(x) = \mathcal{O}(y)$. But then $J = \psi^*(I)S$ and as $\mathcal{O}(y) \xrightarrow{\simeq} \mathcal{O}(x)$ we have $\frac{R}{I} \simeq \frac{S}{J}$ so the result holds for $n = 1$.

Because étale maps are flat, we have $\psi^*(I^n)S = I^n \otimes_R S = J^n$ and an exact sequence

$$0 \longrightarrow I^{n+1} \otimes_R S \longrightarrow I^n \otimes_R S \longrightarrow \frac{I^n}{I^{n+1}} \otimes_R S \longrightarrow 0$$

But then we have

$$\frac{I^n}{I^{n+1}} = \frac{I^n}{I^{n+1}} \otimes_{R/I} \frac{S}{J} = \frac{I^n}{I^{n+1}} \otimes_R S \simeq \frac{J^n}{J^{n+1}}$$

and the result follows from induction on $n$ and the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{I^n}{I^{n+1}} \arrow[r] \arrow[d, "\simeq"'] & \frac{R}{I^{n+1}} \arrow[r] \arrow[d, dotted] & \frac{R}{I^n} \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \frac{J^n}{J^{n+1}} \arrow[r] & \frac{S}{J^{n+1}} \arrow[r] & \frac{S}{J^n} \arrow[r] & 0
\end{tikzcd}$$

□

For an irreducible $GL_n$-representation $s$ and a locally finite $GL_n$-module $X$ we denote its $s$-isotypical component by $X_{(s)}$.

**Lemma 3.9** *Let $s$ be an irreducible $GL_n$-representation. There are natural numbers $m \geq 1$ (independent of $s$) and $n \geq 0$ such that for all $k \in \mathbb{N}$ we have*

$$I^{mk+n} \cap R_{(s)} \hookrightarrow (I^{GL_n})^k R_{(s)} \hookrightarrow I^k \cap R_{(s)}$$

*Proof.* Consider $A = \oplus_{i=0}^{\infty} I^n t^n \hookrightarrow R[t]$, then $A^{GL_n}$ is affine so certainly finitely generated as $R^{GL_n}$-algebra say by

$$\{r_1 t^{m_1}, \ldots, r_z t^{m_z}\} \quad \text{with } r_i \in R \text{ and } m_i \geq 1.$$

Further, $A_{(s)}$ is a finitely generated $A^{GL_n}$-module, say generated by

$$\{s_1 t^{n_1}, \ldots, s_y t^{n_y}\} \quad \text{with } s_i \in R_{(s)} \text{ and } n_i \geq 0.$$

Take $m = max\ m_i$ and $n = max\ n_i$ and $r \in I^{mk+n} \cap R_{(s)}$, then $rt^{mk+n} \in A_{(s)}$ and

$$rt^{mk+n} = \sum_j p_j(r_1 t^{m_1}, \ldots, r_z t^{m_z}) s_j t^{n_j}$$

with $p_j$ a homogeneous polynomial of $t$-degree $mk + n - n_j \geq mk$. But then each monomial in $p_j$ occurs at least with ordinary degree $\frac{mk}{m} = k$ and therefore is contained in $(I^{GL_n})^k R_{(s)} t^{mk+n}$. $\square$

Let $\widehat{R^{GL_n}}$ be the $I^{GL_n}$-adic completion of the invariant ring $R^{GL_n}$ and let $\widehat{S^{GL_n}}$ be the $J^{GL_n}$-adic completion of $S^{GL_n}$.

**Lemma 3.10** *The morphism $\psi^*$ induces an isomorphism*

$$R \otimes_{R^{GL_n}} \widehat{R^{GL_n}} \xrightarrow{\simeq} S \otimes S^{GL_n} \widehat{S^{GL_n}}$$

*Proof.* Let $s$ be an irreducible $GL_n$-module, then the $I^{GL_n}$-adic completion of $R_{(s)}$ is equal to $\widehat{R_{(s)}} = R_{(s)} \otimes_{R^{GL_n}} \widehat{R^{GL_n}}$. Moreover,

$$\widehat{R}_{(s)} = \underset{\leftarrow}{lim}(\frac{R}{I^k})_{(s)} = \underset{\leftarrow}{lim} \frac{R_{(s)}}{(I^k \cap R_{(s)})}$$

which is the $I$-adic completion of $R_{(s)}$. By the foregoing lemma both topologies coincide on $R_{(s)}$ and therefore

$$\widehat{R_{(s)}} = \widehat{R}_{(s)} \quad \text{and similarly} \quad \widehat{S_{(s)}} = \widehat{S}_{(s)}$$

Because $\widehat{R} \simeq \widehat{S}$ it follows that $\widehat{R}_{(s)} \simeq \widehat{S}_{(s)}$ from which the result follows as the foregoing holds for all $s$. $\square$

**Theorem 3.13** *Consider a $GL_n$-equivariant map $\mathtt{Y} \xrightarrow{\psi} \mathtt{X}$, $y \in \mathtt{Y}$, $x = \psi(y)$ and $\psi$ étale in $y$. Assume that the orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are closed and that $\psi$ is injective on $\mathcal{O}(y)$. Then, there is an affine open subset $U \hookrightarrow \mathtt{Y}$ containing $y$ such that*

1. *$U = \pi_Y^{-1}(\pi_Y(U))$ and $\pi_Y(U) = U/GL_n$.*

2. *$\psi$ is étale on $U$ with affine image.*

3. *The induced morphism $U/GL_n \xrightarrow{\overline{\psi}} \mathtt{X}/GL_n$ is étale.*

4. *The diagram below is commutative*

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "\pi_U"'] & \mathtt{X} \arrow[d, "\pi_X"] \\ U/GL_n \arrow[r, "\overline{\psi}"] & \mathtt{X}/GL_n \end{tikzcd}$$

*Proof.* By the foregoing lemma we have $\widehat{R^{GL_n}} \simeq \widehat{S^{GL_n}}$ which means that $\overline{\psi}$ is étale in $\pi_Y(y)$. As étaleness is an open condition, there is an open affine neighborhood $V$ of $\pi_Y(y)$ on which $\overline{\psi}$ is étale. If $\overline{R} = R \otimes_{R^{GL_n}} S^{GL_n}$ then the above lemma implies that

$$\overline{R} \otimes_{S^{GL_n}} \widehat{S^{GL_n}} \simeq S \otimes_{S^{GL_n}} \widehat{S^{GL_n}}$$

Let $S^{GL_n}_{loc}$ be the local ring of $S^{GL_n}$ in $J^{GL_n}$, then as the morphism $S^{GL_n}_{loc} \longrightarrow \widehat{S^{GL_n}}$ is faithfully flat we deduce that

$$\overline{R} \otimes_{S^{GL_n}} S^{GL_n}_{loc} \simeq S \otimes_{S^{GL_n}} S^{GL_n}_{loc}$$

but then there is an $f \in S^{GL_n} - J^{GL_n}$ such that $\overline{R}_f \simeq S_f$. Now, intersect $V$ with the open affine subset where $f \neq 0$ and let $U'$ be the inverse image under $\pi_Y$ of this set. Remains to prove that the image of $\psi$ is affine. As $U' \xrightarrow{\psi} X$ is étale, its image is open and $GL_n$-stable. By the separation property we can find an invariant $h \in R^{GL_n}$ such that $h$ is zero on the complement of the image and $h(x) = 1$. But then we take $U$ to be the subset of $U'$ of points $u$ such that $h(u) \neq 0$. $\square$

**Theorem 3.14 (Slice theorem)** *Let* $\mathtt{X}$ *be an affine* $GL_n$*-variety with quotient map* $\mathtt{X} \overset{\pi}{\twoheadrightarrow} \mathtt{X}/GL_n$*. Let* $x \in X$ *be such that its orbit* $\mathcal{O}(x)$ *is closed and its stabilizer subgroup* $Stab(x) = H$ *is reductive. Then, there is a locally closed affine subscheme* $\mathtt{S} \hookrightarrow \mathtt{X}$ *containing* $x$ *with the following properties*

1. $\mathtt{S}$ *is an affine* $H$*-variety,*

2. *the action map* $GL_n \times \mathtt{S} \longrightarrow \mathtt{X}$ *induces an étale* $GL_n$*-equivariant morphism* $GL_n \times^H \mathtt{S} \xrightarrow{\psi} \mathtt{X}$ *with affine image,*

3. *the induced quotient map* $\psi/GL_n$ *is étale*

$$(GL_n \times^H \mathtt{S})/GL_n \simeq \mathtt{S}/H \xrightarrow{\psi/GL_n} \mathtt{X}/GL_n$$

*and the right hand side of figure 3.11 is commutative.*

$$\begin{tikzcd} GL_n \times^H N_x \arrow[d] & GL_n \times^H \mathtt{S} \arrow[l, "GL_n \times^H \phi"'] \arrow[r, "\psi"] \arrow[d] & \mathtt{X} \arrow[d, "\pi"] \\ N_x/H & \mathtt{S}/H \arrow[l, "\phi/H"'] \arrow[r, "\psi/GL_n"] & \mathtt{X}/GL_n \end{tikzcd}$$

Figure 3.11: Etale slice diagram

*If we assume moreover that* $\mathtt{X}$ *is smooth in* $x$*, then we can choose the slice* $\mathtt{S}$ *such that also the following properties are satisfied*

1. $\mathtt{S}$ *is smooth,*
2. *there is an* $H$*-equivariant morphism* $\mathtt{S} \xrightarrow{\phi} T_x\ \mathtt{S} = N_x$ *with* $\phi(x) = 0$ *having an affine image,*
3. *the induced morphism is étale*

$$\mathtt{S}/H \xrightarrow{\phi/H} N_x/H$$

   *and the left hand side of figure 3.11 is commutative.*

*Proof.* Choose a finite dimensional $GL_n$-subrepresentation $V$ of $\mathbb{C}[\mathtt{X}]$ that generates the coordinate ring as algebra. This gives a $GL_n$-equivariant embedding

$$\mathtt{X} \overset{i}{\hookrightarrow} W = V^*$$

Choose in the vectorspace $W$ an $H$-stable complement $S_0$ of $\mathfrak{gl}_n.i(x) = T_{i(x)}\ \mathcal{O}(x)$ and denote $S_1 = i(x) + S_0$ and $\mathtt{S}_2 = i^{-1}(S_1)$.Then, the diagram below is commutative

$$\begin{tikzcd} GL_n \times^H \mathtt{S}_2 \arrow[r, hook] \arrow[d, "\psi"'] & GL_n \times^H S_1 \arrow[d, "\psi_0"] \\ \mathtt{X} \arrow[r, hook, "i"] & W \end{tikzcd}$$

By construction we have that $\psi_0$ induces an isomorphism between the tangent spaces in $\overline{(1, i(x))} \in GL_n \times^H S_0$ and $i(x) \in W$ which means that $\psi_0$ is étale in $i(x)$, whence $\psi$ is étale in $\overline{(1, x)} \in GL_n \times^H \mathtt{S}_2$. By the fundamental lemma we get an affine neighborhood $U$ which must be of the form $U = GL_n \times^H \mathtt{S}$ giving a slice $\mathtt{S}$ with the required properties.

Assume that $\mathtt{X}$ is smooth in $x$, then $S_1$ is transversal to $\mathtt{X}$ in $i(x)$ as

$$T_{i(x)}\ i(\mathtt{X}) + S_0 = W$$

Therefore, $\mathtt{S}$ is smooth in $x$. Again using the separation property we can find an invariant $f \in \mathbb{C}[\mathtt{S}]^H$ such that $f$ is zero on the singularities of $\mathtt{S}$ (which is a $H$-stable closed subscheme) and $f(x) = 1$. Then replace $\mathtt{S}$ with its affine reduced subvariety of points $s$ such that $f(s) \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}[\mathtt{S}]$ in $x$, then we have an exact sequence of $H$-modules

$$0 \longrightarrow \mathfrak{m}^2 \longrightarrow \mathfrak{m} \xrightarrow{\alpha} N_x^* \longrightarrow 0$$

Choose a $H$-equivariant section $\phi^* : N_x^* \longrightarrow \mathfrak{m} \hookrightarrow \mathbb{C}[\mathtt{S}]$ of $\alpha$ then this gives an $H$-equivariant morphism $\mathtt{S} \xrightarrow{\phi} N_x$ which is étale in $x$. Applying again the fundamental lemma to this setting finishes the proof. $\square$

## References.

More details on étale cohomology can be found in the textbook of J.S. Milne [64] . The material of Tsen and Tate fields is based on the lecture notes of S. Shatz [77]. For more details on the coniveau spectral sequence we refer to the paper [18]. The description of the Brauer group of the functionfield of a surface is due to M. Artin and D. Mumford [6] . The étale slices are due to D. Luna [63] and in the form presented here to F. Knop [45] . For more details we refer to the lecture notes of P. Slodowy [79].

# 4 — Quiver Representations

Having generalized the classical anti-equivalence between commutative algebra and (affine) algebraic geometry to the pair of functors

$$\begin{tikzcd} \texttt{alg@n} \arrow[r, bend left, "\texttt{trep}_n"] & \texttt{GL(n)-affine} \arrow[l, bend left, "\Uparrow^n"] \end{tikzcd}$$

where $\Uparrow^n$ is a left-inverse for $\texttt{trep}_n$, we will define *Cayley-smooth* algebras $A \in \texttt{alg@n}$ which are analogous to smooth commutative algebras. The definition is in terms of a lifting property with respect to nilpotent ideals, following Grothendieck's characterization of regular algebras. We will prove Procesi's result that a degree $n$ Cayley-Hamilton algebra $A$ is Cayley-smooth if and only if $\texttt{trep}_n\ A$ is a smooth (commutative) affine variety.

This result allows us, via the theory of Knop-Luna slices, to describe the étale local structure of Cayley-smooth algebras. We will prove that the local structure of $A$ in a point $\xi \in \texttt{triss}_n\ A$ is determined by a combinatorial gadget : a (marked) quiver $Q$ (given by the simple components of the semi-simple $n$-dimensional representation $M_\xi$ corresponding to $\xi$ and their (self)extensions)and a dimension vector $\alpha$ (given by the multiplicities of the simple factors in $M_{xi}$).

In the second part of this book we will use this description to classify Cayley-smooth orders (as well as their central singularities) in low dimensions. In this study we will need standard results on the representation theory of quivers : the description of the simple (resp. indecomposable) dimension vectors, the canonical decomposition and the notion of semistable representations.

## 4.1 Smoothness

In this section we will introduce smoothness relative to a category of $\mathbb{C}$-algebras. For `commalg` this notion is equivalent to the usual geometric smoothness and we will show that for `alg@n` smoothness of a Cayley-Hamilton algebra $A$ is equivalent to $\texttt{trep}_n\ A$ being a smooth affine variety. Examples of such *Cayley-smooth* algebras arise as level $n$ approximations of smooth algebras in `alg`, called *Quillen smooth algebras.*

**Definition 4.1** *Let* `cat` *be a category of* $\mathbb{C}$*-algebras. An object* $A \in Ob(\texttt{cat})$ *is said to be* `cat`*-smooth if it satisfies the following lifting property. For* $B \in Ob(\texttt{cat})$*, a nilpotent ideal* $I \triangleleft B$ *such* *that $B/I \in Ob(\mathtt{cat})$ and a $\mathbb{C}$-algebra morphism $A \xrightarrow{\kappa} B/I$ in $Mor(\mathtt{cat})$, there exist a lifting*

$$\begin{tikzcd} & A \arrow[dl, dotted, "\exists \lambda"'] \arrow[d, "\kappa"] \\ B & \frac{B}{I} \arrow[l, two heads] \end{tikzcd}$$

*with $\lambda \in Mor(\mathtt{cat})$ making the diagram commutative. An* $\mathtt{alg}$*-smooth algebra is called* Quillen-smooth , $\mathtt{comm}$*-smooth algebras are called* Grothendieck-smooth *and* $\mathtt{alg@n}$*-smooth algebras* Cayley-smooth .

To motivate these definitions, we will show that the categorical notion of $\mathtt{comm}$-smoothness coincides with geometric smoothness. Let $\mathtt{X}$ be a possibly non-reduced affine variety and $x$ a geometric point of $\mathtt{X}$. As we are interested in local properties of $\mathtt{X}$ near $x$, we may assume (after translation) that $x = o$ in $\mathbb{C}^n$ and that we have a presentation

$$\mathbb{C}[\mathtt{X}] = \mathbb{C}[x_1, \ldots, x_n]/I \quad \text{with } I = (f_1, \ldots, f_m) \text{ and } m_x = (x_1, \ldots, x_n)/I.$$

Denote the polynomial algebra $P = \mathbb{C}[x_1, \ldots, x_n]$ and consider the map

$$d \quad : I \longrightarrow (P dx_1 \oplus \ldots \oplus P dx_n) \otimes_P \mathbb{C}[\mathtt{X}] = \mathbb{C}[\mathtt{X}]dx_1 \oplus \ldots \oplus \mathbb{C}[\mathtt{X}]dx_n$$

where the $dx_i$ are a formal basis of the free module of rank $n$ and the map is defined by

$$d(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \ mod \ I.$$

This gives a $\mathbb{C}[\mathtt{X}]$-linear mapping $\frac{I}{I^2} \xrightarrow{d} \mathbb{C}[\mathtt{X}]dx_1 \oplus \ldots \oplus \mathbb{C}[\mathtt{X}]dx_n$. Extending to the local algebra $\mathcal{O}_x$ at $x$ and then quotient out the maximal ideal $\mathfrak{m}_x$ we get a $\mathbb{C} = \mathcal{O}_x/\mathfrak{m}_x$- linear map $\frac{\mathfrak{I}}{\mathfrak{I}^2} \xrightarrow{d(x)} \mathbb{C}dx_1 \oplus \ldots \oplus \mathbb{C}dx_n$ Clearly, $x$ is a nonsingular point of $\mathtt{X}$ *if and only if* the $\mathbb{C}$-linear map $d(x)$ is injective. This is equivalent to the existence of a $\mathbb{C}$-section and by the Nakayama lemma also to the existence of a $\mathcal{O}_x$-linear splitting $s_x$ of the induced $\mathcal{O}_x$-linear map $d_x$

$$\frac{\mathfrak{I}}{\mathfrak{I}^2} \underset{s_x}{\overset{d_x}{\rightleftarrows}} \mathcal{O}_x dx_1 \oplus \ldots \oplus \mathcal{O}_x dx_n$$

satisfying $s_x \circ d_x = id_{\frac{\mathfrak{I}}{\mathfrak{I}^2}}$

A $\mathbb{C}$-algebra epimorphism (between commutative algebras) $R \xrightarrow{\pi} S$ with square zero kernel is called *an infinitesimal extension of $S$* . It is called a *trivial infinitesimal extension* if $\pi$ has an algebra section $\sigma : S \hookrightarrow R$ satisfying $\pi \circ \sigma = id_S$. An infinitesimal extension $R \xrightarrow{\pi} S$ of $S$ is said to be *versal* if for any other infinitesimal extension $R' \xrightarrow{\pi'} S$ of $S$ there is a $\mathbb{C}$-algebra morphism

$$\begin{tikzcd} R \arrow[rr, two heads, "\pi"] \arrow[dr, dotted, "\exists g"'] & & S \\ & R' \arrow[ur, two heads, "\pi'"'] & \end{tikzcd}$$

making the diagram commute. From this universal property it is clear that versal infinitesimal extensions are uniquely determined up to isomorphism. Moreover, if a versal infinitesimal extension is trivial, then so is any infinitesimal extension. By iterating, $S$ is Grothendieck-smooth if and only if it has the lifting property with respect to nilpotent ideals $I$ with square zero. Therefore, assume we have a *test object* $(T, I)$ with $I^2 = 0$, then we have a commuting diagram

$$\begin{tikzcd} S \times_{T/I} T \arrow[r, two heads, "pr_1"] \arrow[d, two heads, "pr_2"'] & S \arrow[d, "\kappa"] \\ T \arrow[r, two heads, "p"'] & T/I \end{tikzcd}$$

where we define the *pull-back algebra* $S \times_{T/I} T = \{(s,t) \in S \times T \mid \kappa(s) = p(t)\}$. Observe that $pr_1 : S \times_{T/I} T \twoheadrightarrow S$ is a $\mathbb{C}$-algebra epimorphism with kernel $0 \times_{T/I} I$ having square zero, that is, it is an infinitesimal extension of $S$. Moreover, the existence of a lifting $\lambda$ of $\kappa$ is equivalent to the existence of a $\mathbb{C}$-algebra section

$$\sigma : S \longrightarrow S \times_{T/I} T \quad \text{defined by } s \mapsto (s, \lambda(s)).$$

Hence, $S$ is Grothendieck-smooth if and only if a versal infinitesimal extension of $S$ is trivial.

Returning to the situation of interest to us, we claim that the algebra epimorphism $\mathcal{O}_x(\mathbb{C}^n)/I_x^2 \xrightarrow{c_x} \mathcal{O}_x$ is a versal infinitesimal extension of $\mathcal{O}_x$. Indeed, consider any other infinitesimal extension $R \xrightarrow{\pi} \mathcal{O}_x$ then we define a $\mathbb{C}$-algebra morphism $\mathcal{O}_x(\mathbb{C}^n)/I_x^2 \longrightarrow R$ as follows : let $r_i \in R$ such that $\pi(r_i) = c_x(x_i)$ and define an algebra morphism $\mathbb{C}[x_1, \ldots, x_n] \longrightarrow R$ by sending the variable $x_i$ to $r_i$. As the image of any polynomial non-vanishing in $x$ is a unit in $R$, this algebra map extends to one from the local algebra $\mathcal{O}_x(\mathbb{C}^n)$ and it factors over $\mathcal{O}_x(\mathbb{C}^n)/I_x^2$ as the image of $I_x$ lies in the kernel of $\pi$ which has square zero, proving the claim. Hence, $\mathcal{O}_x$ is Grothendieck-smooth if and only if there is a $\mathbb{C}$-algebra section

$$\begin{tikzcd} \mathcal{O}_x(\mathbb{C}^n)/I_x^2 \arrow[r, two heads, shift left, "c_x"] & \mathcal{O}_x \arrow[l, hook, shift left, "r_x"] \end{tikzcd}$$

satisfying $c_x \circ r_x = id_{\mathcal{O}_x}$.

**Proposition 4.1** *The affine scheme* $\mathtt{X}$ *is non-singular at the geometric point* $x$ *if and only if the local algebra* $\mathcal{O}_x(\mathtt{X})$ *is Grothendieck-smooth.*

*Proof.* The result will follow once we prove that there is a natural one-to-one correspondence between $\mathcal{O}_x$-module splittings $s_x$ of $d_x$ and $\mathbb{C}$-algebra sections $r_x$ of $c_x$. This correspondence is given by assigning to an algebra section $r_x$ the map $s_x$ defined by

$$s_x(dx_i) = (x_i - r_x \circ c_x(x_i)) \ mod \ I_x^2$$

□

If $\mathtt{X}$ is an affine scheme which is smooth in *all* of its geometric points, then we have seen before that $\mathtt{X} = X$ must be reduced, that is, an affine variety. Restricting to its disjoint irreducible components we may assume that

$$\mathbb{C}[\mathtt{X}] = \cap_{x \in X} \mathcal{O}_x.$$

Clearly, if $\mathbb{C}[\mathtt{X}]$ is Grothendieck-smooth, so is any of the local algebras $\mathcal{O}_x$. Conversely, if all $\mathcal{O}_x$ are Grothendieck-smooth and $\mathbb{C}[\mathtt{X}] = \mathbb{C}[x_1, \ldots, x_n]/I$ one knows that the algebra epimorphism

$$\mathbb{C}[x_1, \ldots, x_n]/I^2 \xrightarrow{c} \mathbb{C}[\mathtt{X}]$$

has local sections in every $x$, but then there is an algebra section. Because $c$ is clearly a versal infinitesimal deformation of $\mathbb{C}[\mathtt{X}]$, it follows that $\mathbb{C}[\mathtt{X}]$ is Grothendieck-smooth.

**Proposition 4.2** *Let* $\mathtt{X}$ *be an affine scheme. Then,* $\mathbb{C}[\mathtt{X}]$ *is Grothendieck-smooth if and only if* $\mathtt{X}$ *is non-singular in all of its geometric points. In this case,* $\mathtt{X}$ *is a reduced affine variety.*

However, Grothendieck-smooth algebras do not have to be `cat`-smooth for more general categories of $\mathbb{C}$-algebras.

**Example 4.1** Consider the polynomial algebra $\mathbb{C}[x_1, \ldots, x_d]$ and the 4-dimensional noncommutative local algebra

$$T = \frac{\mathbb{C}\langle x, y\rangle}{(x^2, y^2, xy + yx)} = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy$$

Consider the one-dimensional nilpotent ideal $I = \mathbb{C}(xy - yx)$ of $T$, then the 3-dimensional quotient $\frac{T}{I}$ is commutative and we have a morphism $\mathbb{C}[x_1, \ldots, x_d] \xrightarrow{\phi} \frac{T}{I}$ by $x_1 \mapsto x, x_2 \mapsto y$ and $x_i \mapsto 0$ for $i \geq 2$. This morphism admits no lift to $T$ as for any potential lift the commutator

$$[\tilde{\phi}(x), \tilde{\phi}(y)] \neq 0 \quad \text{in } T.$$

Therefore, $\mathbb{C}[x_1, \ldots, x_d]$ can only be Quillen smooth if $d = 1$.

Because $\texttt{comm} = \texttt{alg@1}$, it is natural to generalize the foregoing to Cayley-smooth algebras. Let $B$ be a Cayley-Hamilton algebra of degree $n$ with trace map $tr_B$ and trace generated by $m$ elements say $\{b_1, \ldots, b_m\}$. Then, we can write

$$B = \mathbb{T}_n^m / T_B \qquad \text{with } T_B \text{ closed under traces.}$$

Now, consider the extended ideal

$$E_B = M_n(\mathbb{C}[M_n^m]).T_B.M_n(\mathbb{C}[M_n^m]) = M_n(N_B)$$

and we have seen that $\mathbb{C}[\texttt{trep}_n\ B] = \mathbb{C}[M_n^m]/N_B$. We need the following technical result.

**Lemma 4.1** *With notations as above, we have for all $k$ that*

$$E_B^{kn^2} \cap \mathbb{T}_n^m \subset T_B^k.$$

*Proof.* Let $\mathbb{T}_n^m$ be the trace algebra on the generic $n \times n$ matrices $\{X_1, \ldots, X_m\}$ and $\mathbb{T}_n^{l+m}$ the trace algebra on the generic matrices $\{Y_1, \ldots, Y_l, X_1, \ldots, X_m\}$. Let $\{U_1, \ldots, U_l\}$ be elements of $\mathbb{T}_n^m$ and consider the trace preserving map $\mathbb{T}_n^{l+m} \xrightarrow{u} \mathbb{T}_n^m$ induced by the map defined by sending $Y_i$ to $U_i$. Then, by the universal property we have a commutative diagram of Reynold operators

$$\begin{tikzcd}
M_n(\mathbb{C}[M_n^{l+m}]) \arrow[r, "\tilde{u}"] \arrow[d, "R"] & M_n(\mathbb{C}[M_n^m]) \arrow[d, "R"] \\
\mathbb{T}_n^{l+m} \arrow[r, "u"] & \mathbb{T}_n^m
\end{tikzcd}.$$

Now, let $A_1, \ldots, A_{l+1}$ be elements from $M_n(\mathbb{C}[M_n^m])$, then we can calculate $R(A_1U_1A_2U_2A_3 \ldots A_lU_lA_{l+1})$ by first computing

$$r = R(A_1Y_1A_2Y_2A_3 \ldots A_lY_lA_{l+1})$$

and then substituting the $Y_i$ with $U_i$. The Reynolds operator preserves the degree in each of the generic matrices, therefore $r$ will be linear in each of the $Y_i$ and is a sum of trace algebra elements. By our knowledge of the generators of necklaces and the trace algebra we can write each term of the sum as an expression

$$tr(M_1)tr(M_2) \ldots tr(M_z)M_{z+1}$$

where each of the $M_i$ is a monomial of degree $\leq n^2$ in the generic matrices $\{Y_1, \ldots, Y_l, X_1, \ldots, X_m\}$. Now, look at how the generic matrices $Y_i$ are distributed among the monomials $M_j$. Each $M_j$ contains *at most* $n^2$ of the $Y_i$'s, hence the monomial $M_{z+1}$ contains *at least* $l - vn^2$ of the $Y_i$ where $v \leq z$ is the number of $M_i$ with $i \leq z$ containing at least one $Y_j$.

Now, assume all the $U_i$ are taken from the ideal $T_B \lhd \mathbb{T}_n^m$ which is closed under taking traces, then it follows that

$$R(A_1U_1A_2U_2A_3\ldots A_lU_lA_{l+1}) \in T_B^{v+(l-vn^2)} \subset T_B^k$$

if we take $l = kn^2$ as $v + (k-v)n^2 \geq k$. But this finishes the proof of the required inclusion. $\square$

Let $B$ be a Cayley-Hamilton algebra of degree $n$ with trace map $tr_B$ and $I$ a twosided ideal of $B$ which is closed under taking traces. We will denote by $E(I)$ the extended ideal with respect to the universal embedding, that is,

$$E(I) = M_n(\mathbb{C}[\mathtt{trep}_n\ B])IM_n(\mathbb{C}[\mathtt{trep}_n\ B]).$$

Then, for all powers $k$ we have the inclusion $E(I)^{kn^2} \cap B \subset I^k$.

**Theorem 4.1** *Let $A$ be a Cayley-Hamilton algebra of degree $n$ with trace map $tr_A$. Then, $A$ is Cayley-smooth if and only if the trace preserving representation variety* $\mathtt{trep}_n\ A$ *is non-singular in all points (in particular,* $\mathtt{trep}_n\ A$ *is reduced).*

*Proof.* Let $A$ be Cayley-smooth, then we have to show that $\mathbb{C}[\mathtt{trep}_n\ A]$ is Grothendieck-smooth. Take a commutative test-object $(T, I)$ with $I$ nilpotent and an algebra map $\kappa : \mathbb{C}[\mathtt{trep}_n\ A] \longrightarrow T/I$. Composing with the universal embedding $i_A$ we obtain a trace preserving morphism $\mu_0$

$$\begin{tikzcd}
A \arrow[rr, dotted, "\mu_1"] \arrow[d, hook, "i_A"'] \arrow[drr, "\mu_0"] & & M_n(T) \arrow[d] \\
M_n(\mathbb{C}[\mathtt{trep}_n\ A]) \arrow[rr, "M_n(\kappa)"'] & & M_n(T/I)
\end{tikzcd}$$

Because $M_n(T)$ with the usual trace is a Cayley-Hamilton algebra of degree $n$ and $M_n(I)$ a trace stable ideal and $A$ is Cayley-smooth there is a trace preserving algebra map $\mu_1$. But then, by the universal property of the embedding $i_A$ there exists a $\mathbb{C}$-algebra morphism

$$\lambda : \mathbb{C}[\mathtt{trep}_n\ A] \longrightarrow T$$

such that $M_n(\lambda)$ completes the diagram. The morphism $\lambda$ is the required lift.

Conversely, assume that $\mathbb{C}[\mathtt{trep}_n\ A]$ is Grothendieck-smooth. Assume we have a Cayley-Hamilton algebra of degree $n$ with trace map $tr_T$ and a trace-stable nilpotent ideal $I$ of $T$ and a trace preserving $\mathbb{C}$-algebra map $\kappa : A \longrightarrow T/I$. If we combine this test-data with the universal embeddings we obtain a diagram

$$\begin{tikzcd}
 & T \arrow[r, hook, "i_T"] \arrow[d, two heads] & M_n(\mathbb{C}[\mathtt{trep}_n\ T]) \arrow[d, two heads] \\
A \arrow[ur, dotted, "\exists\lambda ?"] \arrow[r, "\kappa"] \arrow[d, hook, "i_A"'] & T/I \arrow[r, hook, "i_{T/I}"] & M_n(\mathbb{C}[\mathtt{trep}_n\ T/I]){=}M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J) \\
M_n(\mathbb{C}[\mathtt{trep}_n\ A]) \arrow[urr, dotted, "M_n(\alpha)"'] & &
\end{tikzcd}$$

Here, $J = M_n(\mathbb{C}[\mathtt{trep}_n\ T])IM_n(\mathbb{C}[\mathtt{trep}_n\ T])$ and we know already that $J \cap T = I$. By the universal property of the embedding $i_A$ we obtain a $\mathbb{C}$-algebra map

$$\mathbb{C}[\mathtt{trep}_n\ A] \xrightarrow{\alpha} \mathbb{C}[\mathtt{trep}_n\ T]/J$$

which we would like to lift to $\mathbb{C}[\mathtt{trep}_n\ T]$. This does *not* follow from Grothendieck-smoothness of $\mathbb{C}[\mathtt{trep}_n\ A]$ as $J$ is usually not nilpotent. However, as $I$ is a nilpotent ideal of $T$ there is some $h$ such that $I^h = 0$. As $I$ is closed under taking traces we know by the remark preceding the theorem that

$$E(I)^{hn^2} \cap T \subset I^h = 0.$$

Now, by definition $E(I) = M_n(\mathbb{C}[\mathtt{trep}_n\ T])IM_n(\mathbb{C}[\mathtt{trep}_n\ T])$ which is equal to $M_n(J)$. That is, the inclusion can be rephrased as $M_n(J)^{hn^2} \cap T = 0$, whence there is a trace preserving embedding $T \hookrightarrow M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2})$. Now, we are in the situation of figure 4.1 This time we *can* lift $\alpha$ to a $\mathbb{C}$-algebra morphism

$$\mathbb{C}[\mathtt{trep}_n\ A] \longrightarrow \mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2}.$$

This in turn gives us a trace preserving morphism

$$A \xrightarrow{\lambda} M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2})$$

the image of which is contained in the algebra of $GL_n$-invariants. Because $T \hookrightarrow M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2})$ and by surjectivity of invariants under surjective maps, the $GL_n$-equivariants are equal to $T$, giving the required lift $\lambda$. $\square$

For an affine $\mathbb{C}$-algebra $A$ recall the construction of its *level $n$ approximation*

$$\int_n A = \frac{\int A}{(tr(1) - n, \chi_a^{(n)}(a)\ \forall a \in A)} = M_n(\mathbb{C}[\mathtt{rep}_n\ A])^{GL_n}$$

$$\begin{tikzcd}
 & T \arrow[r, hook] \arrow[d] & M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J^{kn^2}) \arrow[d] \\
A \arrow[ur, dotted, "\lambda"] \arrow[r, "\kappa"] \arrow[d, hook, "i_A"'] & T/I \arrow[r, hook, "i_{T/I}"] & M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J) \\
M_n(\mathbb{C}[\mathtt{trep}_n\ A]) \arrow[uurr, dotted] \arrow[urr, "M_n(\alpha)"'] & &
\end{tikzcd}$$

Figure 4.1:

In general, it may happen that $\int_n A = 0$ for example if $A$ has no $n$-dimensional representations. The characteristic feature of $\int_n A$ is that any $\mathbb{C}$-algebra map $A \longrightarrow B$ with $B$ a Cayley-Hamilton algebra of degree $n$ factors through $\int_n A$

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr, "can"'] & & B \\
 & \int_n A \arrow[ur, dotted, "\exists \phi_n"'] &
\end{tikzcd}$$

with $\phi_n$ a trace preserving algebra morphism. From this universal property we deduce

**Proposition 4.3** *If $A$ is Quillen-smooth, then for every integer $n$, the Cayley-Hamilton algebra of degree $n$, $\int_n A$, is Cayley-smooth. Moreover,*

$$\mathtt{rep}_n\ A \simeq \mathtt{trep}_n\ \int_n A$$

*is a smooth affine $GL_n$-variety.*

This result allows us to study a Quillen-smooth algebra locally in the étale topology. We know that the algebra $\int_n A$ is given by the $GL_n$-equivariant maps from $\mathtt{rep}_n\ A = \mathtt{trep}_n\ \int_n A$ to $M_n(\mathbb{C})$. As this representation variety is smooth we can apply the full strength of the slice theorem to determine the local structure of the $GL_n$-variety $\mathtt{trep}_n \int_n A$ and hence of $\int_n A$. In the next section we will prove that this local structure is fully determined by a quiver setting.

Therefore, let us recall the definition of *quivers* and their path algebras and show that these algebras are all Quillen-smooth.

**Definition 4.2** *A* quiver *$Q$ is a directed graph determined by*

- *a finite set $Q_v = \{v_1, \ldots, v_k\}$ of* vertices *, and*
- *a finite set $Q_a = \{a_1, \ldots, a_l\}$ of* arrows *where we allow multiple arrows between vertices and loops in vertices.*

*Every arrow* $\textcircled{j} \xleftarrow{\;a\;} \textcircled{i}$ *has a* starting vertex *$s(a) = i$ and a* terminating vertex *$t(a) = j$. Multiplication in the* path algebra *$\mathbb{C}Q$ is induced by (left) concatenation of paths. More precisely, $1 = v_1 + \ldots + v_k$ is a decomposition of $1$ into mutually orthogonal idempotents and further we define*

- *$v_j.a$ is always zero unless* $\textcircled{j} \xleftarrow{\;a\;} \bigcirc$ *in which case it is the path $a$,*
- *$a.v_i$ is always zero unless* $\bigcirc \xleftarrow{\;a\;} \textcircled{i}$ *in which case it is the path $a$,*
- *$a_i.a_j$ is always zero unless* $\bigcirc \xleftarrow{\;a_i\;} \bigcirc \xleftarrow{\;a_j\;} \bigcirc$ *in which case it is the path $a_i a_j$.*

Consider the commutative $\mathbb{C}$-algebra

$$C_k = \mathbb{C}[e_1, \ldots, e_k]/(e_i^2 - e_i, e_i e_j, \sum_{i=1}^k e_i - 1).$$

$C_k$ is the universal $\mathbb{C}$-algebra in which 1 is decomposed into $k$ orthogonal idempotents, that is, if $R$ is any $\mathbb{C}$-algebra such that $1 = r_1 + \ldots + r_k$ with $r_i \in R$ idempotents satisfying $r_i r_j = 0$, then there is an embedding $C_k \hookrightarrow R$ sending $e_i$ to $r_i$.

**Proposition 4.4** *$C_k$ is Quillen smooth. That is, if $I$ be a nilpotent ideal of a $\mathbb{C}$-algebra $T$ and if $1 = \overline{e}_1 + \ldots + \overline{e}_k$ is a decomposition of $1$ into orthogonal idempotents $\overline{e}_i \in T/I$. Then, we can lift this decomposition to $1 = e_1 + \ldots + e_k$ for orthogonal idempotents $e_i \in T$ such that $\pi(e_i) = \overline{e}_i$ where $T \overset{\pi}{\twoheadrightarrow} T/I$ is the canonical projection.*

*Proof.* Assume that $I^l = 0$, clearly any element $1 - i$ with $i \in I$ is invertible in $T$ as

$$(1 - i)(1 + i + i^2 + \ldots + i^{l-1}) = 1 - i^l = 1.$$

If $\overline{e}$ is an idempotent of $T/I$ and $x \in T$ such that $\pi(x) = \overline{e}$. Then, $x - x^2 \in I$ whence

$$0 = (x - x^2)^l = x^l - l x^{l+1} + \binom{l}{2} x^{l+2} - \ldots + (-1)^l x^{2l}$$

and therefore $x^l = ax^{l+1}$ where $a = l - \binom{l}{2} x + \ldots + (-1)^{l-1}x^{l-1}$ and so $ax = xa$. If we take $e = (ax)^l$, then $e$ is an idempotent in $T$ as

$$e^2 = (ax)^{2l} = a^l(a^l x^{2l}) = a^l x^l = e$$

the next to last equality follows from $x^l = ax^{l+1} = a^2x^{l+2} = \ldots = a^l x^{2l}$. Moreover,

$$\pi(e) = \pi(a)^l \pi(x)^l = \pi(a)^l \pi(x)^{2l} = \pi(a^l x^{2l}) = \pi(x)^l = \overline{e}.$$

If $\overline{f}$ is another idempotent in $T/I$ such that $\overline{e}\overline{f} = 0 = \overline{f}\overline{e}$ then as above we can lift $\overline{f}$ to an idempotent $f'$ of $T$. As $f'e \in I$ we can form the element

$$f = (1-e)(1-f'e)^{-1}f'(1-f'e).$$

Because $f'(1-f'e) = f'(1-e)$ one verifies that $f$ is idempotent, $\pi(f) = \overline{f}$ and $e.f = 0 = f.e$. Assume by induction that we have already lifted the pairwise orthogonal idempotents $\overline{e}_1, \ldots, \overline{e}_{k-1}$ to pairwise orthogonal idempotents $e_1, \ldots, e_{k-1}$ of $R$, then $e = e_1 + \ldots + e_{k-1}$ is an idempotent of $T$ such that $\overline{e}\overline{e}_k = 0 = \overline{e}_k\overline{e}$. Hence, we can lift $\overline{e_k}$ to an idempotent $e_k \in T$ such that $ee_k = 0 = e_ke$. But then also

$$e_ie_k = (e_ie)e_k = 0 = e_k(ee_i) = e_ke_i.$$

Finally, as $e_1 + \ldots + e_k - 1 = i \in I$ we have that

$$e_1 + \ldots + e_k - 1 = (e_1 + \ldots + e_k - 1)^l = i^l = 0$$

finishing the proof. $\square$

**Proposition 4.5** *For any quiver $Q$, the path algebra $\mathbb{C}Q$ is Quillen smooth.*

*Proof.* Take an algebra $T$ with a nilpotent twosided ideal $I \triangleleft T$ and consider

$$\begin{tikzcd} T \arrow[r, two heads] & \frac{T}{I} \\ & \mathbb{C}Q \arrow[u, "\phi"'] \arrow[ul, dotted, "\tilde{\phi}\,?"] \end{tikzcd}$$

The decomposition $1 = \phi(v_1) + \ldots + \phi(v_k)$ into mutually orthogonal idempotents in $\frac{T}{I}$ can be lifted up the nilpotent ideal $I$ to a decomposition $1 = \tilde{\phi}(v_1) + \ldots + \tilde{\phi}(v_k)$ into mutually orthogonal idempotents in $T$. But then, taking for every arrow $a$

$$j \xleftarrow{a} i \qquad \text{an arbitrary element} \quad \tilde{\phi}(a) \in \tilde{\phi}(v_j)(\phi(a) + I)\tilde{\phi}(v_i)$$

gives a required lifted algebra morphism $\mathbb{C}Q \xrightarrow{\tilde{\phi}} T$. □

Recall that a *representation* $V$ of the quiver $Q$ is given by

- a finite dimensional $\mathbb{C}$-vector space $V_i$ for each vertex $v_i \in Q_v$, and
- a linear map $V_j \xleftarrow{V_a} V_i$ for every arrow $(j) \xleftarrow{a} (i)$ in $Q_a$.

If $dim\ V_i = d_i$ we call the integral vector $\alpha = (d_1, \ldots, d_k) \in \mathbb{N}^k$ the *dimension vector* of $V$ and denote it with $dim\ V$. A *morphism* $V \xrightarrow{\phi} W$ between two representations $V$ and $W$ of $Q$ is determined by a set of linear maps

$$V_i \xrightarrow{\phi_i} W_i \qquad \text{for all vertices } v_i \in Q_v$$

satisfying the following compatibility conditions for every arrow $(j) \xleftarrow{a} (i)$ in $Q_a$

$$\begin{tikzcd} V_i \arrow[r, "V_a"] \arrow[d, "\phi_i"] & V_j \arrow[d, "\phi_j"] \\ W_i \arrow[r, "W_a"] & W_j \end{tikzcd}$$

Clearly, *composition* of morphisms $V \xrightarrow{\phi} W \xrightarrow{\psi} X$ is given by the rule that $(\psi \circ \phi)_i = \psi_i \circ \psi_i$ and one readily verifies that this is again a morphism of representations of $Q$. In this way we form a *category* `rep` $Q$ of all finite dimensional representations of the quiver $Q$.

**Proposition 4.6** *The category* `rep` $Q$ *is equivalent to the category of finite dimensional* $\mathbb{C}Q$*-representations* $\mathbb{C}Q - \mathtt{mod}$.

*Proof.* Let $M$ be an $n$-dimensional $\mathbb{C}Q$-representation. Then, we construct a representation $V$ of $Q$ by taking

- $V_i = v_i M$, and for any arrow $(j) \xleftarrow{a} (i)$ in $Q_a$ define
- $V_a : V_i \longrightarrow V_j$ by $V_a(x) = v_j a x$.

Observe that the dimension vector $dim(V) = (d_1, \ldots, d_k)$ satisfies $\sum d_i = n$. If $\phi : M \longrightarrow N$ is $\mathbb{C}Q$-linear, then we have a linear map $V_i = v_i M \xrightarrow{\phi_i} W_i = v_i N$ which clearly satisfies the compatibility condition.

Conversely, let $V$ be a representation of $Q$ with dimension vector $dim(V) = (d_1, \ldots, d_k)$. Then, consider the $n = \sum d_i$-dimensional space $M = \oplus_i V_i$ which we turn into a $\mathbb{C}Q$-representation as follows. Consider the canonical injection and projection maps $V_j \overset{i_j}{\hookrightarrow} M \overset{\pi_j}{\twoheadrightarrow} V_j$. Then, define the action of $\mathbb{C}Q$ by fixing the action of the algebra generators $v_j$ and $a_l$ to be

$$\begin{cases} v_j m & = i_j(\pi_j(m)) \\ a_l m & = i_j(V_a(\pi_i(m))) \end{cases}$$

for all arrows $\textcircled{j} \xleftarrow{a_l} \textcircled{i}$. A computation verifies that these two operations are inverse to each other and induce an equivalence of categories. $\square$

## 4.2 Local structure

In this section we give some applications of the slice theorem to the local structure of quotient varieties of representation spaces. We will first handle the case of an affine $\mathbb{C}$-algebra $A$ leading to a local description of $\int_n A$. Next, we will refine this slightly to prove similar results for an arbitrary affine $\mathbb{C}$-algebra $B$ in $\texttt{alg@n}$.

When $A$ is an affine $\mathbb{C}$-algebra generated by $m$ elements $\{a_1, \ldots, a_m\}$, its level $n$ approximation $\int_n A$ is trace generated by $m$ determining a trace preserving epimorphism $\mathbb{T}^m_n \twoheadrightarrow \int_n A$. Thus we have a $GL_n$-equivariant closed embedding of affine schemes

$$\texttt{rep}_n\ A = \texttt{trep}_n \int_n A \overset{\psi}{\hookrightarrow} \texttt{trep}_n\ \mathbb{T}^m_n = M^m_n$$

Take a point $\xi$ of the quotient scheme $\texttt{iss}_n\ A = \texttt{trep}_n \int_n A/GL_n$. We know that $\xi$ determines the isomorphism class of a semi-simple $n$-dimensional representation of $A$, say

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the $S_i$ are distinct simple $A$-representations, say of dimension $d_i$ and occurring in $M_\xi$ with multiplicity $e_i$. These numbers determine the *representation type* $\tau(\xi)$ of $\xi$ (or of the semi-simple representation $M_\xi$), that is

$$\tau(\xi) = (e_1, d_1; e_2, d_2; \ldots; e_k, d_k)$$

Choosing a basis of $M_\xi$ adapted to this decomposition gives us a point $x = (X_1, \ldots, X_m)$ in the orbit $\mathcal{O}(M_\xi)$ such that each $n \times n$ matrix $X_i$ is of the form

$$X_i = \begin{bmatrix} m_1^{(i)} \otimes \mathbb{1}_{e_1} & 0 & \ldots & 0 \\ 0 & m_2^{(i)} \otimes \mathbb{1}_{e_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_k^{(i)} \otimes \mathbb{1}_{e_k} \end{bmatrix}$$

where each $m_j^{(i)} \in M_{d_j}(\mathbb{C})$. Using this description we can compute the stabilizer subgroup $Stab(x)$ of $GL_n$ consisting of those invertible matrices $g \in GL_n$ commuting with every $X_i$. That is, $Stab(x)$ is the multiplicative group of units of the centralizer of the algebra generated by the $X_i$. It is easy to verify that this group is isomorphic to

$$Stab(x) \simeq GL_{e_1} \times GL_{e_2} \times \ldots \times GL_{e_k} = GL(\alpha_\xi)$$

for the dimension vector $\alpha_\xi = (e_1, \ldots, e_k)$ determined by the multiplicities and with embedding $Stab(x) \hookrightarrow GL_n$ given by

$$\begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{d_1}) & 0 & \ldots & 0 \\ 0 & GL_{e_2}(\mathbb{C} \otimes \mathbb{1}_{d_2}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{d_k}) \end{bmatrix}$$

A different choice of point in the orbit $\mathcal{O}(M_\xi)$ gives a subgroup of $GL_n$ conjugated to $Stab(x)$.

We know that the normal space $N_x^{sm}$ can be identified with the self-extensions $Ext_A^1(M, M)$ and we will give a quiver-description of this space. The idea is to describe first the $GL(\alpha)$-module structure of $N_x^{big}$, the normal space to the orbit $\mathcal{O}(M_\xi)$ in $M_n^m$ (see figure 4.2) and then to identify the direct summand $N_x^{sm}$. The description of $N_x^{big}$ follow from a book-keeping operation involving $GL(\alpha)$-representations. For $x = (X_1, \ldots, X_m)$, the tangent space $T_x\ \mathcal{O}(M_{xi})$ in $M_n^m$ to the orbit is equal to the image of the linear map

$$\begin{array}{ccc} \mathfrak{gl}_n = M_n & \longrightarrow & M_n \oplus \ldots \oplus M_n = T_x\ M_n^m \\ A & \mapsto & ([A, X_1], \ldots, [A, X_m]) \end{array}$$

Observe that the kernel of this map is the centralizer of the subalgebra generated by the $X_i$, so we have an exact sequence of $Stab(x) = GL(\alpha)$-modules

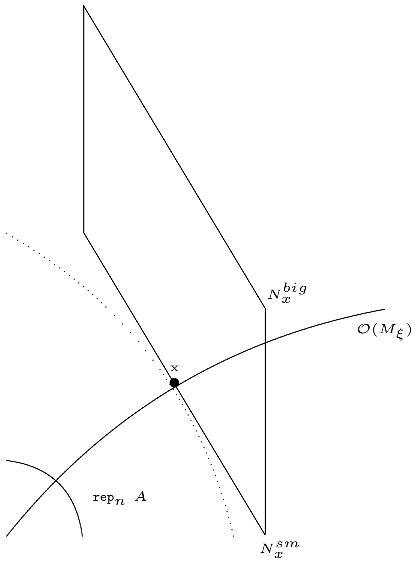
$$0 \longrightarrow \mathfrak{gl}(\alpha) = Lie\ GL(\alpha) \longrightarrow \mathfrak{gl}_n = M_n \longrightarrow T_x\ \mathcal{O}(x) \longrightarrow 0$$

Because $GL(\alpha)$ is a reductive group every $GL(\alpha)$-module is completely reducible and so the sequence splits. But then, the normal space in $M_n^m = T_x\ M_n^m$ to the orbit is isomorphic as $GL(\alpha)$-module to

$$N_x^{big} = \underbrace{M_n \oplus \ldots \oplus M_n}_{m-1} \oplus \mathfrak{gl}(\alpha)$$

with the action of $GL(\alpha)$ (embedded as above in $GL_n$) is given by simultaneous conjugation. If we consider the $GL(\alpha)$-action on $M_n$ depicted in figure 4.2 we see that it decomposes into a direct sum of subrepresentations

- for each $1 \leq i \leq k$ we have $d_i^2$ copies of the $GL(\alpha)$-module $M_{e_i}$ on which $GL_{e_i}$ acts by conjugation and the other factors of $GL(\alpha)$ act trivially,




$$N_x^{sm} = \frac{T_x\ \mathtt{rep}_n\ A}{T_x\ \mathcal{O}(M_\xi)} \triangleleft N_x^{big} = \frac{T_x\ M_n^m}{T_x\ \mathcal{O}(M_\xi)}$$

Figure 4.2: Big and small normal spaces to the orbit.

- for all $1 \leq i, j \leq k$ we have $d_i d_j$ copies of the $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ acts via $g.m = g_i m g_j^{-1}$ and the other factors of $GL(\alpha)$ act trivially.

These $GL(\alpha)$ components are precisely the modules appearing in representation spaces of quivers.

**Theorem 4.2** *Let $\xi$ be of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ and let $\alpha = (e_1, \ldots, e_k)$. Then, the $GL(\alpha)$-module structure of the normal space $N_x^{big}$ in $M_n^m$ to the orbit of the semi-simple $n$-dimensional representation $\mathcal{O}(M_\xi)$ is isomorphic to*

$$\mathtt{rep}_\alpha\ Q_\xi^{big}$$

*where the quiver $Q_\xi^{big}$ has $k$ vertices (the number of distinct simple summands of $M_\xi$) and the*

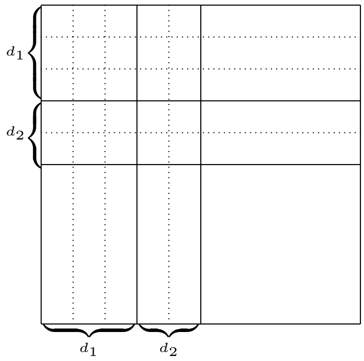
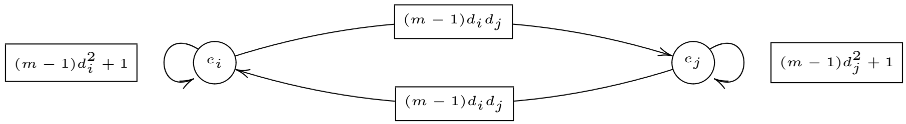


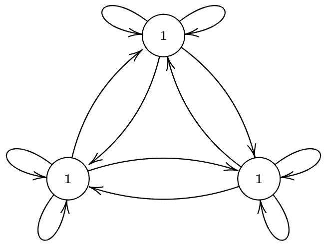

Figure 4.3: The $GL(\alpha)$-action on $M_n$

*subquiver on any two vertices $v_i, v_j$ for $1 \leq i \neq j \leq k$ has the following shape*




*That is, in each vertex $v_i$ there are $(m-1)d_i^2+1$-loops and there are $(m-1)d_id_j$ arrows from vertex $v_i$ to vertex $v_j$ for all $1 \leq i \neq j \leq k$.*

**Example 4.2** If $m = 2$ and $n = 3$ and the representation type is $\tau = (1,1;1,1;1,1)$ (that is, $M_\xi$ is the direct sum of three distinct one-dimensional simple representations) then the quiver $Q_\xi$ is



We have $GL_n$-equivariant embeddings $\mathcal{O}(M_\xi) \hookrightarrow \mathtt{trep}_n \int_n A \hookrightarrow M_n^m$ and corresponding embeddings of the tangent spaces in $x$

$$T_x\ \mathcal{O}(M_\xi) \hookrightarrow T_x\ \mathtt{trep}_n \int_n A \hookrightarrow T_x\ M_n^m$$

Because $GL(\alpha)$ is reductive we then obtain that the normal spaces to the orbit is a direct summand of $GL(\alpha)$-modules.

$$N_x^{sm} = \frac{T_x\ \mathtt{trep}_n \int_n A}{T_x\ \mathcal{O}(M_\xi)} \lhd N_x^{big} = \frac{T_x\ M_n^m}{T_x\ \mathcal{O}(M_\xi)}$$

As we know the isotypical decomposition of $N_x^{big}$ as the $GL(\alpha)$-module $\mathtt{rep}_\alpha\ Q_\xi$ this allows us to control $N_x^{sm}$. We only have to observe that arrows in $Q_\xi$ correspond to simple $GL(\alpha)$-modules, whereas a loop at vertex $v_i$ decomposes as $GL(\alpha)$-module into the simples

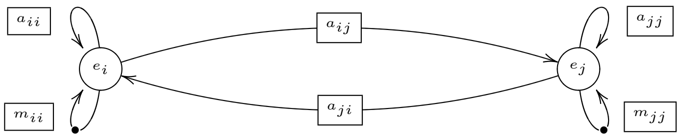
$$M_{e_i} = M_{e_i}^0 \oplus \mathbb{C}_{triv}$$

where $\mathbb{C}_{triv}$ is the one-dimensional simple with trivial $GL(\alpha)$-action and $M_{e_i}^0$ is the space of trace zero matrices in $M_{e_i}$. Any $GL(\alpha)$-submodule of $N_x^{big}$ can be represented by a *marked quiver* using the dictionary

- a loop at vertex $v_i$ corresponds to the $GL(\alpha)$-module $M_{e_i}$ on which $GL_{e_i}$ acts by conjugation and the other factors act trivially,
- a marked loop at vertex $v_i$ corresponds to the simple $GL(\alpha)$-module $M_{e_i}^0$ on which $GL_{e_i}$ acts by conjugation and the other factors act trivially,
- an arrow from vertex $v_i$ to vertex $v_j$ corresponds to the simple $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ acts via $g.m = g_i m g_j^{-1}$ and the other factors act trivially,

Combining this with the calculation that the normalspace is the space of self-extensions $Ext_A^1(M_\xi, M_\xi)$ or the trace preserving self-extensions $Ext_B^{tr}(M_\xi, M_{xi})$ (in case $B \in Ob(\mathtt{alg@n})$) we have.

**Theorem 4.3** *Consider the marked quiver on $k$ vertices such that the full marked subquiver on any two vertices $v_i \neq v_j$ has the form*




*where these numbers satisfy $a_{ij} \leq (m-1)d_i d_j$ and $a_{ii} + m_{ii} \leq (m-1)d_i^2 + 1$. Then,*

$$\begin{tikzcd}
GL_n \times^{GL(\alpha)} N_x^{sm} \arrow[d] & GL_n \times^{GL(\alpha)} S_x \arrow[l, "GL_n \times^{GL(\alpha)} \phi"'] \arrow[r, "\psi"] \arrow[d] & \mathtt{rep}_n\ A \arrow[d] \\
N_x^{sm}/GL(\alpha) & S_x/GL(\alpha) \arrow[l, "\phi/GL(\alpha)"'] \arrow[r, "\psi/GL(\alpha)"] & \mathtt{iss}_n\ A
\end{tikzcd}$$

Figure 4.4: Slice diagram for representation space.

1. *Let $A$ be an affine $\mathbb{C}$-algebra generated by $m$ elements, let $M_\xi$ be an $n$-dimensional semisimple $A$-module of representation-type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ and let $\alpha = (e_1, \ldots, e_k)$. Then, the normal space $N_x^{sm}$ in a point $x \in \mathcal{O}(M_\xi)$ to the orbit with respect to the representation space $\mathtt{rep}_n\ A$ is isomorphic to the $GL(\alpha)$-module of quiver-representations $\mathtt{rep}_\alpha\ Q_\xi$ of above type with*
   - $a_{ii} = dim_{\mathbb{C}}\ Ext^1_A(S_i, S_i)$ *and* $m_{ii} = 0$ *for all* $1 \leq i \leq k$.
   - $a_{ij} = dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j)$ *for all* $1 \leq i \neq j \leq n$.
2. *Let $B$ be a Cayley-Hamilton algebra of degree $n$, trace generated by $m$ elements, let $M_\xi$ be a trace preserving $n$-dimensional semisimple $B$-module of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ and let $\alpha = (e_1, \ldots, e_k)$. Then, the normal space $N_x^{tr}$ in a point $x \in \mathcal{O}(M_\xi)$ to the orbit with respect to the trace preserving representation space $\mathtt{trep}_n\ B$ is isomorphic to the $GL(\alpha)$-module of marked quiver-representations $\mathtt{rep}_\alpha\ Q_\xi^\bullet$ of above type with*
   - $a_{ij} = dim_{\mathbb{C}}\ Ext^1_B(S_i, S_j)$ *for all* $1 \leq i \neq j \leq k$.

   *and the (marked) vertex loops further determine the structure of* $Ext^{tr}_B(M_\xi, M_\xi)$.

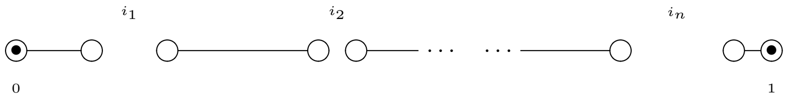
By a *marked quiver-representation* we mean a representation of the underlying quiver (that is, forgetting the marks) subject to the condition that the matrices corresponding to marked loops have trace zero.

Consider the slice diagram of figure 4.4 for the representation space $\mathtt{rep}_n\ A$. The left hand side exists when $x$ is a smooth point of $\mathtt{rep}_n\ A$, the right hand side exists always. The horizontal maps are étale and the upper ones $GL_n$-equivariant.

**Definition 4.3** *A point $\xi \in \mathtt{iss}_n\ A$ is said to belong to the $n$-*smooth locus *of $A$ iff the representation space $\mathtt{rep}_n\ A$ is smooth in $x \in \mathcal{O}(M_\xi)$. The $n$-smooth locus of $A$ will be denoted by $Sm_n(A)$.*

To determine the étale local structure of Cayley-Hamilton algebras in their $n$-smooth locus, we need to investigate the special case of *quiver orders*. We will do this in the next section and, at its end, draw some consequences about the étale local structure. We end this section by explaining the remarkable success of these local quiver settings and suggest that one can extend this using the theory of $A_\infty$-algebras.

The category `alg` has a topological origin. Consider the *tiny interval operad* $D_1$, that is, let $D_1(n)$ be the collection of all configurations




consisting of the unit interval with $n$ closed intervals removed, each gap given a label $i_j$ where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$. Clearly, $D_1(n)$ is a real $2n$-dimensional $\mathcal{C}^\infty$-manifold having $n!$ connected components, each of which is a contractible space. The operad structure comes from the collection of composition maps

$$D_1(n) \times (D_1(m_1) \times \ldots D_1(m_n)) \xrightarrow{m_{(n,m_1,\ldots,m_n)}} D_1(m_1 + \ldots + m_n)$$

defined by resizing the configuration in the $D_1(m_i)$-component such that it fits precisely in the $i$-th gap of the configuration of the $D_1(n)$-component, see figure 4.5. We obtain a unit interval having $m_1 + \ldots + m_n$ gaps which are labeled in the natural way, that is the first $m_1$ labels are for the gaps in the $D_1(m_1)$-configuration fitted in gap 1, the next $m_2$ labels are for the gaps in the $D_1(m_2)$-configuration fitted in gap 2 and so on. The tiny interval operad $D_1$ consists of

- a collection of topological spaces $D_1(n)$ for $n \geq 0$,
- a continuous action of $S_n$ on $D_1(n)$ by relabeling, for every $n$,
- an identity element $id \in D_1(1)$,
- the continuous composition maps $m_{(n,m_1,\ldots,m_n)}$ which satisfy associativity and equivariance with respect to the symmetric group actions.

By taking the *homology groups* of these manifolds $D_1(n)$ we obtain a *linear operad* `assoc`. Because $D_1(n)$ has $n!$ contractible components we can identify $\mathtt{assoc}(n)$ with the subspace of the free algebra $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ spanned by the *multilinear monomials*. $\mathtt{assoc}(n)$ has dimension $n!$ with basis $x_{\sigma(1)} \ldots x_{\sigma(n)}$ for $\sigma \in S_n$. Each $\mathtt{assoc}(n)$ has a natural action of $S_n$ and as $S_n$-representation it is isomorphic to the regular representation. The composition maps $m_{(n,m_1,\ldots,m_n)}$ induce on the homology level linear composition maps

$$\mathtt{assoc}(n) \otimes \mathtt{assoc}(m_1) \otimes \ldots \otimes \mathtt{assoc}(m_n) \xrightarrow{\gamma_{(n,m_1,\ldots,m_n)}} \mathtt{assoc}(m_1 + \ldots + m_n)$$

obtained by substituting the multilinear monomials $\phi_i \in \mathtt{assoc}(m_i)$ in the place of the variable $x_i$ into the multilinear monomial $\psi \in \mathtt{assoc}(n)$.

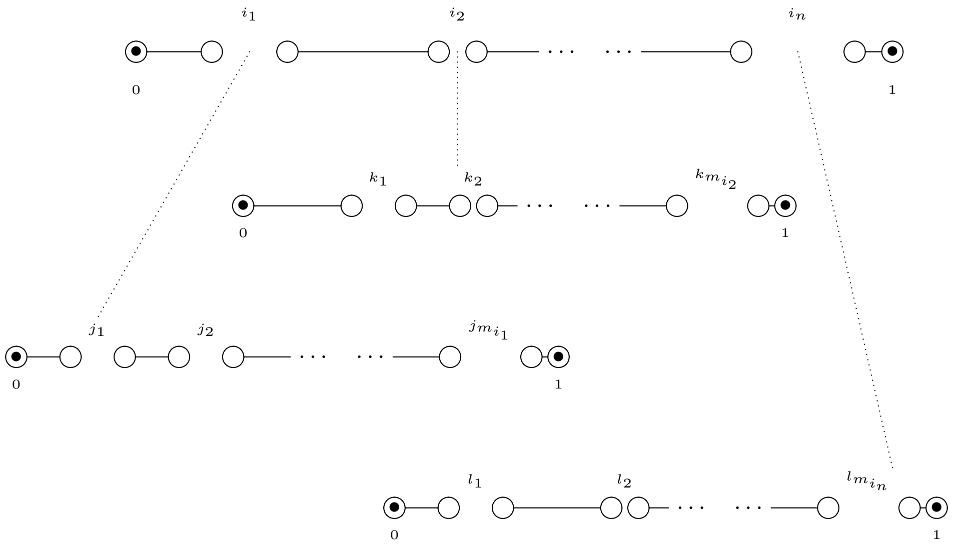



Figure 4.5: The tiny interval operad.

In general, a $\mathbb{C}$-linear operad $\mathtt{P}$ consists of a family of vectorspaces $\mathtt{P}(n)$ each equipped with an $S_n$-action, $\mathtt{P}(1)$ contains an identity element and there are composition linear morphisms

$$\mathtt{P}(n) \otimes \mathtt{P}(m_1) \otimes \ldots \otimes \mathtt{P}(m_n) \xrightarrow{c_{(n,m_1,\ldots,m_n)}} \mathtt{P}(m_1 + \ldots + m_n)$$

satisfying the same compatibility relations as the maps $\gamma_{(n,m_1,\ldots,m_n)}$ above. An example is the *endomorphism operad* $\mathtt{end}_V$ for a vectorspace $V$ defined by taking

$$\mathtt{end}_V(n) = Hom_{\mathbb{C}}(V^{\otimes n}, V)$$

with compositions and $S_n$-action defined in the obvious way and unit element $\mathbb{1}_V \in \mathtt{end}_V(1) = End(V)$. A *morphism* of linear operads $\mathtt{P} \xrightarrow{f} \mathtt{P}'$ is a collection of linear maps which are equivariant with respect to the $S_n$-action, commute with the composition maps and take the identity element of $\mathtt{P}$ to the identity element of $\mathtt{P}'$.

**Definition 4.4** *Let* $\mathtt{P}$ *be a* $\mathbb{C}$*-linear operad. A* $\mathtt{P}$-algebra *is a vectorspace* $A$ *equipped with a morphism of operads* $\mathtt{P} \xrightarrow{f} \mathtt{end}_A$.

For example, `assoc`-algebras are just associative $\mathbb{C}$-algebras, explaining the topological origin of `alg`. Instead of considering the *homology operad* `assoc` of the tiny intervals $D_1$ we can consider its *chain operad* `chain`. For a topological space $X$, let `chains`$(X)$ be the complex concentrated in non-positive degrees, whose $-k$-component consists of the finite formal additive combinations $\sum c_i.f_i$ where $c_i \in \mathbb{C}$ and $f_i : [0,1]^k \longrightarrow X$ is a continuous map (a *singular cube* in $X$ ) modulo the following relations

- For any $\sigma \in S_k$ acting on $[0,1]^k$ by permutation, we have $f \circ \sigma = sg(\sigma)f$.
- For $pr^k_{k-1} : [0,1]^k \overset{k-1}{\twoheadrightarrow}$ the projection on the first $k-1$ coordinates and any continuous map $[0,1]^{k-1} \xrightarrow{f'} X$ we have $f' \circ pr^k_{k-1} = 0$.

Then, `chain` is the collection of complexes `chains`$(D_1(n))$ and is an operad in the category of complexes of vectorspaces with cohomology the homology operad `assoc`. Again, we can consider `chain`-algebras, this time as complexes of vectorspaces. These are the $A_\infty$-algebras.

**Definition 4.5** *An $A_\infty$-algebra is a $\mathbb{Z}$-graded complex vectorspace*

$$B = \oplus_{p \in \mathbb{Z}} B_p$$

*endowed with homogeneous $\mathbb{C}$-linear maps*

$$m_n : B^{\otimes n} \longrightarrow B$$

*of degree $2-n$ for all $n \geq 1$, satisfying the following relations*

- *We have $m_1 \circ m_1 = 0$, that is $(B, m_1)$ is a differential complex*

$$\dots \xrightarrow{m_1} B_{i-1} \xrightarrow{m_1} B_i \xrightarrow{m_1} B_{i+1} \xrightarrow{m_1} \dots$$

- *We have the equality of maps $B \otimes B \longrightarrow B$*

$$m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1)$$

  *where $\mathbb{1}$ is the identity map on the vectorspace $B$. That is, $m_1$ is a derivation with respect to the multiplication $B \otimes B \xrightarrow{m_2} B$.*

- *We have the equality of maps $B \otimes B \otimes B \longrightarrow B$*

$$\begin{aligned} & m_2 \circ (\mathbb{1} \otimes m_2 - m_2 \otimes \mathbb{1}) \\ & = m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_1 \otimes 1 + \mathbb{1} \otimes \mathbb{1} \otimes m_1) \end{aligned}$$

  *where the right second expression is the associator for the multiplication $m_2$ and the first is a boundary of $m_3$, implying that $m_2$ is associative up to homology.*

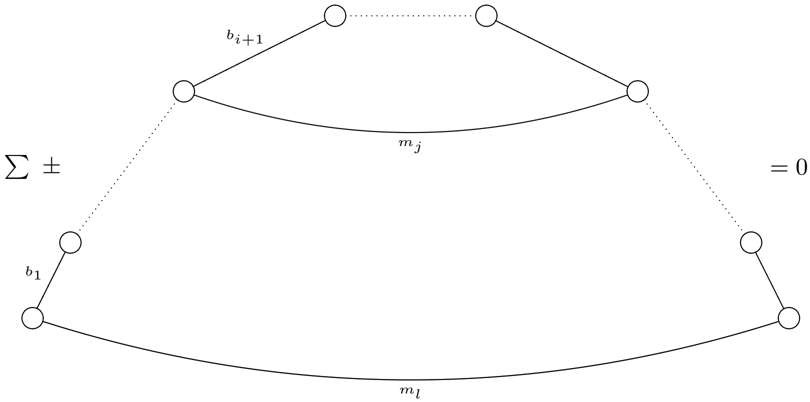



Figure 4.6: $A_\infty$-identities.

- *More generally, for $n \geq 1$ we have the relations*

$$\sum (-1)^{i+j+k} m_l \circ (\mathbb{1}^{\otimes i} \otimes m_j \otimes \mathbb{1}^{\otimes k}) = 0$$

*where the sum runs over all decompositions $n = i + j + k$ and where $l = i + 1 + k$. These identities are pictorially represented in figure 4.6.*

Observe that an $A_\infty$-algebra $B$ is in general not associative for the multiplication $m_2$, but its homology

$$H^* \ B = H^*(B, m_2)$$

is an associative graded algebra for the multiplication induced by $m_2$. Further, if $m_n = 0$ for all $n \geq 3$, then $B$ is an associative *differentially graded algebra* and conversely every differentially graded algebra yields an $A_\infty$-algebra with $m_n = 0$ for all $n \geq 3$.

Let $A$ be an associative $\mathbb{C}$-algebra and $M$ a left $A$-module. Choose an *injective resolution* of $M$

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ \dots$$

with the $I^k$ injective left $A$-modules and denote by $I^\bullet$ the complex

$$I^\bullet \ : \ 0 \longrightarrow I^0 \xrightarrow{d} I^1 \xrightarrow{d} \ \dots$$

Let $B = HOM^\bullet_A(I^\bullet, I^\bullet)$ be the morphism complex. That is, its $n$-th component are the graded $A$-linear maps $I^\bullet \longrightarrow I^\bullet$ of degree $n$. This space can be equipped with a differential

$$d(f) = d \circ f - (-1)^n f \circ d \quad \text{for } f \text{ in the } n\text{-th part}$$

Then, $B$ is a differentially graded algebra where the multiplication is the natural composition of graded maps. The homology algebra

$$H^* \; B = Ext^*_A(M,M)$$

is the *extension algebra* of $M$. Generalizing the description of $Ext^1_A(M,M)$ given in section 4.3, an element of $Ext^k_A(M,M)$ is an equivalence class of exact sequences of $A$-modules

$$0 \longrightarrow M \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_k \longrightarrow M \longrightarrow 0$$

and the algebra structure on the extension algebra is induced by concatenation of such sequences. This extension algebra has a *canonical* structure of $A_\infty$-algebra with $m_1 = 0$ and $m_2$ he usual multiplication.

Now, let $M_1, \ldots, M_k$ be $A$-modules (for example, finite dimensional representations) and with $filt(M_1, \ldots, M_k)$ we denote the full subcategory of all $A$-modules whose objects admit finite filtrations with subquotients among the $M_i$. We have the following result, for a proof and more details we refer to the excellent notes by B. Keller [40, §6].

**Theorem 4.4** *Let $M = M_1 \oplus \ldots \oplus M_k$. The canonical $A_\infty$-structure on the extension algebra $Ext^*_A(M,M)$ contains enough information to reconstruct the category $filt(M_1, \ldots, M_k)$.*

If we specialize to the case when $M$ is a semi-simple $n$-dimensional representation of $A$ of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ say with decomposition

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

Then, the first two terms of the extension algebra $Ext^*_A(M_\xi, M_\xi)$ are

- $Ext^0_A(M_\xi, M_\xi) = End_A(M_\xi) = M_{e_1}(\mathbb{C}) \oplus \ldots \oplus M_{e_k}(\mathbb{C})$ because by *Schur's lemma* $Hom_A(S_i, S_j) = \delta_{ij}\mathbb{C}$. Hence, the 0-th part of $Ext^*_A(M_\xi, M_\xi)$ determine the dimension vector $\alpha = (e_1, \ldots, e_k)$.

- $Ext^1_A(M_\xi, M_\xi) = \oplus_{i,j=1}^k M_{e_j \times e_i}(Ext^1_A(S_i, S_j))$ and we have seen that $dim_{\mathbb{C}} \; Ext^1_A(S_i, S_j)$ is the number of arrows from vertex $v_i$ to $v_j$ in the local quiver $Q_\xi$.

Summarizing the results of the previous section, we have :

**Proposition 4.7** *Let $\xi \in Sm_n(A)$, then the first two terms of the extension algebra $Ext^*_A(M_\xi, M_\xi)$ contain enough information to determine the étale local structure of* $\mathtt{rep}_n$ *$A$ and* $\mathtt{iss}_n$ *$A$ near $M_\xi$.*

If one wants to extend this result to noncommutative singular points $\xi \notin Sm_n(A)$, one will have to consider the canonical $A_\infty$-structure on the full extension algebra $Ext^*_A(M_\xi, M_\xi)$.

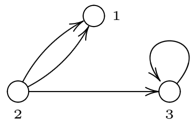
## 4.3 Quiver orders

In this section and the next we will construct a large class of central simple algebras controlled by combinatorial data, using the setting of proposition 3.3.

The description of the quiver $Q$ can be encoded in an integral $k \times k$ matrix

$$\chi_Q = \begin{bmatrix} \chi_{11} & \dots & \chi_{1k} \\ \vdots & & \vdots \\ \chi_{k1} & \dots & \chi_{kk} \end{bmatrix} \quad \text{with} \quad \chi_{ij} = \delta_{ij} - \# \{ \, j \longleftarrow i \, \}$$

**Example 4.3** Consider the quiver $Q$




Then, with the indicated ordering of the vertices we have that the integral matrix is

$$\chi_Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the path algebra of $Q$ is isomorphic to the block-matrix algebra

$$\mathbb{C}Q' \simeq \begin{bmatrix} \mathbb{C} & \mathbb{C} \oplus \mathbb{C} & 0 \\ 0 & \mathbb{C} & 0 \\ 0 & \mathbb{C}[x] & \mathbb{C}[x] \end{bmatrix}$$

where $x$ is the loop in vertex $v_3$.

The subspace $\mathbb{C}Qv_i$ has as basis the paths starting in vertex $v_i$ and because $\mathbb{C}Q = \oplus_i \mathbb{C}Qv_i$, $\mathbb{C}Qv_i$ is a projective left ideal of $\mathbb{C}Q$. Similarly, $v_i\mathbb{C}Q$ has as basis the paths ending at $v_i$ and is a projective right ideal of $\mathbb{C}Q$. The subspace $v_i\mathbb{C}Qv_j$ has as basis the paths starting at $v_j$ and ending at $v_i$ and $\mathbb{C}Qv_i\mathbb{C}Q$ is the twosided ideal of $\mathbb{C}Q$ having as basis all paths passing through $v_i$. If $0 \neq f \in \mathbb{C}Qv_i$ and $0 \neq g \in v_i\mathbb{C}Q$, then $f.g \neq 0$ for let $p$ be a longest path occurring in $f$ and $q$ a longest path in $g$, then the coefficient of $p.q$ in $f.g$ cannot be zero. As a consequence we have

**Lemma 4.2** *The projective left ideals $\mathbb{C}Qv_i$ are indecomposable and pairwise non-isomorphic.*

*Proof.* If $\mathbb{C}Qv_i$ is not indecomposable, then there exists a projection idempotent $f \in Hom_{\mathbb{C}Q}(\mathbb{C}Qv_i, \mathbb{C}Qv_i) \simeq v_i\mathbb{C}Qv_i$. But then, $f^2 = f = f.v_i$ whence $f.(f - v_i) = 0$, contradicting the remark above. Further, for any left $\mathbb{C}Q$-module $M$ we have that $Hom_{\mathbb{C}Q}(\mathbb{C}Qv_i, M) \simeq v_iM$. So, if $\mathbb{C}Qv_i \simeq \mathbb{C}Qv_j$ then the isomorphism gives elements $f \in v_i\mathbb{C}Qv_j$ and $g \in v_j\mathbb{C}Qv_i$ such that $f.g = v_i$ and $g.f = v_j$. But then, $v_i \in \mathbb{C}Qv_j\mathbb{C}Q$, a contradiction unless $i = j$ as this space has basis all paths passing through $v_j$. □

**Example 4.4** Let $Q$ be a quiver, then the following properties hold :

1. $\mathbb{C}Q$ is finite dimensional if and only if $Q$ has no oriented cycles.
2. $\mathbb{C}Q$ is prime (that is, $I.J \neq 0$ for all twosided ideals $I, J \neq 0$) if and only if $Q$ is *strongly connected*, that is, for all vertices $v_i$ and $v_j$ there is a path from $v_i$ to $v_j$.
3. $\mathbb{C}Q$ is Noetherian (that is, satisfies the ascending chain condition on left (or right) ideals) if and only if for every vertex $v_i$ belonging to an oriented cycle there is only one arrow starting at $v_i$ and only one arrow terminating at $v_i$.
4. The radical of $\mathbb{C}Q$ has as basis all paths from $v_i$ to $v_j$ for which there is no path from $v_j$ to $v_i$.
5. The center of $\mathbb{C}Q$ is of the form $\mathbb{C}\times\ldots\times\mathbb{C}\times\mathbb{C}[x]\times\ldots\times\mathbb{C}[x]$ with one factor for each connected component $C$ of $Q$ (that is, connected component for the underlying graph forgetting the orientation) and this factor is isomorphic to $\mathbb{C}[x]$ if and only if $C$ is one oriented cycle.

The *Euler form* of the quiver $Q$ is the bilinear form on $\mathbb{Z}^k$

$$\chi_Q(.,.) : \mathbb{Z}^k \times \mathbb{Z}^k \longrightarrow \mathbb{Z} \quad \text{defined by} \quad \chi_Q(\alpha, \beta) = \alpha.\chi_Q.\beta^\tau$$

for all row vectors $\alpha, \beta \in \mathbb{Z}^k$.

**Theorem 4.5** *Let $V$ and $W$ be two representations of $Q$, then*

$$dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, W) - dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V, W) = \chi_Q(dim(V), dim(W))$$

*Proof.* We claim that there exists an exact sequence of $\mathbb{C}$-vectorspaces

$$0 \longrightarrow Hom_{\mathbb{C}Q}(V, W) \xrightarrow{\gamma} \oplus_{v_i \in Q_v} Hom_{\mathbb{C}}(V_i, W_i) \xrightarrow{d^V_W}$$

$$\xrightarrow{d^V_W} \oplus_{a \in Q_a} Hom_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) \xrightarrow{\epsilon} Ext^1_{\mathbb{C}Q}(V, W) \longrightarrow 0$$

Here, $\gamma(\phi) = (\phi_1, \ldots, \phi_k)$ and $d^V_W$ maps a family of linear maps $(f_1, \ldots, f_k)$ to the linear maps $\mu_a = f_{t(a)}V_a - W_a f_{s(a)}$ for any arrow $a$ in $Q$, that is, to the obstruction of the following diagram to be commutative

$$\begin{tikzcd}
V_{s(a)} \arrow[r, "V_a"] \arrow[d, "f_{s(a)}"'] \arrow[dr, dotted, "\mu_a"] & V_{t(a)} \arrow[d, "f_{t(a)}"] \\
W_{s(a)} \arrow[r, "W_a"] & W_{t(a)}
\end{tikzcd}$$

By the definition of morphisms between representations of $Q$ it is clear that the kernel of $d^V_W$ coincides with $Hom_{\mathbb{C}Q}(V, W)$.

Further, the map $\epsilon$ is defined by sending a family of maps $(g_1, \ldots, g_s) = (g_a)_{a \in Q_a}$ to the equivalence class of the exact sequence

$$0 \longrightarrow W \xrightarrow{i} E \xrightarrow{p} V \longrightarrow 0$$

where for all $v_i \in Q_v$ we have $E_i = W_i \oplus V_i$ and the inclusion $i$ and projection map $p$ are the obvious ones and for each generator $a \in Q_a$ the action of $a$ on $E$ is defined by the matrix

$$E_a = \begin{bmatrix} W_a & g_a \\ 0 & V_a \end{bmatrix} : E_{s(a)} = W_{s(a)} \oplus V_{s(a)} \longrightarrow W_{t(a)} \oplus V_{t(a)} = E_{t(a)}$$

Clearly, this makes $E$ into a $\mathbb{C}Q$-module and one verifies that the above short exact sequence is one of $\mathbb{C}Q$-modules. Remains to prove that the cokernel of $d^V_W$ can be identified with $Ext^1_{\mathbb{C}Q}(V, W)$.

A set of algebra generators of $\mathbb{C}Q$ is given by $\{v_1, \ldots, v_k, a_1, \ldots, a_l\}$. A cycle is given by a linear map $\lambda : \mathbb{C}Q \longrightarrow Hom_{\mathbb{C}}(V, W)$ such that for all $f, f' \in \mathbb{C}Q$ we have the condition

$$\lambda(ff') = \rho(f)\lambda(f') + \lambda(f)\sigma(f')$$

where $\rho$ determines the action on $W$ and $\sigma$ that on $V$. First, consider $v_i$ then the condition says $\lambda(v_i^2) = \lambda(v_i) = p_i^W \lambda(v_i) + \lambda(v_i) p_i^V$ whence $\lambda(v_i) : V_i \longrightarrow W_i$ but then applying again the condition we see that $\lambda(v_i) = 2\lambda(v_i)$ so $\lambda(v_i) = 0$. Similarly, using the condition on $a = v_{t(a)}a = av_{s(a)}$ we deduce that $\lambda(a) : V_{s(a)} \longrightarrow W_{t(a)}$. That is, we can identify $\oplus_{a \in Q_a} Hom_{\mathbb{C}}(V_{s(a)}, W_{t(a)})$ with $Z(V, W)$ under the map $\epsilon$. Moreover, the image of $\delta$ gives rise to a family of morphisms $\lambda(a) = f_{t(a)}V_a - W_a f_{s(a)}$ for a linear map $f = (f_i) : V \longrightarrow W$ so this image coincides precisely to the subspace of boundaries $B(V, W)$ proving that indeed the cokernel of $d^V_W$ is $Ext^1_{\mathbb{C}Q}(V, W)$ finishing the proof of exactness of the long sequence of vectorspaces. But then, if $dim(V) = (r_1, \ldots, r_k)$ and

$dim(W) = (s_1, \ldots, s_k)$, we have that $dim\ Hom(V,W) - dim\ Ext^1(V,W)$ is equal to

$$\begin{aligned}
&\sum_{v_i \in Q_v} dim\ Hom_{\mathbb{C}}(V_i, W_i) - \sum_{a \in Q_a} dim\ Hom_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) \\
&= \sum_{v_i \in Q_v} r_i s_i - \sum_{a \in Q_a} r_{s(a)} s_{t(a)} \\
&= (r_1, \ldots, r_k) M_Q (s_1, \ldots, s_k)^\tau = \chi_Q(dim(V), dim(W))
\end{aligned}$$

finishing the proof. $\square$

Fix a dimension vector $\alpha = (d_1, \ldots, d_k) \in \mathbb{N}^k$ and consider the set $\mathtt{rep}_\alpha\ Q$ of all representations $V$ of $Q$ such that $dim(V) = \alpha$. Because $V$ is completely determined by the linear maps

$$V_a : V_{s(a)} = \mathbb{C}^{d_{s(a)}} \longrightarrow \mathbb{C}^{d_{t(a)}} = V_{t(a)}$$

we see that $\mathtt{rep}_\alpha\ Q$ is the affine space

$$\mathtt{rep}_\alpha\ Q = \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} M_{d_j \times d_i}(\mathbb{C}) \simeq \mathbb{C}^r$$

where $r = \sum_{a \in Q_a} d_{s(a)} d_{t(a)}$. On this affine space we have an action of the algebraic group $GL(\alpha) = GL_{d_1} \times \ldots \times GL_{d_k}$ by conjugation. That is, if $g = (g_1, \ldots, g_k) \in GL(\alpha)$ and if $V = (V_a)_{a \in Q_a}$ then $g.V$ is determined by the matrices

$$(g.V)_a = g_{t(a)} V_a g_{s(a)}^{-1}.$$

If $V$ and $W$ in $\mathtt{rep}_\alpha\ Q$ are isomorphic as representations of $Q$, such an isomorphism is determined by invertible matrices $g_i : V_i \longrightarrow W_i \in GL_{d_i}$ such that for every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ we have a commutative diagram

$$\begin{tikzcd}
V_i \arrow[r, "V_a"] \arrow[d, "g_i"] & V_j \arrow[d, "g_j"] \\
W_i \arrow[r, "W_a"] & W_j
\end{tikzcd}$$

or equivalently, $g_j V_a = W_a g_i$. That is, two representations are isomorphic if and only if they belong to the same orbit under $GL(\alpha)$. In particular, we see that

$$Stab_{GL(\alpha)}\ V \simeq Aut_{\mathbb{C}Q}\ V$$

and the latter is an open subvariety of the affine space $End_{\mathbb{C}Q}(V) = Hom_{\mathbb{C}Q}(V,V)$ whence they have the same dimension. The dimension of the orbit $\mathcal{O}(V)$ of $V$ in $\mathtt{rep}_\alpha\ Q$ is equal to

$$dim\ \mathcal{O}(V) = dim\ GL(\alpha) - dim\ Stab_{GL(\alpha)}\ V.$$

But then we have a geometric reformulation of the above theorem.

**Lemma 4.3** *Let $V \in \mathtt{rep}_\alpha\ Q$, then*

$$dim\ \mathtt{rep}_\alpha\ Q - dim\ \mathcal{O}(V) = dim\ End_{\mathbb{C}Q}(V) - \chi_Q(\alpha,\alpha) = dim\ Ext^1_{\mathbb{C}Q}(V,V)$$

*Proof.* We have seen that $dim\ \mathtt{rep}_\alpha\ Q - dim\ \mathcal{O}(V)$ is equal to

$$\sum_a d_{s(a)}d_{t(a)} - (\sum_i d_i^2 - dim\ End_{\mathbb{C}Q}(V)) = dim\ End_{\mathbb{C}Q}(V) - \chi_Q(\alpha,\alpha)$$

and the foregoing theorem asserts that the latter term is equal to $dim\ Ext^1_{\mathbb{C}Q}(V,V)$. □

In particular it follows that the orbit $\mathcal{O}(V)$ is *open* in $\mathtt{rep}_\alpha\ Q$ if and only if $V$ has no self-extensions. Moreover, as $\mathtt{rep}_\alpha\ Q$ is irreducible there can be at most one isomorphism class of a representation without self-extensions.

For every dimension vector $\alpha = (d_1,\ldots,d_k)$ we will construct a *quiver order* $\mathbb{T}_\alpha Q$ which is a Cayley-Hamilton algebra of degree $n$ where $n = d_1 + \ldots + d_k$. First, we describe the $n$-dimensional representations of the Quillen-smooth algebra $C_k$.

**Proposition 4.8** *Let $C_k = \mathbb{C}[e_1,\ldots,e_k]/(e_i^2 - e_i, e_ie_j, \sum_{i=1}^k e_i - 1)$, then $\mathtt{rep}_n\ C_k$ is reduced and is the disjoint union of the homogeneous varieties*

$$\mathtt{rep}_n\ C_k = \bigcup_\alpha GL_n/(GL_{d_1} \times \ldots \times GL_{d_k})$$

*where the union is taken over all $\alpha = (d_1,\ldots,d_k)$ such that $n = \sum_i d_i$.*

*Proof.* As $C_k$ is Quillen smooth we will see in section 4.1 that all its representation spaces $\mathtt{rep}_n\ C_k$ are smooth varieties hence in particular reduced. Therefore, it suffices to describe the points. For any $n$-dimensional representation

$$C_k \xrightarrow{\phi} M_n(\mathbb{C})$$

the image is a commutative semi-simple algebra with orthogonal idempotents $f_i = \phi(e_i)$ of rank $d_i$. Because $\sum_i e_i = \mathbb{1}_n$ we must have that $\sum_i d_i = n$. Alternatively, the corresponding $n$-dimensional representation $M = \oplus_i M_i$ where $M_i = e_i\mathbb{C}^n$ has dimension $d_i$. The stabilizer subgroup of $M$ is equal to $GL(\alpha) = GL_{d_1} \times \ldots \times GL_{d_k}$, proving the claim. □

The algebra embedding $C_k \xrightarrow{\phi} \mathbb{C}Q$ obtained by $\phi(e_i) = v_i$ determines a morphism

$$\mathtt{rep}_n\ \mathbb{C}Q \xrightarrow{\pi} \mathtt{rep}_n\ C_k = \cup_\alpha \mathcal{O}(\alpha) = \cup_\alpha GL_n/GL(\alpha)$$

where the disjoint union is taken over all the dimension vectors $\alpha = (d_1, \ldots, d_k)$ such that $n = \sum d_i$. Consider the point $p_\alpha \in \mathcal{O}(\alpha)$ determined by sending the idempotents $e_i$ to the canonical diagonal idempotents

$$\sum_{j=\sum_{l=1}^{i-1} d_l+1}^{\sum_{l=1}^{i} d_i} e_{jj} \in M_n(\mathbb{C})$$

We denote by $C_k(\alpha)$ this semi-simple commutative subalgebra of $M_n(\mathbb{C})$. As $\mathtt{rep}_\alpha\ Q$ can be identified with the variety of $n$-dimensional representations of $\mathbb{C}Q$ in block form determined by these idempotents we see that $\mathtt{rep}_\alpha\ Q = \pi^{-1}(p)$.

We define the *quiver trace algebra* $\mathbb{T}Q$ to be the path algebra of $Q$ over the polynomial algebra $R$ in the variables $t_p$ where $p$ is a word in the arrows $a_j \in Q_a$ and is determined only up to cyclic permutation. As a consequence we only retain the variables $t_p$ where $p$ is an oriented cycle in $Q$ (as all the others have a cyclic permutation which is the zero element in $\mathbb{C}Q$). We define a formal trace map $tr$ on $\mathbb{T}Q$ by $tr(p) = t_p$ if $p$ is an oriented cycle in $Q$ and $tr(p) = 0$ otherwise.

For a fixed dimension vector $\alpha = (d_1, \ldots, d_k)$ with $\sum_i d_i = n$ we define $\mathbb{T}_\alpha\ Q$ to be the quotient

$$\mathbb{T}_\alpha Q = \frac{\mathbb{T}Q}{(\chi_a^{(n)}(a), tr(v_i) - d_i)}$$

by dividing out the substitution invariant twosided ideal generated by all the evaluations of the formal Cayley-Hamilton algebras of degree $n$, $\chi_a^{(n)}(a)$ for $a \in \mathbb{T}Q$ together with the additional relations that $tr(v_i) = d_i$. $\mathbb{T}_\alpha\ Q$ is a Cayley-Hamilton algebra of degree $n$ with a decomposition $1 = e_1 + \ldots + e_k$ into orthogonal idempotents such that $tr(e_i) = d_i$.

More generally, let $A$ be a Cayley-Hamilton algebra of degree $n$ with decomposition $1 = a_1 + \ldots + a_n$ into orthogonal idempotents such that $tr(a_i) = d_i \in \mathbb{N}_+$ and $\sum d_i = n$. Then, we have a trace preserving embedding $C_k(\alpha) \xhookrightarrow{i} A$ making $A$ into a $C_k(\alpha) = \times_{i=1}^k \mathbb{C}$-algebra. We have a trace preserving embedding $C_k(\alpha) \xhookrightarrow{i'} M_n(\mathbb{C})$ by sending the idempotent $e_i$ to the diagonal idempotent $E_i \in M_n(\mathbb{C})$ with ones on the diagonal from position $\sum_{j=1}^{i-1} d_j - 1$ to $\sum_{j=1}^{i} d_i$. This calls for the introduction of a *restricted representation space* of all trace preserving algebra morphisms $\chi$ such that the diagram below is commutative

$$\begin{tikzcd}
A \arrow[r, "\chi"] & M_n(\mathbb{C}) \\
C_k(\alpha) \arrow[u, hook, "i"] \arrow[ur, hook, "i'"'] &
\end{tikzcd}$$

that is, such that $\chi(a_i) = E_i$. This again determines an affine scheme $\mathtt{rep}_\alpha^{res}\ A$ which is in fact a closed subscheme of $\mathtt{trep}_n\ A$. The functorial description of the restricted module scheme is as follows. Let $C$ be any commutative $\mathbb{C}$-algebra, then $M_n(C)$ is a $C_k(\alpha)$-algebra and the idempotents $E_i$ allow for a block decomposition

$$M_n(C) = \oplus_{i,j} E_i M_n(C) E_j = \begin{bmatrix} E_1 M_n(C) E_1 & \dots & E_1 M_n(C) E_k \\ \vdots & & \vdots \\ E_k M_n(C) E_1 & \dots & E_k M_n(C) E_k \end{bmatrix}.$$

The scheme $\mathtt{rep}_\alpha^{res}\ A$ assigns to the algebra $C$ the set of all trace preserving algebra maps

$$A \xrightarrow{\phi} M_n(B) \quad \text{such that } \phi(a_i) = E_i.$$

Equivalently, the idempotents $a_i$ decompose $A$ into block form $A = \oplus_{i,j} a_i A a_j$ and then $\mathtt{rep}_\alpha^{res}\ A(C)$ are the trace preserving algebra morphisms $A \longrightarrow M_n(B)$ compatible with the block decompositions.

Still another description of the restricted representation scheme is therefore that $\mathtt{rep}_\alpha^{res}\ A$ is the scheme theoretic fiber $\pi^{-1}(p_\alpha)$ of the point $p_\alpha$ under the $GL_n$-equivariant morphism

$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{trep}_n\ C_k(\alpha).$$

Hence, the stabilizer subgroup of $p$ acts on $\mathtt{rep}_\alpha^{res}\ A$. This stabilizer is the subgroup $GL(\alpha) = GL_{m_1} \times \dots \times GL_{m_k}$ embedded in $GL_n$ along the diagonal

$$GL(\alpha) = \begin{bmatrix} GL_{m_1} & & \\ & \ddots & \\ & & GL_{m_k} \end{bmatrix} \hookrightarrow GL_n.$$

Clearly, $GL(\alpha)$ acts via this embedding by conjugation on $M_n(\mathbb{C})$.

**Theorem 4.6** *Let $A$ be a Cayley-Hamilton algebra of degree $n$ such that $1 = a_1 + \dots + a_k$ is a decomposition into orthogonal idempotents with $tr(a_i) = m_i \in \mathbb{N}_+$. Then, $A$ is isomorphic to the ring of $GL(\alpha)$-equivariant maps*

$$\mathtt{rep}_\alpha^{res}\ A \longrightarrow M_n.$$

*Proof.* We know that $A$ is the ring of $GL_n$-equivariant maps $\mathtt{trep}_n\ A \longrightarrow M_n$. Further, we have a $GL_n$-equivariant map

$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{rep}_n tr\ C_k(\alpha) = GL_n.p \simeq GL_n/GL(\alpha)$$

Thus, the $GL_n$-equivariant maps from $\mathtt{trep}_n\ A$ to $M_n$ coincide with the $Stab(p) = GL(\alpha)$-equivariant maps from the fiber $\pi^{-1}(p) = \mathtt{rep}_\alpha^{res}\ A$ to $M_n$. $\square$

That is, we have a block matrix decomposition for $A$. Indeed, we have

$$A \simeq (\mathbb{C}[\mathtt{rep}_\alpha^{res} \ A] \otimes M_n(\mathbb{C}))^{GL(\alpha)}$$

and this isomorphism is clearly compatible with the block decomposition and thus we have for all $i, j$ that

$$a_i A a_j \simeq (\mathbb{C}[\mathtt{rep}_\alpha^{res} \ A] \otimes M_{m_i \times m_j}(\mathbb{C}))^{GL(\alpha)}$$

where $M_{m_i \times m_j}(\mathbb{C})$ is the space of rectangular $m_i \times m_j$ matrices $M$ with coefficients in $\mathbb{C}$ on which $GL(\alpha)$ acts via

$$g.M = g_i M g_j^{-1} \quad \text{where } g = (g_1, \ldots, g_k) \in GL(\alpha).$$

If we specialize this result to the case of quiver orders we have

$$\mathtt{rep}_\alpha^{res} \ \mathbb{T}_\alpha Q \simeq \mathtt{rep}_\alpha \ Q$$

as $GL(\alpha)$-varieties and we deduce

**Theorem 4.7** *With notations as before,*

1. $\mathbb{T}_\alpha \ Q$ *is the algebra of $GL(\alpha)$-equivariant maps from $\mathtt{rep}_\alpha \ Q$ to $M_n$, that is,*

$$\mathbb{T}_\alpha \ Q = M_n(\mathbb{C}[\mathtt{rep}_\alpha \ Q])^{GL(\alpha)}$$

2. *The* quiver necklace algebra

$$\mathbb{N}_\alpha \ Q = \mathbb{C}[\mathtt{rep}_\alpha \ Q]^{GL(\alpha)}$$

*is generated by traces along oriented cycles in the quiver $Q$ of length bounded by $n^2 + 1$.*

A concrete realization of these algebras is as follows. To an arrow $(j) \xleftarrow{a} (i)$ corresponds a $d_j \times d_i$ matrix of variables from $\mathbb{C}[\mathtt{rep}_\alpha \ Q]$

$$\boxed{M_a} = \begin{bmatrix} x_{11}(a) & \ldots & \ldots & x_{1d_i}(a) \\ \vdots & & & \vdots \\ x_{d_j 1}(a) & \ldots & \ldots & x_{d_j d_i}(a) \end{bmatrix}$$

where $x_{ij}(a)$ are the coordinate functions of the entries of $V_a$ of a representation $V \in \mathtt{rep}_\alpha \ Q$. Let $p = a_1 a_2 \ldots a_r$ be an oriented cycle in $Q$, then we can compute the following matrix

$$M_p = M_{a_r} \ldots M_{a_2} M_{a_1}$$

over $\mathbb{C}[\mathtt{rep}_\alpha \ Q]$. As we have that $s(a_r) = t(a_1) = v_i$, this is a square $d_i \times d_i$ matrix with coefficients in $\mathbb{C}[\mathtt{rep}_\alpha \ Q]$ and we can take its ordinary trace

$$Tr(M_p) \in \mathbb{C}[\mathtt{rep}_\alpha \ Q].$$

Then, $\mathbb{N}_\alpha\ Q$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}[\mathtt{rep}_\alpha\ Q]$ generated by these elements. Consider the block structure of $M_n(\mathbb{C}[\mathtt{rep}_\alpha\ Q])$ with respect to the idempotents $e_i$

$$\begin{bmatrix} M_{d_1}(S) & \dots & \dots & M_{d_1 \times d_k}(S) \\ \vdots & & & \vdots \\ \vdots & M_{d_j \times d_i}(S) & & \vdots \\ M_{d_k \times d_i}(S) & \dots & \dots & M_{d_k}(S) \end{bmatrix}$$

where $S = \mathbb{C}[\mathtt{rep}_\alpha\ Q]$. Then, we can also view the matrix $M_a$ for an arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ as a block matrix in $M_n(\mathbb{C}[\mathtt{rep}_\alpha\ Q])$

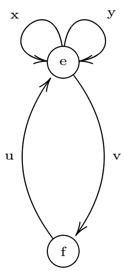
$$\begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ \vdots & \boxed{M_a} & & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}$$

Then, $\mathbb{T}_\alpha\ Q$ is the $C_k(\alpha)$-subalgebra of $M_n(\mathbb{C}[\mathtt{rep}_\alpha\ Q])$ generated by $\mathbb{N}_\alpha\ Q$ and these block matrices for all arrows $a \in Q_a$. $\mathbb{T}_\alpha\ Q$ itself has a block decomposition

$$\mathbb{T}_\alpha\ Q = \begin{bmatrix} P_{11} & \dots & \dots & P_{1k} \\ \vdots & & & \vdots \\ \vdots & P_{ij} & & \vdots \\ P_{k1} & \dots & \dots & P_{kk} \end{bmatrix}$$

where $P_{ij}$ is the $\mathbb{N}_\alpha\ Q$-module spanned by all matrices $M_p$ where $p$ is a path from $v_i$ to $v_j$ of length bounded by $n^2$.

**Example 4.5** Consider the path algebra $\mathbb{M}$ of the quiver which we will encounter in chapter 8 in connection with the Hilbert scheme of points in the plane and with the Calogero-Moser system

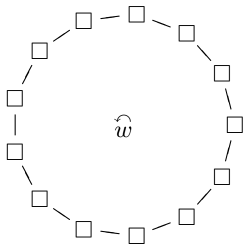


and take as dimension vector $\alpha = (n, 1)$. The total dimension is in this case $\overline{n} = n + 1$ and we fix the embedding $C_2 = \mathbb{C} \times \mathbb{C} \hookrightarrow \mathbb{M}$ given by the decomposition $1 = e + f$. Then, the above realization of $\mathbb{T}_\alpha\ \mathbb{M}$ consists in taking the following $\overline{n} \times \overline{n}$ matrices

$$e_n = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \qquad f_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \qquad x_n = \begin{bmatrix} x_{11} & \dots & x_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \dots & x_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$y_n = \begin{bmatrix} y_{11} & \dots & y_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ y_{n1} & \dots & y_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \qquad u_n = \begin{bmatrix} 0 & \dots & 0 & u_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & u_n \\ 0 & \dots & 0 & 0 \end{bmatrix} \qquad v_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ v_1 & \dots & v_n & 0 \end{bmatrix}$$

In order to determine the ring of $GL(\alpha)$-polynomial invariants of $\mathtt{rep}_\alpha\ \mathbb{M}$ we have to consider the traces along oriented cycles in the quiver. Any nontrivial such cycle must pass through the vertex $e$ and then we can decompose the cycle into factors $x$, $y$ and $uv$ (observe that if we wanted to describe circuits based at the vertex $f$ they are of the form $c = vc'u$ with $c'$ a circuit based at $e$ and we can use the cyclic property of traces to bring it into the claimed form). That is, all relevant oriented cycles in the quiver can be represented by a necklace word $w$




where each bead is one of the elements

$$\boxed{\bullet} = x \qquad \boxed{\circ} = y \quad \text{and} \quad \boxed{\blacktriangledown} = uv$$

In calculating the trace, we first have to replace each occurrence of $x, y, u$ or $v$ by the relevant $\overline{n} \times \overline{n}$-matrix above. This results in replacing each of the beads in the necklace by one of the following $n \times n$ matrices

$$\boxed{\bullet} = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \qquad \boxed{\circ} = \begin{bmatrix} y_{11} & \dots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \dots & y_{nn} \end{bmatrix} \qquad \boxed{\blacktriangledown} = \begin{bmatrix} u_1 v_1 & \dots & u_1 v_n \\ \vdots & & \vdots \\ u_n v_1 & \dots & u_n v_n \end{bmatrix}$$

and taking the trace of the $n \times n$ matrix obtained after multiplying these bead-matrices cyclically in the indicated orientation. This concludes the description of the invariant ring $\mathbb{N}_\alpha\ \mathbb{Q}$. The algebra $\mathbb{T}_\alpha\ \mathbb{M}$ of $GL(\alpha)$-equivariant maps from $\mathtt{rep}_\alpha\ \mathbb{M}$ to $M_{\overline{n}}$ is then the subalgebra of $M_{\overline{n}}(\mathbb{C}[\mathtt{rep}_\alpha\ \mathbb{M}])$ generated as $C_2(\alpha)$-algebra (using the idempotent $\overline{n} \times \overline{n}$ matrices corresponding to $e$ and $f$) by $\mathbb{N}_\alpha\ \mathbb{M}$ and the $\overline{n} \times \overline{n}$-matrices corresponding to $x, y, u$ and $v$.

After these preliminaries, let us return to the local quiver setting $(Q_\xi, \alpha)$ associated to a point $\xi \in Sm_n(A)$ as described in the previous section. Above, we have seen that quiver necklace algebra $\mathbb{N}_\alpha\ Q_\xi$ is the coordinate ring of $N_x/GL(\alpha)$. $\mathbb{N}_\alpha\ Q_\xi$ is a graded algebra and is generated by all traces along oriented cycles in the quiver $Q_\xi$. Let $\mathfrak{m}_0$ be the graded maximal ideal of $\mathbb{N}_\alpha\ Q_\xi$, that is corresponding to the closed orbit of the trivial representation. With $\widehat{\mathbb{T}_\xi}$ (respectively $\widehat{\mathbb{N}_\alpha}$) we will denote the $\mathfrak{m}_0$-adic filtration of the quiver-order $\mathbb{T}_\alpha\ Q_\xi$ (respectively of the quiver necklace algebra $\mathbb{N}_\alpha\ Q_\xi$). Recall that the quiver-order $\mathbb{T}_\alpha\ Q_\xi$ has a block-decomposition determined by oriented paths in the quiver $Q_\xi$. A consequence of the slice theorem and the description of Cayley-Hamilton algebras and their algebra of traces by geometric data we deduce.

**Theorem 4.8** *Let $\xi \in Sm_n(A)$. Let $N = tr \int_n A$, let $\mathfrak{m}$ be the maximal ideal of $N$ corresponding to $\xi$ and denote $T = \int_n A$, then we have the isomorphism and Morita equivalence*

$$\widehat{N}_{\mathfrak{m}} \simeq \widehat{\mathbb{N}_\alpha} \qquad \text{and} \qquad \widehat{T}_{\mathfrak{m}} \underset{Morita}{\sim} \widehat{\mathbb{T}_\alpha}$$

*We have an explicit description of the algebras on the right in terms of the quiver setting $(Q_\xi, \alpha)$ and the Morita equivalence is determined by the embedding $GL(\alpha) \hookrightarrow GL_n$.*

Let $Q^\bullet$ be a marked quiver with underlying quiver $Q$ and let $\alpha = (d_1, \ldots, d_k)$ be a dimension vector. We define the *marked quiver-necklace algebra* $\mathbb{N}_\alpha\ Q^\bullet$ to be the ring of $GL(\alpha)$-polynomial invariants on the representation space $\mathtt{rep}_\alpha\ Q^\bullet$, that is, $\mathbb{N}_\alpha\ Q^\bullet$ is the coordinate ring of the quotient variety $\mathtt{rep}_\alpha\ Q^\bullet/GL(\alpha)$. The *marked quiver-order* $\mathbb{T}_\alpha\ Q^\bullet$ is defined to be the algebra of $GL(\alpha)$-equivariant polynomial maps from $\mathtt{rep}_\alpha\ Q^\bullet$ to $M_d(\mathbb{C})$ where $d = \sum_i d_i$. Because we can separate traces, it follows that

$$\mathbb{N}_\alpha\ Q^\bullet = \frac{\mathbb{N}_{\alpha\ Q}}{(tr(m_1), \ldots, tr(m_l))} \qquad \text{and} \qquad \mathbb{T}_\alpha\ Q^\bullet = \frac{\mathbb{T}_\alpha\ Q}{(tr(m_1), \ldots, tr(m_l))}$$

where $\{m_1, \ldots, m_l\}$ is the set of all marked loops in $Q^\bullet$.

Let $B$ be a Cayley-Hamilton algebra of degree $n$ and let $M_\xi$ be a trace preserving semi-simple $B$-representation of type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ corresponding to the point $\xi$ in the quotient variety $\mathtt{iss}_n^{tr}\ B$.

**Definition 4.6** *A point $\xi \in \mathtt{iss}_n^{tr}\ B$ is said to belong to the* smooth locus *of $B$ iff the trace preserving representation space $\mathtt{trep}_n\ B$ is smooth in $x \in \mathcal{O}(M_\xi)$. The smooth locus of the Cayley-Hamilton algebra $B$ of degree $n$ will be denotes by $Sm_{tr}(B)$.*

Applying the slice theorem to the trace preserving representation space, we obtain with the obvious modifications in notation.

**Theorem 4.9** *Let $\xi \in Sm_{tr}(B)$ and $N = tr\ B$. Let $\mathfrak{m}$ be the maximal ideal of $N$ corresponding to $\xi$, then we have the isomorphism and Morita equivalence*

$$\widehat{N}_{\mathfrak{m}} \simeq \widehat{\mathbb{N}^{\bullet}_{\alpha}} \qquad \text{and} \qquad \widehat{B}_{\mathfrak{m}} \underset{Morita}{\sim} \widehat{\mathbb{T}^{\bullet}_{\alpha}}$$

*where we have an explicit description of the algebras on the right in terms of the quiver setting $(Q_\xi, \alpha)$ and where the Morita equivalence is determined by the embedding $GL(\alpha) \hookrightarrow GL(n)$.*

Even if the left hand sides of the slice diagrams are not defined when $\xi$ is not contained in the smooth locus, the dimension of the normal spaces (that is, the (trace preserving) self-extensions of $M_\xi$) allow us to have a numerical measure of the 'badness' of this noncommutative singularity.

**Definition 4.7** *Let $A$ be an affine $\mathbb{C}$-algebra and $\xi \in \mathtt{iss}_n\ A$ of type $\tau = (e_1, d_1; \ldots ; e_k, d_k)$. The* measure of singularity *in $\xi$ is given by the non-negative number*

$$ms(\xi) = n^2 + dim_{\mathbb{C}}\ Ext^1_A(M_\xi, M_\xi) - e_1^2 - \ldots - e_k^2 - dim_{M_\xi}\ \mathtt{rep}_n\ A$$

*Let $B$ be a Cayley-Hamilton algebra of degree $n$ and $\xi \in \mathtt{iss}^{tr}_n\ B$ of type $\tau = (e_1, d_1; \ldots ; e_k, d_k)$. The measure of singularity in $\xi$ is given by the non-negative number*

$$ms(\xi) = n^2 + dim_{\mathbb{C}}\ Ext^{tr}_B(M_\xi, M_\xi) - e_1^2 - \ldots - e_k^2 - dim_{M_\xi}\ \mathtt{trep}_n\ A$$

*Clearly, $\xi \in Sm_n(A)$ (respectively, $\xi \in Sm_{tr}(B)$) if and only if $ms(\xi) = 0$.*

As an application to the slice theorem, let us prove the connection between Azumaya algebras and principal fibrations. The *Azumaya locus* of an algebra $A$ will be the open subset $U_{Az}$ of $\mathtt{iss}_n\ A$ consisting of the points $\xi$ of type $(1, n)$. Let $\mathtt{rep}_n\ A \xrightarrow{\pi} \mathtt{iss}_n\ A$ be the quotient map.

**Proposition 4.9** *The quotient $\pi^{-1}(U_{Az}) \longrightarrow U_{Az}$ is a principal $PGL_n$-fibration in the étale topology, that is determines an element in $H^1_{et}(U_{Az}, PGL_n)$.*

*Proof.* Let $\xi \in U_{Az}$ and $x = M_\xi$ a corresponding simple representation. Let $S_x$ be the slice in $x$ for the $PGL_n$-action on $\mathtt{rep}_n\ A$. By taking traces of products of a lifted basis from $M_n(\mathbb{C})$ we find a $PGL_n$-affine open neighborhood $U_\xi$ of $\xi$ contained in $U_{Az}$ and hence by the slice result a commuting diagram

$$\begin{tikzcd} PGL_n \times S_x \arrow[r, "\psi"] \arrow[d] & \pi^{-1}(U_\xi) \arrow[d, "\pi"] \\ S_x \arrow[r, "\psi/PGL_n"'] & U_\xi \end{tikzcd}$$

where $\psi$ and $\psi/PGL_n$ are étale maps. That is, $\psi/PGL_n$ is an étale neighborhood of $\xi$ over which $\pi$ is trivialized. As this holds for all points $\xi \in U_{Az}$ the result follows. $\square$

## 4.4 Simple roots

In this section we will use proposition 3.3 to construct quiver orders $\mathbb{T}_\alpha Q$ which determine central simple algebras over the functionfield of the quotient variety $\mathtt{iss}_\alpha\ Q = \mathtt{rep}_\alpha\ Q/GL(\alpha)$. With $\mathtt{PGL}(\alpha)$ we denote the groupscheme corresponding to the algebraic group

$$PGL(\alpha) = GL(\alpha)/\mathbb{C}^*(\mathbb{1}_{d_1}, \ldots, \mathbb{1}_{d_k})$$

If $C$ is a commutative $\mathbb{C}$-algebra, then using the embedding $PGL(\alpha) \hookrightarrow PGL_n$, the pointed cohomology set

$$H^1_{et}(C, \mathtt{PGL}(\alpha)) \hookrightarrow H^1_{et}(C, \mathtt{PGL}_n)$$

classifies Azumaya algebras $A$ over $C$ with a distinguished embedding $C_k \hookrightarrow A$ that are split by an étale cover such that on this cover the embedding of $C_k$ in matrices is conjugate to the standard embedding $C_k(\alpha)$. Modifying the argument of proposition 3.3 we have

**Proposition 4.10** *If $\alpha$ is the dimension vector of a simple representation of $Q$, then*

$$\mathbb{T}_\alpha Q \otimes_{\mathbb{N}_\alpha Q} \mathbb{C}(\mathtt{iss}_\alpha\ Q)$$

*is a central simple algebra over the function field of the quotient variety $\mathtt{iss}_\alpha\ Q$.*

Remains to classify the *simple roots* $\alpha$, that is, the dimension vectors of simple representations of the quiver $Q$. Consider the vertex set $Q_v = \{v_1, \ldots, v_k\}$. To a subset $S \hookrightarrow Q_v$ we associate the full subquiver $Q_S$ of $Q$, that is, $Q_S$ has as set of vertices the subset $S$ and as set of arrows all arrows $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in $Q_a$ such that $v_i$ and $v_j$ belong to $S$. A full subquiver $Q_S$ is said to be *strongly connected* if and only if for all $v_i, v_j \in V$ there is an oriented cycle in $Q_S$ passing through $v_i$ and $v_j$. We can partition

$$Q_v = S_1 \sqcup \ldots \sqcup S_s$$

such that the $Q_{S_i}$ are maximal strongly connected components of $Q$. Clearly, the direction of arrows in $Q$ between vertices in $S_i$ and $S_j$ is the same by the maximality assumption and can be used to define an orientation between $S_i$ and $S_j$. The *strongly connected component quiver* $SC(Q)$ is then the quiver on $s$ vertices $\{w_1, \ldots, w_s\}$ with $w_i$ corresponding to $S_i$ and there is one arrow from $w_i$ to $w_j$ if and only if there is an arrow in $Q$ from a vertex in $S_i$ to a vertex in $S_j$. Observe that when the underlying graph of $Q$ is connected, then so is the underlying graph of $SC(Q)$ and $SC(Q)$ is a quiver without oriented cycles.

Vertices with specific in- and out-going arrows are given names as in figure 4.7 If $\alpha = (d_1, \ldots, d_k)$ is a dimension vector, we define the *support* of $\alpha$ to be $supp(\alpha) = \{v_i \in Q_v \mid d_i \neq 0\}$.

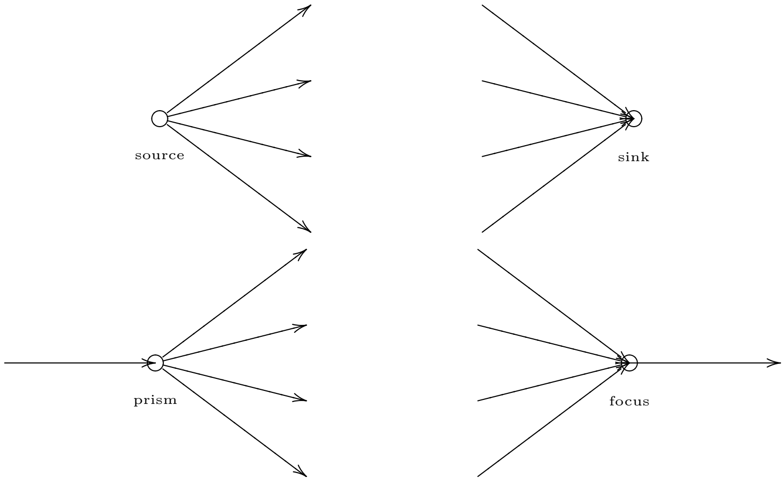



Figure 4.7: Vertex terminology

**Lemma 4.4** *If $\alpha$ is the dimension vector of a simple representation of $Q$, then $Q_{supp(\alpha)}$ is a strongly connected subquiver.*

*Proof.* If not, we consider the strongly connected component quiver $SC(Q_{supp(\alpha)})$ and by assumption there must be a sink in it corresponding to a proper subset $S \overset{\neq}{\hookrightarrow} Q_v$. If $V \in \mathtt{rep}_\alpha\, Q$ we can then construct a representation $W$ by

- $W_i = V_i$ for $v_i \in S$ and $W_i = 0$ if $v_i \notin S$,
- $W_a = V_a$ for an arrow $a$ in $Q_S$ and $W_a = 0$ otherwise.

One verifies that $W$ is a proper subrepresentation of $V$, so $V$ cannot be simple, a contradiction. $\square$

The second necessary condition involves the Euler form of $Q$. With $\epsilon_i$ be denote the dimension vector of the simple representation having a one-dimensional space at vertex $v_i$ and zero elsewhere and all arrows zero matrices.

**Lemma 4.5** *If $\alpha$ is the dimension vector of a simple representation of $Q$, then*

$$\begin{cases} \chi_Q(\alpha, \epsilon_i) & \leq 0 \\ \chi_Q(\epsilon_i, \alpha) & \leq 0 \end{cases}$$

*for all* $v_i \in supp(\alpha)$.

*Proof.* Let $V$ be a simple representation of $Q$ with dimension vector $\alpha = (d_1, \ldots, d_k)$. One verifies that

$$\chi_Q(\epsilon_i, \alpha) = d_i - \sum_{\textcircled{j} \longleftarrow \textcircled{i}} d_j$$

Assume that $\chi_Q(\epsilon_i, \alpha) > 0$, then the natural linear map

$$\bigoplus_{\textcircled{j} \xleftarrow[a]{} \textcircled{i}} V_a \; : \; V_i \longrightarrow \bigoplus_{\textcircled{j} \xleftarrow[a]{} \textcircled{i}} V_j$$

has a nontrivial kernel, say $K$. But then we consider the representation $W$ of $Q$ determined by

- $W_i = K$ and $W_j = 0$ for all $j \neq i$,
- $W_a = 0$ for all $a \in Q_a$.

It is clear that $W$ is a proper subrepresentation of $V$, a contradiction.

Similarly, assume that $\chi_Q(\alpha, \epsilon_i) = d_i - \sum_{\textcircled{i} \longleftarrow \textcircled{j}} d_j > 0$, then the linear map

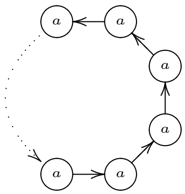
$$\bigoplus_{\textcircled{i} \xleftarrow[a]{} \textcircled{j}} V_a \; : \; \bigoplus_{\textcircled{i} \xleftarrow[a]{} \textcircled{j}} V_j \longrightarrow V_i$$

has an image $I$ which is a proper subspace of $V_i$. The representation $W$ of $Q$ determined by

- $W_i = I$ and $W_j = V_j$ for $j \neq i$,
- $W_a = V_a$ for all $a \in Q_a$.

is a proper subrepresentation of $V$, a contradiction finishing the proof. $\square$

**Example 4.6** The necessary conditions of the foregoing two lemmas are not sufficient. Consider the extended Dynkin quiver of type $\tilde{A}_k$ with cyclic orientation.



and dimension vector $\alpha = (a, \ldots, a)$. For a simple representation all arrow matrices must be invertible but then, under the action of $GL(\alpha)$, they can be diagonalized. Hence, the only simple representations (which are not the trivial simples concentrated in a vertex) have dimension vector $(1, \ldots, 1)$.

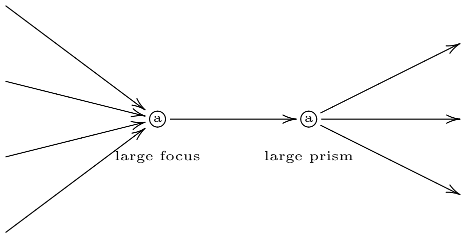
Nevertheless, we will show that these are the only exceptions. A vertex $v_i$ is said to be *large* with respect to a dimension vector $\alpha = (d_1, \ldots, d_k)$ whenever $d_i$ is maximal among the $d_j$. The vertex $v_i$ is said to be *good* if $v_i$ is large and has no direct successor which is a large prism nor a direct predecessor which is a large focus.

**Lemma 4.6** *Let $Q$ be a strongly connected quiver, not of type $\tilde{A}_k$, then one of the following hold*

1. *$Q$ has a good vertex, or,*
2. *$Q$ has a large prism having no direct large prism successors, or*
3. *$Q$ has a large focus having no direct large focus predecessors.*

*Proof.* If neither of the cases hold, we would have an oriented cycle in $Q$ consisting of prisms (or consisting of focusses). Assume $(v_{i_1}, \ldots, v_{i_l})$ is a cycle of prisms, then the unique incoming arrow of $v_{i_j}$ belongs to the cycle. As $Q \neq \tilde{A}_k$ there is at least one extra vertex $v_a$ not belonging to the cycle. But then, there can be no oriented path from $v_a$ to any of the $v_{i_j}$, contradicting the assumption that $Q$ is strongly connected. $\square$

If we are in one of the two last cases, let $a$ be the maximum among the components of the dimension vector $\alpha$ and assume that $\alpha$ satisfies $\chi_Q(\alpha, \epsilon_i) \leq 0$ and $\chi_Q(\epsilon_i, \alpha) \leq 0$ for all $1 \leq i \leq k$, then we have the following subquiver in $Q$

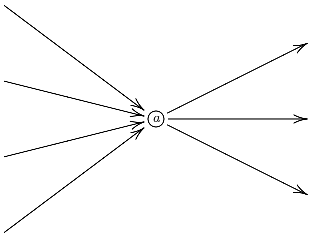



We can reduce to a quiver situation with strictly less vertices.

**Lemma 4.7** *Assume $Q$ is strongly connected and we have a vertex $v_i$ which is a prism with unique predecessor the vertex $v_j$, which is a focus. Consider the dimension vector $\alpha = (d_1, \ldots, d_k)$ with $d_i = d_j = a \neq 0$. Then, $\alpha$ is the dimension of a simple representation of $Q$ if and only if*

$$\alpha' = (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_k) \in \mathbb{N}^{k-1}$$

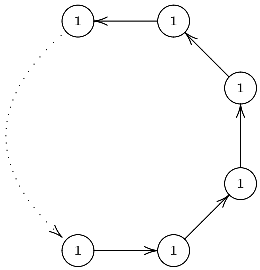
*is the dimension vector of a simple representation of the quiver $Q'$ on $k-1$ vertices, obtained from $Q$ by identifying the vertices $v_i$ and $v_j$, that is, the above subquiver in $Q$ is simplified to the one below in $Q'$*




*Proof.* If $b$ is the unique arrow from $v_j$ to $v_i$ and if $V \in \mathtt{rep}_\alpha\ Q$ is a simple representation then $V_b$ is an isomorphism, so we can identify $V_i$ with $V_j$ and obtain a simple representation of $Q'$. Conversely, if $V' \in \mathtt{rep}_{\alpha'}\ Q'$ is a simple representation, define $V \in \mathtt{rep}_\alpha\ Q$ by $V_i = V'_j$ and $V_z = V'_z$ for $z \neq i$, $V_{b'} = V'_{b'}$ for all arrows $b' \neq b$ and $V_b = \mathbb{1}_a$. Clearly, $V$ is a simple representation of $Q$. □

**Theorem 4.10** *$\alpha = (d_1, \ldots, d_k)$ is the dimension vector of a simple representation of $Q$ if and only if one of the following two cases holds*

1. *$supp(\alpha) = \tilde{A}_k$, the extended Dynkin quiver on $k$ vertices with cyclic orientation and $d_i = 1$ for all $1 \leq i \leq k$*




2. *$supp(\alpha) \neq \tilde{A}_k$. Then, $supp(\alpha)$ is strongly connected and for all $1 \leq i \leq k$ we have*

$$\begin{cases} \chi_Q(\alpha, \epsilon_i) & \leq 0 \\ \chi_Q(\epsilon_i, \alpha) & \leq 0 \end{cases}$$

*Proof.* We will use induction, both on the number of vertices $k$ in $supp(\alpha)$ and on the total dimension $n = \sum_i d_i$ of the representation. If $supp(\alpha)$ does not possess a good vertex, then the above lemma finishes the proof by induction on $k$. Observe that the Euler-form conditions are preserved in passing from $Q$ to $Q'$ as $d_i = d_j$.

Hence, assume $v_i$ is a good vertex in $supp(\alpha)$. If $d_i = 1$ then all $d_j = 1$ for $v_j \in supp(\alpha)$ and we can construct a simple representation by taking $V_b = 1$ for all arrows $b$ in $supp(\alpha)$. Simplicity follows from the fact that $supp(\alpha)$ is strongly connected.

If $d_i > 1$, consider the dimension vector $\alpha' = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_k)$. Clearly, $supp(\alpha') = supp(\alpha)$ is strongly connected and we claim that the Euler-form conditions still hold for $\alpha'$. the only vertices $v_l$ where things might go wrong are direct predecessors or direct successors of $v_i$. Assume for one of them $\chi_Q(\epsilon_l, \alpha) > 0$ holds, then

$$d_l = d'_l > \sum_{m \xleftarrow{a} l} d'_m \geq d'_i = d_i - 1$$

But then, $d_l = d_i$ whence $v_l$ is a large vertex of $\alpha$ and has to be also a focus with end vertex $v_i$ (if not, $d_l > d_i$), contradicting goodness of $v_i$.

Hence, by induction on $n$ we may assume that there is a simple representation $W \in \mathtt{rep}_{\alpha'}\ Q$. Consider the space $\mathtt{rep}_W$ of representations $V \in \mathtt{rep}_\alpha\ Q$ such that $V \mid \alpha' = W$. That is, for every arrow

$$j \xleftarrow{a} i \qquad V_a = \left[\begin{array}{ccc} & W_a & \\ \hline v_1 & \ldots & v_{d_j} \end{array}\right]$$

$$i \xleftarrow{a} j \qquad V_a = \left[\begin{array}{c|c} & v_1 \\ W_a & \vdots \\ & v_{d_j} \end{array}\right]$$

Hence, $\mathtt{rep}_W$ is an affine space consisting of all representations degenerating to $W \oplus S_i$ where $S_i$ is the simple one-dimensional representation concentrated in $v_i$. As $\chi_Q(\alpha', \epsilon_i) < 0$ and $\chi_Q(\epsilon_i, \alpha') < 0$ we have that $Ext^1(W, S_i) \neq 0 \neq Ext^1(S_i, W)$ so there is an open subset of representations which are not isomorphic to $W \oplus S_i$.

As there are simple representations of $Q$ having a one-dimensional component at each vertex in $supp(\alpha)$ and as the subset of simple representations in $\mathtt{rep}_{\alpha'}\ Q$ is open, we can choose $W$ such that $\mathtt{rep}_W$ contains representations $V$ such that a trace of an oriented cycle differs from that of $W \oplus S_i$. Hence, by the description of the invariant ring $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)}$ as being generated by traces along oriented cycles and by the identification of points in the quotient variety as isomorphism classes of semi-simple representations, it follows that the Jordan-Hölder factors of $V$ are different from $W$ and $S_i$. In view of the definition of $\mathtt{rep}_W$, this can only happen if $V$ is a simple representation, finishing the proof of the theorem. $\square$

Still, the central simple algebras constructed from quivers are very special examples as we will see in section 4.6.

## 4.5 Indecomposable roots

Throughout, $Q$ will be a quiver on $k$ vertices $\{v_1, \ldots, v_k\}$ with Euler form $\chi_Q$. For a dimension vector $\alpha = (d_1, \ldots, d_k)$, any $V \in \mathtt{rep}_\alpha\ Q$ decomposes uniquely into

$$V = W_1^{\oplus f_1} \oplus \ldots \oplus W_z^{\oplus f_z}$$

where the $W_i$ are *indecomposable* representations . This follows from the fact that $End(V)$ is finite dimensional. Recall also that a representation $W$ of $Q$ is indecomposable if and only if $End(W)$ is a *local algebra* , that is, the nilpotent endomorphisms in $End_{\mathbb{C}Q}(W)$ form an ideal of codimension one. Equivalently, the maximal torus of the stabilizer subgroup $Stab_{GL(\alpha)}(W) = Aut_{\mathbb{C}Q}(W)$ is one-dimensional, which means that every semisimple element of $Aut_{\mathbb{C}Q}(W)$ lies in $\mathbb{C}^*(\mathbb{1}_{d_1}, \ldots, \mathbb{1}_{d_k})$. More generally, decomposing a representation $V$ into indecomposables corresponds to choosing a maximal torus in the stabilizer subgroup $Aut_{\mathbb{C}Q}(V)$. Let $T$ be such a maximal torus, we define a decomposition of the vertexspaces

$$V_i = \oplus_\chi V_i(\chi) \qquad \text{where} \qquad V_i(\chi) = \{v \in V_i \mid t.v = \chi(t)v\ \forall t \in T\}$$

where $\chi$ runs over all characters of $T$. One verifies that each $V(\chi) = \oplus_i V_i(\chi)$ is a subrepresentation of $V$ giving a decomposition $V = \oplus_\chi V(\chi)$. Because $T$ acts by scalar multiplication on each component $V(\chi)$, we have that $\mathbb{C}^*$ is the maximal torus of $Aut_{\mathbb{C}Q}(V(\chi))$, whence $V(\chi)$ is indecomposable. Conversely, if $V = W_1 \oplus \ldots \oplus W_r$ is a decomposition with the $W_i$ indecomposable, then the product of all the one-dimensional maximal tori in $Aut_{\mathbb{C}Q}(W_i)$ is a maximal torus of $Aut_{\mathbb{C}Q}(V)$.

In this section we will give a classification of the *indecomposable roots* , that is, the dimension vectors of indecomposable representations. As the name suggests, these dimension vectors will form a *root system* .

The *Tits form* of a quiver $Q$ is the symmetrization of its Euler form, that is,

$$T_Q(\alpha, \beta) = \chi_Q(\alpha, \beta) + \chi_Q(\beta, \alpha)$$

This symmetric bilinear form is described by the *Cartan matrix*

$$C_Q = \begin{bmatrix} c_{11} & \ldots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \ldots & c_{kk} \end{bmatrix} \qquad \text{with} c_{ij} = 2\delta_{ij} - \#\ \{\ \textcircled{j}\rule[0.5ex]{4em}{0.4pt}\textcircled{i}\ \}$$

where we count all arrows connecting $v_i$ with $v_j$ *forgetting the orientation.* The corresponding *quadratic form* $q_Q(\alpha) = \frac{1}{2}\chi_Q(\alpha, \alpha)$ on $\mathbb{Q}^k$ is defined to be

$$q_Q(x_1, \ldots, x_k) = \sum_{i=1}^k x_i^2 - \sum_{a \in Q_a} x_{t(a)} x_{h(a)}$$

Hence, $q_Q(\alpha) = dim\ GL(\alpha) - dim\ \mathtt{rep}_\alpha\ Q$. With $\Gamma_Q$ we denote the underlying *graph* of $Q$, that is, forgetting the orientation of the arrows. The following classification result is classical, see for

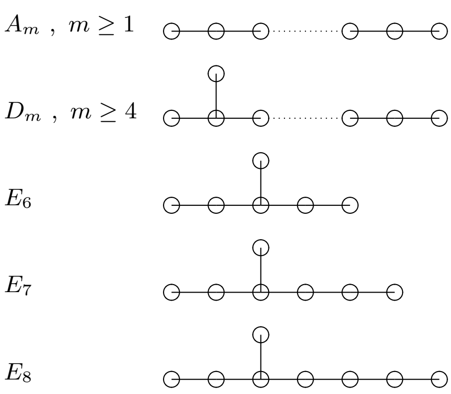



Figure 4.8: The Dynkin diagrams.

example [14]. A quadratic form $q$ on $\mathbb{Z}^k$ is said to be *positive definite* if $0 \neq \alpha \in \mathbb{Z}^k$ implies $q(\alpha) > 0$. It is called positive *semi-definite* if $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^k$. The *radical* of $q$ is $rad(q) = \{\alpha \in \mathbb{Z}^k \ \mid \ T(\alpha, -) = 0\}$. Recall that when $Q$ is a connected and $\alpha \geq \underline{0}$ is a non-zero radical vector, then $\alpha$ is *sincere* (that is, all components of $\alpha$ are non-zero) and $q_Q$ is positive semi-definite. There exist a minimal $\delta_Q \geq \underline{0}$ with the property that $q_Q(\alpha) = 0$ if and only if $\alpha \in \mathbb{Q}\delta_Q$ if and only if $\alpha \in rad(q_Q)$. If the quadratic form $q$ is neither positive definite nor semi-definite, it is called *indefinite*.

**Theorem 4.11** *Let $Q$ be a connected quiver with Tits form $q_Q$, Cartan matrix $C_Q$ and underlying graph $\Gamma_Q$. Then,*

1. *$q_Q$ is positive definite if and only if $\Gamma_Q$ is a* Dynkin diagram , *that is one of the graphs of figure 4.8. The number of vertices is $m$.*

2. *$q_Q$ is semidefinite if and only if $\Gamma_Q$ is an* extended Dynkin diagram*, that is one of the graphs of figure 4.9 and $\delta_Q$ is the indicated dimension vector. The number of vertices is $m+1$.*

Let $V \in \mathtt{rep}_\alpha\ Q$ be decomposed into indecomposables

$$V = W_1^{\oplus f_1} \oplus \ldots \oplus W_z^{\oplus f_z}$$

If $dim(W_i) = \gamma_i$ we say that $V$ is of type $(f_1, \gamma_1; \ldots; f_z, \gamma_z)$.

**Proposition 4.11** *For any dimension vector $\alpha$, there exists a unique type $\tau_{can} = (e_1, \beta_1; \ldots; e_l, \beta_l)$ with $\alpha = \sum_i e_i \beta_i$ such that the set $\mathtt{rep}_\alpha(\tau_{can}) =$*

$$\{V \in \mathtt{rep}_\alpha\ Q \ \mid \ V \simeq W_1^{\oplus e_1} \oplus \ldots \oplus W_l^{\oplus e_l},\ dim(W_i) = \beta_i,\ W_i \textit{ is indecomposable } \}$$

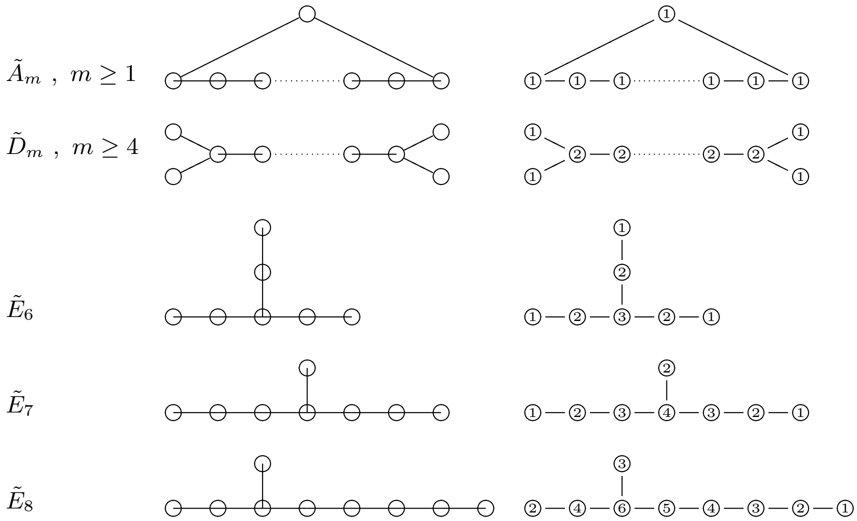



Figure 4.9: The extended Dynkin diagrams.

*contains a dense open set of* $\mathtt{rep}_\alpha\ Q$.

*Proof.* Recall from example 2.4 that for any dimension vector $\beta$ the subset $\mathtt{rep}^{ind}_\beta\ Q$ of *indecomposable* representations of dimension $\beta$ is constructible. Consider for a type $\tau = (f_1, \gamma_1, ; \ldots ; f_z, \gamma_z)$ the subset $\mathtt{rep}_\alpha(\tau) =$

$$\{V \in \mathtt{rep}_\alpha\ Q \mid V \simeq W_1^{\oplus f_1} \oplus \ldots \oplus W_z^{\oplus f_z},\ dim(W_i) = \gamma_i, W_i \text{ indecomposable } \}$$

then $\mathtt{rep}_\alpha(\tau)$ is a constructible subset of $\mathtt{rep}_\alpha\ Q$ as it is the image of the constructible set

$$GL(\alpha) \times \mathtt{rep}^{ind}_{\gamma_1}\ Q \times \ldots \times \mathtt{rep}^{ind}_{\gamma_z}\ Q$$

under the map sending $(g, W_1, \ldots, W_z)$ to $g.(W_1^{\oplus f_1} \oplus \ldots \oplus W_z^{\oplus f_z})$. Because of the uniqueness of the decomposition into indecomposables we have a finite disjoint decomposition

$$\mathtt{rep}_\alpha\ Q = \bigsqcup_\tau \mathtt{rep}_\alpha(\tau)$$

and by irreducibility of $\mathtt{rep}_\alpha\ Q$ precisely one of the $\mathtt{rep}_\alpha(\tau)$ contains a dense open set of $\mathtt{rep}_\alpha\ Q$. $\square$

We call $\tau_{can}$ the *canonical decomposition* of $\alpha$. In the next section we will give an algorithm to compute the canonical decomposition. Consider the action morphisms $GL(\alpha) \times \mathtt{rep}_\alpha\ Q \xrightarrow{\phi} \mathtt{rep}_\alpha\ Q$. By Chevalley's theorem 2.1 we know that the function

$$V \mapsto dim\ Stab_{GL(\alpha)}(V)$$

is upper semi-continuous. Because $dim\ GL(\alpha) = dim\ Stab_{GL(\alpha)}(V) + dim\ \mathcal{O}(V)$ we conclude that for all $m$, the subset

$$\mathtt{rep}_\alpha(m) = \{V \in \mathtt{rep}_\alpha\ Q \mid dim\ \mathcal{O}(V) \geq m\}$$

is Zariski open. In particular, $\mathtt{rep}_\alpha(max)$ the union of all orbits of maximal dimension is open and dense in $\mathtt{rep}_\alpha\ Q$. A representation $V \in \mathtt{rep}_\alpha\ Q$ lying in the intersection

$$\mathtt{rep}_\alpha(\tau_{can}) \cap \mathtt{rep}_\alpha(max)$$

is called a *generic representation* of dimension $\alpha$.

Assume that $Q$ is a connected quiver of *finite representation type* , that is, there are only a finite number of isomorphism classes of indecomposable representations. Let $\alpha$ be an arbitrary dimension vector. Since any representation of $Q$ can be decomposed into a direct sum of indecomposables, $\mathtt{rep}_\alpha\ Q$ contains only finitely many orbits. Hence, one orbit $\mathcal{O}(V)$ must be dense and have th same dimension as $\mathtt{rep}_\alpha\ Q$, but then

$$dim\ \mathtt{rep}_\alpha\ Q = dim\ \mathcal{O}(V) \leq dim\ GL(\alpha) - 1$$

as any representation has $\mathbb{C}^*(\mathbb{1}_{a_1}, \ldots, \mathbb{1}_{a_k})$ in its stabilizer subgroup. That is, for every $\alpha \in \mathbb{N}^k$ we have $q_Q(\alpha) \geq 1$. Because all off-diagonal entries of the Cartan matrix $C_Q$ are non-positive, it follows that $q_Q$ is positive definite on $\mathbb{Z}^k$ whence $\Gamma_Q$ must be a Dynkin diagram. It is well known that to a Dynkin diagram one associates a simple Lie algebra and a corresponding *root system* . We will generalize the notion of a root system to an arbitrary quiver $Q$.

Let $\epsilon_i = (\delta_{1i}, \ldots, \delta_{ki})$ be the standard basis of $\mathbb{Q}^k$. The *fundamental set of roots* is defined to be the following set of dimension vectors

$$F_Q = \{\alpha \in \mathbb{N}^k - \underline{0} \mid T_Q(\alpha, \epsilon_i) \leq 0 \text{ and } supp(\alpha) \text{ is connected } \}$$

Recall that it follows from the description of dimension vectors of simple representations given in section 4.4 that any simple root lies in the fundamental set.

**Lemma 4.8** *Let $\alpha = \beta_1 + \ldots + \beta_s \in F_Q$ with $\beta_i \in \mathbb{N}^k - \underline{0}$ for $1 \leq i \leq s \geq 2$. If $q_Q(\alpha) \geq q_Q(\beta_1) + \ldots + q_Q(\beta_s)$, then $supp(\alpha)$ is a* tame quiver *(that is, its underlying graph is an extended Dynkin diagram) and $\alpha \in \mathbb{N}\delta_{supp(\alpha)}$.*

*Proof.* Let $s = 2$, $\beta_1 = (c_1, \ldots, c_k)$ and $\beta_2 = (d_1, \ldots, d_k)$ and we may assume that $supp(\alpha) = Q$. By assumption $T_Q(\beta_1, \beta_2) = q_Q(\alpha) - q_Q(\beta_1) - q_Q(\beta_2) \geq 0$. Using that $C_Q$ is symmetric and $\alpha = \beta_1 + \beta_2$ we have

$$0 \leq T_Q(\beta_1, \beta_2) = \sum_{i,j} c_{ij} c_i d_i$$

$$= \sum_j \frac{c_j d_j}{a_j} \sum_i c_{ij} a_i + \frac{1}{2} \sum_{i \neq j} c_{ij} (\frac{c_i}{a_i} - \frac{c_j}{a_j})^2 a_i a_j$$

and because $T_Q(\alpha, \epsilon_i) \leq 0$ and $c_{ij} \leq 0$ for all $i \neq j$, we deduce that

$$\frac{c_i}{a_i} = \frac{c_j}{a_j} \qquad \text{for all } i \neq j \text{ such that } c_{ij} \neq 0$$

Because $Q$ is connected, $\alpha$ and $\beta_1$ are proportional. But then, $T_Q(\alpha, \epsilon_i) = 0$ and hence $C_Q \alpha = \underline{0}$. By the classification result, $q_Q$ is semidefinite whence $\Gamma_Q$ is an extended Dynkin diagram and $\alpha \in \mathbb{N}\delta_Q$. Finally, if $s > 2$, then

$$T_Q(\alpha, \alpha) = \sum_i T_Q(\alpha, \beta_i) \geq \sum_i T_Q(\beta_i, \beta_i)$$

whence $T_Q(\alpha - \beta_i, \beta_i) \geq 0$ for some $i$ and then we can apply the foregoing argument to $\beta_i$ and $\alpha - \beta_i$. □

**Definition 4.8** *If $G$ is an algebraic group acting on a variety $Y$ and if $X \hookrightarrow Y$ is a $G$-stable subset, then we can decompose $X = \bigcup_d X_{(d)}$ where $X_{(d)}$ is the union of all orbits $\mathcal{O}(x)$ of dimension $d$. The* number of parameters *of $X$ is*

$$\mu(X) = \max_d \; (dim \; X_{(d)} - d)$$

*where dim $X_{(d)}$ denotes the dimension of the Zariski closure of $X_{(d)}$.*

*In the special case of $GL(\alpha)$ acting on $\mathtt{rep}_\alpha \; Q$, we denote $\mu(\mathtt{rep}_\alpha(max)) = p_Q(\alpha)$ and call it the* number of parameters *of $\alpha$. For example, if $\alpha$ is a Schur root, then $p(\alpha) = dim \; \mathtt{rep}_\alpha \; Q - (dim \; GL(\alpha) - 1) = 1 - q_Q(\alpha)$.*

Recall that a matrix $m \in M_n(\mathbb{C})$ is *unipotent* if some power $m^k = \mathbb{1}_n$. It follows from the Jordan normal form that $GL(\alpha)$ and $PGL(\alpha) = GL(\alpha)/C^*$ contain only finitely many conjugacy classes of unipotent matrices.

**Theorem 4.12** *If $\alpha$ lies in the fundamental set and $supp(\alpha)$ is not tame, then*

$$p_Q(\alpha) = \mu(\mathtt{rep}_\alpha(max)) = \mu(\mathtt{rep}^{ind}_\alpha \; Q) = 1 - q_Q(\alpha) > \mu(\mathtt{rep}^{ind}_\alpha(d))$$

*for all $d > 1$ where $\mathtt{rep}^{ind}_\alpha(d)$ is the union of all indecomposable orbits of dimension $d$.*

*Proof.* A representation $V \in \mathtt{rep}_\alpha\ Q$ is indecomposable if and only if its stabilizer subgroup $Stab_{GL(\alpha)}(V)$ is a *unipotent group* , that is all its elements are unipotent elements. By proposition 4.14 we know that $\mathtt{rep}_\alpha(max) \hookrightarrow \mathtt{rep}^{ind}_\alpha\ Q$ and that $p_Q(\alpha) = \mu(\mathtt{rep}_\alpha(max)) = 1 - q_Q(\alpha)$. Denote $\mathtt{rep}_\alpha(sub) = \mathtt{rep}_\alpha\ Q - \mathtt{rep}_\alpha(max)$. We claim that for any unipotent element $u \neq \mathbb{1}$ we have that

$$dim\ \mathtt{rep}_\alpha(sub)(u) - dim\ cen_{GL(\alpha)}(u) + 1 < 1 - q_Q(\alpha)$$

where $\mathtt{rep}_\alpha(sub)(g)$ denotes the representations in $\mathtt{rep}_\alpha(sub)$ having $g$ in their stabilizer subgroup. In fact, for any $g \in GL(\alpha) - \mathbb{C}^*$ we have

$$dim\ cen_{GL(\alpha)}(g) - dim\ \mathtt{rep}_\alpha(g) > q_Q(\alpha)$$

Indeed, we may reduce to $g$ being a semisimple element, see [49, lemma 3.4]. then, if $\alpha = \alpha_1 + \ldots + \alpha_s$ is the decomposition of $\alpha$ obtained from the eigenspace decompositions of $g$ (we have $s \geq 2$ as $g \notin \mathbb{C}^*$), then

$$cen_{GL(\alpha)}(g) = \prod_i GL(\alpha_i) \quad \text{and} \quad \mathtt{rep}_\alpha(g) = \prod_i \mathtt{rep}_{\alpha_i}(g)$$

whence $dim\ cen_{GL(\alpha)}(g) - dim\ \mathtt{rep}_\alpha(g) = \sum_i q_Q(\alpha_i) > q_Q(\alpha)$, proving the claim. Further, we claim that

$$\mu(\mathtt{rep}_\alpha(sub)) \leq \max_u\ (dim\ \mathtt{rep}_\alpha(sub)(u) - dim\ cen_{GL(\alpha)}(u) + 1)$$

Let $Z = \mathtt{rep}_\alpha(sub)$ and consider the closed subvariety of $PGL(\alpha) \times Z$

$$L = \{(g,z)\ \mid\ g.z = z\}$$

For $z \in Z$ we have $pr_1^{-1}(z) = Stab_{PGL(\alpha)}(z) \times \{z\}$ and if $z$ is indecomposable with orbit dimension $d$ then $dim\ Stab_{PGL(\alpha)}(z) = dim\ PGL(\alpha) - d$, whence

$$dim\ pr_1^{-1}(\mathtt{rep}^{ind}_\alpha)_{(d)} = dim\ (\mathtt{rep}^{ind}_\alpha)_{(d)} + dim\ PGL(\alpha) - d$$

But then,

$$\begin{aligned} p_Q(\alpha) &= \max_d (dim\ (\mathtt{rep}^{ind}_\alpha)_{(d)} - d) \\ &= -dim\ PGL(\alpha) + \max_d\ dim\ pr_1^{-1}((\mathtt{rep}^{ind}_\alpha)_{(d)}) \\ &= -dim\ PGL(\alpha) + dim\ pr_1^{-1}(\mathtt{rep}^{ind}_\alpha\ Q) \end{aligned}$$

By the characterization of indecomposables, we have $pr_1^{-1}(\mathtt{rep}^{ind}_\alpha\ Q) \subset pr_2^{-1}(U)$ where $U$ consists of the (finitely many) conjugacy classes $C_u$ of conjugacy classes of unipotent $u \in PGL(\alpha)$. But then,

$$\begin{aligned} p_Q(\alpha) &\leq -dim\ PGL(\alpha) + \max_u\ dim\ pr_2^{-1}(C_u) \\ &= -dim\ PGL(\alpha) + \max_u dim\ \mathtt{rep}_\alpha(sub)(u) + dim\ PGL(\alpha) - dim\ cen_{PGL(\alpha)}(u) \end{aligned}$$

proving the claim. Finally, as $dim\ \mathtt{rep}_\alpha(sub) - dim\ PGL(\alpha) < dim\ \mathtt{rep}_\alpha\ Q - dim\ GL(\alpha) + 1 < 1 - q_Q(\alpha)$, we are done. $\square$

We will now extend this result to arbitrary *roots* using *reflection functors* . Let $v_i$ be a *source vertex* of $Q$ and let $\alpha = (a_1, \ldots, a_k)$ be a dimension vector such that $\sum_{t(a)=v_i} a_{h(a)} \geq a_i$, then we can consider the subset

$$\mathtt{rep}_\alpha^{mono}(i) = \{V \in \mathtt{rep}_\alpha\ Q \ \mid\ \oplus V_a : V_i \longrightarrow \oplus_{t(a)=v_i} V_{s(a)} \ \text{ is injective } \}$$

Clearly, all indecomposable representations are contained in $\mathtt{rep}_\alpha^{mono}(i)$. Construct the *reflected quiver* $R_iQ$ obtained from $Q$ by reversing the direction of all arrows with tail $v_i$. The *reflected dimension vector* $R_i\alpha = (r1, \ldots, r_k)$ is defined to be

$$r_j = \begin{cases} a_j & \text{if } j \neq i \\ \sum_{t(a)=i} a_{s(a)} - a_i & \text{if } j = i \end{cases}$$

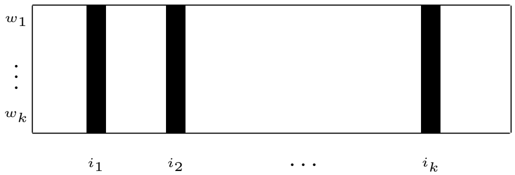
then clearly we have in the reflected quiver $R_iQ$ that $\sum_{h(a)=i} r_{t(a)} \geq r_i$ and we define the subset

$$\mathtt{rep}_{R_i\alpha}^{epi}(i) = \{V \in \mathtt{rep}_{R_i\alpha}\ R_iQ \ \mid\ \oplus V_a : \oplus_{s(a)=i} V_{t(a)} \longrightarrow V_i \text{ is surjective } \}$$

Before stating the main result on reflection functors, we need to recall the definition of the Grassmann manifolds.

Let $k \leq l$ be integers, then the points of the Grassmannian $Grass_k(l)$ are in one-to-one correspondence with $k$-dimensional subspaces of $\mathbb{C}^l$. For example, if $k = 1$ then $Grass_1(l) = \mathbb{P}^{l-1}$. We know that projective space can be covered by affine spaces defining a manifold structure on it. Also Grassmannians admit a cover by affine spaces.

Let $W$ be a $k$-dimensional subspace of $\mathbb{C}^l$ then fixing a basis $\{w_1, \ldots, w_k\}$ of $W$ determines an $k \times l$ matrix $M$ having as $i$-th row the coordinates of $w_i$ with respect to the standard basis of $\mathbb{C}^l$. Linear independence of the vectors $w_i$ means that there is a barcode design $I$ on $M$




where $I = 1 \leq i_1 < i_2 < \ldots < i_k \leq l$ such that the corresponding $k \times k$ minor $M_I$ of $M$ is invertible. Observe that $M$ can have several such designs.

Conversely, given a $k \times l$ matrix $M$ of rank $k$ determines a $k$-dimensional subspace of $l$ spanned by the transposed rows. Two $k \times l$ $M$ and $M'$ matrices of rank $k$ determine the same subspace provided there is a basechange matrix $g \in GL_k$ such that $gM = M'$. That is, we can identify $Grass_k(l)$ with the orbit space of the linear action of $GL_k$ by left multiplication on the open set $M^{max}_{k\times l}(\mathbb{C})$ of $M_{k\times l}(\mathbb{C})$ of matrices of maximal rank. Let $I$ be a barcode design and consider the subset of $Grass_k(l)(I)$ of subspaces having a matrix representation $M$ having $I$ as barcode design. Multiplying on the left with $M_I^{-1}$ the $GL_k$-orbit $\mathcal{O}_M$ has a unique representant $N$ with $N_I = \mathbb{1}_k$. Conversely, any matrix $N$ with $N_I = \mathbb{1}_k$ determines a point in $Grass_k(l)(I)$. Thus, $Grass_k(l)(I)$ depends on $k(l-k)$ free parameters (the entries of the negative of the barcode)


and we have an identification $Grass_k(l)(I) \xrightarrow{\pi_I} \mathbb{C}^{k(l-k)}$. For a different barcode design $I'$ the image $\pi_I(Grass_k(l)(I) \cap Grass_k(l)(I'))$ is an open subset of $\mathbb{C}^{k(l-k)}$ (one extra nonsingular minor condition) and $\pi_{I'} \circ \pi_I^{-1}$ is a diffeomorphism on this set. That is, the maps $\pi_I$ provide us with an atlas and determine a manifold structure on $Grass_k(l)$.

**Theorem 4.13** *For the quotient Zariski topology, we have an homeomorphism*

$$\mathtt{rep}^{mono}_\alpha(i)/GL(\alpha) \xrightarrow{\simeq} \mathtt{rep}^{epi}_{R_i\alpha}(i)/GL(R_i\alpha)$$

*such that corresponding representations have isomorphic endomorphism rings.*

*In particular, the number of parameters as well as the number of irreducible components of maximal dimension coincide for* $(\mathtt{rep}^{ind}_\alpha\ Q)_{(d)}$ *and* $\mathtt{rep}^{ind}_{R_i\alpha}\ R_iQ)_{(d)}$ *for all dimensions* $d$*.*

*Proof.* Let $m = \sum_{t(a)=i} a_i$, $\overline{rep} = \oplus_{t(a)\neq i} M_{a_{s(a)}\times a_{t(a)}}(\mathbb{C})$ and $\overline{GL} = \prod_{j\neq i} GL_{a_j}$. We have the following isomorphisms

$$\mathtt{rep}^{mono}_\alpha(i)/GL_{a_i} \xrightarrow{\simeq} \overline{rep} \times Gass_{a_i}(m)$$

defined by sending a representation $V$ to its restriction to $\overline{rep}$ and $im\ \oplus_{t(a)=i} V_a$. In a similar way, sending a representation $V$ to its restriction and $ker\ \oplus_{s(a)=i} V_a$ we have

$$\mathtt{rep}^{epi}_{R_i\alpha}(i)/GL_{r_i} \xrightarrow{\simeq} \overline{rep} \times Grass_{a_i}(m)$$

But then, the first claim follows from the diagram of figure 4.10. If $V \in \mathtt{rep}_\alpha\ Q$ and $V' \in \mathtt{rep}_{R_i\alpha}\ R_iQ$ with images respectively $v$ and $v'$ in $\overline{rep} \times Grass_{a_i}(m)$, we have isomorphisms

$$\begin{cases} Stab_{\overline{GL}\times GL_{a_i}}(V) & \xrightarrow{\simeq} Stab_{\overline{GL}}(v) \\ Stab_{\overline{GL}\times GL_{r_i}}(V') & \xrightarrow{\simeq} Stab_{\overline{GL}}(v') \end{cases}$$

$$\begin{tikzcd}
\mathtt{rep}_\alpha^{mono}(i) \arrow[d] & & \mathtt{rep}_{R_i\alpha}^{epi}(i) \arrow[d] \\
\mathtt{rep}_\alpha^{mono}(i)/GL(\alpha) \arrow[dr, "\simeq"] \arrow[dd, "/\overline{GL}"'] & & \mathtt{rep}_{R_i\alpha}^{epi}(i)/GL(R_i\alpha) \arrow[dl, "\simeq"'] \arrow[dd, "/\overline{GL}"'] \\
& \overline{rep} \times Grass_{a_i}(m) \arrow[dl] \arrow[dr] & \\
\mathtt{rep}_\alpha^{mono}(i)/GL(\alpha) \arrow[rr, dotted] & & \mathtt{rep}_{R_i\alpha}^{epi}(i)/GL(R_i\alpha)
\end{tikzcd}$$

Figure 4.10: Reflection functor diagram.

from which the claim about endomorphisms follows. □

A similar results holds for *sink vertices*, hence we can apply these *Bernstein-Gelfand- Ponomarev reflection functors* iteratively using a sequence of *admissible vertices* (that is, either a source or a sink).

To a vertex $v_i$ in which $Q$ has no loop, we define a *reflection* $\mathbb{Z}^k \xrightarrow{r_i} \mathbb{Z}^k$ by

$$r_i(\alpha) = \alpha - T_Q(\alpha, \epsilon_i)$$

The *Weyl group of the quiver* $Q$ $Weyl_Q$ is the subgroup of $GL_k(\mathbb{Z})$ generated by all reflections $r_i$.

A *root* of the quiver $Q$ is a dimension vector $\alpha \in \mathbb{N}^k$ such that $\mathtt{rep}_\alpha\ Q$ contains indecomposable representations. All roots have connected support. A root is said to be

$$\begin{cases} \textit{real} & \text{if } \mu(\mathtt{rep}_\alpha^{ind}\ Q) = 0 \\ \textit{imaginary} & \text{if } \mu(\mathtt{rep}_\alpha^{ind}\ Q) \geq 1 \end{cases}$$

For a fixed quiver $Q$ we will denote the set of all roots, real roots and imaginary roots respectively by $\Delta, \Delta_{re}$ and $\Delta_{im}$. With $\Pi$ we denote the set $\{\epsilon_i \mid v_i \text{ has no loops } \}$. the main result on indecomposable representations is due to V. Kac .

**Theorem 4.14** *With notations as before, we have*

1. $\Delta_{re} = Weyl_Q.\Pi \cap \mathbb{N}^k$ *and if* $\alpha \in \Delta_{re}$*, then* $\mathtt{rep}_\alpha^{ind}\ Q$ *is one orbit.*

2. $\Delta_{im} = Weyl.F_Q \cap \mathbb{N}^k$ *and if* $\alpha \in \Delta_{im}$ *then*

$$p_Q(\alpha) = \mu(\mathtt{rep}_\alpha^{ind}\ Q) = 1 - q_Q(\alpha)$$

For a sketch of the proof we refer to [28, §7], full details can be found in the lecture notes [49].

## 4.6 Canonical decomposition

In this section we will determine the canonical decomposition. We need a technical result.

**Lemma 4.9** *Let* $W \twoheadrightarrow W'$ *be an epimorphism of* $\mathbb{C}Q$*-representations. Then, for any* $\mathbb{C}Q$*-representation* $V$ *we have that the canonical map*

$$Ext^1_{\mathbb{C}Q}(V, W) \twoheadrightarrow Ext^1_{\mathbb{C}Q}(V, W')$$

*is surjective. If* $W \hookrightarrow W'$ *is a monomorphism of* $\mathbb{C}Q$*-representations, then the canonical map*

$$Ext^1_{\mathbb{C}Q}(W', V) \twoheadrightarrow Ext^1_{\mathbb{C}Q}(W, V)$$

*is surjective.*

*Proof.* From the proof of theorem 4.5 we have the exact diagram

$$\begin{tikzcd}
\bigoplus_{v_i \in Q_v} Hom_{\mathbb{C}}(V_i, W_i) \arrow[r, "d_W^V"] \arrow[d] & \bigoplus_{a \in Q_a} Hom_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) \arrow[r] \arrow[d] & Ext^1_{\mathbb{C}Q}(V, W) \arrow[r] \arrow[d, dotted] & 0 \\
\bigoplus_{v_i \in Q_v} Hom_{\mathbb{C}}(V_i, W'_i) \arrow[r, "d_{W'}^V"] & \bigoplus_{a \in Q_a} Hom_{\mathbb{C}}(V_{s(a)}, W'_{t(a)}) \arrow[r] & Ext^1_{\mathbb{C}Q}(V, W') \arrow[r] & 0
\end{tikzcd}$$

and applying the *snake lemma* gives the result. The second part is proved similarly. □

**Lemma 4.10** *If* $V = V' \oplus V" \in \mathtt{rep}_\alpha(max)$*, then* $Ext^1_{\mathbb{C}Q}(V', V") = 0$.

*Proof.* Assume $Ext^1(V', V") \neq 0$, that is, there is a non-split exact sequence

$$0 \longrightarrow V" \longrightarrow W \longrightarrow V' \longrightarrow 0$$

then it follows from section 2.3 that $\mathcal{O}(V) \subset \overline{\mathcal{O}(W)} - \mathcal{O}(W)$, whence $dim\ \mathcal{O}(W) > dim\ \mathcal{O}(V)$ contradicting the assumption that $V \in \mathtt{rep}_\alpha(max)$. □

**Lemma 4.11** *If $W, W'$ are indecomposable representation with $Ext^1_{\mathbb{C}Q}(W, W') = 0$, then any non-zero map $W' \xrightarrow{\phi} W$ is an epimorphism or a monomorphism. In particular, if $W$ is indecomposable with $Ext^1_{\mathbb{C}Q}(W, W) = 0$, then $End_{\mathbb{C}Q}(W) \simeq \mathbb{C}$.*

*Proof.* Assume $\phi$ is neither mono- nor epimorphism then decompose $\phi$ into

$$W' \overset{\epsilon}{\twoheadrightarrow} U \overset{\mu}{\hookrightarrow} W$$

As $\epsilon$ is epi, we get a surjection from lemma 4.9

$$Ext^1_{\mathbb{C}Q}(W/U, W') \twoheadrightarrow Ext^1_{\mathbb{C}Q}(W/U, U)$$

giving a representation $V$ fitting into the exact diagram of extensions

$$\begin{tikzcd}
0 \arrow[r] & W' \arrow[r, "\mu'"] \arrow[d, "\epsilon"'] & V \arrow[r] \arrow[d, "\epsilon'"'] & W'/U \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & U \arrow[r, "\mu"] & W \arrow[r] & W'/U \arrow[r] & 0
\end{tikzcd}$$

from which we construct an exact sequence of representations

$$0 \longrightarrow W' \xrightarrow{\begin{bmatrix} \epsilon \\ -\mu' \end{bmatrix}} U \oplus V \xrightarrow{\begin{bmatrix} \mu & \epsilon' \end{bmatrix}} W \longrightarrow 0$$

This sequence cannot split as otherwise we would have $W \oplus W' \simeq U \oplus V$ contradicting uniqueness of decompositions, whence $Ext^1_{\mathbb{C}Q}(W, W') \neq 0$, a contradiction.

For the second part, as $W$ is finite dimensional it follows that $End_{\mathbb{C}Q}(W)$ is a (finite dimensional) division algebra whence it must be $\mathbb{C}$. □

**Definition 4.9** *A representation $V \in \mathtt{rep}_\alpha\, Q$ is said to be a* Schur representation *if $End_{\mathbb{C}Q}(V) = \mathbb{C}$. The dimension vector $\alpha$ of a Schur representation is said to be a* Schur root *.*

**Theorem 4.15** *$\alpha$ is a Schur root if and only if there is a Zariski open subset of $\mathtt{rep}_\alpha\, Q$ consisting of indecomposable representations.*

*Proof.* If $V \in \mathtt{rep}_\alpha\, Q$ is a Schur representation, $V \in \mathtt{rep}_\alpha(max)$ and therefore all representations in the dense open subset $\mathtt{rep}_\alpha(max)$ have endomorphism ring $\mathbb{C}$ and are therefore indecomposable. Conversely, let $Ind \hookrightarrow \mathtt{rep}_\alpha\, Q$ be an open subset of indecomposable representations and assume that for $V \in Ind$ we have $Stab_{GL(\alpha)}(V) \neq \mathbb{C}^*$ and consider $\phi_0 \in Stab_{GL(\alpha)}(V) - \mathbb{C}^*$. For any $g \in GL(\alpha)$ we define the *set of fixed elements*

$$\mathtt{rep}_\alpha(g) = \{W \in \mathtt{rep}_\alpha\ Q \mid g.W = W\}$$

Define the subset of $GL(\alpha)$

$$S = \{g \in GL(\alpha) \mid dim\ \mathtt{rep}_\alpha(g) = dim\ \mathtt{rep}_\alpha(\phi_0)$$

which has no intersection with $\mathbb{C}^*(\mathbb{1}_{d_1}, \ldots, \mathbb{1}_{d_k})$ as $\phi_0 \notin \mathbb{C}^*$. Consider the subbundle of the trivial vectorbundle over $S$

$$\mathcal{B} = \{(s, W) \in S \times \mathtt{rep}_\alpha\ Q \mid s.W = W\} \hookrightarrow S \times \mathtt{rep}_\alpha\ Q \xrightarrow{p} S$$

As all fibers have equal dimension, the restriction of $p$ to $\mathcal{B}$ is a *flat morphism* whence *open* . In particular, the image of the open subset $\mathcal{B} \cap S \times Ind$

$$S' = \{g \in S \mid \exists W \in Ind\ :\ g.W = W\}$$

is an open subset of $S$. Now, $S$ contains a dense set of semisimple elements, see for example [49, (2.5)], whence so does $S' = \cup_{W \in Ind} End_{\mathbb{C}Q}(W) \cap S$. But then one of the $W \in Ind$ must have a torus of rank greater than one in its stabilizer subgroup contradicting indecomposability. □

Schur roots give rise to principal $PGL(\alpha) = GL(\alpha)/C^*$-fibrations, and hence to quiver orders and division algebras.

**Proposition 4.12** *If $\alpha = (a_1, \ldots, a_k)$ is a Schur root, then there is a $GL(\alpha)$-stable affine open subvariety $U_\alpha$ of $\mathtt{rep}_\alpha\ Q$ such that generic orbits are closed in $U$.*

*Proof.* Let $T_k = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ the $k$-dimensional torus in $GL(\alpha)$. Consider the semisimple subgroup $SL(\alpha) = SL_{a_1} \times \ldots \times SL_{a_k}$ and consider the corresponding quotient map

$$\mathtt{rep}_\alpha\ Q \xrightarrow{\pi_s} \mathtt{rep}_\alpha\ Q/SL(\alpha)$$

As $GL(\alpha) = T_k SL(\alpha)$, $T_k$ acts on $\mathtt{rep}_\alpha\ Q/SL(\alpha)$ and the generic stabilizer subgroup is trivial by the Schurian condition. Hence, there is a $T_k$-invariant open subset $U_1$ of $\mathtt{rep}_\alpha\ Q/SL(\alpha)$ such that $T_k$-orbits are closed. But then, according to [41, §2, Thm.5] there is a $T_k$-invariant *affine* open $U_2$ in $U_1$. Because the quotient map $\psi_s$ is an affine map, $U = \psi_s^{-1}(U_2)$ is an affine $GL(\alpha)$-stable open subvariety of $\mathtt{rep}_\alpha\ Q$. Let $x$ be a generic point in $U$, then its orbit

$$\mathcal{O}(x) = GL(\alpha).x = T_k SL(\alpha).x = T_k(\psi_s^{-1}(\psi_s(x))) = \psi_s^{-1}(T_k.\psi_s(x))$$

is the inverse image under the quotient map of a closed set, hence is itself closed. □

If we define $\mathbb{T}^s_\alpha\ Q$ to be the ring of $GL(\alpha)$-equivariant maps from $U_\alpha$ to $M_n(\mathbb{C})$, then this *Schurian quiver order* has *simple* $\alpha$-dimensional representations. Then, extending the argument of proposition 4.9 we have that the quotient map $\mathtt{rep}_\alpha\ Q \longrightarrow\!\!\!\!\rightarrow \mathtt{iss}_\alpha\ Q$ is a principal $PGL(\alpha)$-fibration in the étale topology over the Azumaya locus of the Schurian quiver order $\mathbb{T}^s_\alpha\ Q$. Recall that $H^1_{et}(X, PGL(\alpha))$ classifies twisted forms of $M_n(\mathbb{C})$ (where $n = \sum_a a_i$) as $C_k$-algebra. That is, Azumaya algebras over $X$ with a distinguished embedding of $C_k$ that are split by an étale cover on which this embedding is conjugate to the standard $\alpha$-embedding of $C_k$ in $M_n(\mathbb{C})$. The class in the Brauer group of the functionfield of $\mathtt{iss}_\alpha\ \mathbb{T}^s_\alpha\ Q$ determined by the quiver order $\mathbb{T}^s_\alpha\ Q$ is rather special.

**Proposition 4.13** *If $\alpha = (a_1, \ldots, a_k)$ is a Schur root of $Q$ such that $gcd(a_1, \ldots, a_k) = 1$, then $\mathbb{T}^s_\alpha\ Q$ determines the trivial class in the Brauer group.*

*Proof.* Let $A$ be an Azumaya localization of $\mathbb{T}^s_\alpha\ Q$. By assumption, the natural map between the $K$-groups $K_0(C_k) \longrightarrow K_0(M_n(\mathbb{C}))$ is surjective, whence the same is true for $A$ proving that the class of $A$ is split by a Zariski cover, that is $\mathtt{rep}_\alpha\ Q \simeq X \times PGL(\alpha)$ where $X = \mathtt{iss}_\alpha\ A$. □

**Proposition 4.14** *If $\alpha$ lies in the fundamental region $F_Q$ and supp$(\alpha)$ is not a tame quiver. then, $\alpha$ is a Schur root.*

*Proof.* Let $\alpha = \beta_1 + \ldots + \beta_s$ be the canonical decomposition of $\alpha$ (some $\beta_i$ may occur with higher multiplicity) and assume that $s \geq 2$. By definition, the image of

$$GL(\alpha) \times (\mathtt{rep}_{\beta_1}\ Q \times \ldots \times \mathtt{rep}_{\beta_s}\ Q) \xrightarrow{\phi} \mathtt{rep}_\alpha\ Q$$

is dense and $\phi$ is constant on orbits of the *free action* of $GL(\alpha)$ on the left hand side given by $h.(g,V) = (gh^{-1}, h.V)$. But then,

$$dim\ GL(\alpha) + \sum_i dim\ \mathtt{rep}_{\beta_i}\ Q - \sum_i dim\ GL(\beta_i) \geq dim\ \mathtt{rep}_\alpha\ Q$$

whence $q_Q(\alpha) \geq \sum_i q_Q(\beta_i)$ and lemma 4.8 finishes the proof. □

Next, we want to describe morphisms between quiver-representations. Let $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$ and $V \in \mathtt{rep}_\alpha\ Q$, $W \in \mathtt{rep}_\beta\ Q$. Consider the closed subvariety

$$Hom_Q(\alpha, \beta) \hookrightarrow M_{a_1 \times b_1} \oplus \ldots \oplus M_{a_k \times b_k} \oplus \mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$$

consisting of the triples $(\phi, V, W)$ where $\phi = (\phi_1, \ldots, \phi_k)$ is a morphism of quiver-representations $V \longrightarrow W$. Projecting to the two last components we have an onto morphism between affine varieties

$$Hom_Q(\alpha, \beta) \xrightarrow{h} \mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$$

In theorem 2.1 we have proved that the dimension of fibers is an upper-semicontinuous function. That is, for every natural number $d$, the set

$$\{\Phi \in Hom_Q(\alpha,\beta) \mid dim_\Phi\ h^{-1}(h(\Phi)) \leq d\}$$

is a Zariski open subset of $Hom_Q(\alpha,\beta)$. As the target space $\mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$ is irreducible, it contains a non-empty open subset $hom_{min}$ where the dimension of the fibers attains a minimal value. This minimal fiber dimension will be denoted by $hom(\alpha,\beta)$.

Similarly, we could have defined an affine variety $Ext_Q(\alpha,\beta)$ where the fiber over a point $(V,W) \in \mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$ is given by the extensions $Ext^1_{\mathbb{C}Q}(V,W)$. If $\chi_Q$ is the Euler-form of $Q$ we recall that for all $V \in \mathtt{rep}_\alpha\ Q$ and $W \in \mathtt{rep}_\beta\ Q$ we have

$$dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V,W) - dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V,W) = \chi_Q(\alpha,\beta)$$

Hence, there is also an open set $ext_{min}$ of $\mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$ where the dimension of $Ext^1(V,W)$ attains a minimum. This minimal value we denote by $ext(\alpha,\beta)$. As $hom_{min} \cap ext_{min}$ is a non-empty open subset we have the numerical equality

$$hom(\alpha,\beta) - ext(\alpha,\beta) = \chi_Q(\alpha,\beta).$$

In particular, if $hom(\alpha,\alpha+\beta) > 0$, there will be an open subset where the morphism $V \xrightarrow{\phi} W$ is a monomorphism. Hence, there will be an open subset of $\mathtt{rep}_{\alpha+\beta}\ Q$ consisting of representations containing a subrepresentation of dimension vector $\alpha$. We say that $\alpha$ is a general subrepresentation of $\alpha+\beta$ and denote this with $\alpha \hookrightarrow \alpha+\beta$. We want to characterize this property. To do this, we introduce the quiver-Grassmannians

$$Grass_\alpha(\alpha+\beta) = \prod_{i=1}^{k} Grass_{a_i}(a_i+b_i)$$

which is a projective manifold.

Consider the following diagram of morphisms of reduced varieties

$$\begin{tikzcd}
& \mathtt{rep}_{\alpha+\beta}\ Q \\
\mathtt{rep}_\alpha^{\alpha+\beta}\ Q \arrow[ur, "s"] \arrow[r, hook] \arrow[dr, "p"'] & \mathtt{rep}_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta) \arrow[u, "pr_1"'] \arrow[d, "pr_2"'] \\
& Grass_\alpha(\alpha+\beta)
\end{tikzcd}$$

with the following properties

- $\mathtt{rep}_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta)$ is the trivial vectorbundle with fiber $\mathtt{rep}_{\alpha+\beta}\ Q$ over the projective smooth variety $Grass_\alpha(\alpha+\beta)$ with structural morphism $pr_2$.
- $\mathtt{rep}_\alpha^{\alpha+\beta}\ Q$ is the subvariety of $\mathtt{rep}_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta)$ consisting of couples $(W,V)$ where $V$ is a subrepresentation of $W$ (observe that this is for fixed $W$ a linear condition). Because $GL(\alpha+\beta)$ acts transitively on the Grassmannian $Grass_\alpha(\alpha+\beta)$ (by multiplication on the right) we see that $\mathtt{rep}_\alpha^{\alpha+\beta}\ Q$ is a sub-vectorbundle over $Grass_\alpha(\alpha+\beta)$ with structural morphism $p$. In particular, $\mathtt{rep}_\alpha^{\alpha+\beta}\ Q$ is a reduced variety.
- The morphism $s$ is a projective morphism, that is, can be factored via the natural projection

$$\begin{tikzcd} & \mathtt{rep}_{\alpha+\beta}\ Q \times \mathbb{P}^N \arrow[d, "\pi_2"] \\ \mathtt{rep}_\alpha^{\alpha+\beta}\ Q \arrow[ur, "f"] \arrow[r, "s"] & \mathtt{rep}_{\alpha+\beta}\ Q \end{tikzcd}$$

  where $f$ is the composition of the inclusion $\mathtt{rep}_\alpha^{\alpha+\beta}\ Q \hookrightarrow \mathtt{rep}_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta)$ with the natural inclusion of Grassmannians in projective spaces recalled in the previous section $Grass_\alpha(\alpha+\beta) \hookrightarrow \prod_{i=1}^k \mathbb{P}^{n_i}$ with the Segre embedding $\prod_{i=1}^k \mathbb{P}^{n_i} \hookrightarrow \mathbb{P}^N$. In particular, $s$ is proper by [33, Thm. II.4.9], that is, maps closed subsets to closed subsets.

We are interested in the scheme-theoretic fibers of $s$. If $W \in \mathtt{rep}_{\alpha+\beta}\ Q$ lies in the image of $s$, we denote the fiber $s^{-1}(W)$ by $\mathtt{Grass}_\alpha(W)$. Its geometric points are couples $(W,V)$ where $V$ is an $\alpha$-dimensional subrepresentation of $W$. Whereas $\mathtt{Grass}_\alpha(W)$ is a projective scheme, it is in general neither smooth, nor irreducible nor even reduced. Therefore, in order to compute the tangent space in a point $(W,V)$ of $\mathtt{Grass}_\alpha(W)$ we have to clarify the functor it represents on the category *commalg* of commutative $\mathbb{C}$-algebras.

Let $C$ be a commutative $\mathbb{C}$-algebra, a representation $\mathcal{R}$ of the quiver $Q$ over $C$ consists of a collection $\mathcal{R}_i = P_i$ of projective $C$-modules of finite rank and a collection of $C$-module morphisms for every arrow $a$ in $Q$

$$\textcircled{j} \xleftarrow{a} \textcircled{i} \qquad\qquad \mathcal{R}_j = P_j \xleftarrow{\mathcal{R}_a} P_i = \mathcal{R}_i$$

The dimension vector of the representation $\mathcal{R}$ is given by the $k$-tuple $(rk_C\ \mathcal{R}_1, \ldots, rk_C\ \mathcal{R}_k)$. A subrepresentation $\mathcal{S}$ of $\mathcal{R}$ is determined by a collection of projective sub-summands (and not merely sub-modules) $\mathcal{S}_i \triangleleft \mathcal{R}_i$. In particular, for $W \in \mathtt{rep}_{\alpha+\beta}\ Q$ we define the representation $\mathcal{W}_C$ of $Q$ over the commutative ring $C$ by

$$\begin{cases} (\mathcal{W}_C)_i &= C \otimes_{\mathbb{C}} W_i \\ (\mathcal{W}_C)_a &= id_C \otimes_{\mathbb{C}} W_a \end{cases}$$

With these definitions, we can now define the functor represented by $\mathtt{Grass}_\alpha(W)$ as the functor assigning to a commutative $\mathbb{C}$-algebra $C$ the set of all subrepresentations of dimension vector $\alpha$ of the representation $\mathcal{W}_C$.

**Lemma 4.12** *Let $x = (W,V)$ be a geometric point of $\mathtt{Grass}_\alpha(W)$, then*

$$T_x\ \mathtt{Grass}_\alpha(W) = Hom_{\mathbb{C}Q}(V, \frac{W}{V})$$

*Proof.* The tangent space in $x = (W,V)$ are the $\mathbb{C}[\epsilon]$-points of $\mathtt{Grass}_\alpha(W)$ lying over $(W,V)$. To start, let $V \xrightarrow{\psi} \frac{W}{V}$ be a homomorphism of representations of $Q$ and consider a $\mathbb{C}$-linear lift of this map $\tilde{\psi} : V \longrightarrow W$. Consider the $\mathbb{C}$-linear subspace of $\mathcal{W}_{\mathbb{C}[\epsilon]} = \mathbb{C}[\epsilon] \otimes W$ spanned by the sets

$$\{v + \epsilon \otimes \tilde{\psi}(v) \ \mid \ v \in V\} \quad \text{and} \quad \epsilon \otimes V$$

This determines a $\mathbb{C}[\epsilon]$-subrepresentation of dimension vector $\alpha$ of $\mathcal{W}_{\mathbb{C}[\epsilon]}$ lying over $(W,V)$ and is independent of the chosen linear lift $\tilde{\psi}$.

Conversely, if $\mathcal{S}$ is a $\mathbb{C}[\epsilon]$-subrepresentation of $\mathcal{W}_{\mathbb{C}[\epsilon]}$ lying over $(W,V)$, then $\frac{\mathcal{S}}{\epsilon\mathcal{S}} = V \hookrightarrow W$. But then, a $\mathbb{C}$-linear complement of $\epsilon\mathcal{S}$ is spanned by elements of the form $v + \epsilon\psi(v)$ where $\psi(v) \in W$ and $\epsilon \otimes \psi$ is determined modulo an element of $\epsilon \otimes V$. But then, we have a $\mathbb{C}$-linear map $\tilde{\psi} : V \longrightarrow \frac{W}{V}$ and as $\mathcal{S}$ is a $\mathbb{C}[\epsilon]$-subrepresentation, $\tilde{\psi}$ must be a homomorphism of representations of $Q$. $\square$

**Theorem 4.16** *The following are equivalent*

1. $\alpha \hookrightarrow \alpha + \beta$.
2. *Every representation $W \in \mathtt{rep}_{\alpha+\beta}\ Q$ has a subrepresentation $V$ of dimension $\alpha$.*
3. $ext(\alpha, \beta) = 0$.

*Proof.* Assume 1. , then the image of the proper map $s : \mathtt{rep}_\alpha^{\alpha+\beta}\ Q \longrightarrow \mathtt{rep}_{\alpha+\beta}\ Q$ contains a Zariski open subset. As properness implies that the image of $s$ must also be a closed subset of $\mathtt{rep}_{\alpha+\beta}\ Q$ it follows that $Im\ s = \mathtt{rep}_{\alpha+\beta}\ Q$, that is 2. holds. Conversely, 2. clearly implies 1. so they are equivalent.

We compute the dimension of the vectorbundle $\mathtt{rep}_\alpha^{\alpha+\beta}\ Q$ over $Grass_\alpha(\alpha+\beta)$. Using that the dimension of a Grassmannians $Grass_k(l)$ is $k(l-k)$ we know that the base has dimension $\sum_{i=1}^k a_i b_i$. Now, fix a point $V \hookrightarrow W$ in $Grass_\alpha(\alpha+\beta)$, then the fiber over it determines all possible ways in which this inclusion is a subrepresentation of quivers. That is, for every arrow in $Q$ of the form

$\textcircled{j} \xleftarrow{a} \textcircled{i}$ we need to have a commuting diagram

$$\begin{tikzcd} V_i \arrow[r] \arrow[d, hook] & V_j \arrow[d, hook] \\ W_i \arrow[r] & W_j \end{tikzcd}$$

Here, the vertical maps are fixed. If we turn $V \in \mathtt{rep}_\alpha\ Q$, this gives us the $a_ia_j$ entries of the upper horizontal map as degrees of freedom, leaving only freedom for the lower horizontal map determined by a linear map $\frac{W_i}{V_i} \longrightarrow W_j$, that is, having $b_i(a_j + b_j)$ degrees of freedom. Hence, the dimension of the vectorspace-fibers is

$$\sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_ia_j + b_i(a_j + b_j))$$

giving the total dimension of the reduced variety $\mathtt{rep}_\alpha^{\alpha+\beta}\ Q$. But then,

$$\begin{aligned} dim\ \mathtt{rep}_\alpha^{\alpha+\beta}\ Q - dim\ \mathtt{rep}_{\alpha+\beta}\ Q &= \sum_{i=1}^k a_ib_i + \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_ia_j + b_i(a_j + b_j)) \\ &\quad - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i + b_i)(a_j + b_j) \\ &= \sum_{i=1}^k a_ib_i - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_ib_j = \chi_Q(\alpha, \beta) \end{aligned}$$

Assume that 2. holds, then the proper map $\mathtt{rep}_\alpha^{\alpha+\beta} \overset{s}{\twoheadrightarrow} \mathtt{rep}_{\alpha+\beta}\ Q$ is onto and as both varieties are reduced, the general fiber is a reduced variety of dimension $\chi_Q(\alpha, \beta)$, whence the general fiber contains points such that their tangentspaces have dimension $\chi_Q(\alpha, \beta)$. By the foregoing lemma we can compute the dimension of this tangentspace as $dim\ Hom_{\mathbb{C}Q}(V, \frac{W}{V})$. But then, as

$$\chi_Q(\alpha, \beta) = dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, \frac{W}{V}) - dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V, \frac{W}{V})$$

it follows that $Ext^1(V, \frac{W}{V}) = 0$ for some representation $V$ of dimension vector $\alpha$ and $\frac{W}{V}$ of dimension vector $\beta$. But then, $ext(\alpha, \beta) = 0$, that is, 3. holds.

Conversely, assume that $ext(\alpha,\beta)=0$. Then, for a general point $W \in \mathtt{rep}_{\alpha+\beta}\ Q$ in the image of $s$ and for a general point in its fiber $(W,V) \in \mathtt{rep}_{\alpha}^{\alpha+\beta}\ Q$ we have $dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V,\frac{W}{V})=0$ whence $dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V,\frac{W}{V})=\chi_Q(\alpha,\beta)$. But then, the general fiber of $s$ has dimension $\chi_Q(\alpha,\beta)$ and as this is the difference in dimension between the two irreducible varieties, the map is generically onto. Finally, properness of $s$ then implies that it is onto, giving 2. and finishing the proof. □

**Proposition 4.15** *Let $\alpha$ be a Schur root such that $\chi_Q(\alpha,\alpha)<0$, then for any integer $n$ we have that $n\alpha$ is a Schur root.*

*Proof.* There are infinitely many non-isomorphic Schur representations of dimension vector $\alpha$. Pick $n$ of them $\{W_1,\ldots,W_n\}$ and from $\chi_Q(\alpha,\alpha)<0$ we deduce

$$Hom_{\mathbb{C}Q}(W_i,W_j)=\delta_{ij}\mathbb{C} \qquad \text{and} \qquad Ext^1_{\mathbb{C}Q}(W_i,W_j)\neq 0$$

By lemma 4.9 we can construct a representation $V_n$ having a filtration

$$0=V_0\subset V_1\subset\ldots\subset V_n \quad \text{with} \quad \frac{V_j}{V_{j-1}}\simeq W_j$$

and such that the short exact sequences $0 \longrightarrow V_{j-1} \longrightarrow V_j \longrightarrow W_j \longrightarrow 0$ do not split. By induction on $n$ we may assume that $End_{\mathbb{C}Q}(V_{n-1})=\mathbb{C}$ and we have that $Hom_{\mathbb{C}Q}(V_{n-1},W_n)=0$. But then, the restriction of any endomorphism $\phi$ of $V_n$ to $V_{n-1}$ must be an endomorphism of $V_{n-1}$ and therefore a scalar $\lambda\mathbb{1}$. Hence, $\phi-\lambda\mathbb{1}\in End_{\mathbb{C}Q}(V_n)$ is trivial on $V_{n-1}$. As $Hom_{\mathbb{C}Q}(W_n,V_{n-1})=0$, $End_{\mathbb{C}Q}(W_n)=\mathbb{C}$ and non-splitness of the sequence $0 \longrightarrow V_{n-1} \longrightarrow V_n \longrightarrow W_n \longrightarrow 0$ we must have $\phi-\lambda\mathbb{1}=0$ whence $End_{\mathbb{C}Q}(V_n)=\mathbb{C}$, that is, $n\alpha$ is a Schur root. □

We say that a dimension vector $\alpha$ is *left orthogonal* to $\beta$ if $hom(\alpha,\beta)=0$ and $ext(\alpha,\beta)=0$.

**Definition 4.10** *An ordered sequence $C=(\beta_1,\ldots,\beta_s)$ of dimension vectors is said to be a* compartment *for $Q$ if and only if*

1. *for all $i$, $\beta_i$ is a Schur root,*
2. *for al $i<j$, $\beta_i$ is left orthogonal to $\beta_j$,*
3. *for all $i<j$ we have $\chi_Q(\beta_j,\beta_i)\geq 0$.*

**Theorem 4.17** *Suppose that $C=(\beta_1,\ldots,\beta_s)$ is a compartment for $Q$ and that there are non-negative integers $e_1,\ldots,e_s$ such that $\alpha=e_1\beta_1+\ldots+e_s\beta_s$. Assume that $e_i=1$ whenever $\chi_Q(\beta_i,\beta_i)<0$. Then,*

$$\tau_{can}=(e_1,\beta_1;\ldots;e_s,\beta_s)$$

*is the canonical decomposition of the dimension vector $\alpha$.*

*Proof.* Let $V$ be a generic representation of dimension vector $\alpha$ with decomposition into indecomposables

$$V = W_1^{\oplus e_1} \oplus \ldots \oplus W_s^{\oplus e_s} \qquad \text{with} \quad dim(W_i) = \beta_i$$

we will show that (after possibly renumbering the factors $(\beta_1, \ldots, \beta_s)$ is a compartment for $Q$. To start, it follows from lemma 4.10 that for all $i \neq j$ we have $Ext^1_{\mathbb{C}Q}(W_i, W_j) = 0$. From lemma 4.11 we deduce a partial ordering $i \rightarrow j$ on the indices whenever $Hom_{\mathbb{C}Q}(W_i, W_j) \neq 0$. Indeed, any non-zero morphism $W_i \longrightarrow W_j$ is either a mono- or an epimorphism, assume $W_i \twoheadrightarrow W_j$ then there can be no monomorphism $W_j \hookrightarrow W_k$ as the composition $W_i \longrightarrow W_k$ would be neither mono nor epi. That is, all non-zero morphisms from $W_j$ to factors must be (proper) epi and we cannot obtain cycles in this way by counting dimensions. If $W_i \hookrightarrow W_j$, a similar argument proves the claim. From now on we assume that the chosen index-ordering of the factors is (reverse) compatible with the partial ordering $i \rightarrow j$, that is $Hom(W_i, W_j) = 0$ whenever $i < j$, that is, $\beta_i$ is left orthogonal to $\beta_j$ whenever $i < j$. As $Ext^1_{\mathbb{C}Q}(W_j, W_i) = 0$, it follows that $\chi_Q(\beta_j, \beta_i) \geq 0$. As generic representations are open it follows that all $\mathtt{rep}_{\beta_i}\ Q$ have an open subset of indecomposables, proving that the $\beta_i$ are Schur roots. Finally, it follows from proposition 4.15 that a Schur root $\beta_i$ with $\chi_Q(\beta_i, \beta_i)$ can occur only with multiplicity one in any canonical decomposition.

Conversely, assume that $(\beta_1, \ldots, \beta_s)$ is a compartment for $Q$, $\alpha = \sum_i e_i \beta_i$ satisfying the requirements on multiplicities. Choose Schur representations $W_i \in \mathtt{rep}_{\beta_i}\ Q$, then we have to prove that

$$V = W_1^{\oplus e_1} \oplus \ldots \oplus W_s^{\oplus e_s}$$

is a generic representation of dimension vector $\alpha$. In view of the properties of the compartment we already know that $Ext^1_{\mathbb{C}Q}(W_i, W_j) = 0$ for all $i < j$ and we need to show that $Ext^1_{\mathbb{C}Q}(W_j, W_i) = 0$. Indeed, if this condition is satisfied we have

$$\begin{aligned} dim\ \mathtt{rep}_\alpha\ Q - dim\ \mathcal{O}(V) &= dim_{\mathbb{C}} Ext^1(V, V) \\ &= \sum_i e_i^2 dim_{\mathbb{C}} Ext^1(W_i, W_i) = \sum_i e_i^2 (1 - q_Q(\beta_i) \end{aligned}$$

We know that the Schur representations of dimension vector $\beta_i$ depend on $1 - q_Q(\beta_i)$ parameters by Kac s theorem 4.14 and $e_i = 1$ unless $q_Q(\beta_i) = 1$. Therefore, the union of all orbits of representations with the same Schur-decomposition type as $V$ contain a dense open set of $\mathtt{rep}_\alpha\ Q$ and so this must be the canonical decomposition.

If this extension space is nonzero, $Hom_{\mathbb{C}Q}(W_j, W_i) \neq 0$ as $\chi_Q(\beta_j, \beta_i) \geq 0$. But then by lemma 4.11 any non-zero homomorphism from $W_j$ to $W_i$ must be either a mono or an epi. Assume it is a mono, so $\beta_j < \beta_i$, so in particular a general representation of dimension $\beta_i$ contains a subrepresentation of dimension $\beta_j$ and hence by theorem 4.16 we have $ext(\beta_j, \beta_i - \beta_j) = 0$. Suppose that $\beta_j$ is a real Schur root, then $Ext^1_{\mathbb{C}Q}(W_j, W_j) = 0$ and therefore also $ext(\beta_j, \beta_i) = 0$ as $Ext^1_{\mathbb{C}Q}(W_j, W_j \oplus (W_j/W_i)) = 0$. If $\beta$ is not a real root, then for a general representation $S \in \mathtt{rep}_{\beta_j}\ Q$ take a representation $R \in \mathtt{rep}_{\beta_i}\ Q$ in the open set where $Ext^1_{\mathbb{C}Q}(S, R) = 0$, then there is a monomorphism $S \hookrightarrow R$. Because $Ext^1_{\mathbb{C}Q}(S, S) \neq 0$ we deduce from lemma 4.9 that

$Ext^1_{\mathbb{C}Q}(R,S) \neq 0$ contradicting the fact that $ext(\beta_i,\beta_j) = 0$. If the nonzero morphism $W_j \longrightarrow W_i$ is epi one has a similar argument. $\square$

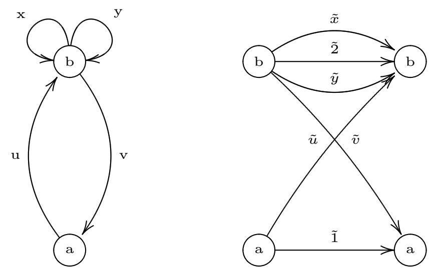
This result can be used to obtain a fairly efficient algorithm to compute the canonical decomposition in case the quiver $Q$ has *no oriented cycles*. Fortunately, one can reduce the general problem to that of quiver without oriented cycles using the *bipartite double* $Q^b$ of $Q$. We double the vertex-set of $Q$ in a left and right set of vertices, that is

$$Q^b_v = \{v^l_1,\ldots,v^l_k,v^r_1,\ldots,v^r_k\}$$

To every arrow $a \in Q_a$ from $v_i$ to $v_j$ we assign an arrow $\tilde{a} \in Q^b_a$ from $v^l_i$ to $v^r_j$. In addition, we have for each $1 \leq i \leq k$ one extra arrow $\tilde{i}$ in $Q^b_a$ from $v^l_i$ to $v^r_i$. If $\alpha = (a_1,\ldots,a_k)$ is a dimension vector for $Q$, the associated dimension vector $\tilde{\alpha}$ for $Q^b$ has components

$$\tilde{\alpha} = (a_1,\ldots,a_k,a_1,\ldots,a_k).$$

**Example 4.7** Consider the quiver $Q$ and dimension vector $\alpha = (a,b)$ on the left hand side, then




the bipartite quiver situation $Q^b$ and $\tilde{\alpha}$ is depicted on the right hand side.

If the canonical decomposition of $\alpha$ for $Q$ is $\tau_{can} = (e_1,\beta_1;\ldots;e_s,\beta_s)$, then the canonical decomposition of $\tilde{\alpha}$ for $Q^b$ is $(e_1,\tilde{\beta}_1;\ldots;e_s,\tilde{\beta}_s)$ as for a general representation of $Q^b$ of dimension vector $\tilde{\alpha}$ the morphisms corresponding to $\tilde{i}$ for $1 \leq i \leq k$ are all invertible matrices and can be used to identify the left and right vertex sets, that is, there is an equivalence of categories between representations of $Q^b$ where all the maps $\tilde{i}$ are invertible and representations of the quiver $Q$. That is, the algorithm below can be applied to $(Q^b,\tilde{\alpha})$ to obtain the canonical decomposition of $\alpha$ for an arbitrary quiver $Q$.

Let $Q$ be a quiver *without oriented cycles* then we can order the vertices $\{v_1,\ldots,v_k\}$ such that there are no oriented paths from $v_i$ to $v_j$ whenever $i < j$ (start with a sink of $Q$, drop it and continue recursively). For example, for the bipartite quiver $Q^b$ we first take all the right vertices and then the left ones.

input : quiver $Q$, ordered set of vertices as above, dimension vector $\alpha = (a_1, \ldots, a_k)$ and type $\tau = (a_1, \vec{v_1}; \ldots; a_k, \vec{v_k})$ where $\vec{v_i} = (\delta_{ij})_j = dim\ v_i$ is the canonical basis. By the assumption on the ordering of vertices we have that $\tau$ is a *good type* for $\alpha$. We say that a type $(f_1, \gamma_1; \ldots; f_s, \gamma_s)$ is a good type for $\alpha$ if $\alpha = \sum_i f_i \gamma_i$ and the following properties are satisfies

1. $f_i \geq 0$ for all $i$,
2. $\gamma_i$ is a Schur root,
3. for each $i < j$, $\gamma_i$ is left orthogonal to $\gamma_j$,
4. $f_i = 1$ whenever $\chi_Q(\gamma_i, \gamma_i) < 0$.

A type is said to be *excellent* provided that, in addition to the above, we also have that for all $i < j$, $\chi_Q(\alpha_j, \alpha_i) \geq 0$. In view of theorem 4.17 the purpose of the algorithm is to transform the good type $\tau$ into the excellent type $\tau_{can}$. We will describe the main loop of the algorithm on a good type $(f_1, \gamma_1; \ldots; f_s, \gamma_s)$.

step 1 : Omit all couples $(f_i, \gamma_i)$ with $f_i = 0$ and verify whether the remaining type is excellent. If it is, stop and output this type. If not, proceed.
step 2 : Reorder the type as follows, choose $i$ and $j$ such that $j - i$ is minimal and $\chi_Q(\beta_j, \beta_i) < 0$. Partition the intermediate entries $\{i+1, \ldots, j-1\}$ into the sets

- $\{k_1, \ldots, k_a\}$ such that $\chi_Q(\gamma_j, \gamma_{k_m}) = 0$,
- $\{l_1, \ldots, l_b\}$ such that $\chi_Q(\gamma_j, \gamma_{l_m}) > 0$.

Reorder the couples in the type in the sequence

$$(1, \ldots, i-1, k_1, \ldots, k_a, i, j, l_1, \ldots, l_b, j+1, \ldots, s)$$

define $\mu = \gamma_i$, $\nu = \gamma_j$, $p = f_i$, $q = f_j$, $\zeta = p\mu + q\nu$ and $t = -\chi_Q(\nu, \mu)$, then proceed.
step 3 : Change the part $(p, \mu; q, \nu)$ of the type according to the following scheme

- If $\mu$ and $\nu$ are real Schur roots, consider the subcases
  1. $\chi_Q(\zeta, \zeta) > 0$, replace $(p, \mu, q, \nu)$ by $(p', \mu'; q'; \nu')$ where $\nu'$ and $\nu'$ are non-negative combinations of $\nu$ and $\mu$ such that $\mu'$ is left orthogonal to $\nu'$, $\chi_Q(\nu', \mu') = t \geq 0$ and $\zeta = p'\mu' + q'\nu'$ for non-negative integers $p', q'$.
  2. $\chi_Q(\zeta, \zeta) = 0$, replace $(p, \mu; q, \nu)$ by $(k, \zeta')$ with $\zeta = k\zeta'$, $k$ positive integer, and $\zeta'$ an indivisible root.
  3. $\chi_Q(\zeta, \zeta) < 0$, replace $(p, \mu; q, \nu)$ by $(1, \zeta)$.
- If $\mu$ is a real root and $\nu$ is imaginary, consider the subcases
  1. If $p + q\chi_Q(\nu, \mu) \geq 0$, replace $(p, \mu; q, \nu)$ by $(q, \nu - \chi_Q(\nu, \mu)\mu; p + q\chi_Q(\nu, \mu), \mu)$.

2. If $p + q\chi_Q(\nu,\mu) < 0$, $r$eplace $(p,\mu;q,\nu)$ by $(1,\zeta)$.

- If $\mu$ is an imaginary root and $\nu$ is real, consider the subcases

  1. If $q + p\chi_Q(\nu,\mu) \geq 0$, $r$eplace $(p,\mu;q,\nu)$ by $(q + p\chi_Q(\nu,\mu),\nu;p,\mu - \chi_Q(\nu,\mu)\nu)$.
  2. If $q + p\chi_Q(\nu,\mu) < 0$, $r$eplace $(p,\mu;q,\nu)$ by $(1,\zeta)$.

- If $\mu$ and $\nu$ are imaginary roots, $r$eplace $(p,\mu;q,\nu)$ by $(1,\zeta)$.

then go to `step` 1.

One can show that in every loop of the algorithm the number $\sum_i f_i$ decreases, so the algorithm must stop, giving the canonical decomposition of $\alpha$. A consequence of this algorithm is that $r(\alpha) + 2i(\alpha) \leq k$ where $r(\alpha)$ is the number of real Schur roots occurring in the canonical decomposition of $\alpha$, $i(\alpha)$ the number of imaginary Schur roots and $k$ the number of vertices of $Q$. For more details we refer to [24].

## 4.7 General subrepresentations

Often, we will need to determine the dimension vectors of *general subrepresentations* . It follows from theorem 4.16 that this problem is equivalent to the calculation of $ext(\alpha,\beta)$. An inductive algorithm to do this was discovered by A. Schofield [73].

Recall that $\alpha \hookrightarrow \beta$ iff a general representation $W \in \mathtt{rep}_\beta\ Q$ contains a subrepresentation $S \hookrightarrow W$ of dimension vector $\alpha$. Similarly, we denote $\beta \twoheadrightarrow \gamma$ if and only if a general representation $W \in \mathtt{rep}_\beta\ Q$ has a quotient-representation $W \twoheadrightarrow T$ of dimension vector $\gamma$. As before, $Q$ will be a quiver on $k$-vertices $\{v_1,\ldots,v_k\}$ and we denote dimension vectors $\alpha = (a_1,\ldots,a_k)$, $\beta = (b_1,\ldots,b_k)$ and $\gamma = (c_1,\ldots,c_k)$. We will first determine the *rank of a general homomorphism* $V \longrightarrow W$ between representations $V \in \mathtt{rep}_\alpha\ Q$ and $W \in \mathtt{rep}_\beta\ Q$. We denote

$$Hom(\alpha,\beta) = \oplus_{i=1}^k M_{b_i \times a_i} \quad \text{and} \quad Hom(V,\beta) = Hom(\alpha,\beta) = Hom(\alpha,W)$$

for any representations $V$ and $W$ as above. With these conventions we have

**Lemma 4.13** *There is an open subset $Hom_m(\alpha,\beta) \hookrightarrow \mathtt{rep}_\alpha\ Q \times \mathtt{rep}_\beta\ Q$ and a dimension vector $\gamma \overset{def}{=} rk\ hom(\alpha,\beta)$ such that for all $(V,W) \in Hom_{min}(\alpha,\beta)$*

- *$dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V,W)$ is minimal, and*
- *$\{\phi \in Hom_{\mathbb{C}Q}(V,W) \mid rk\ \phi = \gamma\}$ is a non-empty Zariski open subset of $Hom_{\mathbb{C}Q}(V,W)$.*

*Proof.* Consider the subvariety $Hom_Q(\alpha, \beta)$ of the trivial vectorbundle

$$\begin{tikzcd} Hom_Q(\alpha,\beta) \arrow[r, hook] \arrow[dr, "\Phi"'] & Hom(\alpha,\beta) \times \mathtt{rep}_\alpha\, Q \times \mathtt{rep}_\beta\, Q \arrow[d, "pr"] \\ & \mathtt{rep}_\alpha\, Q \times \mathtt{rep}_\beta\, Q \end{tikzcd}$$

of triples $(\phi, V, W)$ such that $V \xrightarrow{\phi} W$ is a morphism of representations of $Q$. The fiber $\Phi^{-1}(V, W) = Hom_{\mathbb{C}Q}(V, W)$. As the fiber dimension is upper semi-continuous, there is an open subset $Hom_{min}(\alpha, \beta)$ of $\mathtt{rep}_\alpha\, Q \times \mathtt{rep}_\beta\, Q$ consisting of points $(V, W)$ where $dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, W)$ is minimal. For given dimension vector $\delta = (d_1, \ldots, d_k)$ we consider the subset

$$Hom_Q(\alpha, \beta, \delta) = \{(\phi, V, W) \in Hom_Q(\alpha, \beta) \mid rk\ \phi = \delta\} \hookrightarrow Hom_Q(\alpha, \beta)$$

This is a constructible subset of $Hom_Q(\alpha, \beta)$ and hence there is a dimension vector $\gamma$ such that $Hom_Q(\alpha, \beta, \gamma) \cap \Phi^{-1}(Hom_{min}(\alpha, \beta))$ is constructible and dense in $\Phi^{-1}(Hom_{min}(\alpha, \beta))$. But then,

$$\Phi(Hom_Q(\alpha, \beta, \gamma) \cap \Phi^{-1}(Hom_{min}(\alpha, \beta)))$$

is constructible and dense in $Hom_{min}(V, W)$. Therefore it contains an open subset $Hom_m(V, W)$ satisfying the requirements of the lemma. □

**Lemma 4.14** *Assume we have short exact sequences of representations of $Q$*

$$\begin{cases} 0 \longrightarrow S \longrightarrow V \longrightarrow X \longrightarrow 0 \\ 0 \longrightarrow Y \longrightarrow W \longrightarrow T \longrightarrow 0 \end{cases}$$

*then there is a natural onto map*

$$Ext^1_{\mathbb{C}Q}(V, W) \twoheadrightarrow Ext^1_{\mathbb{C}Q}(S, T)$$

*Proof.* By lemma 4.9 we have surjective maps

$$Ext^1_{\mathbb{C}Q}(V, W) \twoheadrightarrow Ext^1_{\mathbb{C}Q}(V, T) \twoheadrightarrow Ext^1_{\mathbb{C}Q}(S, T)$$

from which the assertion follows. □

**Theorem 4.18** *Let $\gamma = rk\ hom(\alpha, \beta)$ (with notations as in lemma 4.13), then*

1. $\alpha - \gamma \hookrightarrow \alpha \twoheadrightarrow \gamma \hookrightarrow \beta \twoheadrightarrow \beta - \gamma$

2. $ext(\alpha,\beta) = -\chi_Q(\alpha-\gamma,\beta-\gamma) = ext(\alpha-\gamma,\beta-\gamma)$

*Proof.* The first statement is obvious from the definitions, for if $\gamma = rk\ hom(\alpha,\beta)$, then a general representation of dimension $\alpha$ will have a quotient-representation of dimension $\gamma$ (and hence a subrepresentation of dimension $\alpha-\gamma$) and a general representation of dimension $\beta$ will have a subrepresentation of dimension $\gamma$ (and hence a quotient-representation of dimension $\beta-\gamma$.

The strategy of the proof of the second statement is to compute the dimension of the subvariety of $Hom(\alpha,\beta) \times \mathtt{rep}_\alpha \times \mathtt{rep}_\beta \times \mathtt{rep}_\gamma$ defined by

$$H^{factor} = \{(\phi, V, W, X) \mid \begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr] & & W \\ & X = Im\ \phi \arrow[ur, hook] & \end{tikzcd} \text{ factors as representations } \}$$

in two different ways. Consider the intersection of the open set $Hom_m(\alpha,\beta)$ determined by lemma 4.13 with the open set of couples $(V,W)$ such that $dim\ Ext(V,W) = ext(\alpha,\beta)$ and let $(V,W)$ lie in this intersection. In the previous section we have proved that

$$dim\ \mathtt{Grass}_\gamma(W) = \chi_Q(\gamma,\beta-\gamma)$$

Let $H$ be the subbundle of the trivial vectorbundle over $\mathtt{Grass}_\gamma(W)$

$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr] & Hom(\alpha,W) \times \mathtt{Grass}_\gamma(W) \arrow[d] \\ & \mathtt{Grass}_\gamma(W) \end{tikzcd}$$

consisting of triples $(\phi, W, U)$ with $\phi : \oplus_i \mathbb{C}^{\oplus a_i} \longrightarrow W$ a linear map such that $Im(\phi)$ is contained in the subrepresentation $U \hookrightarrow W$ of dimension $\gamma$. That is, the fiber over $(W,U)$ is $Hom(\alpha,U)$ and therefore has dimension $\sum_{i=1}^k a_i c_i$. With $H^{full}$ we consider the open subvariety of $H$ of triples $(\phi, W, U)$ such that $Im\ \phi = U$. We have

$$dim\ H^{full} = \sum_{i=1}^k a_i c_i + \chi_Q(\gamma,\beta-\gamma)$$

But then, $H^{factor}$ is the subbundle of the trivial vectorbundle over $H^{full}$

$$\begin{tikzcd}
H^{factor} \arrow[r, hook] \arrow[dr, "\pi"] & \mathtt{rep}_\alpha\ Q \times H^{full} \arrow[d] \\
& H^{full}
\end{tikzcd}$$

consisting of quadruples $(V, \phi, W, X)$ such that $V \xrightarrow{\phi} W$ is a morphism of representations, with image the subrepresentation $X$ of dimension $\gamma$. The fiber of $\pi$ over a triple $(\phi, W, X)$ is determined by the property that for each arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ the following diagram must be commutative, where we decompose the vertex spaces $V_i = X_i \oplus K_i$ for $K = Ker\ \phi$

$$\begin{tikzcd}
X_i \oplus K_i \arrow[r, "\begin{bmatrix} A & B \\ C & D \end{bmatrix}"] \arrow[d, "\begin{bmatrix} \mathbb{1}_{c_i} & 0 \end{bmatrix}"'] & X_j \oplus K_j \arrow[d, "\begin{bmatrix} \mathbb{1}_{c_j} & 0 \end{bmatrix}"] \\
X_i \arrow[r, "A"'] & X_j
\end{tikzcd}$$

where $A$ is fixed, giving the condition $B = 0$ and hence the fiber has dimension equal to

$$\sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i - c_i)(a_j - c_j) + \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} c_i(a_j - c_j) = \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i(a_j - c_j)$$

This gives our first formula for the dimension of $H^{factor}$

$$H^{factor} = \sum_{i=1}^{k} a_i c_i + \chi_Q(\gamma, \beta - \gamma) + \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i(a_j - c_j)$$

On the other hand, we can consider the natural map $H^{factor} \xrightarrow{\Phi} \mathtt{rep}_\alpha\ Q$ defined by sending a quadruple $(V, \phi, W, X)$ to $V$. the fiber in $V$ is given by all quadruples $(V, \phi, W, X)$ such that $V \xrightarrow{\phi} W$ is a morphism of representations with $Im\ \phi = X$ a representation of dimension vector $\gamma$, or equivalently

$$\Phi^{-1}(V) = \{V \xrightarrow{\phi} W \mid rk\ \phi = \gamma\}$$

Now, recall our restriction on the couple $(V,W)$ giving at the beginning of the proof. There is an open subset $max$ of $\mathtt{rep}_\alpha\ Q$ of such $V$ and by construction $max \hookrightarrow Im\ \Phi$, $\Phi^{-1}(max)$ is open and dense in $H^{factor}$ and the fiber $\Phi^{-1}(V)$ is open and dense in $Hom_{\mathbb{C}Q}(V,W)$. This provides us with the second formula for the dimension of $H^{factor}$

$$dim\ H^{factor} = dim\ \mathtt{rep}_\alpha\ Q + hom(\alpha, W) = \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i a_j + hom(\alpha,\beta)$$

Equating both formulas we obtain the equality

$$\chi_Q(\gamma,\beta-\gamma) + \sum_{i=1}^k a_i c_i - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i c_j = hom(\alpha,\beta)$$

which is equivalent to

$$\chi_Q(\gamma,\beta-\gamma) + \chi_Q(\alpha,\gamma) - \chi_Q(\alpha,\beta) = ext(\alpha,\beta)$$

Now, for our $(V,W)$ we have that $Ext(V,W) = ext(\alpha,\beta)$ and we have exact sequences of representations

$$0 \longrightarrow S \longrightarrow V \longrightarrow X \longrightarrow 0 \qquad 0 \longrightarrow X \longrightarrow W \longrightarrow T \longrightarrow 0$$

and using lemma 4.14 this gives a surjection $Ext(V,W) \twoheadrightarrow Ext(S,T)$. On the other hand we always have from the homological interpretation of the Euler form the first inequality

$$\begin{aligned} dim_{\mathbb{C}}\ Ext(S,T) &\geq -\chi_Q(\alpha-\gamma,\beta-\gamma) = \chi_Q(\gamma,\beta-\gamma) - \chi_Q(\alpha,\beta) + \chi_Q(\alpha,\gamma) \\ &= ext(\alpha,\beta) \end{aligned}$$

As the last term is $dim_{\mathbb{C}}\ Ext(V,W)$, this implies that the above surjection must be an isomorphism and that

$$dim_{\mathbb{C}}\ Ext(S,T) = -\chi_Q(\alpha-\gamma,\beta-\gamma) \quad \text{whence} \quad dim_{\mathbb{C}}\ Hom(S,T) = 0$$

But this implies that $hom(\alpha-\gamma,\beta-\gamma) = 0$ and therefore $ext(\alpha-\gamma,\beta-\gamma) = -\chi_Q(\alpha-\gamma,\beta-\gamma)$. Finally,

$$ext(\alpha-\gamma,\beta-\gamma) = dim\ Ext(S,T) = dim\ Ext(V,W) = ext(\alpha,\beta)$$

finishing the proof. $\square$

**Theorem 4.19** *For all dimension vectors $\alpha$ and $\beta$ we have*

$$\begin{aligned} ext(\alpha,\beta) &= \max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} -\chi_Q(\alpha',\beta') \\ &= \max_{\beta \twoheadrightarrow \beta''} -\chi_Q(\alpha,\beta'') \\ &= \max_{\alpha'' \hookrightarrow \alpha} -\chi_Q(\alpha'',\beta) \end{aligned}$$

*Proof.* Let $V$ and $W$ be representation of dimension vector $\alpha$ and $\beta$ such that $dim\ Ext(V,W) = ext(\alpha,\beta)$. Let $S \hookrightarrow V$ be a subrepresentation of dimension $\alpha'$ and $W \twoheadrightarrow T$ a quotient representation of dimension vector $\beta'$. Then, we have

$$ext(\alpha,\beta) = dim_{\mathbb{C}}\ Ext(V,W) \geq dim_{\mathbb{C}}\ Ext(S,T) \geq -\chi_Q(\alpha',\beta')$$

where the first inequality is lemma 4.14 and the second follows from the interpretation of the Euler form. Therefore, $ext(\alpha,\beta)$ is greater or equal than all the terms in the statement of the theorem. The foregoing theorem asserts the first equality, as for $rk\ hom(\alpha,\beta) = \gamma$ we do have that $ext(\alpha,\beta) = -\chi_Q(\alpha-\gamma,\beta-\gamma)$.

In the proof of the above theorem, we have found for sufficiently general $V$ and $W$ an exact sequence of representations

$$0 \longrightarrow S \longrightarrow V \longrightarrow W \longrightarrow T \longrightarrow 0$$

where $S$ is of dimension $\alpha-\gamma$ and $T$ of dimension $\beta-\gamma$. Moreover, we have a commuting diagram of surjections

$$\begin{tikzcd} Ext(V,W) \arrow[r, two heads] \arrow[d] \arrow[dr, dotted] & Ext(V,T) \arrow[d] \\ Ext(S,W) \arrow[r, two heads] & Ext(S,T) \end{tikzcd}$$

and the dashed map is an isomorphism, hence so are all the epimorphisms. Therefore, we have

$$\begin{cases} ext(\alpha,\beta-\gamma) & \leq dim\ Ext(V,T) = dim\ Ext(V,W) = ext(\alpha,\beta) \\ ext(\alpha-\gamma,\beta) & \leq dim\ Ext(S,W) = dim\ Ext(V,W) = ext(\alpha,\beta) \end{cases}$$

Further, let $T'$ be a sufficiently general representation of dimension $\beta-\gamma$, then it follows from $Ext(V,T') \twoheadrightarrow Ext(S,T)$ that

$$ext(\alpha-\gamma,\beta-\gamma) \leq dim\ Ext(S,T') \leq dim\ Ext(V,T') = ext(\alpha,\beta-\gamma)$$

but the left term is equal to $ext(\alpha,\beta)$ by the above theorem. But then, we have $ext(\alpha,\beta) = ext(\alpha,\beta-\gamma)$. Now, we may assume by induction that the theorem holds for $\beta-\gamma$. That is, there exists $\beta-\gamma \twoheadrightarrow \beta''$ such that $ext(\alpha,\beta-\gamma) = -\chi_Q(\alpha,\beta'')$. Whence, $\beta \twoheadrightarrow \beta''$ and $ext(\alpha,\beta) = -\chi_Q(\alpha,\beta'')$ and the middle equality of the theorem holds. By a dual argument so does the last. □

This gives us the following inductive algorithm to find all the dimension vectors of general subrepresentations. Take a dimension vector $\alpha$ and assume by induction we know for all $\beta < \alpha$ the set of general subrepresentations $\beta' \hookrightarrow \beta$. Then, $\beta \hookrightarrow \alpha$ if and only if

$$0 = ext(\beta,\alpha-\beta) = \max_{\beta' \hookrightarrow \beta} -\chi_Q(\beta',\alpha-\beta)$$

where the first equality comes from theorem 4.16 and the last from the above theorem.

## 4.8 Semistable representations

Let $Q$ be a quiver on $k$ vertices $\{v_1, \ldots, v_k\}$ and fix a dimension vector $\alpha$. So far, we have considered the algebraic quotient map

$$\mathtt{rep}_\alpha\ Q \longrightarrow\!\!\!\!\!\rightarrow \mathtt{iss}_\alpha\ Q$$

classifying closed $GL(\alpha)$-orbits in $\mathtt{rep}_\alpha\ Q$, that is, isomorphism classes of semi-simple representations of dimension $\alpha$. We have seen that the invariant polynomial maps are generated by traces along oriented cycles in the quiver. Hence, if $Q$ has no oriented cycles, the quotient variety $\mathtt{iss}_\alpha\ Q$ is reduced to one point corresponding to the semi-simple

$$S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k}$$

where $S_i$ is the trivial one-dimensional simple concentrated in vertex $v_i$. Still, in these cases one can often classify nice families of representations.

**Example 4.8** Consider the quiver setting

$$\begin{tikzcd} 1 \arrow[r, bend left, "x"] \arrow[r, "y"] \arrow[r, bend right, "z"'] & 1 \end{tikzcd}$$

Then, $\mathtt{rep}_\alpha\ Q = \mathbb{C}^3$ and the action of $GL(\alpha) = \mathbb{C}^* \times \mathbb{C}^*$ is given by $(\lambda, \mu).(x, y, z) = (\frac{\lambda}{\mu}x, \frac{\lambda}{\mu}y, \frac{\lambda}{\mu}z)$. The only closed $GL(\alpha)$-orbit in $\mathbb{C}^3$ is $(0,0,0)$ as the one-parameter subgroup $\lambda(t) = (t, 1)$ has the property

$$\lim_{t \rightarrow 0} \lambda(t).(x, y, z) = (0,0,0)$$

so $(0,0,0) \in \overline{\mathcal{O}(x,y,z)}$ for any representation $(x, y, z)$. Still, if we trow away the zero-representation, then we have a nice quotient map

$$\mathbb{C}^3 - \{(0,0,0)\} \overset{\pi}{\longrightarrow\!\!\!\!\!\rightarrow} \mathbb{P}^2 \qquad (x, y, z) \mapsto [x : y : z]$$

and as $\mathcal{O}(x,y,z) = \mathbb{C}^*(x,y,z)$ we see that every $GL(\alpha)$-orbit is closed in this complement $\mathbb{C}^3 - \{(0,0,0)\}$. We will generalize such settings to arbitrary quivers.

A *character* of $GL(\alpha)$ is an algebraic group morphism $\chi : GL(\alpha) \longrightarrow \mathbb{C}^*$. They are fully determined by an integral $k$-tuple $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ where

$$GL(\alpha) \xrightarrow{\chi_\theta} \mathbb{C}^* \qquad (g_1, \ldots, g_k) \mapsto det(g_1)^{t_1}. \ \ldots \ .det(g_k)^{t_k}$$

For a fixed $\theta$ we can extend the $GL(\alpha)$-action to the space $\texttt{rep}_\alpha \oplus \mathbb{C}$ by

$$GL(\alpha) \times \texttt{rep}_\alpha\ Q \oplus \mathbb{C} \longrightarrow \texttt{rep}_\alpha\ Q \oplus \mathbb{C} \qquad g.(V,c) = (g.V, \chi_\theta^{-1}(g)c)$$

The coordinate ring $\mathbb{C}[\texttt{rep}_\alpha\ Q \oplus \mathbb{C}] = \mathbb{C}[\texttt{rep}_\alpha][t]$ can be given a $\mathbb{Z}$-gradation by defining $deg(t) = 1$ and $deg(f) = 0$ for all $f \in \mathbb{C}[\texttt{rep}_\alpha\ Q]$. The induced action of $GL(\alpha)$ on $\mathbb{C}[\texttt{rep}_\alpha\ Q \oplus \mathbb{C}]$ preserves this gradation. Therefore, the ring of invariant polynomial maps

$$\mathbb{C}[\texttt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = \mathbb{C}[\texttt{rep}_\alpha\ Q][t]^{GL(\alpha)}$$

is also graded with homogeneous part of degree zero the ring of invariants $\mathbb{C}[\texttt{rep}_\alpha]^{GL(\alpha)}$. An invariant of degree $n$, say $ft^n$ with $f \in \mathbb{C}[\texttt{rep}_\alpha\ Q]$ has the characteristic property that

$$f(g.V) = \chi_\theta^n(g) f(V)$$

that is, $f$ is a *semi-invariant* of weight $\chi_\theta^n$. That is, the graded decomposition of the invariant ring is

$$\mathbb{C}[\texttt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = R_0 \oplus R_1 \oplus \ldots \quad \text{with} \quad R_i = \mathbb{C}[\texttt{rep}_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$$

**Definition 4.11** *With notations as above, the* moduli space of semi-stable quiver representations *of dimension* $\alpha$ *is the projective variety*

$$M_\alpha^{ss}(Q,\theta) = \texttt{proj}\ \mathbb{C}[\texttt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = \texttt{proj}\ \oplus_{n=0}^\infty \mathbb{C}[\texttt{rep}_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$$

Recall that for a positively graded affine commutative $\mathbb{C}$-algebra $R = \oplus_{i=0}^\infty R_i$, the geometric points of the projective scheme $\texttt{proj}\ R$ correspond to graded-maximal ideals $\mathfrak{m}$ not containing the positive part $R_+ = \oplus_{i=1}^\infty R_i$. Intersecting $\mathfrak{m}$ with the part of degree zero $R_0$ determines a point of $\texttt{spec}\ R_0$, the affine variety with coordinate ring $R_0$ and defines a structural morphism

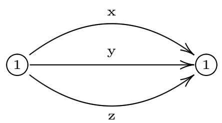
$$\texttt{proj}\ R \longrightarrow \texttt{spec}\ R_0$$

The *Zariski closed subsets* of $\texttt{proj}\ R$ are of the form

$$\mathbb{V}(I) = \{\mathfrak{m} \in \texttt{proj}\ R \mid I \subset \mathfrak{m}\}$$

for a homogeneous ideal $I \triangleleft R$. Further, recall that $\texttt{proj}\ R$ can be covered by affine varieties of the form $\mathbb{X}(f)$ with $f$ a homogeneous element in $R_+$. The coordinate ring of this affine variety is the part of degree zero of the *graded localization* $R_f^g$. We refer to [33, II.2] for more details.

**Example 4.9** Consider again the quiver-situation



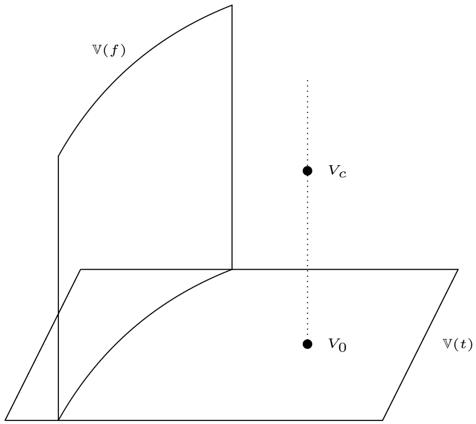



Figure 4.11:

and character $\theta = (-1, 1)$, then the three coordinate functions $x, y$ and $z$ of $\mathbb{C}[\mathtt{rep}_\alpha\ Q]$ are semi-invariants of weight $\chi_\theta$. It is clear that the invariant ring is equal to

$$\mathbb{C}[\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = \mathbb{C}[xt, yt, zt]$$

where the three generators all have degree one. That is,

$$M_\alpha^{ss}(Q,\theta) = \mathtt{proj}\ \mathbb{C}[xt, yt, zt] = \mathbb{P}^2$$

as desired.

We will now investigate which orbits in $\mathtt{rep}_\alpha\ Q$ are parameterized by the moduli space $M_\alpha^{ss}(Q,\theta)$.

**Definition 4.12** *We say that a representation $V \in \mathtt{rep}_\alpha\ Q$ is $\chi_\theta$-*semistable *if and only if there is a semi-invariant $f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$ for some $n \geq 1$ such that $f(V) \neq 0$.*

*The subset of $\mathtt{rep}_\alpha\ Q$ consisting of all $\chi_\theta$-semistable representations will be denoted by $\mathtt{rep}_\alpha^{ss}(Q,\theta)$.*

Observe that $\mathtt{rep}_\alpha^{ss}(Q,\theta)$ is Zariski open (but it may be empty for certain $(\alpha,\theta)$). We can lift a representation $V \in \mathtt{rep}_\alpha\ Q$ to points $V_c = (V, c) \in \mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ and use $GL(\alpha)$-invariant theory on this larger $GL(\alpha)$-module see figure 4.8 Let $c \neq 0$ and assume that the orbit closure $\overline{\mathcal{O}(V_c)}$ does not intersect $\mathbb{V}(t) = \mathtt{rep}_\alpha\ Q \times \{0\}$. As both are $GL(\alpha)$-stable closed subsets of $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ we know from the separation property of invariant theory, proposition 2.10, that this is equivalent to the existence of a $GL(\alpha)$-invariant function $g \in \mathbb{C}[\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)}$ such that $g(\overline{\mathcal{O}(V_c)}) \neq 0$ but $g(\mathbb{V}(t)) = 0$.

We have seen that the invariant ring is graded, hence we may assume $g$ to be homogeneous, that is, of the form $g = ft^n$ for some $n$. But then, $f$ is a semi-invariant on $\mathtt{rep}_\alpha\ Q$ of weight $\chi_\theta^n$ and we see that $V$ must be $\chi_\theta$-semistable. Moreover, we must have that $\theta(\alpha) = \sum_{i=1}^k t_i a_i = 0$, for the one-dimensional central torus of $GL(\alpha)$

$$\mu(t) = (t\mathbb{1}_{a_1}, \ldots, t\mathbb{1}_{a_k}) \hookrightarrow GL(\alpha)$$

acts trivially on $\mathtt{rep}_\alpha\ Q$ but acts on $\mathbb{C}$ via multiplication with $\prod_{i=1}^k t^{-a_i t_i}$ hence if $\theta(\alpha) \neq 0$ then $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) \neq \emptyset$.

More generally, we have from the strong form of the Hilbert criterium proved in theorem 2.2 that $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) = \emptyset$ if and only if for every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ we must have that $\underset{t\to 0}{lim}\ \lambda(t).V_c \notin \mathbb{V}(t)$. We can also formulate this in terms of the $GL(\alpha)$-action on $\mathtt{rep}_\alpha\ Q$. The composition of a one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ with the character

$$\mathbb{C}^* \xrightarrow{\lambda(t)} GL(\alpha) \xrightarrow{\chi\theta} \mathbb{C}^*$$

is an algebraic group morphism and is therefore of the form $t \longrightarrow t^m$ for some $m \in \mathbb{Z}$ and we denote this integer by $\theta(\lambda) = m$. Assume that $\lambda(t)$ is a one-parameter subgroup such that $\underset{t\to 0}{lim}\ \lambda(t).V = V'$ exists in $\mathtt{rep}_\alpha\ Q$, then as

$$\lambda(t).(V, c) = (\lambda(t).V, t^{-m}c)$$

we must have that $\theta(\lambda) \geq 0$ for the orbitclosure $\overline{\mathcal{O}(V_c)}$ not to intersect $\mathbb{V}(t)$.

That is, we have the following characterization of $\chi_\theta$-semistable representations.

**Proposition 4.16** *The following are equivalent*

1. *$V \in \mathtt{rep}_\alpha\ Q$ is $\chi_\theta$-semistable.*
2. *For $c \neq 0$, we have $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) = \emptyset$.*
3. *For every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ we have $\underset{t\to 0}{lim}\ \lambda(t).V_c \notin \mathbb{V}(t) = \mathtt{rep}_\alpha\ Q \times \{0\}$.*
4. *For every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ such that $\underset{t\to 0}{lim}\ \lambda(t).V$ exists in $\mathtt{rep}_\alpha\ Q$ we have $\theta(\lambda) \geq 0$.*

*Moreover, these cases can only occur if $\theta(\alpha) = 0$.*

Assume that $g = ft^n$ is a homogeneous invariant function for the $GL(\alpha)$-action on $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ and consider the affine open $GL(\alpha)$-stable subset $\mathbb{X}(g)$. The construction of the algebraic quotient and the fact that the invariant rings here are graded asserts that the closed $GL(\alpha)$-orbits in $\mathbb{X}(g)$ are classified by the points of the graded localization at $g$ which is of the form

$$(\mathbb{C}[\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)})_g = R_f[h, h^{-1}]$$

for some homogeneous invariant $h$ and where $R_f$ is the coordinate ring of the affine open subset $\mathbb{X}(f)$ in $M^{ss}_\alpha(Q,\theta)$ determined by the semi-invariant $f$ of weight $\chi^n_\theta$. Because the moduli space is covered by such open subsets we have

**Proposition 4.17** *The moduli space of $\theta$-semistable representations of $\mathtt{rep}_\alpha\ Q$*

$$M^{ss}_\alpha(Q,\theta)$$

*classifies closed $GL(\alpha)$-orbits in the open subset $\mathtt{rep}^{ss}_\alpha(Q,\theta)$ of all $\chi_\theta$-semistable representations of $Q$ of dimension vector $\alpha$.*

**Example 4.10** In the foregoing example $\mathtt{rep}^{ss}_\alpha(Q,\theta) = \mathbb{C}^3 - \{(0,0,0)\}$ as for all these points one of the semi-invariant coordinate functions is non-zero. For $\theta = (-1,1)$ the lifted $GL(\alpha) = \mathbb{C}^* \times \mathbb{C}^*$-action to $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C} = \mathbb{C}^4$ is given by

$$(\lambda,\mu).(x,y,z,t) = (\frac{\mu}{\lambda}x, \frac{\mu}{\lambda}y, \frac{\mu}{\lambda}z, \frac{\lambda}{\mu}t)$$

We have seen that the ring of invariants is $\mathbb{C}[xt,yt,zt]$. Consider the affine open set $\mathbb{X}(xt)$ of $\mathbb{C}^4$, then the closed orbits in $\mathbb{X}(xt)$ are classified by

$$\mathbb{C}[xt,yt,zt]^g_{xt} = \mathbb{C}[\frac{y}{x}, \frac{z}{x}][xt, \frac{1}{xt}]$$

and the part of degree zero $\mathbb{C}[\frac{y}{x}, \frac{z}{x}]$ is the coordinate ring of the open set $\mathbb{X}(x)$ in $\mathbb{P}^2$.

We have seen that closed $GL_n$-orbits in $\mathtt{rep}_n\ A$ correspond to semi-simple $n$-dimensional representations. We will now give a representation theoretic interpretation of closed $GL(\alpha)$-orbits in $\mathtt{rep}^{ss}_\alpha(Q,\theta)$.

Again, the starting point is that one-parameter subgroups $\lambda(t)$ of $GL(\alpha)$ correspond to filtrations of representations. Let us go through the motions one more time. For $\lambda : \mathbb{C}^* \longrightarrow GL(\alpha)$ a one-parameter subgroup and $V \in \mathtt{rep}_\alpha\ Q$ we can decompose for every vertex $v_i$ the vertex-space in weight spaces

$$V_i = \oplus_{n\in\mathbb{Z}} V_i^{(n)}$$

where $\lambda(t)$ acts on the weight space $V_i^{(n)}$ as multiplication by $t^n$. This decomposition allows us to define a filtration

$$V_i^{(\geq n)} = \oplus_{m\geq n} V_i^{(m)}$$

For every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$, $\lambda(t)$ acts on the components of the arrow maps

$$V_i^{(n)} \xrightarrow{V_a^{m,n}} V_j^{(m)}$$

by multiplication with $t^{m-n}$. That is, a limit $\underset{t \to 0}{lim}\ V_a$ exists if and only if $V_a^{m,n} = 0$ for all $m < n$, that is, if $V_a$ induces linear maps

$$V_i^{(\geq n)} \xrightarrow{V_a} V_j^{(\geq n)}$$

Hence, a limiting representation exists if and only if the vertex-filtration spaces $V_i^{(\geq n)}$ determine a subrepresentation $V_n \hookrightarrow V$ for all $n$. That is, a one-parameter subgroup $\lambda$ such that $\underset{t \to}{lim}\ \lambda(t).V$ exists determines a decreasing filtration of $V$ by subrepresentations

$$\ldots \hookleftarrow V_n \hookleftarrow V_{n+1} \hookleftarrow \ldots$$

Further, the limiting representation is then the associated graded representation

$$\underset{t \to 0}{lim}\ \lambda(t).V = \oplus_{n \in \mathbb{Z}} \frac{V_n}{V_{n+1}}$$

where of course only finitely many of these quotients can be nonzero. For the given character $\theta = (t_1, \ldots, t_k)$ and a representation $W \in \texttt{rep}_\beta\ Q$ we denote

$$\theta(W) = t_1 b_1 + \ldots + t_k b_k \quad \text{where} \quad \beta = (b_1, \ldots, b_k)$$

Assume that $\theta(V) = 0$, then with the above notations, we have an interpretation of $\theta(\lambda)$ as

$$\theta(\lambda) = \sum_{i=1}^{k} t_i \sum_{n \in Z} n dim_{\mathbb{C}}\ V_i^{(n)} = \sum_{n \in \mathbb{Z}} n\theta(\frac{V_n}{V_{n+1}}) = \sum_{n \in \mathbb{Z}} \theta(V_n)$$

**Definition 4.13** *A representation $V \in \texttt{rep}_\alpha\ Q$ is said to be*

- $\theta$-semistable *if $\theta(V) = 0$ and for all subrepresentations $W \hookrightarrow V$ we have $\theta(W) \geq 0$.*
- $\theta$-stable *if $V$ is $\theta$-semistable and if the only subrepresentations $W \hookrightarrow V$ such that $\theta(W) = 0$ are $V$ and $0$.*

**Proposition 4.18** *For $V \in \texttt{rep}_\alpha\ Q$ the following are equivalent*

1. *$V$ is $\chi_\theta$-semistable.*
2. *$V$ is $\theta$-semistable.*

*Proof.* (1) $\Rightarrow$ (2) : Let $W$ be a subrepresentation of $V$ and let $\lambda$ be the one-parameter subgroup associated to the filtration $V \hookleftarrow W \hookleftarrow 0$, then $\underset{t \to 0}{lim}\ \lambda(t).V$ exists whence by proposition 4.16.4 we have $\theta(\lambda) \geq 0$, but we have

$$\theta(\lambda) = \theta(V) + \theta(W) = \theta(W)$$

(2) $\Rightarrow$ (1) : Let $\lambda$ be a one-parameter subgroup of $GL(\alpha)$ such that $\underset{t \to 0}{lim}\ \lambda(t).V$ exists and consider the induced filtration by subrepresentations $V_n$ defined above. By assumption all $\theta(V_n) \geq 0$, whence

$$\theta(\lambda) = \sum_{n \in Z} \theta(V_n) \geq 0$$

and again proposition 4.16.4 finishes the proof. $\square$

**Lemma 4.15** *Let $V \in \mathtt{rep}_\alpha\ Q$ and $W \in \mathtt{rep}_\beta\ Q$ be both $\theta$-semistable and*

$$V \xrightarrow{f} W$$

*a morphism of representations. Then, $Ker\ f$, $Im\ f$ and $Coker\ f$ are $\theta$-semistable representations.*

*Proof.* Consider the two short exact sequences of representations of $Q$

$$\begin{cases} 0 \longrightarrow Ker\ f \longrightarrow V \longrightarrow Im\ f \longrightarrow 0 \\ 0 \longrightarrow Im\ f \longrightarrow W \longrightarrow Coker\ f \longrightarrow 0 \end{cases}$$

As $\theta(-)$ is additive, we have $0 = \theta(V) = \theta(Ker\ f) + \theta(Im\ f)$ and as both are subrepresentations of $\theta$-semistable representations $V$ resp. $W$, the right-hand terms are $\geq 0$ whence are zero. But then, from the second sequence also $\theta(Coker\ f) = 0$. Being submodules of $\theta$-semistable representations, $Ker\ f$ and $Im\ f$ also satisfy $\theta(S) \geq 0$ for all their subrepresentations $U$. Finally, a subrepresentation $T \hookrightarrow Coker\ f$ can be lifted to a subrepresentation $T' \hookrightarrow W$ and $\theta(T) \geq 0$ follows from the short exact sequence $0 \longrightarrow Im\ f \longrightarrow T' \longrightarrow T \longrightarrow 0$. $\square$

That is, the full subcategory $rep^{ss}(Q,\theta)$ of $rep\ Q$ consisting of all $\theta$-semistable representations is an Abelian subcategory and clearly the simple objects in $rep^{ss}(Q,\theta)$ are precisely the $\theta$-stable representations. As this Abelian subcategory has the necessary finiteness conditions, one can prove a version of the Jordan-Hölder theorem. That is, every $\theta$-semistable representation $V$ has a finite filtration

$$V = V_0 \hookleftarrow V_1 \hookleftarrow \dots \hookleftarrow V_z = 0$$

of subrepresentation such that every factor $\frac{V_i}{V_{i+1}}$ is $\theta$-stable. Moreover, the unordered set of these $\theta$-stable factors are uniquely determined by $V$.

**Theorem 4.20** *For a $\theta$-semistable representation $V \in \mathtt{rep}_\alpha\ Q$ the following are equivalent*

1. *The orbit $\mathcal{O}(V)$ is closed in $\mathtt{rep}^{ss}_\alpha(Q,\alpha)$.*
2. *$V \simeq W_1^{\oplus e_1} \oplus \ldots \oplus W_l^{\oplus e_l}$ with every $W_i$ a $\theta$-stable representation.*

*That is, the geometric points of the moduli space $M^{ss}_\alpha(Q,\theta)$ are in natural one-to-one correspondence with isomorphism classes of $\alpha$-dimensional representations which are direct sums of $\theta$-stable subrepresentations. The quotient map*

$$\mathtt{rep}^{ss}_\alpha(Q,\theta) \twoheadrightarrow M^{ss}_\alpha(Q,\theta)$$

*maps a $\theta$-semistable representation $V$ to the direct sum of its Jordan-Hölder factors in the Abelian category $rep^{ss}(Q,\theta)$.*

*Proof.* Assume that $\mathcal{O}(V)$ is closed in $\mathtt{rep}^{ss}_\alpha(Q,\theta)$ and consider the $\theta$-semistable representation $W = gr_{ss}\ V$, the direct sum of the Jordan-Hölder factors in $rep^{ss}(Q,\theta)$. As $W$ is the associated graded representation of a filtration on $V$, there is a one-parameter subgroup $\lambda$ of $GL(\alpha)$ such that $\underset{t\to 0}{lim}\ \lambda(t).V \simeq W$, that is $\mathcal{O}(W) \subset \overline{\mathcal{O}(V)} = \mathcal{O}(V)$, whence $W \simeq V$ and 2. holds.

Conversely, assume that $V$ is as in 2. and let $\mathcal{O}(W)$ be a closed orbit contained in $\overline{\mathcal{O}(V)}$ (one of minimal dimension). By the Hilbert criterium there is a one-parameter subgroup $\lambda$ in $GL(\alpha)$ such that $\underset{t\to 0}{lim}\ \lambda(t).V \simeq W$. Hence, there is a finite filtration of $V$ with associated graded $\theta$-semistable representation $W$. As none of the $\theta$-stable components of $V$ admits a proper quotient which is $\theta$-semistable (being a direct summand of $W$), this shows that $V \simeq W$ and so $\mathcal{O}(V) = \mathcal{O}(W)$ is closed. The other statements are clear from this. □

Remains to determine the situations $(\alpha,\theta)$ such that the corresponding moduli space $M^{ss}_\alpha(Q,\theta)$ is non-empty, or equivalently, such that the Zariski open subset $\mathtt{rep}^{ss}_\alpha(Q,\theta) \hookrightarrow \mathtt{rep}_\alpha\ Q$ is non-empty.

**Theorem 4.21** *Let $\alpha$ be a dimension vector such that $\theta(\alpha) = 0$. Then,*

1. *$\mathtt{rep}^{ss}_\alpha(Q,\alpha)$ is a non-empty Zariski open subset of $\mathtt{rep}_\alpha\ Q$ if and only if for every $\beta \hookrightarrow \alpha$ we have that $\theta(\beta) \geq 0$.*
2. *The $\theta$-stable representations $\mathtt{rep}^{s}_\alpha(Q,\alpha)$ are a non-empty Zariski open subset of $\mathtt{rep}_\alpha\ Q$ if and only if for every $0 \neq \beta \hookrightarrow \alpha$ we have that $\theta(\beta) > 0$*

The algorithm at the end of the last section gives an inductive procedure to calculate these conditions.

The graded algebra $\mathbb{C}[\mathtt{rep}_\alpha \oplus \mathbb{C}]^{GL(\alpha)}$ of all semi-invariants on $\mathtt{rep}_\alpha\ Q$ of weight $\chi^n_\theta$ for some $n \geq 0$ has as degree zero part the ring of polynomial invariants $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)}$. This embedding determines a *proper morphism*

$$M^{ss}_\alpha(Q,\theta) \xrightarrow{\pi} \mathtt{iss}_\alpha\ Q$$

which is onto whenever $\mathtt{rep}_\alpha^{ss}(Q,\alpha)$ is non-empty. In particular, if $Q$ is a quiver without oriented cycles, then the moduli space of $\theta$-semistable representations of dimension vector $\alpha$, $M_\alpha^{ss}(Q,\theta)$, is a projective variety.

## References.

The étale slices are due to D. Luna [63] and in the form presented here to F. Knop [45] . For more details we refer to the lecture notes of P. Slodowy [79]. The geometric interpretation of Cayley-smoothness is due to C. Procesi [68]. The local structure description with marked quivers is due to L. Le Bruyn [58]. The description of the division algebras of quiver orders is due to L. Le Bruyn and A. Schofield [61]. The material on $A_\infty$-algebras owes to the notes by B. Keller [40] and the work of M. Kontsevich [48] and [47]. The determination of indecomposable dimension vectors is due to V. Kac [37], [38]. The present description owes to the lecture notes of H. Kraft and Ch. Riedtmann [49], W. Crawley-Boevey [19] and the book by P. Gabriel and A.V. Roiter [28]. The main results on the canonical decomposition are due to A. Schofield [73], lemma 4.10 and lemma 4.11 are due to D. Happel and C.M. Ringel [32]. The algorithm is due to H. Derksen and J. Weyman [24] leaning on work of A. Schofield [74].