5 — Semi-Simple Representations

For a Cayley-Hamilton algebra $A \in \mathtt{alg@n}$ we have seen in the first volume that the quotient scheme

$$\mathtt{triss}_n\ A = \mathtt{trep}_n\ A/GL_n$$

classifies isomorphism classes of (trace preserving) semi-simple n-dimensional representations. A point $\xi \in \mathtt{triss}_n\ A$ is said to lie in the Cayley-smooth locus of A if $\mathtt{trep}_n\ A$ is a smooth variety in the semi-simple module M_ξ determined by ξ. In this case, the étale local structure of A and its central subalgebra $tr(A)$ are determined by a marked quiver setting.

We will extend some results on quotient varieties of representations of quivers to the setting of marked quivers. We will give a computational method to verify whether ξ belongs to the Cayley-smooth locus of A and develop reduction steps for the corresponding marked quiver setting which preserve geometric information, such as the type of singularity.

In low dimensions we can give a complete classification of all marked quiver settings which can arise for a Cayley-smooth order, allowing us to determine the classes in the Brauer group of the function field of a projective smooth surface which allow a noncommutative smooth model.

In arbitrary (central) dimension we are able to determine the smooth locus of the center as well as to classify the occurring singularities up to smooth equivalence.

5.1 Representation types

In this section we will determine the étale local structure of quotient varieties of marked quivers, characterize their dimension vectors of simples and introduce the *representation type* of a representation.

We fix a quiver Q and dimension vector α. Closed $GL(\alpha)$-orbits is $\mathtt{rep}_\alpha\ Q$ correspond to isomorphism classes of semi-simple representations of Q of dimension vector α. We have a quotient map

$$\mathtt{rep}_\alpha\ Q \xrightarrow{\pi} \mathtt{rep}_\alpha\ Q/GL(\alpha) = \mathtt{iss}_\alpha\ Q$$

and we know that the coordinate ring $\mathbb{C}[\mathtt{iss}_\alpha\ Q]$ is generated by traces along oriented cycles in the quiver Q. Consider a point $\xi \in \mathtt{iss}_\alpha\ Q$ and assume that the corresponding semi-simple representation V_ξ has a decomposition

$$V_\xi = V_1^{\oplus e_1} \oplus \ldots \oplus V_z^{\oplus e_z}$$

into distinct simple representations V_i of dimension vector say α_i and occurring in V_ξ with multiplicity e_i. We then say that ξ is a point of *representation-type*

$$\tau = t(\xi) = (e_1, \alpha_1; \ldots, e_z, \alpha_z) \quad \text{with} \quad \alpha = \sum_{i=1}^{z} e_i \alpha_i$$

We want to apply the slice theorem to obtain the étale $GL(\alpha)$-local structure of the representation space $\mathtt{rep}_\alpha\, Q$ in a neighborhood of V_ξ and the étale local structure of the quotient variety $\mathtt{iss}_\alpha\, Q$ in a neighborhood of ξ. We have to calculate the normal space N_ξ to the orbit $\mathcal{O}(V_\xi)$ as a representation over the stabilizer subgroup $GL(\alpha)_\xi = Stab_{GL(\alpha)}(V_\xi)$.

Denote $a_i = \sum_{j=1}^{k} a_{ij}$ where $\alpha_i = (a_{i1}, \ldots, a_{ik})$, that is, $a_i = dim\ V_i$. We will choose a basis of the underlying vectorspace

$$\oplus_{v_i \in Q_v} \mathbb{C}^{\oplus e_i a_i} \quad \text{of} \quad V_\xi = V_1^{\oplus e_1} \oplus \ldots \oplus V_z^{\oplus e_z}$$

as follows : the first $e_1 a_1$ vectors give a basis of the vertex spaces of all simple components of type V_1, the next $e_2 a_2$ vectors give a basis of the vertex spaces of all simple components of type V_2, and so on. If $n = \sum_{i=1}^{k} e_i d_i$ is the total dimension of V_ξ, then with respect to this basis, the subalgebra of $M_n(\mathbb{C})$ generated by the representation V_ξ has the following block-decomposition

$$\begin{bmatrix} M_{a_1}(\mathbb{C}) \otimes \mathbb{1}_{e_1} & 0 & \ldots & 0 \\ 0 & M_{a_2}(\mathbb{C}) \otimes \mathbb{1}_{e_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & M_{a_z}(\mathbb{C}) \otimes \mathbb{1}_{e_z} \end{bmatrix}$$

But then, the stabilizer subgroup

$$Stab_{GL(\alpha)}(V_\xi) \simeq GL_{e_1} \times \ldots \times GL_{e_z}$$

embedded in $GL(\alpha)$ with respect to this particular basis as

$$\begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{a_1}) & 0 & \ldots & 0 \\ 0 & GL_{e_2}(\mathbb{C} \otimes \mathbb{1}_{a_2}) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & GL_{e_z}(\mathbb{C} \otimes \mathbb{1}_{a_z}) \end{bmatrix}$$

The tangentspace to the $GL(\alpha)$-orbit in V_ξ is equal to the image of the natural linear map

$$Lie\ GL(\alpha) \longrightarrow \mathtt{rep}_\alpha\, Q$$

sending a matrix $m \in Lie\ GL(\alpha) \simeq M_{e_1} \oplus \ldots \oplus M_{e_k}$ to the representation determined by the commutator $[m, V_\xi] = mV_\xi - V_\xi m$. By this we mean that the matrix $[m, V_\xi]_a$ corresponding to an arrow a is obtained as the commutator in $M_n(\mathbb{C})$ using the canonical embedding with respect to the above choice of basis. The kernel of this linear map is the centralizer subalgebra. That is, we have an exact sequence of $GL(\alpha)_\xi$-modules

$$0 \longrightarrow C_{M_n(\mathbb{C})}(V_\xi) \longrightarrow Lie\ GL(\alpha) \longrightarrow T_{V_\xi}\ \mathcal{O}(V_\xi) \longrightarrow 0$$

where

$$C_{M_n(\mathbb{C})}(V_\xi) = \begin{bmatrix} M_{e_1}(\mathbb{C} \otimes \mathbb{1}_{a_1}) & 0 & \dots & 0 \\ 0 & M_{e_2}(\mathbb{C} \otimes \mathbb{1}_{a_2}) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & M_{e_z}(\mathbb{C} \otimes \mathbb{1}_{a_z}) \end{bmatrix}$$

and the action of $GL(\alpha)_{V_\xi}$ is given by conjugation on $M_n(\mathbb{C})$ via the above embedding. We will now engage in a book-keeping operation counting the factors of the relevant $GL(\alpha)_\xi$-spaces. We identify the factors by the action of the GL_{e_i}-components of $GL(\alpha)_\xi$

1. The centralizer $C_{M_n(\mathbb{C})}(V_\xi)$ decomposes as a $GL(\alpha)_\xi$-module into

   - one factor M_{e_i} on which GL_{e_1} acts via conjugation and the other factors act trivially,

     $\vdots$

   - one factor M_{e_z} on which GL_{e_z} acts via conjugation and the other factors act trivially.

2. Recall the notation $\alpha_i = (a_{i1}, \dots, a_{ik})$,then the Lie algebra *Lie* $GL(\alpha)$ decomposes as a $GL(\alpha)_\xi$-module into

   - $\sum_{j=1}^k a_{1j}^2$ factors M_{e_1} on which GL_{e_1} acts via conjugation and the other factors act trivially,

     $\vdots$

   - $\sum_{j=1}^k a_{zj}^2$ factors M_{e_z} on which GL_{e_z} acts via conjugation and the other factors act trivially,
   - $\sum_{j=1}^k a_{1j}a_{2j}$ factors $M_{e_1 \times e_2}$ on which $GL_{e_1} \times GL_{e_2}$ acts via $\gamma_1.m.\gamma_2^{-1}$ and the other factors act trivially,

     $\vdots$

   - $\sum_{j=1}^k a_{zj}a_{z-1\ j}$ factors $M_{e_z \times e_{z-1}}$ on which $GL_{e_z} \times GL_{e_{z-1}}$ acts via $\gamma_z.m.\gamma_{z-1}^{-1}$ and the other factors act trivially.

3. The representation space $\mathtt{rep}_\alpha\ Q$ decomposes as a $GL(\alpha)_\xi$-modulo into the following factors, for every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in Q (or every loop in v_i by setting $i = j$ in the expressions below) we have
   - $a_{1i}a_{1j}$ factors M_{e_1} on which GL_{e_1} acts via conjugation and the other factors act trivially,
   - $a_{1i}a_{2j}$ factors $M_{e_1 \times e_2}$ on which $GL_{e_1} \times GL_{e_2}$ acts via $\gamma_1.m.\gamma_2^{-1}$ and the other factors act trivially,

   $$\vdots$$

   - $a_{zi}a_{z-1\ j}$ factors $M_{e_z \times e_{z-1}}$ on which $GL_{e_z} \times GL_{e_{z-1}}$ act via $\gamma_z.m.\gamma_{z-1}^{-1}$ and the other factors act trivially,
   - $a_{zi}a_{zj}$ factors M_{e_z} on which GL_{e_z} acts via conjugation and the other factors act trivially.

Removing the factors of 1. from those of 2. we obtain a description of the tangentspace to the orbit $T_{V_\xi}\ \mathcal{O}(V_\xi)$. But then, removing these factors from those of 3. we obtain the description of the normal space N_{V_ξ} as a $GL(\alpha)_\xi$-module as there is an exact sequence of $GL(\alpha)_\xi$-modules

$$0 \longrightarrow T_{V_\xi}\ \mathcal{O}(V_\xi) \longrightarrow \mathtt{rep}_\alpha\ Q \longrightarrow N_{V_\xi} \longrightarrow 0$$

This proves that the normal space to the orbit in V_ξ depends only on the representation type $\tau = t(\xi)$ of the point ξ and can be identified with the representation space of a *local quiver* Q_τ.

Theorem 5.1 *Let $\xi \in \mathtt{iss}_\alpha\ Q$ be a point of representation type*

$$\tau = t(\xi) = (e_1, \alpha_1; \ldots, e_z, \alpha_z)$$

Then, the normal space N_{V_ξ} to the orbit, as a module over the stabilizer subgroup, is identical to the representation space of a local quiver situation

$$N_{V_\xi} \simeq \mathtt{rep}_{\alpha_\tau}\ Q_\tau$$

where Q_τ is the quiver on z vertices (the number of distinct simple components of V_ξ) say $\{w_1, \ldots, w_z\}$ such that in Q_τ

$$\#\ \textcircled{j} \xleftarrow{a} \textcircled{i} \quad = \quad -\chi_Q(\alpha_i, \alpha_j) \quad \text{for } i \neq j, \text{ and}$$

$$\#\ \textcircled{i}\circlearrowleft \quad = \quad 1 - \chi_Q(\alpha_i, \alpha_i)$$

and such that the dimension vector $\alpha_\tau = (e_1, \ldots, e_z)$ (the multiplicities of the simple components in V_ξ).

We can repeat this argument in the case of a marked quiver $Q^\bullet$. The only difference is the description of the factors of $\texttt{rep}_\alpha\ Q^\bullet$ where we need to replace the factors M_{e_j} in the description of a loop in v_i by $M^0_{e_i}$ (trace zero matrices) in case the loop gets a mark in $Q^\bullet$. We define the *Euler form* of the marked quiver $Q^\bullet$

$$\chi^1_{Q^\bullet} = \begin{bmatrix} 1-a_{11} & \chi_{12} & \dots & \chi_{1k} \\ \chi_{21} & 1-a_{22} & \dots & \chi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{k1} & \chi_{k2} & \dots & 1-a_{kk} \end{bmatrix} \qquad \chi^2_{Q^\bullet} = \begin{bmatrix} -m_{11} & & & \\ & -m_{22} & & \\ & & \ddots & \\ & & & -m_{kk} \end{bmatrix}$$

such that $\chi_Q = \chi^1_{Q^\bullet} + \chi^2_{Q^\bullet}$ where Q is the underlying quiver of $Q^\bullet$.

Theorem 5.2 *Let $\xi \in \texttt{iss}_\alpha\ Q^\bullet$ be a point of representation type*

$$\tau = t(\xi) = (e_1, \alpha_1; \dots, e_z, \alpha_z)$$

Then, the normal space N_{V_ξ} to the orbit, as a module over the stabilizer subgroup, is identical to the representation space of a local marked quiver situation

$$N_{V_\xi} \simeq \texttt{rep}_{\alpha_\tau}\ Q^\bullet_\tau$$

where $Q^\bullet_\tau$ is the quiver on z vertices (the number of distinct simple components of V_ξ) say $\{w_1, \dots, w_z\}$ such that in $Q^\bullet_\tau$

$$\# \; \textcircled{\scriptsize j} \xleftarrow{\;a\;} \textcircled{\scriptsize i} \quad = \quad -\chi_Q(\alpha_i, \alpha_j) \quad \text{for } i \neq j, \text{ and}$$

$$\# \begin{tikzcd} \textcircled{\scriptsize i} \arrow[loop above] \end{tikzcd} \quad = \quad 1 - \chi^1_{Q^\bullet}(\alpha_i, \alpha_i)$$

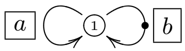
$$\# \begin{tikzcd} \textcircled{\scriptsize i} \arrow[loop above, "\bullet"] \end{tikzcd} \quad = \quad -\chi^2_{Q^\bullet}(\alpha_i, \alpha_i)$$

and such that the dimension vector $\alpha_\tau = (e_1, \dots, e_z)$ (the multiplicities of the simple components in V_ξ).

Proposition 5.1 *If $\alpha = (d_1, \dots, d_k)$ is the dimension vector of a simple representation of $Q^\bullet$, then the dimension of the quotient variety $\texttt{iss}_\alpha\ Q^\bullet$ is equal to*

$$1 - \chi^1_{Q^\bullet}(\alpha, \alpha)$$

Proof. There is a Zariski open subset of $\mathtt{iss}_\alpha\ Q^\bullet$ consisting of points ξ such that the corresponding semi-simple module V_ξ is simple, that is, ξ has representation type $\tau = (1, \alpha)$. But then the local quiver setting (Q_τ, α_τ) is



where $a = 1 - \chi^1_{Q^\bullet}(\alpha, \alpha)$ and $b = -\chi^2_{Q^\bullet}(\alpha, \alpha)$. The corresponding representation space has coordinate ring

$$\mathbb{C}[\mathtt{rep}_{\alpha_\tau}\ Q^\bullet_\tau] = \mathbb{C}[x_1, \ldots, x_a]$$

on which $GL(\alpha_\tau) = \mathbb{C}^*$ acts trivially. That is, the quotient variety is

$$\mathtt{rep}_{\alpha_\tau}\ Q^\bullet_\tau / GL(\alpha_\tau) = \mathtt{rep}_{\alpha_\tau}\ Q^\bullet_\tau \simeq \mathbb{C}^a$$

By the slice theorem, $\mathtt{iss}_\alpha\ Q^\bullet$ has the same local structure near ξ as this quotient space near the origin and the result follows. □

We can extend the classifications of simple roots of a quiver to the setting of marked quivers. Let Q be the underlying quiver of a marked quiver $Q^\bullet$. If $\alpha = (a_1, \ldots, a_k)$ is a simple root of Q and if l is a marked loop in a vertex v_i with $a_i > 1$, then we can replace the matrix V_l of a simple representation $V \in \mathtt{rep}_\alpha\ Q$ by $V'_l = V_l - \frac{1}{d_i}\mathbb{1}_{d_i}$ and retain the property that V' is a simple representation. Things are different, however, for a marked loop in a vertex v_i with $a_i = 1$ as this 1×1-matrix factor is removed from the representation space. That is, we have the following characterization result.

Theorem 5.3 *$\alpha = (a_1, \ldots, a_k)$ is the dimension vector of a simple representation of a marked quiver $Q^\bullet$ if and only if $\alpha = (a_1, \ldots, a_k)$ is the dimension vector of a simple representation of the quiver Q' obtained from the underlying quiver Q of $Q^\bullet$ after removing the loops in Q which are marked in $Q^\bullet$ in all vertices v_i where $a_i = 1$.*

We draw some consequences from the description of the local quiver. We state all results in the setting of marked quivers. Often, the quotient varieties $\mathtt{iss}_\alpha\ Q^\bullet = \mathtt{rep}_\alpha\ Q^\bullet / GL(\alpha)$ classifying isomorphism classes of semi-simple α-dimensional representations have singularities. Still, we can decompose these quotient varieties in smooth pieces according to representation types.

Proposition 5.2 *Let $\mathtt{iss}_\alpha\ Q^\bullet(\tau)$ be the set of points $\xi \in \mathtt{iss}_\alpha\ Q^\bullet$ of representation type*

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$$

Then, $\mathtt{iss}_\alpha\ Q^\bullet(\tau)$ is a locally closed smooth subvariety of $\mathtt{iss}_\alpha\ Q^\bullet$ and

$$\mathtt{iss}_\alpha\ Q^\bullet = \bigsqcup_\tau \mathtt{iss}_\alpha\ Q^\bullet(\tau)$$

is a finite smooth stratification *of the quotient variety.*

Proof. Let $Q^\bullet_\tau$ be the local marked quiver in ξ. Consider a nearby point ξ'. If some trace of an oriented cycles of length > 1 in $Q^\bullet_\tau$ is non-zero in ξ', then ξ' cannot be of representation type τ as it contains a simple factor composed of vertices of that cycle. That is, locally in ξ the subvariety $\mathtt{iss}_\alpha\ Q^\bullet(\tau)$ is determined by the traces of unmarked loops in vertices of the local quiver $Q^\bullet_\tau$ and hence is locally in the étale topology an affine space whence smooth. All other statements are direct. □

Given a *stratification of a topological space* , one wants to determine which strata make up the boundary of a given stratum. For the above stratification of $\mathtt{iss}_\alpha\ Q^\bullet$ we have a combinatorial solution to this problem. Two representation types

$$\tau = (e_1, \alpha_1; \ldots ; e_z, \alpha_z) \quad \text{and} \quad \tau' = (e'_1, \alpha'_1; \ldots ; e'_{z'}, \alpha'_{z'})$$

are said to be direct *successors* $\tau < \tau'$ if and only if one of the following two cases occurs

- (splitting of one simple) : $z' = z + 1$ and for all but one $1 \leq i \leq z$ we have that $(e_i, \alpha_i) = (e'_j, \alpha'_j)$ for a uniquely determined j and for the remaining i_0 we have that the remaining couples of τ' are
$$(e_i, \alpha'_u; e_i, \alpha'_v) \quad \text{with} \quad \alpha_i = \alpha'_u + \alpha'_v$$
- (combining two simple types) : $z' = z - 1$ and for all but one $1 \leq i \leq z'$ we have that $(e'_i, \alpha'_i) = (e_j, \alpha_j)$ for a uniquely determined j and for the remaining i we have that the remaining couples of τ are
$$(e_u, \alpha'_i; e_v, \alpha'_i) \quad \text{with} \quad e_u + e_v = e'_i$$

This direct successor relation $<$ induces an ordering which we will denote with $<<$. Observe that $\tau << \tau'$ if and only if the stabilizer subgroup $GL(\alpha)_\tau$ is conjugated to a subgroup of $GL(\alpha)_{\tau'}$. The following result either follows from general theory, see for example [76, lemma 5.5], or from the description of the local marked quivers.

Proposition 5.3 *The stratum* $\mathtt{iss}_\alpha\ Q^\bullet(\tau')$ *lies in the closure of the stratum* $\mathtt{iss}_\alpha\ Q^\bullet$ *if and only if* $\tau << \tau'$.

Proposition 5.1 gives us the dimensions of the different strata $\mathtt{iss}_\alpha\ Q^\bullet(\tau)$.

Proposition 5.4 *Let* $\tau = (e_1, \alpha_1; \ldots ; e_z, \alpha_z)$ *a representation type of* α. *Then,*

$$dim\ \mathtt{iss}_\alpha\ Q^\bullet(\tau) = \sum_{j=1}^{z} (1 - \chi^1_{Q^\bullet}(\alpha_j, \alpha_j))$$

Because $\mathtt{rep}_\alpha\ Q^\bullet$ and hence $\mathtt{iss}_\alpha\ Q^\bullet$ is an irreducible variety, there is a unique representation type τ^{ss}_{gen} such that $\mathtt{iss}_\alpha\ Q^\bullet(\tau^{ss}_{gen})$ is Zariski open in the quotient variety $\mathtt{iss}_\alpha\ Q^\bullet$. We call τ^{ss}_{gen} the *generic semi-simple representation type* for α. The generic semi-simple representation type can be determined by the following algorithm.

input : A quiver Q, a dimension vector $\alpha = (a_1, \ldots, a_k)$ and a semi-simple representation type

$$\tau = (e_1, \alpha_1; \ldots; e_l, \alpha_l)$$

with $\alpha = \sum +i = 1^l e_i\alpha_i$ and all α_i simple roots for Q. For example, ne can always start with the type $(a_1, \vec{v_1}; \ldots; a_k, \vec{v_k})$.

step 1 : Compute the local quiver Q_τ on l vertices and the dimension vector α_τ. If the only oriented cycles in Q_τ are vertex-loops, stop and output this type. If not, proceed.
step 2 : Take a proper oriented cycle $C = (j_1, \ldots, j_r)$ with $r \geq 2$ in Q_τ where j_s is the vertex in Q_τ determined by the dimension vector α_{j_s}. Set $\beta = \alpha_{j_1} + \ldots + \alpha_{j_r}$, $e'_i = e_i - \delta_{iC}$ where $\delta_{iC} = 1$ if $i \in C$ and is 0 otherwise. replace τ by the new semi-simple representation type

$$\tau' = (e'_1, \alpha_1; \ldots; e'_l, \alpha_l; 1, \beta)$$

delete the terms (e'_i, α_i) with $e'_i = 0$ and set τ to be the resulting type. goto step 1.

The same algorithm extends to marked quivers with the modified construction of the local marked quiver $Q^\bullet_\tau$ in that case. We can give an A_∞-interpretation of the characterization of the *canonical decomposition* and the generic semi-simple representation type . Let

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z) \qquad \alpha = \sum_{i=1}^{z} e_i\alpha_i$$

be a decomposition of α with all the α_i *roots*. We define $\alpha_\tau = (e_1, \ldots, e_z)$ and construct two quivers Q^0_τ and Q^1_τ on z vertices determined by the rules

$$\text{in } Q^0_\tau : \qquad \#\ \textcircled{j} \xleftarrow{a} \textcircled{i} \quad = \quad dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V_i, V_j)$$

$$\text{in } Q^1_\tau : \qquad \#\ \textcircled{j} \xleftarrow{a} \textcircled{i} \quad = \quad dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V_i, V_j)$$

where V_i is a general representation of Q of dimension vector α_i.

Theorem 5.4 *With notations as above, we have :*

1. *The* canonical decomposition τ_{can} *is the unique type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$ such that all α_i are Schur roots, Q^0_τ has no (non-loop) oriented cycles and Q^1_τ has no arrows and loops only in vertices where $e_i = 1$.*

2. *The* generic semi-simple representation type τ^{ss}_{gen} *is the unique type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$ such that all α_i are simple roots, Q^0_τ has only loops and Q^1_τ has no (non-loop) oriented cycles.*

5.2 Cayley-smooth locus

Let A be a Cayley-Hamilton algebra of degree n equipped with a trace map $A \xrightarrow{tr} A$ and consider the quotient map

$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{triss}_n\ A$$

Let ξ be a geometric point of he quotient scheme $\mathtt{triss}_n\ A$ with corresponding n-dimensional trace preserving semi-simple representation V_ξ with decomposition

$$V_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i are distinct simple representations of A of dimension d_i such that $n = \sum_{i=1}^k d_i e_i$.

Definition 5.1 *The Cayley-smooth locus of A is the subset of $\mathtt{triss}_n\ A$*

$$Sm_{tr}\ A = \{\xi \in \mathtt{triss}_n\ A \mid \mathtt{triss}_n\ A \ \ \textit{is smooth along} \ \ \pi^{-1}(\xi)\ \}$$

As the singular locus of $\mathtt{triss}_n\ A$ is a GL_n-stable closed subscheme of $\mathtt{triss}_n\ A$ this is equivalent to

$$Sm_{tr}\ A = \{\xi \in \mathtt{triss}_n\ A \mid \mathtt{triss}_n\ A \ \ \textit{is smooth in} \ \ V_\xi\ \}$$

We will give some numerical conditions on ξ to be in the smooth locus $Sm_{tr}\ A$. To start, $\mathtt{trep}_n\ A$ is smooth in V_ξ if and only if the dimension of the tangent space in V_ξ is equal to the local dimension of $\mathtt{trep}_n\ A$ in V_ξ. From example 3.11 we know that the tangent space is the set of trace preserving derivations $A \xrightarrow{D} M_n(\mathbb{C})$ satisfying

$$D(aa') = D(a)\rho(a') + \rho(a)D(a')$$

where $A \xrightarrow{\rho} M_n(\mathbb{C})$ is the $\mathbb{C}$-algebra morphism determined by the action of A on V_ξ. The $\mathbb{C}$-vectorspace of such derivations is denoted by $Der^t_\rho\ A$. Therefore,

$$\xi \in Sm_{tr}\ A \Longleftrightarrow dim_{\mathbb{C}}\ Der^t_\rho\ A = dim_{V_\xi}\ \mathtt{trep}_n\ A$$

Next, if $\xi \in Sm_{tr}\ A$, then we know from the slice theorem that the local GL_n-structure of $\mathtt{trep}_n\ A$ near V_ξ is determined by a local marked quiver setting $(Q^\bullet_\xi, \alpha_\xi)$ as defined in theorem 4.3. We have local étale isomorphisms between the varieties

$$GL_n \times^{GL(\alpha_\xi)} \mathtt{rep}_{\alpha_\xi}\ Q^\bullet_\xi \xleftrightarrow{et} \mathtt{trep}_n\ A \quad \text{and} \quad \mathtt{rep}_{\alpha_\xi}\ Q^\bullet_\xi/GL(\alpha_\xi) \xleftrightarrow{et} \mathtt{triss}_n\ A$$

Which gives us the following numerical restrictions on $\xi \in Sm_{tr}\ A$:

Proposition 5.5 *$\xi \in Sm_{tr}\ A$ if and only if the following two equalities hold*

$$\begin{cases} dim_{V_\xi}\ \mathtt{trep}_n\ A & = n^2 - (e_1^2 + \ldots + e_k^2) + dim_{\mathbb{C}}\ Ext_A^{tr}(V_\xi, V_\xi) \\ dim_\xi\ \mathtt{triss}_n\ A & = dim_{\overline{0}}\ \mathtt{rep}_{\alpha_\xi}\ Q_\xi^\bullet / GL(\alpha_\xi) = dim_{\underline{0}}\ \mathtt{iss}_{\alpha_\xi} Q_\xi^\bullet \end{cases}$$

Moreover, if $\xi \in Sm_{tr}\ A$, then $\mathtt{trep}_n\ A$ is a normal variety (that is, the coordinate ring is integrally closed) in a neighborhood of ξ

Proof. The last statement follows from the fact that $\mathbb{C}[\mathtt{rep}_{\alpha_\xi}\ Q_\xi^\bullet]^{GL(\alpha_\xi)}$ is integrally closed and this property is preserved under the étale map. $\square$

In general, the difference between these numbers gives a measure for the noncommutative singularity of A in ξ.

Example 5.1 (Quantum plane of order 2) Consider the affine $\mathbb{C}$-algebra $A = \frac{\mathbb{C}\langle x,y\rangle}{(xy+yx)}$ then $u = x^2$ and $v = y^2$ are central elements of A and A is a free module of rank 4 over $\mathbb{C}[u,v]$. In fact, A is a $\mathbb{C}[u,v]$-order in the quaternion division algebra

$$\Delta = \begin{pmatrix} u & & v \\ & \mathbb{C}(u,v) & \end{pmatrix}$$

and the reduced trace map on Δ makes A into a Cayley-Hamilton algebra of degree 2. More precisely, tr is the linear map on A such that

$$\begin{cases} tr(x^iy^j) = 0 & \text{if either } i \text{ or } j \text{ are odd, and} \\ tr(x^iy^j) = 2x^iy^j & \text{if } i \text{ and } j \text{ are even.} \end{cases}$$

In particular, a trace preserving 2-dimensional representation is determined by a couple of 2×2 matrices

$$\rho = (\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} , \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix}) \quad \text{with} \quad tr(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} . \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix}) = 0$$

That is,$\mathtt{trep}_2\ A$ is the hypersurface in $\mathbb{C}^6$ determined by the equation

$$\mathtt{trep}_2\ A = \mathbb{V}(2x_1x_4 + x_2x_6 + x_3x_5) \hookrightarrow \mathbb{C}^6$$

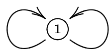
and is therefore irreducible of dimension 5 with an isolated singularity at $p = (0, \ldots, 0)$. The image of the trace map is equal to the center of A which is $\mathbb{C}[u,v]$ and the quotient map

$$\mathtt{trep}_2\ A \xrightarrow{\pi} \mathtt{triss}_2\ A = \mathbb{C}^2 \qquad \pi(x_1, \ldots, x_6) = (x_1^2 + x_2x_3, x_4^2 + x_5x_6)$$

There are three different representation types to consider. Let $\xi = (a,b) \in \mathbb{C}^2 = \mathtt{triss}_2\ A$ with $ab \neq 0$, then $\pi^{-1}(\xi)$ is a closed GL_2-orbit and a corresponding simple A-module is given by the matrixcouple

$$(\begin{bmatrix} i\sqrt{a} & 0 \\ 0 & -i\sqrt{a} \end{bmatrix} , \begin{bmatrix} 0 & \sqrt{b} \\ -\sqrt{b} & 0 \end{bmatrix})$$

That is, ξ is of type $(1,2)$ and the stabilizer subgroup are the scalar matrixes $\mathbb{C}^*\mathbb{1}_2 \hookrightarrow GL_2$. So, the action on both the tangentspace to $\mathtt{trep}_2\ A$ and the tangent space to the orbit are trivial. As they have respectively dimension 5 and 3, the normalspace corresponds to the quiver setting

$N_\xi =$ 

which is compatible with the numerical restrictions. Next, consider a point $\xi = (0,b)$ (or similarly, $(a,0)$), then ξ is of type $(1,1;1,1)$ and the corresponding semi-simple representation is given by the matrices

$$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} i\sqrt{b} & 0 \\ 0 & -i\sqrt{b} \end{bmatrix})$$

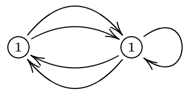
The stabilizer subgroup is in this case the maximal torus of diagonal matrices $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow GL_2$. The tangent space in this point to $\mathtt{trep}_2\ A$ are the 6-tuples $(a_1, \ldots, a_6)$ such that

$$tr\ (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{bmatrix}).(\begin{bmatrix} i\sqrt{b} & 0 \\ 0 & -i\sqrt{b} \end{bmatrix} + \epsilon \begin{bmatrix} b_4 & b_5 \\ b_6 & -b_4 \end{bmatrix}) = 0 \quad \text{where } \epsilon^2 = 0$$

This leads to the condition $a_1 = 0$, so the tangentspace are the matrix couples

$$(\begin{bmatrix} 0 & a_2 \\ a_3 & 0 \end{bmatrix} , \begin{bmatrix} a_4 & a_5 \\ a_6 & -a_4 \end{bmatrix}) \quad \text{on which the stabilizer } \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

acts via conjugation. That is, the tangentspace corresponds to the quiver setting



Moreover, the tangentspace to the orbit is the image of the linear map

$$(\mathbb{1}_2 + \epsilon \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}).(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{bmatrix}).(\mathbb{1}_2 - \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix})$$

which is equal to

$$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{bmatrix} + \epsilon \begin{bmatrix} 0 & -2m_2\sqrt{b} \\ 2m_3\sqrt{b} & 0 \end{bmatrix})$$

on which the stabilizer acts again via conjugation giving the quiver setting

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left] & \textcircled{1} \arrow[l, bend left] \end{tikzcd}$$

Therefore, the normal space to the orbit corresponds to the quiver setting

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left] & \textcircled{1} \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

which is again compatible with the numerical restrictions. Finally, consider $\xi = (0,0)$ which is of type $(2,1)$ and whose semi-simple representation corresponds to the zero matrix-couple. The action fixes this point, so the stabilizer is GL_2 and the tangent space to the orbit is the trivial space. Hence, the tangent space to $\mathtt{trep}_2\ A$ coincides with the normalspace to the orbit and both spaces are acted on by GL_2 via simultaneous conjugation leading to the quiver setting

$$N_\xi = \begin{tikzcd} \textcircled{2} \arrow[loop above] \arrow[loop below] \end{tikzcd}$$

This time, the data is not compatible with the numerical restriction as

$$5 = dim\ \mathtt{trep}_2\ A \neq n^2 - e^2 + dim\ \mathtt{rep}_\alpha\ Q^\bullet = 4 - 4 + 6$$

consistent with the fact that the zero matrix-couple is a (in fact, the only) singularity on $\mathtt{trep}_2\ A$.

We will put additional conditions on the Cayley-Hamilton algebra A. Let X be a normal affine variety with coordinate ring $\mathbb{C}[X]$ and functionfield $\mathbb{C}(X)$. Let Δ be a central simple $\mathbb{C}(X)$-algebra of dimension n^2 which is a Cayley-Hamilton algebra of degree n using the reduced trace map tr. Let A be a $\mathbb{C}[X]$-order in Δ, that is, the center of A is $\mathbb{C}[X]$ and $A \otimes_{\mathbb{C}[X]} \mathbb{C}(X) \simeq \Delta$. Because $\mathbb{C}[X]$ is integrally closed, the restriction of the reduced trace tr to A has its image in $\mathbb{C}[X]$, that is, A is a Cayley-Hamilton algebra of degree n and

$$tr(A) = \mathbb{C}[X]$$

Consider the quotient morphism for the representation variety

$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{triss}_n\ A$$

then the above argument shows that $X \simeq \mathtt{triss}_n\ A$ and in particular the quotient scheme is reduced.

Proposition 5.6 *Let A be a Cayley-Hamilton order of degree n over $\mathbb{C}[X]$. Then, its smooth locus $Sm_{tr}\ A$ is a nonempty Zariski open subset of X. In particular, the set X_{az} of Azumaya points, that is, of points $x \in X = \mathtt{triss}_n\ A$ of representation type $(1,n)$ is a non-empty Zariski open subset of X and its intersection with the Zariski open subset X_{reg} of smooth points of X satisfies*

$$X_{az} \cap X_{reg} \hookrightarrow Sm_{tr}\ A$$

Proof. Because $A\mathbb{C}(X) = \Delta$, there is an $f \in \mathbb{C}[X]$ such that $A_f = A \otimes_{\mathbb{C}[X]} \mathbb{C}[X]_f$ is a free $\mathbb{C}[X]_f$-module of rank n^2 say with basis $\{a_1, \ldots, a_{n^2}\}$. Consider the $n^2 \times n^2$ matrix with entries in $\mathbb{C}[X]_f$

$$R = \begin{bmatrix} tr(a_1a_1) & \ldots & tr(a_1a_{n^2}) \\ \vdots & & \vdots \\ tr(a_{n^2}a_1) & \ldots & tr(a_{n^2}a_{n^2}) \end{bmatrix}$$

The determinant $d = det\ R$ is nonzero in $\mathbb{C}[X]_f$. For, let $\mathbb{K}$ be the algebraic closure of $\mathbb{C}(X)$ then $A_f \otimes_{\mathbb{C}[X]_f} \mathbb{K} \simeq M_n(\mathbb{K})$ and for any $\mathbb{K}$-basis of $M_n(\mathbb{K})$ the corresponding matrix is invertible (for example, verify this on the matrixes e_{ij}). As $\{a_1, \ldots, a_{n^2}\}$ is such a basis, $d \neq 0$. Next, consider the Zariski open subset $U = \mathbb{X}(f) \cap \mathbb{X}(d) \hookrightarrow X$. For any $x \in X$ with maximal ideal $\mathfrak{m}_x \triangleleft \mathbb{C}[X]$ we claim that

$$\frac{A}{A\mathfrak{m}_x A} \simeq M_n(\mathbb{C})$$

Indeed, the images of the a_i give a $\mathbb{C}$-basis in the quotient such that the $n^2 \times n^2$-matrix of their product-traces is invertible. This property is equivalent to the quotient being $M_n(\mathbb{C})$. The corresponding semi-simple representation of A is simple, proving that X_{az} is a non-empty Zariski open subset of X. But then, over U the restriction of the quotient map

$$\mathtt{trep}_n\ A \mid \pi^{-1}(U) \twoheadrightarrow U$$

is a principal PGL_n-fibration. In fact, this restricted quotient map determines an element in $H^1_{et}(U, PGL_n)$ determining the class of the central simple $\mathbb{C}(X)$-algebra Δ in $H^1_{et}(\mathbb{C}(X), PGL_n)$. Restrict this quotient map further to $U \cap X_{reg}$, then the PGL_n-fibration

$$\mathtt{trep}_n\ A \mid \pi^{-1}(U \cap X_{reg}) \twoheadrightarrow U \cap X_{reg}$$

has a smooth base and therefore also the total space is smooth. But then, $U \cap X_{reg}$ is a non-empty Zariski open subset of $Sm_{tr}\ A$. $\square$

Observe that the normality assumption on X is no restriction as the quotient scheme is locally normal in a point of $Sm_{tr}\ A$. Our next result limits the local dimension vectors α_ξ.

Proposition 5.7 *Let A be a Cayley-Hamilton order and $\xi \in Sm_{tr}\ A$ such that the normal space to the orbit of the corresponding semi-simple n-dimensional representation is*

$$N_\xi = \mathtt{rep}_{\alpha_\xi}\ Q_\xi^\bullet$$

Then, α_ξ is the dimension vector of a simple representation of $Q^\bullet_\xi$.

Proof. Let V_ξ be the semi-simple representation of A determined by ξ. Let S_ξ be the slice variety in V_ξ then by the slice theorem we have the following diagram of étale GL_n-equivariant maps

$$\begin{tikzcd} & GL_n \times^{GL(\alpha_\xi)} S_\xi \arrow[dl, "et"'] \arrow[dr, "et"] & \\ GL_n \times^{GL(\alpha_\xi)} \mathtt{rep}_{\alpha_\xi}\ Q^\bullet_\xi & & \mathtt{trep}_n\ A \end{tikzcd}$$

linking a neighborhood of V_ξ with one of $\overline{(\mathbb{1}_n, 0)}$. Because A is an order, every Zariski neighborhood of V_ξ in $\mathtt{trep}_n\ A$ contains simple n-dimensional representations, that is, closed GL_n-orbits with stabilizer subgroup isomorphic to $\mathbb{C}^*$. Transporting this property via the GL_n-equivariant étale maps, every Zariski neighborhood of $\overline{(\mathbb{1}_n, 0)}$ contains closed GL_n-orbits with stabilizer $\mathbb{C}^*$. By the correspondence of orbits is associated fiber bundles, every Zariski neighborhood of the trivial representation $0 \in \mathtt{rep}_{\alpha_\xi}\ Q^\bullet_\xi$ contains closed $GL(\alpha_\xi)$-orbits with stabilizer subgroup $\mathbb{C}^*$. We have seen that closed $GL(\alpha_\xi)$-orbits correspond to semi-simple representations of $Q^\bullet_\xi$. However, if the stabilizer subgroup of a semi-simple representation is $\mathbb{C}^*$ this representation must be simple. □

Theorem 5.5 *Let A be a Cayley-Hamilton order of degree n with center $\mathbb{C}[X]$, X a normal variety of dimension d. For $\xi \in X = \mathtt{triss}_n\ A$ with corresponding semi-simple representation*

$$V_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

and normal space to the orbit $\mathcal{O}(V_\xi)$ isomorphic to $\mathtt{rep}_{\alpha_\xi}\ Q^\bullet_\xi$ as $GL(\alpha_\xi)$-modules where $\alpha_\xi = (e_1, \ldots, e_k)$. Then, $\xi \in Sm_{tr}\ A$ if and only if the following two conditions are met

$$\begin{cases} \alpha_\xi & \text{is the dimension vector of a simple representation of } Q^\bullet\text{, and} \\ d & = 1 - \chi_Q(\alpha_\xi, \alpha_\xi) - \sum_{i=1}^k m_{ii} \end{cases}$$

where Q is the underlying quiver of $Q^\bullet_\xi$ and m_{ii} is the number of marked loops in $Q^\bullet_\xi$ in vertex v_i.

Proof. By the slice theorem we have étale maps

$$\mathtt{rep}_{\alpha_\xi}\ Q^\bullet_\xi / GL(\alpha_\xi) \xleftarrow{et} S_\xi / GL(\alpha_\xi) \xrightarrow{et} \mathtt{triss}_n\ A = X$$

connecting a neighborhood of $\xi \in X$ with one of the trivial semi-simple representation $\overline{0}$. By definition of the Euler-form of Q we have that

$$\chi_Q(\alpha_\xi, \alpha_\xi) = -\sum_{i \neq j} e_i e_j \chi_{ij} + \sum_i e_i^2 (1 - a_{ii} - m_{ii})$$

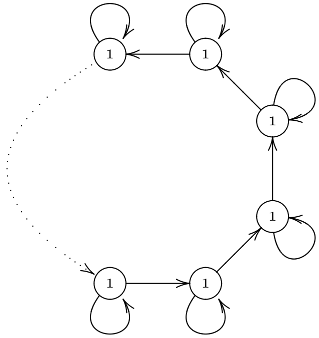


Figure 5.1: Ext-quiver of quantum plane

On the other hand we have

$$dim\ \mathtt{rep}_{\alpha}\ Q^{\bullet}_{\alpha_\xi} = \sum_{i \neq j} e_i e_j \chi_{ij} + \sum_i e_i^2 (a_{ii} + m_{ii}) - \sum_i m_{ii}$$

$$dim\ GL(\alpha_\xi) = \sum_i e_i^2$$

As any Zariski open neighborhood of ξ contains an open set where the quotient map is a $PGL(\alpha_\xi) = \frac{GL(\alpha_\xi)}{\mathbb{C}^*}$-fibration we see that the quotient variety $\mathtt{rep}_{\alpha_\xi}\ Q^{\bullet}_{\xi}$ has dimension equal to

$$dim\ \mathtt{rep}_{\alpha_\xi}\ Q^{\bullet}_{\xi} - dim\ GL(\alpha_\xi) + 1$$

and plugging in the above information we see that this is equal to $1 - \chi_Q(\alpha_\xi, \alpha_\xi) - \sum_i m_{ii}$. $\square$

Example 5.2 (Quantum plane) We will generalize the discussion of example 5.1 to the algebra

$$A = \frac{\mathbb{C}\langle x, y \rangle}{(yx - qxy)}$$

where q is a primitive n-th root of unity. Let $u = x^n$ and $v = y^n$ then it is easy to see that A is a free module of rank n^2 over its center $\mathbb{C}[u, v]$ and is a Cayley-Hamilton algebra of degree n with the trace determined on the basis

$$tr(x^i y^j) = \begin{cases} 0 & \text{when either } i \text{ or } j \text{ is not a multiple of } n, \\ n x^i y^j & \text{when } i \text{ and } j \text{ are multiples of } n, \end{cases}$$

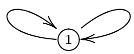
Let $\xi \in \mathtt{iss}_n\ A = \mathbb{C}^2$ be a point (a^n, b) with $a.b \neq 0$, then ξ is of representation type $(1, n)$ as the corresponding (semi)simple representation V_ξ is determined by (if m is odd, for even n we replace a by ia and b by $-b$)

$$\rho(x) = \begin{bmatrix} a & & & \\ & qa & & \\ & & \ddots & \\ & & & q^{n-1}a \end{bmatrix} \quad \text{and} \quad \rho(y) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ b & 0 & 0 & \dots & 0 \end{bmatrix}$$

One computes that $Ext^1_A(V_\xi, V_\xi) = \mathbb{C}^2$ where the algebra map $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ corresponding to (α, β) is given by

$$\begin{cases} \phi(x) &= \rho(x) + \varepsilon\ \alpha \mathbb{1}_n \\ \phi(y) &= \rho(y) + \varepsilon\ \beta \mathbb{1}_n \end{cases}$$

and all these algebra maps are trace preserving. That is, $Ext^1_A(V_\xi, V_\xi) = Ext^{tr}_A(V_\xi, V_\xi)$ and as the stabilizer subgroup is $\mathbb{C}^*$ the marked quiver-setting $(Q^\bullet_\xi, \alpha_\xi)$ is



and $d = 1 - \chi_Q(\alpha, \alpha) - \sum_i m_{ii}$ as $2 = 1 - (-1) + 0$, compatible with the fact that over these points the quotient map is a principal PGL_n-fibration.

Next, let $\xi = (a^n, 0)$ with $a \neq 0$ (or, by a similar argument $(0, b^n)$ with $b \neq 0$). Then, the representation type of ξ is $(1, 1; \dots; 1, 1)$ because

$$V_\xi = S_1 \oplus \dots \oplus S_n$$

where the simple one-dimensional representation S_i is given by

$$\begin{cases} \rho(x) &= q^i a \\ \rho(y) &= 0 \end{cases}$$

One verifies that

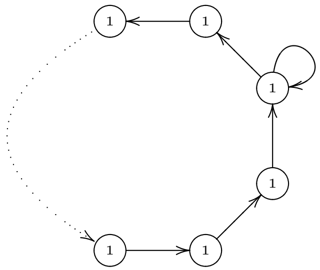
$$Ext^1_A(S_i, S_i) = \mathbb{C} \quad \text{and} \quad Ext^1_A(S_i, S_j) = \delta_{i+1,j}\ \mathbb{C}$$

and as the stabilizer subgroup is $\mathbb{C}^* \times \dots \times \mathbb{C}^*$, the Ext-quiver setting is depicted in figure 5.1. The algebra map $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ corresponding to the extension $(\alpha_1, \beta_1, \dots, \alpha_n, \beta_n) \in Ext^1_A(V_\xi, V_\xi)$

is given by

$$\begin{cases} \phi(x) &= \begin{bmatrix} a+\varepsilon\,\alpha_1 & & & \\ & qa+\varepsilon\,\alpha_2 & & \\ & & \ddots & \\ & & & q^{n-1}a+\varepsilon\,\alpha_n \end{bmatrix} \\ \phi(y) &= \varepsilon \begin{bmatrix} 0 & \beta_1 & 0 & \dots & 0 \\ 0 & 0 & \beta_2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & \beta_{n-1} \\ \beta_n & 0 & 0 & \dots & 0 \end{bmatrix} \end{cases}$$

The conditions $tr(x^j) = 0$ for $1 \leq i < n$ impose $n-1$ linear conditions among the α_j, whence the space of trace preserving extensions $Ext^{tr}_A(V_\xi, V_\xi)$ corresponds to the quiver setting



The Euler-form of this quiver $Q^\bullet$ is given by the $n \times n$ matrix

$$\begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ & 1 & -1 & & 0 \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{bmatrix}$$

giving the numerical restriction as $\alpha_\xi = (1, \dots, 1)$

$$1 - \chi_Q(\alpha, \alpha) - \sum_i m_{ii} = 1 - (-1) - 0 = 2 = dim\ \mathtt{triss}_n\ A$$

so $\xi \in Sm_{tr}\ A$. Finally, the only remaining point is $\xi = (0,0)$. This has representation type $(n,1)$ as the corresponding semi-simple representation V_ξ is the trivial one. The stabilizer subgroup is

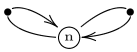
GL_n and the (trace preserving) extensions are given by

$$Ext^1_A(V_\xi, V_\xi) = M_n \oplus M_n \quad \text{and} \quad Ext^{tr}_A(V_\xi, V_\xi) = M_n^0 \oplus M_n^0$$

determined by the algebra maps $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ given by

$$\begin{cases} \phi(x) & = \varepsilon\ m_1 \\ \phi(y) & = \varepsilon\ m_2 \end{cases}$$

That is, the relevant quiver setting $(Q^\bullet_\xi, \alpha_\xi)$ is in this point



This time, $\xi \notin Sm_{tr}\ A$ as the numerical condition fails

$$1 - \chi_Q(\alpha,\alpha) - \sum_i m_{ii} = 1 - (-n^2) - 0 \neq 2 = dim\ \mathtt{triss}_n\ A$$

unless $n = 1$. That is, $Sm_{tr}\ A = \mathbb{C}^2 - \{(0,0)\}$.

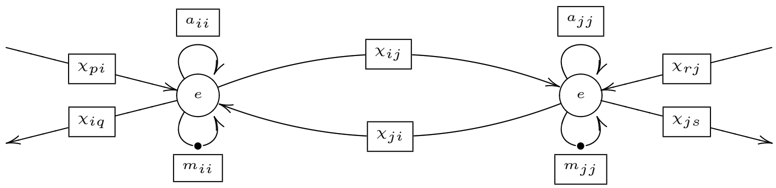
5.3 Reduction steps

If we want to study the local structure of Cayley-Hamilton orders A of degree n over a central normal variety X of dimension d, we have to compile a list of admissible marked quiver settings, that is couples $(Q^\bullet, \alpha)$ satisfying the two properties

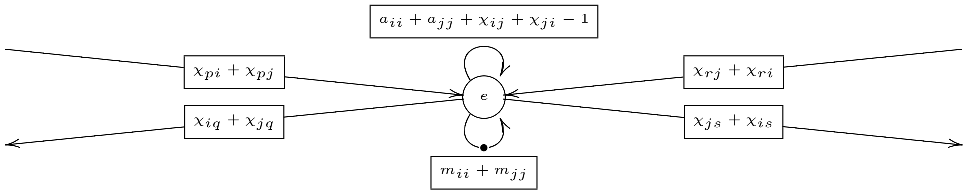
$$\begin{cases} \alpha & \text{is the dimension vector of a simple representation of } Q^\bullet\text{, and} \\ d & = 1 - \chi_Q(\alpha,\alpha) - \sum_i m_i \end{cases}$$

In this section, we will give two methods to start this classification project.

The first idea is to shrink a marked quiver-setting to its simplest form and classify these simplest forms for given d. By *shrinking* we mean the following process. Assume $\alpha = (e_1, \dots, e_k)$ is the dimension vector of a simple representation of $Q^\bullet$ and let v_i and v_j be two vertices connected with an arrow such that $e_i = e_j = e$. That is, locally we have the following situation



We will use one of the arrows connecting v_i with v_j to identify the two vertices. That is, we form the shrinked marked quiver-setting $(Q_s^\bullet, \alpha_s)$ where $Q_s^\bullet$ is the marked quiver on $k-1$ vertices $\{v_1, \ldots, \hat{v_i}, \ldots, v_k\}$ and α_s is the dimension vector with e_i removed. $Q_s^\bullet$ has the following form in a neighborhood of the contracted vertex



In $Q_s^\bullet$ we have for all $k, l \neq i, j$ that $\chi_{kl}^s = \chi_{kl}$, $a_{kk}^s = a_{kk}$, $m_{kk}^s = m_{kk}$ and the number of arrows and (marked) loops connected to v_j are determined as follows

- $\chi_{jk}^s = \chi_{ik} + \chi_{jk}$
- $\chi_{kj}^s = \chi_{ki} + \chi_{kj}$
- $a_{jj}^s = a_{ii} + a_{jj} + \chi_{ij} + \chi_{ji} - 1$
- $m_{jj}^s = m_{ii} + m_{jj}$

Lemma 5.1 *α is the dimension vector of a simple representation of $Q^\bullet$ if and only if α_s is the dimension vector of a simple representation of $Q_s^\bullet$. Moreover,*

$$dim\ \mathtt{rep}_\alpha\ Q^\bullet/GL(\alpha) = dim\ \mathtt{rep}_{\alpha_s}\ Q_s^\bullet/GL(\alpha_s)$$

Proof. Fix an arrow $(j) \xleftarrow{a} (i)$. As $e_i = e_j = e$ there is a Zariski open subset $U \hookrightarrow \mathtt{rep}_\alpha\ Q^\bullet$ of points V such that V_a is invertible. By basechange in either v_i or v_j we can find a point W in its orbit such that $W_a = \mathbb{1}_e$. If we think of W_a as identifying $\mathbb{C}^{e_i}$ with $\mathbb{C}^{e_j}$ we can view the remaining maps of W as a representation in $\mathtt{rep}_{\alpha_s}\ Q_s^\bullet$ and denote it by W^s. The map $U \longrightarrow \mathtt{rep}_{\alpha_s}\ Q_s^\bullet$ is well-defined and maps $GL(\alpha)$-orbits to $GL(\alpha_s)$-orbits. Conversely, given a representation $W' \in \mathtt{rep}_{\alpha_s}\ Q_s^\bullet$ we can uniquely determine a representation $W \in U$ mapping to W'. Both claims follow immediately from this observation. □

A marked quiver-setting can uniquely be shrinked to its *simplified form* , which has the characteristic property that no arrow-connected vertices can have the same dimension. The shrinking process has a converse operation which we will call *splitting of a vertex* . However, this splitting operation is usually not uniquely determined.

Before compiling a lists of marked-quiver settings in simplified form for a specific base-dimension d, we bound the components of α.

Proposition 5.8 *Let $\alpha = (e_1, \ldots, e_k)$ be the dimension vector of a simple representation of Q and let $1 - \chi_Q(\alpha, \alpha) = d = dim\ \mathtt{rep}_\alpha\ Q/GL(\alpha)$. Then, if $e = max\ e_i$, we have that $d \geq e + 1$.*

Proof. By lemma 5.1 we may assume that (Q, α) is brought in its simplified form, that is, no two arrow-connected vertices have the same dimension. Let χ_{ii} denote the number of loops in a vertex v_i, then

$$-\chi_Q(\alpha, \alpha) = \begin{cases} \sum_i\ e_i\ (\sum_j\ \chi_{ij} e_j - e_i) \\ \sum_i\ e_i\ (\sum_j\ \chi_{ji} e_j - e_i) \end{cases}$$

and observe that the bracketed terms are positive by the requirement that α is the dimension vector of a simple representation. We call them the incoming in_i, respectively outgoing out_i, contribution of the vertex v_i to d. Let v_m be a vertex with maximal vertex-dimension e.

$$in_m = e(\sum_{j \neq m} \chi_{jm} e_j + (\chi_{ii} - 1)e) \quad \text{and} \quad out_m = e(\sum_{j \neq m} \chi_{ij} e_j + (\chi_{ii} - 1)e)$$

If there are loops in v_m, then $in_m \geq 2$ or $out_m \geq 2$ unless the local structure of Q is

$$\begin{tikzcd} \textcircled{1} \arrow[r] & \textcircled{e} \arrow[r] \arrow[loop above] & \textcircled{1} \end{tikzcd}$$

in which case $in_m = e = out_m$. Let v_i be the unique incoming vertex of v_m, then we have $out_i \geq e - 1$. But then,

$$d = 1 - \chi_Q(\alpha, \alpha) = 1 + \sum_j out_j \geq 2e$$

If v_m has no loops, consider the incoming vertices $\{v_{i_1}, \ldots, v_{i_s}\}$, then

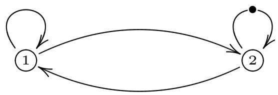
$$in_m = e(\sum_{j=1}^{s} \chi_{i_j m} e_{i_j} - e)$$

which is $\geq e$ unless $\sum \chi_{i_j m} e_{i_j} = e$, but in that case we have

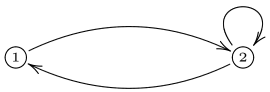
$$\sum_{j=1}^{s} out_{i_j} \geq e^2 - \sum_{j=1}^{s} e_{i_j}^2 \geq e$$

the last inequality because all $e_{i_j} < e$. In either case we have that $d = 1 - \chi_Q(\alpha, \alpha) = 1 + \sum_i out_i = 1 + \sum_i in_i \geq e + 1$. $\square$

Example 5.3 In a list of simplified marked quivers we are only interested in $\mathtt{rep}_\alpha\ Q^\bullet$ as $GL(\alpha)$-module and we call two setting *equivalent* if they determine the same $GL(\alpha)$-module. For example, the marked quiver-settings



and



determine the same $\mathbb{C}^* \times GL_2$-module, hence are equivalent.

Theorem 5.6 *Let A be a Cayley-Hamilton order of degree n over a central normal variety X of degree d. Then, the local quiver of A in a point $\xi \in X = \mathtt{triss}_n\ A$ belonging to the smooth locus $Sm_{tr}\ A$ can be shrinked to one of a finite list of equivalence classes of simplified marked quiver-settings. For $d \leq 4$, the complete list is given in figure 5.2 where the boxed value is the dimension d of X.*

An immediate consequence is a noncommutative analog of the fact that commutative smooth varieties have only one type of analytic (or étale) local behavior.

Theorem 5.7 *There are only finitely many types of étale local behavior of smooth Cayley-Hamilton orders of degree n over a central variety of dimension d.*

Proof. The foregoing reduction shows that for fixed d there are only a finite number of marked quiver-settings shrinked to their simplified form. As $\sum e_i \leq n$, we can apply the splitting operations on vertices only a finite number of times. □

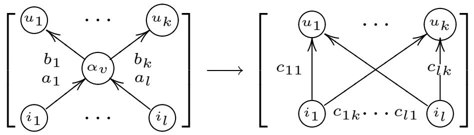
The second set of reduction steps is due to Raf Bocklandt who found them to prove his theorem, see section 5.7, which is crucial to study the smooth locus and the singularities of $\mathtt{triss}_n\ A$. In essence the reduction steps relate quiver settings which have invariants rings which are isomorphic (up to adding variables).

Theorem 5.8 *We have the following reductions :*

1. `b1` *: Let (Q, α) be a quiver setting and v a vertex without loops such that*

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \text{ or } \chi_Q(\epsilon_v, \alpha) \geq 0.$$

Define the quiver setting (Q', α') by composing arrows through v :



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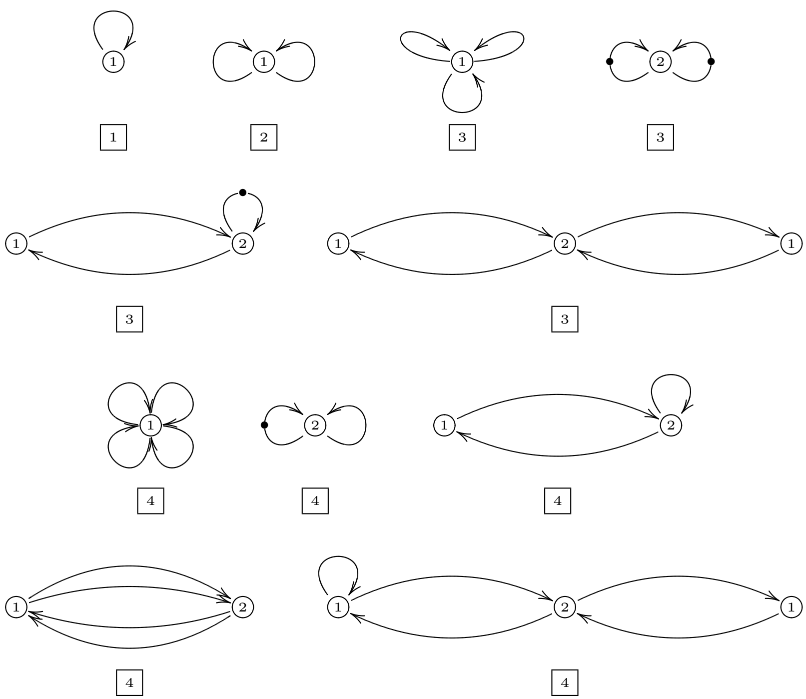


Figure 5.2: The simplified local quivers for $d \leq 4$

(some of the vertices may be the same). Then,

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] \simeq \mathbb{C}[\mathtt{iss}_{\alpha'}\ Q']$$

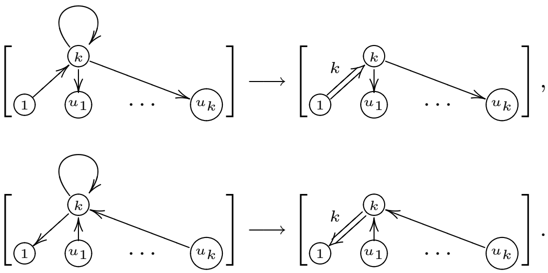
2. `b2` *: Let (Q, α) be a quiver setting and v a vertex with k loops such that $\alpha_v = 1$. Let (Q', α) be the quiver setting where Q' is the quiver obtained by removing the loops in v, then*

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] \simeq \mathbb{C}[\mathtt{iss}_\alpha\ Q'] \otimes \mathbb{C}[X_1, \cdots, X_k]$$

3. `b3` *: Let (Q, α) be a quiver setting and v a vertex with one loop such that $\alpha_v = k \geq 2$ and*

$$\chi_Q(\alpha, \epsilon_v) = -1 \text{ or } \chi_Q(\epsilon_v, \alpha) = -1.$$

Define the quiver setting (Q', α) by changing the quiver as below :



Then,

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] \simeq \mathbb{C}[\mathtt{iss}_\alpha Q'] \otimes \mathbb{C}[X_1, \dots, X_k]$$

Proof. (1) : $\mathtt{rep}_\alpha\ Q$ can be decomposed as

$$\begin{aligned}\mathtt{rep}_\alpha\ Q &= \underbrace{\bigoplus_{a,\ s(a)=v} M_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})}_{\text{arrows starting in } v} \oplus \underbrace{\bigoplus_{a,\ t(a)=v} M_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})}_{\text{arrows terminating in } v} \oplus \mathtt{rest} \\ &= M_{\sum_{s(a)=v} \alpha_{t(a)} \times \alpha_v}(\mathbb{C}) \oplus M_{\alpha_v \times \sum_{t(a)=v} \alpha_{s(a)}}(\mathbb{C}) \oplus \mathtt{rest} \\ &= M_{\alpha_v - \chi(\alpha, \epsilon_v) \times \alpha_v}(\mathbb{C}) \oplus M_{\alpha_v \times \alpha_v - \chi(\epsilon_v, \alpha)}(\mathbb{C}) \oplus \mathtt{rest}\end{aligned}$$

$GL_{\alpha_v}(\mathbb{C})$ only acts on the first two terms and not on $\mathtt{rest}$. Taking the quotient corresponding to $GL_{\alpha_v}(\mathbb{C})$ involves only the first two terms.

We recall the *first fundamental theorem* for GL_n-invariants , see for example [51, II.4.1]. The quotient variety

$$(M_{l \times n}(\mathbb{C}) \oplus M_{n \times m})/GL_n$$

where GL_n acts in the natural way, is for all $l, n, m \in \mathbb{N}$ isomorphic to the space of all $l \times m$ matrices of rank $\leq n$. The projection map is induced by multiplication

$$M_{l \times n}(\mathbb{C}) \oplus M_{n \times m}(\mathbb{C}) \xrightarrow{\pi} M_{l \times m}(\mathbb{C}) \qquad (A, B) \mapsto A.B$$

In particular, if $n \geq l$ and $n \geq m$ then π is surjective and the quotient variety is isomorphic to $M_{l \times m}(\mathbb{C})$.

By this fundamental theorem and the fact that either $\chi_Q(\alpha, \epsilon_v) \geq 0$ or $\chi_Q(\epsilon_v, \alpha) \geq 0$, the above quotient variety is isomorphic to

$$M_{\alpha_v - \chi(\alpha, \epsilon_v) \times \alpha_v - \chi(\epsilon_v, \alpha)}(\mathbb{C}) \oplus \mathtt{rest}$$

This space can be decomposed as

$$\bigoplus_{a,\ t(a)=vb,\ s(b)=v} M_{\alpha_{t(b)}\times\alpha_{s(a)}}(\mathbb{C}) \oplus \mathtt{rest} = \mathtt{rep}_{\alpha'}\ Q'$$

Taking quotients for $GL(\alpha')$ then proves the claim.

(2) : Trivial as $GL(\alpha)$ acts trivially on the loop-representations in v.

(3) : We only prove this for the first case. Call the loop in the first quiver ℓ and the incoming arrow a. Call the incoming arrows in the second quiver $c_i, i = 0, \ldots, k-1$.

There is a map

$$\pi : \mathtt{rep}_\alpha\ Q \rightarrow \mathtt{rep}_{\alpha'}\ Q' \times \mathbb{C}^k : V \mapsto (V', Tr(V_\ell), \ldots, Tr(V_{\ell^k})) \text{ with } V'_{c_i} := V_\ell^i V_a$$

Suppose $(V', x_1, \ldots, x_k) \in \mathtt{rep}_{\alpha'} Q' \times \mathbb{C}^k \in$ such that $(x_1, \ldots, x_k)$ correspond to the traces of powers of an invertible diagonal matrix D with k different eigenvalues $(\lambda_i, i = 1, \ldots, k)$ and the matrix A made of the columns $(V_{c_i}, i = 0, \ldots, k-1)$ is invertible. The image of the representation

$$V \in \mathtt{rep}_\alpha\ Q : V_a = V'_{c_0}, V_\ell = A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1}$$

under π is $(V', x_1, \ldots, x_k)$ because

$$\begin{aligned} V_\ell^i V_a &= A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D^i \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1} V'_{c_0} \\ &= A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1^i \\ \vdots \\ \lambda_k^i \end{pmatrix} \\ &= V_{c_i} \end{aligned}$$

and the traces of V_ℓ are the same as those of D. The conditions on $(V', x_1, \ldots, x_k)$, imply that the image of π, U, is dense, and hence π is a dominant map.

There is a bijection between the generators of $\mathbb{C}[\mathtt{iss}_\alpha Q]$ and $\mathbb{C}[\mathtt{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k]$ by identifying

$$f_{\ell^i} \mapsto X_i, i = 1, \ldots, k\ , f_{\ldots a\ell^i\ldots} \mapsto f_{\cdots c_i \cdots}, i = 0, \ldots, k-1$$

Notice that higher orders of ℓ don't occur by the Caley Hamilton identity on V_ℓ. If n is the number of generators of $\mathbb{C}[\mathtt{iss}_\alpha Q]$, we have two maps

$$\phi : \mathbb{C}[Y_1, \cdots Y_n] \to \mathbb{C}[\mathtt{iss}_\alpha Q] \subset \mathbb{C}[\mathtt{rep}_\alpha Q],$$
$$\phi' : \mathbb{C}[Y_1, \cdots Y_n] \to \mathbb{C}[\mathtt{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k] \subset \mathbb{C}[\mathtt{rep}_{\alpha'} Q' \times \mathbb{C}^k].$$

Note that $\phi'(f) \circ \pi \equiv \phi(f)$ and $\phi(f) \circ \pi^{-1}|_U \equiv \phi'(f)|_U$. So if $\phi(f) = 0$ then also $\phi'(f)|_U = 0$. Because U is zariski-open and dense in $\mathtt{rep}_{\alpha'} Q' \times \mathbb{C}^2$, $\phi'(f) \equiv 0$. A similar argument holds for the inverse implication whence $Ker(\phi) = Ker(\phi')$. $\square$

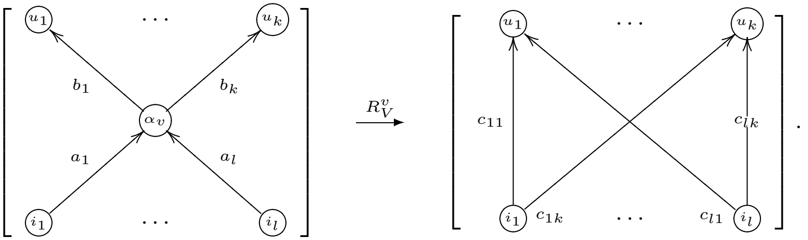
We have to work with *marked* quiver settings and therefore we need slightly more general reduction steps. The proofs of the claims below follow immediately from the above theorem by separating traces.

With ϵ_v we denote the basevector concentrated in vertex v and α_v will denote the vertex dimension component of α in vertex v. There are three types of reduction moves, each with their own condition and effect on the ring of invariants.

Vertex removal (b1) : Let $(Q^\bullet, \alpha)$ be a marked quiver setting and v a vertex satisfying the condition C_V^v, that is, v is without (marked) loops and satisfies

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \quad \text{or} \quad \chi_Q(\epsilon_v, \alpha) \geq 0$$

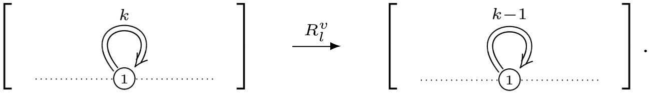
Define the new quiver setting $(Q^{\bullet'}, \alpha')$ obtained by the operation R_V^v which removes the vertex v and composes all arrows through v, the dimensions of the other vertices are unchanged :



where $c_{ij} = a_i b_j$ (observe that some of the incoming and outgoing vertices may be the same so that one obtains loops in the corresponding vertex). In this case we have

$$\mathbb{C}[\mathtt{rep}_\alpha\ Q^\bullet]^{GL(\alpha)} \simeq \mathbb{C}[\mathtt{rep}_{\alpha'}\ Q^{\bullet'}]^{GL(\alpha')}$$

loop removal (b2) : Let $(Q^\bullet, \alpha)$ be a marked quiver setting and v a vertex satisfying the condition C_l^v that the vertex-dimension $\alpha_v = 1$ and there are $k \geq 1$ loops in v. Let $(Q^{\bullet'}, \alpha)$ be the quiver setting obtained by the loop removal operation R_l^v



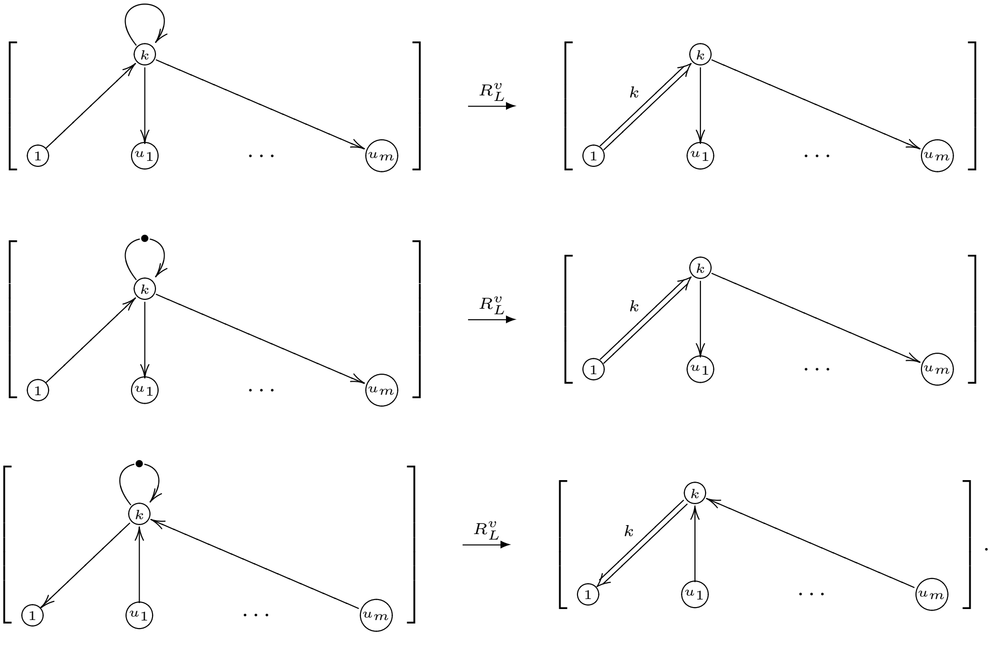
removing one loop in v and keeping the dimension vector the same, then

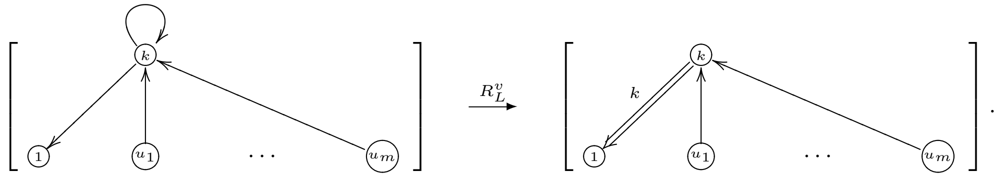
$$\mathbb{C}[\mathtt{rep}_\alpha\ Q^\bullet]^{GL(\alpha)} \simeq \mathbb{C}[\mathtt{rep}_\alpha\ Q^{\bullet'}]^{GL(\alpha)}[x]$$

Loop removal (b3) : Let $(Q^\bullet, \alpha)$ be a marked quiver setting and v a vertex satisfying condition C_L^v, that is, the vertex dimension $\alpha_v \geq 2$, v has precisely one (marked) loop in v and

$$\chi_Q(\epsilon_v, \alpha) = -1 \quad \text{or} \quad \chi_Q(\alpha, \epsilon_v) = -1$$

(that is, there is exactly one other incoming or outgoing arrow from/to a vertex with dimension 1). Let $(Q^{\bullet'}, \alpha)$ be the marked quiver setting obtained by changing the quiver as indicated below (depending on whether the incoming or outgoing condition is satisfied and whether there is a loop or a marked loop in v)





and the dimension vector is left unchanged, then we have

$$\mathbb{C}[\mathtt{rep}_\alpha\ Q^\bullet]^{GL(\alpha)} = \begin{cases} \mathbb{C}[\mathtt{rep}_\alpha\ Q^{\bullet\prime}]^{GL(\alpha)}[x_1,\ldots,x_k] & \text{(loop)} \\ \mathbb{C}[\mathtt{rep}_\alpha\ Q^{\bullet\prime}]^{GL(\alpha)}[x_1,\ldots,x_{k-1}] & \text{(marked loop)} \end{cases}$$

Definition 5.2 *A marked quiver $Q^\bullet$ is said to be strongly connected if for every pair of vertices $\{v,w\}$ there is an oriented path from v to w and an oriented path from w to v.*

A marked quiver setting $(Q^\bullet,\alpha)$ is said to be reduced *if and only if there is* no *vertex v such that one of the conditions C_V^v, C_l^v or C_L^v is satisfied.*

Lemma 5.2 *Every marked quiver setting $(Q_1^\bullet,\alpha_1)$ can be reduced by a sequence of operations R_V^v, R_l^v and R_L^v to a* reduced *quiver setting $(Q_2^\bullet,\alpha_2)$ such that*

$$\mathbb{C}[\mathtt{rep}_{\alpha_1}\ Q_1^\bullet]^{GL(\alpha_1)} \simeq \mathbb{C}[\mathtt{rep}_{\alpha_2}\ Q_2^\bullet]^{GL(\alpha_2)}[x_1,\ldots,x_z]$$

Moreover, the number z of extra variables is determined by the reduction sequence

$$(Q_2^\bullet,\alpha_2) = R_{X_u}^{v_{i_u}} \circ \ldots \circ R_{X_1}^{v_{i_1}}(Q_1^\bullet,\alpha_1)$$

where for every $1 \leq j \leq u$, $X_j \in \{V,l,L\}$. More precisely,

$$z = \sum_{X_j=l} 1 + \sum_{X_j=L}^{(unmarked)} \alpha_{v_{i_j}} + \sum_{X_j=L}^{(marked)} (\alpha_{v_{i_j}} - 1)$$

Proof. As any reduction step removes a (marked) loop or a vertex, any sequence of reduction steps starting with $(Q_1^\bullet,\alpha_1)$ must eventually end in a reduced marked quiver setting. The statement then follows from the discussion above. □

As the reduction steps have no uniquely determined inverse, there is no a priori reason why the reduced quiver setting of the previous lemma should be unique. Nevertheless this is true.

We will say that a vertex v is *reducible* if one of the conditions C_V^v (vertex removal), C_l^v (loop removal in vertex dimension one) or C_L^v (one (marked) loop removal) is satisfied. If we let the specific condition unspecified we will say that v satisfies C_X^v and denote R_X^v for the corresponding marked quiver setting reduction. The resulting marked quiver setting will be denoted by

$$R_X^v(Q^\bullet, \alpha)$$

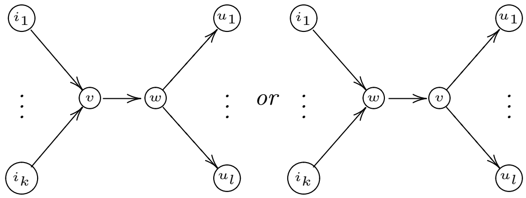
If $w \neq v$ is another vertex in $Q^\bullet$ we will denote the corresponding vertex in $R_X^v(Q^\bullet)$ also with w. The proof of the uniqueness result relies on three claims :

1. If $w \neq v$ satisfies R_Y^w in $(Q^\bullet, \alpha)$, then w virtually always satisfies R_Y^w in $R_X^v(Q^\bullet, \alpha)$.
2. If v satisfies R_X^v and w satisfies R_Y^w, then $R_X^v(R_Y^w(Q^\bullet, \alpha)) = R_Y^w(R_X^v(Q^\bullet, \alpha))$.
3. The previous two facts can be used to prove the result by induction on the minimal length of the reduction chain.

By the *neighborhood* of a vertex v in $Q^\bullet$ we mean the (marked) subquiver on the vertices connected to v. A neighborhood of a set of vertices is the union of the vertex-neighborhoods. *Incoming* resp. *outgoing* neighborhoods are defined in the natural manner.

Lemma 5.3 *Let $v \neq w$ be vertices in $(Q^\bullet, \alpha)$.*

1. *If v satisfies C_V^v in $(Q^\bullet, \alpha)$ and w satisfies C_X^w, then v satisfies C_V^w in $R_X^w(Q^\bullet, \alpha)$ unless the neighborhood of $\{v, w\}$ looks like*



   and $\alpha_v = \alpha_w$. Observe that in this case $R_V^v(Q^\bullet, \alpha) = R_V^w(Q^\bullet, \alpha)$.

2. *If v satisfies C_l^v and w satisfies C_X^w then then v satisfies C_l^v in $R_X^w(Q^\bullet, \alpha)$.*
3. *If v satisfies C_V^v and w satisfies C_X^w then then v satisfies C_V^v in $R_X^w(Q^\bullet, \alpha)$.*

Proof. (1) : If $X = l$ then R_X^w does not change the neighborhood of v so C_V^v holds in $R_l^w(Q^\bullet, \alpha)$. If $X = L$ then R_X^w does not change the neighborhood of v unless $\alpha_v = 1$ and $\chi_Q(\epsilon_w, \epsilon_v) = -1$ (resp. $\chi_Q(\epsilon_v, \epsilon_w) = -1$) depending on whether w satisfies the in- or outgoing condition C_L^w. We only consider the first case, the latter is similar. Then v cannot satisfy the outgoing form of C_V^v in $(Q^\bullet, \alpha)$ so the incoming condition is satisfied. Because the R_L^w-move does not change the incoming neighborhood of v, C_V^v still holds for v in $R_L^w(Q^\bullet, \alpha)$.

If $X = V$ and v and w have disjoint neighborhoods then C_V^v trivially remains true in $R_V^w(Q^\bullet, \alpha)$. Hence assume that there is at least one arrow from v to w (the case where there are only arrows from

w to v is similar). If $\alpha_v < \alpha_w$ then the incoming condition C_V^v must hold (outgoing is impossible) and hence w does not appear in the incoming neighborhood of v. But then R_V^w preserves the incoming neighborhood of v and C_V^v remains true in the reduction. If $\alpha_v > \alpha_w$ then the outgoing condition C_V^w must hold and hence w does not appear in the incoming neighborhood of v. So if the incoming condition C_V^v holds in $(Q^\bullet, \alpha)$ it will still hold after the application of R_V^w. If the outgoing condition C_V^v holds, the neighborhoods of v and w in $(Q^\bullet, \alpha)$ and v in $R_V^w(Q^\bullet, \alpha)$ are depicted in figure 5.3. Let A be the set of arrows in $Q^\bullet$ and A' the set of arrows in the reduction, then because $\sum_{a\in A, s(a)=w} \alpha_{t(a)} \leq \alpha_w$ (the incoming condition for w) we have

$$\begin{aligned}\sum_{a\in A', s(a)=v} \alpha'_{t(a)} &= \sum_{\substack{a\in A,\\ s(a)=v, t(a)\neq w}} \alpha_{t(a)} + \sum_{\substack{a\in A\\ t(a)=w, s(a)=v}} \sum_{a\in A, s(a)=w} \alpha_{t(a)} \\ &\leq \sum_{\substack{a\in A,\\ s(a)=v, t(a)\neq w}} \alpha_{t(a)} + \sum_{\substack{a\in A\\ t(a)=w, s(a)=w}} \alpha_w \\ &= \sum_{a\in A, s(a)=v} \alpha_{t(a)} \leq \alpha_v\end{aligned}$$

and therefore the outgoing condition C_V^v also holds in $R_V^w(Q^\bullet, \alpha)$. Finally if $\alpha_v = \alpha_w$, it may be that C_V^v does not hold in $R_V^w(Q^\bullet, \alpha)$. In this case $\chi(\epsilon_v, \alpha) < 0$ and $\chi(\alpha, \epsilon_w) < 0$ (C_V^v is false in $R_V^w(Q^\bullet, \alpha)$). Also $\chi(\alpha, \epsilon_v) \geq 0$ and $\chi(\epsilon_w, \alpha) \geq 0$ (otherwise C_V does not hold for v or w in $(Q^\bullet, \alpha)$). This implies that we are in the situation described in the lemma and the conclusion follows.
(2) : None of the R_X^w-moves removes a loop in v nor changes $\alpha_v = 1$.
(3) : Assume that the incoming condition C_L^v holds in $(Q^\bullet, \alpha)$ but not in $R_X^w(Q^\bullet, \alpha)$, then w must be the unique vertex which has an arrow to v and $X = V$. Because $\alpha_w = 1 < \alpha_v$, the incoming condition C_V^w holds. This means that there is also only one arrow arriving in w and this arrow is coming from a vertex with dimension 1. Therefore after applying R_V^w, v will still have only one incoming arrow starting in a vertex with dimension 1. A similar argument holds for the outgoing condition C_L^v. □

Lemma 5.4 *Suppose that $v \neq w$ are vertices in $(Q^\bullet, \alpha)$ and that C_X^v and C_Y^w are satisfied. If C_X^v holds in $R_Y^w(Q^\bullet, \alpha)$ and C_Y^w holds in $R_X^v(Q^\bullet, \alpha)$ then*

$$R_X^v R_Y^w(Q^\bullet, \alpha) = R_Y^w R_X^v(Q^\bullet, \alpha)$$

Proof. If $X, Y \in \{l, L\}$ this is obvious, so let us assume that $X = V$. If $Y = V$ as well, we can calculate the Euler form $\chi_{R_V^w R_V^v Q}(\epsilon_x, \epsilon_y)$. Because

$$\chi_{R_V^v Q}(\epsilon_x, \epsilon_y) = \chi_Q(\epsilon_x, \epsilon_y) - \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y)$$

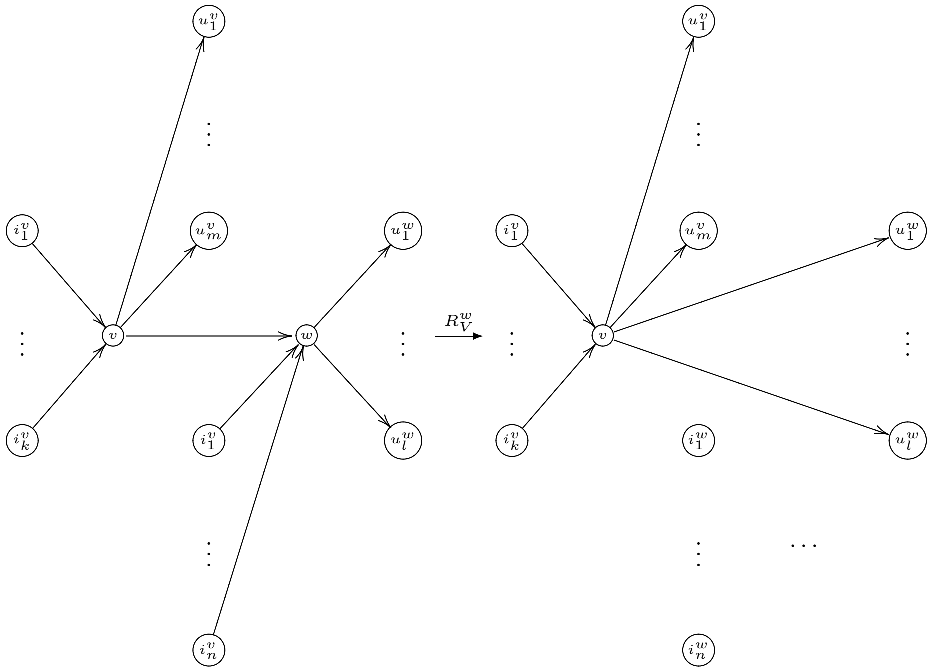


Figure 5.3: Neighborhoods of v and w

it follows that

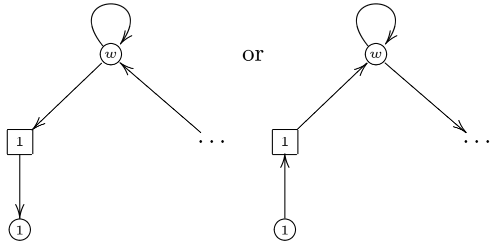
$$\begin{aligned}
\chi_{R_V^w R_V^v Q}(\epsilon_x, \epsilon_y) &= \chi_{R_V^v Q}(\epsilon_x, \epsilon_y) - \chi_{R_V^v Q}(\epsilon_x, \epsilon_w)\chi_{R_V^v Q}(\epsilon_v, \epsilon_y) \\
&= \chi_Q(\epsilon_x, \epsilon_y) - \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y) \\
&- \left(\chi_Q(\epsilon_x, \epsilon_w) - \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_w)\right)\left(\chi_Q(\epsilon_w, \epsilon_y) - \chi_Q(\epsilon_w, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y)\right) \\
&= \chi_Q(\epsilon_x, \epsilon_y) - \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y) - \chi_Q(\epsilon_x, \epsilon_w)\chi_Q(\epsilon_w, \epsilon_y) \\
&- \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_w)\chi_Q(\epsilon_w, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y) \\
&+ \chi_Q(\epsilon_x, \epsilon_w)\chi_Q(\epsilon_w, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y) + \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_w)\chi_Q(\epsilon_w, \epsilon_y)
\end{aligned}$$

This is symmetric in v and w and therefore the ordering of R_V^v and R_V^w is irrelevant.

If $Y = l$ we have the following equalities

$$\begin{aligned}\chi_{R_l^w R_V^v Q}(\epsilon_x, \epsilon_y) &= \chi_{R_V^v Q}(\epsilon_x, \epsilon_y) - \delta_{wx}\delta_{wy} \\ &= \chi_Q(\epsilon_x, \epsilon_y) - \chi_Q(\epsilon_x, \epsilon_v)\chi_Q(\epsilon_v, \epsilon_y) - \delta_{wx}\delta_{wy} \\ &= \chi_Q(\epsilon_x, \epsilon_y) - \delta_{wx}\delta_{wy} - (\chi_Q(\epsilon_x, \epsilon_v) - \delta_{wx}\delta_{wv})(\chi_Q(\epsilon_v, \epsilon_y) - \delta_{wv}\delta_{wy}) \\ &= \chi_{R_l^w Q}(\epsilon_x, \epsilon_y) - \chi_{R_l^w Q}(\epsilon_x, \epsilon_v)\chi_{R_l^w Q}(\epsilon_v, \epsilon_y) \\ &= \chi_{R_V^v R_l^w Q}.\end{aligned}$$

If $Y = L$, an R_L^w-move commutes with the R_V^v move because it does not change the neighborhood of v except when v is the unique vertex of dimension 1 connected to w. In this case the neighborhood of v looks like



In this case the reduction at v is equivalent to a reduction at v' (i.e. the lower vertex) which certainly commutes with R_L^w. □

We are now in a position to prove the claimed uniqueness result.

Theorem 5.9 *If $(Q^\bullet, \alpha)$ is a strongly connected marked quiver setting and $(Q_1^\bullet, \alpha_1)$ and $(Q_2^\bullet, \alpha_2)$ are two reduced marked quiver setting obtained by applying reduction moves to $(Q^\bullet, \alpha)$ then*

$$(Q_1^\bullet, \alpha_1) = (Q_2^\bullet, \alpha_2)$$

Proof. We do induction on the length l_1 of the reduction chain R_1 reducing $(Q^\bullet, \alpha)$ to $(Q_1^\bullet, \alpha_1)$. If $l_1 = 0$, then $(Q^\bullet, \alpha)$ has no reducible vertices so the result holds trivially. Assume the result holds for all lengths $< l_1$. There are two cases to consider.

There exists a vertex v satisfying a loop removal condition $C_X^v, X = l$ or L. Then, there is a R_X^v-move in both reduction chains R_1 and R_2. This follows from lemma 5.3 and the fact that none of the vertices in $(Q_1^\bullet, \alpha_1)$ and $(Q_2^\bullet, \alpha_2)$ are reducible. By the commutation relations from lemma 5.4, we can bring this reduction to the first position in both chains and use induction.

If there is a vertex v satisfying condition C_V^v, either both chains will contain an R_V^v-move or the neighborhood of v looks like the figure in lemma 5.3 (1). Then, R_1 can contain an R_V^v-move and R_2 an R_V^w-move. But then we change the R_V^w move into a R_V^v move, because they have the same effect. The concluding argument is similar to that above. □

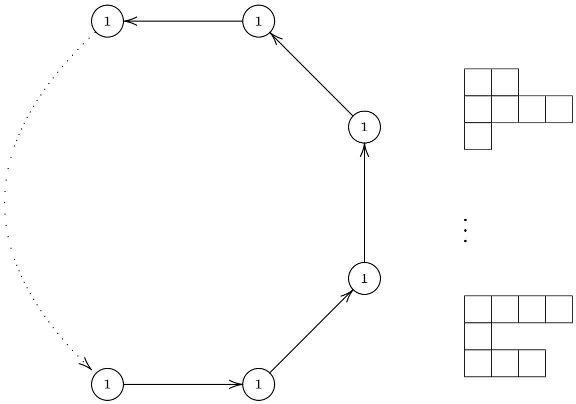


Figure 5.4: Cayley-smooth curve types.

5.4 Curves and surfaces

W. Schelter has proved in [71] that in dimension one, Cayley-smooth orders are hereditary. We give an alternative proof of this result using the étale local classification. The next result follows also by splitting the dimension 1 case in figure 5.2. We give a direct proof illustrating the type-stratification result of section 5.1.

Theorem 5.10 *Let A be a Cayley-Hamilton order of degree n over an affine curve $X = \mathtt{triss}_n\ A$. If $\xi \in Sm_{tr}\ A$, then the étale local structure of A in ξ is determined by a marked quiver-setting which is an oriented cycle on k vertices with $k \leq n$ and an unordered partition $p = (d_1, \ldots, d_k)$ having precisely k parts such that $\sum_i d_i = n$ determining the dimensions of the simple components of V_ξ, see figure 5.4.*

Proof. Let $(Q^\bullet, \alpha)$ be the corresponding local marked quiver-setting. Because $Q^\bullet$ is strongly connected, there exist oriented cycles in $Q^\bullet$. Fix one such cycle of length $s \leq k$ and renumber the vertices of $Q^\bullet$ such that the first s vertices make up the cycle. If $\alpha = (e_1, \ldots, e_k)$, then there exist semi-simple representations in $\mathtt{rep}_\alpha\ Q^\bullet$ with composition

$$\alpha_1 = (\underbrace{1, \ldots, 1}_{s}, \underbrace{0, \ldots, 0}_{k-s}) \oplus \epsilon_1^{\oplus e_1 - 1} \oplus \ldots \oplus \epsilon_s^{\oplus e_s - 1} \oplus \epsilon_{s+1}^{\oplus e_{s+1}} \oplus \ldots \oplus \epsilon_k^{\oplus e_k}$$

where ϵ_i stands for the simple one-dimensional representation concentrated in vertex v_i. There is a one-dimensional family of simple representations of dimension vector α_1, hence the stratum of semi-simple representations in $\mathtt{iss}_\alpha\ Q^\bullet$ of representation type $\tau = (1, \alpha_1; e_1 - 1, \epsilon_1; \ldots ; e_s - 1, \epsilon_s; e_{s+1}, \epsilon_{s+1}; e_k, \epsilon_k)$ is at least one-dimensional. However, as $dim\ \mathtt{iss}_\alpha\ Q^\bullet = 1$ this can only happen if this semi-simple representation is actually simple. That is, when $\alpha = \alpha_1$ and $k = s$. $\square$

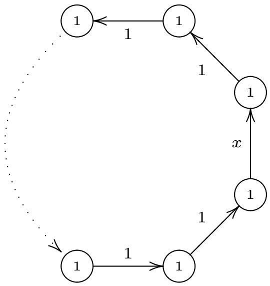
If V_ξ is the semi-simple n-dimensional representation of A corresponding to ξ, then

$$V_\xi = S_1 \oplus \ldots \oplus S_k \quad \text{with} \quad dim\ S_i = d_i$$

and the stabilizer subgroup is $GL(\alpha) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ embedded in GL_n via the diagonal embedding

$$(\lambda_1, \ldots, \lambda_k) \longrightarrow \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{d_k})$$

Further, using basechange in $\mathtt{rep}_\alpha\ Q^\bullet$ we can bring every simple α-dimensional representation of Q^α in standard form



where $x \in \mathbb{C}^*$ is the arrow from v_k to v_1. That is, $\mathbb{C}[\mathtt{rep}_\alpha\ Q^\bullet]^{GL(\alpha)} \simeq \mathbb{C}[x]$ proving that the quotient (or central) variety X must be smooth in ξ by the slice result. Moreover, as $\widehat{A_\xi} \simeq \widehat{\mathbb{T}_\alpha}$ we have, using the numbering conventions of the vertices) the following block decomposition

$$\widehat{A_\xi} \simeq \left[\begin{array}{c|c|c|c} M_{d_1}(\mathbb{C}[[x]]) & M_{d_1 \times d_2}(\mathbb{C}[[x]]) & \ldots & M_{d_1 \times d_k}(\mathbb{C}[[x]]) \\ \hline M_{d_2 \times d_1}(x\mathbb{C}[[x]]) & M_{d_2}(\mathbb{C}[[x]]) & \ldots & M_{d_2 \times d_k}(\mathbb{C}[[x]]) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_{d_k \times d_1}(x\mathbb{C}[[x]]) & M_{d_k \times d_2}(x\mathbb{C}[[x]]) & \ldots & M_{d_k}(\mathbb{C}[[x]]) \end{array} \right]$$

From the local description of hereditary orders given in [70, Thm. 39.14] we deduce that A_ξ is an hereditary order. That is, we have the following characterization of the smooth locus

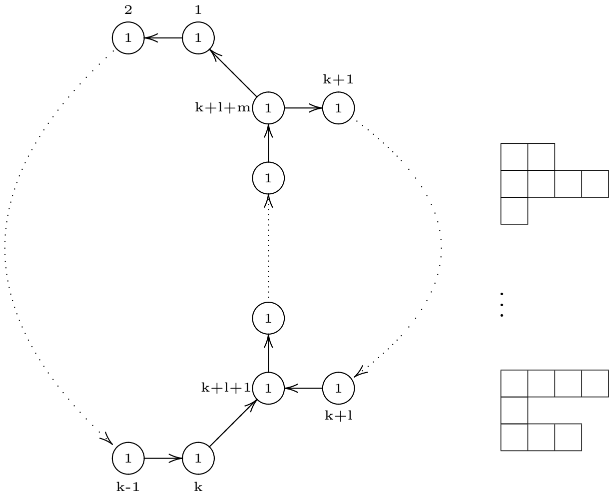


Figure 5.5: Cayley-smooth surface types.

Proposition 5.9 *Let A be a Cayley-Hamilton order of degree n over a central affine curve X. Then, $Sm_{tr}\ A$ is the locus of points $\xi \in X$ such that A_ξ is an hereditary order (in particular, ξ must be a smooth point of X).*

Theorem 5.11 *Let $\mathcal{A}$ be a Cayley-Hamilton central $\mathcal{O}_X$-order of degree n where X is a projective curve. Equivalent are*

1. *$\mathcal{A}$ is a sheaf of Cayley-smooth orders*

2. *X is smooth and $\mathcal{A}$ is a sheaf of hereditary $\mathcal{O}_X$-orders*

We now turn to orders over surfaces. The next result can equally be proved using splitting and the classification of figure 5.2.

Theorem 5.12 *Let A be a Cayley-Hamilton order of degree n over an affine surface $X = \mathtt{triss}_n\ A$. If $\xi \in Sm_{tr}\ A$, then the étale local structure of A in ξ is determined by a marked local quiver-setting A_{klm} on $k+l+m \leq n$ vertices and an unordered partition $p = (d_1, \ldots, d_{k+l+m})$ of n with $k+l+m$ non-zero parts determined by the dimensions of the simple components of V_ξ as in figure 5.5.*

Proof. Let $(Q^\bullet, \alpha)$ be the marked quiver-setting on r vertices with $\alpha = (e_1, \ldots, e_r)$ corresponding to ξ. As $Q^\bullet$ is strongly connected and the quotient variety is two-dimensional, $Q^\bullet$ must contain more than one oriented cycle, hence it contains a sub-quiver of type A_{klm}, possibly degenerated with k or l equal to zero. Order the first $k+l+m$ vertices of $Q^\bullet$ as indicated. One verifies that A_{klm} has simple representations of dimension vector $(1, \ldots, 1)$. Assume that A_{klm} is a proper subquiver and denote $s = k+l+m+1$ then $Q^\bullet$ has semi-simple representations in $\mathtt{rep}_\alpha\ Q^\bullet$ with dimension-vector decomposition

$$\alpha_1 = (\underbrace{1, \ldots, 1}_{k+l+m}, 0, \ldots, 0) \oplus \epsilon_1^{\oplus e_1 - 1} \oplus \ldots \oplus \epsilon_{k+l+m}^{\oplus e_{k+l+m} - 1} \oplus \epsilon_s^{\oplus e_s} \oplus \ldots \oplus \epsilon_r^{\oplus e_r}$$

Applying the formula for the dimension of the quotient variety shows that $\mathtt{iss}_{(1,\ldots,1)}\ A_{klm}$ has dimension 2 so there is a two-dimensional family of such semi-simple representation in the two-dimensional quotient variety $\mathtt{iss}_\alpha\ Q^\bullet$. This is only possible if this semi-simple representation is actually simple, whence $r = k+l+m$, $Q^\bullet = A_{klm}$ and $\alpha = (1, \ldots, 1)$. $\square$

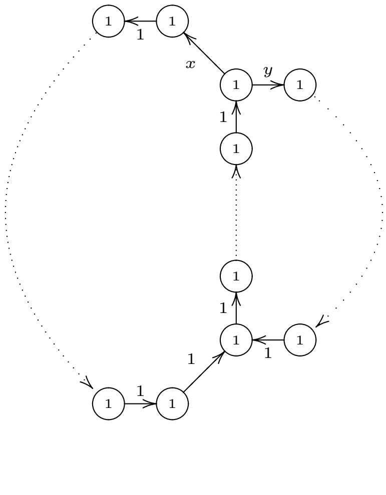
If V_ξ is the semi-simple n-dimensional representation of A corresponding to ξ, then

$$V_\xi = S_1 \oplus \ldots \oplus S_r \quad \text{with} \quad dim\ S_i = d_i$$

and the stabilizer subgroup $GL(\alpha) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ embedded diagonally in GL_n

$$(\lambda_1, \ldots, \lambda_r) \mapsto \text{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{d_r})$$

By basechange in $\mathtt{rep}_\alpha\ A_{klm}$ we can bring every simple α-dimensional representation in the following standard form



with $x, y \in \mathbb{C}^*$ and as $\mathbb{C}[\mathtt{iss}_\alpha\ A_{klm}] = \mathbb{C}[\mathtt{rep}_\alpha\ A_{klm}]^{GL(\alpha)}$ is the ring generated by traces along oriented cycles in A_{klm}, it is isomorphic to $\mathbb{C}[x, y]$. From the slice result one deduces that ξ must be a smooth point of X and because $\widehat{A_\xi} \simeq \widehat{\mathbb{T}_\alpha}$ we deduce it must have the following block-decomposition

$$\hat{A}_\xi \simeq \begin{array}{c} \left[\begin{array}{c|c|c} \begin{smallmatrix} & (1) \\ (x) & \end{smallmatrix} & (y) & (1) \\ \hline (x) & \begin{smallmatrix} & (1) \\ (y) & \end{smallmatrix} & (1) \\ \hline (x) & (y) & \begin{smallmatrix} & (1) \\ (x,y) & \end{smallmatrix} \end{array}\right] \\ \begin{array}{ccc} \underbrace{\qquad}_{k} & \underbrace{\qquad}_{l} & \underbrace{\qquad}_{m} \end{array} \end{array} \hookrightarrow M_n(\mathbb{C}[[x, y]])$$

where at spot (i, j) with $1 \leq i, j \leq k + l + m$ there is a block of dimension $d_i \times d_j$ with entries the indicated ideal of $\mathbb{C}[[x, y]]$.

Definition 5.3 *Let A be a Cayley-Hamilton central $\mathbb{C}[X]$-order of degree n in a central simple $\mathbb{C}(X)$- algebra Δ of dimension n^2.*

1. *A is said to be* étale locally split *in ξ if and only if $\widehat{A_\xi}$ is a central $\hat{\mathcal{O}}_{X,x}$-order in $M_n(\hat{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}(X))$.*
2. *The* ramification locus *ram_A of A is the locus of points $\xi \in X$ such that*

$$\frac{A}{\mathfrak{m}_\xi A \mathfrak{m}_\xi} \not\simeq M_n(\mathbb{C})$$

The complement $X - ram_A$ is the Azumaya locus X_{az} of A.

Theorem 5.13 *Let $\mathcal{A}$ be a Cayley-smooth central $\mathcal{O}_X$-order of degree n over a projective surface X. Then,*

1. *X is smooth.*
2. *$\mathcal{A}$ is étale locally split in all points of X.*

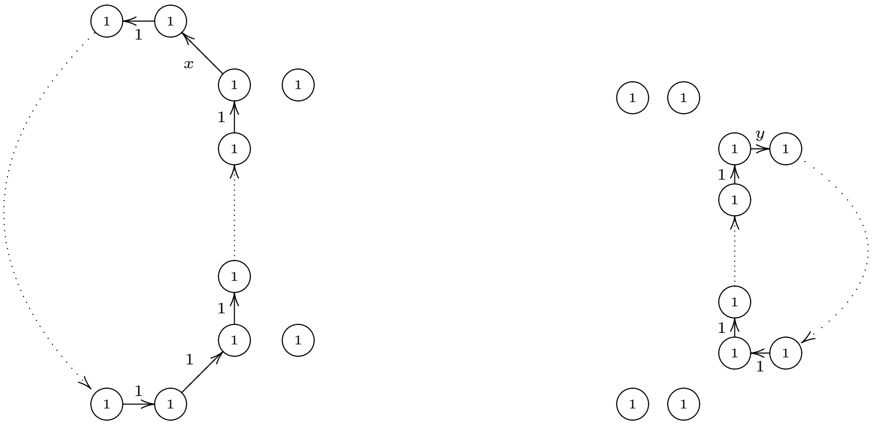
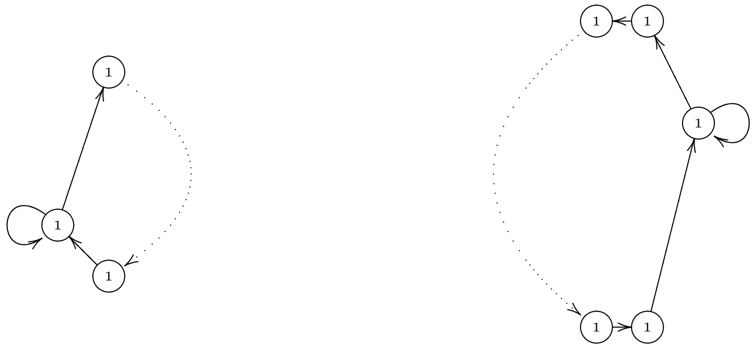


Figure 5.6: Proper semi-simples of A_{klm}.

3. *The ramification divisor* $ram_{\mathcal{A}} \hookrightarrow X$ *is either empty or consists of a finite number of isolated (possibly embedded) points and a reduced divisor having as its worst singularities normal crossings.*

Proof. (1) and (2) follow from the above local description of $\mathcal{A}$. As for (3) we have to compute the local quiver-settings in proper semi-simple representations of $\mathtt{rep}_\alpha\ A_{klm}$. As simples have a strongly connected support, the decomposition types of these proper semi-simples are depicted in figure 5.6. with $x, y \in \mathbb{C}^*$. By the description of local quivers given in section 3 we see that they are respectively of the forms in figure 5.7. The associated unordered partitions are defined in the obvious way, that is, to the looped vertex one assigns the sum of the d_i belonging to the loop-contracted circuit and the other components of the partition are preserved. Using the étale local isomorphism between X in a neighborhood of ξ and of $\mathtt{iss}_\alpha\ A_{klm}$ in a neighborhood of the trivial representation, we see that the local picture of quiver-settings of $\mathcal{A}$ in a neighborhood of ξ is described in figure 5.8. The Azumaya points are the points in which the quiver-setting is A_{001} (the two-loop quiver). From this local description the result follows if we take care of possibly degenerated cases. □

An isolated point in ξ can occur if the quiver-setting in ξ is of type A_{00m} with $m \geq 2$. In the case of curves and surfaces, the central variety X of a Cayley-smooth model $\mathcal{A}$ had to be smooth and that $\mathcal{A}$ is étale locally split in every point $\xi \in X$. Both of these properties are no longer valid in higher dimensions.



A_{0l1} A_{k01}

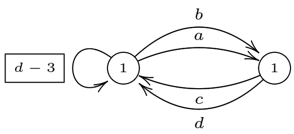
Figure 5.7: Local quivers for A_{klm}.

Lemma 5.5 *For dimension $d \geq 3$, the center Z of a Cayley-smooth order of degree n can have singularities.*

Proof. Consider the marked quiver-setting of figure 5.9 which is allowed for dimension $d = 3$ and degree $n = 2$. The quiver-invariants are generated by the traces along oriented cycles, that is by ac, ad, bc and bd. The coordinate ring is

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] \simeq \frac{\mathbb{C}[x, y, z, v]}{(xv - yz)}$$

having a singularity in the origin. This example can be extended to dimensions $d \geq 3$ by adding loops in one of the vertices.



□

Lemma 5.6 *For dimension $d \geq 3$, a Cayley-smooth algebra does not have to be locally étale split in every point of its central variety.*

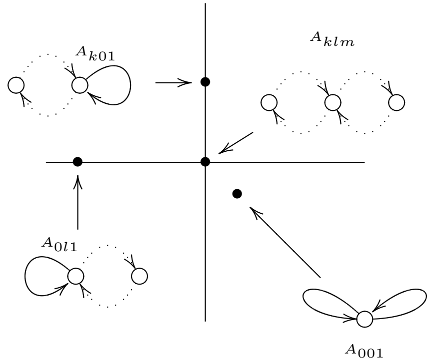


Figure 5.8: Local picture for A_{klm}.

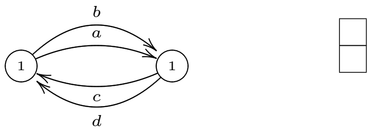


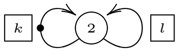
Figure 5.9: Central singularities can arise.

Proof. Consider the following allowable quiver-setting for $d = 3$ and $n = 2$

The corresponding Cayley-smooth algebra A is generated by two generic 2×2 trace zero matrices, say A and B. From the description of the trace algebra $\mathbb{T}_2^2$ we see that its center is generated by $A^2 = x$, $B^2 = z$ and $AB + BA = z$. Alternatively, we can identify A with the Clifford-algebra over $\mathbb{C}[x, y, z]$ of the non-degenerate quadratic form

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

This is a noncommutative domain and remains to be so over the formal power series $\mathbb{C}[[x, y, z]]$. That is, A cannot be split by an étale extension in the origin. More generally, whenever the local marked quiver contains vertices with dimension ≥ 2, the corresponding Cayley-smooth algebra cannot be split by an étale extension as the local quiver-setting does not change and for a split algebra all vertex-dimensions have to be equal to 1. In particular, the Cayley-smooth algebra of degree 2 corresponding to the quiver-setting



□

cannot be split by an étale extension in the origin. Its corresponding dimension is

$$d = 3k + 4l - 3$$

whenever $k + l \geq 2$ and all dimensions $d \geq 3$ are obtained. □

Let X be a projective surface. We will characterize the central simple $\mathbb{C}(X)$-algebras Δ allowing a *Cayley-smooth model*. We first need to perform a local calculation. Consider the ring of algebraic functions in two variables $\mathbb{C}\{x, y\}$ and let $X_{loc} = Spec\ \mathbb{C}\{x, y\}$. There is only one codimension two subvariety : $m = (x, y)$. Let us compute the coniveau spectral sequence for X_{loc}. If K is the field of fractions of $\mathbb{C}\{x, y\}$ and if we denote with k_p the field of fractions of $\mathbb{C}\{x, y\}/p$ where p is a height one prime, we have as its first term

0	0	0	0	...
$H^2(K, \mu_n)$	$\oplus_p\ H^1(k_p, \mathbb{Z}_n)$	μ_n^{-1}	0	...
$H^1(K, \mu_n)$	$\oplus_p \mathbb{Z}_n$	0	0	...
μ_n	0	0	0	...

Because $\mathbb{C}\{x, y\}$ is a unique factorization domain, we see that the map

$$H^1_{et}(K, \mu_n) = K^*/(K^*)^n \xrightarrow{\gamma} \oplus_p\ \mathbb{Z}_n$$

is surjective. Moreover, all fields k_p are isomorphic to the field of fractions of $\mathbb{C}\{z\}$ whose only cyclic extensions are given by adjoining a root of z and hence they are all ramified in m. Therefore, the component maps

$$\mathbb{Z}_n = H^1_{et}(k_p, \mathbb{Z}_n) \xrightarrow{\beta_L} \mu^{-1}$$

are isomorphisms. But then, the second (and limiting) term of the spectral sequence has the form

0	0	0	0	$\dots$
$Ker\ \alpha$	$Ker\ \beta / Im\ \alpha$	0	0	$\dots$
$Ker\ \gamma$	0	0	0	$\dots$
μ_n	0	0	0	$\dots$

Finally, we use the fact that $\mathbb{C}\{x, y\}$ is strict Henselian whence has no proper étale extensions. But then,

$$H^i_{et}(X_{loc}, \mu_n) = 0 \text{ for } i \geq 1$$

and substituting this information in the spectral sequence we obtain that the top sequence of the coniveau spectral sequence

$$0 \longrightarrow Br_n\ K \xrightarrow{\alpha} \oplus_p \mathbb{Z}_n \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

is exact. From this sequence we immediately obtain the following

Lemma 5.7 *With notations as before, we have*

1. *Let $U = X_{loc} - V(x)$, then $Br_n\ U = 0$*
2. *Let $U = X_{loc} - V(xy)$, then $Br_n\ U = \mathbb{Z}_n$ with generator the quantum-plane algebra*

$$\mathbb{C}_\zeta[u, v] = \frac{\mathbb{C}\langle u, v\rangle}{(vu - \zeta uv)}$$

where ζ is a primitive n-th root of one

Let Δ be a central simple algebra of dimension n^2 over a field L of transcendence degree 2. We want to determine when Δ admits a *Cayley-smooth model* $\mathcal{A}$, that is, a sheaf of Cayley-smooth $\mathcal{O}_X$-algebras where X is a projective surface with functionfield $\mathbb{C}(X) = L$. It follows from theorem 5.13 that, if such a model exists, X must be a smooth projective surface. We may assume that X is a (commutative) smooth model for L. By the Artin-Mumford exact sequence 3.11 the class of Δ in $Br_n\ \mathbb{C}(X)$ is determined by the following geo-combinatorial data

- A finite collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of *irreducible curves* in X.
- A finite collection $\mathcal{P} = \{P_1, \ldots, P_l\}$ of *points* of X where each P_i is either an intersection point of two or more C_i or a singular point of some C_i.
- For each $P \in \mathcal{P}$ the *branch-data* $b_P = (b_1, \ldots, b_{i_P})$ with $b_i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\{1, \ldots, i_P\}$ the different branches of $\mathcal{C}$ in P. These numbers must satisfy the admissibility condition

  $$\sum_i b_i = 0 \in \mathbb{Z}_n$$

  for every $P \in \mathcal{P}$
- for each $C \in \mathcal{C}$ we fix a cyclic $\mathbb{Z}_n$-cover of smooth curves

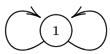
  $$D \twoheadrightarrow \tilde{C}$$

  of the desingularization $\tilde{C}$ of C which is compatible with the branch-data.

If $\mathcal{A}$ is a maximal $\mathcal{O}_X$-order in Δ, then the ramification locus $ram_{\mathcal{A}}$ coincides with the collection of curves $\mathcal{C}$. We fix such a maximal $\mathcal{O}_X$-order $\mathcal{A}$ and investigate its Cayley-smooth locus.

Proposition 5.10 *Let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-order in Δ with X a projective smooth surface and with geo-combinatorial data $(\mathcal{C}, \mathcal{P}, b, \mathcal{D})$ determining the class of Δ in $Br_n\ \mathbb{C}(X)$.*
If $\xi \in X$ lies in $X - \mathcal{C}$ or if ξ is a non-singular point of $\mathcal{C}$, then $\mathcal{A}$ is Cayley-smooth in ξ.

Proof. If $\xi \notin \mathcal{C}$, then $\mathcal{A}_\xi$ is an Azumaya algebra over $\mathcal{O}_{X,x}$. As X is smooth in ξ, $\mathcal{A}$ is Cayley-smooth in ξ. Alternatively, we know that Azumaya algebras are split by étale extensions, whence $\hat{\mathcal{A}}_\xi \simeq M_n(\mathbb{C}[[x,y]])$ which shows that the behavior of $\mathcal{A}$ near ξ is controlled by the local data



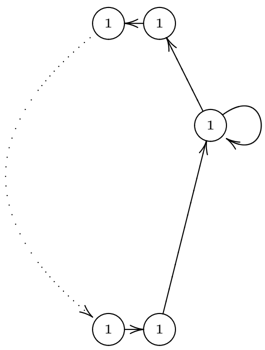
$$\underbrace{\square \ldots \square}_{n}$$

and hence $\xi \in Sm_{tr}\ \mathcal{A}$. Next, assume that ξ is a nonsingular point of the ramification divisor $\mathcal{C}$. Consider the pointed spectrum $X_\xi = \text{Spec}\ \mathcal{O}_{X,\xi} - \{\mathfrak{m}_\xi\}$. The only prime ideals are of height one, corresponding to the curves on X passing through ξ and hence this pointed spectrum is a Dedekind scheme. Further, $\mathcal{A}$ determines a maximal order over X_ξ. But then, tensoring $\mathcal{A}$ with the strict henselization $\mathcal{O}^{sh}_{X,\xi} \simeq \mathbb{C}\{x,y\}$ determines a sheaf of hereditary orders on the pointed spectrum $\hat{X}_\xi = \text{Spec}\ \mathbb{C}\{x,y\} - \{(x,y)\}$ and we may choose the local variable x such that x is a local parameter of the ramification divisor $\mathcal{C}$ near ξ.

Using the characterization result for hereditary orders over discrete valuation rings, given in [70, Thm. 39.14] we know the structure of this extended sheaf of hereditary orders over any height one prime of $\hat{X}_\xi$. Because $\mathcal{A}_\xi$ is a reflexive (even a projective) $\mathcal{O}_{X,\xi}$-module, this height one information determines $\mathcal{A}_\xi^{sh}$ or $\widehat{\mathcal{A}}_\xi$. This proves that $\mathcal{A}_\xi^{sh}$ must be isomorphic to the following blockdecomposition

$$\left[\begin{array}{c|c|c|c} M_{d_1}(\mathbb{C}\{x,y\}) & M_{d_1\times d_2}(\mathbb{C}\{x,y\}) & \ldots & M_{d_1\times d_k}(\mathbb{C}\{x,y\}) \\ \hline M_{d_2\times d_1}(x\mathbb{C}\{x,y\}) & M_{d_2}(\mathbb{C}\{x,y\}) & \ldots & M_{d_2\times d_k}(\mathbb{C}\{x,y\}) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_{d_k\times d_1}(x\mathbb{C}\{x,y\}) & M_{d_k\times d_2}(x\mathbb{C}\{x,y\}) & \ldots & M_{d_k}(\mathbb{C}\{x,y\}) \end{array}\right]$$

for a certain partition $p = (d_1, \ldots, d_k)$ of n having k parts. In fact, as we started out with a maximal order $\mathcal{A}$ one can even show that all these integers d_i must be equal. This local form corresponds to the following quiver-setting



$p = (d_1, \ldots, d_k)$

A_{k01}

whence $\xi \in Sm_{tr}\ A$ as this is one of the allowed surface settings. $\square$

A maximal $\mathcal{O}_X$-order in Δ can have at worst noncommutative singularities in the singular points of the ramification divisor $\mathcal{C}$. Theorem 5.13 a Cayley-smooth order over a surface has as ramification-singularities at worst normal crossings. We are always able to reduce to normal crossings by the following classical result on commutative surfaces, see for example [33, V.3.8].

Theorem 5.14 (Embedded resolution of curves in surfaces) *Let $\mathcal{C}$ be any curve on the surface X. Then, there exists a finite sequence of blow-ups*

$$X' = X_s \longrightarrow X_{s-1} \longrightarrow \ldots \longrightarrow X_0 = X$$

and, if $f : X' \twoheadrightarrow X$ is their composition, then the total inverse image $f^{-1}(\mathcal{C})$ is a divisor with normal crossings.

Fix a series of blow-ups $X' \overset{f}{\twoheadrightarrow} X$ such that the inverse image $f^{-1}(\mathcal{C})$ is a divisor on X' having as worst singularities normal crossings. We will replace the Cayley-Hamilton $\mathcal{O}_X$-order $\mathcal{A}$ by a Cayley-Hamilton $\mathcal{O}_{X'}$-order $\mathcal{A}'$ where $\mathcal{A}'$ is a sheaf of $\mathcal{O}_{X'}$-maximal orders in Δ. In order to determine the ramification divisor of $\mathcal{A}'$ we need to be able to keep track how the ramification divisor $\mathcal{C}$ of Δ changes if we blow up a singular point $p \in \mathcal{P}$.

Lemma 5.8 *Let $\tilde{X} \twoheadrightarrow X$ be the blow-up of X at a singular point p of $\mathcal{C}$, the ramification divisor of Δ on X. Let $\tilde{\mathcal{C}}$ be the strict transform of $\mathcal{C}$ and E the exceptional line on $\tilde{X}$. Let $\mathcal{C}'$ be the ramification divisor of Δ on the smooth model $\tilde{X}$. Then,*

1. *Assume the local branch data at p distribute in an admissible way on $\tilde{\mathcal{C}}$, that is,*

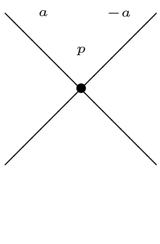
$$\sum_{i \ at\ q} b_{i,p} = 0 \ \text{for all } q \in E \cap \tilde{\mathcal{C}}$$

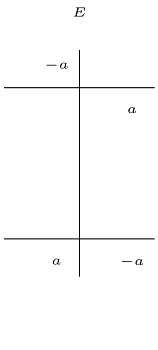
where the sum is taken only over the branches at q. Then, $\mathcal{C}' = \tilde{\mathcal{C}}$.

2. *Assume the local branch data at p do not distribute in an admissible way, then $\mathcal{C}' = \tilde{\mathcal{C}} \cup E$.*

Proof. Clearly, $\tilde{\mathcal{C}} \hookrightarrow \mathcal{C}' \hookrightarrow \tilde{\mathcal{C}} \cup E$. By the Artin-Mumford sequence applied to X' we know that the branch data of $\mathcal{C}'$ must add up to zero at all points q of $\tilde{\mathcal{C}} \cap E$. We investigate the two cases

1. : Assume $E \subset \mathcal{C}'$. Then, the E-branch number at q must be zero for all $q \in \tilde{\mathcal{C}} \cap E$. But there are no non-trivial étale covers of $\mathbb{P}^1 = E$ so $ram(\Delta)$ gives the trivial element in $H^1_{et}(\mathbb{C}(E), \mu_n)$, a contradiction. Hence $\mathcal{C}' = \tilde{\mathcal{C}}$.





2. : If at some $q \in \tilde{\mathcal{C}} \cap E$ the branch numbers do not add up to zero, the only remedy is to include E in the ramification divisor and let the E-branch number be such that the total sum is zero in $\mathbb{Z}_n$. □

Theorem 5.15 *Let Δ be a central simple algebra of dimension n^2 over a field L of transcendence degree two. Then, there exists a smooth projective surface S with functionfield $\mathbb{C}(S) = L$ such that any maximal $\mathcal{O}_S$-order $\mathcal{A}_S$ in Δ has at worst a finite number of isolated noncommutative singularities. Each of these singularities is locally étale of quantum-plane type.*

Proof. We take any projective smooth surface X with functionfield $\mathbb{C}(X) = L$. By the Artin-Mumford exact sequence, the class of Δ determines a geo-combinatorial set of data

$$(\mathcal{C}, \mathcal{P}, b, \mathcal{D})$$

as before. In particular, $\mathcal{C}$ is the ramification divisor $ram(\Delta)$ and $\mathcal{P}$ is the set of singular points of $\mathcal{C}$. We can separate $\mathcal{P}$ in two subsets

- $\mathcal{P}_{unr} = \{P \in \mathcal{P}$ where all the branch-data $b_P = (b_1, \ldots, b_{i_P})$ are trivial, that is, all $b_i = 0$ in $\mathbb{Z}_n\}$
- $\mathcal{P}_{ram} = \{P \in \mathcal{P}$ where some of the branch-data $b_P = (b_1, \ldots, b_{i_P})$ are non-trivial, that is, some $b_i \neq 0$ in $\mathbb{Z}_n\}$

After a finite number of blow-ups we get a birational morphism $S_1 \xrightarrow{\pi} X$ such that $\pi^{-1}(\mathcal{C})$ has as its worst singularities normal crossings and all branches in points of $\mathcal{P}$ are separated in S. Let $\mathcal{C}_1$ be the ramification divisor of Δ in S_1. By the foregoing argument we have

- If $P \in \mathcal{P}_{unr}$, then we have that $\mathcal{C}' \cap \pi^{-1}(P)$ consists of smooth points of $\mathcal{C}_1$,
- If $P \in \mathcal{P}_{ram}$, then $\pi^{-1}(P)$ contains at least one singular points Q of $\mathcal{C}_1$ with branch data $b_Q = (a, -a)$ for some $a \neq 0$ in $\mathbb{Z}_n$.

In fact, after blowing-up singular points Q' in $\pi^{-1}(P)$ with trivial branch-data we obtain a smooth surface $S \twoheadrightarrow S_1 \twoheadrightarrow X$ such that the only singular points of the ramification divisor $\mathcal{C}'$ of Δ have non-trivial branch-data $(a, -a)$ for some $a \in \mathbb{Z}_n$. Then, take a maximal $\mathcal{O}_S$-order $\mathcal{A}$ in Δ. By the local calculation of $Br_n\ \mathbb{C}\{x, y\}$ performed in the last section we know that locally étale $\mathcal{A}$ is of quantum-plane type in these remaining singularities. As the quantum-plane is not étale locally split, $\mathcal{A}$ is not Cayley-smooth in these finite number of singularities. □

In fact, the above proof gives a complete classification of the central simple algebras admitting a Cayley-smooth model.

Theorem 5.16 *Let Δ be a central simple $\mathbb{C}(X)$-algebra of dimension n^2 determined by the geo-combinatorial data $(\mathcal{C}, \mathcal{P}, b, \mathcal{D})$ given by the Artin-Mumford sequence. Then, Δ admits a Cayley-smooth model if and only if all branch-data are trivial.*

Proof. If all branch-data are trivial, the foregoing proof constructs a Cayley-smooth model of Δ. Conversely, if $\mathcal{A}$ is a Cayley-smooth $\mathcal{O}_S$-order in Δ with S a smooth projective model of $\mathbb{C}(X)$, then $\mathcal{A}$ is locally étale split in every point $s \in S$. But then, so is any maximal $\mathcal{O}_S$-order $\mathcal{A}_{max}$ containing $\mathcal{A}$. By the foregoing arguments this can only happen if all branch-data are trivial. $\square$

5.5 Complex moment map

We fix a quiver Q on k vertices $\{v_1, \ldots, v_k\}$ and define the *opposite quiver* Q^o the quiver on $\{v_1, \ldots, v_k\}$ obtained by reversing all arrows in Q. That is, there is an arrow $\textcircled{j} \xrightarrow{a^*} \textcircled{i}$ in Q^o for each arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in the quiver Q. Fix a dimension vector $\alpha = (a_1, \ldots, a_k)$, using the *trace pairings*

$$M_{a_i \times a_j} \times M_{a_j \times a_i} \longrightarrow \mathbb{C} \qquad (V_{a^*}, V_a) \mapsto tr(V_{a^*} V_a)$$

we can identify the representation space $\mathtt{rep}_\alpha\ Q^o$ with the *dual space* $(\mathtt{rep}_\alpha\ Q)^* = Hom_{\mathbb{C}}(\mathtt{rep}_\alpha\ Q, \mathbb{C})$. Observe that the base change action of $GL(\alpha)$ on $\mathtt{rep}_\alpha\ Q^o$ coincides with the action dual to that of $GL(\alpha)$ on $\mathtt{rep}_\alpha\ Q$.

The *dual quiver* Q^d is the superposition of the quivers Q and Q^o. Clearly, for an dimension vector α we have

$$\mathtt{rep}_\alpha\ Q^d = \mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\alpha\ Q^o = \mathtt{rep}_\alpha\ Q \oplus (\mathtt{rep}_\alpha\ Q)^*$$

whence $\mathtt{rep}_\alpha\ Q^d$ can be viewed as the *cotangent bundle* $T^*\mathtt{rep}_\alpha\ Q$ on $\mathtt{rep}_\alpha\ Q$ with structural morphism projection on the first factor. Cotangent bundles are equipped with a canonical *symplectic structure*, see [17, Example 1.1.3] or chapter 8 for more details. The natural action of $GL(\alpha)$ on $\mathtt{rep}_\alpha\ Q$ extends to an action of $GL(\alpha)$ on $T^*\mathtt{rep}_\alpha\ Q$ preserving the symplectic structure and it coincides with the basechange action of $GL(\alpha)$ on $\mathtt{rep}_\alpha\ Q^d$. Such an action on the cotangent bundle gives rise to a *complex moment map*

$$T^*\mathtt{rep}_\alpha\ Q \xrightarrow{\mu_{\mathbb{C}}} (Lie\ GL(\alpha))^*$$

Recall that $Lie\ GL(\alpha) = M_\alpha(\mathbb{C}) = M_{a_1}(\mathbb{C}) \oplus \ldots \oplus M_{a_k}(\mathbb{C})$. Using the trace pairings on both sides, the complex moment map is the mapping

$$\mathtt{rep}_\alpha\ Q^d \xrightarrow{\mu_{\mathbb{C}}} M_\alpha(\mathbb{C})$$

defined by

$$\mu_{\mathbb{C}}(V)_i = \sum_{\substack{a \in Q_a \\ t(a)=i}} V_a V_{a^*} - \sum_{\substack{a \in Q_a \\ s(a)=i}} V_{a^*} V_a$$

Observe that the image of the complex moment map is contained in $M^0_\alpha(\mathbb{C})$ where

$$M^0_\alpha(\mathbb{C}) = \{(M_1, \ldots, M_k) \in M_\alpha(\mathbb{C}) \mid \sum_i tr(M_i) = 0\} = Lie\ PGL(\alpha)$$

corresponding to the fact that the action of $GL(\alpha)$ on $T^*\mathtt{rep}_\alpha\ Q$ is really a $PGL(\alpha) = GL(\alpha)/C^*$ action.

Definition 5.4 *Elements of* $\mathbb{C}^k = \mathbb{C}^{Q_v}$ *are called* weights . *If* λ *is a weight, one defines the* deformed preprojective algebra *of the quiver* Q *to be*

$$\Pi_\lambda(Q) \overset{dfn}{=} \Pi_\lambda = \frac{\mathbb{C}Q^d}{c - \lambda}$$

where c *is the* commutator element

$$c = \sum_{a \in Q_a} [a, a^*]$$

in $\mathbb{C}Q^d$ *and where* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *is identified with the element* $\sum_i \lambda_i v_i \in \mathbb{C}Q^d$.
the algebra $\Pi(Q) = \Pi$ *is known as the* preprojective algebra *of the quiver* Q.

Lemma 5.9 *The ideal* $(c - \lambda) \lhd \mathbb{C}Q^d$ *is the same as the ideal with a generator*

$$\sum_{\substack{a \in Q_a \\ t(a)=i}} aa^* - \sum_{\substack{a \in Q_a \\ s(a)=i}} a^*a - \lambda_i v_i$$

for each vertex $v_i \in Q_v$.

Proof. These elements are of the form $v_j(c - \lambda)v_i$, so they belong to the ideal $(c - \lambda)$. As $c - \lambda$ is also the sum of them, the ideal they generate contains $c - \lambda$. □

That is, α-dimensional representations of the deformed preprojective algebra Π_λ coincide with representations $V \in \mathtt{rep}_\alpha\ Q^d$ which satisfy

$$\sum_{\substack{a \in Q_a \\ t(a)=i}} V_a V_{a^*} - \sum_{\substack{a \in Q_a \\ s(a)=i}} V_{a^*} V_a = \lambda_i \mathbb{1}_{a_i}$$

for each vertex v_i. That is, we have an isomorphism between the scheme theoretic fiber of the complex moment map and the representation space

$$\mathtt{rep}_\alpha\ \Pi_\lambda = \mu_{\mathbb{C}}^{-1}(\lambda)$$

As the image of $\mu_{\mathbb{C}}$ is contained in $M_\alpha^0(\mathbb{C})$ we have in particular

Lemma 5.10 *If* $\lambda.\alpha = \sum_i \lambda_i a_i \neq 0$, *then there are no* α*-dimensional representations of* Π_λ.

Because we have an embedding $C_k \hookrightarrow \Pi_\lambda$, the n-dimensional representations of the deformed preprojective algebra decompose into disjoint subvarieties

$$\mathtt{rep}_n \ \Pi_\lambda = \bigsqcup_{\alpha : \sum_i a_i = n} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha \ \Pi_\lambda$$

Hence, in studying Cayley-smoothness of Π_λ we may reduce to the distinct components and hence to the study of α-*Cayley-smoothness* , that is, smoothness in the category of $C_k(\alpha)$-algebras which are Cayley-Hamilton algebras of degree $n = \sum_i a_i$. Again, one can characterize this smoothness condition in a geometric way by the property that the restricted representation scheme $\mathtt{rep}_\alpha$ is smooth. In the next section we will investigate this property for the preprojective algebra Π_0, in chapter 8 we will be able to extend these results to arbitrary Π_λ. In this section we will compute the dimension of these representation schemes. First, we will investigate the fibers of the structural map of the cotangent bundle, that is, the projection

$$T^* \mathtt{rep}_\alpha \ Q \simeq \mathtt{rep}_\alpha \ Q^d \longrightarrow \mathtt{rep}_\alpha \ Q$$

Proposition 5.11 *If $V \in \mathtt{rep}_\alpha \ Q$, then there is an exact sequence*

$$0 \longrightarrow Ext^1_{\mathbb{C}Q}(V,V)^* \longrightarrow \mathtt{rep}_\alpha \ Q^o \xrightarrow{c} M_\alpha(\mathbb{C}) \xrightarrow{t} Hom_{\mathbb{C}Q}(V,V)^* \longrightarrow 0$$

where c maps $W = (W_{a^*})_{a^*} \in \mathtt{rep}_\alpha \ Q^o$ to $\sum_{a \in Q_a} [V_a, W_{a^*}]$ and t maps $M = (M_i)_i \in M_{|alpha}(\mathbb{C})$ to the linear map $Hom_{\mathbb{C}Q}(V,V) \longrightarrow \mathbb{C}$ sending a morphism $N = (N_i)_i$ to $\sum_i tr(M_i N_i)$.

Proof. There is an exact sequence

$$0 \longrightarrow Hom_{\mathbb{C}Q}(V,V) \longrightarrow M_\alpha(\mathbb{C}) \xrightarrow{f} \mathtt{rep}_\alpha \ Q \longrightarrow Ext^1_{\mathbb{C}Q}(V,V) \longrightarrow 0$$

where f sends $M = (M_i)_i \in M_\alpha(\mathbb{C})$ to $V' = (V'_a)_a$ with $V'_a = M_{t(a)} V_a - V_a M_{s(a)}$. By definition, the kernel of f is $Hom_{\mathbb{C}Q}(V,V)$ and by the Euler form interpretation of theorem 4.5 we have

$$dim_{\mathbb{C}} \ Hom_{\mathbb{C}Q}(V,V) - dim_{\mathbb{C}} \ Ext^1_{\mathbb{C}Q}(V,V) = \chi_Q(\alpha,\alpha) = dim_{\mathbb{C}} \ M_\alpha(\mathbb{C}) - dim_{\mathbb{C}} \ \mathtt{rep}_\alpha \ Q$$

so the cokernel of f has the same dimension as $Ext^1_{\mathbb{C}Q}(V,V)$ and using the standard projective resolution of V one can show that it is naturally isomorphic to it. The required exact sequence follows by dualizing, using the trace pairing to identify $\mathtt{rep}_\alpha \ Q^o$ with $(\mathtt{rep}_\alpha \ Q)^*$ and $M_\alpha(\mathbb{C})$ with its dual. $\square$

This result allows us to give a characterization of the dimension vectors α such that $\mathtt{rep}_\alpha \ Q \neq \emptyset$.

Theorem 5.17 *For a weight $\lambda \in \mathbb{C}^k$ and a representation $V \in \mathtt{rep}_\alpha \ Q$ the following are equivalent*

1. *V extends to an α-dimensional representation of the deformed preprojective algebra Π_λ.*

2. *For all dimension vectors β of direct summands W of V we have $\lambda.\beta = 0$.*

Moreover, if $V \in \mathtt{rep}_\alpha\ Q$ does lift, then $\pi^{-1}(V) \simeq (Ext^1_{\mathbb{C}Q}(V,V))^*$.

Proof. If V lifts to a representation of Π_λ, then there is a representation $W \in \mathtt{rep}_\alpha\ Q^o$ mapping under c of proposition 5.11 to λ. But then, by exactness of the sequence in proposition 5.11 λ must be in the kernel of t. In particular, for any morphism $N = (N_i)_i \in Hom_{\mathbb{C}Q}(V,V)$ we have that $\sum_i \lambda_i tr(N_i) = 0$. In particular, let W be a direct summand of V (as Q-representation) and let $N = (N_i)_i$ be the projection morphism $V \twoheadrightarrow W \hookrightarrow V$, then $\sum_i \lambda_i tr(N_i) = \sum_i \lambda_i b_i$ where $\beta = (b_1, \ldots, b_k)$ is the dimension vector of W.

Conversely, it suffices to prove the lifting of any indecomposable representation W having a dimension vector β satisfying $\lambda.\beta = 0$. Because the endomorphism ring of W is a local algebra, any endomorphism $N = (N_i)_i$ of W is the sum of a nilpotent matrix and a scalar matrix whence $\sum_i \lambda_i tr(N_i) = 0$. But then considering the sequence of proposition 5.11 for β and considering λ as an element of $M_{|\beta|}(\mathbb{C})$, it lies in the kernel of t whence in the image of c and therefore W can be extended to a representation of Π_λ.

The last statement follows again from the exact sequence of proposition 5.11. □

In particular, if α is a root for Q satisfying $\lambda.\alpha = 0$, then there are α-dimensional representations of Π_λ. Recall the definition of the *number of parameters* given in definition 4.8

$$\mu(X) = \max_d \ (dim\ X_{(d)} - d)$$

where $X_{(d)}$ is the union of all orbits of dimension d. We denote $\mu(\mathtt{rep}^{ind}_\alpha\ Q)$ for the $GL(\alpha)$-action on the indecomposables of $\mathtt{rep}_\alpha\ Q$ by $p_Q(\alpha)$. Recall that part of Kac's theorem 4.14 asserts that

$$p_Q(\alpha) = 1 - \chi_Q(\alpha, \alpha)$$

We will apply these facts to the determination of the dimension of the fibers of the complex moment map.

Lemma 5.11 *Let U be a $GL(\alpha)$-stable constructible subset of $\mathtt{rep}_\alpha\ Q$ contained in the image of the projection map $\mathtt{rep}_\alpha\ Q^d \overset{\pi}{\twoheadrightarrow} \mathtt{rep}_\alpha\ Q$. Then,*

$$dim\ \pi^{-1}(U) = \mu(U) + \alpha.\alpha - \chi_Q(\alpha, \alpha)$$

If in addition $U = \mathcal{O}(V)$ is a single orbit, then $\pi^{-1}(U)$ is irreducible of dimension $\alpha.\alpha - \chi_Q(\alpha,\alpha)$.

Proof. Let $V \in U_{(d)}$, then by theorem 5.17, the fiber $\pi^{-1}(V)$ is isomorphic to $(Ext^1_{\mathbb{C}Q}(V,V))^*$ and has dimension $dim_{\mathbb{C}} End(V) - \chi_Q(\alpha,\alpha)$ by theorem 4.5 and

$$dim_{\mathbb{C}}\ End(V) = dim\ GL(\alpha) - dim\ \mathcal{O}(V) = \alpha.\alpha - d.$$

Hence, $dim\ \pi^{-1}(U_{(d)}) = (dim\ U_{(d)} - d) + \alpha.\alpha - \chi_Q(\alpha,\alpha)$. If we now vary d, the result follows.

For the second assertion, suppose that $\pi^{-1}(U) \hookleftarrow Z_1 \sqcup Z_2$ with Z_i a $GL(\alpha)$-stable open subset, but then $\pi^{-1}(V) \cap Z_i$ are non-empty disjoint open subsets of the irreducible variety $\pi^{-1}(V)$, a contradiction. $\square$

Theorem 5.18 *Let λ be a weight and α a dimension vector such that $\lambda.\alpha = 0$. Then,*

$$dim\ \mathtt{rep}_\alpha\ \Pi_\lambda = dim\ \mu_{\mathbb{C}}^{-1}(\lambda) = \alpha.\alpha - \chi_Q(\alpha,\alpha) + m$$

where m is the maximum number among all

$$p_Q(\beta_1) + \ldots + p_Q(\beta_r)$$

with $r \geq 1$, all β_i are (positive) roots such that $\lambda.\beta_i = 0$ and $\alpha = \beta_1 + \ldots + \beta_r$.

Proof. Decompose $\mathtt{rep}_\alpha\ Q = \bigsqcup_\tau \mathtt{rep}_\alpha(\tau)$ where $\mathtt{rep}_\alpha(\tau)$ are the representations decomposing as a direct sum of indecomposables of dimension vector $\tau = (\beta_1, \ldots, \beta_r)$. By Kac's theorem 4.14 we have that

$$\mu(\mathtt{rep}_\alpha(\tau)) = p_Q(\beta_1) + \ldots + p_Q(\beta_r)$$

If some of the β_i are such that $\lambda.\beta_i \neq 0$, and $\mu_{\mathbb{C}}^{-1}(\lambda) \xrightarrow{\pi} \mathtt{rep}_\alpha\ Q$ is the projection then $\pi^{-1}(\mathtt{rep}_\alpha(\tau)) = \emptyset$ by lemma 5.10. Combining this with lemma 5.11 the result follows. $\square$

Definition 5.5 *The* set of λ-Schur roots S_λ *is defined to be the set of $\alpha \in \mathbb{N}^k$ such that $p_Q(\alpha) \geq p_Q(\beta_1) + \ldots + p_Q(\beta_r)$ for all decompositions $\alpha = \beta_1 + \ldots + \beta_r$ with β_i positive roots satisfying $\lambda.\beta_i = 0$.*

$S_{\underline{0}}$ is the set of $\alpha \in \mathbb{N}^k$ such that $p_Q(\alpha) \geq p_Q(\beta_1) + \ldots + p_Q(\beta_r)$ for all decompositions $\alpha = \beta_1 + \ldots + \beta_r$ with $\beta_i \in \mathbb{N}^k$

Observe that $S_{\underline{0}}$ consists of Schur roots for Q, for if

$$\tau_{can} = (e_1, \beta_1; \ldots ; e_s, \beta_s) = (\gamma_1, \ldots, \gamma_t)$$

(the γ_j possibly occurring with multiplicities) is the canonical decomposition of α with $t \geq 2$ we have

$$\begin{aligned} p_Q(\alpha) &= 1 - \chi_Q(\alpha,\alpha) \\ &= 1 - \sum_{i,j} \chi_Q(\gamma_i,\gamma_j) \\ &= \sum_i (1 - \chi_Q(\gamma_i,\gamma_i)) - \sum_{i \neq j} \chi_Q(\gamma_i,\gamma_j) - (t-1) \\ &> \sum_i p_Q(\gamma_i) \end{aligned}$$

whence $\alpha \notin S_0$. This argument also shows that in the definition of $S_{\underline{0}}$ we could have taken all decompositions in positive roots, replacing the components β_i by their canonical decompositions.

Theorem 5.19 *For $\alpha \in \mathbb{N}^k$, the following are equivalent :*

1. *The complex moment map $\mathtt{rep}_\alpha\ Q^d \xrightarrow{\mu_{\mathbb{C}}} \mathtt{rep}_\alpha\ Q$ is flat.*
2. *$\mathtt{rep}_\alpha\ \Pi_0 = \mu_{\mathbb{C}}^{-1}(\underline{0})$ has dimension $\alpha.\alpha - 1 + 2p_Q(\alpha)$.*
3. *$\alpha \in S_{\underline{0}}$.*

Proof. The dimensions of the relevant representation spaces are

$$\begin{cases} dim\ \mathtt{rep}_\alpha\ Q & = \alpha.\alpha - \chi_Q(\alpha,\alpha) = \alpha.\alpha - 1 + p_Q(\alpha) \\ dim\ \mathtt{rep}_\alpha\ Q^d & = 2\alpha.\alpha - 2\chi_Q(\alpha,\alpha) = 2\alpha.\alpha - 2 + 2p_Q(\alpha) \\ dim\ M^0_\alpha(\mathbb{C}) & = \alpha.\alpha - 1 \end{cases}$$

so the relative dimension of the complex moment map is $d = \alpha.\alpha - 1 + 2p_Q(\alpha)$.

(1) $\Rightarrow$ (2) : Because $\mu_{\mathbb{C}}$ os flat, its image U is an open subset of $M^0_\alpha(\mathbb{C})$ which obviously contains $\underline{0}$, but then the dimension of $\mu_{\mathbb{C}}^{-1}(\underline{0})$ is equal to the relative dimension d.

(2) $\Rightarrow$ (3) : Assume $p_Q(\alpha) < \sum_i p_Q(\beta_i)$ for some decomposition $\alpha = \beta_1 + \ldots + \beta_s$ with $\beta_i \in \mathbb{N}^k$. Replacing each β_i by its canonical decomposition, we may assume that the β_i are actually positive roots. But then, theorem 5.18 implies that $\mu_{\mathbb{C}}^{-1}(\underline{0})$ has dimension greater than d.

(3) $\Rightarrow$ (1) : We have that α is a Schur root. We claim that $\mathtt{rep}_\alpha\ Q^d \xrightarrow{\mu_{\mathbb{C}}} M^0_\alpha(\mathbb{C})$ is surjective. Let $V \in \mathtt{rep}_\alpha\ Q$ be a general representation, then $Hom_{\mathbb{C}Q}(V,V) = \mathbb{C}$. But then, the map c in proposition 5.11 has a one-dimensional cokernel. But as the image of c is contained in $M^0_\alpha(\mathbb{C})$, this shows that

$$\mathtt{rep}_\alpha\ Q^0 \overset{c}{\twoheadrightarrow} M^0_\alpha(\mathbb{C})$$

is surjective from which the claim follows. Let $M = (M_i)_i \in M^0_\alpha(\mathbb{C})$ and consider the projection

$$\mu_{\mathbb{C}}^{-1}(M) \xrightarrow{\tilde{\pi}} \mathtt{rep}_\alpha\ Q$$

If U is a constructible $GL(\alpha)$-stable subset of $\mathtt{rep}_\alpha\ Q$, then by an argument as in lemma 5.11 we have that

$$dim\ \tilde{\pi}^{-1}(U) \leq \mu(U) + \alpha.\alpha - \chi_Q(\alpha,\alpha)$$

But then, decomposing $\mathtt{rep}_\alpha\ Q$ into types τ of direct sums of indecomposables, it follows from the assumption that $\mu_{\mathbb{C}}^{-1}(M)$ has dimension at most d. But then by the dimension formula it must be equidimensional of dimension d whence flat. $\square$

5.6 Preprojective algebras

In this section we will determine the n-smooth locus of the preprojective algebra Π_0. By the étale local description of section 4.2 it is clear that we need to control the Ext^1-spaces of representations of Π_0.

Proposition 5.12 *Let V and W be representations of Π_0 of dimension vectors α and β, then we have*

$$dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(V,W) = dim_{\mathbb{C}}\ Hom_{\Pi_0}(V,W) + dim_{\mathbb{C}}\ Hom_{\Pi_0}(W,V) - T_Q(\alpha,\beta)$$

Proof. It is easy to verify by direct computation that V has a projective resolution as Π_0-module which starts as

$$\ldots \longrightarrow \bigoplus_{i \in Q_v} \Pi_0 v_i \otimes v_i V \xrightarrow{f} \bigoplus_{\substack{\textcircled{j} \xleftarrow{a} \textcircled{i} \\ a \in Q_a^d}} \Pi_0 v_j \otimes v_i V \xrightarrow{g} \bigoplus_{i \in Q_v} \Pi_0 v_i \otimes v_i V \xrightarrow{h} V \longrightarrow 0$$

where f is defined by

$$f(\sum_i p_i \otimes m_i) = \sum_{\substack{\textcircled{j} \xleftarrow{a} \textcircled{i} \\ a \in Q_a}} (p_i a^* \otimes m_i - p_j \otimes a^* m_j)_a - (p_j a \otimes m_j - p_i \otimes a m_i)_{a^*}$$

where $p_i \in \Pi_0 v_i$ and $m_i \in v_i V$. The map g is defined on the summand corresponding to an arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in Q^d by

$$g(pa \otimes m) = (pa \otimes m)_i - (p \otimes am)_j$$

for $p \in \Pi_0 v_j$ and $m \in v_i V$. the map h is the multiplication map. If we compute homomorphisms to W and use the identification

$$Hom_{\Pi_0}(\Pi_0 v_j \otimes v_i V, W) = Hom_{\mathbb{C}}(v_i V, v_j W)$$

we obtain a complex

$$0 \longrightarrow \bigoplus_{i \in Q_v} Hom_{\mathbb{C}}(v_iV, v_iW) \longrightarrow \bigoplus_{\substack{(j) \xleftarrow{a} (i) \\ a \in Q_a^d}} Hom_{\mathbb{C}}(v_iV, v_jW) \longrightarrow \bigoplus_{i \in Q_v} Hom_{\mathbb{C}}(v_iV, v_iW)$$

in which the left hand cohomology is $Hom_{\Pi_0}(V, W)$ and the middle cohomology is $Ext^1_{\Pi_0}(V, W)$. Moreover, the alternating sum of the dimensions of the terms is $T_Q(\alpha, \beta)$. It remains to prove that the cokernel of the right hand side map has the same dimension as $Hom_{\Pi_0}(W, V)$. But using the trace pairing to identify

$$Hom_{\mathbb{C}}(M, N)^* = Hom_{\mathbb{C}}(N, M)$$

we obtain that the dual of this complex is

$$\bigoplus_{i \in Q_v} Hom_{\mathbb{C}}(v_iW, v_iV) \longrightarrow \bigoplus_{\substack{(j) \xleftarrow{a} (i) \\ a \in Q_a^d}} Hom_{\mathbb{C}}(v_iW, v_jV) \longrightarrow \bigoplus_{i \in Q_v} Hom_{\mathbb{C}}(v_iW, v_iV) \longrightarrow 0$$

and, up to changing the sign of components in the second direct sum corresponding to arrows which are not in Q, this is the same complex as the complex arising with V and W interchanged. From this the result follows. □

In order to determine the n-smooth locus we observe that the representation space decomposes into a disjoint union and we have quotient morphisms

$$\begin{tikzcd}
\mathtt{rep}_n\ \Pi_0 \arrow[r, "="] \arrow[d, "\pi_n"'] & \bigsqcup\limits_{\substack{\alpha=(a_1,\ldots,a_k) \\ a_1+\ldots+a_k=n}} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ \Pi_0 \arrow[d, "\sqcup \pi_\alpha"] \\
\mathtt{iss}_n\ \Pi_0 \arrow[r, "="] & \bigsqcup\limits_{\substack{\alpha=(a_1,\ldots,a_k) \\ a_1+\ldots+a_k=n}} \mathtt{iss}_\alpha\ \Pi_0
\end{tikzcd}$$

Hence if $\xi \in \mathtt{iss}_\alpha\ \Pi_0$ for $\xi \in Sm_{tr}\ \Pi_0$ it is necessary and sufficient that $\mathtt{rep}_\alpha\ \Pi_0$ is smooth along $\mathcal{O}(M_\xi)$ where M_ξ is the semi-simple α-dimensional representation of Π_0 corresponding to ξ. Assume that ξ is of type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$, that is,

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z}$$

with S_i a simple Π_0-representation of dimension vector α_i. Again, the normal space to the orbit $\mathcal{O}(M_\xi)$ is determined by $Ext^1_{\Pi_o}(M_\xi, M_\xi)$ and can be depicted by a local quiver setting (Q_ξ, α_ξ) where Q_ξ is a quiver on z vertices and where $\alpha_\xi = \alpha_\tau = (e_1, \ldots, e_z)$. Repeating the arguments of section 4.2 we have

Lemma 5.12 *With notations as above, $\xi \in Sm_n\ \Pi_0$ if and only if*

$$dim\ GL(\alpha) \times^{GL(\alpha_\xi)} Ext^1_{\Pi_0}(M_\xi, M_\xi) = dim_{M_\xi}\ \mathtt{rep}_\alpha\ \Pi_0$$

As we have enough information to compute both sides, we can prove :

Theorem 5.20 *If $\xi \in \mathtt{iss}_\alpha\ \Pi_0$ with $\alpha = (a_1, \ldots, a_k) \in S_{\underline{0}}$ and $\sum_i a_i = n$, then $\xi \in Sm_n\ \Pi_0$ if and only if M_ξ is a simple n-dimensional representation of Π_0.*

Proof. Assume that ξ is a point of semi-simple representation type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$, that is,

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z} \qquad \text{with} \qquad dim(S_i) = \alpha_i$$

and S_i a simple Π_0-representation. Then, by proposition 5.12 we have

$$\begin{cases} dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(S_i, S_j) & = -T_Q(\alpha_i, \alpha_j) \qquad i \neq j \\ dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(S_i, S_i) & = 2 - T_Q(\alpha_i, \alpha_i) \end{cases}$$

But then, the dimension of $Ext^1_{\Pi_0}(M_\xi, M_\xi)$ is equal to

$$\sum_{i=1}^{z} (2 - T_Q(\alpha_i, \alpha_i)) e_i^2 + \sum_{i \neq j} e_i e_j (-T_Q(\alpha_i, \alpha_j) = 2 \sum_{i=1}^{z} e_i - T_Q(\alpha, \alpha)$$

from which it follows immediately that

$$dim\ GL(\alpha) \times^{GL(\alpha_\xi)} Ext^1_{\Pi_0}(M_\xi, M_\xi) = \alpha.\alpha + \sum_{i=1}^{z} e_i^2 - T_Q(\alpha, \alpha)$$

On the other hand, as $\alpha \in S_{\underline{0}}$ we know from theorem 5.19 that

$$dim\ \mathtt{rep}_\alpha\ \Pi_0 = \alpha.\alpha - 1 + 2p_Q(\alpha) = \alpha.\alpha - 1 + 2 + 2\chi_Q(\alpha, \alpha) = \alpha.\alpha + 1 - T_Q(\alpha, \alpha)$$

But then, equality occurs if and only if $\sum_i e_i^2 = 1$, that is, $\tau = (1, \alpha)$ or M_ξ is a simple n-dimensional representation of Π_0. $\square$

In particular it follows that the preprojective algebra Π_0 is *never* Quillen-smooth. Further, as $\vec{v_i} = (0, \ldots, 1, 0, \ldots, 0)$ are dimension vectors of simple representations of Π_0 it follows that Π_0 is α-smooth if and only if $\alpha = \vec{v_i}$ for some i. In chapter 8 we will determine the dimension vectors of simple representations of the (deformed) preprojective algebras.

Example 5.4 Let Q be an extended Dynkin diagram and δ_Q the corresponding dimension vector. Then, we will show that δ_Q is the dimension vector of a simple representation and $\delta_Q \in S_{\underline{0}}$. Then, the dimension of the quotient variety

$$\begin{aligned} dim\ \mathtt{iss}_{\delta_Q}\ \Pi_0 &= dim\ \mathtt{rep}_{\delta_Q}\ \Pi_0 - \delta_Q.\delta_Q + 1 \\ &= 2p_Q(\delta_Q) = 2 \end{aligned}$$

so it is a surface. The only other semi-simple δ_Q-dimensional representation of Π_0 is the trivial representation. By the theorem, this must be an isolated singular point of $\mathtt{iss}_{\delta_Q}\ Q$. In fact, one can show that $\mathtt{iss}_{\delta_Q}\ \Pi_0$ is the Kleinian singularity corresponding to the extended Dynkin diagram Q.

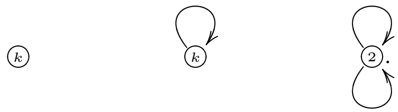
5.7 Central smooth locus

In this section we will prove the characterization, due to Raf Bocklandt, of (marked) quiver settings such that the ring of invariants is smooth. Remark that as the ring of invariants is a positively graded algebra, this is equivalent to being a polynomial algebra.

Definition 5.6 *A quiver setting (Q, α) is said to be* final *iff none of the reduction steps* `b1`, `b2` *or* `b3` *of theorem 5.8 can be applied. Every quiver setting can be reduced to a final quiver setting which we denote $(Q, \alpha) \rightsquigarrow (Q_f, \alpha_f)$.*

Theorem 5.21 *For a quiver setting (Q, α) with $Q = \mathtt{supp}\alpha$ strongly connected, the following are equivalent :*

1. $\mathbb{C}[\mathtt{iss}_\alpha\ Q] = \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)}$ *is* `commalg`*-smooth.*
2. $(Q_f, \alpha_f) \rightsquigarrow (Q_f, \alpha_f)$ *with (Q_f, α_f) one of the following quiver settings*



Proof. $(2) \Rightarrow (1)$: Follows from the foregoing theorem and the fact that the rings of invariants of the three quiver settings are resp. $\mathbb{C}$, $\mathbb{C}[tr(X), tr(X^2), \ldots, tr(X^k)]$ and $\mathbb{C}[tr(X), tr(Y), tr(X^2), tr(Y^2), tr(XY)]$.

$(1) \Rightarrow (2)$: Take a final reduction $(Q, \alpha) \rightsquigarrow (Q_f, \alpha_f)$ and to avoid subscripts rename $(Q_f, \alpha_f) = (Q, \alpha)$ (observe that the condition of the theorem as well as (1) is preserved under the reduction steps by the foregoing theorem). That is, we will assume that (Q, α) is final whence, in particular as `b1` cannot be applied,

$$\chi_Q(\alpha, \epsilon_v) < 0 \qquad \chi_Q(\epsilon_v, \alpha) < 0$$

for all vertices v of Q. With $\mathbf{1}$ we denote the dimension vector $(1, \ldots, 1)$.

claim 1 : Either $(Q, \alpha) = \textcircled{k}$ or Q has loops. Assume neither, then if $\alpha \neq \mathbf{1}$ we can choose a vertex v with maximal α_v. By the above inequalities and theorem 4.10 we have that

$$\tau = (1, \alpha - \epsilon_v; 1, \epsilon_v) \in \mathtt{types}_\alpha Q$$

As there are no loops in v, we have

$$\begin{cases} \chi_Q(\alpha - \epsilon_v, \epsilon_v) & = \chi(\alpha, \epsilon_v) - 1 < -1 \\ \chi_Q(\epsilon_v, \alpha - \epsilon_v) & = \chi(\epsilon_v, \alpha) - 1 < -1 \end{cases}$$

and the local quiver setting (Q_τ, α_τ) contains the subquiver

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left, "k"] & \textcircled{1} \arrow[l, bend left, "l"] \end{tikzcd} \qquad \text{with } k, l \geq 2$$

The invariant ring of the local quiver setting cannot be a polynomial ring as it contains the subalgebra

$$\frac{\mathbb{C}[a, b, c, d]}{(ab - cd)}$$

where $a = x_1 y_1$, $b = x_2 y_2$, $c = x_1 y_2$ and $d = x_2 y_1$ are necklaces of length 2 with x_i arrows from w_1 to w_2 and y_i arrows from w_2 to w_1. This contradicts the assumption (1) by the étale local structure result.

Hence, $\alpha = \mathbf{1}$ and because (Q, α) is final, every vertex must have least have two incoming and two outgoing arrows. Because Q has no loops,

$$dim\ \mathtt{iss}_\mathbf{1}\ Q = 1 - \chi_Q(\mathbf{1}, \mathbf{1}) = \#\mathtt{arrows} - \#\mathtt{vertices} + 1$$

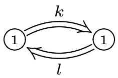
On the other hand, a minimal generating set for $\mathbb{C}[\mathtt{iss}_\mathbf{1}\ Q]$ is the set of *Eulerian necklaces* , that is, those necklaces in Q not re-entering any vertex. By (1) both numbers must be equal, so we will reach a contradiction by showing that $\#\mathtt{euler}$, the number of Eulerian necklaces is strictly larger than $\chi(Q) = \#\mathtt{arrows} - \#\mathtt{vertices} + 1$. We will do this by induction on the number of vertices.

If $\#\mathtt{vertices} = 2$, the statement is true because

$$Q := \begin{tikzcd} \textcircled{1} \arrow[r, bend left, "k"] & \textcircled{1} \arrow[l, bend left, "l"] \end{tikzcd} \qquad \text{whence } \#\mathtt{euler} = kl > \chi(Q) = k + l - 1$$

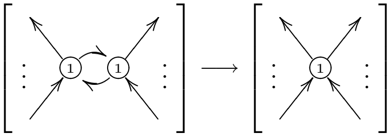
as both k and l are at least 2.

Assume $\#\texttt{vertices} > 2$ and that there is a subquiver of the form

$\texttt{basic} =$ 

If $k > 1$ and $l > 1$ we have seen before that this subquiver and hence Q cannot have a polynomial ring of invariants.

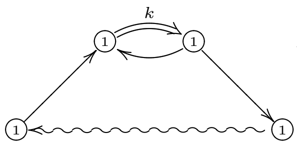
If $k = 1$ and $l = 1$ then substitute this subquiver by one vertex.



The new quiver Q' is again final without loops because there are at least four incoming arrows in the vertices of the subquiver and we only deleted two (the same holds for the outgoing arrows). Q' has one Eulerian necklace less than Q. By induction, we have that

$$\begin{aligned}\#\texttt{euler} &= \#\texttt{euler}' + 1 \\ &> \chi(Q') + 1 \\ &= \chi(Q).\end{aligned}$$

If $k > 1$ then one can look at the subquiver Q' of Q obtained by deleting $k - 1$ of these arrows. If Q' is final, we are in the previous situation and obtain the inequality as before. If Q' is not final, then Q contains a subquiver of the form



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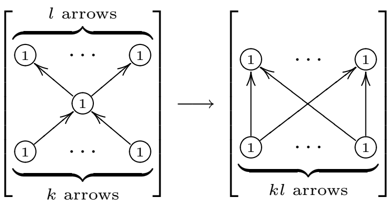
which cannot have a polynomial ring of invariants, as it is reducible to $\texttt{basic}$ with both k and l at least equal to 2.

Finally, if $\#\texttt{vertices} > 2$ and there is no $\texttt{basic}$-subquiver, take an arbitrary vertex v. Construct a new quiver Q' bypassing v



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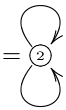
Q' is again final without loops and has the same number of Eulerian necklaces. By induction

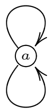
$$\begin{aligned}
\#\texttt{euler} &= \#\texttt{euler}' \\
&> \#\texttt{arrows}' - \#\texttt{vertices}' + 1 \\
&= \#\texttt{arrows} + (kl - k - l) - \#\texttt{vertices} + 1 + 1 \\
&> \#\texttt{arrows} - \#\texttt{vertices} + 1.
\end{aligned}$$

In all cases, we obtain a contradiction with (1) and hence have proved **claim 1**. So we may assume from now on that Q has loops.

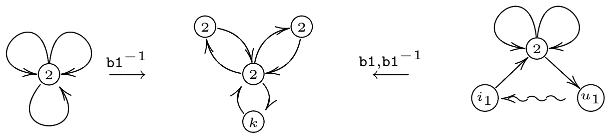
claim 2 : If Q has loops in v, then there is at most one loop in v or (Q, α) is

`2twobytwo` =


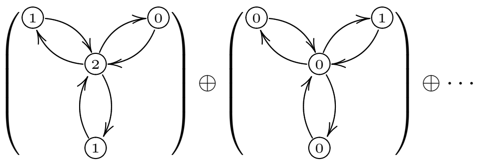
Because (Q, α) is final, we have $\alpha_v \geq 2$. If $\alpha_v = a \geq 3$ then there is only one loop in v. If not, there is a subquiver of the form



and its ring of invariants cannot be a polynomial algebra. Indeed, consider its representation type $\tau = (1, k-1; 1, 1)$ then the local quiver is of type `basic` with $k = l = a - 1 \geq 2$ and we know already that this cannot have a polynomial algebra as invariant ring. If $\alpha_v = 2$ then either we are in the `2twobytwo` case or there is at most one loop in v. If not, we either have at least three loops in v or two loops and a cyclic path through v, but then we can use the reductions



The middle quiver cannot have a polynomial ring as invariants because we consider the type



The number of arrows between the first and the second simple component equals

$$-\begin{pmatrix}2 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & -1 & -1\\ -1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 2$$

whence the corresponding local quiver contains $\mathtt{basic}$ with $k = l = 2$ as subquiver. This proves **claim 2**. From now on we will assume that the quiver setting (Q, α) is such that there is precisely one loop in v and that $k = \alpha_v \geq 2$. Let

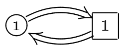
$$\tau = (1, 1; 1, \epsilon_v; \alpha_{v_1} - 1, \epsilon_{v_1}; \ldots; \ldots; \alpha_v - 2, \epsilon_v; \ldots; \alpha_{v_l} - 1, \epsilon_{v_l}) \in \mathtt{types}_\alpha Q$$

Here, the second simple representation, concentrated in v has non-zero trace in the loop whereas the remaining $\alpha_v - 2$ simple representations concentrated in v have zero trace. Further, $\mathbf{1} \in \mathtt{simp}\mathbb{C}Q$ as Q is strongly connected by theorem 4.10. We work out the local quiver setting (Q_τ, α_τ). The number of arrows between the vertices in Q_τ corresponding to simple components concentrated in a vertex is equal to the number of arrows in Q between these vertices. We will denote the vertex (and multiplicity) in Q_τ corresponding to the simple component of dimension vector $\mathbf{1}$ by $\boxed{1}$.

The number of arrows between the vertex in Q_τ corresponding to a simple concentrated in vertex w in Q to $\boxed{1}$ is $-\chi_Q(\epsilon_w, \mathbf{1})$ and hence is one less than the number of outgoing arrows from w in Q. Similarly, the number of arrows from the vertex $\boxed{1}$ to that of the simple concentrated in w is $-\chi_Q(\mathbf{1}, \epsilon_w)$ and is equal to one less than the number of incoming arrows in w in Q. But then we must have for all vertices w in Q that

$$\chi_Q(\epsilon_w, \mathbf{1}) = -1 \qquad \text{or} \qquad \chi_Q(\mathbf{1}, \epsilon_w) = -1$$

Indeed, because (Q, α) is final we know that these numbers must be strictly negative, but they cannot be both ≤ -2 for then the local quiver Q_τ will contain a subquiver of type

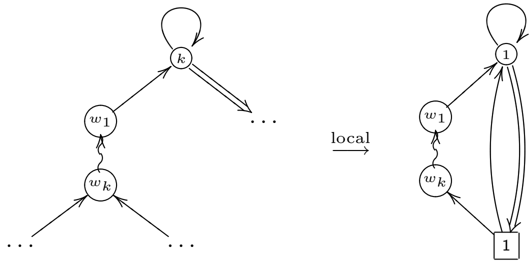


contradicting that the ring of invariants is a polynomial ring. Similarly, we must have

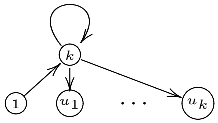
$$\chi_Q(\epsilon_w, \epsilon_v) \geq -1 \qquad \text{or} \qquad \chi_Q(\epsilon_v, \epsilon_v)$$

for all vertices w in Q for which $\alpha_w \geq 2$. Let us assume that $\chi_Q(\epsilon_v, \mathbf{1}) = -1$.

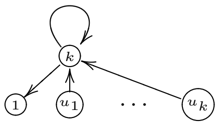
claim 3 : If w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$, then $\alpha_{w_1} = 1$. If this was not the case there is a vertex corresponding to a simple representation concentrated in w_1 in the local quiver Q_τ. If $\chi_Q(\mathbf{1}, \epsilon_{w_1}) = 0$ then the dimension of the unique vertex w_2 with an arrow to w_1 has strictly bigger dimension than w_1, otherwise $\chi_Q(\alpha, \epsilon_{w_1}) \geq 0$ contradicting finality of (Q, α). The vertex w_2 corresponds again to a vertex in the local quiver. If $\chi_Q(\mathbf{1}, \epsilon_{w_2}) = 0$, the unique vertex w_3 with an arrow to w_2 has strictly bigger dimension than w_2. Proceeding this way one can find a sequence of vertices with increasing dimension, which attains a maximum in vertex w_k. Therefore $\chi_Q(\mathbf{1}, \epsilon_{w_k}) \leq -1$. This last vertex is in the local quiver connected with W, so one has a path from $\mathbf{1}$ to ϵ_v.



The subquiver of the local quiver Q_τ consisting of the vertices corresponding to the simple representation of dimension vector $\mathbf{1}$ and the simples concentrated in vertex v resp. w_k is reducible via `b1` to , at least if $\chi_Q(\mathbf{1}, \epsilon_v) \leq -2$, a contradiction finishing the proof of the claim. But then, the quiver setting (Q, α) has the following shape in the neighborhood of v



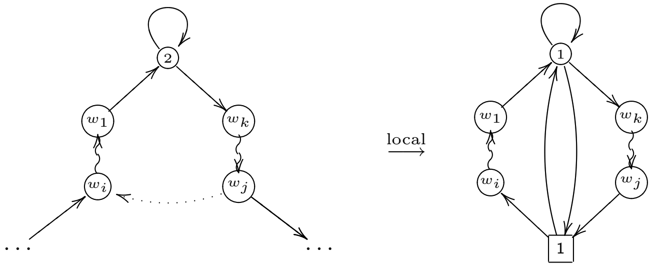
contradicting finality of (Q, α) for we can apply `b3`. In a similar way one proves that the quiver setting (Q, α) has the form

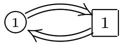


in a neighborhood of v if $\chi_Q(\mathbf{1}, \epsilon_v) = -1$ and $\chi_Q(\epsilon_v, \mathbf{1}) \leq -2$, again contradicting finality.

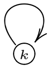
There remains one case to consider : $\chi_Q(\mathbf{1}, \epsilon_v) = -1$ *and* $\chi_Q(\epsilon_v, \mathbf{1}) = -1$. Suppose w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$ and w_k is the unique vertex in Q such that $\chi_Q(\epsilon_{w_k}, \epsilon_v) = -1$, then we claim :

claim 4 : Either $\alpha_{w_1} = 1$ or $\alpha_{w_k} = 1$. If not, consider the path connecting w_k and w_1 and call the intermediate vertices w_i, $1 < i < k$. Starting from w_1 we go back the path until α_{w_i} reaches a maximum. at that point we know that $\chi_Q(\mathbf{1}, \epsilon_{w_k}) \leq -1$, otherwise $\chi_Q(\alpha, \epsilon_{w_k}) \geq 0$. In the local quiver there is a path from the vertex corresponding to the 1-dimensional simple over the ones corresponding to the simples concentrated in w_i to v. Repeating the argument, starting from w_k we also have a path from the vertex of the simple v-representation over the vertices of the w_j-simples to the vertex of the 1-dimensional simple.



The subquiver consisting of $\mathbf{1}$, ϵ_v and the two paths through the ϵ_{w_i} is reducible to  and we again obtain a contradiction.

The only way out of these dilemmas is that the final quiver setting (Q, α) is of the form

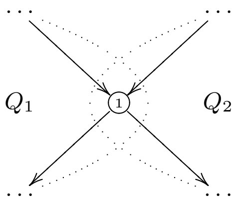


finishing the proof. $\square$

Definition 5.7 *Let (Q, α) and (Q', α') be two quiver settings such that there is a vertex v in Q and a vertex v' in Q' with $\alpha_v = 1 = \alpha'_{v'}$. We define the* connected sum *of the two settings to be the quiver setting*

$$(\; Q \overset{v}{\underset{v'}{\#}} Q' \; , \; \alpha \overset{v}{\underset{v'}{\#}} \alpha' \;)$$

where $Q \overset{v}{\underset{v'}{\#}} Q$ is the quiver obtained by identifying the two vertices v and v'



and where $\alpha \overset{v}{\underset{v'}{\#}} \alpha'$ is the dimension vector which restricts to α (resp. α') on Q (resp. Q').

Example 5.5 With this notation we have

$$\mathbb{C}[\mathtt{iss}_{\alpha \overset{v}{\underset{v'}{\#}} \alpha'} Q \overset{v}{\underset{v'}{\#}} Q'] \simeq \mathbb{C}[\mathtt{iss}_\alpha Q] \otimes \mathbb{C}[\mathtt{iss}_{\alpha'} Q']$$

Because traces of a necklaces passing more than once through a vertex where the dimension vector is equal to 1 can be split as a product of traces of necklaces which pass through this vertex only one time, we see that the invariant ring of the connected sum is generated by Eulerian necklaces fully contained in Q or in Q'.

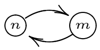
Theorem 5.21 gives a procedure to decide whether a given quiver setting (Q, α) has a regular ring of invariants. However, is is not feasible to give a graphtheoretic description of all such settings in general. Still, in the special (but important) case of *symmetric* quivers, there is a nice graphtheoretic characterization.

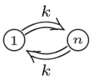
Theorem 5.22 *Let (Q, α) be a symmetric quiver setting such that Q is connected and has no loops. Then, the ring of polynomial invariants*

$$\mathbb{C}[\mathtt{iss}_\alpha Q] = \mathbb{C}[\mathtt{rep}_\alpha Q]^{GL(\alpha)}$$

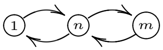
is a polynomial ring if and only if the following conditions are satisfied

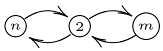
1. *Q is tree-like, that is, if we draw an edge between vertices of Q whenever there is at least one arrow between them in Q, the graph obtained in a tree.*
2. *α is such that in every branching vertex v of the tree we have $\alpha_v = 1$.*
3. *The quiver subsetting corresponding to branches of the tree are connected sums of the following atomic pieces :*

I 

II 

, $k \leq n$

III 

IV 

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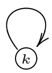
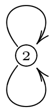
Proof. Using theorem 5.21 any of the atomic quiver settings has a polynomial ring of invariants. Type `I` reduces via `b1` to

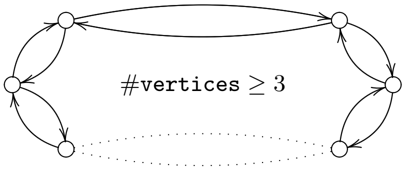


where $k = min(m, n)$, type `II` reduces via `b1` and `b2` to ①, type `III` reduces via `b1`, `b3`, `b1` and `b2` to ① and finally, type `IV` reduces via `b1` to

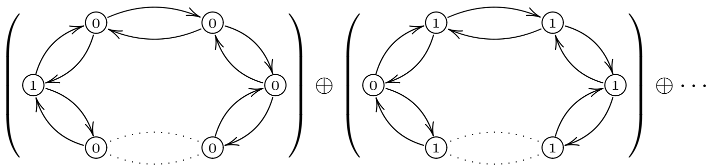


By the previous example, any connected sum constructed out of these atomic quiver settings has a regular ring of invariants. Observe that such connected sums satisfy the first two requirements. Therefore, any quiver setting satisfying the requirements has indeed a polynomial ring of invariants.

Conversely, assume that the ring of invariants $\mathbb{C}[\mathtt{iss}_\alpha Q]$ is a polynomial ring, then there can be no quiver subsetting of the form

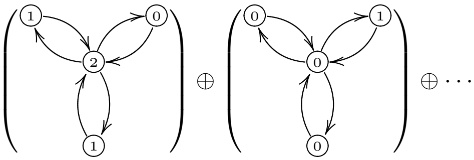


For we could look at a semisimple representation type τ with decomposition



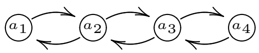
The local quiver contains a subquiver (corresponding to the first two components) of type `basic` with k and $l \geq 2$ whence cannot give a polynomial ring. That is, Q is tree-like.

Further, the dimension vector α cannot have components ≥ 2 at a branching vertex v. For we could consider the semisimple representation type with decomposition

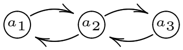


and again the local quiver contains a subquiver setting of type `basic` with $k = 2 = l$ (the one corresponding to the first two components). Hence, α satisfies the second requirement.

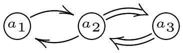
Remains to show that the branches do not contain other subquiver settings than those made of the atomic components. That is, we have to rule out the following subquiver settings :



with $a_2 \geq 2$ and $a_3 \geq 2$,

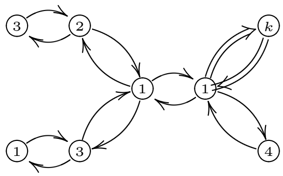


with $a_2 \geq 3$ and $a_1 \geq 2$, $a_3 \geq 2$ and



whenever $a_2 \geq 2$. These situations are easily ruled out by theorem 5.21 and we leave this as a pleasant exercise. $\square$

Example 5.6 The quiver setting



has a polynomial ring of invariants if and only if $k \geq 2$.

Example 5.7 Let $(Q^\bullet, \alpha)$ be a *marked* quiver setting and assume that $\{l_1, \ldots, l_u\}$ are the marked loops in $Q^\bullet$. If Q is the underlying quiver forgetting the markings we have by separating traces that

$$\mathbb{C}[\mathtt{iss}_\alpha Q] \simeq \mathbb{C}[\mathtt{iss}_\alpha Q^\bullet][tr(l_1), \ldots, tr(l_u)]$$

Hence, we do not have to do extra work in the case of marked quivers :

A marked quiver setting $(Q^\bullet, \alpha)$ has a regular ring of invariants if and only if (Q, α) can be reduced to a one of the three final quiver settings of theorem 5.21.

5.8 Central singularities

Surprisingly, the reduction steps of section 5.3 allow us to classify all central singularities of a Cayley-smooth algebra $A \in \mathtt{alg@n}$ up to *smooth equivalence*. Recall that two commutative *local* rings $C_\mathfrak{m}$ and $D_\mathfrak{n}$ are said to ne smooth equivalent if there are numbers k and l such that

$$\hat{C}_\mathfrak{m}[[x_1, \ldots, x_k]] \simeq \hat{D}_\mathfrak{n}[[y_1, \ldots, y_l]]$$

By theorem 5.8 (and its extension to marked quivers) and the étale local classification of Cayley-smooth orders it is enough to classify the rings of invariants of *reduced* marked quiver settings up to smooth equivalence. We can always assume that the quiver Q is strongly connected (if not, the ring of invariants is the tensor product of the rings of invariants of the maximal strongly connected subquivers). Our aim is to classify the reduced quiver singularities up to equivalence, so we need to determine the Krull dimension of the rings of invariants.

Lemma 5.13 *Let $(Q^\bullet, \alpha)$ be a reduced marked quiver setting and Q strongly connected. Then,*

$$dim\ \mathtt{iss}_\alpha\ Q^\bullet = 1 - \chi_Q(\alpha, \alpha) - m$$

where m is the total number of marked loops in $Q^\bullet$.

Proof. Because $(Q^\bullet, \alpha)$ is reduced, none of the vertices satisfies condition C_V^v, whence

$$\chi_Q(\epsilon_v, \alpha) \leq -1 \quad \text{and} \quad \chi_Q(\alpha, \epsilon_v) \leq -1$$

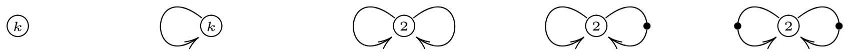
for all vertices v. In particular it follows (because Q is strongly connected) from section 4.3 that α is the dimension vector of a simple representation of Q and that the dimension of the quotient variety

$$dim\ \mathtt{iss}_\alpha\ Q = 1 - \chi_Q(\alpha, \alpha)$$

Finally, separating traces of the loops to be marked gives the required formula. $\square$

Extending theorem 5.21 to the setting of marked quivers, we can classify all smooth points of $\mathtt{triss}_n\ A$ for a Cayley-smooth order A.

Theorem 5.23 *Let $(Q^\bullet, \alpha)$ be a marked quiver setting such that Q is strongly connected. Then $\mathtt{iss}_\alpha\ Q^\bullet$ is smooth if and only if the unique reduced marked quiver setting to which $(Q^\bullet, \alpha)$ can be reduced is one of the following five types*



The next step is to classify for a given dimension d all reduced marked quiver settings $(Q^\bullet, \alpha)$ such that $dim\ \mathtt{iss}_\alpha\ Q^\bullet = d$. The following result limits the possible cases drastically in low dimensions.

Lemma 5.14 *Let $(Q^\bullet, \alpha)$ be a reduced marked quiver setting on $k \geq 2$ vertices. Then,*

$$dim\ \mathtt{iss}_\alpha\ Q^\bullet \geq 1 + \sum^{a\geq 1} a + \sum^{a>1} (2a-1) + \sum^{a>1} (2a) + \sum^{a>1} (a^2+a-2) +$$

$$\sum^{a>1} (a^2+a-1) + \sum^{a>1} (a^2+a) + \ldots + \sum^{a>1} ((k+l-1)a^2+a-k) + \ldots$$

In this sum the contribution of a vertex v with $\alpha_v = a$ is determined by the number of (marked) loops in v. By the reduction steps (marked) loops only occur at vertices where $\alpha_v > 1$.

Proof. We know that the dimension of $\mathtt{iss}_\alpha\ Q^\bullet$ is equal to

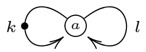
$$1 - \chi_Q(\alpha,\alpha) - m = 1 - \sum_v \chi_Q(\epsilon_v,\alpha)\alpha_v - m$$

If there are no (marked) loops at v, then $\chi_Q(\epsilon_v,\alpha) \leq -1$ (if not we would reduce further) which explains the first sum. If there is exactly one (marked) loop at v then $\chi_Q(\epsilon_v,\alpha) \leq -2$ for if $\chi_Q(\epsilon_v,\alpha) = -1$ then there is just one outgoing arrow to a vertex w with $\alpha_w = 1$ but then we can reduce the quiver setting further. This explains the second and third sums. If there are k marked loops and l ordinary loops in v (and Q has at least two vertices) , then

$$-\chi_Q(\epsilon_v,\alpha)\alpha_v - k \geq ((k+l)\alpha_v - \alpha_v + 1)\alpha_v - k$$

which explains all other sums. $\square$

Observe that the dimension of the quotient variety of the one vertex marked quivers



is equal to $(k+l-1)a^2+1-k$ and is singular (for $a \geq 2$) unless $k+l=2$. We will now classify the reduced singular settings when there are at least two vertices in low dimensions. By the previous lemma it follows immediately that

1. the maximal number of vertices in a reduced marked quiver setting $(Q^\bullet,\alpha)$ of dimension d is $d-1$ (in which case all vertex dimensions must be equal to one)
2. if a vertex dimension in a reduced marked quiver setting is $a \geq 2$, then the dimension $d \geq 2a$.

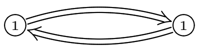
Lemma 5.15 *Let $(Q^\bullet,\alpha)$ be a reduced marked quiver setting such that $\mathtt{iss}_\alpha\ Q^\bullet$ is singular of dimension $d \leq 5$, then $\alpha = (1,\ldots,1)$. Moreover, each vertex must have at least two incoming and two outgoing arrows and no loops.*

Proof. From the lower bound of the sum formula it follows that if some $\alpha_v > 1$ it must be equal to 2 and must have a unique marked loop and there can only be one other vertex w with $\alpha_w = 1$. If there are x arrows from w to v and y arrows from v to w, then

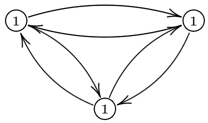
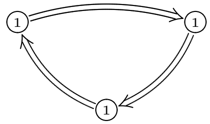
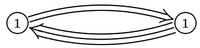
$$dim\ \mathtt{iss}_\alpha\ Q^\bullet = 2(x+y) - 1$$

whence x or y must be equal to 1 contradicting reducedness. The second statement follows as otherwise we could perform extra reductions. $\square$

Proposition 5.13 *The only reduced marked quiver singularity in dimension 3 is*

3_{con} :


The reduced marked quiver singularities in dimension 4 are

4_{3a} :  4_{3b} :  4_2 : 

Proof. All one vertex marked quiver settings with quotient dimension ≤ 5 are smooth, so we are in the situation of lemma 5.15. If the dimension is 3 there must be two vertices each having exactly two incoming and two outgoing arrows, whence the indicated type is the only one. The resulting singularity is the *conifold singularity*

$$\frac{\mathbb{C}[[x,y,u,v]]}{(xy-uv)}$$

In dimension 4 we can have three or two vertices. In the first case, each vertex must have exactly two incoming and two outgoing arrows whence the first two cases. If there are two vertices, then just one of them has three incoming arrows and one has three outgoing arrows. $\square$

Assume that all vertex dimensions are equal to one, then one can write any (trace of an) oriented cycle as a product of (traces of) *primitive* oriented cycles (that is, those that cannot be decomposed further). From this one deduces immediately :

Lemma 5.16 *Let $(Q^\bullet, \alpha)$ be a reduced marked quiver setting such that all $\alpha_v = 1$. Let m be the maximal graded ideal of $\mathbb{C}[\mathtt{rep}_\alpha\ Q^\bullet]^{GL(\alpha)}$, then a vectorspace basis of*

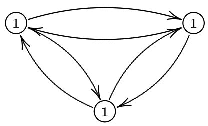
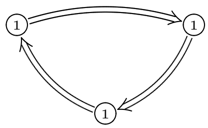
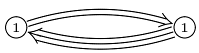
$$\frac{m^i}{m^{i+1}}$$

is given by the oriented cycles in Q which can be written as a product of i primitive cycles but not as a product of $i+1$ such cycles.

Clearly, the dimensions of the quotients m^i/m^{i+1} are (étale) isomorphism invariants. Recall that the first of these numbers m/m^2 is the embedding dimension of the singularity. Hence, for $d \leq 5$ this simple minded counting method can be used to separate quiver singularities.

Theorem 5.24 *There are precisely three reduced quiver singularities in dimension $d = 4$.*

Proof. The number of primitive oriented cycles of the three types of reduced marked quiver settings in dimension four

4_{3a} :  4_{3b} :  4_2 : 

is 5, respectively 8 and 6. Hence, they give nonisomorphic rings of invariants. $\square$

If some of the vertex dimensions are ≥ 2 we have no easy description of the vectorspaces m^i/m^{i+1} and we need a more refined argument. The idea is to answer the question "what other singularities can the reduced singularity see ?" An *α-representation type* is a datum

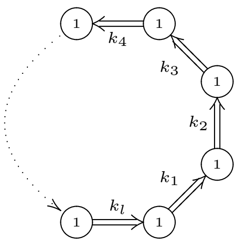
$$\tau = (e_1, \beta_1; \ldots ; e_l, \beta_l)$$

where the e_i are natural numbers ≥ 1, the β_i are dimension vectors of simple representations of Q such that $\alpha = \sum_i e_i \beta_i$. Any neighborhood of the trivial representation contains semi-simple representations of Q of type τ for any α-representation type. Let $(Q_\tau^\bullet, \alpha_\tau)$ be the associated (marked) local quiver setting. Assume that $\mathtt{iss}_{\alpha_\tau}\ Q_\tau$ has a singularity, then the couple

(dimension of strata, type of singularity)

is a characteristic feature of the singularity of $\mathtt{iss}_\alpha\ Q^\bullet$ and one can often distinguish types by these couples. The *fingerprint* of a reduced quiver singularity will be the Hasse diagram of those α-representation types τ such that the local marked quiver setting $(Q_\tau^\bullet, \alpha_\tau)$ can be reduced to a reduced quiver singularity (necessarily occurring in lower dimension and the difference between the two dimensions gives the dimension of the stratum). Clearly, this method fails in case the marked quiver singularity is an *isolated singularity.* Fortunately, we have a complete characterization of these.

Theorem 5.25 *[12] The only reduced marked quiver settings $(Q^\bullet, \alpha)$ such that the quotient variety is an isolated singularity are of the form*



where Q has l vertices and all $k_i \geq 2$. The dimension of the corresponding quotient is

$$d = \sum_i k_i + l - 1$$

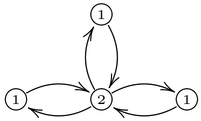
and the unordered l-tuple $\{k_1, \ldots, k_l\}$ *is an (étale) isomorphism invariant of the ring of invariants.*

Not only does this result distinguish among isolated reduced quiver singularities, but it also shows that in all other marked quiver settings we will have additional families of singularities. We will illustrate the method in some detail to separate the reduced marked quiver settings in dimension 6 having one vertex of dimension two.

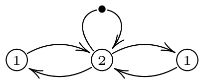
Proposition 5.14 *The reduced singularities of dimension* 6 *such that* α *contains a component equal to* 2 *are pairwise non-equivalent.*

Proof. One can show that the reduced marked quiver setting for $d = 6$ with at least one component ≥ 2 are

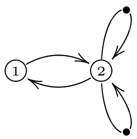
type A



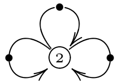
type B



type C

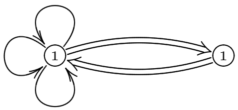


type D



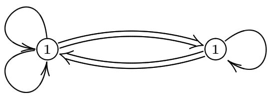
We will order the vertices such that $\alpha_1 = 2$.

type A : There are three different representation types $\tau_1 = (1, (2; 1, 1, 0); 1, (0; 0, 0, 1))$ (and permutations of the 1-vertices). The local quiver setting has the form



because for $\beta_1 = (2; 1, 1, 0)$ and $\beta_2 = (0; 0, 0, 1)$ we have that $\chi_Q(\beta_1, \beta_1) = -2$, $\chi_Q(\beta_1, \beta_2) = -2$, $\chi_Q(\beta_2, \beta_1) = -2$ and $\chi(\beta_2, \beta_2) = 1$. These three representation types each give a three dimensional family of conifold (type 3_{con}) singularities.

Further, there are three different representation types $\tau_2 = (1, (1; 1, 1, 0); 1, (1; 0, 0, 1))$ (and permutations) of which the local quiver setting is of the form

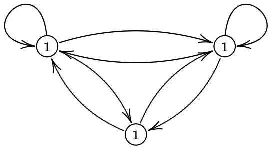


as with $\beta_1 = (1; 1, 1, 0)$ and $\beta_2 = (1; 0, 0, 1)$ we have $\chi_Q(\beta_1, \beta_1) = -1$, $\chi_Q(\beta_1, \beta_2) = -2$, $\chi_Q(\beta_2, \beta_1) = -2$ and $\chi_Q(\beta_2, \beta_2) = 0$. These three representation types each give a three dimensional family of conifold singularities.

Finally, there are the three representation types

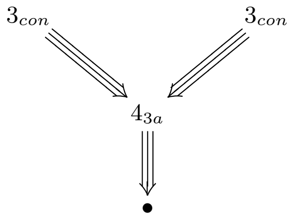
$$\tau_3 = (1, (1; 1, 0, 0); 1, (1; 0, 1, 0); 1, (0; 0, 0, 1))$$

(and permutations) with local quiver setting



These three types each give a two dimensional family of reduced singularities of type 4_{3a}.

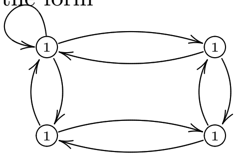
The degeneration order on representation types gives $\tau_1 < \tau_3$ and $\tau_2 < \tau_3$ (but for different permutations) and the *fingerprint* of this reduced singularity can be depicted as



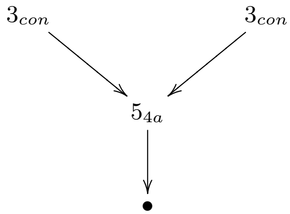
type B : There is one representation type $\tau_1 = (1, (1; 1, 0); 1, (1; 0, 1))$ giving as above a three dimensional family of conifold singularities, one representation type $\tau_2 = (1, (1; 1, 1); 1, (1; 0, 0))$ giving a three dimensional family of conifolds and finally one representation type

$$\tau_3 = (1, (1; 0, 0); 1, (1; 0, 0); 1, (0; 1, 1); 1, (0; 0, 1))$$

of which the local quiver setting has the form



(the loop in the downright corner is removed to compensate for the marking) giving rise to a one-dimensional family of five-dimensional singularities of type 5_{4a}. This gives the fingerprint



type C : We have a three dimensional family of conifold singularities coming from the representation type $(1,(1;1);1,(1;0))$ and a two-dimensional family of type 4_{3a} singularities corresponding to the representation type $(1,(1;0);1,(1,0);1,(0;1))$. Therefore, the fingerprint is depicted as

$$3_{con} \longrightarrow 4_{3a} \longrightarrow \bullet$$

type D : We have just one three-dimensional family of conifold singularities determined by the representation type $(1,(1);1,(1))$ so the fingerprint is $3_{con} \longrightarrow \bullet$. As fingerprints are isomorphism invariants of the singularity, this finishes the proof.

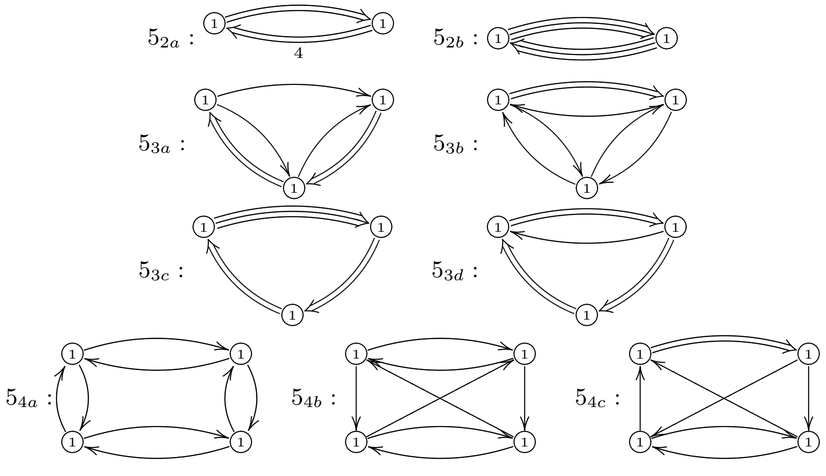
We claim that the minimal number of generators for these invariant rings is 7. The structure of the invariant ring of three 2×2 matrices upto simultaneous conjugation was determined by Ed Formanek [27] who showed that it is generated by 10 elements

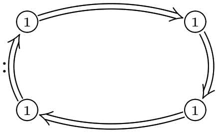
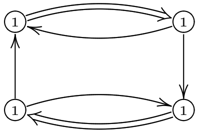
$$\{tr(X_1), tr(X_2), tr(X_3), det(X_1), det(X_2), det(X_3), tr(X_1X_2), tr(X_1X_3), tr(X_2X_3), tr(X_1X_2X_3)\}$$

and even gave the explicit quadratic polynomial satisfied by $tr(X_1X_2X_3)$ with coefficients in the remaining generators. The rings of invariants of the four cases of interest to us are quotients of this algebra by the ideal generated by three of its generators : for type A it is $(det(X_1), det(X_2), det(X_3))$, for type B : $(det(X_1), tr(X_2), det(X_3))$, for type C : $(det(X_1), tr(X_2), tr(X_3))$ and for type D : $(tr(X_1), tr(X_2), tr(X_3))$. $\square$

These two tricks (counting cycles and fingerprinting) are sufficient to classify all central singularities of Cayley-smooth orders for central dimension $d \leq 6$. We will give the details for $d = 5$, the remaining cases for $d = 6$ can be found in the paper [13].

Proposition 5.15 *The reduced marked quiver settings for $d = 5$ are*

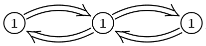


5_{4d} :  5_{4e} : 

Proof. We are in the situation of lemma 5.15 and hence know that all vertex-dimensions are equal to one, every vertex has at least two incoming and two outgoing arrows and the total number of arrows is equal to $5 - 1 + k$ where k is the number of arrows which can be at most 4.

$k = 2$ **:** There are 6 arrows and as there must be at least two incoming arrows in each vertex, the only possibilities are types 5_{2a} and 5_{2b}.

$k = 3$ **:** There are seven arrows. Hence every two vertices are connected, otherwise one needs at least 8 arrows:

.

There is one vertex with 3 incoming arrows and one vertex with 3 outgoing arrows. If these vertices are equal $(= v)$, there are no triple arrows. Call x the vertex with 2 arrows coming from v and y the other one. Because there are already two incoming arrows in x, $\chi_Q(\epsilon_y, \epsilon_x) = 0$. This also implies that $\chi_Q(\epsilon_y, \epsilon_v) = -2$ and $\chi_Q(\epsilon_x, \epsilon_v) = \chi_Q(\epsilon_x, \epsilon_y) = -1$. This gives us setting 5_{3a}. If the two vertices are different, we can delete one arrow between them, which leaves us with a singularity of dimension $d = 4$ (because now all vertices have 2 incoming and 2 outgoing vertices). So starting from the types 4_{3a-b} and adding one extra arrow we obtain three new types 5_{3b-d}.

$k = 4$ **:** There are 8 arrows so each vertex must have exactly two incoming and two outgoing arrows. First consider the cases having no double arrows. Fix a vertex v, there is at least one vertex connected to v in both directions. This is because there are 3 remaining vertices and four arrows connected to v (two incoming and two outgoing). If there are two such vertices, w_1 and w_2, the remaining vertex w_3 is not connected to v. Because there are no double arrows we must be in case 5_{4a}. If there is only one such vertex, the quiver contains two disjoint cycles of length 2. This leads to type 5_{4b}.

If there is precisely one double arrow (from v to w), the two remaining vertices must be contained in a cycle of length 2 (if not, there would be 3 arrows leaving v). This leads to type 5_{4c}.

If there are two double arrows, they can be consecutive or disjoint. In the first case, all arrows must be double (if not, there are three arrows leaving one vertex), so this is type 5_{4d}. In the latter case, let v_1 and v_2 be the starting vertices of the double arrows and w_1 and w_2 the end points. As there are no consecutive double arrows, the two arrows leaving w_1 must go to different vertices not equal to w_2. An analogous condition holds for the arrows leaving w_2 and therefore we are in type 5_{4e}. □

Next, we have to separate the corresponding rings of invariants up to isomorphism.

Theorem 5.26 *There are exactly ten reduced marked quiver singularities in dimension $d = 5$. Only the types 5_{3a} and 5_{4e} have an isomorphic ring of invariants.*

Proof. Recall that the dimension of m/m^2 is given by the number of primitive cycles in Q. These numbers are

type	$dim\ m/m^2$	type	$dim\ m/m^2$
5_{2a}	8	5_{4a}	6
5_{2b}	9	5_{4b}	6
5_{3a}	8	5_{4c}	9
5_{3b}	7	5_{4d}	16
5_{3c}	12	5_{4e}	8
5_{3d}	10		

Type 5_{4a} can be separated from type 5_{4b} because 5_{4a} contains $2+4$ twodimensional families of conifold singularities corresponding to representation types of the form

$$\begin{cases} \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \end{cases} \quad \text{and } 4 \times \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}.$$

whereas type 5_{4b} has only $1+4$ such families as the decomposition

$$\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}$$

is not a valid representation type.

Type 5_{2a} and 5_{2b} are both isolated singularities because we have no non- trivial representation types, whereas types 5_{4c}, and 5_{4e} are not as they have representation types of the form

$$\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}$$

giving local quivers smooth equivalent to type 4_{3b} (in the case of type 5_{4c}) and to type 3_a (in the case of 5_{3e}).

Finally, as we know the algebra generators of the rings of invariants (the primitive cycles) it is not difficult to compute these rings explicitly. Type 5_{3a} and type 5_{4e} have a ring of invariants isomorphic to

$$\frac{\mathbb{C}[X_i, Y_i, Z_{ij} : 1 \leq i, j \leq 2]}{(Z_{11}Z_{22} = Z_{12}Z_{21}, X_1Y_1Z_{22} = X_1Y_2Z_{21} = X_2Y_1Z_{12} = X_2Y_2Z_{11})}$$

$\square$

References

The results of section 5.1 are due to L. Le Bruyn and C. Procesi [60]. The results of sections 5.2, 5.3, 5.4 are due to L. Le Bruyn, [58], [15]. Lemma 5.7 and proposition 5.10 are due to M. Artin [4], [5]. The results of section 5.5 are due to W. Crawley-Boevey and M. Holland [22], [21]. Proposition 5.12 is due to W. Crawley-Boevey [20] and theorem 5.20 is due to R. Bockland and L. Le Bruyn [11]. The results of section 5.7 are due to R. Bocklandt [10], [9] and the classification of central singularities is due to R. Bocklandt, L. Le Bruyn and G. Van de Weyer [13].

6 — Nilpotent Representations

Having obtained some control over the quotient variety $\mathtt{triss}_n\ A$ of a Cayley-smooth algebra A we turn to the study of the *fibers* of the quotient map

$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{triss}_n\ A$$

If $(Q^\bullet, \alpha)$ is the local marked quiver setting of a point $\xi \in \mathtt{triss}_n\ A$ then the GL_n-structure of the fiber $\pi^{-1}(\xi)$ is isomorphic to the $GL(\alpha)$-structure of the *nullcone* $Null_\alpha\ Q^\bullet$ consisting of all nilpotent α-dimensional representations of $Q^\bullet$. In geometric invariant theory, nullcones are investigated by a refinement of the Hilbert criterium : *Hesselink's stratification.*

The main aim of the present chapter is to prove that the different strata in the Hesselink stratification of the nullcone of quiver-representations can be studied via *moduli spaces* of semi-stable quiver-representations. We will illustrate the method first by considering nilpotent m-tuples of $n \times n$ matrices and generalize the results later to quivers and Cayley-smooth orders. The methods allow us to begin to attack the 'hopeless' problem of studying simultaneous conjugacy classes of matrices. We then turn to the description of representation fibers, which can be studied quite explicitly for low-dimensional Cayley-smooth orders, and investigate the fibers of the Brauer-Severi fibration. Before reading the last two sections on Brauer-Severi varieties, it may be helpful to glance through the final chapter where similar, but easier, constructions are studied.

6.1 Cornering matrices

In this section we will outline the main idea of the Hesselink stratification of the nullcone [35] in the generic case, that is, the action of GL_n by simultaneous conjugation on m-tuples of matrices $M_n^m = M_n \oplus \ldots \oplus M_n$. With $Null_n^m$ we denote the nullcone of this action

$$Null_n^m = \{x = (A_1, \ldots, A_m) \in M_n^m \ \mid\ \underline{0} = (0, \ldots, 0) \in \overline{\mathcal{O}(x)}\}$$

It follows from the Hilbert criterium 2.2 that $x = (A_1, \ldots, A_m)$ belongs to the nullcone if and only if there is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that

$$\underset{t \to 0}{lim}\ \lambda(t).(A_1, \ldots, A_m) = (0, \ldots, 0).$$

We recall from proposition 2.5 that any one-parameter subgroup of GL_n is conjugated to one determined by an integral n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ by

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}$$

Moreover, permuting the basis if necessary, we can conjugate this λ to one where the n-tuple if *dominant* , that is, $r_1 \geq r_2 \geq \ldots \geq r_n$. By applying *permutation Jordan-moves* , that is, by simultaneously interchanging certain rows and columns in all A_i, we may therefore assume that the limit-formula holds for a dominant one-parameter subgroup λ of the maximal torus

$$T_n \simeq \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{n} = \{ \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} \mid c_i \in \mathbb{C}^* \} \hookrightarrow GL_n$$

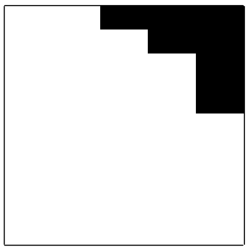
of GL_n. Computing its action on an $n \times n$ matrix A we obtain

$$\begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} t^{-r_1} & & 0 \\ & \ddots & \\ 0 & & r^{-r_n} \end{bmatrix} = \begin{bmatrix} t^{r_1 - r_1} a_{11} & \ldots & t^{r_1 - r_n} a_{1n} \\ \vdots & & \vdots \\ t^{r_n - r_1} a_{n1} & \ldots & t^{r_n - r_n} a_{nn} \end{bmatrix}$$

But then, using dominance $r_i \leq r_j$ for $i \geq j$, we see that the limit is only defined if $a_{ij} = 0$ for $i \geq j$, that is, when A is a strictly upper triangular matrix. We have proved the first 'cornering' result.

Lemma 6.1 *Any m-tuple $x = (A_1, \ldots, A_m) \in Null_n^m$ has a point in its orbit $\mathcal{O}(x)$ under simultaneous conjugation $x' = (A'_1, \ldots, A'_m)$ with all A'_i strictly upper triangular matrices. In fact permutation Jordan-moves suffice to arrive at x'.*

For specific m-tuples $x = (A_1, \ldots, A_m)$ it might be possible to improve on this result. That is, we want to determine the smallest 'corner' C in the upper right hand corner of the matrix, such that all the component matrices A_i can be conjugated simultaneously to matrices A'_i having only non-zero entries in the corner C

$C =$ 

and no strictly smaller corner C' can be found with this property. Our first task will be to compile a list of the relevant corners and to define an order relation on this set. Consider the *weight space decomposition* of M_n^m for the action by simultaneous conjugation of the maximal torus T_n,

$$M_n^m = \oplus_{1 \leq i,j \leq n} M_n^m(\pi_i - \pi_j) = \oplus_{1 \leq i,j \leq n} \mathbb{C}^{\oplus m}_{\pi_i - \pi_j}$$

where $c = diag(c_1, \ldots, c_n) \in T_m$ acts on any element of $M_n^m(\pi_i - \pi_j)$ by multiplication with $c_i c_j^{-1}$, that is, the eigenspace $M_n^m(\pi_i - \pi_j)$ is the space of the (i,j)-entries of the m-matrices. We call

$$\mathcal{W} = \{ \pi_i - \pi_j \mid 1 \leq i,j \leq n \}$$

the set of T_n-*weights* of M_n^m. Let $x = (A_1, \ldots, A_m) \in Null_n^m$ and consider the subset $E_x \subset \mathcal{W}$ consisting of the elements $\pi_i - \pi_j$ such that for at least one of the matrix components A_k the (i,j)-entry is non-zero. Repeating the argument above, we see that if λ is a one-parameter subgroup of T_n determined by the integral n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ such that $lim\ \lambda(t).x = \underline{0}$ we have

$$\forall\ \pi_i - \pi_j \in E_x \quad \text{we have} \quad r_i - r_j \geq 1$$

Conversely, let $E \subset \mathcal{W}$ be a subset of weights, we want to determine the subset

$$\{ s = (s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_i - s_j \geq 1\ \forall\ \pi_i - \pi_j \in E\ \}$$

and determine a point in this set, minimal with respect to the usual norm

$$\| s \| = \sqrt{s_1^2 + \ldots + s_n^2}$$

Let $s = (s_1, \ldots, s_n)$ attain such a minimum. We can partition the entries of s in a disjoint union of *strings*

$$\{ p_i, p_i + 1, \ldots, p_i + k_i \}$$

with $k_i \in \mathbb{N}$ and subject to the condition that all the numbers $p_{ij} \overset{def}{=} p_i + j$ with $0 \leq j \leq k_i$ occur as components of s, possibly with a multiplicity that we denote by a_{ij}. We call a string $string_i = \{ p_i, p_i + 1, \ldots, p_i + k_i \}$ of s *balanced* if and only if

$$\sum_{s_k \in string_i} s_j = \sum_{j=0}^{k_i} a_{ij}(p_i + j) = 0$$

In particular, all balanced strings consists entirely of rational numbers. We have

Lemma 6.2 *Let $E \subset \mathcal{W}$, then the subset of $\mathbb{R}^n$ determined by*

$$\mathbb{R}_E^n = \{\ (r_1, \ldots, r_n) \mid r_i - r_j \geq 1\ \forall\ \pi_i - \pi_j \in E \}$$

has a unique point $s_E = (s_1, \ldots, s_n)$ of minimal norm $\| s_E \|$. This point is determined by the characteristic feature that all its strings are balanced. In particular, $s_E \in \mathbb{Q}^n$.

Proof. Let s be a minimal point for the norm in $\mathbb{R}^n_E$ and consider a string of s and denote with S the indices $k \in \{1, \ldots, n\}$ such that $s_k \in string$. Let $\pi_i - \pi_j \in E$, then if only one of i or j belongs to S we have a strictly positive number a_{ij}

$$s_i - s_j = 1 + r_{ij} \quad \text{with} \quad r_{ij} > 0$$

Take $\epsilon_0 > 0$ smaller than all r_{ij} and consider the n-tuple

$$s_\epsilon = s + \epsilon(\delta_{1S}, \ldots, \delta_{nS}) \quad \text{with} \quad \delta_{kS} = 1 \text{ if } k \in S \text{ and } 0 \text{ otherwise}$$

with $\mid \epsilon \mid \leq \epsilon_0$. Then, $s_\epsilon \in \mathbb{R}^n_E$ for if $\pi_i - \pi_j \in E$ and i and j both belong to S or both do not belong to S then $(s_\epsilon)_i - (s_\epsilon)_j = s_i - s_j \geq 1$ and if one of i or j belong to S, then

$$(s_\epsilon)_i - (s_\epsilon)_j = 1 + r_{ij} \pm \epsilon \geq 1$$

by the choice of ϵ_0. However, the norm of s_ϵ is

$$\parallel s_\epsilon \parallel = \sqrt{\parallel s \parallel + 2\epsilon \sum_{k \in S} s_k + \epsilon^2 \# S}$$

Hence, if the string would not be balanced, $\sum_{k \in S} s_k \neq 0$ and we can choose ϵ small enough such that $\parallel s_\epsilon \parallel < \parallel s \parallel$, contradicting minimality of s. □

For given n we have the following `algorithm` to compile the list $\mathcal{S}_n$ of all dominant n-tuples $(s_1, \ldots, s_n)$ (that is, $s_i \leq s_j$ whenever $i \geq j$) having all its strings balanced.

- List all Young-diagrams $\mathcal{Y}_n = \{Y_1, \ldots\}$ having $\leq n$ boxes.
- For every diagram Y_l fill the boxes with strictly positive integers subject to the rules
  1. the total sum is equal to n
  2. no two rows are filled identically
  3. at most one row has length 1

  This gives a list $\mathcal{T}_n = \{T_1, \ldots\}$ of tableaux.
- For every tableau $T_l \in \mathcal{T}_n$, for each of its rows $(a_1, a_2, \ldots, a_k)$ find a solution p to the linear equation

  $$a_1 x + a_2(x+1) + \ldots + a_k(x+k) = 0$$

  and define the $\sum a_i$-tuple of rational numbers

  $$(\underbrace{p, \ldots, p}_{a_1}, \underbrace{p+1, \ldots, p+1}_{a_2}, \ldots \underbrace{p+k, \ldots, p+k}_{a_k})$$

  Repeating this process for every row of T_l we obtain an n-tuple, which we then order.

The list $\mathcal{S}_n$ will be the combinatorial object underlying the relevant corners and the stratification of the nullcone.

Example 6.1 ($\mathcal{S}_n$ for small n) For $n = 2$, we have $\boxed{1}\boxed{1}$ giving $(\frac{1}{2}, -\frac{1}{2})$ and $\boxed{2}$ giving $(0, 0)$. For $n = 3$ we have five types

$\mathcal{S}_3 =$

tableau	s_1	s_2	s_3	$\lVert s \rVert^2$
$\boxed{1}\boxed{1}\boxed{1}$	1	0	-1	2
$\boxed{1}\boxed{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
$\boxed{2}\boxed{1}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
$\begin{array}{ll}\boxed{1}\boxed{1}\\\boxed{1}\end{array}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
$\boxed{3}$	0	0	0	0

$\mathcal{S}_4$ has eleven types

$\mathcal{S}_4 =$

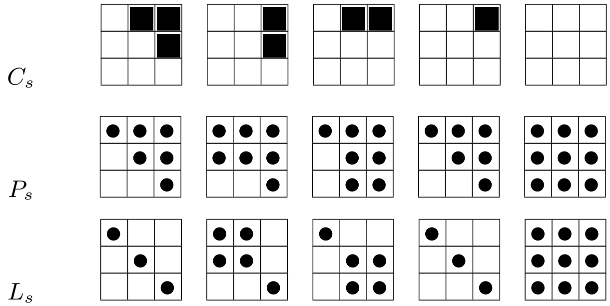
tableau	s_1	s_2	s_3	s_4	$\lVert s \rVert^2$
$\boxed{1}\boxed{1}\boxed{1}\boxed{1}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	5
$\boxed{2}\boxed{1}\boxed{1}$	$\frac{5}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{11}{4}$
$\boxed{1}\boxed{1}\boxed{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$\frac{11}{4}$
$\boxed{1}\boxed{2}\boxed{1}$	1	0	0	-1	2
$\boxed{2}\boxed{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
$\boxed{3}\boxed{1}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$
$\boxed{1}\boxed{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$
$\begin{array}{ll}\boxed{1}\boxed{2}\\\boxed{1}\end{array}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$
$\begin{array}{ll}\boxed{2}\boxed{1}\\\boxed{1}\end{array}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
$\begin{array}{ll}\boxed{1}\boxed{1}\\\boxed{2}\end{array}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
$\boxed{4}$	0	0	0	0	0

Observe that we ordered the elements in $\mathcal{S}_n$ according to $\| s \|$. The reader is invited to verify that $\mathcal{S}_5$ has 28 different types.

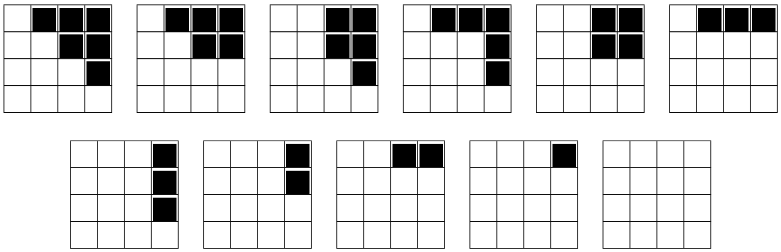
To every $s = (s_1, \ldots, s_n) \in \mathcal{S}_n$ we associate the following data

- the *corner* C_s is the subspace of M_n^m consisting of those m tuples of $n \times n$ matrices with zero entries except perhaps at position (i,j) where $s_i - s_j \geq 1$. A partial ordering is defined on these corners by the rule
$$C_{s'} < C_s \;\Leftrightarrow\; \parallel s' \parallel \;<\; \parallel s \parallel$$
- the *parabolic subgroup* P_s which is the subgroup of GL_n consisting of matrices with zero entries except perhaps at entry (i,j) when $s_i - s_j \geq 0$.
- the *Levi subgroup* L_s which is the subgroup of GL_n consisting of matrices with zero entries except perhaps at entry (i,j) when $s_i - s_j = 0$. Observe that $L_s = \prod GL_{a_{ij}}$ where the a_{ij} are the multiplicities of $p_i + j$.

Example 6.2 Using the sequence of types in the previous example, we have that the relevant corners and subgroup for 3×3 matrices are



For 4×4 matrices the relevant corners are



Returning to the corner-type of an m-tuple $x = (A_1, \ldots, A_m) \in Null_n^m$, we have seen that $E_x \subset \mathcal{W}$ determines a unique $s_{E_x} \in \mathbb{Q}^n$ which up to permuting the entries an element s of $\mathcal{S}_n$. As permuting the entries of s translates into permuting rows and columns in $M_n(\mathbb{C})$ we have

Theorem 6.1 *Every $x = (A_1, \ldots, A_m) \in Null_n^m$ can be brought by permutation Jordan-moves to an m-tuple $x' = (A'_1, \ldots, A'_m) \in C_s$. Here, s is the dominant reordering of s_{E_x} with $E_x \subset \mathcal{W}$ the subset $\pi_i - \pi_j$ determined by the non-zero entries at place (i,j) of one of the components A_k. The permutation of rows and columns is determined by the dominant reordering.*

The m-tuple s (or s_{E_x}) determines a one-parameter subgroup λ_s of T_n where λ corresponds to the unique n-tuple of integers

$$(r_1, \ldots, r_n) \in \mathbb{N}_+ s \cap \mathbb{Z}^n \quad \text{with} \quad gcd(r_i) = 1$$

For any one-parameter subgroup μ of T_n determined by an integral n-tuple $\mu = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and any $x = (A_1, \ldots, A_n) \in Null_n^m$ we define the integer

$$m(x, \mu) = min \; \{ a_i - a_j \; \mid \; x \text{ contains a non-zero entry in } M_n^m(\pi_i - \pi_j) \; \}$$

From the definition of $\mathbb{R}^n_E$ it follows that the minimal value s_E and λ_{s_E} is

$$s_{E_x} = \frac{\lambda_{s_{E_x}}}{m(x, \lambda_{s_{E_x}})} \quad \text{and} \quad s = \frac{\lambda_s}{m(x, \lambda_s)}$$

We can now state to what extend λ_s is an optimal one-parameter subgroup of T_n.

Theorem 6.2 *Let $x = (A_1, \ldots, A_m) \in Null_n^m$ and let μ be a one-parameter subgroup contained in T_n such that $\lim_{t \rightarrow 0} \lambda(t).x = \underline{0}$, then*

$$\frac{\| \lambda_{s_{E_x}} \|}{m(x, \lambda_{s_{E_x}})} \leq \frac{\| \mu \|}{m(x, \mu)}$$

The proof follows immediately from the observation that $\frac{\mu}{m(x,\mu)} \in \mathbb{R}^n_{E_x}$ and the minimality of s_{E_x}. Phrased differently, there is no simultaneous reordering of rows and columns that admit an m-tuple $x" = (A"_1, \ldots, A"_m) \in C_{s'}$ for a corner $C_{s'} < C_s$. In the next section we will improve on this result.

6.2 Optimal corners

We have seen that one can transform an m-tuple $x = (A_1, \ldots, A_m) \in Null_n^m$ by interchanging rows and columns to an m-tuple in corner-form C_s. However, it is possible that another point in the orbit $\mathcal{O}(x)$ say $y = g.x = (B_1, \ldots, B_m)$ can be transformed by permutation Jordan moves in a strictly smaller corner.

Example 6.3 Consider one 3×3 nilpotent matrix of the form

$$x = \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad ab \neq 0$$

Then, $E_x = \{\pi_1 - \pi_2, \pi_1 - \pi_3\}$ and the corresponding $s = s_{E_x} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ so x is clearly of corner type

$$C_s = \begin{array}{|c|c|c|}\hline & \blacksquare & \blacksquare \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

However, x is a nilpotent matrix of rank 1 and by the Jordan-normalform we can conjugate it in standard form, that is, there is some $g \in GL_3$ such that

$$y = g.x = gxg^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For this y we have $E_y = \{\pi_1 - \pi_2\}$ and the corresponding $s_{E_y} = (\frac{1}{2}, -\frac{1}{2}, 0)$, which can be brought into standard dominant form $s' = (\frac{1}{2}, 0, -\frac{1}{2})$ by interchanging the two last entries. Hence, by interchanging the last two rows and columns, y is indeed of corner type

$$C_{s'} = \begin{array}{|c|c|c|}\hline & & \blacksquare \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

and we have that $C_{s'} < C_s$.

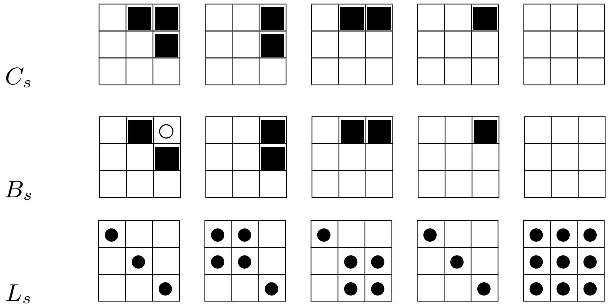
We have used the Jordan-normalform to produce this example. As there are no known canonical forms for m tuples of $n \times n$ matrices, it is a difficult problem to determine the optimal corner type in general.

Definition 6.1 *We say that $x = (A_1, \ldots, A_m) \in Null_n^m$ is of* optimal corner type C_s *if after reordering rows and columns, x is of corner type C_s and there is no point $y = g.x$ in the orbit which is of corner type $C_{s'}$ with $C_{s'} < C_s$.*

We can give an elegant solution to the problem of determining the optimal corner type of an m-tuple in $Null_n^m$ by using results on θ-semistable representations. We assume that $x = (A_1, \ldots, A_m)$ is brought into corner type C_s with $s = (s_1, \ldots, s_n) \in \mathcal{S}_n$. We will associate a quiver-representation to x. As we are interested in checking whether we can transform x to a smaller corner-type, it is intuitively clear that the *border* region of C_s will be important.

- the *border* B_s is the subspace of C_s consisting of those m-tuples of $n \times n$ matrices with zero entries except perhaps at entries (i,j) where $s_i - s_j = 1$.

Example 6.4 For 3×3 matrices we have the following corner-types C_s having border-regions B_s and associated Levi-subgroups L_s



For 4×4 matrices the relevant data are given in figure 6.1

From these examples, it is clear that the action of the Levi-subgroup L_s on the border B_s is a quiver-setting. In general, let $s \in \mathcal{S}_n$ be determined by the tableau T_s, then the *associated quiver-setting* (Q_s, α_s) is

- Q_s is the quiver having as many connected components as there are rows in the tableau T_s. If the i-th row in T_s is
$$(a_{i0}, a_{i1}, \ldots, a_{ik_i})$$
then the corresponding string of entries in s is of the form
$$\{\underbrace{p_i, \ldots, p_i}_{a_{i0}}, \underbrace{p_i + 1, \ldots, p_i + 1}_{a_{i1}}, \ldots, \underbrace{p_i + k_i, \ldots, p_i + k_i}_{a_{ik_i}}\}$$
and the i-th component of Q_s is defined to be the quiver Q_i on $k_i + 1$ vertices having m arrows between the consecutive vertices, that is Q_i is
$$\textcircled{0} \xRightarrow{m} \textcircled{1} \xRightarrow{m} \textcircled{2} \xRightarrow{m} \cdots \xRightarrow{m} \textcircled{k_i}$$

- the dimension vector α_i for the i-th component quiver Q_i is equal to the i-th row of the tableau T_s, that is
$$\alpha_i = (a_{i0}, a_{i1}, \ldots, a_{ik_i})$$
and the total dimension vector α_s is the collection of these component dimension vectors.

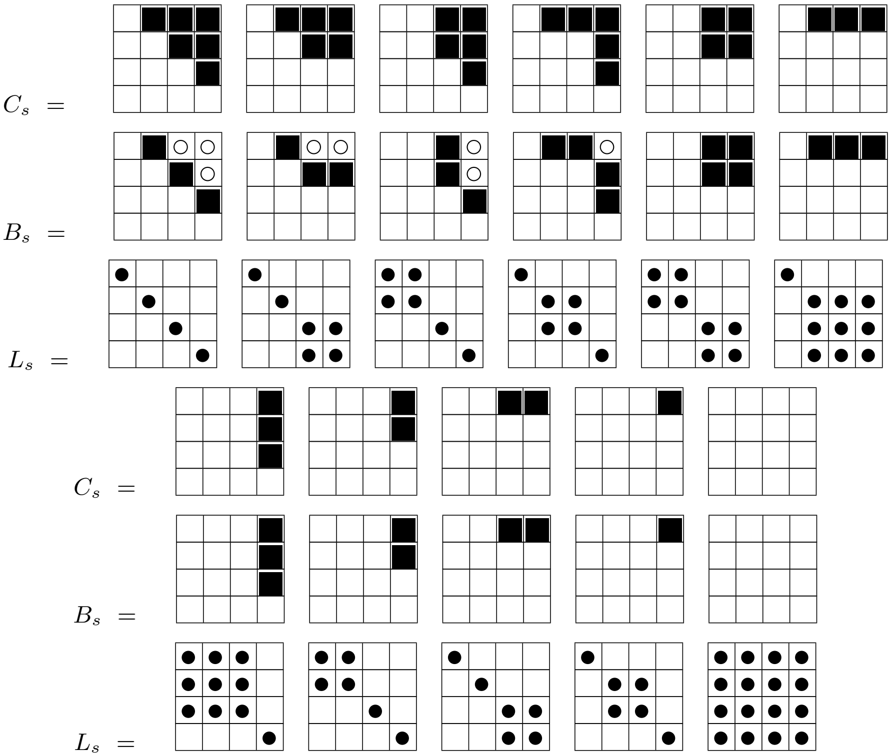


Figure 6.1: Corners and borders for 4×4 matrices.

- the character $GL(\alpha_s) \xrightarrow{\chi_s} \mathbb{C}^*$ is determined by the integral n-tuple $\theta_s = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ where if entry k corresponds to the j-th vertex of the i-th component of Q_s we have

$$t_k = n_{ij} \overset{\text{def}}{=} d.(p_i + j)$$

where d is the least common multiple of the numerators of the p_i's for all i. Equivalently, the n_{ij} are the integers appearing in the description of the one-parameter subgroup $\lambda_s = (r_1, \ldots, r_n)$ grouped together according to the ordering of vertices in the quiver Q_s. Recall that the character χ_s is then defined to be

$$\chi_s(g_1 \ldots, g_n) = \prod_{i=1}^{n} det(g_i)^{t_i}$$

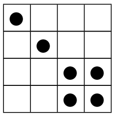
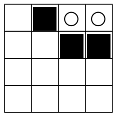
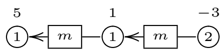
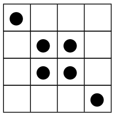
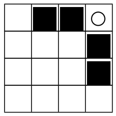
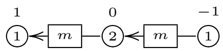
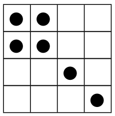
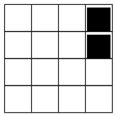
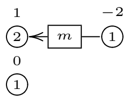
or in terms of $GL(\alpha_s)$ it sends an element $g_{ij} \in GL(\alpha_s)$ to $\prod_{i,j} \; det(g_{ij})^{n_{ij}}$.

Proposition 6.1 *The action of the Levi-subgroup $L_s = \prod_{i,j} GL_{a_{ij}}$ on the border B_s coincides with the base-change action of $GL(\alpha_s)$ on the representation space $\mathtt{rep}_{\alpha_s} \; Q_s$. The isomorphism*

$$B_s \longrightarrow \mathtt{rep}_{\alpha_s} \; Q_s$$

is given by sending an m-tuple of border B_s-matrices $(A_1, \ldots, A_m)$ to the representation in $\mathtt{rep}_{\alpha_s} \; Q_s$ where the j-th arrow between the vertices v_a and v_{a+1} of the i-th component quiver Q_i is given by the relevant block in the matrix A_j.

Example 6.5 We illustrate these definitions with a few examples for 4×4 matrices

tableau	L_s	B_s	θ_s	$(Q_s, \alpha_s, \theta_s)$
2 1 1			$(5, 1, -3, -3)$	
1 2 1			$(1, 0, 0, -1)$	
1 2 1			$(1, 1, 0, -2)$	

Using these conventions we can now state the main result of this section, giving a solution to the problem of optimal corners.

Theorem 6.3 *Let $x = (A_1, \ldots, A_m) \in Null_n^m$ be of corner type C_s. Then, x is of optimal corner type C_s if and only if under the natural maps*

$$C_s \twoheadrightarrow B_s \xrightarrow{\simeq} \mathtt{rep}_{\alpha_s}\ Q_s$$

(the first map forgets the non-border entries) x is mapped to a θ_s-semistable representation in $\mathtt{rep}_{\alpha_s}\ Q_s$.

6.3 Hesselink stratification

Every orbit in $Null_n^m$ has a representative $x = (A_1, \ldots, A_m)$ with all A_i strictly upper triangular matrices. That is, if $N \subset M_n$ is the subspace of strictly upper triangular matrices, then the action map determines a surjection

$$GL_n \times N^m \overset{ac}{\twoheadrightarrow} Null_n^m$$

Recall that the standard *Borel subgroup* B is the subgroup of GL_n consisting of all upper triangular matrices and consider the action of B on $GL_n \times M_n^m$ determined by

$$b.(g, x) = (gb^{-1}, b.x)$$

Then, B-orbits in $GL_n \times N^m$ are mapped under the action map ac to the same point in the nullcone $Null_n^m$. Consider the morphisms

$$GL_n \times M_n^m \overset{\pi}{\twoheadrightarrow} GL_n/B \ \times M_n^m$$

which sends a point (g, x) to $(gB, g.x)$. The quotient GL_n/B is called a *flag variety* and is a projective manifold. Its points are easily seen to correspond to complete *flags*

$$\mathcal{F} \ : \ 0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n \quad \text{with} \quad dim_{\mathbb{C}}\ F_i = i$$

of subspaces of $\mathbb{C}^n$. For example, if $n = 2$ then $GL_2/B \simeq \mathbb{P}^1$. Consider the fiber π^{-1} of a point $(\overline{g}, (B_1, \ldots, B_m)) \in GL_n/B \times M_n^m$. These are the points

$$(h, (A_1, \ldots, A_m)) \quad \text{such that} \quad \begin{cases} g^{-1}h & = b \in B \\ bA_ib^{-1} & = g^{-1}B_ig \quad \text{for all } 1 \leq i \leq m. \end{cases}$$

Therefore, the fibers of π are precisely the B-orbits in $GL_n \times M_n^m$. That is, there exists a quotient variety for the B-action on $GL_n \times M_n^m$ which is the trivial vectorbundle of rank mn^2

$$\mathcal{T} = GL_n/B \times M_n^m \overset{p}{\twoheadrightarrow} GL_n/B$$

$$\begin{tikzcd} GL_n \times^B U \arrow[r, "\simeq"] \arrow[d, hook] & GL_n.U \arrow[d, hook] \\ GL_n \times^B N^m \arrow[r, "ac"] & Null_n^m \end{tikzcd}$$

Figure 6.2: Resolution of the nullcone.

over the flag variety GL_n/B. We will denote with $GL_n \times^B N^m$ the image of the subvariety $GL_n \times N^m$ of $GL_n \times M_n^m$ under this quotient map. That is, we have a commuting diagram

$$\begin{tikzcd} GL_n \times N^m \arrow[r, hook] \arrow[d, two heads] & GL_n \times M_n^m \arrow[d, two heads] \\ GL_n \times^B N^m \arrow[r, hook] & GL_n/B \times M_n^m \end{tikzcd}$$

Hence, $\mathcal{V} = GL_n \times^B N^m$ is a sub-bundle of rank $m.\frac{n(n-1)}{2}$ of the trivial bundle $\mathcal{T}$ over the flag variety. Note however that $\mathcal{V}$ itself is not trivial as the action of GL_n does not map N^m to itself.

Theorem 6.4 *Let U be the open subvariety of m-tuples of strictly upper triangular matrices N^m consisting of those tuples such that one of the component matrices has rank $n-1$. The action map ac induces the commuting diagram of figure 6.2. The upper map is an isomorphism of GL_n-varieties for the action on fiber bundles to be left multiplication in the first component.*

Therefore, there is a natural one-to-one correspondence between GL_n-orbits in $GL_n.U$ and B-orbits in U. Further, ac is a desingularization of the nullcone and $Null_n^m$ is irreducible of dimension

$$(m+1)\frac{n(n-1)}{2}.$$

Proof. Let $A \in N$ be a strictly upper triangular matrix of rank $n-1$ and $g \in GL_n$ such that $gAg^{-1} \in N$, then $g \in B$ as one verifies by first bringing A into Jordan-normal form $J_n(0)$. This implies that over a point $x = (A_1, \ldots, A_m) \in U$ the fiber of the action map

$$GL_n \times N^m \xrightarrow{ac} Null_n^m$$

has dimension $\frac{n(n-1)}{2} = dim\ B$. Over all other points the fiber has at least dimension $\frac{n(n-1)}{2}$.But then, by the dimension formula we have

$$dim\ Null_n^m = dim\ GL_n + dim\ N^m - dim\ B = (m+1)\frac{n(n-1)}{2}$$

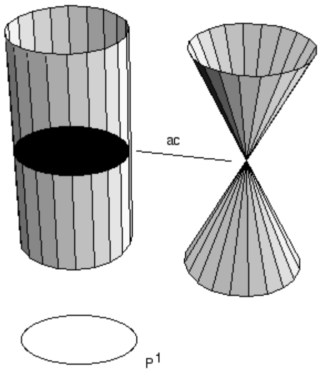


Figure 6.3: Resolution of $Null_2^1$.

Over $GL_n.U$ this map is an isomorphism of GL_n-varieties. Irreducibility of $Null_n^m$ follows from surjectivity of ac as $\mathbb{C}[Null_n^m] \hookrightarrow \mathbb{C}[GL_n] \otimes \mathbb{C}[N^m]$ and the latter is a domain. These facts imply that the induced action map

$$GL_n \times^B N^m \xrightarrow{ac} Null_n^m$$

is birational and as the former is a smooth variety (being a vectorbundle over the flag manifold), this is a desingularization. $\square$

Example 6.6 Let $n = 2$ and $m = 1$. We have seen in chapter 3 that $Null_2^1$ is a cone in 3-space with the singular top the orbit of the zero-matrix and the open complement the orbit of

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

In this case the flag variety is $\mathbb{P}^1$ and the fiber bundle $GL_2 \times^B N$ has rank one. The action map is depicted in figure 6.3 and is a GL_2-isomorphism over the complement of the fiber of the top.

Theorem 6.4 gives us a complexity-reduction, both in the dimension of the acting group and in the dimension of the space acted upon, from

- GL_n-orbits in the nullcone $Null_n^m$, to
- B-orbits in N^m.

at least on the stratum $GL_n.U$ described before. The aim of the *Hesselink stratification* of the nullcone is to extend this reduction also to the complement.

Let $s \in \mathcal{S}_n$ and let C_s be the vectorspace of all m-tuples in M_n^m which are of corner-type C_s. We have seen that there is a Zariski open subset (but, possibly empty) U_s of C_s consisting of m-tuples of optimal corner type C_s. Observe that the action of conjugation of GL_n on M_n^m induces an action of the associated parabolic subgroup P_s on C_s.

Definition 6.2 *The* Hesselink stratum S_s *associated to* s *is the subvariety* $GL_n.U_s$ *where* U_s *is the open subset of* C_s *consisting of the optimal* C_s*-type tuples.*

Theorem 6.5 *With notations as before we have a commuting diagram*

$$\begin{array}{ccc} GL_n \times^{P_s} U_s & \xrightarrow{\simeq} & S_s \\ \cap\big\downarrow & & \cap\big\downarrow \\ GL_n \times^{P_s} C_s & \xrightarrow{ac} & \overline{S_s} \end{array}$$

where ac *is the action map,* $\overline{S_s}$ *is the Zariski closure of* S_s *in* $Null_n^m$ *and the upper map is an isomorphism of* GL_n*-varieties.*

Here, GL_n/P_s *is the flag variety associated to the parabolic subgroup* P_s *and is a projective manifold. The variety* $GL_n \times^{P_s} C_s$ *is a vectorbundle over the flag variety* GL_n/P_s *and is a subbundle of the trivial bundle* $GL_n \times^{P_s} M_n^m$.

Therefore, the Hesselink stratum S_s *is an irreducible smooth variety of dimension*

$$\begin{aligned} dim\ S_s &= dim\ GL_n/P_s + rk\ GL_n \times^{P_s} C_s \\ &= n^2 - dim\ P_s + dim_{\mathbb{C}}\ C_s \end{aligned}$$

and there is a natural one-to-one correspondence between the GL_n*-orbits in* S_s *and the* P_s*-orbits in* U_s.

Moreover, the vectorbundle $GL_n \times^{P_s} C_s$ *is a desingularization of* $\overline{S_s}$ *hence 'feels' the gluing of* S_s *to the remaining strata. Finally, the ordering of corners has the geometric interpretation*

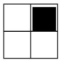
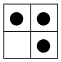
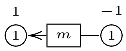
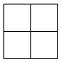
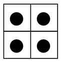
$$\overline{S_s} \subset \bigcup_{\|s'\| \leq \|s\|} S_{s'}$$

We have seen that $U_s = p^{-1}\ \mathtt{rep}^{ss}_{\alpha_s}(Q_s, \theta_s)$ where $C_s \xrightarrow{p} B_s$ is the canonical projection forgetting the non-border entries. As the action of the parabolic subgroup P_s restricts to the action of its Levi-part L_s on $B_s = \mathtt{rep}_{\alpha_s}\ Q$ we have a canonical projection

$$U_s/P_s \xrightarrow{p} M^{ss}_{\alpha_s}(Q_s, \theta_s)$$

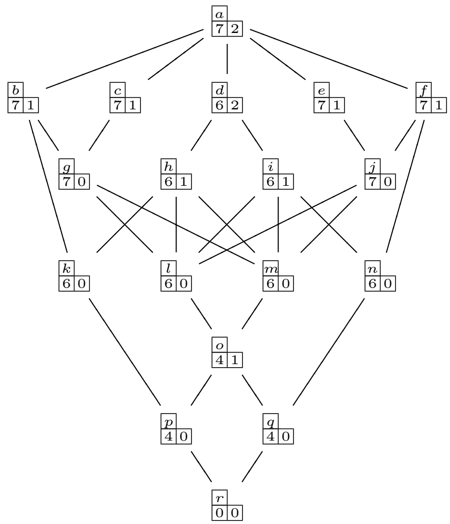
to the moduli space of θ_s-semistable representations in $\mathtt{rep}_{\alpha_s}\ Q_s$. As none of the components of Q_s admits cycles, these moduli spaces are projective varieties. For small values of m and n these moduli spaces give good approximations to the study of the orbits in the nullcone.

Example 6.7 (Nullcone of m-tuples of 2×2 matrices) In the first volume we have seen by a brute force method that the orbits in $Null_2^2$ correspond to points on $\mathbb{P}^1$ together with one extra orbit, the zero representation. For arbitrary m, the relevant strata-information for $Null_2^m$ is contained in the following table

tableau	s	$B_s = C_s$	P_s	$(Q_s, \alpha_s, \theta_s)$
$\boxed{1}\boxed{1}$	$(\frac{1}{2}, -\frac{1}{2})$			
$\boxed{2}$	$(0, 0)$			

Because $B_s = C_s$ we have that the orbit space $U_s/P_s \simeq M^{ss}_{\alpha_s}(Q_s, \theta_s)$. For the first stratum, every representation in $\mathtt{rep}_{\alpha_s}\ Q_s$ is θ_s-semistable except the zero-representation (as it contains a subrepresentation of dimension $\beta = (1, 0)$ and $\theta_s(\beta) = -1 < 0$. The action of $L_s = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^m - \underline{0}$ has as orbit space $\mathbb{P}^{m-1}$, classifying the orbits in the maximal stratum. The second stratum consists of one point, the zero representation.

Example 6.8 A more interesting application, illustrating all of the general phenomena, is the description of orbits in the nullcone of two 3×3 matrices. H. Kraft described them in [50, p. 202] by brute force. The orbit space decomposes as a disjoint union of tori and can be represented by the picture



Here, each node corresponds to a torus of dimension the right-hand side number in the bottom row. A point in this torus represents an orbit with dimension the left-hand side number. The top letter is included for classification purposes. That is, every orbit has a unique representant in the following list of couples of 3×3 matrices (A, B). The top letter gives the torus, the first 2 rows give the first two rows of A and the last two rows give the first two rows of B, $x, y \in \mathbb{C}^*$

a		
0	1	0
0	0	1
0	x	0
0	0	y

b		
0	1	0
0	0	1
0	0	0
0	0	x

c		
0	1	0
0	0	1
0	x	0
0	0	0

d		
0	1	0
0	0	1
0	x	y
0	0	x

e		
0	1	0
0	0	1
0	x	0
0	0	0

f		
0	0	0
0	0	1
0	1	0
0	0	x

g		
0	1	0
0	0	0
0	0	0
0	0	1

h		
0	1	0
0	0	1
0	0	x
0	0	0

i		
0	0	x
0	0	0
0	1	0
0	0	1

j		
0	0	0
0	0	1
0	1	0
0	0	0

k		
0	0	1
0	0	0
0	1	0
0	0	0

l		
0	0	0
0	0	1
0	0	1
0	0	0

m		
0	0	1
0	0	0
0	1	0
0	0	0

n		
0	0	0
0	0	0
0	1	0
0	0	1

o		
0	1	0
0	0	0
0	x	0
0	0	0

p		
0	1	0
0	0	0
0	0	0
0	0	0

q		
0	0	0
0	0	0
0	1	0
0	0	0

r		
0	0	0
0	0	0
0	0	0
0	0	0

We will now derive this result from the above description of the Hesselink stratification. To begin, the relevant data concerning $\mathcal{S}_3$ is summarized in the following table

tableau	s	B_s, C_s	P_s	$(Q_s, \alpha_s, \theta_s)$
$\boxed{1}\boxed{1}\boxed{1}$	$(1, 0, -1)$	$\begin{array}{\|c\|c\|c\|} \hline & \blacksquare & \bullet \\ \hline & & \blacksquare \\ \hline & & \\ \hline \end{array}$	$\begin{array}{\|c\|c\|c\|} \hline \bullet & \bullet & \bullet \\ \hline & \bullet & \bullet \\ \hline & & \bullet \\ \hline \end{array}$	$\begin{tikzcd} \overset{1}{1} & \overset{0}{1} \arrow[l, bend right] \arrow[l, bend left] & \overset{-1}{1} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$
$\boxed{1}\boxed{2}$	$(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$	$\begin{array}{\|c\|c\|c\|} \hline & & \blacksquare \\ \hline & & \blacksquare \\ \hline & & \\ \hline \end{array}$	$\begin{array}{\|c\|c\|c\|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline & & \bullet \\ \hline \end{array}$	$\begin{tikzcd} \overset{1}{2} & \overset{-2}{1} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$
$\boxed{2}\boxed{1}$	$(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$	$\begin{array}{\|c\|c\|c\|} \hline & \blacksquare & \blacksquare \\ \hline & & \\ \hline & & \\ \hline \end{array}$	$\begin{array}{\|c\|c\|c\|} \hline \bullet & \bullet & \bullet \\ \hline & \bullet & \bullet \\ \hline & \bullet & \bullet \\ \hline \end{array}$	$\begin{tikzcd} \overset{2}{1} & \overset{-1}{2} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$
$\begin{array}{ll} \boxed{1} & \boxed{1} \\ \boxed{1} & \end{array}$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\begin{array}{\|c\|c\|c\|} \hline & & \blacksquare \\ \hline & & \\ \hline & & \\ \hline \end{array}$	$\begin{array}{\|c\|c\|c\|} \hline \bullet & \bullet & \bullet \\ \hline & \bullet & \bullet \\ \hline & & \bullet \\ \hline \end{array}$	$\begin{tikzcd} \overset{1}{1} & \overset{-1}{1} \arrow[l, bend right] \arrow[l, bend left] \\ \overset{0}{1} & \end{tikzcd}$
$\boxed{3}$	$(0, 0, 0,)$	$\begin{array}{\|c\|c\|c\|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$	$\begin{array}{\|c\|c\|c\|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$	$\overset{0}{3}$

For the last four corner types, $B_s = C_s$ whence the orbit space U_s/P_s is isomorphic to the moduli space $M^{ss}_{\alpha_s}(Q_s, \theta_s)$. Consider the quiver-setting

$$\begin{tikzcd} \overset{1}{2} & \overset{-2}{1} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

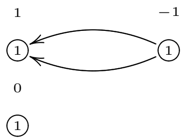
If the two arrows are not linearly independent, then the representation contains a proper sub-representation of dimension-vector $\beta = (1, 1)$ or $(1, 0)$ and in both cases $\theta_s(\beta) < 0$ whence the representation is not θ_s-semistable. If the two arrows are linearly independent, we can use the GL_2-component to bring them in the form $(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$, whence $M^{ss}_{\alpha_s}(Q_s, \alpha_s)$ is reduced to one point, corresponding to the matrix-couple of type l

$$(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

A similar argument, replacing linear independence by common zero-vector shows that also the quiver-setting corresponding to the tableau $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$ has one point as its moduli space, the matrix-tuple of type k. Incidentally, this shows that the corners corresponding to the tableaux $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ cannot be optimal when $m = 1$ as then the row or column vector always has a kernel or cokernel whence cannot be θ_s-semistable. This of course corresponds to the fact that the only orbits in $Null_3^1$ are those corresponding to the Jordan-matrixes

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

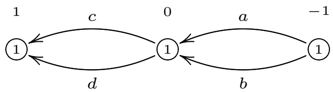
which are respectively of corner type $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|}\hline 3 \\ \hline\end{array}$, whence the two other types do not occur. Next, consider the quiver setting



A representation in $\mathtt{rep}_{\alpha_s}\ Q_s$ is θ_s-semistable if and only if the two maps are not both zero (otherwise, there is a subrepresentation of dimension $\beta = (1,0)$ with $\theta_s(\beta) < 0$). The action of $GL(\alpha_s) = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2 - \underline{0}$ has a s orbit space $\mathbb{P}^1$ and they are represented by matrix-couples

$$(\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

with $[a:b] \in \mathbb{P}^1$ giving the types o,p and q. Clearly, the stratum $\begin{array}{|c|}\hline 3 \\ \hline\end{array}$ consists just of the zero-matrix, which is type r. Remains to investigate the quiver-setting



Again, one easily verifies that a representation in $\mathtt{rep}_{\alpha_s}\ Q_s$ is θ_s-semistable if and only if $(a,b) \neq (0,0) \neq (c,d)$ (for otherwise one would have subrepresentations of dimensions $(1,1,0)$ or $(1,0,0)$). The corresponding $GL(\alpha_s)$-orbits are classified by

$$M^{ss}_{\alpha_s}(Q_s.\theta_s) \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

corresponding to the matrix-couples of types a, b, c, e, f, g, j, k and n

$$\left(\begin{bmatrix} 0 & c & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \right)$$

where $[a : b]$ and $[c : d]$ are points in $\mathbb{P}^1$. In this case, however, $C_s \neq B_s$ and we need to investigate the fibers of the projection

$$U_s/P_s \overset{p}{\longrightarrow\!\!\!\!\!\to} M^{ss}_{\alpha_s}(Q_s, \alpha_s)$$

Now, P_s is the Borel subgroup of upper triangular matrices and one verifies that the following two couples

$$\left(\begin{bmatrix} 0 & c & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \right) \text{ and } \left(\begin{bmatrix} 0 & c & x \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & y \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \right)$$

lie in the same B-orbit if and only if $det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \neq 0$, that is, if and only if $[a : b] \neq [c : d]$ in $\mathbb{P}^1$. Hence, away from the diagonal p is an isomorphism. On the diagonal one can again verify by direct computation that the fibers of p are isomorphic to $\mathbb{C}$, giving rise to the cases d, h and i in the classification.

The connection between this approach and Kraft's result is depicted in figure 6.4. The picture on the left is Kraft's toric degeneration picture where we enclosed all orbits belonging to the same Hesselink strata, that is, having the same optimal corner type. The dashed region enclosed the orbits which do not come from the moduli spaces $M^{ss}_{\alpha_s}(Q_s, \theta_s)$, that is, those coming from the projection $U_s/P_s \longrightarrow\!\!\!\!\!\to M^{ss}_{\alpha_s}(Q_s, \theta_s))$. The picture on the right gives the ordering of the relevant corners.

Example 6.9 We see that we get most orbits in the nullcone from the moduli spaces $M^{ss}_{\alpha_s}(Q_s, \theta_s)$. The reader is invited to work out the orbits in $Null^2_4$. We list here the moduli spaces of the relevant

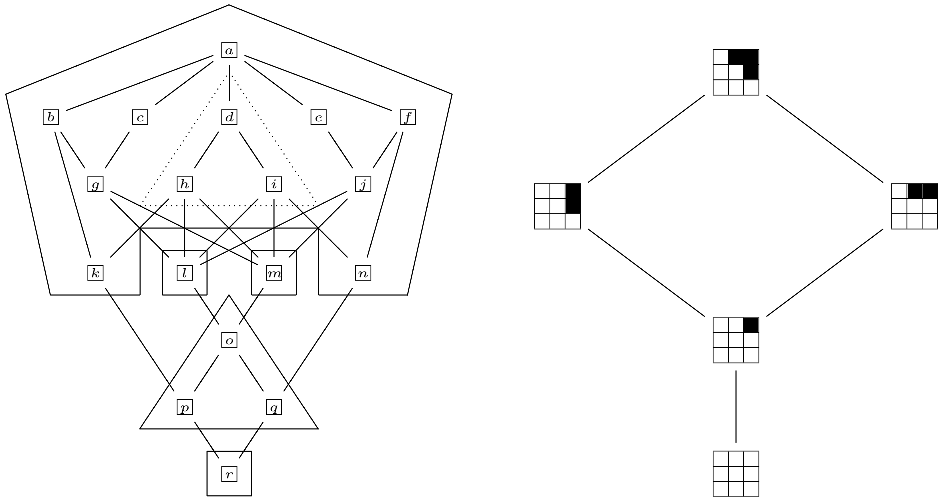
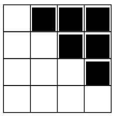
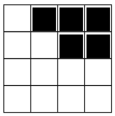
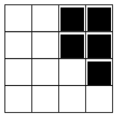
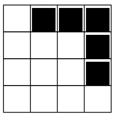
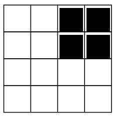
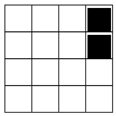
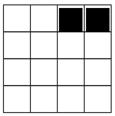
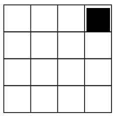
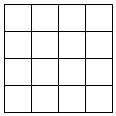
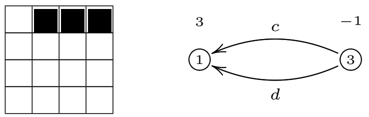


Figure 6.4: Nullcone of couples of 3×3 matrices.

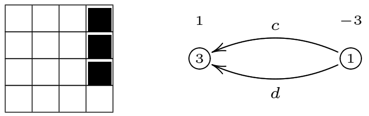
corners

corner	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$	corner	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$	corner	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$
	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$		$\mathbb{P}^1$		$\mathbb{P}^1$
	$\mathbb{P}^3 \sqcup \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1$		$\mathbb{P}^1 \sqcup S^2(\mathbb{P}^1)$		$\mathbb{P}^0$
	$\mathbb{P}^1$		$\mathbb{P}^1$		$\mathbb{P}^0$

Observe that two potential corners are missing in this list. This is because we have the following quiver setting for the corner



and there are no θ_s-semistable representations as the two maps have a common kernel, whence a subrepresentation of dimension $\beta = (1, 0)$ and $\theta_s(\beta) < 0$. A similar argument holds for the other missing corner and quiver setting



For general n, a similar argument proves that the corners associated to the tableaux $\boxed{1}\boxed{n}$ and $\boxed{n}\boxed{1}$ are not optimal for tuples in $Null^m_{n+1}$ unless $m \geq n$. It is also easy to see that with $m \geq n$ all relevant corners appear in $Null^m_{n+1}$, that is all potential Hesselink strata are non-empty.

6.4 Cornering quiver representations

In this section we generalize the results on matrices to representation of arbitrary quivers. Let Q be a quiver on k vertices $\{v_1, \ldots, v_k\}$ and fix a dimension vector $\alpha = (a_1, \ldots, a_k)$ and denote the total dimension $\sum_{i=1}^k a_i$ by a. A representation $V \in \mathtt{rep}_\alpha\ Q$ is said to belong to the nullcone $Null_\alpha\ Q$ if the trivial representation $\underline{0} \in \overline{\mathcal{O}(V)}$. Equivalently, all polynomial invariants are zero when evaluated in V, that is, the traces of all oriented cycles in Q are zero in V. By the Hilbert criterium 2.2 for $GL(\alpha)$, $V \in Null_\alpha\ Q$ if and only if there is a one-parameter subgroup

$$\mathbb{C}^* \xrightarrow{\lambda} GL(\alpha) = \begin{bmatrix} GL_{a_1} & & \\ & \ddots & \\ & & GL_{a_k} \end{bmatrix} \hookrightarrow GL_a$$

such that $\underset{\rightarrow}{lim}\ \lambda(t).V = \underline{0}$. Up to conjugation in $GL(\alpha)$, or equivalently, replacing V by another point in the orbit $\mathcal{O}(V)$, we may assume that λ lies in the maximal torus T_a of $GL(\alpha)$ (and of GL_a) and can be represented by an integral a-tuple $(r_1, \ldots, r_a) \in \mathbb{Z}^a$ such that

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & \\ & \ddots & \\ & & t^{r_a} \end{bmatrix}$$

We have to take the vertices into account, so we decompose the integer interval $[1,2,\ldots,a]$ into *vertex intervals* I_{v_i} such that

$$[1,2,\ldots,a] = \sqcup_{i=1}^{k} I_{v_i} \quad \text{with} \quad I_{v_i} = [\sum_{j=1}^{i-1} a_j + 1, \ldots, \sum_{j=1}^{i} a_j]$$

If we recall that the weights of T_a are isomorphic to $\mathbb{Z}^a$ having canonical generators π_p for $1 \leq p \leq a$ we can decompose the representation space into weight spaces

$$\mathtt{rep}_\alpha\ Q = \bigoplus_{\pi_{pq} = \pi_q - \pi_p} \mathtt{rep}_\alpha\ Q(\pi_{pq})$$

where the eigenspace of π_{pq} is non-zero if and only if for $p \in I_{v_i}$ and $q \in I_{v_j}$, there is an arrow

$$\textcircled{j} \longleftarrow \textcircled{i}$$

in the quiver Q. Call $\pi_\alpha\ Q$ the set of weights π_{pq} which have non-zero eigenspace in $\mathtt{rep}_\alpha\ Q$. Using this weight space decomposition we can write every representation as $V = \sum_{p,q} V_{pq}$ where V_{pq} is a vector of the (p,q)-entries of the maps $V(a)$ for all arrows a in Q from v_i to v_j. Using the fact that the action of T_a on $\mathtt{rep}_\alpha\ Q$ is induced by conjugation, we deduce as before that for λ determined by $(r_1,\ldots,r_a)$

$$\underset{t\to 0}{lim}\ \lambda(t).V = \underline{0} \ \Leftrightarrow\ r_q - r_p \geq 1 \text{ whenever } V_{pq} \neq 0$$

Again, we define the corner type C of the representation V by defining the subset of real a-tuples

$$E_V = \{(x_1,\ldots,x_a) \in \mathbb{R}^a \mid x_q - x_p \geq 1 \ \forall\ V_{pq} \neq 0\}$$

and determine a minimal element s_V in it, minimal with respect to the usual norm on $\mathbb{R}^a$. Similar to the case of matrices considered before, it follows that s_V is a uniquely determined point in $\mathbb{Q}^a$, having the characteristic property that its entries can be partitioned into strings

$$\{\underbrace{p_l,\ldots,p_l}_{a_{l0}}, \underbrace{p_l+1,\ldots,p_l+1}_{a_{l1}}, \ldots, \underbrace{p_l+k_l,\ldots,p_l+k_l}_{a_{lk_l}}\} \quad \text{with all } a_{lm} \geq 1$$

which are balanced, that is $\sum_{m=0}^{k_l} a_{lm}(p_l+m) = 0$.

Note however that this time we are not allowed to bring s_V into dominant form, as we can only permute base-vectors of the vertex-spaces. That is, we can only use the action of the *vertex-symmetric groups*

$$S_{a_1} \times \ldots \times S_{a_k} \hookrightarrow S_a$$

to bring s_V into *vertex dominant form* , that is if $s_V = (s_1,\ldots,s_a)$ then

$$s_q \leq s_p \quad \text{whenever} \quad p,q \in I_{v_i} \text{ for some } i \text{ and } p < q$$

We compile a list $\mathcal{S}_\alpha$ of such rational a-tuples by the following `algorithm`

- Start with the list $\mathcal{S}_a$ of matrix corner types.
- For every $s \in \mathcal{S}_a$ consider all permutations $\sigma \in S_a/(S_{a_1} \times \ldots \times S_{a_k})$ such that $\sigma.s = (s_{\sigma(1)}, \ldots, s_{\sigma(a)})$ is vertex dominant.
- Take $\mathcal{H}_\alpha$ to be the list of the distinct a-tuples $\sigma.s$ which are vertex dominant.
- Remove $s \in \mathcal{H}_\alpha$ whenever there is an $s' \in \mathcal{H}_\alpha$ such that
$$\pi_s\ Q = \{\pi_{pq} \in \pi_\alpha\ Q \mid s_q - s_p \geq 1\} \subset \pi_{s'}\ Q = \{\pi_{pq} \in \pi_\alpha\ Q\ \mid\ s'_q - s'_p \geq 1\}$$
and $\parallel s \parallel > \parallel s' \parallel$.
- The list $\mathcal{S}_\alpha$ are the remaining entries s from $\mathcal{H}_\alpha$.

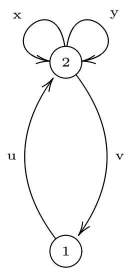
For $s \in \mathcal{S}_\alpha$, we define associated data similar to the case of matrices

- The *corner* C_s is the subspace of $\mathtt{rep}_\alpha\ Q$ such that all arrow matrices V_b, when viewed as $a \times a$ matrices using the partitioning in vertex-entries, have only non-zero entries at spot (p, q) when $s_q - s_p \geq 1$.
- The *border* B_s is the subspace of $\mathtt{rep}_\alpha\ Q$ such that all arrow matrices V_b, when viewed as $a \times a$ matrices using the partitioning in vertex-entries, have only non-zero entries at spot (p, q) when $s_q - s_p = 1$.
- The *parabolic subgroup* $P_s(\alpha)$ is the intersection of $P_s \subset GL_a$ with $GL(\alpha)$ embedded along the diagonal. $P_s(\alpha)$ is a parabolic subgroup of $GL(\alpha)$, that is, contains the product of the Borels $B(\alpha) = B_{a_1} \times \ldots \times B_{a_k}$.
- The *Levi-subgroup* $L_s(\alpha)$ is the intersection of $L_s \subset GL_a$ with $GL(\alpha)$ embedded along the diagonal.

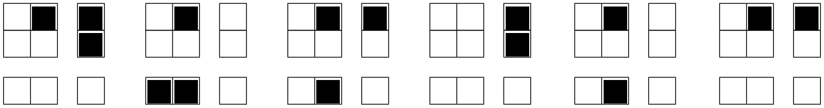
We say that a representation $V \in \mathtt{rep}_\alpha\ Q$ is of *corner type* C_s whenever $V \in C_s$.

Theorem 6.6 *By permuting the vertex-bases, every representation $V \in \mathtt{rep}_\alpha\ Q$ can be brought to a corner type C_s for a uniquely determined s which is a vertex-dominant reordering of s_V.*

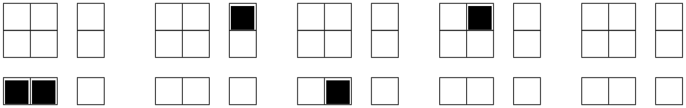
Example 6.10 Consider the following quiver setting



Then, the relevant corners have the following block decomposition



$(1,0,-1)$ $(0,-1,1)$ $(1,-1,0)$ $(\frac{1}{3},\frac{1}{3},-\frac{2}{3})$ $(\frac{1}{3},-\frac{2}{3},\frac{1}{3})$ $(\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$



$(-\frac{1}{3},-\frac{1}{3},\frac{2}{3})$ $(\frac{1}{2},0,-\frac{1}{2})$ $(0,-\frac{1}{2},\frac{1}{2})$ $(\frac{1}{2},-\frac{1}{2},0)$ $(0,0,0)$

Again, we solve the problem of *optimal corner representations* by introducing a new quiver setting.

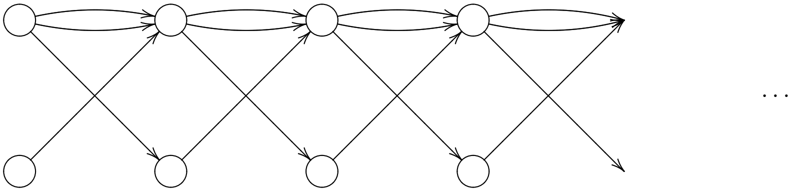
Fix a type $s \in \mathcal{S}_\alpha\ Q$ and let $J_1, \ldots, J_u$ be the distinct strings partitioning the entries of s, say with

$$J_l = \{\underbrace{p_l, \ldots, p_l}_{\sum_{i=1}^k b_{i,l0}}, \underbrace{p_l+1, \ldots, p_l+1}_{\sum_{i=1}^k b_{i,l1}}, \ldots, \underbrace{p_l+k_l, \ldots, p_l+k_l}_{\sum_{i=1}^k b_{i,lk_l}}\}$$

where $b_{i,lm}$ is the number of entries $p \in I_{v_i}$ such that $s_p = p_l + m$. To every string l we will associate a quiver $Q_{s,l}$ and dimension vector $\alpha_{s,l}$ as follows

- $Q_{s,l}$ has $k.(k_l+1)$ vertices labeled (v_i, m) with $1 \leq i \leq k$ and $0 \leq m \leq k_l$.
- In $Q_{s,l}$ there are as many arrows from vertex (v_i, m) to vertex $(v_j, m+1)$ as there are arrows in Q from vertex v_i to vertex v_j. There are no arrows between (v_i, m) and (v_j, m') if $m' - m \neq 1$.
- The dimension-component of $\alpha_{s,l}$ in vertex (v_i, m) is equal to $b_{i,lm}$.

Example 6.11 For the above quiver, all component quivers $Q_{s,l}$ are pieces of the quiver below



Clearly, we only need to consider that part of the quiver $Q_{s,l}$ where the dimensions of the vertex spaces are non-zero.

The quiver-setting (Q_s, α_s) associated to a type $s \in \mathcal{S}_\alpha\ Q$ will be the disjoint union of the string quiver-settings $(Q_{s,l}, \alpha_{s,l})$ for $1 \leq l \leq u$.

Theorem 6.7 *With notations as before, for $s \in \mathcal{S}_\alpha\ Q$ we have isomorphisms*

$$\begin{cases} B_s & \simeq \mathtt{rep}_{\alpha_s}\ Q_s \\ L_s(\alpha) & \simeq GL(\alpha_s) \end{cases}$$

Moreover, the base-change action of $GL(\alpha_s)$ on $\mathtt{rep}_{\alpha_s}\ Q_s$ coincides under the isomorphisms with the action of the Levi-subgroup $L_s(\alpha)$ on the border B_s.

In order to determine the representations in $\mathtt{rep}_{\alpha_s}\ Q_s$ which have *optimal corner type* C_s we define the following character on the Levi-subgroup

$$L_s(\alpha) = \prod_{l=1}^{u} \times_{i=1}^{k} \times_{m=0}^{k_l} GL_{b_{i,lm}} \xrightarrow{\chi_{\theta_s}} \mathbb{C}^*$$

determined by sending a tuple $(g_{i,lm})_{ilm} \longrightarrow \prod_{ilm} det\ g_{i,lm}^{m_{i,lm}}$ where the exponents are determined by

$$\theta_s = (m_{i,lm})_{ilm} \quad \text{where} \quad m_{i,lm} = d(p_l + m)$$

with d the least common multiple of the numerators of the rational numbers p_l for all $1 \leq l \leq u$.

Theorem 6.8 *Consider a representation $V \in Null_\alpha\ Q$ of corner type C_s. Then, V is of optimal corner type C_s if and only if under the natural maps*

$$C_s \overset{\pi}{\twoheadrightarrow} B_s \xrightarrow{\simeq} \mathtt{rep}_{\alpha_s}\ Q_s$$

V is mapped to a θ_s-semistable representation in $\mathtt{rep}_{\alpha_s}\ Q_s$. If U_s is the open subvariety of C_s consisting of all representations of optimal corner type C_s, then

$$U_s = \pi^{-1}\ \mathtt{rep}^{ss}_{\alpha_s}(Q_s, \theta_s)$$

For the corresponding Hesselink stratum $S_s = GL(\alpha).U_s$ we have the commuting diagram

$$\begin{tikzcd} GL(\alpha) \times^{P_s(\alpha)} U_s \arrow[r, "\simeq"] \arrow[d, hook] & S_s \arrow[d, hook] \\ GL(\alpha) \times^{P_s(\alpha)} C_s \arrow[r, "ac"] & \overline{S_s} \end{tikzcd}$$

where ac is the action map, $\overline{S_s}$ is the Zariski closure of S_s in $Null_\alpha\ Q$ and the upper map is an isomorphism as $GL(\alpha)$-varieties.

Here, $GL(\alpha)/P_s(\alpha)$ is the flag variety associated to the parabolic subgroup $P_s(\alpha)$ and is a projective manifold. The variety $GL(\alpha) \times^{P_s(\alpha)} C_s$ is a vectorbundle over the flag variety $GL(\alpha)/P_s(\alpha)$ and is a subbundle of the trivial bundle $GL(\alpha) \times^{P_s(\alpha)} \mathtt{rep}_\alpha\ Q$.

Hence, the Hesselink stratum S_s is an irreducible smooth variety of dimension

$$\begin{aligned} dim\ S_s &= dim\ GL(\alpha)/P_s(\alpha) + rk\ GL(\alpha) \times^{P_s(\alpha)} C_s \\ &= \sum_{i=1}^{k} a_i^2 - dim\ P_s(\alpha) + dim_{\mathbb{C}}\ C_s \end{aligned}$$

and there is a natural one-to-one correspondence between the $GL(\alpha)$-orbits in S_s and the $P_s(\alpha)$-orbits in U_s.

Moreover, the vectorbundle $GL(\alpha) \times^{P_s(\alpha)} C_s$ is a desingularization of $\overline{S_s}$ hence 'feels' the gluing of S_s to the remaining strata. The ordering of corners has the geometric interpretation

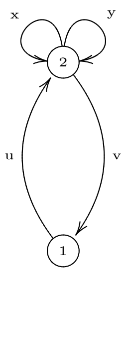
$$\overline{S_s} \subset \bigcup_{\| s' \| \leq \| s \|} S_{s'}$$

Finally, because $P_s(\alpha)$ acts on B_s by the restriction to its subgroup $L_s(\alpha) = GL(\alpha_s)$ we have a projection from the orbit space

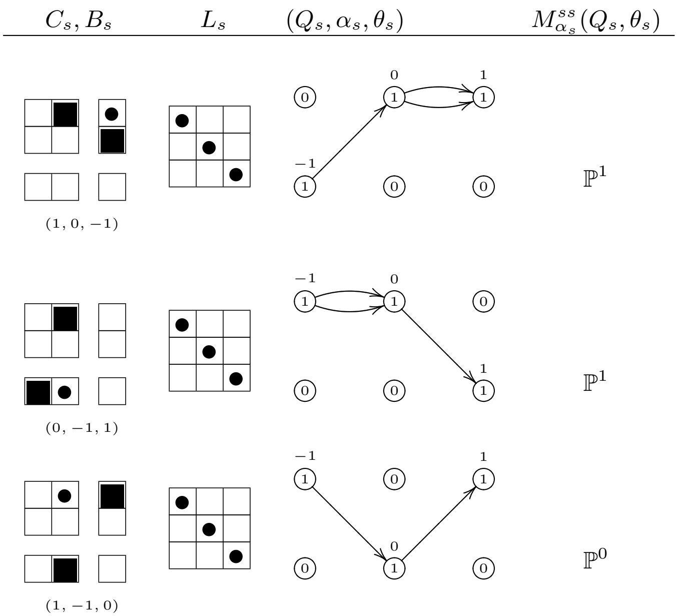
$$U_s/P_s \xrightarrow{p} M^{ss}_{\alpha_s}(Q_s, \theta_s)$$

to the moduli space of θ_s-semistable quiver representations.

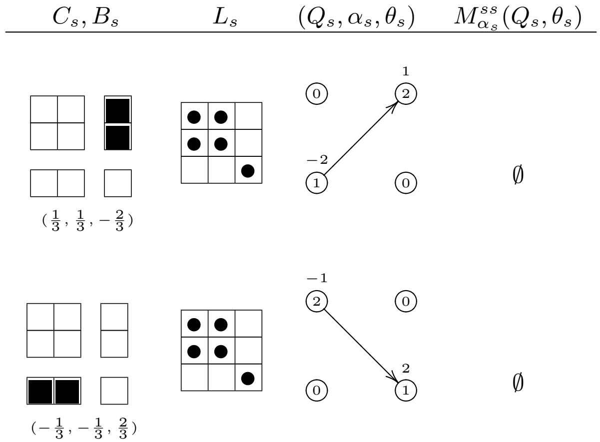
Example 6.12 Above we have listed the relevant corner-types for the nullcone of the quiver-setting



In the table below we list the data of the three irreducible components of $Null_\alpha\ Q/GL(\alpha)$ corresponding to the three maximal Hesselink strata :



There are 6 other Hesselink strata consisting of precisely one orbit. Finally, two possible corner-types do not appear as there are no θ_s-semistable representations for the corresponding quiver setting



6.5 Simultaneous conjugacy classes

We have come a long way from our bare hands description of the simultaneous conjugacy classes of couples of 2×2 matrices in the first chapter of volume 1. In this section we will summarize what we have learned so far to approach the hopeless problem of classifying conjugacy classes of m tuples of $n \times n$ matrices.

First, we show how one can reduce the study of representations of a Quillen-smooth algebra to that of studying nullcones of quiver representations. Let A be an affine $\mathbb{C}$-algebra and M_ξ is a semi-simple n-dimensional module such that the representation variety $\mathtt{rep}_n \int_n A$ is smooth in M_ξ, that is $\xi \in Sm_n\ A$. Let M_ξ be of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$, that is,

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

with distinct simple components S_i of dimension d_i and occurring in M_ξ with multiplicity e_i, then the $GL(\alpha) = Stab\ M_\xi$-structure on the normal space N_ξ to the orbit $\mathcal{O}(M_\xi)$ is isomorphic to that of the representation space

$$\mathtt{rep}_\alpha\ Q^\bullet$$

of a certain marked quiver on k vertices. The slice theorem asserts the existence of a slice $S_\xi \xrightarrow{\phi} N_\xi$

and a commuting diagram

$$\begin{tikzcd}
& GL_n \times^{GL(\alpha)} S_\xi \arrow[dl, "GL_n \times^{GL(\alpha)} \phi"'] \arrow[dd] \arrow[dr, "\psi"] & \\
GL_n \times^{GL(\alpha)} N_\xi \arrow[ddd, "\pi_2"] & & \mathtt{rep}_n \int_n A \arrow[ddd, "\pi_1"] \\
& & \\
& S_\xi/GL(\alpha) \arrow[dl, "\phi/GL(\alpha)"] \arrow[dr, "\psi/GL_n"] & \\
N_\xi/GL(\alpha) & & \mathtt{iss}_n \int_n A
\end{tikzcd}$$

in a neighborhood of $\xi \in \mathtt{iss}_n \int_n A$ on the right and a neighborhood of the image $\underline{0}$ of the trivial representation in $N_\xi/GL(\alpha)$ on the left. In this diagram, the vertical maps are the quotient maps, all diagonal maps are étale and the upper ones are GL_n-equivariant. In particular, there is a GL_n-isomorphism between the fibers

$$\pi_2^{-1}(\underline{0}) \simeq \pi_1^{-1}(\xi)$$

Because $\pi_2^{-1}(\underline{0}) \simeq GL_n \times^{GL(\alpha)} \pi^{-1}(\underline{0})$ with π is the quotient morphism for the marked quiver representations $N_\xi = \mathtt{rep}_\alpha\ Q^\bullet \xrightarrow{\pi} \mathtt{iss}_\alpha\ Q^\bullet = N_x/GL(\alpha)$ we have a GL_n-isomorphism

$$\pi_1^{-1}(\xi) \simeq GL_n \times^{GL(\alpha)} \pi^{-1}(\underline{0})$$

That is, there is a natural one-to-one correspondence between

- GL_n-orbits in the fiber $\pi_1^{-1}(\zeta)$, that is, isomorphism classes of n-dimensional representations of A with Jordan-Hölder decomposition M_ξ, and
- $GL(\alpha)$-orbits in $\pi^{-1}(\underline{0})$, that is, the nullcone of the marked quiver $Null_\alpha\ Q^\bullet$.

Summarizing we have the following

Theorem 6.9 *Let A be an affine Quillen-smooth $\mathbb{C}$-algebra and M_ξ a semi-simple n-dimensional representation of A. Then, the isomorphism classes of n-dimensional representations of A with Jordan-Hölder decomposition isomorphic to M_ξ are given by the $GL(\alpha)$-orbits in the nullcone $Null_\alpha\ Q^\bullet$ of the local marked quiver setting.*

The problem of classifying simultaneous conjugacy classes of m-tuples of $n \times n$ matrices, is the same as n-dimensional representations of the Quillen-smooth algebra $\mathbb{C}\langle x_1, \ldots, x_m \rangle$. To study semi-simple representations, one considers the quotient map

$$M_n^m = \mathtt{rep}_n \mathbb{C}\langle x_1, \ldots, x_m \rangle \xrightarrow{\pi} \mathtt{iss}_n \ \mathbb{C}\langle x_1, \ldots, x_m \rangle = \mathtt{iss}_n^m$$

Fix a point $\xi \in \mathtt{iss}_n^m$ and assume that the corresponding semi-simple n-dimensional representation M_ξ is of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$.

We have shown that the coordinate ring $\mathbb{C}[\mathtt{iss}_n^m] = \mathbb{N}_n^m$ is the *necklace algebra* , that is, is generated by traces of monomials in the generic $n \times n$ matrices $X_1, \ldots, X_m$ of length bounded by $n^2 + 1$. Further, if we collect all M_ξ with representation type τ in the subset $\mathtt{iss}_n^m(\tau)$, then

$$\mathtt{iss}_n = \bigsqcup_\tau \mathtt{iss}_n^m(\tau)$$

is a finite stratification of $\mathtt{iss}_n^m$ into locally closed smooth algebraic subvarieties.

We have an ordering on the representation types $\tau' < \tau$ indicating that the stratum $\mathtt{iss}_n^m(\tau')$ is contained in the Zariski closure of $\mathtt{iss}_n^m(\tau)$. This order relation is induced by the *direct ordering*

$$\tau' = (e'_1, d'_1; \ldots; e'_{k'}, d'_{k'}) <^{dir} \tau = (e_1, d_1; \ldots; e_k, d_k)$$

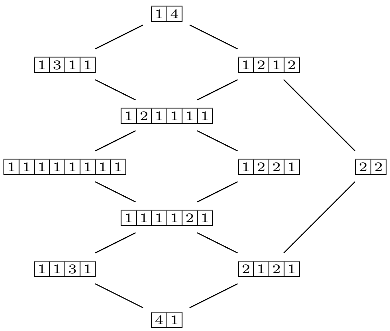
if there is a permutation σ of $[1, 2, \ldots, k']$ and there are numbers

$$1 = j_0 < j_1 < j_2 \ldots < j_k = k'$$

such that for every $1 \leq i \leq k$ we have the following relations

$$\begin{cases} e_i d_i & = \sum_{j=j_{i-1}+1}^{j_i} e'_{\sigma(j)} d'_{\sigma(j)} \\ e_i & \leq e'_{\sigma(j)} \text{ for all } j_{i-1} < j \leq j_i \end{cases}$$

Example 6.13 The order relation on the representation types of dimension $n = 4$ has the following Hasse diagram.



Because $\mathtt{iss}_n^m$ is irreducible, there is an open stratum corresponding to the simple representations, that is type $(1, n)$. The sub-generic strata are all of the form

$$\tau = (1, m_1; 1, m_2) \quad \text{with} \quad m_1 + m_2 = n.$$

The (in)equalities describing the locally closed subvarieties $\mathtt{iss}_n^m(\tau)$ can (in principle) be deduced from the theory of trace identities. Remains to study the local structure of the quotient variety $\mathtt{iss}_n^m$ near ξ and the description of the fibers $\pi^{-1}(\xi)$.

Both problems can be tackled by studying the local quiver setting (Q_ξ, α_ξ) corresponding to ξ which describes the $GL(\alpha_\xi) = Stab(M_\xi)$-module structure of the normal space to the orbit of M_ξ. If ξ is of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ then the local quiver Q_ξ has k-vertices $\{v_1, \ldots, v_k\}$ corresponding the the k distinct simple components $S_1, \ldots, S_k$ of M_ξ and the number of arrows (resp. loops) from v_i to v_j (resp. in v_i) are given by the dimensions

$$dim_{\mathbb{C}} Ext^1(S_i, S_j) \quad \text{resp.} \quad dim_{\mathbb{C}} Ext^1(S_i, S_i)$$

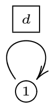
and these numbers can be computed from the dimensions of the simple components,

$$\begin{cases} \# \; \textcircled{j} \xleftarrow{\;a\;} \textcircled{i} & = \; (m-1)d_i d_j \\ \# \; \circlearrowright \textcircled{i} & = (m-1)d_i^2 + 1 \end{cases}$$

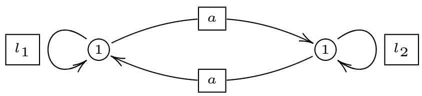
Further, the local dimension vector α_ξ is given by the multiplicities $(e_1, \ldots, e_k)$. The étale local structure of $\mathtt{iss}_n^m$ in a neighborhood of ξ is the same as that of the quotient variety $\mathtt{iss}_{\alpha_\xi}\ Q_\xi$ in a neighborhood of $\underline{0}$. The local algebra of the latter is generated by traces along oriented cycles in the quiver Q_ξ. A direct application is

Proposition 6.2 *For $m \geq 2$, ξ is a smooth point of $\mathtt{iss}_n^m$ if and only if M_ξ is a simple representation, unless $(m, n) = (2, 2)$ in which case $\mathtt{iss}_2^2 \simeq \mathbb{C}^5$ is a smooth variety.*

Proof. If ξ is of representation type $(1, n)$, the local quiver setting (Q_ξ, α_ξ) is



where $d = (m-1)n^2 + 1$, whence the local algebra is the formal power series ring in d variables and so $\mathtt{iss}_n^m$ is smooth in ξ. Because the singularities form a Zariski closed subvariety of $\mathtt{iss}_n^m$, the result follows if we prove that all points ξ lying in sub-generic strata, say of type $(1, m_1; 1, m_2)$ are singular. In this case the local quiver setting is equal to



where $a = (m-1)m_1m_2$ and $l_i = (m-1)m_i^2 + 1$. Let us denote the arrows from v_1 to v_2 by $x_1, \ldots, x_a$ and those from v_2 to v_1 by $y_1, \ldots, y_a$. If $(m,n) \neq (2,2)$ then $a \geq 2$, but then we have traces along cycles

$$\{x_iy_j \mid 1 \leq i,j \leq a\}$$

that is, the polynomial ring of invariants is the polynomial algebra in $l_1 + l_2$ variables (the traces of the loops) over the homogeneous coordinate ring of the Segre embedding

$$\mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \hookrightarrow \mathbb{P}^{a^2-1}$$

which has a singularity at the top (for example we have equations of the form $(x_1y_2)(x_2y_1) - (x_1y_1)(x_2y_2)$). Thus, the local algebra of $\mathtt{iss}_n^m$ cannot be a formal power series ring in ξ whence $\mathtt{iss}_n^m$ is singular in ξ. We have seen in section 1.2 that for the exceptional case $\mathtt{iss}_2^2 \simeq \mathbb{C}^5$. □

To determine the fibers of the quotient map $M_n^m \xrightarrow{\pi} \mathtt{iss}_n^m$ we have to study the nullcone of this local quiver setting, $Null_{\alpha_\xi}\ Q_\xi$. Observe that the quiver Q_ξ has loops in every vertex and arrows connecting each ordered pair of vertices, whence we do not have to worry about potential corner-type removals. Denote $\sum e_i = z \leq n$ and let $\mathcal{C}_z$ be the set of all $s = (s_1, \ldots, s_z) \in \mathbb{Q}^z$ which are disjoint unions of strings of the form

$$\{p_i, p_i + 1, \ldots, p_i + k_i\}$$

where $l_i \in \mathbb{N}$, all intermediate numbers $p_i + j$ with $j \leq k_i$ do occur as components in s with multiplicity $a_{ij} \geq 1$ and p_i satisfies the balance-condition

$$\sum_{j=0}^{k_i} a_{ij}(p_i + j) = 0$$

for every string in s. For fixed $s \in \mathcal{C}_z$ we can distribute the components s_i over the vertices of Q_ξ (e_j of them to vertex v_j) in all possible ways modulo the action of the small Weyl group $S_{e_1} \times \ldots S_{e_k} \hookrightarrow S_z$. That is, we can rearrange the s_i's belonging to a fixed vertex such that they are in decreasing order. This gives us the list $\mathcal{S}_{\alpha_\xi}$ or $\mathcal{S}_\tau$ of all corner-types in $Null_{\alpha_\xi}\ Q_\xi$. For each $s \in \mathcal{S}_{\alpha_\xi}$ we then construct the corner-quiver setting

$$(Q_{\xi\ s}, \alpha_{\xi\ s}, \theta_{\xi\ s})$$

and study the Hesselink strata S_s which actually do appear, which is equivalent to verifying whether there are $\theta_{\xi\ s}$-semistable representations in $\mathtt{rep}_{\alpha_{\xi\ s}}\ Q_{\xi\ s}$. We have given a purely combinatorial way to settle this (in general quite hard) problem of optimal corner-types.

That is, we can determine which Hesselink strata S_s actually occur in $\pi^{-1}(\xi) \simeq Null_{\alpha_{xi}}\ Q_\xi$. The $GL(\alpha_{\xi\ s})$-orbits in the stratum S_s are in natural one-to-one correspondence with the orbits under the associated parabolic subgroup P_s acting on the semistable representations

$$U_s = \pi^{-1}\ \mathtt{rep}^{ss}_{\alpha_{\xi\ s}}(Q_{\xi\ s}, \theta_{\xi\ s})$$

type	τ	(Q_τ, α_τ)
2_a	$(1,2)$	
2_b	$(1,1;1,1)$	
2_c	$(2,1)$	

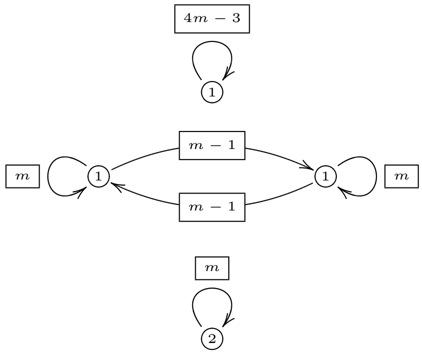


Figure 6.5: Local quiver settings for 2×2 matrices.

and there is a natural projection morphism from the corresponding orbit-space

$$U_s/P_s \xrightarrow{p_s} M^{ss}_{\alpha_{\xi\ s}}(Q_{\xi\ s}, \theta_{\xi\ s})$$

to the moduli space of $\theta_{\xi\ s}$-semistable representations. The remaining (hard) problem in the classification of m-tuples of $n \times n$ matrices under simultaneous conjugation is the description of the fibers of this projection map p_s.

Example 6.14 (m-tuples of 2×2 matrices) There are three different representation types τ of 2-dimensional representations of $\mathbb{C}\langle x_1, \ldots, x_m\rangle$ with corresponding local quiver settings (Q_τ, α_τ) given in figure 6.5 The defining (in)equalities of the strata $\mathtt{iss}_2^m(\tau)$ are given by $k \times k$ minors (with $k \leq 4$ of the symmetric $m \times m$ matrix

$$\begin{bmatrix} tr(x_1^0 x_1^0) & \ldots & tr(x_1^0 x_m^0) \\ \vdots & & \vdots \\ tr(x_m^0 x_1^0) & \ldots & tr(x_m^0 x_m^0) \end{bmatrix}$$

where $x_i^0 = x_i - \frac{1}{2}tr(x_i)$ is the generic trace zero matrix. These facts follow from the description of the trace algebras $\mathbb{T}_2^m$ as polynomial algebras over the generic Clifford algebras of rank ≤ 4 (determined by the above symmetric matrix) and the classical matrix decomposition of Clifford algebras over $\mathbb{C}$. For more details we refer to [53].

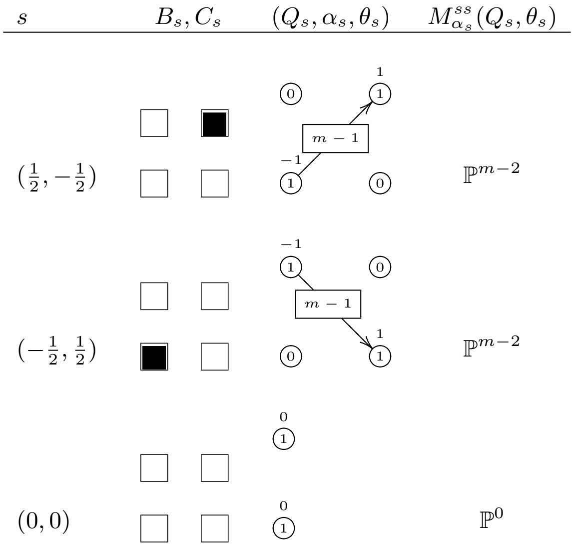
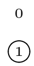


Figure 6.6: Moduli spaces for type 2_b.

To study the fibers $M_2^m \longrightarrow\!\!\!\!\!\rightarrow \mathtt{iss}_2^m$ we need to investigate the different Hesselink strata in the nullcones of these local quiver settings. Type 2_a has just one potential corner type corresponding to $s = (0) \in \mathcal{S}_1$ and with corresponding corner-quiver setting

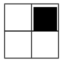
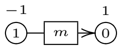
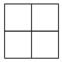


which obviously has $\mathbb{P}^0$ (one point) as corresponding moduli (and orbit) space. This corresponds to the fact that for $\xi \in \mathtt{iss}_2^m(1,2)$, M_ξ is simple and hence the fiber $\pi^{-1}(\xi)$ consists of the closed orbit $\mathcal{O}(M_\xi)$.

For type 2_b the list of figure 6.6 gives the potential corner-types C_s together with their associated corner-quiver settings and moduli spaces (note that as $B_s = C_s$ in all cases, these moduli spaces describe the full fiber) That is, for $\xi \in \mathtt{iss}_2^m(1,1;1,1)$, the fiber $\pi^{-1}(\xi)$ consists of the unique closed orbit $\mathcal{O}(M_\xi)$ (corresponding to the $\mathbb{P}^0$) and two families $\mathbb{P}^{m-2}$ of non-closed orbits. Observe that in the special case $m = 2$ we recover the two non-closed orbits found in section 1.2.

Finally, for type 2_c, the fibers are isomorphic to the nullcones of m-tuples of 2×2 matrices. We have the following list of corner-types, corner-quiver settings and moduli spaces. Again, as $B_s = C_s$

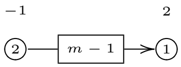
in all cases, these moduli spaces describe the full fiber.

s	B_s, C_s	$(Q_s, \alpha_s, \theta_s)$	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$
$(\frac{1}{2}, -\frac{1}{2})$			$\mathbb{P}^{m-1}$
$(0, 0)$			$\mathbb{P}^0$

whence the fiber $\pi^{-1}(\xi)$ consists of the closed orbit, together wit a $\mathbb{P}^{m-1}$-family of non-closed orbits. Again, in the special case $m = 2$, we recover the $\mathbb{P}^1$-family found in section 1.2.

Example 6.15 (m-tuples of 3×3 matrices) There are 5 different representation-types for 3-dimensional representations. Their associated local quiver settings are given in figure 6.7 For each of these types we can perform an analysis of the nullcones as before. We leave the details to the interested reader and mention only the end-result

- For type 3_a the fiber is one closed orbit.
- For type 3_b the fiber consists of the closed orbit together with two $\mathbb{P}^{2m-3}$-families of non-closed orbits.
- For type 3_c the fiber consists of the closed orbit together with twelve $\mathbb{P}^{m-2} \times \mathbb{P}^{m-2}$-families and one $\mathbb{P}^{m-2}$-family of non-closed orbits.
- For type 3_d the fiber consists of the closed orbit together with four $\mathbb{P}^{m-1} \times \mathbb{P}^{m-2}$-families, one $\mathbb{P}^{m-2} \times \mathbb{P}^{m-2}$-family, two $\mathbb{P}^{m-2}$-families, one $\mathbb{P}^{m-1}$-family and two M-families of non-closed orbits determined by moduli spaces of quivers, where M is the moduli space of the following quiver setting

  together with some additional orbits coming from the projection maps p_s.
- For type 3_e we have to study the nullcone of m-tuples of 3×3 matrices, which can be done as in the case of couples but for $m \geq 3$ the two extra strata do occur.

We see that in this case the only representation-types where the fiber is not fully determined by moduli spaces of quivers are 3_d and 3_e.

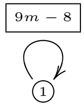
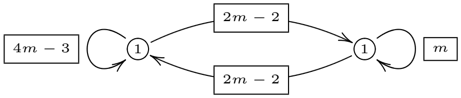
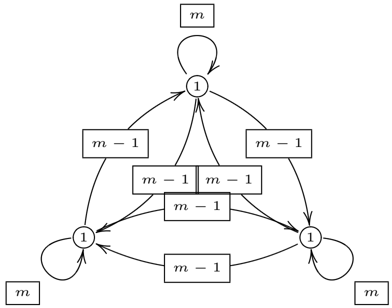
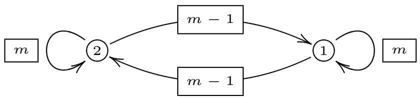
type	τ	(Q_τ, α_τ)
3_a	$(1,3)$	
3_b	$(1,2;1,1)$	
3_c	$(1,1;1,1;1,1)$	
3_d	$(2,1;1,1)$	
3_e	$(3,1)$	

Figure 6.7: Local quiver settings for 3×3 matrices.

6.6 Representation fibers

Let A be a Cayley-Hamilton algebra of degree n and consider the algebraic quotient map

$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{triss}_n\ A$$

from the variety of n-dimensional trace preserving representations to the variety classifying isomorphism classes of trace preserving n-dimensional semi-simple representations. Assume $\xi \in Sm_{tr}\ A \hookrightarrow \mathtt{triss}_n\ A$. That is, the representation variety $\mathtt{trep}_n\ A$ is smooth along the GL_n-orbit of M_ξ where M_ξ is the semi-simple representation determined by $\xi \in \mathtt{triss}_n\ A$. We have seen that the local structure of A and $\mathtt{trep}_n\ A$ near ξ is fully determined by a local marked quiver setting $(Q^\bullet_\xi, \alpha_\xi)$. That is, we have a GL_n-isomorphism between the fiber of the quotient map, that is, the n-dimensional trace preserving representation degenerating to M_ξ

$$\pi^{-1}(\xi) \simeq GL_n \times^{GL(\alpha_\xi)} Null_{\alpha_\xi}\ Q_\xi$$

and the nullcone of the marked quiver-setting. In this section we will apply the results on nullcones to the study of these representation fibers $\pi^{-1}(\xi)$.

Observe that all the facts on nullcones of quivers extend verbatim to marked quivers $Q^\bullet$ using the underlying quiver Q with the proviso that we drop all loops in vertices with vertex-dimension 1 which get a marking in $Q^\bullet$. This is clear as nilpotent quiver representations obviously have zero trace along each oriented cycle, in particular in each loop.

The examples given before illustrate that a complete description of the nullcone is rather cumbersome. For this reason we restrict here to the determination of the number of irreducible components and their dimensions in the representation fibers. Modulo the GL_n-isomorphism above this study amounts to describing the irreducible components of $Null_{\alpha_\xi}\ Q_\xi$ which are determined by the maximal corner-types C_s, that is such that the set of weights in C_s is maximal among subsets of $\pi_{\alpha_{xi}}\ Q_\xi$ (and hence $\parallel s \parallel$ is maximal among $\mathcal{S}_{\alpha_\xi}\ Q_\xi$).

To illustrate our strategy, consider the case of curve orders. In section 5.4 we proved that if A is a Cayley-Hamilton order of degree n over an affine curve $X = \mathtt{triss}_n\ A$ and if $\xi \in Sm_n\ A$, then the local quiver setting (Q, α) is determined by an oriented cycle Q on k vertices with $k \leq n$ being the number of distinct simple components of M_ξ, the dimension vector $\alpha = (1, \ldots, 1)$ as in figure 6.8 and an unordered partition $p = (d_1, \ldots, d_k)$ having precisely k parts such that $\sum_i d_i = n$, determining the dimensions of the simple components of M_ξ. Fixing a cyclic ordering of the k-vertices $\{v_1, \ldots, v_k\}$ we have that the set of weights of the maximal torus $T_k = \mathbb{C}^* \times \ldots \times \mathbb{C}^* = GL(\alpha)$ occurring in $\mathtt{rep}_\alpha\ Q$ is the set

$$\pi_\alpha\ Q = \{\pi_{k1}, \pi_{12}, \pi_{23}, \ldots, \pi_{k-1k}\}$$

Denote $K = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}$ and consider the one string vector

$$s = (\ \ldots, k-2-\frac{K}{k}, k-1-\frac{K}{k}, \underbrace{-\frac{K}{k}}_{i}, 1-\frac{K}{k}, 2-\frac{K}{k}, \ldots\ )$$

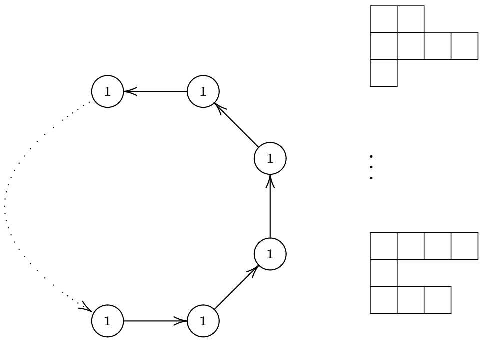
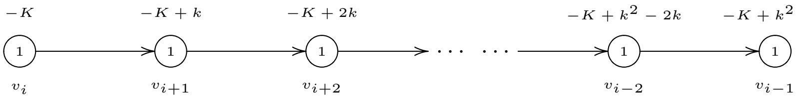


Figure 6.8: Local quiver settings for curve orders.

then s is balanced and vertex-dominant, $s \in \mathcal{S}_\alpha\ Q$ and $\pi_s\ Q = \Pi$. To check whether the corresponding Hesselink strata in $Null_\alpha\ Q$ is nonempty we have to consider the associated quiver-setting $(Q_s, \alpha_s, \theta_s)$ which is



It is well known and easy to verify that $\mathtt{rep}_{\alpha_s}\ Q_s$ has an open orbit with representative all arrows equal to 1. For this representation all proper subrepresentations have dimension vector $\beta = (0, \ldots, 0, 1, \ldots, 1)$ and hence $\theta_s(\beta) > 0$. That is, the representation is θ_s-stable and hence the corresponding Hesselink stratum $S_s \neq \emptyset$. Finally, because the dimension of $\mathtt{rep}_{\alpha_s}\ Q_s$ is $k - 1$ we have that the dimension of this component in the representation fiber $\pi^{-1}(x)$ is equal to

$$dim\ GL_n - dim\ GL(\alpha) + dim\ \mathtt{rep}_{\alpha_s}\ Q_s = n^2 - k + k - 1 = n^2 - 1$$

which completes the proof of the following

Theorem 6.10 *Let A be a Cayley-Hamilton order of degree n over an affine curve X such that A is smooth in $\xi \in X$. Then, the representation fiber $\pi^{-1}(\xi)$ has exactly k irreducible components of dimension $n^2 - 1$, each the closure of one orbit. In particular, if A is Cayley-smooth over X, then the quotient map*

$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{triss}_n\ A = X$$

is flat, that is, all fibers have the same dimension $n^2 - 1$.

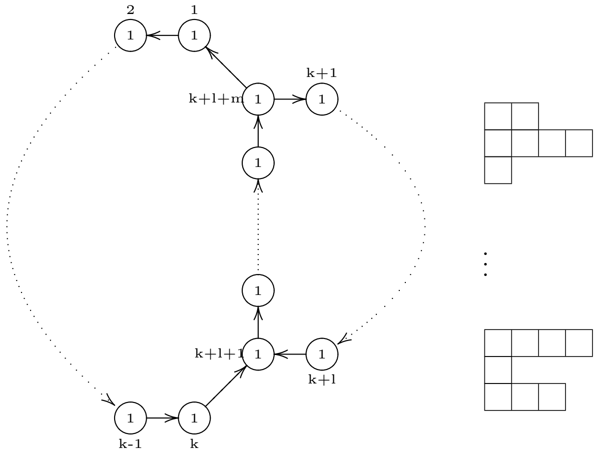


Figure 6.9: Local quiver settings for surface orders.

For Cayley-Hamilton orders over surfaces, the situation is slightly more complicated. From section 5.4 we recall that if A is a Cayley-Hamilton order of degree n over an affine surface $S = \mathtt{triss}_n\ A$ and if A is smooth in $\xi \in X$, then the local structure of A is determined by a quiver setting (Q, α) where $\alpha = (1, \dots, 1)$ and Q is a two-circuit quiver on $k + l + m \leq n$ vertices, corresponding to the distinct simple components of M_ξ as in figure 6.9 and an unordered partition $p = (d_1, \dots, d_{k+l+m})$ of n with $k+l+m$ non-zero parts determined by the dimensions of the simple components of M_ξ. With the indicated ordering of the vertices we have that

$$\pi_\alpha\ Q = \{\pi_{i\ i+1} \ \mid\ \begin{cases} 1 & \leq i \leq k-1 \\ k+1 & \leq i \leq k+l-1 \\ k+l+1 & \leq i \leq k+l+m-1 \end{cases} \}$$

$$\cup \{\pi_{k\ k+l+1}, \pi_{k+l\ k+l+1}, \pi_{k+l+m\ 1}, \pi_{k+l+m\ k+1}\}$$

As the weights of a corner cannot contain all weights of an oriented cycle in Q we have to consider the following two types of potential corner-weights Π of maximal cardinality

- (outer type) : $\Pi = \pi_\alpha\ Q - \{\pi_a, \pi_b\}$ where a is an edge in the interval $[v_1, \dots, v_k]$ and b is an edge in the interval $[v_{k+1}, \dots, v_{k+l}]$.
- (inner type) : $\Pi = \pi_\alpha\ Q - \{\pi_c\}$ where c is an edge in the interval $[v_{k+l+1}, v_{k+l+m}]$.

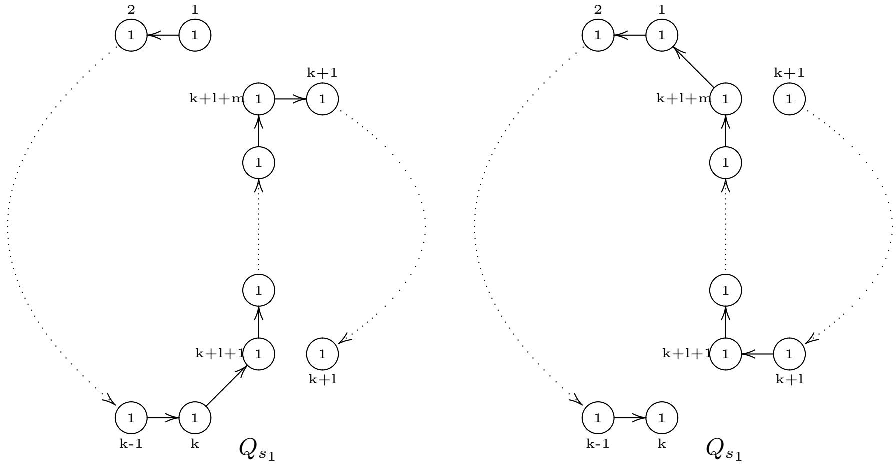


Figure 6.10: Border quiver settings.

There are $2 + (k-1)(l-1)$ different subsets Π of outer type, each occurring as the set of weights of a corner C_s, that is $\Pi = \pi_s\ Q$ for some $s \in \mathcal{S}_\alpha\ Q$. The two exceptional cases correspond to

$$\begin{cases} \Pi_1 & = \pi_\alpha\ Q - \{\pi_{k+l+m\ 1}, \pi_{k+l\ k+l+1}\} \\ \Pi_2 & = \pi_\alpha\ Q - \{\pi_{k+l+m\ k+1}, \pi_{k\ k+l+1}\} \end{cases}$$

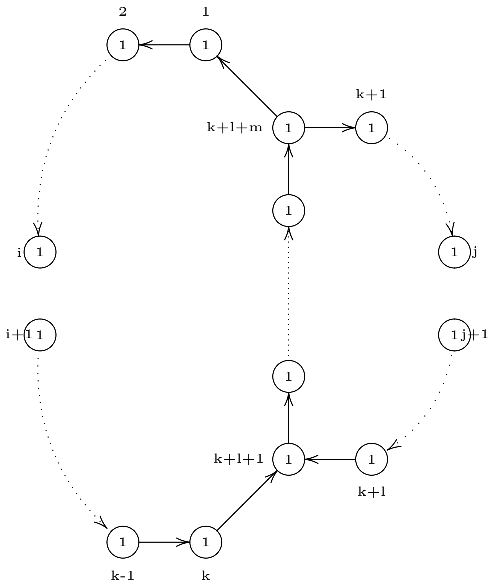
which are of the form $\pi_{s_i}\ Q$ with associated border quiver-setting $(Q_{s_i}, \alpha_{s_i}, \theta_{s_i})$ where $\alpha_{s_i} = (1, \ldots, 1)$, Q_{s_i} are the full line subquivers of Q given in figure 6.10 with starting point v_1 (resp. v_{k+1}). The corresponding $s_i \in \mathcal{S}_\alpha\ Q$ is a single string with minimal entry

$$-\frac{\sum_{i=0}^{k+l+m-1} i}{k+l+m} = -\frac{k+l+m-1}{2} \qquad \text{at place} \qquad \begin{cases} 1 \\ k+1 \end{cases}$$

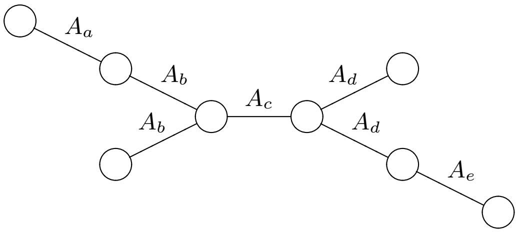
and going with increments equal to one along the unique path. Again, one verifies that $\mathtt{rep}_{\alpha_s}\ Q_s$ has a unique open and θ_s-stable orbit, whence these Hesselink strata do occur and the border B_s is the full corner C_s. The corresponding irreducible component in $\pi^{-1}(\xi)$ has therefore dimension equal to $n^2 - 1$ and is the closure of a unique orbit. The remaining $(k-1)(l-1)$ subsets Π of outer type are of the form

$$\Pi_{ij} = \pi_\alpha\ Q - \{\pi_{i\ i+1}, \pi_{j\ j+1}\}$$

with $1 \leq i \leq k-1$ and $k+1 \leq j \leq k+l-1$. We will see in a moment that they are again of type $\pi_s\ Q$ for some $s \in \mathcal{S}_\alpha\ Q$ with associated border quiver-setting $(Q_s, \alpha_s, \theta_s)$ where $\alpha_s = (1, \ldots, 1)$ and Q_s is the full subquiver of Q



If we denote with A_l the directed line quiver on $l+1$ vertices, then Q_s can be decomposes into full line subquivers



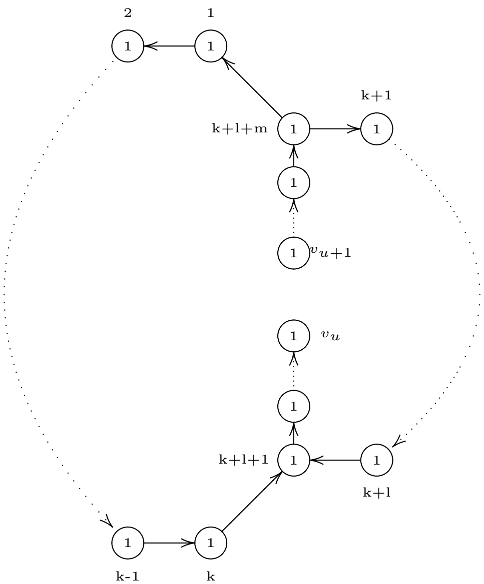
but then we consider the one string $s \in \mathcal{S}_\alpha\ Q$ with minimal entry equal to $-\frac{x}{k+l+m}$ where with notations as above

$$x = \sum_{i=1}^{a} i + 2\sum_{i=1}^{b}(a+i) + \sum_{i=1}^{c}(a+b+i) + 2\sum_{i=1}^{d}(a+b+c+i) + \sum_{i=1}^{e}(a+b+c+d+i)$$

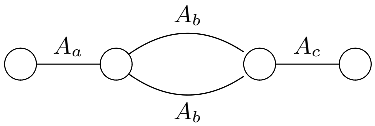
where the components of s are given to the relevant vertex-indices. Again, there is a unique open orbit in $\mathtt{rep}_{\alpha_s}\ Q_s$ which is a θ_s-stable representation and the border B_s coincides with the corner C_s. That is, the corresponding Hesselink stratum occurs and the irreducible component of $\pi^{-1}(\xi)$ it determines had dimension equal to

$$\begin{aligned} dim\ GL_n - dim\ GL(\alpha) + dim\ \mathtt{rep}_{\alpha_s}\ Q_s &= n^2 - (k+l+m) + (k+l+m-1) \\ &= n^2 - 1 \end{aligned}$$

There are $m-1$ different subsets Π_u of inner type, where for $k+l+1 \leq u < k+l+m$ we define $\Pi_u = \pi_\alpha\ Q - \{\pi_{u\ u+1}\}$, that is dropping an edge in the middle



First assume that $k = l$. In this case we can walk through the quiver (with notations as before)



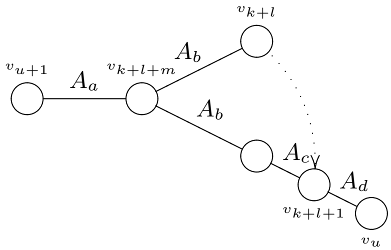
and hence the full subquiver of Q is part of a corner quiver-setting $(Q_s, \alpha_s, \theta_s)$ where $\alpha = (1, \ldots, 1)$ and where s has as its minimal entry $-\frac{x}{k+l+m}$ where

$$x = \sum_{i=1}^{a} i + 2\sum_{i=1}^{b}(a+i) + \sum_{i=1}^{c}(a+b+i)$$

In this case we see that $\mathtt{rep}_{\alpha_s}\ Q_s$ has θ_s-stable representations, in fact, there is a $\mathbb{P}^1$-family of such orbits. The corresponding Hesselink stratum is nonempty and the irreducible component of $\pi^{-1}(\xi)$ determined by it has dimension

$$dim\ GL_n - dim\ GL(\alpha) + dim\ \mathtt{rep}_{\alpha_s}\ Q_s = n^2 - (k+l+m) + (k+l+m) = n^2$$

If $l < k$, then $\Pi_u = \pi_s\ Q$ for some $s \in \mathcal{S}_\alpha\ Q$ but this time the border quiver-setting $(Q_s, \alpha_s, \theta_s)$ is determined by $\alpha_s = (1, \ldots, 1)$ and Q_s the full subquiver of Q by also dropping the arrow corresponding to $\pi_{k+l+1\ k+l}$, that is



If Q_s is this quiver (without the dashed arrow) then $B_s = \mathtt{rep}_{\alpha_s}\ Q_s$ and it contains an open orbit of a θ_s-stable representation. Observe that s is determines as the one string vector with minimal entry $-\frac{x}{k+l+m}$ where

$$x = \sum_{i=1}^{a} i + 2\sum_{i=1}^{b}(a+i) + \sum_{i=1}^{c}(a+b+i) + \sum_{i=1}^{d}(a+b+c+i)$$

However, in this case $B_s \neq C_s$ and we can identify C_s with $\mathtt{rep}_{\alpha_s}\ Q'_s$ where Q'_s is Q_s together with the dashed arrow. There is an $\mathbb{A}^1$-family of orbits in C_s mapping to the θ_s-stable representation. In particular, the Hesselink stratum exists and the corresponding irreducible component in $\pi^{-1}(\xi)$ has dimension equal to

$$dim\ GL_n - dim\ GL(\alpha) + dim\ C_s = n^2 - (k+l+m) + (k+l+m) = n^2.$$

This concludes the proof of the description of the representation fibers of smooth orders over surfaces, summarized in the following result.

Theorem 6.11 *Let A be a Cayley-Hamilton order of degree n over an affine surface $X = \mathtt{triss}_n\ A$ and assume that A is smooth in $\xi \in X$ of local type (A_{klm}, α). Then, the representation fiber $\pi^{-1}(\xi)$ has exactly $2 + (k-1)(l-1) + (m-1)$ irreducible components of which $2 + (k-1)(l-1)$ are of dimension $n^2 - 1$ and are closure of one orbit and the remaining $m-1$ have dimension n^2 and are closures of a one-dimensional family of orbits. In particular, if A is Cayley-smooth, then the algebraic quotient map*

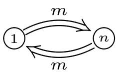
$$\mathtt{trep}_n\ A \xrightarrow{\pi} \mathtt{triss}_n\ A = X$$

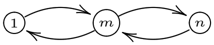
is flat if and only if all local quiver settings of A have quiver A_{klm} with $m = 1$.

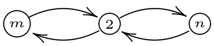
The final example will determine the fibers over *smooth* points in the quotient varieties (or moduli spaces) provided the local quiver is *symmetric*. This computation is due to Geert Van de Weyer.

Example 6.16 (Smooth symmetric settings) Recall from theorem 5.22 that a smooth symmetric quiver setting ($\mathtt{sss}$) if and only if it is a tree constructed as a connected sum of three different types of quivers:

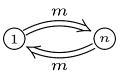
- 

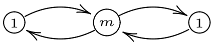
- 

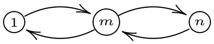
, with $m \leq n$
- 

- 

where the connected sum is taken in the vertex with dimension 1. We call the vertices where the connected sum is taken *connecting vertices* and depict them by a square vertex $\square$. We want to study the nullcone of connected sums composed of more than one of these quivers so we will focus on instances of these four quivers having at least one vertex with dimension 1:

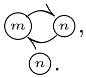
I , with $m \leq n$

II(1) 

II(2) 

We will call the quiver settings of type **I** and **II** forming an `sss` (Q, α) the *terms* of Q.

claim 1 : *Let (Q, α) be an* `sss` *and Q_μ a type quiver for Q, then any string quiver of Q_μ is either a connected sum of string quivers of type quivers for terms of Q or a string quiver of type quivers of*

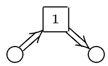


Consider a string quiver $Q_{\mu(i)}$ of Q_μ. By definition vertices in a type quiver are only connected if they originate from the same term in Q. This means we may divide the string quiver $Q_{\mu(i)}$ into segments, each segment either a string quiver of a type quiver of a term of Q (if it contains the connecting vertex) or a level quiver of a type quiver of the quivers listed above (if it does not contain the connecting vertex).

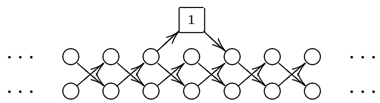
The only vertices these segments may have in common are instances of the connecting vertices. Now note that there is only one instance of each connecting vertex in Q_μ because the dimension of each connecting vertex is 1. Moreover, two segments cannot have more than one connecting vertex in common as this would mean that in the original quiver there is a cycle, proving the claim.

Hence, constructing a type quiver for an `sss` boils down to patching together string quivers of its terms. These string quivers are subquivers of the following two quivers:

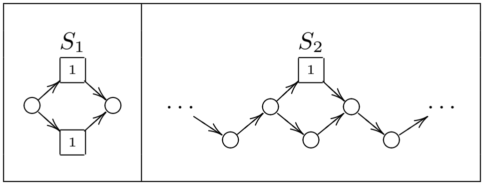
I:



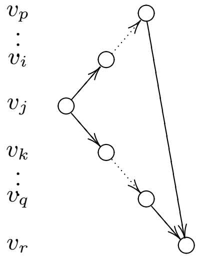
II:



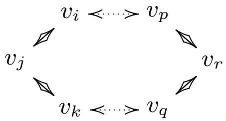
Observe that the second quiver has two components. So a string quiver will either be a tree (possible from all components) or a quiver containing a square. We will distinguish two different types of squares; S_1 corresponding to a term of type **II**(1) and S_2 corresponding to a term of type **II**(2).



These squares are the only polygons that can appear in our type quiver. Indeed, consider a possible polygon



This polygon corresponds to the following subquiver of Q:



But Q is a tree, so this is only a subquiver if it collapses to $v_i \longleftrightarrow v_j \longleftrightarrow v_k$.

claim 2 : *Let (Q,α) be an* `sss` *and Q_μ a type quiver containing (connected) squares. If Q_μ determines a non-empty Hesselink stratum then*

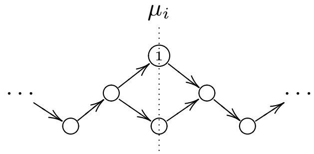
(i) the 0-axis in Q_μ lies between the axes containing the outer vertices of the squares of type S_1;

(ii) squares of type S_1 are connected through paths of maximum length 2;

*(iii) squares of type S_1 that are connected through a path of length 2 are connected to other quivers in top and bottom vertex (and hence originate from type **II**(1) terms that are connected to other terms in both their connecting vertices);*

(iv) the string $\mu(i)$ containing squares of type S_1 connected through a path of length two equals $(\dots,-2,-1,0,1,2,\dots)$.

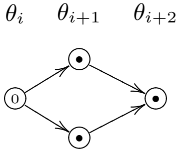
(v) for a square of type S_2:



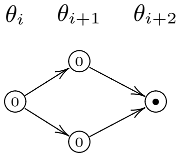
with p vertices on its left branch and q vertices on its right branch we have

$$-\frac{q}{2} \leq \mu_i \leq \frac{p}{2}$$

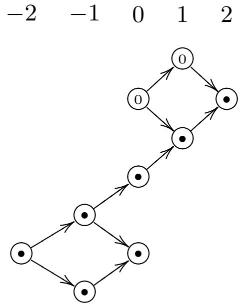
Let us call the string quiver of Q_μ containing the squares $Q_{\mu(i)}$ and let $\theta \in \mu(i)\mathbb{N}_0$ be the character determining this string quiver. Consider the subrepresentation



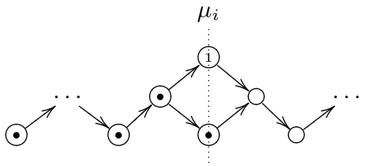
This subrepresentation has character $\theta(\alpha_{\mu(i)}) - \alpha_{\mu(i)}(\mathbf{v})\theta_i \geq 0$ where $\mathbf{v}$ is the vertex which dimension we reduced to 0, so $\theta_i \leq 0$. But then the subrepresentation



gives $\theta_{i+2} \geq 0$, whence (i). Note that the left vertex of one square can never lie on an axis right of the right vertex of another square. At most it can lie on the same axis as the right vertex, in which case this axis is the 0-axis and the squares are connected by a path of length 2. In order to prove (iii) look at the subrepresentation



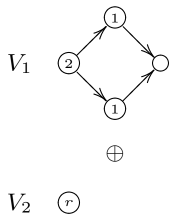
This subrepresentation has negative character and hence the original representation was not semistable. Finally, for (v) we look at the subrepresentation obtained by reducing the dimension of all dotted vertices by 1:



having character $-((p+1)\mu_i - \sum_{j=1}^{p} j) \geq 0$. So $\mu_i \leq \frac{p}{2}$. Mirroring this argument yields the other inequality $\mu_i \geq -\frac{q}{2}$.

claim 3 : *Let (Q, α) be an* sss *and Q_μ be a type quiver determining a non-empty stratum and let $Q_{\mu(i)}$ be a string quiver determined by a segment $\mu(i)$ not containing 0. Then the only possible dimension vectors for squares of type S_1 in $Q_{\mu(i)}$ are those of figure 6.11.*

Top and bottom vertex of the square are constructed from the connecting vertices so can only be one-dimensional. Left and right vertex of the square are constructed from a vertex of dimension n. Claim 2 asserts that the leftmost vertex lies on a negative axis while the rightmost vertex lies on a positive axis. If the left dimension is > 2 then the representation splits



$$\begin{aligned}
\alpha_1 &= \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \\
\alpha_2 &= \begin{pmatrix} & 1 & \\ 1 & & 2 \\ & 1 & \end{pmatrix} \\
\alpha_3 &= \begin{pmatrix} & 1 & \\ 2 & & 2 \\ & 1 & \end{pmatrix}
\end{aligned}$$

Figure 6.11: Possible dimension vectors for squares.

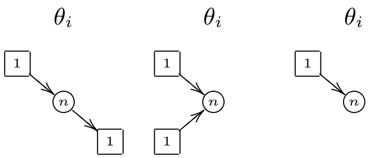
with $r = m - 2$. By semistability the character of V_2 must be zero. A similar argument applies to the right vertex.

claim 4 : *Let μ be a type determining a non-empty stratum.*

*(i) When a vertex (v, i) in Q_μ determined by a term of type **II**(1) has $\alpha(v, i) > 2$ then $\mu_i = 0$.*

*(ii) When a vertex (v, i) in Q_μ determined by a term of type **I** with m arrows has $\alpha(v, i) > m$ then $\mu_i = 0$.*

Suppose we have a vertex v with dimension $\alpha_{\mu(i)}(v) > 2$, then the number of paths running through this vertex is at most 2: would there be at least three paths arriving or departing in the vertex, it would be a connecting vertex which is not possible because of its dimension. Are there two paths arriving and at least one path departing, it must be a central vertex of a type **II**(2) term. But then the only possible subtrees generated from type **II**(1) terms with vertices of dimension at least three are (modulo reversing all arrows)



In the last tree there are no other arrows from the vertex with dimension n. For each of these trees we have a subrepresentation

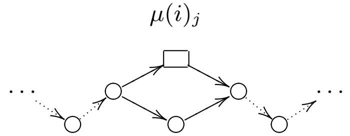
$$\begin{array}{c} \theta_i \\ 1 \end{array}$$

whence $\theta_i \geq 0$. But if $\theta_i > 0$, reducing the dimension of the vertex with dimension ≥ 3 gives a subrepresentation with negative character, so $\theta_i = 0$. The second part is proved similarly.

Summarizing these results we obtain the description of the nullcone of a smooth symmetric quiver-setting.

Let (Q, α) be an $\mathtt{sss}$ *and μ a type determining a non-empty stratum in* $\mathtt{null}_\alpha\ Q$. *Let Q_μ be the corresponding type quiver and α_μ the corresponding dimension vector, then*

(i) every connected component $Q_{\mu(i)}$ of Q_μ is a connected sum of string quivers of either terms of Q or quivers generated from terms of Q by removing the connecting vertex. The connected sum is taken in the instances of the connecting vertices and results in a connected sum of trees and quivers of the form



(ii) For a square of type S_1 we have $\mu(i)_{j-1} \leq 0 \leq \mu(i)_{j+1}$. Moreover, such squares cannot be connected by paths longer than two arrows and can only be connected by paths of this length if $\mu(i)_{j+1} = 0$.

*(iii) For vertices (v, j) constructed from type **II**(1) terms we have $\alpha_{\mu_i}(v, j) \leq 2$ when $\mu_i \neq 0$.*

*(iv) For a vertex (v, j) constructed from a type **I** term with m arrows we have $\alpha_{\mu_i}(v, j) \leq m$ when $\mu_i \neq 0$.*

6.7 Brauer-Severi varieties

In this section we will reconsider the Brauer-Severi scheme $BS_n(A)$ of an algebra A. In the generic case, that is when A is the free algebra $\mathbb{C}\langle x_1, \ldots, x_m\rangle$, we will show that it is a moduli space of a certain quiver situation. This then allows us to give the étale local description of $BS_n(A)$ whenever A is a Cayley-smooth algebra. Again, this local description will be a moduli space.

The generic Brauer-Severi scheme of degree n for m-generators, $BS_n^m(gen)$ is defined as follows. Consider the free algebra on m generators $\mathbb{C}\langle x_1, \ldots, x_m\rangle$ and consider the GL_n-action on $\mathtt{rep}_n\ \mathbb{C}\langle x_1, \ldots, x_m\rangle \times \mathbb{C}^n = M_n^m \oplus \mathbb{C}^n$ given by

$$g.(A_1, \ldots, A_m, v) = (gA_1g^{-1}, \ldots, gA_mg^{-1}, gv)$$

and consider the open subset $Brauer^s(gen)$ consisting of those points $(A_1, \ldots, A_m, v)$ where v is a cyclic vector, that is, there is no proper subspace of $\mathbb{C}^n$ containing v and invariant under left multiplication by the matrices A_i. The GL_n-stabilizer is trivial in every point of $Brauer^s(gen)$ whence we can define the orbit space

$$BS_n^m(gen) = Brauer^s(gen)/GL_n$$

Consider the following quiver situation

on two vertices $\{v_1, v_2\}$ such that there are m loops in v_2 and consider the dimension vector $\alpha = (1, n)$. Then, clearly

$$\mathtt{rep}_\alpha\ Q = \mathbb{C}^n \oplus M_n^m \simeq \mathtt{rep}_n\ \mathbb{C}\langle x_1, \ldots, x_m\rangle \oplus \mathbb{C}^n$$

where the isomorphism is as GL_n-module. On $\mathtt{rep}_\alpha\ Q$ we consider the action of the larger group $GL(\alpha) = \mathbb{C}^* \times GL_n$ acting as

$$(\lambda, g).(v, A_1, \ldots, A_m) = (gv\lambda^{-1}, gA_1g^{-1}, \ldots, gA_mg^{-1})$$

Consider the character χ_θ where $\theta = (-n, 1)$, then $\theta(\alpha) = 0$ and consider the open subset of θ-semistable representations in $\mathtt{rep}_\alpha\ Q$.

Lemma 6.3 *The following are equivalent for $V = (v, A_1, \ldots, A_m) \in \mathtt{rep}_\alpha\ Q$*

1. *V is θ-semistable.*
2. *V is θ-stable.*
3. *$V \in Brauer^s(gen)$.*

Consequently,

$$M_\alpha^{ss}(Q, \alpha) \simeq BS_n^m(gen)$$

Proof. 1. $\Rightarrow$ 2. : If V is θ-semistable it must contain a largest θ-stable subrepresentation W (the first term in the Jordan-Hölder filtration for θ-semistables). In particular, if the dimension vector of W is $\beta = (a, b) < (1, n)$, then $\theta(\beta) = 0$ which is impossible unless $\beta = \alpha$ whence $W = V$ is θ-stable.

2. $\Rightarrow$ 3. : Observe that $v \neq 0$, for otherwise V would contain a subrepresentation of dimension vector $\beta = (1, 0)$ but $\theta(\beta) = -n$ is impossible. Assume that v is non-cyclic and let $U \hookrightarrow \mathbb{C}^n$ be a proper subspace say of dimension $l < n$ containing v and stable under left multiplication by the A_i, then V has a subrepresentation of dimension vector $\beta' = (1, l)$ and again $\theta(\beta') = l - n < 0$ is impossible.

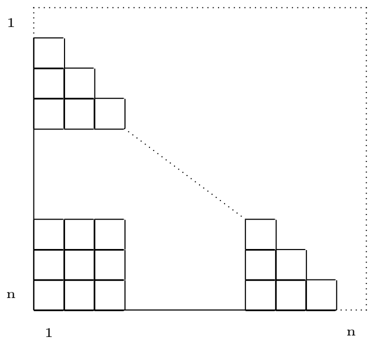
3. $\Rightarrow$ 1. : By cyclicity of v, the only proper subrepresentations of V have dimension vector $\beta = (0, l)$ for some $0 < l \leq n$, but they satisfy $\theta(\beta) > 0$, whence V is θ-(semi)stable.

As for the last statement, recall that geometric points of $M^{ss}_\alpha(Q, \alpha)$ classify isomorphism classes of direct sums of θ-stable representations. As there are no proper θ-stable subrepresentations, $M^{ss}_\alpha(Q, \alpha)$ classifies the $GL(\alpha)$-orbits in $Brauer^s(gen)$. Finally, as in chapter 1, there is a one-to-one orbits between the GL_n-orbits as described in the definition of the Brauer-Severi variety and the $GL(\alpha)$-orbits on $\mathtt{rep}_\alpha\ Q$. $\square$

By definition, $M^{ss}_\alpha(Q, \theta) = \mathtt{proj}\ \oplus_{n=0}^{\infty} \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha), \chi^n \theta}$ and we can either use the results of section 3 or the previous section to show that these semi-invariants f are generated by brackets, that is,

$$f(V) = det\ \begin{bmatrix} w_1(A_1, \ldots, A_m)v & \ldots & w_n(A_1, \ldots, A_m)v \end{bmatrix}$$

where the w_i are words in the noncommuting variables $x_1, \ldots, x_m$. As in section I.3 we can restrict these n-tuples of words $\{w_1, \ldots, w_n\}$ to sequences arising from $m$ulticolored Hilbert n-stairs. That is, the lower triangular part of a square $n \times n$ array

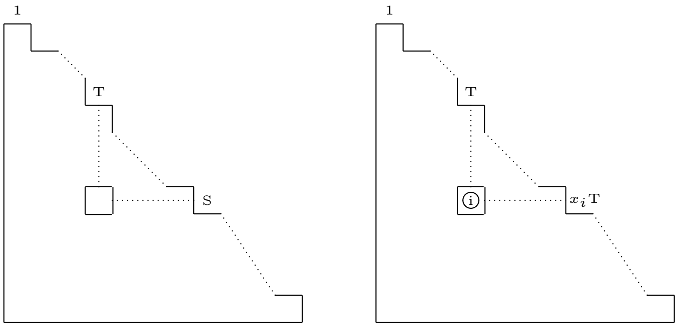


this time filled with colored stones ⓘ where $1 \leq i \leq m$ subject to the two coloring rules

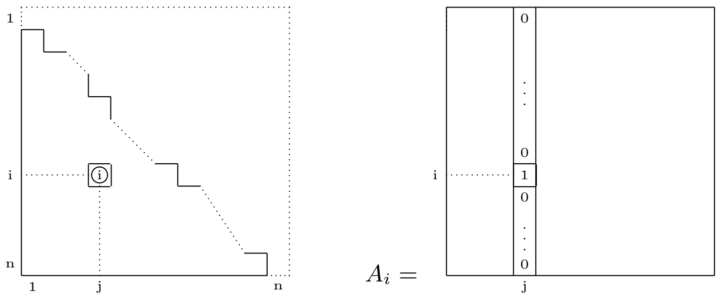
- each row contains exactly one stone
- each column contains at most one stone of each color

The relevant sequences $W(\sigma) = \{1, w_2, \ldots, w_n\}$ of words are then constructed by placing the identity element 1 at the top of the stair, and descend according to the rule

- Every go-stone has a *top word* T which we may assume we have constructed before and a *side word* S and they are related as indicated below



In a similar way to the argument in chapter 1 we can cover $M^{ss}_{\alpha}(Q,\alpha) = BS^m_n(gen)$ by open sets determined by Hilbert stairs and find representatives of the orbits in σ-standard form, that is replacing every i-colored stone in σ by a 1 at the same spot in A_i and fill the remaining spots in the same column of A_i by zeroes



As this fixes $(n-1)n$ entries of the mn^2+n entries of V, one recovers the following result of M. Van den Bergh [81]

Theorem 6.12 *The generic Brauer-Severi variety $BS^m_n(gen)$ of degree n in m generators is a smooth variety which can be covered by affine open subsets each isomorphic to $\mathbb{C}^{(m-1)n^2+n}$.*

For an arbitrary affine $\mathbb{C}$-algebra A, one defines the Brauer stable points to be the open subset of $\mathtt{rep}_n\ A \times \mathbb{C}^n$
$$Brauer^s_n(A) = \{(\phi, v) \in \mathtt{rep}_n\ A \times \mathbb{C}^n \ \mid\ \phi(A)v = \mathbb{C}^n\}$$

As Brauer stable points have trivial stabilizer in GL_n all orbits are closed and we can define the Brauer-Severi variety of A of degree n to be the orbit space

$$BS_n(A) = Brauer_n^s(A)/GL_n$$

We claim that Quillen-smooth algebras have smooth Brauer-Severi varieties. Indeed, as the quotient morphism

$$Brauer_n^s(A) \twoheadrightarrow BS_n(A)$$

is a principal GL_n-fibration, the base is smooth whenever the total space is smooth. The total space is an open subvariety of $\mathtt{rep}_n A \times \mathbb{C}^n$ which is smooth whenever A is Quillen-smooth.

Proposition 6.3 *If A is Quillen-smooth, then for every n we have that the Brauer-Severi variety of A at degree n is smooth.*

Next, we bring in the approximation at level n. Observe that for every affine $\mathbb{C}$-algebra A we have a GL_n-equivariant isomorphism

$$\mathtt{rep}_n\ A \simeq \mathtt{trep}_n\ \int_n A$$

More generally, we can define for every Cayley-Hamilton algebra A of degree n the trace preserving Brauer-Severi variety to be the orbit space of the Brauer stable points in $\mathtt{trep}_n\ A \times \mathbb{C}^n$. We denote this variety with $BS_n^{tr}(A)$. Again, the same argument applies

Proposition 6.4 *If A is Cayley-smooth of degree n , then the trace preserving Brauer-Severi variety $BS_n^{tr}(A)$ is smooth.*

We have seen that the moduli spaces are projective fiber bundles over the variety determined by the invariants,

$$M_\alpha^{ss}(Q,\theta) \twoheadrightarrow \mathtt{iss}_\alpha\ Q$$

Similarly, the (trace preserving) Brauer-Severi variety is a projective fiber bundle over the quotient variety of $\mathtt{rep}_n\ A$, that is, there is a proper map

$$BS_n(A) \overset{\pi}{\twoheadrightarrow} \mathtt{iss}_n\ A$$

and we would like to study the fibers of this map. Recall that when A is an order in a central simple algebra of degree n, then the general fiber will be isomorphic to the projective space $\mathbb{P}^{n-1}$ embedded in a higher dimensional $\mathbb{P}^N$. Over non-Azumaya points we expect this $\mathbb{P}^{n-1}$ to degenerate to more complex projective varieties which we would like to describe. To perform this study we need to control the étale local structure of the fiber bundle π in a neighborhood of $\xi \in \mathtt{iss}_n\ A$. Again, it is helpful to consider first the generic case, that is when $A = \mathbb{C}\langle x_1, \ldots, x_m\rangle$ or $\mathbb{T}_n^m$. In this case, we have seen that the following two fiber bundles are isomorphic

$$BS_n^m(gen) \twoheadrightarrow \mathtt{iss}_n\ \mathbb{T}_n^m \quad \text{and} \quad M_\alpha^{ss}(Q,\theta) \twoheadrightarrow \mathtt{iss}_\alpha\ Q$$

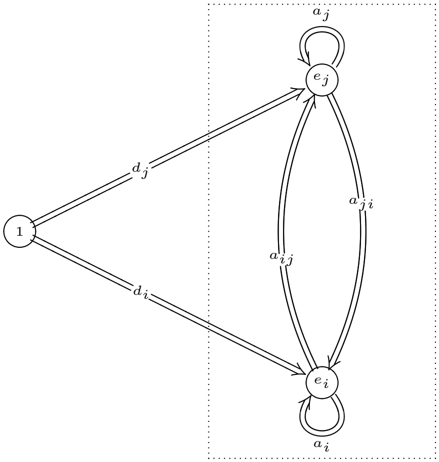
where $\alpha = (1, n)$, $\theta = (-n, 1)$ and the quiver

has Euler form $\chi_Q = \begin{bmatrix} 1 & -1 \\ 0 & 1-m \end{bmatrix}$

A semi-simple α-dimensional representation V_ζ of Q has representation type

$$(1,0) \oplus (0, d_1)^{\oplus e_1} \oplus \ldots \oplus (0, d_k)^{\oplus e_k} \quad \text{with} \quad \sum_i d_i e_i = n$$

and hence corresponds uniquely to a point $\xi \in \mathtt{iss}_n\ \mathbb{T}^m_n$ of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$. The étale local structure of $\mathtt{rep}_\alpha\ Q$ and of $\mathtt{iss}_\alpha\ Q$ near ζ is determined by the local quiver Q_ζ on $k+1$-vertices, say $\{v_0, v_1, \ldots, v_k\}$ with dimension vector $\alpha_\zeta = (1, e_1, \ldots, e_k)$ and where Q_ζ has the following local form for every triple (v_0, v_i, v_j) as can be verified from the Euler-form



where $a_{ij} = (m-1)d_i d_j = a_{ji}$ and $a_i = (m-1)d_i^2 + 1$, $a_j = (m-1)d_j^2 + 1$. The dashed part of Q_ζ is the same as the local quiver Q_ξ describing the étale local structure of $\mathtt{iss}_n\ \mathbb{T}^m_n$ near ξ. Hence, we see that the fibration $BS^m_n(gen) \twoheadrightarrow \mathtt{iss}_n\ \mathbb{T}^m_n$ is étale isomorphic in a neighborhood of ξ to the fibration of the moduli space

$$M^{ss}_{\alpha_\zeta}(Q_\zeta, \theta_\zeta) \twoheadrightarrow \mathtt{iss}_{\alpha_\zeta}\ Q_\zeta \simeq \mathtt{iss}_{\alpha_\xi}\ Q_\xi$$

in a neighborhood of the trivial representation and where $\theta_\zeta = (-n, d_1, \ldots, d_k)$. Another application of the Luna slice results gives the following

Theorem 6.13 *Let A be a Cayley-smooth algebra of degree n. Let $\xi \in \mathtt{triss}_n\ A$ correspond to the trace preserving n-dimensional semi-simple representation*

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

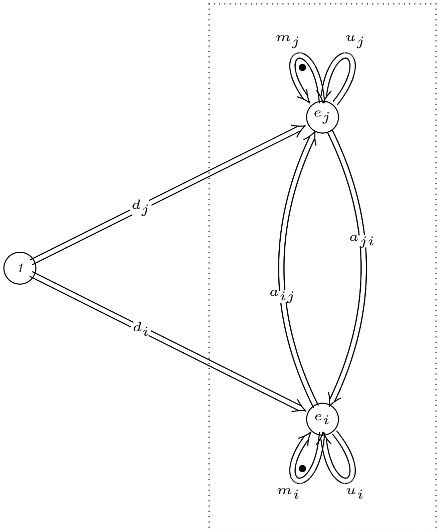
where the S_i are distinct simple representations of dimension d_i and occurring with multiplicity e_i. Then, the projective fibration

$$BS_n^{tr}(A) \xrightarrow{\pi} \mathtt{triss}_n\ A$$

is étale isomorphic in a neighborhood of ξ to the fibration of the moduli space

$$M^{ss}_{\alpha_\zeta}(Q^\bullet_\zeta, \theta_\zeta) \longrightarrow \mathtt{iss}_{\alpha_\zeta}\ Q^\bullet_\zeta \simeq \mathtt{iss}_{\alpha_\xi}\ Q^\bullet_\xi$$

in a neighborhood of the trivial representation. Here, $Q^\bullet_\xi$ is the local marked quiver describing the étale local structure of $\mathtt{trep}_n\ A$ near ξ, where $Q^\bullet_\zeta$ is the extended marked quiver situation, which locally for every triple (v_0, v_i, v_j) has the following shape where the dashed region is the local marked quiver $Q^\bullet_\xi$ describing $Ext^{tr}_A(M_\xi, M_\xi)$ and where $\alpha_\zeta = (1, e_1, \ldots, e_k)$ and $\theta_\zeta = (-n, d_1, \ldots, d_k)$.



6.8 Brauer-Severi fibers

In the foregoing section we have given a description of the generic Brauer-Severi variety $BS_n^m(gen)$ as a moduli space of quiver representation as well as a local description of the fibration

$$BS_n^m(gen) \xrightarrow{\psi} \mathtt{iss}_n^m$$

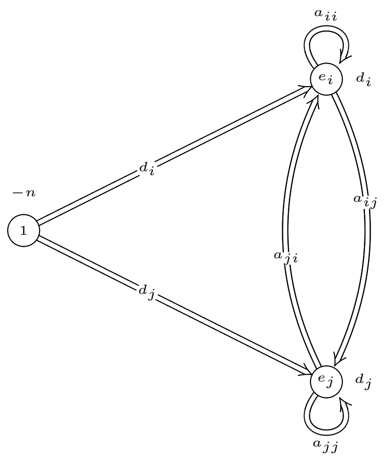
in an étale neighborhood of a point $\xi \in \mathtt{iss}_n^m$ of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$. We proved that it is étale locally isomorphic to the fibration

$$M^{ss}_{\alpha_\zeta}(Q_\zeta, \theta_\zeta) \longrightarrow\!\!\!\!\rightarrow \mathtt{iss}_{\alpha_\zeta}\ Q_\zeta$$

in a neighborhood of the trivial representation. That is, we can obtain the generic Brauer-Severi fiber $\psi^{-1}(\xi)$ from the description of the nullcone $Null_{\alpha_\zeta}\ Q_\zeta$ provided we can keep track of θ_ζ-semistable representations. Let us briefly recall the description of the quiver-setting $(Q_\zeta, \alpha_\zeta, \theta_\zeta)$.

- The quiver Q_ζ has $k+1$ vertices $\{v_0, v_1, \ldots, v_k\}$ such that there are d_i arrows from v_0 to v_i for $1 \leq i \leq k$. For $1 \leq i, j \leq k$ there are $a_{ij} = (m-1)d_i d_j + \delta_{ij}$ directed arrows from v_i to v_j.
- The dimension vector $\alpha_\zeta = (1, e_1, \ldots, e_k)$.
- The character θ_ζ is determined by the integral $k+1$-tuple $(-n, d_1, \ldots, d_k)$.

That is, for any triple (v_0, v_i, v_j) of vertices, the full subquiver of Q_ζ on these three vertices has the following form



Let $E = \sum_{i=1}^k e_i$ and T the usual (diagonal) maximal torus of dimension $1+E$ in $GL(\alpha_\zeta) \hookrightarrow GL_E$ and let $\{\pi_0, \pi_1, \ldots, \pi_E\}$ be the obvious basis for the weights of T.. As there are loops in every v_i for $i \geq 1$ and there are arrows from v_i to v_j for all $i, j \geq 1$ we see that the set of weights of $\mathtt{rep}_{\alpha_\zeta}\ Q_\zeta$ is

$$\pi_{\alpha_\zeta}\ Q_\zeta = \{\pi_{ij} = \pi_j - \pi_i \mid 0 \leq i \leq E, 1 \leq j \leq E\}$$

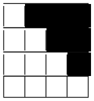
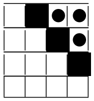
The maximal sets $\pi_s\ Q_\zeta$ for $s \in \mathcal{S}_{\alpha_\zeta}\ Q_\zeta$ are of the form

$$\pi_s\ Q_\zeta \overset{dfn}{=} \pi_\sigma = \{\pi_{ij} \mid i = 0 \text{ or } \sigma(i) < \sigma(j)\}$$

for some fixed permutation $\sigma \in S_E$ of the last E entries. To begin, there can be no larger subset as this would imply that for some $1 \leq i, j \leq E$ both π_{ij} and π_{ji} would belong to it which cannot be the case for a subset $\pi_{s'}\ Q_\zeta$. Next, $\pi_\sigma = \pi_s\ Q_\zeta$ where

$$s = (p, p+\sigma(1), p+\sigma(2), \ldots, p+\sigma(E)) \quad \text{where} \quad p = -\tfrac{E}{2}$$

If we now make s vertex-dominant, or equivalently if we only take a σ in the factor $S_E/(S_{e_1} \times S_{e_2} \times \ldots \times S_{e_k})$, then s belongs to $\mathcal{S}_{\alpha_\zeta}\ Q_\zeta$. For example, if $E = 3$ and $\sigma = id \in S_3$, then the corresponding border and corner regions for π_s are

$C_s =$  and $B_s =$ 

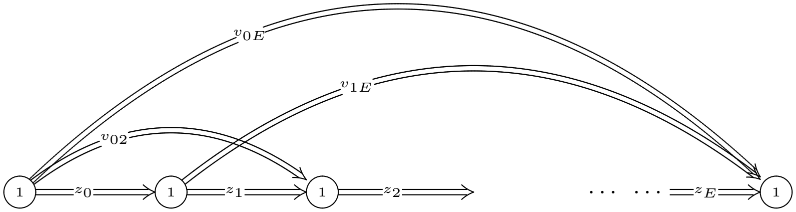
We have to show that the corresponding Hesselink stratum is non-empty in $Null_{\alpha_\zeta}\ Q_\zeta$ and that it contains θ_ζ-semistable representations. For s corresponding to a fixed $\sigma \in S_E$ the border quiver-setting $(Q_s, \alpha_s, \theta_s)$ is equal to

$$\begin{array}{ccccccccccc} -E & & -E+2 & & -E+4 & & & & E-2 & & E \\ ① & \xrightarrow{z_0} & ① & \xrightarrow{z_1} & ① & \xrightarrow{z_2} & \cdots & \cdots \xrightarrow{z_{E-1}} & ① & \xrightarrow{z_E} & ① \end{array}$$

where the number of arrows z_i are determined by

$$\begin{cases} z_0 & = p_u \text{ if } \sigma(1) \in I_{v_u} \\ z_i = a_{uv} \text{ if } \sigma(i) \in I_{v_u} \text{ and } \sigma(i+1) \in I_{v_v} \end{cases}$$

where we recall that I_{v_i} is the interval of entries in $[1, \ldots, E]$ belonging to vertex v_i. As all the $z_i \geq 1$ it follows that $\texttt{rep}_{\alpha_s}\ Q_s$ contains θ_s-stable representations, so the stratum in $Null_{\alpha_\zeta}\ Q_\zeta$ determined by the corner-type C_s is non-empty. We can depict the $L_s = T$-action on the corner as a representation space of the extended quiver-setting



Translating representations of this extended quiver back to the original quiver-setting (Q_ζ, α_ζ) we see that the corner C_s indeed contains θ_ζ-semistable representations and hence that this stratum in the nullcone determines an irreducible component in the Brauer-Severi fiber $\psi(\xi)$ of the generic Brauer-Severi variety.

Theorem 6.14 *Let $\xi \in \mathtt{iss}_n^m$ be of representation type $\tau = (e_1, d_1; \ldots ; e_k, d_k)$ and let $E = \sum_{i=1}^k e_i$. Then, the fiber $\pi^{-1}(\xi)$ of the Brauer-Severi fibration*

$$\begin{tikzcd} Brauer^s(gen) \arrow[d] \arrow[dr, "\psi"] & \\ BS_n^m(gen) \arrow[r, two heads, "\pi"'] & \mathtt{iss}_n^m \end{tikzcd}$$

has exactly $\frac{E!}{e_1!e_2!\ldots e_k!}$ irreducible components, all of dimension

$$n + (m-1)\sum_{i<j} e_i e_j d_i d_j + (m-1)\sum_i \frac{e_i(e_i - 1)}{2} - \sum_i e_i$$

Proof. In view of the foregoing remarks we only have to compute the dimension of the irreducible components. For a corner type C_s as above we have that the corresponding irreducible component in $Null_{\alpha_\zeta}\ Q_\zeta$ has dimension

$$dim\ GL(\alpha_\zeta) - dim\ P_s + dim\ C_s$$

and from the foregoing description of C_s as a quiver-representation space we see that

- $dim\ P_s = 1 + \frac{e_i(e_i+1)}{2}$.
- $dim\ C_s = n + \sum_i \frac{e_i(e_i-1)}{2}((m-1)d_i^2 + 1) + \sum_{i<j}(m-1)e_i e_j d_i d_j$.

as we can identify $P_s \simeq \mathbb{C}^* \times B_{e_1} \times \ldots \times B_{e_k}$ where B_e is the Borel subgroup of GL_e. Moreover, as $\psi^{-1}(\xi)$ is a Zariski open subset of

$$(\mathbb{C}^* \times GL_n) \times^{GL(\alpha_\zeta)} Null_{\alpha_\zeta}\ Q_\zeta$$

we see that the corresponding irreducible component of $\psi^{-1}(\xi)$ has dimension

$$1 + dim\ GL_n - dim\ P_s + dim\ C_s$$

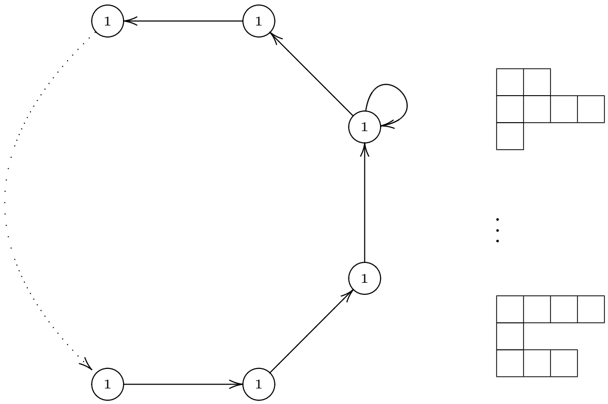
As the quotient morphism $\psi^{-1}(\xi) \twoheadrightarrow \pi^{-1}(\xi)$ is surjective, we have that the Brauer-Severi fiber $\pi^{-1}(\xi)$ has the same number of irreducible components of $\psi^{-1}(\xi)$. As the quotient

$$\psi^{-1}(\xi) \twoheadrightarrow \pi^{-1}(\xi)$$

is by Brauer-stability of all point a principal $PGL(1,n)$-fibration, substituting the obtained dimensions finishes the proof. $\square$

In particular, we deduce that the Brauer-Severi fibration $BS_n^m(gen) \xrightarrow{\pi} \mathtt{iss}_n^m$ is a flat morphism if and only if $(m,n) = (2,2)$ in which case all Brauer-Severi fibers have dimension one.

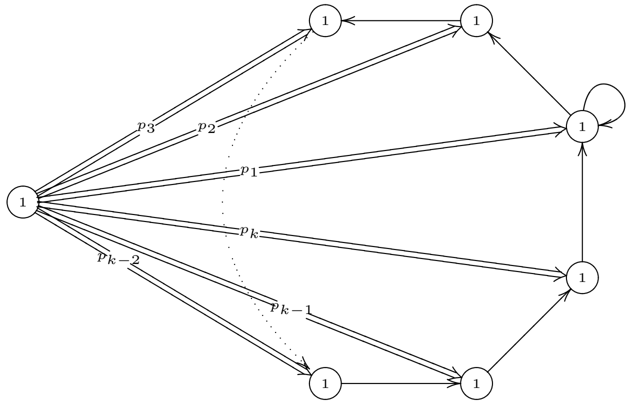
As a final application, let us compute the Brauer-Severi fibers in a point $\xi \in X = \mathtt{triss}_n\ A$ of the smooth locus $Sm_n\ A$ of a Cayley-Hamilton order of degree n which is of local quiver type (Q,α) where $\alpha = (1,\dots,1)$ and Q is the quiver



where the cycle has k vertices and $p = (p_1,\dots,p_k)$ is an unordered partition of n having exactly k parts. That is, A is a local Cayley-smooth order over a surface of type A_{k-101}. These are the only types that can occur for smooth surface orders which are maximal orders and have a non-singular ramification divisor. Observe also that in the description of nullcones, the extra loop will play no role, so the discussion below also gives the Brauer-Severi fibers of smooth curve orders. The Brauer-Severi fibration is étale locally isomorphic to the fibration

$$M_{\alpha'}^{ss}(Q',\theta') \xrightarrow{\pi} \mathtt{iss}_\alpha\ Q = \mathtt{iss}_{\alpha'}\ Q'$$

in a neighborhood of the trivial representation. Here, Q' is the extended quiver by one vertex v_0



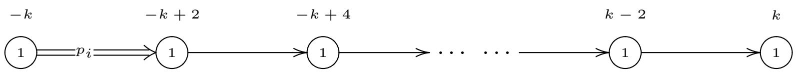
the extended dimension vector is $\alpha' = (1, 1, \ldots, 1)$ and the character is determined by the integral $k+1$-tuple $(-n, p_1, p_2, \ldots, p_k)$. The weights of the maximal torus $T = GL(\alpha')$ of dimension $k+1$ that occur in representations in the nullcone are

$$\pi_{\alpha'}\ Q' = \{\pi_{0\ i}, \pi_{i\ i+1}, 1 \leq i \leq k\}$$

Therefore, maximal corners C_s are associated to $s \in \mathcal{S}_{\alpha'}\ Q'$ where

$$\pi_s\ Q' = \{\pi_{0\ j}, 1 \leq j \leq k\} \cup \{\pi_{i\ i+1}, \pi_{i+1\ i+2}, \ldots, \pi_{i-2\ i-1}\}$$

for some fixed i. For such a subset the corresponding s is a one string $k+1$-tuple having as minimal value $-\frac{k}{2}$ at entry 0, $-\frac{k}{2}+1$ at entry i, $-\frac{k}{2}+2$ at entry $i+1$ and so on. To verify that this corner-type occurs in $Null_{\alpha'}\ Q'$ we have to consider the corresponding border quiver-setting $(Q'_s, \alpha'_s, \theta'_s)$ which is



which clearly has θ'_s-semistable representations, in fact, the corresponding moduli space $M^{ss}_{\alpha'_s}(Q'_s, \theta'_s) \simeq \mathbb{P}^{p_1-1}$. In this case we have that $L_s = P_s = GL(\alpha'_s)$ and therefore we can also interpret the corner as an open subset of the representation space

$$C_s \hookrightarrow \mathtt{rep}_{\alpha'_s}\ Q''_s$$

where the embedding is $P_s = GL(\alpha'_s)$-equivariant and the extended quiver $Q"_s$ is

$$
\begin{tikzcd}
1 \arrow[r, Rightarrow, "p_i"] \arrow[rr, bend left, Rightarrow, "p_{i+1}"] \arrow[rrrrr, bend left=60, Rightarrow, "p_{i-1}"] & 1 \arrow[r] & 1 \arrow[r] & {} & \cdots \quad \cdots \arrow[r] & 1
\end{tikzcd}
$$

Translating corner representations back to $\mathtt{rep}_{\alpha'}\ Q'$ we see that C_s contains θ'-semistable representations, so will determine an irreducible component in the Brauer-Severi fiber $\pi^{-1}(\xi)$. Let us calculate its dimension. The irreducible component N_s of $Null_{\alpha'}\ Q'$ determined by the corner C_s has dimension

$$
\begin{aligned}
dim\ GL(\alpha') - dim\ P_s + dim\ C_s &= (k+1) - (k+1) + \sum_i p_i + (k-1) \\
&= n + k - 1
\end{aligned}
$$

But then, the corresponding component in the Brauer-stable is an open subvariety of $(\mathbb{C}^* \times GL_n) \times^{GL(\alpha')} N_s$ and therefore has dimension

$$
\begin{aligned}
dim\ \mathbb{C}^* \times GL_n - dim\ GL(\alpha') + dim\ N_s &= 1 + n^2 - (k+1) + n + k - 1 \\
&= n^2 + n - 1
\end{aligned}
$$

But then, as the stabilizer subgroup of all Brauer-stable points is one dimensional in $\mathbb{C}^* \times GL_n$ the corresponding irreducible component in the Brauer-Severi fiber $\pi^{-1}(\xi)$ has dimension $n-1$. This completes the proof of the

Theorem 6.15 *Let A be a Cayley-Hamilton order of degree n over a surface $X = \mathtt{triss}_n\ A$ and let A be Cayley-smooth in $\xi \in X$ of type A_{k-101} and p as before. Then,the fiber of the Brauer-Severi fibration*

$$BS^t_n(A) \twoheadrightarrow X$$

in ξ has exactly k irreducible components, each of dimension $n-1$. In particular, if A is a Cayley-smooth order over the surface X such that all local types are $(A_{k-101}.p)$ for some $k \geq 1$ and partition p of n in having k-parts, then the Brauer-Severi fibration is a flat morphism.

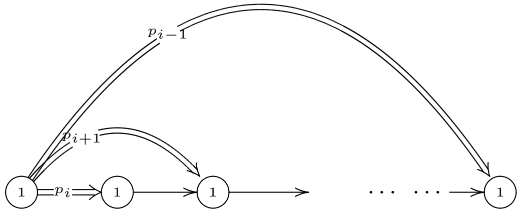
In fact, one can give a nice geometric interpretation to the different components. Consider the component corresponding to the corner C_s with notations as before. Consider the sequence of $k-1$ rational maps

$$\mathbb{P}^{n-1} \twoheadrightarrow \mathbb{P}^{n-1-p_{i-1}} \twoheadrightarrow \mathbb{P}^{n-1-p_{i-1}-p_{i-2}} \twoheadrightarrow \dots \twoheadrightarrow \mathbb{P}^{p_i-1}$$

defined by killing the right hand coordinates

$$[x_1 : \ldots : x_n] \mapsto [x_1 : \ldots : x_{n-p_{i-1}} : \underbrace{0 : \ldots : 0}_{p_{i-1}}] \mapsto \ldots \mapsto [x_1 : \ldots : x_{p_i} : \underbrace{0 : \ldots : 0}_{n-p_i}]$$

that is in the extended corner-quiver setting



we subsequently set all entries of the arrows from v_0 to v_{i-j} zero for $j \geq 1$, the extreme projection $\mathbb{P}^{n-1} \twoheadrightarrow \mathbb{P}^{p_i-1}$ corresponds to the projection $C_s/P_s \twoheadrightarrow B_s/L_s = M^{ss}_{\alpha'_s}(Q'_s, \theta'_s)$. Let V_i be the subvariety in $\times_{j=1}^{k} \mathbb{P}^{n-1}$ be the closure of the graph of this sequence of rational maps. If we label the coordinates in the $k-j$-th component $\mathbb{P}^{n-1}$ as $x(j) = [x_1(j) : \ldots : x_n(j)]$, then the multi-homogeneous equations defining V_i are

$$\begin{cases} x_a(j) & = 0 \text{ if } a > p_i + p_{i+1} + \ldots + p_{i+j} \\ x_a(j)x_b(j-1) & = x_b(j)x_a(j-1) \text{ if } 1 \leq a < b \leq p_i + \ldots + p_{i+l-1} \end{cases}$$

One verifies that V_i is a smooth variety of dimension $n-1$. If we would have the patience to work out the whole nullcone (restricting to the θ'-semistable representations) rather than just the irreducible components, we would see that the Brauer-Severi fiber $\pi^{-1}(\xi)$ consists of the varieties $V_1, \ldots, V_k$ intersecting transversally. The reader is invited to compare our description of the Brauer-Severi fibers with that of M. Artin [3] in the case of Cayley-smooth maximal curve orders.

References

The stratification of the nullcone used in this chapter is due to W. Hesselink [35]. The proof of theorem 6.3 uses results of F. Kirwan [44]. Theorem 6.4 is due to H-P. Kraft [50], theorem 6.5 and theorem 6.8 are special cases of Hesselinks result [35]. Example 6.16 is due to G. Van de Weyer . The definition of Brauer-Severi varieties of orders used here is due to M. Van den Bergh [81]. All other results are due to L. Le Bruyn [55], [56] and [59].

7 — Noncommutative Manifolds

By now we have developed enough machinery to study the representation varieties $\mathtt{trep}_n\ A$ and $\mathtt{triss}_n\ A$ of a Cayley-smooth algebra $A \in \mathtt{alg@n}$. In particular, we now understand the varieties

$$\mathtt{rep}_n\ A = \mathtt{trep}_n\ \int_n A \qquad \text{and} \qquad \mathtt{iss}_n\ A = \mathtt{triss}_n\ \int_n A$$

for the level n approximation $\int_n A$ of a Quillen-smooth algebra A, for all n. In this chapter we begin to study noncommutative manifolds, that is, *families* $(X_n)_n$ of commutative varieties which are locally controlled by Quillen-smooth algebras. Observe that for every $\mathbb{C}$-algebra A, the direct sum of representations induces *sum maps*

$$\mathtt{rep}_n\ A \times \mathtt{rep}_m\ A \longrightarrow \mathtt{rep}_{n+m}\ A \quad \text{and} \quad \mathtt{iss}_n\ A \times \mathtt{iss}_m\ A \longrightarrow \mathtt{iss}_{n+m}\ A$$

The characteristic feature of a family $(X_n)_n$ of varieties defining a noncommutative variety is that they are connected by sum-maps

$$X_n \times X_m \longrightarrow X_{n+m}$$

and that these morphisms are locally of the form $\mathtt{iss}_n\ A \times \mathtt{iss}_m\ A \longrightarrow \mathtt{iss}_{n+m}\ A$ for a Quillen-smooth algebra A. An important class of examples of such noncommutative manifolds is given by moduli spaces of quiver representations. In order to prove that they are indeed of the above type, we have to recall results on semi-invariants of quiver representations and on universal localization.

Next, we turn to the study of noncommutative differential forms. The idea being that noncommutative functions,vectorfields and differential forms on an algebra A induce ordinary functions, vectorfields and differential forms on *all* of the representation varieties $\mathtt{rep}_n\ A$ and $\mathtt{iss}_n\ A$. This approach is especially important in case A is a *symplectic* algebra, for example the path algebra of a double quiver. In this case we will define an infinite dimensional Lie algebra, the *necklace Lie algebra*, which induces flows on all the varieties $\mathtt{iss}_n\ A$ providing a dynamic aspect to noncommutative geometry.

7.1 Formal structure

Objects in $\mathtt{noncommutative\ geometry@n}$ are families of varieties $(X_i)_i$ which are locally controlled by a set of noncommutative algebras $\mathcal{A}$. That is, X_i is locally the quotient variety of a representation variety $\mathtt{rep}_n\ A$ for some n and some $\mathbb{C}$-algebra $A \in \mathcal{A}$. In section 2.7 we have seen that the representation varieties form a somewhat mysterious subclass of the category of all (affine) GL_n-varieties. For this reason it is important to equip them with additional structures that may make them stand out among the GL_n-varieties. In this section we define the *formal structure* on representation varieties, extending in a natural way the formal structure introduced by M. Kapranov on smooth affine varieties. Let us give an illustrative example of this structure.

Example 7.1 (Formal structure on $\mathbb{A}^d$) Consider the affine space $\mathbb{A}^d$ with coordinate ring $\mathbb{C}[x_1,\ldots,x_d]$ and order the coordinate functions $x_1 < x_2 < \ldots < x_d$. Let $\mathfrak{f}_d$ be the free Lie algebra on $\mathbb{C}x_1 \oplus \ldots \oplus \mathbb{C}x_d$ which has an ordered basis $B = \cup_{k\geq 1} B_k$ defined as follows. B_1 is the ordered set $\{x_1,\ldots,x_d\}$ and $B_2 = \{[x_i,x_j] \mid j < i\}$, ordered such that $B_1 < B_2$ and $[x_i,x_j] < [x_k,x_l]$ iff $j < l$ or $j = l$ and $i < k$. Having constructed the ordered sets B_l for $l < k$ we define

$$B_k = \{[t,w] \mid t = [u,v] \in B_l, w \in B_{k-l} \text{ such that } \ v \leq w < t \text{ for } \ l < k\}.$$

For $l < k$ we let $B_l < B_k$ and B_k is ordered by $[t,w] < [t'.w']$ iff $w < w'$ or $w = w'$ and $t < t'$.

It is well known that B is an ordered $\mathbb{C}$-basis of the Lie algebra $\mathfrak{f}_d$ and that its enveloping algebra

$$U(\mathfrak{f}_d) = \mathbb{C}\langle x_1,\ldots,x_d\rangle$$

is the free associative algebra on the x_i. We number the elements of $\cup_{k\geq 2} B_k$ according to the order $\{b_1,b_2,\ldots\}$ and for $b_i \in B_k$ we define $ord(b_i) = k-1$ (the number of brackets needed to define b_i). Let Λ be the set of all functions with finite support $\lambda : \cup_{k\geq 2} B_k \longrightarrow \mathbb{N}$ and define $ord(\lambda) = \sum \lambda(b_i) ord(b_i)$. Rephrasing the *Poincaré-Birkhoff-Witt* result for $U(\mathfrak{f}_d)$ we have that any noncommutative polynomial $p \in \mathbb{C}\langle x_1,\ldots,x_d\rangle$ can be written uniquely as a finite sum

$$p = \sum_{\lambda \in \Lambda} [\![f_\lambda]\!]\, M_\lambda$$

where $[\![f_\lambda]\!] \in \mathbb{C}[x_1,\ldots,x_d] = S(B_1)$ and $M_\lambda = \prod_i b_i^{\lambda(b_i)}$. In particular, for every $\lambda,\mu,\nu \in \Lambda$, there is a unique bilinear differential operator with polynomial coefficients

$$C^\nu_{\lambda\mu} : \mathbb{C}[x_1,\ldots,x_d] \otimes_{\mathbb{C}} \mathbb{C}[x_1,\ldots,x_d] \longrightarrow \mathbb{C}[x_1,\ldots,x_d]$$

defined by expressing the product $[\![f]\!]\, M_\lambda .\ [\![g]\!]\, M_\mu$ in $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ uniquely as $\sum_{\nu\in\Lambda} [\![C^\nu_{\lambda\mu}(f,g)]\!]\, M_\nu$.

By associativity of $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ the $C^\nu_{\lambda\mu}$ satisfy the *associativity constraint* , that is, we have equality of the trilinear differential operators

$$\sum_{\mu_1} C^\nu_{\mu_1\lambda_3} \circ (C^{\mu_1}_{\lambda_1\lambda_2} \otimes id) = \sum_{\mu_2} C^\nu_{\lambda_1\mu_2} \circ (id \otimes C^{\mu_2}_{\lambda_2\lambda_3})$$

for all $\lambda_1,\lambda_2,\lambda_3,\nu \in \Lambda$. That is, one can define the algebra $\mathbb{C}\langle x_1,\ldots,x_d\rangle_{[[\mathsf{ab}]]}$ to be the $\mathbb{C}$-vectorspace of possibly *infinite formal sums* $\sum_{\lambda\in\Lambda} [\![f_\lambda]\!]\, M_\lambda$ with multiplication defined by the operators $C^\nu_{\lambda\mu}$.

Let $A_d(\mathbb{C})$ be the d-th *Weyl algebra* , that is, the ring of differential operators with polynomial coefficients on $\mathbb{A}^d$. Let $\mathcal{O}_{\mathbb{A}^d}$ be the structure sheaf on $\mathbb{A}^d$ then it is well-known that the ring of sections $\mathcal{O}_{\mathbb{A}^d}(U)$ on any Zariski open subset $U \hookrightarrow \mathbb{A}^d$ is a left $A_d(\mathbb{C})$-module. Define a sheaf $\mathcal{O}^f_{\mathbb{A}^d}$ of noncommutative algebras on $\mathbb{A}^d$ by taking as its sections over U the algebra

$$\mathcal{O}^f_{\mathbb{A}^d}(U) = \mathbb{C}\langle x_1, \ldots, x_d\rangle_{[[\mathtt{ab}]]} \underset{\mathbb{C}[x_1,\ldots,x_d]}{\otimes} \mathcal{O}_{\mathbb{A}^d}(U)$$

that is the $\mathbb{C}$-vectorspace of possibly infinite formal sums $\sum_{\lambda \in \Lambda} [\![f_\lambda]\!]\, M_\lambda$ with $f_\lambda \in \mathcal{O}_{\mathbb{A}^d}(U)$ and the multiplication is given as before by the action of the bilinear differential operators $C^\nu_{\lambda\mu}$ on the left $A_d(\mathbb{C})$-module $\mathcal{O}_{\mathbb{A}^d}(U)$, that is, for all $f, g \in \mathcal{O}_{\mathbb{A}^d}(U)$ we have

$$[\![f]\!]\, M_\lambda . [\![g]\!]\, M_\mu = \sum_\nu [\![C^\nu_{\lambda\mu}(f,g)]\!]\, M_\nu$$

This sheaf of noncommutative algebras $\mathcal{O}^f_{\mathbb{A}^d}$ is called *the formal structure* on $\mathbb{A}^d$.

We will now define formal structures on arbitrary affine smooth varieties. Let R be an associative $\mathbb{C}$-algebra, R^{Lie} its Lie structure and R^{Lie}_m the subspace spanned by the expressions $[r_1, [r_2, \ldots, [r_{m-1}, r_m] \ldots]$ containing $m-1$ instances of Lie brackets. The *commutator filtration* of R is the (increasing) filtration by ideals $(F^k\ R)_{k \in \mathbb{Z}}$ with $F^k\ R = R$ for $d \in \mathbb{N}$ and

$$F^{-k}\ R = \sum_m \sum_{i_1 + \ldots + i_m = k} R R^{Lie}_{i_1} R \ldots R R^{Lie}_{i_m} R$$

Observe that all $\mathbb{C}$-algebra morphisms preserve the commutator filtration. The *associated graded algebra* $gr_F\ R$ is a (negatively) graded commutative *Poisson algebra* with part of degree zero, the *abelianization* $R_{ab} = \frac{R}{[R,R]}$. If $R = \mathbb{C}\langle x_1, \ldots, x_d\rangle$, then the commutator filtration has components

$$F^{-k}\ \mathbb{C}\langle x_1, \ldots, x_d\rangle = \{\sum_\lambda [\![f_\lambda]\!]\, M_\lambda, \forall \lambda : ord(\lambda) \geq k\}$$

Definition 7.1 *Denote with* $\mathtt{nil}_k$ *the category of associative* $\mathbb{C}$*-algebras* R *such that* $F^{-k} R = 0$ *(in particular,* $\mathtt{nil}_1 = \mathtt{commalg}$ *the category of commutative* $\mathbb{C}$*-algebras). An algebra* $A \in Ob(\mathtt{nil}_k)$ *is said to be* k-smooth *if and only if for all* $T \in Ob(\mathtt{nil}_k)$*, all nilpotent twosided ideals* $I \triangleleft T$ *and all* $\mathbb{C}$*-algebra morphisms* $A \xrightarrow{\phi} \frac{T}{I}$ *there exist a lifted* $\mathbb{C}$*-algebra morphism*

$$\begin{tikzcd} T \arrow[r, two heads] & \frac{T}{I} \\ & A \arrow[u, "\phi"'] \arrow[ul, dotted, "\exists \tilde{\phi}"] \end{tikzcd}$$

making the diagram commutative. Alternatively, an algebra is k*-smooth if and only if it is* $\mathtt{nil}_k$*-smooth.*

For example, the quotient $\frac{\mathbb{C}\langle x_1,\ldots,x_d\rangle}{F^{-k}\ \mathbb{C}\langle x_1,\ldots,x_d\rangle}$ is k-smooth using the lifting property of free algebras and the fact that algebra morphisms preserve the commutator filtration. Generalizing this, if A is Quillen-smooth then the quotient

$$A_{(k)} = \frac{A}{F^{-k}\ A}$$

is k-smooth.

Kapranov proves [39, Thm 1.6.1] that an affine commutative *Grothendieck-smooth* algebra C has a *unique* (upto $\mathbb{C}$-algebra isomorphism identical on C) k-smooth *thickening* $C^{(k)}$ with $C^{(k)}_{ab} \simeq C$. The inverse limit (connecting morphisms are given by the unicity result)

$$C^f = \lim_{\leftarrow} C^{(k)}$$

is then called the *formal completion* of C. Clearly, one has $C^f_{ab} = C$. For example,

$$\mathbb{C}[x_1,\ldots,x_d]^f = \lim_{\leftarrow} \frac{\mathbb{C}\langle x_1,\ldots,x_d\rangle}{F^{-k}\ \mathbb{C}\langle x_1,\ldots,x_d\rangle} \simeq \mathbb{C}\langle x_1,\ldots,x_d\rangle_{[[\mathsf{ab}]]}.$$

If X is an affine smooth (commutative) variety, one can use the formal completion $\mathbb{C}[X]^f$ to define a sheaf of noncommutative algebras $\mathcal{O}^f_X$ defining the *formal structure* on X.

The fact that C is Grothendieck-smooth is essential to construct and prove uniqueness of the formal completion. At present, no sufficiently functorial extension of formal completion is known for arbitrary commutative $\mathbb{C}$-algebras. It is not true that any (non affine) smooth variety can be equipped with a formal structure. In fact, the obstruction gives important new invariants of a smooth variety related to *Atiyah classes* . We refer to [39, §4] for more details.

We recall briefly the algebraic construction of *microlocalization*. Let R be a filtered algebra with a *separated filtration* $\{F_n\}_{n\in\mathbb{Z}}$ and let S be a multiplicatively closed subset of R containing 1 but not 0. For any $r \in F_n - F_{n-1}$ we denote its *principal character* $\sigma(r)$ to be the image of r in the associated graded algebra $gr(R)$. We assume that the set $\sigma(S)$ is a multiplicatively closed subset of $gr(R)$. We define the *Rees ring* $\tilde{R}$ to be the graded algebra

$$\tilde{R} = \oplus_{n\in Z} F_n t^n \hookrightarrow R[t,t^{-1}]$$

where t is an extra central variable. If $\sigma(s) \in gr(R)_n$ then we define the element $\tilde{s} = st^n \in \tilde{R}_n$. The set $\tilde{S} = \{\tilde{s}, s \in S\}$ is a multiplicatively closed subset of homogeneous elements in $\tilde{R}$.

Assume that $\sigma(S)$ is an Ore set in $gr(R) = \frac{\tilde{R}}{(t)}$, then for every $n \in \mathbb{N}_0$ the image $\pi_n(\tilde{S})$ is an Ore set in $\frac{\tilde{R}}{(t^n)}$ where $\tilde{R} \twoheadrightarrow \frac{\tilde{R}}{(t^n)}$ is the quotient morphism. Hence, we have an inverse system of graded localizations and can form the inverse limit in the graded sense

$$Q^{\mu}_{\tilde{S}}(\tilde{R}) = \lim_{\leftarrow}{}^{g}\ \pi_n(\tilde{S})^{-1}\frac{\tilde{R}}{(t^n)}$$

The element t acts torsionfree on this limit and hence we can form the filtered algebra

$$Q^{\mu}_{S}(R) = \frac{Q^{\mu}_{\tilde{S}}(\tilde{R})}{(t-1)Q^{\mu}_{\tilde{S}}(\tilde{R})}$$

which is the *micro-localization* of R at the multiplicatively closed subset S. We recall that the associated graded algebra of the microlocalization can be identified with the graded localization

$$gr(Q^{\mu}_{S}(R)) = \sigma(S)^{-1} gr(R).$$

Let R be a $\mathbb{C}$-algebra with $R_{ab} = \frac{R}{[R,R]} = C$. We assume that the commutator filtration $(F^k)_{k \in \mathbb{Z}}$ is a separated filtration on R. Observe that this is not always the case (for example consider $U(\mathfrak{g})$ for $\mathfrak{g}$ a semi-simple Lie algebra) but often one can repeat the argument below replacing R with $\frac{R}{\cap F^n}$.

Observe that $gr(R)$ is a negatively graded commutative algebra with part of degree zero C. Take a multiplicatively closed subset S_c of C, then $S = S_c + [R,R]$ is a multiplicatively closed subset of R with the property that $\sigma(S) = S_c$ and clearly S_c is an Ore set in $gr(R)$. Therefore, $\tilde{S}$ is a multiplicatively closed set of the Rees ring $\tilde{R}$ consisting of homogeneous elements of degree zero. Observing that $(t^n)_0 = F^{-n}t^n$ for all $n \in \mathbb{N}_0$ we see that

$$Q^{\mu}_{S}(R) = \lim_{\leftarrow} \pi_n(S)^{-1} \frac{R}{F^{-n}}$$

where $R \overset{\pi_n}{\twoheadrightarrow} \frac{R}{F^n}$ is the quotient morphism and Q^{μ}_{S} is filtered again by the commutator filtration and has as associated graded algebra

$$gr(Q^{\mu}_{S}(R)) = S_c^{-1} gr(R).$$

One can define a *microstructure sheaf* $\mathcal{O}^{\mu}_{R}$ on the affine scheme X of C by taking as its sections over the affine Zariski open set $X(f)$

$$\Gamma(X(f), \mathcal{O}^{\mu}_{R}) = Q^{\mu}_{S_f}(R)$$

where $S = \{1, f, f^2, \ldots\} + [R,R]$. For C a Grothendieck-smooth affine commutative algebra this sheaf of noncommutative algebras is the *formal structure* on X introduced by M. Kapranov.

An important remark to make is that one really needs microlocalization to construct a sheaf of noncommutative algebras on X. If by some fluke we would have that all the S_f are already Ore sets in R, we might optimistically assume that taking as sections over $X(f)$ the Ore localization $S_f^{-1}R$ we would define a sheaf $\mathcal{O}_R$ over X. This is in general *not* the case as the Ore set S_g need no longer be Ore in a localization $S_f^{-1}R$!

Still one can remedy this by defining a *noncommutative Zariski topology* on X using *words* in the Ore sets S_f, see [82, §1.3]. Whereas we do not need this to define formal structures it seems to me inevitable that at a later stage in the development of noncommutative geometry we will need to resort to such *noncommutative Grothendieck topologies* on usual commutative schemes.

Having define a formal structure on affine smooth varieties, we will now extend it to arbitrary representation varieties. The starting point is that for every associative algebra A the functor

$$\mathtt{alg} \xrightarrow{Hom_{\mathtt{alg}}(A, M_n(-))} \mathtt{sets}$$

is *representable* in $\mathtt{alg}$. That is, there exists an associative $\mathbb{C}$-algebra $\sqrt[n]{A}$ such that there is a natural equivalence between the functors

$$Hom_{\mathtt{alg}}(A, M_n(-)) \underset{n.e.}{\sim} Hom_{\mathtt{alg}}(\sqrt[n]{A}, -).$$

In other words, for every associative $\mathbb{C}$-algebra B, there is a functorial one-to-one correspondence between the sets

$$\begin{cases} \text{algebra maps} & A \longrightarrow M_n(B) \\ \text{algebra maps} & \sqrt[n]{A} \longrightarrow B \end{cases}$$

We call $\sqrt[n]{A}$ the *n-th root algebra* of A .

Example 7.2 If $A = \mathbb{C}\langle x_1, \ldots, x_d \rangle$, then it is easy to see that $\sqrt[n]{A}$ is the free algebra $\mathbb{C}\langle x_{11,1}, \ldots, x_{nn,d} \rangle$ on dn^2 variables. For, given an algebra map $A \xrightarrow{\phi} M_n(B)$ we obtain an algebra map $\sqrt[n]{A} \longrightarrow B$ by sending the free variable $x_{ij,k}$ to the (i,j)-entry of the matrix $\phi(x_k) \in M_n(B)$. Conversely, to an algebra map $\sqrt[n]{A} \xrightarrow{\psi} B$ we assign the algebra map $A \longrightarrow M_n(B)$ by sending x_k to the matrix $(\psi(x_{ij,k}))_{i,j} \in M_n(B)$. Clearly, these operations are each others inverses.

To define $\sqrt[n]{A}$ in general, consider the free algebra product $A * M_n(\mathbb{C})$ and consider the subalgebra

$$\sqrt[n]{A} = A * M_n(\mathbb{C})^{M_n(\mathbb{C})} = \{p \in A * M_n(\mathbb{C}) \mid p.(1 * m) = (1 * m).p \ \forall m \in M_n(\mathbb{C})\}$$

Before we can prove the universal property of $\sqrt[n]{A}$ we need to recall a property that $M_n(\mathbb{C})$ shares with any Azumaya algebra : if $M_n(\mathbb{C}) \xrightarrow{\phi} R$ is an algebra morphism and if $R^{M_n(\mathbb{C})} = \{r \in R \mid r.\phi(m) = \phi(m).r \ \forall m \in M_n(\mathbb{C})\}$, then we have $R \simeq M_n(\mathbb{C}) \otimes_{\mathbb{C}} R^{M_n(\mathbb{C})}$.

In particular, if we apply this to $R = A * M_n(\mathbb{C})$ and the canonical map $M_n(\mathbb{C}) \xrightarrow{\phi} A * M_n(\mathbb{C})$ where $\phi(m) = 1 * m$ we obtain that $M_n(\sqrt[n]{A}) = M_n(\mathbb{C}) \otimes_{\mathbb{C}} \sqrt[n]{A} = A * M_n(\mathbb{C})$.

Hence, if $\sqrt[n]{A} \xrightarrow{f} B$ is an algebra map we can consider the composition

$$A \xrightarrow{id_A * 1} A * M_n(\mathbb{C}) \simeq M_n(\sqrt[n]{A}) \xrightarrow{M_n(f)} M_n(B)$$

to obtain an algebra map $A \longrightarrow M_n(B)$. Conversely, consider an algebra map $A \xrightarrow{g} M_n(B)$ and the canonical map $M_n(\mathbb{C}) \xrightarrow{i} M_n(B)$ which centralizes B in $M_n(B)$. Then, by the universal property of free algebra products we have an algebra map $A * M_n(\mathbb{C}) \xrightarrow{g*i} M_n(B)$ and restricting to $\sqrt[n]{A}$ we see that this maps factors

$$\begin{tikzcd} A * M_n(\mathbb{C}) \arrow[r, "g*i"] & M_n(B) \\ \sqrt[n]{A} \arrow[u, hook] \arrow[r, dotted] & B \arrow[u, hook] \end{tikzcd}$$

and one verifies that these two operations are each others inverses.

It follows from the functoriality of the $\sqrt[n]{.}$ construction that $\mathbb{C}\langle x_1, \ldots, x_d\rangle \twoheadrightarrow A$ implies that $\sqrt[n]{\mathbb{C}\langle x_1, \ldots, x_d\rangle} \twoheadrightarrow \sqrt[n]{A}$. Therefore, if A is affine and generated by $\leq d$ elements, then $\sqrt[n]{A}$ is also affine and generated by $\leq dn^2$ elements.

These properties allow us define a *formal completion* of $\mathbb{C}[\mathtt{rep}_n\ A]$ in a functorial way for any associative algebra A. Equip $\sqrt[n]{A}$ with the commutator filtration

$$\ldots \hookrightarrow F_{-2}\ \sqrt[n]{A} \hookrightarrow F_{-1}\ \sqrt[n]{A} \hookrightarrow \sqrt[n]{A} \quad = \quad \sqrt[n]{A} \quad = \quad \ldots$$

Because algebra morphisms are commutator filtration preserving, it follows from the universal property of $\sqrt[n]{A}$ that $\frac{\sqrt[n]{A}}{F_{-k}\sqrt[n]{A}}$ is the object in $\mathtt{nil}_k$ representing the functor

$$\mathtt{nil}_k \xrightarrow{Hom_{\mathtt{alg}}(A, M_n(-))} \mathtt{sets}.$$

In particular, because the categories $\mathtt{commalg}$ and $\mathtt{nil}_1$ are naturally equivalent, we deduce that

$$\sqrt[n]{A}_{ab} = \frac{\sqrt[n]{A}}{[\sqrt[n]{A}, \sqrt[n]{A}]} = \frac{\sqrt[n]{A}}{F_{-1}\sqrt[n]{A}} \simeq \mathbb{C}[\mathtt{rep}_n\ A]$$

because both algebras represent the same functor. We now define

$$\sqrt[n]{A}_{[[\mathtt{ab}]]} = \lim_{\leftarrow} \frac{\sqrt[n]{A}}{F_{-k}\ \sqrt[n]{A}}.$$

Assume that A is Quillen-smooth, then so is $\sqrt[n]{A}$ because we have seen before that

$$M_n(\sqrt[n]{A}) \simeq A * M_n(\mathbb{C})$$

and the class of Quillen-smooth algebras is easily seen to be closed under free products and matrix algebras.

As a consequence, we have for every $k \in \mathbb{N}$ that the quotient $\frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}}$ is k-smooth. Moreover, we have that

$$(\frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}})_{ab} \simeq \frac{\sqrt[n]{A}}{[\sqrt[n]{A}, \sqrt[n]{A}]} \simeq \mathbb{C}[\mathtt{rep}_n\ A].$$

Because $\mathbb{C}[\mathtt{rep}_n\ A]$ is an affine commutative Grothendieck-smooth algebra, we deduce from the uniqueness of k-smooth thickenings that

$$\mathbb{C}[\mathtt{rep}_n\ A]^{(k)} \simeq \frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}}$$

and consequently that the formal completion of $\mathbb{C}[\mathtt{rep}_n\ A]$ can be identified with

$$\mathbb{C}[\mathtt{rep}_n\ A]^f \simeq \sqrt[n]{A}_{[[\mathtt{ab}]]}.$$

Therefore, if we define for an *arbitrary* $\mathbb{C}$-algebra A the *formal completion* of $\mathbb{C}[\mathtt{rep}_n\ A]$ to be $\sqrt[n]{A}_{[[\mathtt{ab}]]}$ we have a canonical extension of the formal structure on affine Grothendieck-smooth commutative algebras to the class of coordinate rings of representation spaces on which it is functorial in the algebras.

There is a natural action of GL_n by algebra automorphisms on $\sqrt[n]{A}$. Let u_A denote the universal morphism $A \xrightarrow{u_A} M_n(\sqrt[n]{A})$ corresponding to the identity map on $\sqrt[n]{A}$. For $g \in GL_n$ we can consider the composed algebra map

$$\begin{tikzcd}
A \arrow[r, "u_A"] \arrow[dr, "\psi_g"'] & M_n(\sqrt[n]{A}) \arrow[d, "g.g^{-1}"] \\
& M_n(\sqrt[n]{A})
\end{tikzcd}$$

Then g acts on $\sqrt[n]{A}$ via the automorphism $\sqrt[n]{A} \xrightarrow{\phi_g} \sqrt[n]{A}$ corresponding the the composition ψ_g. It is easy to verify that this defines indeed a GL_n-action on $\sqrt[n]{A}$.

The *formal structure sheaf* $\mathcal{O}^f_{\mathtt{rep}_n\ A}$ defined over $\mathtt{rep}_n\ A$ constructed from $\sqrt[n]{A}$ will be denoted by $\mathcal{O}^f_{\sqrt[n]{A}}$. We see that it actually has a GL_n-structure which is compatible with the GL_n-action on $\mathtt{rep}_n\ A$.

7.2 Semi invariants

An important class of examples of noncommutative varieties are *moduli spaces* of θ-semistable representations of quivers. Because the moduli space $M^{ss}_\alpha(Q,\theta)$ is by definition the projective scheme of the graded *algebra of semi-invariants* of weight χ_θ^n for some n

$$M_\alpha^{ss}(Q,\theta) = \mathtt{proj}\ \oplus_{n=0}^{\infty} \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$$

we need some control on these semi-invariants of quivers.

In this section we will give a generating set of semi-invariants. The strategy of proof should be clear by now. First, we will describe a large set of semi-invariants. Then we use classical invariant theory to describe all multilinear semi-invariants of $GL(\alpha)$, or equivalently, all multilinear invariants of $SL(\alpha) = SL_{a_1} \times \ldots \times SL_{a_k}$ and describe them in terms of these determinantal semi-invariants. Finally, we show by polarization and restitution that these semi-invariants do indeed generate all semi-invariants.

Let Q be a quiver on k vertices $\{v_1, \ldots, v_k\}$. We introduce the *additive* $\mathbb{C}$-*category* $\mathtt{add}\ Q$ generated by the quiver. For every vertex v_i we introduce an indecomposable object which we denote by i. An arbitrary object in $\mathtt{add}\ Q$ is then a sum of these

$$1^{\oplus e_1} \oplus \ldots \oplus k^{\oplus e_k}$$

That is we can identify $\mathtt{add}\ Q$ with $\mathbb{N}^k$. Morphisms in the category $\mathtt{add}\ Q$ are defined by the rules

$$\begin{cases} Hom_{\mathtt{add}\ Q}(\ i\ ,\ j\ ) = j \longleftarrow \cdots\cdots i \\[2ex] Hom_{\mathtt{add}\ Q}(\ i\ ,\ i\ ) = \ i \circlearrowleft \end{cases}$$

where the right hand sides are the $\mathbb{C}$-vectorspaces spanned by all oriented paths from v_i to v_j in the quiver Q, including the idempotent (trivial) path e_i when $i = j$.

Clearly, for any k-tuples of positive integers $\alpha = (u_1, \ldots, u_k)$ and $\beta = (v_1, \ldots, v_k)$

$$Hom_{\mathtt{add}\ Q}(\ 1^{\oplus u_1} \oplus \ldots \oplus k^{\oplus u_k}\ ,\ 1^{\oplus v_1} \oplus \ldots \oplus k^{\oplus v_k}\ )$$

is defined by matrices and composition arises via matrix multiplication

$$\begin{bmatrix} M_{v_1\times u_1}(\ 1 \circlearrowleft\ ) & \ldots & M_{v_1\times u_k}(\ 1 \longleftarrow \cdots\cdots k\ ) \\ \vdots & \ddots & \vdots \\ M_{v_k\times u_1}(\ k \longleftarrow \cdots\cdots 1\ ) & \ldots & M_{v_k\times u_k}(\ k \circlearrowleft\ ) \end{bmatrix}$$

Fix a dimension vector $\alpha = (a_1, \ldots, a_k)$ and a morphism ϕ in `add` Q

$$\textcircled{1}^{\oplus u_1} \oplus \ldots \oplus \textcircled{k}^{\oplus u_k} \xrightarrow{\phi} \textcircled{1}^{\oplus v_1} \oplus \ldots \oplus \textcircled{k}^{\oplus v_k}$$

For any representation $V \in \mathtt{rep}_\alpha\ Q$ we can replace each occurrence of an arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ of Q in ϕ by the $a_j \times a_i$-matrix V_a. This way we obtain a rectangular matrix

$$V(\phi) \in M_{\sum_{i=1}^k a_i v_i \times \sum_{i=1}^k a_i u_i}(\mathbb{C})$$

If we are in a situation where $\sum a_i v_i = \sum a_i u_i$, then we can define a *semi-invariant polynomial function* on $\mathtt{rep}_\alpha\ Q$ by

$$P_{\alpha,\phi}(V) = det\ V(\phi)$$

We call such semi-invariants *determinantal semi-invariants* . One verifies that $P_{\phi,\alpha}$ is a semi-invariant of weight χ_θ where $\theta = (u_1 - v_1, \ldots, u_k - v_k)$. We will show that such determinantal semi-invariant together with traces along oriented cycles in the quiver Q generate all semi-invariants.

Because semi-invariants for the $GL(\alpha)$-action on $\mathtt{rep}_\alpha\ Q$ are the same as invariants for the restricted action of $SL(\alpha) = SL_{a_1} \times \ldots \times SL_{a_k}$, we will describe the multilinear $SL(\alpha)$-invariants from classical invariant theory. Because

$$\begin{aligned} \mathtt{rep}_\alpha\ Q &= \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} M_{a_j \times a_i}(\mathbb{C}) \\ &= \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} \mathbb{C}^{a_i} \otimes \mathbb{C}^{*a_j} \end{aligned}$$

we have to consider multilinear $SL(\alpha)$-invariants of

$$\bigotimes_{\textcircled{j} \leftarrow \textcircled{i}} \mathbb{C}^{a_i} \otimes \mathbb{C}^{*a_j} = \bigotimes_{\textcircled{i}} \Big[\bigotimes_{\bigcirc \leftarrow \textcircled{i}} \mathbb{C}^{a_i} \otimes \bigotimes_{\textcircled{i} \leftarrow \bigcirc} \mathbb{C}^{*a_i} \Big]$$

Hence, any multilinear $SL(\alpha)$-invariant can be written as $f = \prod_{i=1}^k f_i$ where f_i is a SL_{a_i}-invariant of

$$\bigotimes_{\bigcirc \leftarrow \textcircled{i}} \mathbb{C}^{a_i} \otimes \bigotimes_{\textcircled{i} \leftarrow \bigcirc} \mathbb{C}^{*a_i}$$

To increase our cultural luggage let us recall the classical description of multilinear SL_n-invariants on $M_n^{\oplus i} \oplus V_n^{\oplus j} \oplus V_n^{*\oplus z}$, that is, the SL_n-invariant linear maps

$$\underbrace{M_n \otimes \ldots \otimes M_n}_{i} \otimes \underbrace{V_n \otimes \ldots \otimes V_n}_{j} \otimes \underbrace{V_n^* \otimes \ldots \otimes V_n^*}_{z} \xrightarrow{f} \mathbb{C}$$

By the identification $M_n = V_n \otimes V_n^*$ we have to determine the SL_n-invariant linear maps

$$V_n^{\otimes i+j} \otimes V_n^{*\otimes i+z} \xrightarrow{f} \mathbb{C}$$

The description of such invariants is given by classical invariant theory, see [84, II.5,Thm. 2.5.A].

Theorem 7.1 *The multilinear SL_n-invariants f are linear combinations of invariants of one of the following two types*

1. *For $(i_1, \ldots, i_n, h_1, \ldots, h_n, \ldots, t_1, \ldots, t_n, s_1, \ldots, s_r)$ a permutation of the $i+j$ vector indices and $(u_1, \ldots, u_r)$ a permutation of the $i+z$ covector indices, consider the SL_n-invariant*

$$[v_{i_1}, \ldots, v_{i_n}]\ [v_{h_1}, \ldots, v_{h_n}]\ \ldots\ [v_{t_1}, \ldots, v_{t_n}]\ \phi_{u_1}(v_{s_1}) \ldots \phi_{u_r}(v_{s_r})$$

   where the brackets *are the determinantal invariants*

$$[v_{a_1}, \ldots, v_{a_n}] = det\ \begin{bmatrix} v_{a_1} & v_{a_2} & \ldots & v_{a_n} \end{bmatrix}$$

2. *For $(i_1, \ldots, i_n, h_1, \ldots, h_n, \ldots, t_1, \ldots, t_n, s_1, \ldots, s_r)$ a permutation of the $i+z$ covector indices and $(u_1, \ldots, u_r)$ a permutation of the $i+j$ vector indices, consider the SL_n-invariant*

$$[\phi_{i_1}, \ldots, \phi_{i_n}]^*\ [\phi_{h_1}, \ldots, \phi_{h_n}]^*\ \ldots\ [\phi_{t_1}, \ldots, \phi_{t_n}]^*\ \phi_{u_1}(v_{s_1}) \ldots \phi_{u_r}(v_{s_r})$$

   where the cobrackets *are the determinantal invariants*

$$[\phi_{a_1}, \ldots, \phi_{a_n}]^* = det\ \begin{bmatrix} \phi_{a_1} \\ \vdots \\ \phi_{a_n} \end{bmatrix}$$

Observe that we do not have at the same time brackets and cobrackets, due to the relation

$$[v_1, \ldots, v_n]\ [\phi_1, \ldots, \phi_n] = \det \begin{bmatrix} \phi_1(v_1) & \ldots & \phi_1(v_n) \\ \vdots & & \vdots \\ \phi_n(v_1) & \ldots & \phi_n(v_n) \end{bmatrix}$$

We can give a matrix-interpretation of these basic invariants. Let us consider the case of a bracket of vectors (the case of cobrackets is similar)

$$[v_{i_1}, \ldots, v_{i_n}]$$

If all the indices $\{i_1, \ldots, i_n\}$ are original vector-indices (and so do not come from the matrix-terms) we save this term and go to the next factor. Otherwise, if say i_1 is one of the matrix indices,

$A_{i_1} = \phi_{i_1} \otimes v_{i_1}$, then the covector ϕ_{i_1} must be paired up in a scalar product $\phi_{i_1}(v_{u_1})$ with a vector v_{u_1}. Again, two cases can occur. If u_1 is a vector index, we have that

$$\phi_{i_1}(v_{u_1})[v_{i_1}, \ldots, v_{i_n}] = [A_{i_1} v_{u_1}, v_{i_2}, \ldots, v_{i_n}] = [v'_{i_1}, v_{i_2}, \ldots, v_{i_n}]$$

Otherwise, we can keep on matching the matrix indices and get an expression

$$\phi_{i_1}(v_{u_1})\ \phi_{u_1}(v_{u_2})\ \phi_{u_2}(v_{u_3})\ \ldots$$

until we finally hit again a vector index, say u_l, but then we have the expression

$$\phi_{i_1}(v_{u_1})\ \phi_{u_1}(v_{z_1})\ \ldots\ \phi_{u_{l-1}}(v_{u_l})\ [v_{i_1}, \ldots, v_{i_n}] = [M v_{u_l}, v_{i_2}, \ldots, v_{i_n}]$$

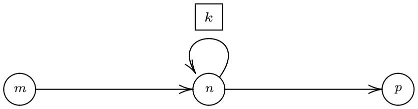
where $M = A_{i_1} A_{u_1} \ldots A_{u_{l-1}}$. One repeats the same argument for all vectors in the brackets. As for the remaining scalar product terms, we have a similar procedure of matching up the matrix indices and one verifies that in doing so one obtains factors of the type

$$\phi(Mv) \quad \text{and} \quad tr(M)$$

where M is a monomial in the matrices. As we mentioned, the case of covector-brackets is similar except that in matching the matrix indices with a covector ϕ, one obtains a monomial in the transposed matrices.

Having found these interpretations of the basic SL_n-invariant linear terms, we can proceed by polarization and restitution processes to prove

Theorem 7.2 *The SL_n-invariants of $W = \mathtt{rep}_\alpha\ Q'$ where Q' is the quiver*



are generated by the following four types of functions, where we write a typical element in W as

$$(\underbrace{A_1, \ldots, A_k}_{k}, \underbrace{v_1, \ldots, v_m}_{m}, \underbrace{\phi_1, \ldots, \phi_p}_{p})$$

with the A_i the matrices corresponding to the loops, the v_j making up the rows of the $n \times m$ matrix and the ϕ_j the columns of the $p \times n$ matrix.

- *$tr(M)$ where M is a monomial in the matrices A_i,*
- scalar products *$\phi_j(Mv_i)$ where M is a monomial in the matrices A_i,*
- brackets *$[M_1 v_{i_1}, M_2 v_{i_2}, \ldots, M_n v_{i_n}]$ where the M_j are monomials in the matrices A_i,*

- cobrackets $[M_1\phi^\tau_{i_1},\ldots,M_n\phi^\tau_{i_n}]$ *where the* M_j *are monomials in the matrices* A_i,

Returning to the special case under consideration, that is, of SL_m-invariants on $\otimes_B \mathbb{C}^m \otimes \otimes_C \mathbb{C}^{*m}$, it follows from this that the linear SL_m-invariants are determined by the following three sets

- *traces*, that is, for each pair (b,c) we have $\mathbb{C}^m \otimes \mathbb{C}^{*m} = M_m(\mathbb{C}) \xrightarrow{Tr} \mathbb{C}$.
- *brackets*, that is, for each m-tuple $(b_1,\ldots,b_m)$ we have an invariant $\otimes_{b_j}\mathbb{C}^m \longrightarrow \mathbb{C}$ defined by
$$v_{b_1} \otimes \ldots \otimes v_{b_m} \mapsto det \begin{bmatrix} v_{b_1} & \ldots & v_{b_m} \end{bmatrix}$$
- *cobrackets*, that is, for each m-tuple $(c_1,\ldots,c_m)$ we have an invariant $\otimes_{c_i}\mathbb{C}^{*m} \longrightarrow \mathbb{C}$ defined by
$$\phi_{c_1} \otimes \ldots \otimes \phi_{c_m} \mapsto det \begin{bmatrix} \phi_{c_1} \\ \vdots \\ \phi_{c_m} \end{bmatrix}$$

Multilinear SL_m-invariants of $\otimes_B \mathbb{C}^m \otimes \otimes_C \mathbb{C}^{*m}$ are then spanned by invariants constructed from the following data. Take three disjoint index-sets I, J and K and consider surjective maps

$$\begin{cases} B & \overset{\mu}{\twoheadrightarrow} I \sqcup K \\ C & \overset{\nu}{\twoheadrightarrow} J \sqcup K \end{cases}$$

subject to the following conditions

$$\begin{cases} \#\, \mu^{-1}(k) = 1 = \#\, \nu^{-1}(k) & \text{for all } k \in K. \\ \#\, \mu^{-1}(i) = m = \#\, \nu^{-1}(j) & \text{for all } i \in I \text{ and } j \in J. \end{cases}$$

To this data $\gamma = (\mu,\nu,I,J,K)$ we can associate a multilinear SL_m-invariant $f_\gamma(\otimes_B v_b \otimes \otimes_C \phi_c)$ defined by

$$\prod_{k\in K} \phi_{\nu^{-1}(k)}(v_{\mu^{-1}(k)}) \prod_{i\in I} det \begin{bmatrix} v_{b_1} & \ldots & v_{b_m} \end{bmatrix} \prod_{j \in J} det \begin{bmatrix} \phi_{c_1} \\ \vdots \\ \phi_{c_m} \end{bmatrix}$$

where $\mu^{-1}(i) = \{b_1,\ldots,b_m\}$ and $\nu^{-1}(j) = \{c_1,\ldots,c_m\}$. Observe that f_γ is determined only up to a sign by the data γ.

But then, we also have a spanning set for the multilinear $SL(\alpha)$-invariants on

$$\mathtt{rep}_\alpha\, Q = \bigotimes_{(v)} \Big[\bigotimes_{\circ \leftarrow (v)} \mathbb{C}^{a_v} \otimes \bigotimes_{(v) \leftarrow \circ} \mathbb{C}^{*a_v} \Big]$$

determined by quintuples $\Gamma = (\mu, \nu, I, J, K)$ where we have disjoint index-sets partitioned over the vertices $v \in \{v_1, \ldots, v_k\}$ of Q

$$\begin{cases} I & = \bigsqcup_v \; I_v \\ J & = \bigsqcup_v \; J_v \\ K & = \bigsqcup_v \; K_v \end{cases}$$

together with surjective maps from the set of all arrows A of Q

$$\begin{cases} A & \overset{\mu}{\twoheadrightarrow} \; I \sqcup K \\ A & \overset{\nu}{\twoheadrightarrow} \; J \sqcup K \end{cases}$$

where we have for every arrow $w \xleftarrow{a} v$ that

$$\begin{cases} \mu(a) & \in I_v \sqcup K_v \\ \nu(a) & \in J_w \sqcup K_w \end{cases}$$

and these maps μ and ν are subject to the numerical restrictions

$$\begin{cases} \# \; \mu^{-1}(k) = 1 = \# \; \nu^{-1}(k) & \text{for all } k \in K. \\ \# \; \mu^{-1}(i) = a_v = \# \; \nu^{-1}(j) & \text{for all } i \in I_v \text{ and all } j \in J_v. \end{cases}$$

Such a quintuple $\Gamma = (\mu, \nu, I, J, K)$ determines for every vertex v a quintuple

$$\gamma_v = (\mu_v = \mu \;\mid\; \{ \bigcirc \xleftarrow{\quad a \quad} v \;\},\; \nu_v = \nu \;\mid\; \{ v \xleftarrow{\quad a \quad} \bigcirc \}, I_v, J_v, K_v)$$

satisfying the necessary numerical restrictions to define the SL_{a_v}-invariant f_{γ_v} described before. Then, the multilinear $SL(\alpha)$-invariant on $\mathtt{rep}_\alpha\; Q$ determined by Γ is defined to be

$$f_\gamma = \prod_v \; f_{\gamma_v}$$

and we have to show that these semi-invariants lie in the linear span of the determinantal semi-invariants.

First, consider the case where the index set K is empty. If we denote the total number of arrows in Q by n, then the numerical restrictions imposed give us two expressions for n

$$\sum_v \; a_v.\# \; I_v = n = \sum_v \; a_v.\# \; J_v$$

Every arrow $\textcircled{w} \xleftarrow{a} \textcircled{v}$ determines a pair of indices $\mu(a) \in I_v$ and $\nu(a) \in J_w$. To the quintuple Γ we assign a map Φ_Γ in $\mathtt{add}\ Q$

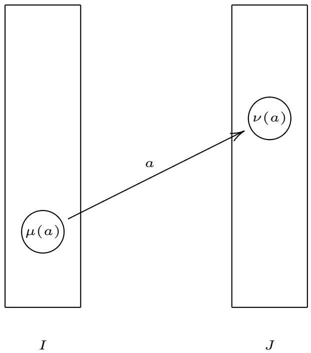
$$\circledcirc_1^{\oplus I_1} \oplus \ldots \oplus \circledcirc_k^{\oplus I_k} \xrightarrow{\Phi_\Gamma} \circledcirc_1^{\oplus J_1} \oplus \ldots \oplus \circledcirc_k^{\oplus J_k}$$

which decomposes as a block-matrix in blocks $M_{v,w} \in Hom(\circledcirc_v^{\oplus I_v}, \circledcirc_w^{\oplus J_w})$ of which the (i,j) entry is given by the sum of arrows

$$\sum_{\substack{\mu(a)=i \\ \nu(a)=j}} \textcircled{w} \xleftarrow{a} \textcircled{v}$$

For a representation $V \in \mathtt{rep}_\alpha\ Q$, $V(\Phi_\Gamma)$ is an $n \times n$ matrix and the determinant defines the determinantal semi-invariant $P_{\Phi_{\alpha,\Gamma}}$ which we claim to be equal to the basic invariant f_Γ possibly up to a sign.

We introduce a new quiver situation. Let Q' be the quiver with vertices the elements of $I \sqcup J$ and with arrows the set A of arrows of Q, but this time w take the starting point of the arrow $\circ \xleftarrow{a} \circ$ in Q to be $\mu(a) \in I$ and the terminating vertex to be $\nu(a) \in J$. That is, Q' is a bipartite quiver



On Q' we have the quintuple $\Gamma' = (\mu', \nu', I', J', K')$ where $K' = \emptyset$,

$$I' = \bigsqcup_{i \in I} I'_i = \bigsqcup_{i \in I} \{i\} \qquad J' = \bigsqcup_{j \in J} J'_j = \bigsqcup_{j \in J} \{j\}$$

and $\mu' = \mu$, $\nu' = \nu$. We define an additive functor $\mathtt{add}\ Q' \xrightarrow{s} \mathtt{add}\ Q$ by

$$\circledcirc_i \xrightarrow{s} \circledcirc_v \qquad \circledcirc_j \xrightarrow{s} \circledcirc_w \qquad \circ \xleftarrow{a} \circ \xrightarrow{s} \circ \xleftarrow{a} \circ$$

for all $i \in I_v$ and all $j \in J_w$. The functor s induces a functor $rep\ Q \xrightarrow{s} rep\ Q'$ defined by $V \xrightarrow{s} V \circ s$. If $V \in \mathtt{rep}_\alpha\ Q$ then $s(V) \in \mathtt{rep}_{\alpha'}\ Q'$ where

$$\alpha' = (\underbrace{c_1, \ldots, c_p}_{\#\ I}, \underbrace{d_1, \ldots, d_q}_{\#\ J}) \quad \text{with} \quad \begin{cases} c_i = a_v & \text{if } i \in I_v \\ d_j = a_w & \text{if } j \in J_w \end{cases}$$

That is, the characteristic feature of Q' is that every vertex $i \in I$ is the source of exactly c_i arrows (follows from the numerical condition on μ) and that every vertex $j \in J$ is the sink of exactly d_j arrows in Q'. That is, locally Q' has the following form

$$\textcircled{c} \longrightarrow \boxed{c} \longrightarrow \quad \text{or} \quad \longrightarrow \boxed{d} \longrightarrow \textcircled{d}$$

There are induced maps

$$\mathtt{rep}_\alpha\ Q \xrightarrow{s} \mathtt{rep}_{\alpha'}\ Q' \qquad GL(\alpha) \xrightarrow{s} GL(\alpha')$$

where the latter follows from functoriality by considering $GL(\alpha)$ as the automorphism group of the trivial representation in $\mathtt{rep}_\alpha\ Q$. These maps are compatible with the actions as one checks that $s(g.V) = s(g).s(V)$. Also s induces a map on the coordinate rings $\mathbb{C}[\mathtt{rep}_\alpha\ Q] \xrightarrow{s} \mathbb{C}[\mathtt{rep}_{\alpha'}\ Q']$ by $s(f) = f \circ s$. In particular, for the determinantal semi-invariants we have

$$s(P_{\alpha',\phi'}) = P_{\alpha,s(\phi')}$$

and from the compatibility of the action it follows that when f is a semi-invariant the $GL(\alpha')$ action on $\mathtt{rep}_{\alpha'}\ Q'$ with character χ', then $s(f)$ is a semi-invariant for the $GL(\alpha)$-action on $\mathtt{rep}_\alpha\ Q$ with character $s(\chi) = \chi' \circ s$. In particular we have that

$$s(P_{\alpha',\Phi_{\Gamma'}}) = P_{\alpha,s(\Phi_{\Gamma'})} = P_{\alpha,\Phi_\Gamma} \quad \text{and} \quad s(f_{\Gamma'}) = f_\Gamma$$

Hence in order to prove our claim, we may replace the triple (Q, α, Γ) by the triple (Q', α', Γ'). We will do this and forget the dashes from here on.

In order to verify that $f_\Gamma = \pm P_{\alpha,\Phi_\Gamma}$ it suffices to check this equality on the image of

$$W = \bigoplus_{\textcircled{i} \xrightarrow{a} \textcircled{j}} \mathbb{C}^{c_i} \oplus \mathbb{C}^{*d_j} \quad \text{in} \quad \bigotimes_{\textcircled{i} \xrightarrow{a} \textcircled{j}} \mathbb{C}^{c_i} \otimes \mathbb{C}^{*d_j}$$

One verifies that both f_Γ and P_{α,Φ_Γ} are $GL(\alpha)$-semi-invariants on W of weight χ_θ where

$$\theta = (\underbrace{1, \ldots, 1}_{\#\ I}, \underbrace{-1, \ldots, -1}_{\#\ J})$$

Using the characteristic local form of $Q = Q'$, we see that W is isomorphic to the $GL(\alpha)$- module

$$W \simeq \bigoplus_{i \in I} (\underbrace{\mathbb{C}^{c_i} \oplus \ldots \oplus \mathbb{C}^{c_i}}_{c_i}) \oplus \bigoplus_{j \in J} (\underbrace{\mathbb{C}^{*d_j} \oplus \ldots \oplus \mathbb{C}^{*d_j}}_{d_j}) \simeq \bigoplus_{i \in I} M_{c_i}(\mathbb{C}) \oplus \bigoplus_{j \in J} M_{d_j}(\mathbb{C})$$

and the i factors of $GL(\alpha)$ act by inverse right-multiplication on the component M_{c_i} (and trivially on all others) and the j factors act by left-multiplication on the component M_{d_j} (and trivially on the others). That is, $GL(\alpha)$ acts on W with an open orbit, say that of the element

$$w = (\mathbb{1}_{c_1}, \ldots, \mathbb{1}_{c_p}, \mathbb{1}_{d_1}, \ldots, \mathbb{1}_{d_q}) \in W$$

One verifies immediately from the definitions that that both f_Γ and P_{α,Φ_Γ} evaluate to ± 1 in w. Hence, indeed, f_Γ can be expressed as a determinantal semi-invariant.

Remains to consider the case when K is non-empty. For $k \in K$ two situations can occur

- $\mu^{-1}(k) = a$ and $\nu^{-1}(k) = b$ are distinct, then k corresponds to replacing the arrows a and b by their concatenation

$$\circ \xleftarrow{a} \textcircled{k} \xleftarrow{b} \circ$$

- $\mu^{-1}(k) = a = \nu^{-1}(k)$ then a is a loop in Q and k corresponds

$$\begin{tikzcd} \textcircled{k} \arrow[loop above, "a"] \end{tikzcd}$$

  to taking the trace of a.

This time we construct a new quiver Q" with vertices $\{w_1, \ldots, w_n\}$ corresponding to the set A of arrows in Q. The arrows in Q" will correspond to elements of K, that is if $k \in K$ we have the arrow (or loop) in Q" with notations as before

$$\textcircled{b} \xleftarrow{k} \textcircled{a} \quad \text{or} \quad \begin{tikzcd} \textcircled{a} \arrow[loop above, "k"] \end{tikzcd}$$

We consider the connected components of Q". They are of the following three types

- (oriented cycle) : To an oriented cycle C in Q" corresponds an oriented cycle C'_C in the original quiver Q. We associate to it the trace $tr(C'_C)$ of this cycle.
- (open paths) : An open path P in Q" corresponds to an oriented path P'_P in Q which may be a cycle. To P we associate the corresponding path P'_P in Q.

- (isolated points) : They correspond to arrows in Q.

We will now construct a new quiver Q' having the same vertex set $\{v_1, \ldots, v_k\}$ as Q but with arrows corresponding to the set of paths P'_P described above. The starting and ending vertex of the arrow corresponding to P'_P are of course the starting and ending vertex of the path P_P in Q. Again, we define an additive functor $\mathtt{add}\ Q' \xrightarrow{s} \mathtt{add}\ Q$ by the rules

$$\begin{tikzcd} v \arrow[r, "s"] & v \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} j & i \arrow[l, "P'_P"'] \arrow[r, "s"] & j & i \arrow[l, dotted, bend right, "P'_P"'] \end{tikzcd}$$

If the path P'_P is the concatenation of the arrows $a_d \circ \ldots \circ a_1$ in Q, we define the maps

$$\begin{cases} \mu'(P'_P) & = \mu(a_1) \\ \nu'(P'_P) & = \nu(a_d) \end{cases} \quad \text{whence} \quad \begin{cases} \{P'_P\} & \overset{\mu}{\twoheadrightarrow} I' \\ \{P'_P\} & \overset{\nu}{\twoheadrightarrow} J' \end{cases}$$

that is, a quintuple $\Gamma' = (\mu', \nu', I', J', K' = \emptyset)$ for the quiver Q'. One then verifies that

$$\begin{aligned} f_\Gamma = s(f_{\Gamma'}) \prod_C tr(C'_C) &= s(P_{\alpha, \Phi_{\Gamma'}}) \prod_C tr(C'_C) \\ &= P_{\alpha, s(\Phi_{\Gamma'})} \prod_C tr(C'_C) \end{aligned}$$

finishing the proof of the fact that multilinear semi-invariants lie in the linear span of determinantal semi-invariants (and traces of oriented cycles).

The arguments above can be reformulated in a more combinatorial form which is often useful in constructing semi-invariants of a specific weight, as is necessary in the study of the moduli spaces $M^{ss}_\alpha(Q, \theta)$. Let Q be a quiver on the vertices $\{v_1, \ldots, v_k\}$, fix a dimension vector $\alpha = (a_1, \ldots, a_k)$ and a character χ_θ where $\theta = (t_1, \ldots, t_k)$ such that $\theta(\alpha) = 0$. We will call a bipartite quiver Q' as in figure 7.1 on left vertex-set $L = \{l_1, \ldots, l_p\}$ and right vertex-set $R = \{r_1, \ldots, r_q\}$ and a dimension vector $\beta = (c_1, \ldots, c_p; d_1, \ldots, d_q)$ to be of type (Q, α, θ) if the following conditions are met

- All left and right vertices correspond to vertices of Q, that is, there are maps

  $$\begin{cases} L & \xrightarrow{l} \{v_1, \ldots, v_k\} \\ R & \xrightarrow{r} \{v_1, \ldots, v_k\} \end{cases}$$

  possibly occurring with multiplicities, that is there is a map

  $$L \cup R \xrightarrow{m} \mathbb{N}_+$$

  such that $c_i = m(l_i) a_z$ if $l(l_i) = v_z$ and $d_j = m(r_j) a_z$ if $r(r_j) = v_z$.

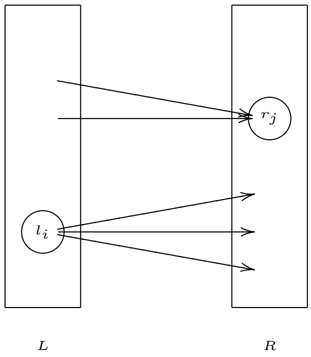


Figure 7.1: Left-right bipartite quiver.

- There can only be an arrow $l_i \longrightarrow r_j$ if for $v_k = l(l_i)$ and $v_l = r(r_i)$ there is an oriented path

$$v_k \dashrightarrow v_l$$

  in Q allowing the trivial path and loops if $v_k = v_l$.

- Every left vertex l_i is the source of exactly c_i arrows in Q' and every right-vertex r_j is the sink of precisely d_j arrows in Q'.

- Consider the $u \times u$ matrix where $u = \sum_i c_i = \sum_j d_j$ (both numbers are equal to the total number of arrows in Q') where the i-th row contains the entries of the i-th arrow in Q' with respect to the obvious left and right bases. Observe that this is a $GL(\beta)$ semi-invariant on $\mathtt{rep}_\beta\ Q'$ with weight determined by the integral $k+l$-tuple $(-1,\dots,-1;1,\dots,1)$. If we fix for every arrow a from l_i to r_j in Q' an $m(r_j) \times m(l_i)$ matrix p_a of linear combinations of paths in Q from $l(l_i)$ to $r(r_j)$, we obtain a morphism

$$\mathtt{rep}_\alpha\ Q \longrightarrow \mathtt{rep}_\beta\ Q'$$

  sending a representation $V \in \mathtt{rep}_\alpha\ Q$ to the representation W of Q' defined by $W_a = p_a(V)$. Composing this map with the above semi-invariant we obtain a $GL(\alpha)$ semi-invariant of $\mathtt{rep}_\alpha\ Q$ with weight determined by the k-tuple $\theta = (t_1, \dots, t_k)$ where

$$t_i = \sum_{j \in r^{-1}(v_i)} m(r_j) - \sum_{j \in l^{-1}(v_i)} m(l_j)$$

  .

We call such semi-invariants *standard determinantal* . Summarizing the arguments of this section we have proved after applying polarization and restitution processes

Theorem 7.3 *The semi-invariants of the $GL(\alpha)$-action on* $\texttt{rep}_\alpha\ Q$ *are generated by traces of oriented cycles and by standard determinantal semi-invariants.*

7.3 Universal localization

In order to prove that the moduli spaces $M^{ss}_\alpha(Q,\theta)$ are locally controlled by Quillen-smooth algebras, we need to recall the notion of *universal localization* . We refer to the monograph by A. Schofield [72] for full details.

Let A be a $\mathbb{C}$-algebra and $\texttt{projmod}\ A$ the category of finitely generated projective left A-modules. Let Σ be some class of maps in this category (that is some left A-module morphisms between certain projective modules). Then, there exists an algebra map $A \xrightarrow{j_\Sigma} A_\Sigma$ with *the universal property* that the maps $A_\Sigma \otimes_A \sigma$ have an inverse for all $\sigma \in \Sigma$. A_Σ is called the *universal localization* of A with respect to the set of maps Σ.

Proposition 7.1 *When A is Quillen-smooth, then so is A_Σ.*

Proof. Consider a test-object (T, I) in $\texttt{alg}$, then we have the following diagram

$$\begin{tikzcd}
T \arrow[r, two heads] & \frac{T}{I} \\
A \arrow[u, dotted, "\psi"] \arrow[r, "j_\Sigma"'] & A_\Sigma \arrow[u, "\phi"'] \arrow[ul, dotted, "\tilde{\phi}"]
\end{tikzcd}$$

where ψ exists by Quillen-smoothness of A. By *Nakayama's lemma* all maps $\sigma \in \Sigma$ become isomorphisms under tensoring with ψ. Then, $\tilde{\phi}$ exists by the universal property of A_Σ. $\square$

Consider the special case when A is the path algebra $\mathbb{C}Q$ of a quiver on k vertices. Then, we can identify the isomorphism classes in $\texttt{projmod}\ \mathbb{C}Q$ with the opposite category of $\texttt{add}\ Q$ introduced in the foregoing section. To each vertex v_i corresponds an *indecomposable projective* left $\mathbb{C}Q$-ideal $P_i = \mathbb{C}Qe_i$ having as $\mathbb{C}$-vectorspace basis all paths in Q *starting at* v_i. For the homomorphisms we have

$$Hom_{\mathbb{C}Q}(P_i, P_j) = \bigoplus_{(i) \overset{p}{\leftsquigarrow} (j)} \mathbb{C}p = Hom_{\texttt{add}\ Q}((j) , (i))$$

where p is an oriented path in Q starting at v_j and ending at v_i. Therefore, any A-module morphism σ between two projective left modules

$$P_{i_1} \oplus \ldots \oplus P_{i_u} \xrightarrow{\sigma} P_{j_1} \oplus \ldots \oplus P_{j_v}$$

can be represented by an $u \times v$ matrix M_σ whose (p,q)-entry m_{pq} is a linear combination of oriented paths in Q starting at v_{j_q} and ending at v_{i_p}.

Now, form an $v \times u$ matrix N_σ of free variables y_{pq} and consider the algebra $\mathbb{C}Q_\sigma$ which is the quotient of the free product $\mathbb{C}Q * \mathbb{C}\langle y_{11}, \ldots, y_{uv} \rangle$ modulo the ideal of relations determined by the matrix equations

$$M_\sigma.N_\sigma = \begin{bmatrix} v_{i_1} & & 0 \\ & \ddots & \\ 0 & & v_{i_u} \end{bmatrix} \qquad N_\sigma.M_\sigma = \begin{bmatrix} v_{j_1} & & 0 \\ & \ddots & \\ 0 & & v_{j_v} \end{bmatrix}$$

Equivalently, $\mathbb{C}Q_\sigma$ is the path algebra of a quiver *with relations* where the quiver is Q extended with arrows y_{pq} from v_{i_p} to v_{j_q} for all $1 \leq p \leq u$ and $1 \leq q \leq v$ and the relations are the above matrix entry relations.

Repeating this procedure for every $\sigma \in \Sigma$ we obtain the universal localization $\mathbb{C}Q_\Sigma$. This proves

Proposition 7.2 *If Σ is a finite set of maps, then the universal localization $\mathbb{C}Q_\Sigma$ is an affine $\mathbb{C}$-algebra.*

It is easy to verify that the representation space $\mathtt{rep}_n\ \mathbb{C}Q_\sigma$ is an affine Zariski open subscheme (but possibly empty) of $\mathtt{rep}_n\ \mathbb{C}Q$. Indeed, if $V = (V_a)_a \in \mathtt{rep}_\alpha\ Q$, then V determines a point in $\mathtt{rep}_n\ \mathbb{C}Q_\Sigma$ if and only if the matrices $M_\sigma(V)$ in which the arrows are all replaced by the matrices V_a are invertible for all $\sigma \in \Sigma$.

In particular, this induces numerical conditions on the dimension vectors α such that $\mathtt{rep}_\alpha\ Q_\Sigma \neq \emptyset$. Let $\alpha = (a_1, \ldots, a_k)$ be a dimension vector such that $\sum a_i = n$ then every $\sigma \in \Sigma$ say with

$$P_1^{\oplus e_1} \oplus \ldots \oplus P_k^{\oplus e_k} \xrightarrow{\sigma} P_1^{\oplus f_1} \oplus \ldots \oplus P_k^{\oplus f_k}$$

gives the numerical condition

$$e_1 a_1 + \ldots + e_k a_k = f_1 a_1 + \ldots + f_k a_k.$$

These numerical restrictions will be used to relate θ-stable representations of Q to simple representations of universal localizations of $\mathbb{C}Q$.

Fix a character $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ and divide the set of indices $1 \leq i \leq k$ into the *left set* $L = \{i_1, \ldots, i_u\}$ consisting of those i such that $t_i \leq 0$ and the *right set* $R = \{j_1, \ldots, j_v\}$ consisting of those j such that $t_j \geq 0$. Consider a dimension vector α such that $\theta.\alpha = 0$, then θ determines the *character*

$$GL(\alpha) \xrightarrow{\chi_\theta} \mathbb{C}^* \qquad (g_1, \ldots, g_k) \mapsto \prod_i det(g_i)^{t_i}$$

Next, consider the sets of morphisms

$$\Sigma_\theta = \bigcap_{z \in \mathbb{N}_+} \Sigma_\theta(z)$$

where $\Sigma_\theta(z)$ is the set of all morphisms

$$P_{i_1}^{\oplus -zt_{i_1}} \oplus \ldots \oplus P_{i_u}^{\oplus -zt_{i_u}} \xrightarrow{\sigma} P_{j_1}^{\oplus zt_{j_1}} \oplus \ldots \oplus P_{j_v}^{\oplus zt_{j_v}}$$

With notation as before, it follows that

$$d_\sigma(V) = det\ M_\sigma(V) \qquad V \in \mathtt{rep}_\alpha\ Q$$

is a semi-invariant on $\mathtt{rep}_\alpha\ Q$ of weight $z\chi_\theta$. This semi-invariant determines the Zariski open subset of $\mathtt{rep}_\alpha\ Q$

$$X_\sigma(\alpha) = \{V \in \mathtt{rep}_\alpha\ Q \mid d_\sigma(V) \neq 0\}$$

It is clear from the results of section 4.8 that $X_\sigma(\alpha)$ consists of θ-semistable representations. We can characterize the θ-stable representations in this open set.

Lemma 7.1 *For $V \in X_\sigma(\alpha)$ the following are equivalent*

1. *V is a θ-stable representation.*
2. *V is a simple α-dimensional representation of the universal localization $\mathbb{C}Q_\sigma$.*

Proof. Let W be a β-dimensional subrepresentation of V with $\beta = (b_1, \ldots, b_k)$, then for W to be a β-dimensional representation of the universal localization $\mathbb{C}Q_\sigma$ it must satisfy the numerical restriction

$$-t_{i_1}b_{i_1} - \ldots - t_{i_u}b_{i_u} = t_{j_1}b_{j_1} + \ldots + t_{j_v}b_{j_v} \quad \text{that is} \quad \theta.\beta = 0$$

Hence, if V is θ-stable, there are no proper subrepresentations of V as a $\mathbb{C}Q_\sigma$-representation. Conversely, if V is an α-dimensional subrepresentation of $\mathbb{C}Q_\sigma$ we must have that $d_\sigma(V) \neq 0$. But then, if W is a β-dimensional Q-subrepresentation of V we must have that $\sum_a -t_{i_a}b_{i_a} \leq \sum_b t_{j_b}b_{i_b}$ (if not, $\sigma(V)$ would have a kernel) whence $\theta.\beta \geq 0$. If W is a subrepresentation such that $\theta.\beta = 0$, then W would be a proper $\mathbb{C}Q_\sigma$ subrepresentation of V, a contradiction. Therefore, V is θ-stable. □

Theorem 7.4 *The moduli space of θ-semistable representations of the quiver Q*

$$M_\alpha^{ss}(Q, \theta)$$

is locally controlled by the set of Quillen-smooth algebras $\{\mathbb{C}Q_\sigma \mid \sigma \in \Sigma_\theta\}$.

Proof. By the results of the foregoing section we know that the quotient varieties of the Zariski open affine subsets $X_\sigma(\alpha)$ cover the moduli space $M^{ss}_\alpha(Q,\theta)$. Further, by lemma 7.1 we have a canonical isomorphism

$$X_\sigma(\alpha)/GL(\alpha) \simeq \mathtt{iss}_\alpha \; \mathbb{C}Q_\sigma$$

Finally, because

$$\mathtt{rep}_n \; \mathbb{C}Q_\sigma = \sqcup_\alpha GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha \; \mathbb{C}Q_\sigma$$

where the disjoint union is taken over all $\alpha = (a_1, \ldots, a_k)$ such that $\sum_i a_i = n$, we have that $\mathtt{iss}_\alpha \; \mathbb{C}Q_\sigma$ is an irreducible component of $\mathtt{iss}_n \; \mathbb{C}Q_\sigma$ finishing the proof. $\square$

Lemma 7.1 also allows us to study the moduli spaces $M^{ss}_\alpha(Q,\theta)$ locally by the local quiver settings associated to semi-simple representations. That is, let $\xi \in M^{ss}_\alpha(Q,\theta)$ be the point corresponding to

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_z^{\oplus e_z}$$

where S_i is a θ-stable representation of dimension vector β_i occurring in M_ξ with multiplicity e_i.

Theorem 7.5 *With notations as above, the étale local structure of the moduli space $M^{ss}_\alpha(Q,\theta)$ near ξ is that of the quotient variety $\mathtt{iss}_\beta \; Q_\xi$ where $\beta = (e_1, \ldots, e_z)$ and Q_ξ is the quiver on z vertices such that*

$$\begin{cases} \# \; \textcircled{j} \xleftarrow{\;a\;} \textcircled{i} & = -\chi_Q(\beta_i, \beta_j) \\ \# \; \overset{\circlearrowright}{\textcircled{i}} & = 1 - \chi_Q(\beta_i, \beta_i) \end{cases}$$

near the trivial representation.

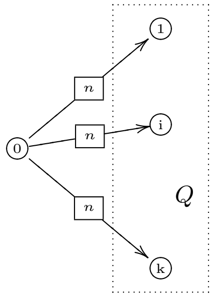
Proof. In view of the above results and the slice theorems, we only have to compute the ext-spaces $Ext^1_{\mathbb{C}Q_\sigma}(S_i, S_j)$. From [72, Thm. 4.7] we recall that the category of $\mathbb{C}Q_\sigma$ representations is closed under extensions in the category of representations of Q. Therefore, we have for all $\mathbb{C}Q_\sigma$-representations V and W that

$$Ext^1_{\mathbb{C}Q}(V,W) \simeq Ext^1_{\mathbb{C}Q_\sigma}(V,W)$$

from which the result follows using theorem 4.5. $\square$

In the following section we will give some applications of this result. Universal localizations can also be used to determine the *formal structure* on representation spaces of quivers.

Let Q be a quiver on k vertices and consider the extended quiver $Q^{(n)}$



That is, we add to the vertices and arrows of Q one extra vertex v_0 and for every vertex v_i in Q we add n directed arrows from v_0 to v_i. We will denote the j-th arrow $1 \leq j \leq n$ from v_0 to v_i by x_{ij}.

Consider the morphism between projective left $\mathbb{C}Q^{(n)}$-modules

$$P_1 \oplus P_2 \oplus \ldots \oplus P_k \xrightarrow{\sigma} \underbrace{P_0 \oplus \ldots \oplus P_0}_{n}$$

determined by the matrix

$$M_\sigma = \begin{bmatrix} x_{11} & \ldots & \ldots & x_{1n} \\ \vdots & & & \vdots \\ x_{k1} & \ldots & \ldots & x_{kn} \end{bmatrix}.$$

We consider the universal localization $\mathbb{C}Q^{(n)}_\sigma$, that is, we add for each vertex v_i in Q another n arrows y_{ij} with $1 \leq j \leq n$ from v_i to v_0.

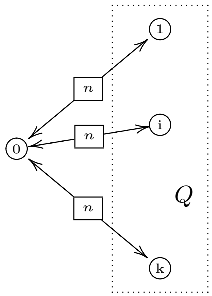
With these arrows y_{ij} one forms the $n \times k$ matrix

$$N_\sigma = \begin{bmatrix} y_{11} & \ldots & y_{k1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ y_{1n} & \ldots & y_{kn} \end{bmatrix}$$

and the universal localization $\mathbb{C}Q^{(n)}_\sigma$ is described by the relations

$$M_\sigma . N_\sigma = \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_k \end{bmatrix} \quad \text{and} \quad N_\sigma . M_\sigma = \begin{bmatrix} v_0 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & v_1 \end{bmatrix}.$$

We will depict this quiver with relations by the picture $Q_\sigma^{(n)}$



From the discussion above it follows that there is a canonical isomorphism

$$\mathtt{rep}_m\ \sqrt[n]{\mathbb{C}Q} \simeq \mathtt{rep}_m\ \mathbb{C}Q_\sigma^{(n)}.$$

In fact we can even identify

$$\sqrt[n]{\mathbb{C}Q} = v_0\ \mathbb{C}Q_\sigma^{(n)}\ v_0.$$

Indeed, the right hand side is generated by all the oriented cycles in $Q_\sigma^{(n)}$ starting and ending at v_0 and is therefore generated by the $y_{ip}x_{iq}$ and the $y_{ip}ax_{jq}$ where a is an arrow in Q starting in v_j and ending in v_i. If we have an algebra morphism

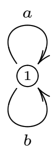
$$\mathbb{C}Q \xrightarrow{\phi} M_n(B)$$

then we have an associated algebra morphism

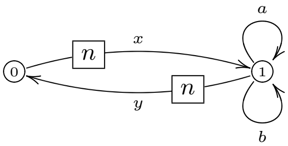
$$v_0\ \mathbb{C}Q_\sigma^{(n)}\ v_0 \xrightarrow{\psi} B$$

defined by sending $y_{ip}ax_{jq}$ to the (p,q)-entry of the $n \times n$ matrix $\phi(a)$ and $y_{ip}x_{iq}$ to the (p,q)-entry of $\phi(v_i)$. The defining relations among the x_{ip} and y_{iq} introduced before imply that ψ is indeed an algebra morphism.

Example 7.3 Let $A = \mathbb{C}\langle a, b\rangle$, that is A is the path algebra of the quiver



In order to describe $\sqrt[n]{A}$ we consider the quiver with relations



$$: \quad y_i x_j = \delta_{ij} v_0, \qquad \sum_i x_i y_i = v_1.$$

We see that the algebra of oriented cycles in v_0 in this quiver with relations is isomorphic to the free algebra in $2n^2$ free variables

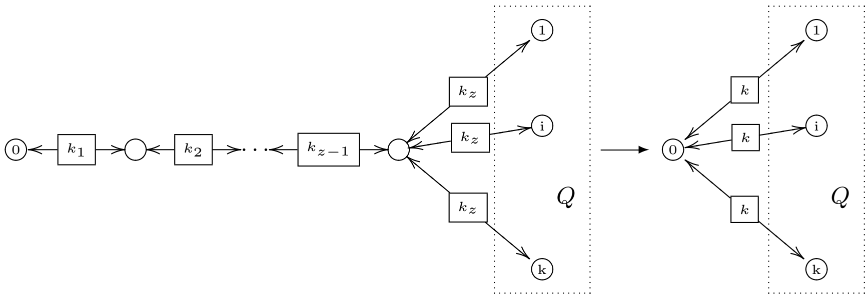
$$\mathbb{C}\langle y_1 a x_1, \ldots, y_n a x_n, y_1 b x_1, \ldots, y_n b x_n \rangle$$

which coincides with our knowledge of $\sqrt[n]{\mathbb{C}\langle a, b\rangle}$.

There is some elementary calculus among the n-th roots of algebras. For example, it follows from the universal property of $\sqrt[n]{A}$ that there is a natural morphism

$$\sqrt[k_1]{\sqrt[k_2]{\ldots \sqrt[k_z]{A}}} \longleftarrow \sqrt[k]{A}$$

where $k = \prod k_i$. When $A = \mathbb{C}Q$ we can represent this morphism graphically by the picture



where the map is given by composing paths from v_0 to v_i. Also observe that we used the isomorphisms in the rightmost part of the left quiver to remove additional arrows from the extra vertices to v_i at each stage.

Probably more important are the *connecting morphisms*

$$\sqrt[k_1]{A} * \sqrt[k_2]{A} * \ldots * \sqrt[k_z]{A} \xleftarrow{c(k_1,\ldots,k_z)} \sqrt[k]{A}$$

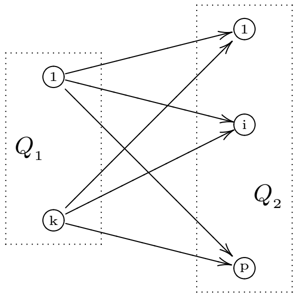


Figure 7.2: Free product of quivers.

with $k = \sum k_i$ obtained from the universal property of $\sqrt[n]{A}$ by composing algebra morphisms $A \xrightarrow{\phi_i} M_{k_i}(B)$ to an algebra morphism

$$A \xrightarrow{\begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_z \end{bmatrix}} M_k(B).$$

Observing that the ordering of the factors is important (but only up to isomorphism of the representations).

We need to have a quiver interpretation of the *free product* $\mathbb{C}Q_1 * \mathbb{C}Q_2$ of two path algebras (at least as far as finite dimensional representations are concerned). Let Q_1 be a quiver on k vertices $\{v_1, \ldots, v_k\}$ and Q_2 a quiver on p vertices $\{w_1, \ldots, w_p\}$ and consider the extended quiver $Q_1 * Q_2$ of figure 7.2. That is, we add one extra arrow from each vertex of Q_1 to each arrow of Q_2.

Let $\{P_1, \ldots, P_k\}$ be the projective left $\mathbb{C}Q_1 * Q_2$-modules corresponding to the vertices of Q_1 and $\{P'_1, \ldots, P'_p\}$ those corresponding to the vertices of Q_2 and consider the morphism

$$P'_1 \oplus \ldots \oplus P'_p \xrightarrow{\sigma} P_1 \oplus \ldots \oplus P_k$$

determined by the $p \times k$ matrix

$$M_\sigma = \begin{bmatrix} x_{11} & \ldots & x_{1k} \\ \vdots & & \vdots \\ x_{p1} & \ldots & x_{pk} \end{bmatrix}$$

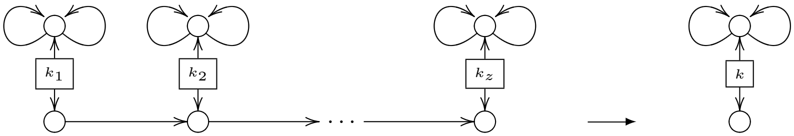
where x_{ij} denotes the extra arrow from vertex v_j to vertex w_i.

Let $Q_1 * Q_{2\sigma}$ denote the quiver with relations one obtains by inverting this map (as above). Then, it is fairly easy to see that

$$\mathtt{rep}_n\ \mathbb{C}Q_1 * Q_2 \simeq \mathtt{rep}_n\ Q_1 * Q_{2\sigma}$$

where the right-hand side denote the subscheme of n-dimensional representations of the quiver Q_1 times the n-dimensional representations of Q_2 where the extra arrows determine an isomorphism of the representations.

Using this interpretation of the free product one can now give a graphical interpretation of the connecting morphisms in the case of the two loop quiver (the general case is similar).



obtained by 'grafting' the bottom tree. Observe that again we used the isomorphisms given by the k_i bundles to eliminate adding extra arrows in the free products.

7.4 Compact manifolds

`noncommutative geometry@n` is the study of *families* of algebraic varieties (with specified connecting morphisms) which are local controlled by a set of noncommutative algebras. If this set of algebras consists of Quillen-smooth algebras we say that the family of varieties is a *noncommutative manifold* . If all varieties in the family are in addition projective (possibly with singularities) we say that the family is a *compact noncommutative manifold* .

So far, we have mot specified the properties of the connecting morphisms. In this section we present a first class of examples, the *sum families*. In the next chapter we will encounter another possibility coming from the theory of completely integrable dynamical systems.

Definition 7.2 *A* sum family *is an object* $(X_n)_n$ *in* `noncommutative geometry@n` *indexed over the positive integers such that for each n there is a* GL_n*-variety* Y_n *and a quotient morphism*

$$Y_n \twoheadrightarrow Y_n/GL_n \simeq X_n$$

and Y_n *is locally of the form* $\mathtt{rep}_n\ A$ *for an affine* $\mathbb{C}$*-algebra* A *belonging to a set* $\mathcal{A}$ *of algebras. Moreover, there are* equivariant connecting sum-maps

$$Y_m \times Y_n \xrightarrow{\oplus} Y_{m+n}$$

for all $m, n \in \mathbb{N}_+$ *where equivariance means with respect to the group* $GL_m \times GL_n$ *embedded diagonally in* GL_{m+n}*. If the set* $\mathcal{A}$ *consists of Quillen-smooth algebras, we call* $(X_n)_n$ *a* sum manifold .

Theorem 7.6 *For a quiver Q on k vertices and a fixed character $\theta \in \mathbb{Z}^k$, the family of varieties*

$$(\bigsqcup_{\substack{\alpha=(a_1,\ldots,a_k) \\ \sum_i a_i = n}} M_\alpha^{ss}(Q,\theta) \ )_n$$

is a sum manifold in `noncommutative geometry@n`. *If Q has no oriented cycles, then this family is a compact sum manifold.*

Proof. In view of theorem 7.5 we only need to construct equivariant-sum maps. They are induced from the direct sums of representations

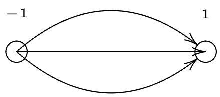
$$\mathtt{rep}_\alpha\ Q \times \mathtt{rep}_\beta\ Q \xrightarrow{\oplus} \mathtt{rep}_{\alpha+\beta}\ Q \qquad (V,W) \mapsto V \oplus W$$

and the required properties are clearly satisfied. □

Example 7.4 Let $M_{\mathbb{P}^2}(n;0,n)$ be the moduli space of semi-stable vectorbundles of rank n over the projective plane $\mathbb{P}^2$ with *Chern numbers* $c_1 = 0$ and $c_2 = n$. Using results of K. Hulek [36] one can identify this moduli space with

$$M_{\mathbb{P}^2}(n;0,n) \simeq M_{(n,n)}^{ss}(Q,\theta)$$

where Q and θ are the following quiver-setting



Therefore, the family of moduli spaces $(M_{\mathbb{P}^2}(n;0,n))_n$ is a compact sum manifold in `geo @n`.

Let C be a smooth projective curve of genus g and let $M_C(n,0)$ be the moduli space of semi-stable vectorbundles of rank n and degree 0 over C. We expect that the family of moduli spaces $(M_C(n,0))_n$ is a compact sum manifold.

In this section we will investigate another class of examples : representations of the *torus knot groups* . Consider a slid cylinder C with m line segments on its curved face, equally spaced and parallel to the axis. If the ends of C are identified with a twist of $2\pi\frac{n}{m}$ where n is an integer relatively prime to m, we obtain a single curve $K_{m,n}$ on the surface of a solid torus T. If we assume that the torus T lies in $\mathbb{R}^3$ in the standard way, the curve $K_{m,n}$ is called the (m,n) *torus knot* .

Computing the *fundamental group* of the complement $\mathbb{R}^3 - K_{m,n}$ one obtains the (m,n)-*torus knot group*

$$\pi_1(\mathbb{R}^3 - K_{m,n}) = G_{m,n} \simeq \langle\ a, b\ \mid\ a^m = b^n\ \rangle$$

An important example is the *three string braid group.*

Example 7.5 Consider Artin's braid group B_3 on three strings. B_3 has the presentation

$$B_3 \simeq \langle L, R \ \mid \ LR^{-1}L = R^{-1}LR^{-1}\rangle$$

where L and R are the fundamental 3-braids

L $\qquad$ R

If we let $S = LR^{-1}L$ and $T = R^{-1}L$, an algebraic manipulation shows that

$$B_3 = \langle S, T \ \mid \ T^3 = S^2\rangle$$

is an equivalent presentation for B_3. The center of B_3 is the infinite cyclic group generated by the braid

$$Z = S^2 = (LR^{-1}L)^2 = (R^{-1}L)^3 = T^3$$

It follows from the second presentation of B_3 that the quotient group modulo the center is isomorphic to

$$\frac{B_3}{\langle Z\rangle} \simeq \langle s, t \ \mid \ s^2 = 1 = t^3\rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_3$$

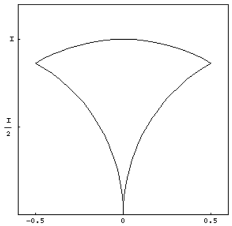
the free product of the cyclic group of order 2 (with generator s) and the cyclic group of order 3 (with generator t). This group is isomorphic to the modular group $PSL_2(\mathbb{Z})$ via

$$\overline{L} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \overline{R} \longrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

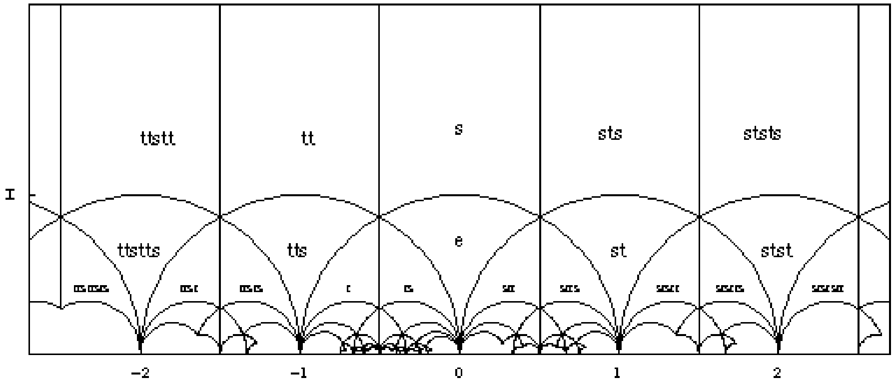
It is well known that the modular group $PSL_2(\mathbb{Z})$ acts on the upper half-plane H^2 by left multiplication in the usual way, that is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ H^2 \longrightarrow H^2 \quad \text{given by} \quad z \longrightarrow \frac{az+b}{cz+d}$$

The fundamental domain $H^2/PSL_2(\mathbb{Z})$ for this action is the hyperbolic triangle



and the action defines a quilt-tiling on the hyperbolic plane, indexed by elements of $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$



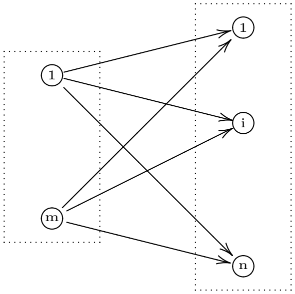
We want to study the irreducible representations of the torus knot group $G_{m,n}$. We recall that the *center* of $G_{m,n}$ is generated by a^m and that the quotient group is the *free product group*

$$\overline{G}_{m,n} = \frac{G_{m,n}}{\langle\ a^m\ \rangle} = \langle\ x, y\ \mid\ x^m = 1 = y^n\ \rangle = \mathbb{Z}_m * \mathbb{Z}_n$$

of the cyclic groups of order m and n. As the center acts by scalar multiplication on an irreducible representation by *Schur's lemma* the representation theory of $G_{m,n}$ essentially reduces to that of the quotient $\overline{G}_{m,n}$. The latter can be studied b noncommutative geometry as the *group algebra* $\mathbb{C}\overline{G}_{m,n}$ is Quillen-smooth. This follows from

$$\mathbb{C}\overline{G}_{m,n} = \mathbb{C}\mathbb{Z}_m * \mathbb{Z}_n \simeq \mathbb{C}\mathbb{Z}_m * \mathbb{C}Z_n \simeq \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{m} * \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{n}$$

and as both factors of the *free algebra product* on the right are Quillen-smooth (in fact, semisimple) so is the product by the universal property. Further, as both factors are the path algebras of quivers on m resp. n vertices without arrows, we know that the representation theory of the free algebra product, and hence of $\mathbb{C}\overline{G}_{m,n}$ can be reduced to θ-semistable representations the quiver $Q_{m,n}$

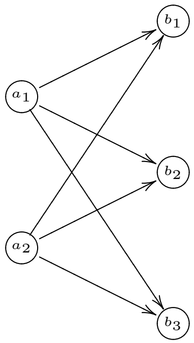


where $\theta = (\underbrace{-1,\ldots,-1}_{m},\underbrace{1,\ldots,1}_{n})$, by the results of the foregoing section. The left vertex spaces S_i, $1 \leq i \leq m$ for a $\overline{G}_{m,n}$-representation are the eigenspaces for the restricted $\mathbb{Z}_m$-action and the left vertex spaces T_j, $1 \leq j \leq n$ are the eigenspaces for the restricted $\mathbb{Z}_n$-action.

Example 7.6 Consider the modular group $PSL_2(\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$, the free product of the cyclic groups of order two and three with generators σ resp. τ. Let S be an n-dimensional simple representation of $PSL_2(\mathbb{Z})$. Let ξ be a 3-rd root of unity, then restricting S to these finite Abelian subgroups we have

$$\begin{cases} S\downarrow_{\mathbb{Z}_2} & \simeq S_1^{\oplus a_1} \oplus S_{-1}^{\oplus a_2} \\ S\downarrow_{\mathbb{Z}_3} & \simeq T_1^{\oplus b_1} \oplus T_{\xi}^{\oplus b_2} \oplus T_{\xi^2}^{\oplus b_3} \end{cases}$$

where S_x resp. T_x are the one-dimensional representations on which σ resp. τ acts via multiplication with x. Observe that $a_1 + a_2 = b_1 + b_2 + b_3 = n$ and we associate to S a representation V of the quiver situation



with $V_{1i} = S_i^{\oplus a_i}$ and $V_{2j} = T_j^{\oplus b_j}$ and where the linear map corresponding to an arrow $b_j \xrightarrow{a_{ij}} a_i$ is the composition of

$$V_{a_{ij}} \quad : \qquad S_i^{\oplus a_i} \hookrightarrow S \downarrow_{\mathbb{Z}_2} = V \downarrow_{\mathbb{Z}_3} \twoheadrightarrow T_j^{\oplus b_j}$$

of the canonical injections and projections. If $\alpha = (a_1, a_2, b_1, b_2, b_3)$ then we take as $\theta = (-1, -1, +1, +1, +1)$. Observe that $\oplus_{i,j} V_{a_{ij}} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a linear isomorphism. If $W \hookrightarrow V$ is a subrepresentation, then $\theta(W) \geq 0$. Indeed, if the dimension vector of W is $\beta = (c_1, c_2, d_1, d_2, d_3)$ and assume that $\theta(W) < 0$, then $k = c_1 + c_2 > l = d_1 + d_2 + d_3$, but then the restriction of $\oplus V_{a_{ij}}$ to W gives a linear map $\mathbb{C}^k \twoheadrightarrow \mathbb{C}^l$ having a kernel which is impossible. Hence, V is a θ-semistable representation of the quiver. In fact, V is even θ-stable, for consider a subrepresentation $W \hookrightarrow V$ with dimension vector β as before and $\theta(W) = 0$, that is, $c_1 + c_2 = d_1 + d_2 + d_3 = m$, then the isomorphism $\oplus_{i,j} V_{a_{ij}} \mid W$ and the decomposition into eigenspaces of $\mathbb{C}^m$ with respect to the $\mathbb{Z}_2$ and $\mathbb{Z}_3$-action, makes $\mathbb{C}^m$ into an m-dimensional representation of $PSL_2(\mathbb{Z})$ which is a subrepresentation of S. S being simple then implies that $W = V$ or $W = 0$, whence V is θ-stable. The underlying reason is that the group algebra $\mathbb{C}PSL_2(\mathbb{Z})$ is a universal localization of the path algebra $\mathbb{C}Q$ of the above quiver.

As irreducible $\overline{G}_{m,n}$-representations correspond to θ-stable representations of the quiver $Q_{m,n}$ we need to determine the dimension vectors α of θ-stables. In section 4.8 we have given an inductive algorithm to determine them. However, using the fact that the moduli spaces are locally controlled and hence are determined locally by local quivers we can apply the easier classification of simple roots given in section 4.4 so solve this problem.

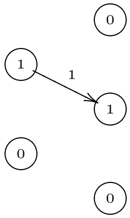
Example 7.7 With S_{ij} we denote the simple 1-dimensional representation of $PSL_2(\mathbb{Z})$ determined by

$$S_{ij} \downarrow_{\mathbb{Z}_2} = S_i \quad \text{and} \quad S_{ij} \downarrow_{\mathbb{Z}_3}$$

Let $n = x_1 + \ldots + x_6$ and we aim to study the local structure of $\texttt{rep}_n\ \mathbb{C}PSL_2(\mathbb{Z})$ in a neighborhood of the semi-simple n-dimensional representation

$$V_\xi = S_{11}^{\oplus x_1} \oplus S_{12}^{\oplus x_2} \oplus S_{13}^{\oplus x_3} \oplus S_{21}^{\oplus x_4} \oplus S_{22}^{\oplus x_5} \oplus S_{23}^{\oplus x_6}$$

To determine the structure of Q_ξ we have to compute $dim\ Ext^1(S_{ij}, S_{kl})$. To do this we view the S_{ij} as representations of the quiver $Q_{2,3}$ in the example above. For example S_{12} is the representation



of dimension vector $(1, 0; 0, 1, 0)$.For representations of $Q_{2,3}$, the dimensions of Hom and Ext-groups are determined by the bilinear form

$$\chi_Q = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

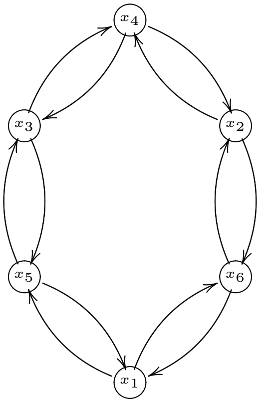
If $V \in \texttt{rep}_\alpha\ Q$ and $W \in \texttt{rep}_\beta\ Q$ where $\alpha = (a_1, a_2; b_1, b_2, b_3)$ with $a_1 + a_2 = b_1 + b_2 + b_3 = k$ and $\beta = (c_1, c_2; d_1, d_2, d_3)$ with $c_1 + c_2 = d_1 + d_2 + d_3 = l$ we have

$$dim\ Hom(V, W) - dim\ Ext^1(V, W) = \chi_Q(\alpha, \beta) = kl - (a_1c_1 + a_2c_2 + b_1d_1 + b_2d_2 + b_3d_3)$$

As $Hom(S_{ij}, S_{kl}) = \mathbb{C}^{\oplus \delta_{ik}\delta_{jl}}$ we have that

$$dim\ Ext^1(S_{ij}, S_{kl}) = \begin{cases} 1 & \text{if } i \neq k \text{ and } j \neq l \\ 0 & \text{otherwise} \end{cases}$$

But then, the local quiver setting (Q_ξ, α_ξ) is



We want to determine whether the irreducible component of $\texttt{rep}_n\ \mathbb{C}PSL_2(\mathbb{Z})$ containing V_ξ contains simple $PSL_2(\mathbb{Z})$-representations, or equivalently, whether α_ξ is the dimension vector of a simple representation of Q_ξ, that is,

$$\chi_{Q_\xi}(\alpha_\xi, \epsilon_j) \leq 0 \quad \text{and} \quad \chi_{Q_\xi}(\epsilon_j, \alpha_\xi) \quad \text{for all } 1 \leq j \leq 6$$

The Euler-form of Q_ξ is determined by the matrix where we number the vertices cyclically

$$\chi_{Q_\xi^\bullet} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

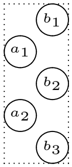
leading to the following set of inequalities

$$\begin{cases} x_1 & \leq x_5 + x_6 \\ x_2 & \leq x_4 + x_6 \\ x_3 & \leq x_4 + x_5 \end{cases} \qquad \begin{cases} x_4 & \leq x_2 + x_3 \\ x_5 & \leq x_1 + x_3 \\ x_6 & \leq x_1 + x_2 \end{cases}$$

Finally, observe that V_ξ corresponds to a $Q_{2,3}$-representation of dimension vector $(x_1+x_2+x_3, x_4+x_5+x_6; x_1+x_4, x_2+x_5, x_3+x_6)$. If we write this dimension vector as $(a_1, a_2; b_1, b_2, b_3)$ then the inequalities are equivalent to the conditions

$$a_i \geq b_j \quad \text{for all } 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 3$$

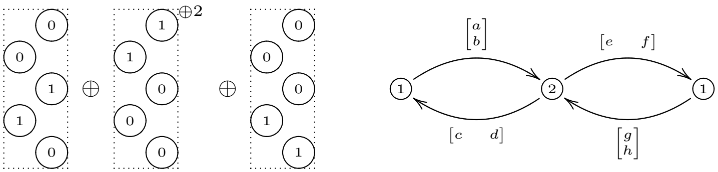
which gives us the desired restriction on the quintuples



at least when $a_i \geq 3$ and $b_j \geq 2$. The remaining cases are handled similarly.

Observe that we can use a similar strategy to determine the restrictions on irreducible representations of any torus knot group quotient $\overline{G}_{m,n} \simeq \mathbb{Z}_m * \mathbb{Z}_n$. Having the classification of the dimension vectors α of θ-semistable representations of $Q_{m,n}$ we can use the local quiver settings to study these projective varieties $M_\alpha^{ss}(Q_{m,n}, \theta)$, in particular to determine the α for which this moduli space is a projective smooth variety.

Example 7.8 For example, $\mathtt{iss}_4\ PSL_2(\mathbb{Z})$ has several components of dimension 3 and 2. For one of the three 3-dimensional components, the one corresponding to $\alpha = (2, 2; 2, 1, 1)$, the different types of semi-simples M_ξ and corresponding local quivers Q_ξ are listed in figure 7.3. To verify whether $\mathtt{iss}_n\ PSL_2(\mathbb{Z})$ is smooth in ξ it suffices to prove that the traces along oriented cycle for the quiver-setting (Q_ξ, α_ξ) generate a polynomial algebra. For example, consider a point $\xi \in \mathtt{iss}_4\ PSL_2(\mathbb{Z})$ of type



Then, the traces along oriented cycles in Q_ξ are generated by the following three algebraic independent polynomials

$$\begin{cases} x &= ac + bd \\ y &= eg + fh \\ z &= (cg + dh)(ea + fb) \end{cases}$$

and hence $\mathtt{iss}_4\ PSL_2(\mathbb{Z})$ is smooth in ξ. The other cases being easier, we see that this component of $\mathtt{iss}_4\ PSL_2(\mathbb{Z})$ is a smooth compact manifold.

A further application of our local quiver-settings (Q_ξ, α_ξ) is that one can often describe large families of irreducible $G_{m,n}$-representations, starting from knowing only rather trivial ones.

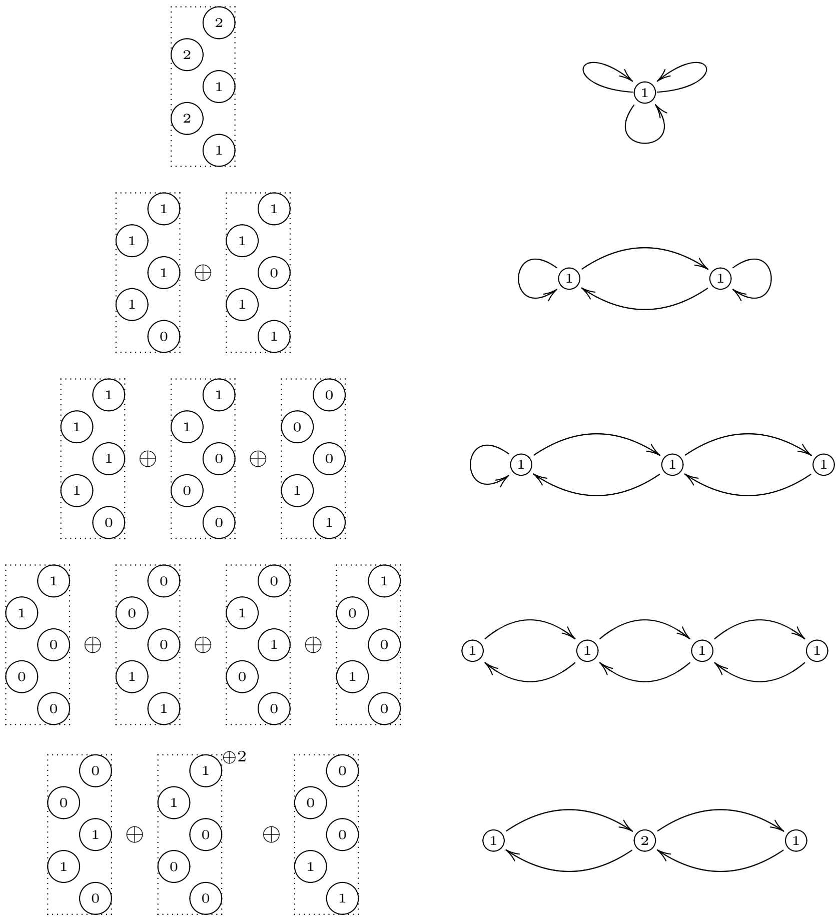
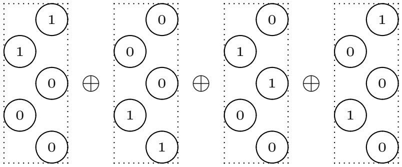
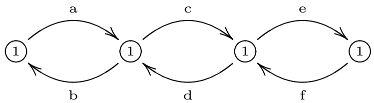


Figure 7.3: Local quiver settings for $M^{ss}_{\alpha}(Q_{2,3},\theta)$ for $\alpha = (2,2;2,1,1)$.

Example 7.9 Consider the semisimple $PSL_2(\mathbb{Z})$-representation ξ of type





Then, M_ξ is determined by the following matrices

$$(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix})$$

The quiver-setting (Q_ξ, α_ξ) implies that any nearby orbit is determined by a matrix-couple

$$(\begin{bmatrix} 1 & b_1 & 0 & 0 \\ a_1 & -1 & d_1 & 0 \\ 0 & c_1 & 1 & f_1 \\ 0 & 0 & e_1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & b_2 & 0 & 0 \\ a_2 & \zeta^2 & d_2 & 0 \\ 0 & c_2 & \zeta & f_2 \\ 0 & 0 & e_2 & 1 \end{bmatrix})$$

and as there is just one arrow in each direction these entries must satisfy

$$0 = a_1a_2 = b_1b_2 = c_1c_2 = d_1d_2 = e_1e_2 = f_1f_2$$

As the square of the first matrix must be the identity matrix $\mathbb{1}_4$, we have in addition that

$$0 = a_1b_1 = c_1d_1 = e_1f_1$$

Hence, we get several sheets of 3-dimensional families of representations (possibly, matrix-couples lying on different sheets give isomorphic $PSL_2(\mathbb{Z})$-representations, as the isomorphism holds in the étale topology and not necessarily in the Zariski topology). One of the sheets has representatives

$$(\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & -1 & d & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e & -1 \end{bmatrix}, \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & c & \zeta & f \\ 0 & 0 & 0 & 1 \end{bmatrix})$$

From the description of dimension vectors of semi-simple quiver representations it follows that such a representation is simple if and only if

$$ab \neq 0 \quad cd \neq 0 \quad \text{and} \quad ef \neq 0$$

Moreover, these simples are not-isomorphic unless their traces ab, cd and ef evaluate to the same numbers.

Finally, one can use the local quiver-settings (Q_ξ, α_ξ) to determine the isomorphism classes of $G_{m,n}$-representations having a specified Jordan-Hölder sequence. For this we apply the theory on nullcones developed in the foregoing chapter.

Example 7.10 In the above example, this nullcone problem is quite trivial. A representation has M_ξ as Jordan-Hölder sum if and only if all traces vanish, that is,

$$ab = cd = ef = 0$$

Under the action of the group $GL(\alpha_\xi) = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$, these orbits are easily seen to be classified by the arrays

a	c	e
b	d	f

filled with zeroes and ones subject to the rule that no column can have two 1's, giving $27 = 3^3$-orbits.

7.5 Differential forms

In this section we will define the complex of *noncommutative differential forms* of an arbitrary $\mathbb{C}$-algebra A and deduce some extra features in case A is Quillen-smooth. In the following section we will compute the *noncommutative deRham cohomology* spaces which will be of crucial importance in the final chapter.

Let us recall briefly the classical (commutative) case. When A is a commutative $\mathbb{C}$-algebra, the A-module of *Kähler differentials* Ω^1_A is generated by the $\mathbb{C}$-linear symbols da for $a \in A$ satisfying the relations

$$d(ab) = adb + bda \qquad \forall a, b \in A$$

and the map $A \xrightarrow{d} \Omega^1_A$ is the universal derivation. By convention we define

$$\begin{cases} \Omega^0_A = & A \\ \Omega^n_A = & \wedge^n_A \, \Omega^1_A \end{cases}$$

where the *exterior product* is taken over A (not over $\mathbb{C}$). Observe that it is spanned by the elements $a_0 da_1 \wedge \ldots \wedge da_n$ that we usually write $a_0 da_1 \ldots da_n$.

The *exterior differential operator*

$$\Omega^n_A \xrightarrow{d} \Omega^{n+1}_A$$

is defined by

$$d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n$$

and gives rise to a sequence

$$A = \Omega^0_A \xrightarrow{d} \Omega^1_A \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_A \xrightarrow{d} \Omega^{n+1}_A \xrightarrow{d} \ldots$$

which is a complex (that is, $d \circ d = 0$) called the *deRham complex* . The *homology groups* of this complex

$$\mathtt{H}^n_{dR}\ \mathtt{A} = \frac{\mathtt{Ker}\ \Omega^n_A \xrightarrow{d} \Omega^{n+1}_A}{\mathtt{Im}\ \Omega^{n-1}_A \xrightarrow{d} \Omega^n_A}$$

are called the *de Rham cohomology groups* of A (over $\mathbb{C}$).

We will extend this to noncommutative $\mathbb{C}$-algebras. We denote by `dgalg` the category of *differential graded* $\mathbb{C}$*-algebras* , that is, an object $R \in$ `dgalg` is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra

$$R = \oplus_{i \in \mathbb{Z}} R_i$$

endowed with a differential d of degree one

$$\ldots \xrightarrow{d} R_{i-1} \xrightarrow{d} R_i \xrightarrow{d} R_{i+1} \xrightarrow{d} \ldots$$

such that $d \circ d = 0$ and for all $r \in R_i$ and $s \in R$ we have

$$d(rs) = (dr)s + (-1)^i r(ds).$$

Clearly, morphisms in `dgalg` are $\mathbb{C}$-algebra morphisms $R \xrightarrow{\phi} S$ which are graded and commute with the differentials.

To a $\mathbb{C}$-algebra A we will now associate the differential graded algebra $\Omega\ A$ of *noncommutative differential forms* . Denote the quotient vector space $A/\mathbb{C}.1$ with $\overline{A}$ and define

$$\Omega^n\ A = A \otimes \underbrace{\overline{A} \otimes \ldots \otimes \overline{A}}_{n}$$

for $n \geq 0$ and $\Omega^n\ A = 0$ for $n < 0$. For $a_i \in A$ we denote the image of $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ in $\Omega^n\ A$ by

$$(a_0, \ldots, a_n).$$

Consider the vectorspace $\Omega\ A = \oplus_{n \in Z}\ \Omega^n\ A$ and define a product on it by

$$(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_m) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_m).$$

Further, define an operator d of degree one

$$\ldots \xrightarrow{d} \Omega^{n-1}\ A \xrightarrow{d} \Omega^n\ A \xrightarrow{d} \Omega^{n+1}\ A \xrightarrow{d} \ldots$$

by the rule

$$d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n).$$

Theorem 7.7 *These formulas define the unique* `dgalg` *structure on* $\Omega\ A$ *such that*

$$a_0 da_1 \ldots da_n = (a_0, a_1, \ldots, a_n).$$

Proof. In *any* $R = \oplus_i R_i \in$ `dgalg` containing A as an even degree subalgebra we have the following identities

$$\begin{array}{rcl} d(a_0 da_1 \ldots da_n) = & & da_0 da_1 \ldots da_n \\ (a_0 da_1 \ldots da_n)(a_{n+1} da_{n+2} \ldots da_m) = & & (-1)^n a_0 a_1 da_2 \ldots da_m \\ & & + \sum_{i=1}^{n} (-1)^{n-i} a_0 da_1 \ldots d(a_i a_{i+1}) \ldots da_m \end{array}$$

which proves uniqueness.

To prove existence, we define d on $\Omega\ A$ as above making the $\mathbb{Z}$-graded $\mathbb{C}$-vectorspace $\Omega\ A$ into a complex as $d \circ d = 0$. Consider the *graded endomorphism ring* of the complex

$$\mathtt{End} = \oplus_{n \in \mathbb{Z}} \mathtt{End}_n = \oplus_{n \in \mathbb{Z}} \mathtt{Hom}_{complex}(\Omega^\bullet\ A, \Omega^{\bullet + n}\ A).$$

With the composition as multiplication, `End` is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra and we make it into an object in `dgalg` by defining a differential

$$\ldots \xrightarrow{D} \mathtt{End}_{n-1} \xrightarrow{D} \mathtt{End}_n \xrightarrow{D} \mathtt{End}_{n+1} \xrightarrow{D} \ldots$$

by the formula on any homogeneous ϕ

$$D\phi = d \circ \phi - (-1)^{deg\ \phi} \phi \circ d.$$

Now define the morphism $A \xrightarrow{l} \mathtt{End}_0$ which assigns to $a \in A$ the left multiplication operator

$$la(a_0, \ldots, a_n) = (aa_0, \ldots, a_n)$$

and extend it to a map

$$\Omega\ A \xrightarrow{l_*} \mathtt{End} \qquad \text{by} \qquad l_*(a_0, \ldots, a_n) = la_0 \circ D\ la_1 \circ \ldots \circ D\ la_n.$$

Applying the general formulae given at the beginning of the proof to the subalgebra $l(A) \hookrightarrow$ `End` we see that the image of l_* is a differential graded subalgebra of `End` and is the differential graded subalgebra generated by $l(A)$.

Define an evaluation map $\mathtt{End} \xrightarrow{ev} \Omega\ A$ by $ev(\phi) = \phi(1)$. Because

$$\begin{array}{rcl} D\ la_i(1, a_{i+1}, \ldots, a_n) = & d(a_i, a_{i-1}, \ldots, a_n) - la_i d(1, a_{i+1}, \ldots, a_n) \\ = & (1, a_i, \ldots, a_n) \end{array}$$

we have that

$$ev(la_0 \circ D\ la_1 \circ \ldots \circ D\ la_n) = (a_0, \ldots, a_n)$$

showing that ev is a left inverse for l_* whence l_* in injective.

Hence we can use the isomorphism $\Omega\ A \simeq Im(l_*)$ to transport the `dgalg` structure to $\Omega\ A$ finishing the proof. $\square$

Example 7.11 (Noncommutative differential forms of $\mathbb{C}\times\mathbb{C}$) Let $A=\mathbb{C}\times\mathbb{C}$ and e and f the idempotents corresponding to the two factors. The quotient space $\overline{A}=A/\mathbb{C}1$ can be identified with $\mathbb{C}\overline{e}$ and therefore

$$\Omega^n\ \mathbb{C}\times\mathbb{C}=(\mathbb{C}\times\mathbb{C})\otimes\mathbb{C}\overline{e}^{\otimes n}=(\mathbb{C}\times\mathbb{C})de^n.$$

The differential d is defined by the formula

$$d((\alpha e+\beta f)de^n)=(\alpha-\beta)de^{n+1}$$

and the product of $\Omega\ \mathbb{C}\times\mathbb{C}$ is defined by the rule

$$(\alpha e+\beta f)de^n(\gamma e+\delta f)de^m=\begin{cases}(\alpha\gamma e+\beta\delta f)de^{n+m} & \text{when } n \text{ is even}\\ (\alpha\delta e+\beta\gamma f)de^{n+m} & \text{when } n \text{ is odd}\end{cases}$$

We will relate the algebra structure of $\Omega\ A$ to that of A. The trick is to define *another* multiplication on $\Omega\ A$ making it only into a *filtered* algebra. We then prove that this filtered algebra is isomorphic to the I-adic filtration of an algebra constructed from A and we recover the `dgalg` multiplication on $\Omega\ A$ by taking the associated graded algebra.

We introduce the universal algebra $\mathbb{L}_A$ with respect to *based linear maps* from A to $\mathbb{C}$-algebras. A based linear map is a $\mathbb{C}$-linear map

$$A\xrightarrow{\rho}R$$

where R is a $\mathbb{C}$-algebra and $\rho(1)=1$. The *curvature* of ρ is then defined to be the bilinear map $A\times A\xrightarrow{\omega}R$ defined by

$$\omega(a,a')=\rho(aa')-\rho(a)\rho(a')$$

that is, it is a measure for the failure of ρ to be an algebra map. Observe that ω vanishes if either a or a' is 1 so it can be viewed as a linear map

$$\overline{A}\otimes\overline{A}\xrightarrow{\omega}R.$$

Let $T(A)=\oplus_{n\geq 0}A^{\otimes n}$ be the *tensor algebra* of the vectorspace A and define

$$\mathbb{L}_A=\frac{T(A)}{T(A)(1-1_A)T(A)}$$

where 1_A is the identity of A consider as a 1-tensor in $T(A)$, then we have a based linear map

$$A\xrightarrow{\rho^{un}}\mathbb{L}_A\qquad a\mapsto\overline{a}$$

where $\overline{a}$ is the image in $\mathbb{L}_A$ of the 1-tensor a in $T(A)$. The map ρ^{un} is universal for based linear maps $A\xrightarrow{\rho}R$, that is, there is a unique *algebra morphism* $\mathbb{L}_A\xrightarrow{\phi_\rho}R$ making the diagram commute

$$\begin{tikzcd} & \mathbb{L}_A \arrow[d, "\exists \phi_\rho"] \\ A \arrow[ur, "\rho^{un}"] \arrow[r, "\rho"] & R \end{tikzcd}$$

In particular, there is a canonical algebra map $\mathbb{L}_A \xrightarrow{\phi_{id}} A$ corresponding to the identity map on A. We define

$$\mathbb{I}_A = \mathtt{Ker}\ \phi_{id} \triangleleft \mathbb{L}_A$$

and equip $\mathbb{L}_A$ with the $\mathbb{I}_A$-adic filtration.

For an arbitrary $R \in \mathtt{dgalg}$ we define the *Fedosov product* on R to be the one induced by defining on homogeneous $r, s \in R$ the product

$$r.s = rs - (-1)^{deg\ r} drds$$

One easily checks that the Fedosov product is associative. Observe that if we decompose $R = R^{ev} \oplus R^{odd}$ into its homogeneous components of even (resp. odd) degree, then this new multiplication is compatible with this decomposition and makes R into a $\mathbb{Z}/2\mathbb{Z}$-graded algebra.

We will now investigate the Fedosov product on $\Omega\ A$. Let ω^{un} be the curvature of the universal based linear map $A \xrightarrow{\rho^{un}} \mathbb{L}_A$.

Theorem 7.8 *There is an isomorphism of algebras*

$$\mathbb{L}_A \simeq (\Omega^{ev}\ A\ ,\ .\ )$$

between $\mathbb{L}_A$ and the even forms $\Omega^{ev}\ A$ equipped with the Fedosov product given by

$$\rho^{un}(a_0)\omega^{un}(a_1, a_2) \dots \omega^{un}(a_{2n-1}, a_{2n}) \longrightarrow a_0 da_1 \dots da_{2n}$$

Under this isomorphism we have the correspondence

$$\mathbb{I}_A^n \simeq \oplus_{k \geq n} \Omega^{2k}\ A$$

The associated graded algebra gives an isomorphism

$$\mathtt{gr}_{\mathbb{I}_A}\ \mathbb{L}_A = \oplus \frac{\mathbb{I}_A^n}{\mathbb{I}_A^{n+1}} \simeq \Omega^{ev}\ A$$

with even forms equipped with the `dgalg` *structure.*

Proof. Consider the based linear map $A \xrightarrow{\rho} \Omega^{ev}\ A$ given by inclusion, then its curvature is given by

$$\omega(a, a') = aa' - a.a' = dada'.$$

By the universal property of $\mathbb{L}_A$ there is an algebra morphism

$$\mathbb{L}_A \xrightarrow{\phi} (\Omega^{ev}\ A\ ,\ .\ )$$

such that $\phi(\rho^{un}(a)) = a$ and $\phi(\omega^{un}(a, a')) = dada'$. Observe that the Fedosov product coincides with the usual `dgalg` product when one of the terms is *closed* , that is $d\ r = 0$. Therefore, we have

$$\phi(\rho^{un}(a_0)\omega^{un}(a_1, a_2) \dots \omega^{un}(a_{2n-1}, a_{2n})) = a_0 da_1 \dots da_{2n}$$

On the other hand, as $\Omega^{2n}\ A = A \otimes \overline{A}^{\otimes 2n}$ we have a well defined map $\Omega^{ev}\ A \xrightarrow{\psi} \mathbb{L}_A$ given by

$$\psi(a_0 da_1 \dots da_{2n}) = \rho^{un}(a_0)\omega^{un}(a_1, a_2) \dots \rho^{un}(a_{2n-1}, a_{2n})$$

and it remains to prove that this map is surjective. The image of ψ is closed under left multiplication as it is closed under left multiplication by elements $\rho^{un}(a)$ (and they generate $\mathbb{L}_A$) as

$$\begin{aligned} & \rho^{un}(a).\rho^{un}(a_0)\omega^{un}(a_1, a_2) \dots \omega^{un}(a_{2n-1}, a_{2n}) \\ = \quad & \rho^{un}(aa_0)\omega^{un}(a_1, a_2) \dots \omega^{un}(a_{2n-1}, a_{2n}) - \omega^{un}(a, a_0)\omega^{un}(a_1, a_2) \dots \omega^{un}(a_{2n-1}, a_{2n}) \end{aligned}$$

Because the image contains 1 this proves the claim and the isomorphism.

Identify via this isomorphism $\mathbb{L}_A$ with $\Omega^{ev}\ A$. Because $dada' \in \mathbb{I}_A$ we have $\Omega^{2k}\ A \hookrightarrow \mathbb{I}_A^n$ for all $k \geq n$. Thus, $F_n = \oplus_{k \geq n}\Omega^{2k}\ A \hookrightarrow \mathbb{I}_A$. Conversely, $\mathbb{I}_A = F_1$ and hence

$$\mathbb{I}_A^n = (F_1)^n \hookrightarrow F_n$$

by the definition of the Fedosov product. Therefore, $\mathbb{I}_A^n = F_n$ and the claim over the associated graded follows. $\square$

Example 7.12 (Even differential forms of $\mathbb{C} \times \mathbb{C}$) As before, let e and f be the idempotents of $A = \mathbb{C} \times \mathbb{C}$ corresponding to the two components. By definition,

$$\mathbb{L}_{\mathbb{C}\times\mathbb{C}} = \frac{T(\mathbb{C}e + \mathbb{C}f)}{(1 - e - f)} = \frac{\mathbb{C}\langle E, F\rangle}{(1 - E - F)} \simeq \mathbb{C}[E]$$

The universal based linear map is given by

$$\mathbb{C} \times \mathbb{C} \xrightarrow{\rho^{un}} \mathbb{C}[E] \qquad \begin{cases} e & \mapsto E \\ f & \mapsto 1 - E \end{cases}$$

and the curvature on $\overline{A} = \mathbb{C}\overline{e}$ is given by

$$\omega^{un}(e,e) = E - E^2$$

Therefore the isomorphism between $\Omega^{ev}\ A$ and $\mathbb{L}_A = \mathbb{C}[E]$ is given by

$$(\alpha e + \beta f)de^{2n} \xrightarrow{\psi} (\alpha E + \beta(1-E))(E-E^2)^n.$$

The Fedosov product on $\Omega^{ev}\ A$ is given by the formula (using the multiplication formulas we found above)

$$(\alpha e + \beta f)de^{2n}.(\gamma e + \delta f)de^{2m} = (\alpha\gamma e + \beta\delta f)de^{2n+2m} - (\alpha - \beta)(\gamma - \delta)de^{2n+2m+2}$$

In order to check that ψ is indeed an algebra morphism we need to verify that in $\mathbb{C}[E]$ we have the equality

$$\begin{aligned} & (\alpha E + \beta(1-E))(E-E^2)^n(\gamma E + \delta(1-E))(E-E^2)^m \\ = \quad & (\alpha\gamma E + \beta\delta(1-E))(E-E^2)^{n+m} - (\alpha-\beta)(\gamma-\delta)(E-E^2)^{n+m+1} \end{aligned}$$

which is indeed the case.

Further, $\mathbb{I}_A = \mathbb{C}[E](E-E^2)$ and indeed $\frac{\mathbb{C}[E]}{(E-E^2)} \simeq \mathbb{C} \times \mathbb{C}$. Finally, under the identification ψ we obtain the usual multiplication of noncommutative differential forms from

$$\Omega^{2n}\ A \times \Omega^{2m}\ A = \frac{(E-E^2)^n}{(E-E^2)^{n+1}} \times \frac{(E-E^2)^m}{(E-E^2)^{m+1}} \longrightarrow \frac{(E-E^2)^{n+m}}{(E-E^2)^{n+m+1}} = \Omega^{2n+2m}\ A.$$

We now turn to *all* noncommutative differential forms $\Omega\ A$. Observe that this algebra has an involution σ which is the identity on even forms and is minus the identity on odd forms. σ is an algebra automorphism both for the usual `dgalg`-algebra structure as for the Fedosov product. Algebras with an involution are called *super-algebras* .

We want to construct an algebra universal for algebra morphisms from A to a super-algebra. Consider the *free product* $A * A$ which is defined as follows. Let $\mathcal{B}_1$ be a vectorspace basis for $A - \mathbb{C}.1$ and $\mathcal{B}_2$ a duplicate of it. As a $\mathbb{C}$-vectorspace $A * A$ has a basis consisting of words

$$w = a_1 b_1 a_2 b_2 \ldots a_k b_k \quad \text{or} \quad w = a_1 b_1 a_2 b_2 \ldots a_k$$

for some k where the a_i's all belong to $\mathcal{B}_1$ or all to $\mathcal{B}_2$ and the b_j's all belong to the other base set. On this vectorspace one defines a $\mathbb{C}$-algebra structure in the obvious way, that is by concatenating words and if necessary (if the end term of the first word lies in the same base-set as the beginning term of the second) use the multiplication table in A to reduce to a linear combination of allowed words.

The algebra $A * A$ is universal with respect to *pairs of algebra maps* $A \overset{f}{\underset{g}{\rightrightarrows}} R$ from A to R. That is, there is a unique algebra map γ

$$\begin{tikzcd}
& R & \\
& A * A \arrow[u, "\exists\gamma"] & \\
A \arrow[uur, "f"] \arrow[ur, "i_1"] & & A \arrow[uul, "g"'] \arrow[ul, "i_2"']
\end{tikzcd}$$

making the diagram commute. Here, i_1 is the inclusion of A in $A * A$ using only syllables in $\mathcal{B}_1$ and i_2 is defined similarly. The construction of γ clearly is induced by sending $a \in \mathcal{B}_1$ to $f(a)$ and $b \in \mathcal{B}_2$ to $g(b)$.

Further, interchanging the bases $\mathcal{B}_1 \xrightarrow{\tau} \mathcal{B}_2$ equips $A * A$ with an involution, or if you prefer, makes $A * A$ a super-algebra. Now, let S be a super-algebra with involution σ_S and let $A \xrightarrow{f} S$ be an algebra morphism, then there is a unique morphism of super-algebras ψ making the diagram commute

$$\begin{tikzcd}
& A * A \arrow[d, "\exists\psi"] \\
A \arrow[ur, "i_1"] \arrow[r, "f"] & S
\end{tikzcd}$$

ψ is the universal map corresponding to the pair of algebra maps $A \overset{f}{\underset{\sigma_S \circ f}{\rightrightarrows}} S$.

For any $a \in A$ we define the elements in $A * A$:

$$\begin{cases} p(a) & = \frac{1}{2}(i_1(a) + i_2(a)) \\ q(a) & = \frac{1}{2}(i_1(a) - i_2(a)) \end{cases}$$

and we define $\mathbb{Q}_A \triangleleft A * A$ to be the ideal of $A * A$ generated by the elements $q(a)$ for $a \in A$, then clearly

$$A \simeq \frac{A * A}{\mathbb{Q}_A}$$

We now have an analog of the previous theorem for all differential forms.

Theorem 7.9 *There is an isomorphism of super-algebras*

$$A * A \simeq (\Omega\ A\ ,\ .\ )$$

between $A * A$ and the noncommutative differential forms $\Omega\ A$ equipped with the Fedosov product given by

$$p(a_0)q(a_1)\dots q(a_n) \longrightarrow a_0da_1\dots da_n$$

Under this isomorphism we have the correspondence

$$\mathbb{Q}^n_A \simeq \oplus_{k\geq n}\Omega^n\ A$$

and the associated graded algebra is isomorphic to $\Omega\ A$ with the usual `dgalg` *structure.*

Proof. We have an algebra map $A \xrightarrow{u} \Omega\,A$ equipped with the Fedosov product given by $a \mapsto a+da$ because

$$\begin{aligned}(a+da).(a'+da') = &\quad aa' - dada' + ada' + daa' + dada' \\ = &\quad aa' + d(aa')\end{aligned}$$

By the universal property of $A * A$ there is a super-algebra morphism

$$A * A \xrightarrow{\psi} \Omega\ A \qquad \psi(p(a)) = a \quad \text{and} \quad \psi(q(a)) = da$$

But then using that the Fedosov product coincides with the usual product when one of the forms is closed we have

$$\psi(p(a_0)q(a_1)\dots q(a_n)) = a_0da_1\dots da_n$$

Conversely, we have a section to ψ defined by

$$\Omega\ A \xrightarrow{\phi} A * A \qquad a_0da_1\dots da_n \mapsto p(a_0)q(a_1)\dots q(a_n)$$

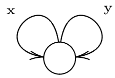
and we only have to prove that ϕ is surjective. The image `Im` ϕ is closed under left multiplication by $p(a)$ and $q(a)$ as $p(1) = 1$ and

$$\begin{cases} p(a)p(a_0)q(a_1)\dots q(a_n) = & p(aa_0)q(a_1)\dots q(a_n) - q(a)q(a_0)q(a_1)\dots q(a_n) \\ q(a)p(a_0)q(a_1)\dots q(a_n) = & q(aa_0)q(a_1)\dots q(a_n) - p(a)q(a_0)q(a_1)\dots q(a_n)\end{cases}$$

Because the elements $p(a)$ and $q(a)$ generate $A * A$, the image `Im` ϕ is a left ideal containing 1, whence ψ is surjective.

The claims about the ideals $\mathbb{Q}^n_A$ and about the associated graded algebra follow as in the proof for even forms. $\square$

Example 7.13 (Noncommutative differential forms of $\mathbb{C}\langle x, y\rangle$) The noncommutative free algebra in two variables $\mathbb{C}\langle x, y\rangle$ is the path algebra of the quiver



Clearly we have $\mathbb{C}\langle x, y\rangle * \mathbb{C}\langle x, y\rangle = \mathbb{C}\langle x_1, y_1, x_2, y_2\rangle$ and the maps

$$\begin{cases} p(x) = \frac{1}{2}(x_1 + x_2) & q(x) = \frac{1}{2}(x_1 - x_2) \\ p(y) = \frac{1}{2}(y_1 + y_2) & q(y) = \frac{1}{2}(y_1 - y_2) \end{cases}$$

It is easy to compute the maps p and q on any monomial in x and y using the formulae holding in any $A * A$

$$\begin{cases} p(aa') = & p(a)p(a') + q(a)q(a') \\ q(aa') = & p(a)q(a') + q(a)p(a') \end{cases}$$

Further note that it follows from this that $\mathbb{Q}_{\mathbb{C}\langle x,y\rangle} = (x_1 - x_2, y_1 - y_2)$ and we have all the required tools to calculate (in principle) with $\Omega\ \mathbb{C}\langle x, y\rangle$.

Example 7.14 (Noncommutative differential forms of $\mathbb{C} \times \mathbb{C}$) The *infinite dihedral group* D_∞ is the group with presentation

$$D_\infty = \langle a, b \mid a^2 = 1 = b^2\rangle$$

that is, an arbitrary element in D_∞ is a word of the form

$$a^i babab \ldots abab^j$$

where $i, j = 0$ or 1. Multiplication is given by concatenation of words, using the relations $a^2 = 1 = b^2$ when necessary.

The *group algebra* $\mathbb{C}[D_\infty]$ is the vectorspace with basis D_∞ and with multiplication induced by the groupmultiplication in D_∞. We now claim that

$$(\mathbb{C} \times \mathbb{C}) * (\mathbb{C} \times \mathbb{C}) \simeq \mathbb{C}[D_\infty]$$

Indeed, $\mathbb{C} \times \mathbb{C} \simeq \mathbb{C}[\mathbb{Z}_2]$ the group algebra of the cyclic group of order two, that is $\mathbb{C}[\mathbb{Z}_2] = \mathbb{C}[x]/(x^2 - 1)$, the isomorphism being given by

$$e \longrightarrow \frac{1}{2}(1 + x) \qquad f \longrightarrow \frac{1}{2}(1 - x)$$

One also has the obvious notion of a free product in the category of groups and from the definition it is clear that

$$\mathbb{Z}_2 * \mathbb{Z}_2 \simeq D_\infty$$

and therefore also on the level of group algebras

$$\mathbb{C}[\mathbb{Z}_2] * \mathbb{C}[\mathbb{Z}_2] \simeq \mathbb{C}[D_\infty]$$

The relevant maps $\mathbb{C} \times \mathbb{C} \underset{q}{\overset{p}{\rightrightarrows}} \mathbb{C}[D_\infty]$ are given by

$$\begin{cases} p(e) = \frac{1}{2} + \frac{1}{4}(a+b) & q(e) = \frac{1}{4}(a-b) \\ p(f) = \frac{1}{2} - \frac{1}{4}(a+b) & q(f) = -\frac{1}{4}(a-b) \end{cases}$$

and so $\mathbb{Q}_{\mathbb{C}\times\mathbb{C}} = (a-b) \triangleleft \mathbb{C}[D_\infty]$. Again, this information allows us to calculate with $\Omega\ \mathbb{C} \times \mathbb{C}$ by referring all computations to the more familiar group algebra $\mathbb{C}[D_\infty]$.

The above definitions and results are valid for every $\mathbb{C}$-algebra A. We will indicate a few extra properties provided the algebra A is Quillen-smooth.

We have the universal lifting algebra $\mathbb{L}_A$ for based *linear* maps from A to $\mathbb{C}$-algebras and the ideal $\mathbb{I}_A$ such that

$$A \underset{\simeq}{\xleftarrow{\overline{\phi_{id}}}} \frac{\mathbb{L}_A}{\mathbb{I}_A}.$$

The $\mathbb{I}_A$-*adic completion* of $\mathbb{L}_A$ is by definition the inverse limit

$$\hat{\mathbb{L}}_A = \underset{\underset{n}{\longleftarrow}}{\mathtt{lim}} \ \frac{\mathbb{L}_A}{\mathbb{I}_A}$$

Assume that A is formally smooth, then for every k we have an algebra map lifting $\overline{\phi_{id}}^{-1}$

$$\begin{tikzcd} & \frac{\mathbb{L}_A}{\mathbb{I}_A^k} \arrow[d] \\ A \arrow[ur, "\phi_k"] \arrow[r, "\overline{\phi_{id}}^{-1}"'] & \frac{\mathbb{L}_A}{\mathbb{I}_A} \end{tikzcd}$$

These compatible lifts define an algebra lift $A \xrightarrow{l^{un}} \hat{\mathbb{L}}_A$. This map can be used to construct algebra lifts modulo nilpotent ideals in a systematic way. Assume $I \triangleleft R$ is such that $I^k = 0$ and there is an algebra map $A \xrightarrow{\mu} \frac{R}{I}$. We can lift μ to R as a based linear map, say ρ. Now we have the following situation

$$\begin{tikzcd}
\mathbb{L}_A \arrow[r, "can"] \arrow[dr, "\phi_\rho"] & \hat{\mathbb{L}}_A \arrow[d, "\hat{\phi_\rho}"] \\
 & R \arrow[d, two heads] \\
A \arrow[uu, "\rho^{un}"] \arrow[uur, "l^{un}"] \arrow[ur, "\rho"] \arrow[r, "\mu"] & \frac{R}{I}
\end{tikzcd}$$

Here, ϕ_ρ is the algebra map coming from the universal lifting property of $\mathbb{L}_A$ and $\hat{\phi_\rho}$ is its extension to the completion. But then, $\tilde{\mu} = \hat{\phi_\rho} \circ l^{un}$ is an *algebra* lift of μ. That is,

Proposition 7.3 *A is formally smooth if and only if there is an algebra section $A \longrightarrow \hat{\mathbb{L}}_A$ to the projection $\hat{\mathbb{L}}_A \twoheadrightarrow A$ defined by mapping out $\mathbb{I}_A$.*

We will give an explicit construction of the embedding $A \xrightarrow{l^{un}} \hat{\mathbb{L}}_A$. By formal smoothness we have an algebra lift

$$\begin{tikzcd}
 & \mathtt{A} \oplus \Omega^2\, A & = & \frac{\mathbb{L}_A}{\mathbb{I}_A^2} \arrow[d, two heads] \\
A \arrow[ur, "l_2"] \arrow[r, "id"'] & A & = & \frac{\mathbb{L}_A}{\mathbb{I}_A}
\end{tikzcd}$$

which is of the form $l_2(a) = a - \phi(a)$ for a linear map $A \xrightarrow{\phi} \Omega^2\, A$. As $\mathbb{L}_A$ is freely generated by the $a \in A - \mathbb{C}1$, we can define a *derivation* on $\mathbb{L}_A$ defined by

$$\mathbb{L}_A \xrightarrow{D} \mathbb{L}_A \qquad D(a) = \phi(a) \quad \forall a \in A.$$

This derivation is called the *Yang-Mills derivation* of A.

Clearly $D(\mathbb{L}_A) \hookrightarrow \mathbb{I}_A$ and we have

$$\begin{aligned} D(dada') &= D(aa' - a.a') \\ &= D(aa') - D(a).a' - a.D(a') \\ &= \phi(aa') - \phi(a).a' - a.\phi(a') \\ &\equiv \; aa' - a.a' \ \mathtt{mod}\ \mathbb{I}_A^2 \\ &\equiv \; dada' \ \mathtt{mod}\ \mathbb{I}_A^2 \end{aligned}$$

the next to last equality coming from the fact that l_2 is an algebra map. Hence, $D = id$ on $\frac{\mathbb{I}_A}{\mathbb{I}_A^2} = \Omega^2\ A$.

Further, $D(\mathbb{I}_A^n) \hookrightarrow \mathbb{I}_A^n$ and so D induces a derivation on the associated graded $gr_{\mathbb{I}_A}\ \mathbb{L}_A$. As this derivation is zero on $A = \frac{\mathbb{L}_A}{\mathbb{I}_A}$ and one on $\frac{\mathbb{I}_A}{\mathbb{I}_A^2}$ it is n on $\frac{\mathbb{I}_A^n}{\mathbb{I}_A^{n+1}}$. But then we have by induction

$$(D-n)...(D-1)D(\mathbb{L}_A) \hookrightarrow \mathbb{I}_A^{n+1}$$

Therefore, $\frac{\mathbb{L}_A}{\mathbb{I}_A^{n+1}}$ decomposes into eigenspaces of D corresponding to the eigenvalues $0, 1, \ldots, n$ and because D is a derivation this decomposition defines a grading compatible with the product.

Hence, we obtain an isomorphism of $\frac{\mathbb{L}_A}{\mathbb{I}_A^{n+1}}$ with its associated graded algebra by lifting $\frac{\mathbb{I}_A^k}{\mathbb{I}_A^{k+1}}$ to the eigenspace of D on $\frac{\mathbb{I}_A^k}{\mathbb{I}_A^{n+1}}$ corresponding to the eigenvalue k.

Taking the inverse limit as $n \longrightarrow \infty$ we obtain an algebra isomorphism of $\hat{\mathbb{L}}_A$ with the completion of its associated graded algebra, that is,

$$\hat{\Omega^{ev}}\ \mathtt{A} = \prod_n \Omega^{2n}\ A \simeq \hat{\mathbb{L}}_A$$

In particular, the kernel of D is a subalgebra of $\hat{\mathbb{L}}_A$ mapped isomorphically onto A by the canonical surjection $\hat{\mathbb{L}}_A \twoheadrightarrow A$. Hence, this subalgebra gives the desired universal lift $A \xhookrightarrow{l^{un}} \hat{\mathbb{L}}_A$.

We can even give an explicit formula for l^{un}. Let L be the degree two operator on $\Omega^{ev}\ A$ defined by

$$L(a_0da_1\ldots da_{2n}) = \phi(a_0)da_1\ldots da_{2n} + \sum_{j=1}^{2n} a_0da_1\ldots da_{j-1}d\phi(a_j)da_{j+1}\ldots da_{2n}$$

and let H denote the degree zero operator on even forms which is multiplication by n on $\Omega^{2n}\ A$. Then, we have the relations

$$[H, L] = L \qquad \text{and} \qquad D = H + L$$

whence we have on $\hat{\Omega}^{ev}\ A$ that

$$e^{-L}He^{L} = H + e^{-L}[H,e^{L}] = H + \int_0^1 e^{-tL}[H,L]e^{tL}dt = D$$

Therefore, the universal lift for all $a \in A$ is given by

$$l^{un}(a) = e^{-L}a = a - \phi(a) + \frac{1}{2}L\phi(a) - \dots$$

Example 7.15 (The universal lift for $\mathbb{C}\times\mathbb{C}$) Recall the correspondence between $\Omega^{ev}\ \mathbb{C}\times\mathbb{C}$ and $\mathbb{L}_{\mathbb{C}\times\mathbb{C}} = \mathbb{C}[E]$ given by

$$(\alpha e + \beta f)de^{2n} \longrightarrow (\alpha E + \beta(1-E))(E-E^2)^n$$

Lifting e to $\frac{\mathbb{L}}{\mathbb{I}^2}$ we have to compute

$$(2-E)^2E^2 = E + (2E-1)(E-E^2) + (E-E^2)^2$$

whence $\phi(e) = (1-2E)(E-E^2)$ and as $f = 1-e$ we have $\phi(f) = (2E-1)(E-E^2)$. The Yang-Mills derivation D on $\mathbb{C}[E]$ is hence the one determined by

$$\mathbb{C}[E] \xrightarrow{D} \mathbb{C}[E] \qquad D(E) = (1-2E)(E-E^2).$$

To determine the universal lift of e we have to compute

$$l^{un}(e) = e - Le + \frac{1}{2}L^2e - \frac{1}{6}L^3e + \dots$$

and we have

$$\begin{aligned}
L(e) &= \phi(e) = (f-e)de^2 \\
L^2(e) &= L(f-e)de^2 = -6(f-e)de^4 \\
L^3(e) &= -6L(f-e)de^4 = 60(f-e)de^6 \\
L^4(e) &= ...
\end{aligned}$$

and therefore

$$l^{un}(e) = E + (2E-1)(E-E^2) + 3(2E-1)(E-E^2)^2 + 10(2E-1)(E-E^2)^3 + \dots$$

Another characteristic feature of formally smooth algebras is the existence of *connections* on $\Omega^1\ A$. If E is an A-bimodule, then a connection on E consists of two operators

- A *right connection* : $E \xrightarrow{\nabla_r} E \otimes_A \Omega^1\ A$ satisfying

$$\nabla_r(aea') = a(\nabla_r e)a' + aeda',$$

- A *left connection* : $E \xrightarrow{\nabla_l} \Omega^1\ A \otimes_A E$ satisfying

$$\nabla_l(aea') = a(\nabla_l e)a' + daea'$$

Given a right connection ∇_r there is a bimodule splitting s_r of the right multiplication map m_r

$$E \otimes_A A \underset{s_r}{\overset{m_r}{\rightleftarrows}} E$$

given by the formula

$$s_r(e) = e \otimes 1 - j(\nabla_r e) \quad \text{where} \quad j(e \otimes da) = ea \otimes 1 - e \otimes a$$

Similarly, a left connection gives a bimodule splitting s_l to the left multiplication map. Consequently, if a connection exists on E, then E must be a *projective* bimodule.

Consider the special bimodule of noncommutative 1-forms $\Omega^1\ A$, then as $\Omega^1\ A \otimes_A \Omega^1\ A = \Omega^2\ A$ a connection on $\Omega^1\ A$ is the datum of three maps

$$\begin{tikzcd} \Omega^1\ A \arrow[r, shift left=4, "\nabla_l"] \arrow[r, "d"] \arrow[r, shift right=4, "\nabla_r"'] & \Omega^2\ A \end{tikzcd}$$

satisfying the following properties

$$\begin{array}{rclll} \nabla_l(aea') & = & a\nabla_l(e)a' & +(da)ea' & \\ d(aea') & = & a(de)a' & +(da)ea' & -ae(da') \\ \nabla_r(aea') & = & a\nabla_r(e)a' & & +ae(da') \end{array}$$

Hence, if ∇_r is a right connection then $d + \nabla_r$ is a left connection and if ∇_l is a left connection then $\nabla_l - d$ is a right connection. Therefore, onesided connections exist on $\Omega^1\ A$ if and only if connections exist and hence if and only if $\Omega^1\ A$ is a projective bimodule.

But then we have an A-bimodule splitting of the exact sequence

$$0 \longrightarrow \Omega^2\ A \xrightarrow{j} \Omega^1\ A \otimes \mathtt{A} \xrightarrow{m} \Omega^1\ A \longrightarrow 0$$

where $j(\omega da) = \omega a \otimes 1 - \omega \otimes a$ and $m(\omega \otimes a) = \omega a$.

Proposition 7.4 *A connection exists on $\Omega^1\ A$ if and only if A is formally smooth.*

Proof. A bimodule splitting of the above map is determined by a retraction bimodule map p for j. As $\Omega^1\ A \otimes A \simeq A \otimes \overline{A} \otimes A$, a bimodule map p

$$\Omega^{`}\ A \otimes \mathtt{A} \xrightarrow{p} \Omega^2\ A$$

is equivalent to a map $\overline{A} \xrightarrow{\phi} \Omega^2\ A$ via $p(a_0 da_1 \otimes a_2) = a_0\phi(a_1 a_2)$. But then we have

$$\begin{aligned} pj(da_1 da_2) =& p((da_1)a_2 \otimes 1 - da_1 \otimes da_2) \\ =& p(d(a_1 a_2) \otimes 1 - a_1(da_2) \otimes 1 - da_1 \otimes a_2) \\ =& \phi(a_1 a_2) - a_1\phi(a_2) - \phi(a_1)a_2 \end{aligned}$$

and splitting of the map means $pj = id$ that is that ϕ satisfies

$$\phi(aa') = a\phi(a') + \phi(a)a' + dada'$$

which is equivalent to an algebra lift

$$A \xrightarrow{\phi^*} \frac{\mathbb{L}_A}{\mathbb{I}_A} = \mathtt{A} \oplus \Omega^2\ A$$

Now, assume we have an algebra morphism

$$A \xrightarrow{f} \frac{R}{I} \qquad \text{with} \qquad I^2 = 0$$

and lift f to a based linear map $A \xrightarrow{\rho} R$. By the universal property of $\mathbb{L}_A$ we have an algebra lift

$$\mathbb{L}_A \xrightarrow{\rho^*} R$$

living over f. Therefore $\rho^*(\mathbb{I}_A) \subset I$ and therefore ρ^* is zero on $\mathbb{I}_A^2$ giving an algebra morphism

$$\frac{\mathbb{L}_A}{\mathbb{I}_A^2} \xrightarrow{f^*} R$$

living over f. But then the existence of an algebra map ϕ^* as above gives a desired lifting $f^* \circ \phi^*$ of f, finishing the proof. $\square$

For a map $\overline{A} \xrightarrow{\phi} \Omega^2\ A$ as above, a connection is given by the formulae

$$\nabla_r(ada') = a\phi(a') \quad \text{and} \quad \nabla_r(ada') = a\phi(a') + dada'$$

Example 7.16 (Connection on $\mathbb{C}\langle x,y\rangle$) Clearly we have $\Omega^1\ \mathbb{C}\langle x,y\rangle = \mathbb{C}\langle x,y\rangle \otimes \mathbb{C}x + \mathbb{C}y \otimes \mathbb{C}\langle x,y\rangle$ which is the free bimodule generated by dx and dy. There is a canonical connection with

$$\begin{cases} \phi(x) = 0 \quad \text{and} \quad \nabla_l(dx) = \nabla_r(dx) = 0 \\ \phi(y) = 0 \quad \text{and} \quad \nabla_l(dy) = \nabla_r(dy) = 0 \end{cases}$$

The image of ϕ on any word $z_1 \dots z_n$ with $z_i = x$ or y is given by the formula

$$\begin{aligned} \phi(z_1 \dots z_n) =& \nabla_r d(z_1 \dots z_n) \\ =& \nabla_r(\sum_{i=1}^{n} z_1 \dots z_{i-1}(dz_i)z_{i+1} \dots z_n) \\ =& \sum_{i=1}^{n-1} z_1 \dots z_{i-1}(dz_i)d(z_{i+1} \dots z_n) \end{aligned}$$

Example 7.17 (Connection on $\mathbb{C} \times \mathbb{C}$) We have calculated above that the lifting map ϕ is determined by

$$\phi(e) = (1-2E)(E-E^2) = (f-e)de^2$$

Therefore the corresponding left and right connections are given by

$$\begin{cases} \nabla_r((\alpha e + \beta f)de) = & (\beta f - \alpha e)de^2 \\ \nabla_l((\alpha e + \beta f)de) = & (\alpha f - \beta e)de^2 \end{cases}$$

7.6 deRham cohomology

In this section we will compute various sorts of *noncommutative deRham cohomology* . We have for an arbitrary $\mathbb{C}$-algebra A the complex of *noncommutative* differential forms

$$A = \Omega^0\ A \xrightarrow{d} \Omega^1\ A \xrightarrow{d} \dots \xrightarrow{d} \Omega^n\ A \xrightarrow{d} \Omega^{n+1}\ A \xrightarrow{d} \dots$$

A first attempt to define noncommutative de Rham cohomology is to take the homology groups of this complex, we call these the *big* noncommutative de Rham cohomology

$$H^n_{big}\ A = \frac{Ker\ \Omega^n\ A \xrightarrow{d} \Omega^{n+1}\ A}{Im\ \Omega^{n-1}\ A \xrightarrow{d} \Omega^n\ A}$$

Example 7.18 (Big de Rham cohomology of $\mathbb{C}\times\mathbb{C}$) We have seen before that $\Omega^n\ \mathbb{C}\times\mathbb{C} = (\mathbb{C}\times\mathbb{C})de^n$ and that the differential is given by

$$\begin{array}{ccc} \Omega^n\ \mathbb{C}\times\mathbb{C} & \xrightarrow{d} & \Omega^{n+1}\ \mathbb{C}\times\mathbb{C} \\ (\alpha e+\beta f)de^n & \mapsto & (\alpha-\beta)de^{n+1} \end{array}$$

From which it is immediately clear that

$$\begin{cases} H^0_{big}\ \mathbb{C}\times\mathbb{C} = & \mathbb{C} \\ H^n_{big}\ \mathbb{C}\times\mathbb{C} = & 0 \end{cases}$$

for all $n \geq 1$. This is not quite the answer $H^0\ \mathbb{C}\times\mathbb{C} = \mathbb{C}\oplus C$ we would expect from the commutative case.

For a general $\mathbb{C}$-algebra A it is usually very difficult to compute these cohomology groups. In case of free algebras we can use the graded structure of the complex together with the *Euler derivation* to compute them, a trick we will use later in greater generality.

Example 7.19 (Big de Rham cohomology of $\mathbb{C}\langle x,y\rangle$) Define the *Euler derivation* E on $\mathbb{C}\langle x,y\rangle$ by

$$E(x) = x \qquad \text{and} \qquad E(y) = y$$

Observe that if w is a *word* in x and y of degree k, then we have the Eulerian property that

$$E(w) = kw$$

as one easily verifies.

We can define a degree preserving *derivation* L_E on the differentially graded algebra $\Omega\ \mathbb{C}\langle x,y\rangle$ by the rules

$$L_E(a) = E(a) \qquad \text{and} \qquad L_E(da) = dE(a) \qquad \forall a \in \mathbb{C}\langle x,y\rangle$$

Further we introduce the degree -1 *contraction operator* i_E which is the *super-derivation* on $\Omega\ \mathbb{C}\langle x,y\rangle$, that is,

$$i_E(\omega\omega') = i_E(\omega)\omega' + (-1)^i\omega i_E(\omega') \quad \text{for } \omega \in \Omega^i\ \mathbb{C}\langle x,y\rangle$$

defined by the rules

$$i_E(a) = 0 \qquad i_E(da) = E(a) \qquad \forall a \in \mathbb{C}\langle x,y\rangle.$$

That is, we have the following situation

$$\begin{tikzcd} \Omega^{n-1} \arrow[r, bend left, "d"] \arrow[loop below, "L_E"] & \Omega^n \arrow[l, bend left, "i_E"] \arrow[r, bend left, "d"] \arrow[loop below, "L_E"] & \Omega^{n+1} \arrow[l, bend left, "i_E"] \arrow[loop below, "L_E"] \end{tikzcd}$$

These operators satisfy the equation

$$L_E = i_E \circ d + d \circ i_E$$

as both sides are *derivations* on $\Omega\ \mathbb{C}\langle x, y\rangle$ and coincide on the generators a and da for $a \in \mathbb{C}\langle x, y\rangle$ of this differentially graded algebra.

We claim that L_E is a total degree preserving linear automorphism on

$$\Omega^n\ \mathbb{C}\langle x, y\rangle \qquad \text{for } n \geq 1.$$

For if w_i for $0 \leq i \leq n$ are words in x and y of degree k_i with $k_i \geq 1$ for $i \geq 1$, then we have

$$L_E(w_0 dw_1 \dots dw_n) = (k_0 + \dots + k_n) w_0 dw_1 \dots dw_n.$$

Using the words in x and y as a basis for $\overline{A}$ we see that the kernel and image of the differential d must be homogeneous. But then, if ω is a multi-homogeneous element in $\Omega^n\ \mathbb{C}\langle x, y\rangle$ and in $Ker\ d$ we have for some integer $k \neq 0$ that

$$k\omega = L_E(\omega) = (i_E \circ d + d \circ i_E)\omega = d(i_E\ \omega)$$

and hence ω lies in $Im\ d$. Therefore, we have proved

$$\begin{cases} H^0_{big}\ \mathbb{C}\langle x, y\rangle = & \mathbb{C} \\ H^n_{big}\ \mathbb{C}\langle x, y\rangle = & 0 \end{cases}$$

for all $n \geq 1$.

The examples show that the differentially graded algebra $\Omega\ A$ is *formal* for $A = \mathbb{C} \times \mathbb{C}$ or $\mathbb{C}\langle x, y\rangle$. Recall that for an arbitrary A_∞-algebra Ω (in particular for $\Omega \in \mathtt{dgalg}$), the homology algebra $H^*\ \Omega$ has a *canonical A_∞-structure* . That is, we have $m_1 = 0$, m_2 is induced by the 'multiplication' m_2 on Ω and there is a quasi-isomorphism of A_∞-algebras $H^*\ \Omega \longrightarrow \Omega$ lifting the identity of $H^*\ \Omega$.

The A_∞-algebra Ω is said to be *formal* if the canonical structure makes $H^*\ \Omega$ into an ordinary associative graded algebra (that is, such that all $m_n = 0$ for $n \geq 3$). In particular, if $\Omega = \Omega\ A$ and if the big deRham cohomology is concentrated in degree zero, then the degree properties of m_n imply that $m_n = 0$ for $n \geq 3$ and hence that $\Omega\ A$ is formal.

Let A be an arbitrary $\mathbb{C}$-algebra and $\theta \in Der_{\mathbb{C}}\ A$, the Lie algebra of $\mathbb{C}$-algebra *derivations* of A, then we define a degree preserving derivation L_θ and a degree -1 super-derivation i_θ on $\Omega\ A$

$$\begin{tikzcd}
\Omega^{n-1}\ A \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^n\ A \arrow[l, bend left, "i_\theta"] \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^{n+1}\ A \arrow[l, bend left, "i_\theta"] \arrow[loop below, "L_\theta"]
\end{tikzcd}$$

defined by the rules

$$\begin{cases} L_\theta(a) = \theta(a) & L_\theta(da) = d\,\theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

for all $a \in A$. In this generality we again have the fundamental identity

$$L_\theta = i_\theta \circ d + d \circ i_\theta$$

as both sides are degree preserving derivations on $\Omega\ A$ and they agree on all the generators a and da for $a \in A$.

Lemma 7.2 *Let $\theta, \gamma \in Der_{\mathbb{C}}\ A$, then we have on $\Omega\ A$ the following identities of operators*

$$\begin{cases} L_\theta \circ i_\gamma - i_\gamma \circ L_\theta = [L_\theta, i_\gamma] & = i_{[\theta,\gamma]} = i_{\theta\circ\gamma-\gamma\circ\theta} \\ L_\theta \circ L_\gamma - L_\gamma \circ L_\theta = [L_\theta, L_\gamma] & = L_{[\theta,\gamma]} = L_{\theta\circ\gamma-\gamma\circ\theta} \end{cases}$$

Proof. Consider the first identity. By definition both sides are degree -1 super-derivations on $\Omega\ A$ so it suffices to check that they agree on generators. Clearly, both sides give 0 when evaluated on $a \in A$ and for da we have

$$(L_\theta \circ i_\gamma - i_\gamma \circ L_\theta)da = L_\theta\ \gamma(a) - i_\gamma\ d\ \theta(a) = \theta\ \gamma(a) - \gamma\ \theta(a) = i_{[\theta,\gamma]}(da)$$

A similar argument proves the second identity. □

Let Q be a quiver on k vertices $\{v_1, \ldots, v_k\}$, then we can define an *Euler derivation* E on $\mathbb{C}Q$ by the rules that

$$E(v_i) = 0\ \forall 1 \leq i \leq k \qquad \text{and} \qquad E(a) = a\ \forall a \in Q_a$$

By induction on the length $l(p)$ of an oriented path p in the quiver Q one easily verifies that $E(p) = l(p)p$. By the lemma above we have all the necessary ingredients to redo the argument in example 7.19.

Theorem 7.10 *For a quiver Q on k vertices, the noncommutative differential forms $\Omega\ \mathbb{C}Q$ is formal. In fact, we have for the big deRham cohomology*

$$\begin{cases} H^0_{big}\ \mathbb{C}Q & \simeq \mathbb{C} \times \ldots \times \mathbb{C}\ (k\ \textit{factors}) \\ H^n_{big}\ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

For $\omega \in \Omega^i\ A$ and $\omega' \in \Omega^j\ A$ we define the *super-commutator* to be

$$[\omega, \omega'] = \omega\omega' - (-1)^{ij}\omega'\omega$$

That is, it is the usual commutator unless both i and j are odd in which case it is the sum $\omega\omega' + \omega'\omega$.

As the differential d is a super-derivation on $\Omega\ A$ we have that

$$d([\omega, \omega']) = [d\omega, \omega'] + (-1)^i [\omega, d\omega']$$

and therefore the differential maps the *subspaces* of super-commutators to subspaces of super-commutators. Therefore, if we define

$$\mathtt{DR}^n\ A = \frac{\Omega^n\ A}{\sum_{i=0}^n [\Omega^i\ A, \Omega^{n-i}\ A]}$$

Then the `dgalg`-structure on $\Omega\ A$ induces one on the complex

$$\mathtt{DR}^0\ \mathtt{A} \xrightarrow{d} \mathtt{DR}^1\ \mathtt{A} \xrightarrow{d} \mathtt{DR}^2\ \mathtt{A} \xrightarrow{d} \dots$$

which is called the *Karoubi complex* of A.

We define the *noncommutative de Rham cohomology groups* of A to be the homology of the Karoubi complex, that is

$$H^n_{dR}\ A = \frac{Ker\ \mathtt{DR}^n\ A \xrightarrow{d} \mathtt{DR}^{n+1}\ A}{Im\ \mathtt{DR}^{n-1}\ A \xrightarrow{d} \mathtt{DR}^n\ A}$$

Example 7.20 (Noncommutative de Rham cohomology of $\mathbb{C} \times \mathbb{C}$) Recall that the product on $\Omega\ \mathbb{C} \times \mathbb{C}$ is given by the formula

$$(\alpha e + \beta f) de^n (\gamma e + \delta f) de^m = \begin{cases} (\alpha\gamma e + \beta\delta f) de^{n+m} & \text{when } n \text{ is even} \\ (\alpha\delta e + \beta\gamma f) de^{n+m} & \text{when } n \text{ is odd} \end{cases}$$

If m is odd, then we deduce from this that the commutator

$$[\alpha e + \beta f, (\gamma e + \delta f) de^m] = (\alpha - \beta)(\gamma e - \delta f) de^m$$

and hence we can write any element of $\Omega^m\ \mathbb{C} \times \mathbb{C} = (\mathbb{C} \times \mathbb{C}) de^m$ as a (super) commutator, whence

$$\mathtt{DR}^m\ \mathbb{C} \times \mathbb{C} = 0 \qquad \text{when } m \text{ is odd.}$$

On the other hand, if m is even then any commutator with k even

$$[(\alpha e + \beta f) de^k, (\gamma e + \delta f) de^{m-k}] = 0$$

whereas if k is odd we have

$$[(\alpha e + \beta f) de^k, (\gamma e + \delta f) de^{m-k}] = (\alpha\delta + \beta\gamma) de^m$$

As a consequence the space of super-commutators in $\Omega^m\ \mathbb{C}\times\mathbb{C}$ is one dimensional and therefore

$$\mathtt{DR}^m\ \mathbb{C}\times\mathbb{C}=\mathbb{C} \qquad \text{when } m \text{ is even and } > 0.$$

Thus, the Karoubi complex of $\mathbb{C}\times\mathbb{C}$ has the following form

$$\mathbb{C}\times\mathbb{C} \xrightarrow{d} 0 \xrightarrow{d} \mathbb{C} \xrightarrow{d} 0 \xrightarrow{d} \mathbb{C} \xrightarrow{d} 0 \xrightarrow{d} \ \dots$$

and therefore we have for the noncommutative de Rham cohomology groups

$$\mathtt{H}^n_{dR}\ \mathbb{C}\times\mathbb{C} = \begin{cases} \mathbb{C}\times\mathbb{C} & \text{when } n=0 \\ 0 & \text{when } n \text{ is odd} \\ \mathbb{C} & \text{when } n \text{ is even and } > 0. \end{cases}$$

Example 7.21 (Noncommutative de Rham cohomology of $\mathbb{C}\langle x,y\rangle$) Consider again the Eulerian derivation E on $\mathbb{C}\langle x,y\rangle$ and the operators L_E and i_E on $\Omega\ \mathbb{C}\langle x,y\rangle$. Repeating the above argument that d is compatible with the subspaces of super-commutators for i_E and L_E we see that we have induced operations

$$\begin{tikzcd} \mathtt{DR}^{n-1} \arrow[r, bend left, "d"] \arrow[loop below, "L_E"] & \mathtt{DR}^{n} \arrow[l, bend left, "i_E"] \arrow[r, bend left, "d"] \arrow[loop below, "L_E"] & \mathtt{DR}^{n+1} \arrow[l, bend left, "i_E"] \arrow[loop below, "L_E"] \end{tikzcd}$$

We have again that L_E is an isomorphism on $\mathtt{DR}^n\ \mathbb{C}\langle x,y\rangle$ whenever $n\geq 1$ and again we deduce from the equality $L_E = i_E\circ d + d\circ i_E$ that

$$\mathtt{H}^n_{dR}\ \mathbb{C}\langle x,y\rangle = \begin{cases} \mathbb{C} & \text{when } n=0\ , \\ 0 & \text{when } n\geq 1. \end{cases}$$

Theorem 7.11 *Let Q be a quiver on k vertices, then the Karoubi complex of $\mathbb{C}Q$ is acyclic. In particular,*

$$\begin{cases} H^0_{dR}\ \mathbb{C}Q & \simeq \mathbb{C}\times\ldots\times\mathbb{C} \text{ } (k \text{ factors}) \\ H^n_{dR}\ \mathbb{C}Q & \simeq 0 \qquad \forall n\geq 1 \end{cases}$$

So far we have considered differential forms with respect to the basefield $\mathbb{C}$. Sometimes it is useful to consider only the *relative differential forms* on A with respect to a subalgebra B. These can be defined as follows.

Let $\overline{A}_B$ be the cokernel of the inclusion $B \hookrightarrow A$ in the category $B-\texttt{bimod}$ of *bimodules* over B. We define the space of relative differential forms of degree n with respect to B to be

$$\Omega_B^n\ A = A \otimes_B \underbrace{\overline{A}_B \otimes_B \ldots \otimes_B \overline{A}_B}_{n}$$

By definition $\Omega_B^n\ A$ is the quotient space of $\Omega^n\ A$ by the relations

$$\begin{aligned}(a_0, \ldots, a_{i-1}b, a_i, \ldots, a_n) =& (a_0, \ldots, a_{i-1}, ba_i, \ldots, a_n)\\ (a_0, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n) =& 0\end{aligned}$$

for all $b \in B$ and $1 \leq i \leq n$. One verifies that the multiplication and differential defined on $\Omega\ A$ are compatible with these relations, making $\Omega_B\ A$ an object in $\texttt{dgalg}$. Moreover, there is a canonical epimorphism

$$\Omega\ A \twoheadrightarrow \Omega_B\ A \qquad \text{in } \texttt{dgalg}.$$

We will now determine the kernel. First we give the universal property for $\Omega_B\ A$. Given $\Gamma = \oplus \Gamma^n$ in $\texttt{dgalg}$ and an algebra map $A \xrightarrow{f} \Gamma^0$ such that $d(f\ B) = 0$, then there is a unique morphism in $\texttt{dgalg}$ making the diagram commute

$$\begin{tikzcd}
\Omega_B\ A \arrow[r, dotted, "\exists f_*"] & \Gamma \\
A \arrow[u, hook] \arrow[r, "f"] & \Gamma^0 \arrow[u, hook]
\end{tikzcd}$$

Indeed, by the universal property of $\Omega\ A$ there is a unique morphism $\Omega\ A \xrightarrow{f_*} \Gamma$ in $\texttt{dgalg}$ extending f given by

$$f_*(a_0 da_1 \ldots da_n) = f(a_0)d(f(a_1)) \ldots d(f(a_n)).$$

If $d(f\ B) = 0$ then one verifies that f_* is compatible with the relations defining $\Omega_B\ A$, proving the universal property.

Proposition 7.5 *For a subalgebra B of A we have an isomorphism in* $\texttt{dgalg}$

$$\Omega_B\ A = \frac{\Omega\ A}{\Omega\ A\ d(B)\ \Omega\ A}$$

Proof. The ideal generated by $d(B)$ is closed under d and therefore the quotient is an object in $\texttt{dgalg}$ with the same universal property as $\Omega_B\ A$. $\square$

An important special case is when $B = \mathbb{C} \times \ldots \times \mathbb{C}$ is the subalgebra of $\mathbb{C}Q$ generated by the vertex-idempotents. In this case we will denote

$$\Omega_{rel}\ \mathbb{C}Q = \Omega_B\ \mathbb{C}Q$$

and call it the relative differential forms on Q.

Lemma 7.3 *Let Q be a quiver on k vertices, then a basis for $\Omega^n_{rel}\ \mathbb{C}Q$ is given by the elements*

$$p_0 dp_1 \ldots dp_n$$

where p_i is an oriented path in the quiver such that length $p_0 \geq 0$ and length $p_i \geq 1$ for $1 \leq i \leq n$ and such that the starting point of p_i is the endpoint of p_{i+1} for all $1 \leq i \leq n-1$.

Proof. Clearly $l(p_i) \geq 1$ when $i \geq 1$ or p_i would be a vertex-idempotent whence in B. Let v be the starting point of p_i and w the end point of p_{i+1} and assume that $v \neq w$, then

$$p_i \otimes_B p_{i+1} = p_i v \otimes_B w p_{i+1} = p_i v w \otimes_B p_{i+1} = 0$$

from which the assertion follows. $\square$

We define the *big relative de Rham* cohomology groups of A with respect to B to be the cohomology of the complex

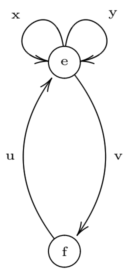
$$\Omega^0_B\ A \xrightarrow{d} \Omega^1_B\ A \xrightarrow{d} \Omega^2_B\ A \xrightarrow{d} \ \ldots$$

that is,

$$H^n_B\ A = \frac{Ker\ \Omega^n\ A \xrightarrow{d} \Omega^{n+1}\ A}{Im\ \Omega^{n-1}\ A \xrightarrow{d} \Omega^n\ A}$$

In the case of path algebras of quivers, we can use the grading by length op paths and the Eulerian derivation to compute these relative de Rham groups.

Example 7.22 (Big relative de Rham cohomology) Let $\mathbb{M}$ (resp. $\mathbb{C}\times\mathbb{C}$) be the path algebras of the quivers



resp.

The Eulerian derivation E on $\mathbb{M}$ is defined by

$$E(e) = E(f) = 0 \quad E(x) = x \quad E(y) = y \quad E(u) = u \quad \text{and} \quad E(v) = v.$$

Observe that E respects all relation holding in $\mathbb{M}$ and so is indeed a $\mathbb{C} \times \mathbb{C}$- derivation of $\mathbb{M}$.

As before we define a degree preserving derivation L_E and a degree -1 super-derivation i_E on $\Omega_{rel}\ \mathbb{M} = \Omega_{\mathbb{C}\times\mathbb{C}}\ \mathbb{M}$ by the rules

$$\begin{cases} L_E(a) = E(a) & L_E(da) = dE(a) \\ i_E(a) = 0 & i_E(da) = E(a) \end{cases}$$

for all $a \in \mathbb{M}$. We have the equality

$$L_E = i_E \circ d + d \circ i_E$$

and arguing as before we obtain that

$$\mathtt{H}^n_{rel}\ \mathbb{M} = \begin{cases} \mathbb{C} \times \mathbb{C} & \text{when } n = 0, \\ 0 & \text{when } n \geq 1. \end{cases}$$

Theorem 7.12 *Let Q be a quiver on k vertices, then the relative differential forms $\Omega_{rel}\ \mathbb{C}Q$ is a formal differentially graded algebra. In fact,*

$$\begin{cases} H^0_{rel}\ \mathbb{C}Q & \simeq \mathbb{C} \times \ldots \times \mathbb{C} \ \textit{(k factors)} \\ H^n_{rel}\ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

We can repeat the construction of the Karoubi complex verbatim for relative differential operators and define a *relative Karoubi complex*

$$\mathtt{DR}^0_B\ A \xrightarrow{d} \mathtt{DR}^1_B\ A \xrightarrow{d} \mathtt{DR}^2_B\ A \xrightarrow{d} \ldots$$

where

$$\mathtt{DR}^n_B\ A = \frac{\Omega^n_B\ A}{\sum_{i=0}^n [\ \Omega^i_B\ A, \Omega^{n-i}_B\ A\ ]}$$

Clearly , we then define the *noncommutative relative de Rham cohomology groups* of A with respect to B to be the homology of this complex

$$\mathtt{H}^n_{B,dR}\ A = \frac{Ker\ \mathtt{DR}^n_B\ A \xrightarrow{d} \mathtt{DR}^{n+1}_B\ A}{Im\ \mathtt{DR}^{n-1}_B\ A \xrightarrow{d} \mathtt{DR}^n_B\ A}$$

Let $\theta \in Der_B\ \mathtt{A}$, that is θ is a $\mathbb{C}$-derivation on A such that $\theta(b) = 0$ for every $b \in B$. Then, as

$$L_\theta(db) = d\ \theta(b) = 0 \qquad \text{and} \qquad i_\theta(db) = \theta(b) = 0$$

we see that the operators L_θ and i_θ can be defined on the relative forms

$$\Omega_B\ A = \frac{\Omega\ A}{\Omega\ A\ dB\ \Omega\ A}$$

and also on the relative Karoubi complex. Again, these induced operators satisfy the identities of lemma 7.2. In the special case of the Eulerian derivation E on the path algebra $\mathbb{C}Q$ we see that $E \in Der_B\ \mathbb{C}Q$ and hence we have the following result.

Theorem 7.13 *Let Q be a quiver on k vertices. Then, the relative Karoubi complex is acyclic. That is,*

$$\begin{cases} H^0_{rel,dR}\ \mathbb{C}Q & \simeq \mathbb{C} \times \ldots \times \mathbb{C}\ \textit{(k factors)} \\ H^n_{rel,dR}\ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

7.7 Symplectic structure

Let Q be a quiver on k vertices $\{v_1, \ldots, v_k\}$. We will determine the first terms in the relative Karoubi complex. Define

$$\mathtt{dR}^n_{rel}\ \mathbb{C}Q = \frac{\Omega^n_{rel}\ \mathbb{C}Q}{\sum_{i=0}^n [\ \Omega^i_{rel}\ \mathbb{C}Q, \Omega^{n-i}\ \mathbb{C}Q\ ]}$$

In the commutative case, dR^0 are the functions on the manifold and dR^1 the 1-forms. We will characterize the *noncommutative functions* and *noncommutative 1-forms* in the case of quivers.

Recall that a *necklace word* w in the quiver Q is an equivalence class of an oriented cycle $c = a_1 \ldots a_l$ of length $l \geq 0$ in Q, where $c \sim c'$ if c' is obtained from c by cyclically permuting the composing arrows a_i.

Lemma 7.4 *A $\mathbb{C}$-basis for the noncommutative functions*

$$\mathtt{dR}^0_{rel}\ \mathbb{C}Q \simeq \frac{\mathbb{C}Q}{[\ \mathbb{C}Q, \mathbb{C}Q\ ]}$$

are the necklace words in the quiver Q.

Proof. Let $\mathbb{W}$ be the $\mathbb{C}$-space spanned by all necklace words w in Q and define a linear map

$$\mathbb{C}Q \xrightarrow{n} \mathbb{W} \qquad \begin{cases} p \mapsto w_p & \text{if } p \text{ is a cycle} \\ p \mapsto 0 & \text{if } p \text{ is not} \end{cases}$$

for all oriented paths p in the quiver Q, where w_p is the necklace word in Q determined by the oriented cycle p. Because $w_{p_1 p_2} = w_{p_2 p_1}$ it follows that the commutator subspace $[\mathbb{C}Q, \mathbb{C}Q]$ belongs to the kernel of this map. Conversely, let

$$x = x_0 + x_1 + \ldots + x_m$$

be in the kernel where x_0 is a linear combination of non-cyclic paths and x_i for $1 \leq i \leq m$ is a linear combination of cyclic paths mapping to the same necklace word w_i, then $n(x_i) = 0$ for all $i \geq 0$. Clearly, $x_0 \in [\mathbb{C}Q, \mathbb{C}Q]$ as we can write every noncyclic path $p = a.p' = a.p' - p'.a$ as a commutator. If $x_i = a_1 p_1 + a_2 p_2 + \ldots + a_l p_l$ with $n(p_i) = w_i$, then $p_1 = q.q'$ and $p_2 = q'.q$ for some paths q, q' whence $p_1 - p_2$ is a commutator. But then, $x_i = a_1(p_1 - p_2) + (a_2 - a_1)p_2 + \ldots + a_l p_l$ is a sum of a commutator and a linear combination of strictly fewer elements. By induction, this shows that $x_i \in [\mathbb{C}Q, \mathbb{C}Q]$. $\square$

If we fix a dimension vector α, then taking traces defines a map

$$dR^0\ \mathbb{C}Q \xrightarrow{tr} \mathbb{C}[\mathtt{rep}_\alpha\ Q]$$

whence noncommutative functions determine $GL(\alpha)$-invariant commutative functions on the representation space $\mathtt{rep}_\alpha\ Q$ and hence commutative functions on the quotient varieties $\mathtt{iss}_\alpha\ Q$. In fact, we have seen that the image $tr(dR^0\ \mathbb{C}Q)$ generates the ring of polynomial invariants $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)} = \mathbb{C}[\mathtt{iss}_\alpha\ Q]$.

Lemma 7.5 $\mathtt{dR}^1_{rel}\ \mathbb{C}Q$ *is isomorphic as* $\mathbb{C}$*-space to*

$$\bigoplus_{j \xleftarrow{a} i} v_i.\mathbb{C}Q.v_j\ da = \bigoplus_{j \xleftarrow{a} i} \left(i \overset{\cdots}{\longleftarrow} j \right) d \left(j \xleftarrow{a} i \right)$$

Proof. If $p.q$ is not a cycle, then $pdq = [p, dq]$ and so vanishes in $\mathtt{dR}^1_{rel}\ \mathbb{C}Q$ so we only have to consider terms pdq with $p.q$ an oriented cycle in Q. For any three paths p, q and r in Q we have the equality

$$[p.qdr] = pqdr - qd(rp) + qrdp$$

whence in $\mathtt{dR}^1_{rel}\ \mathbb{C}Q$ we have relations allowing to reduce the length of the differential part

$$qd(rp) = pqdr + qrdp$$

so $\mathtt{dR}^1_{rel}\ \mathbb{C}Q$ is spanned by terms of the form pda with $a \in Q_a$ and $p.a$ an oriented cycle in Q. Therefore, we have a surjection

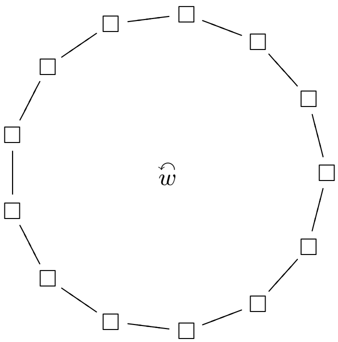
$$\Omega^1_{rel}\ \mathbb{C}Q \twoheadrightarrow \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} v_i.\mathbb{C}Q.v_j\ da$$

By construction, it is clear that $[\Omega^0_{rel}\ \mathbb{C}Q, \Omega^1_{rel}\ \mathbb{C}Q]$ lies in the kernel of this map and using an argument as in the lemma above one shows also the converse inclusion. $\square$

Example 7.23 **($\mathtt{dR}^i_{rel}\ \mathbb{M}$)** Take the path algebra $\mathbb{M}$ of the quiver of example 7.22. Noncommutative *functions* on $\mathbb{M}$ are the 0-forms, which is by definition the quotient space

$$dR^0_{rel}\ \mathbb{M} = \frac{\mathbb{M}}{[\ \mathbb{M}, \mathbb{M}\ ]}$$

If p is an oriented path of length ≥ 1 in the quiver with different begin- and endpoint, then we can write p as a concatenation $p = p_1p_2$ with p_i an oriented path of length ≥ 0 such that $p_2p_1 = 0$ in $\mathbb{M}$. As $[p_1, p_2] = p_1p_2 - p_2p_1 = 0$ in $dR^0_{rel}\ \mathbb{M}$ we deduce that the space of noncommutative functions on $\mathbb{M}$ has as $\mathbb{C}$-basis the *necklace words* w



where each bead is this time one of the elements

$$\boxed{\bullet} = x \qquad \boxed{\circ} = y \quad \text{and} \quad \boxed{\blacktriangledown} = uv$$

together with the necklace words of length zero e and f. Each necklace word w corresponds to the equivalence class of the words in $\mathbb{M}$ obtained from multiplying the beads in the indicated orientation and and two words in $\{x, y, u, v\}$ in $\mathbb{M}$ are said to be equivalent if they are identical up to cyclic permutation of the terms.

Substituting each bead with the $n \times n$ matrices specified before and taking traces we get a map

$$dR^0_{rel}\ \mathbb{M} = \frac{\mathbb{M}}{[\ \mathbb{M}, \mathbb{M}\ ]} \xrightarrow{tr} \mathbb{C}[\mathtt{rep}_\alpha\ \mathbb{M}]$$

Hence, noncommutative functions on $\mathbb{M}$ induce ordinary functions on *all* the representation spaces $\mathtt{rep}_\alpha\ \mathbb{M}$ and these functions are $GL(\alpha)$-invariant. Moreover, the image of this map generates the ring of polynomial invariants as we mentioned before.

Next, we consider *noncommutative* 1-*forms* on $\mathbb{M}$ which are by definition elements of the space

$$dR^1_{rel}\ \mathbb{M} = \frac{\Omega^1_{rel}\ \mathbb{M}}{[\ \mathbb{M}, \Omega^1_{rel}\ \mathbb{M}\ ]}$$

Recall that $\Omega^1_{rel}\ \mathbb{M}$ is spanned by the expressions $p_0 dp_1$ with p_0 resp. p_1 oriented paths in the quiver of length ≥ 0 resp. ≥ 1 and such that the starting point of p_0 is the end point of p_1. To form $dR^1_{rel}\ \mathbb{M}$ we have to divide out expressions such as

$$[\ p, p_0 dp_1\ ] = pp_0 dp_1 + p_0 p_1 dp - p_0 d(p_1 p)$$

That is, if we have connecting oriented paths p_2 and p_1 both of length ≥ 1 we have in $dR^1_{rel}\ \mathbb{M}$

$$p_0 d(p_1 p_2) = p_2 p_0 dp_1 + p_0 p_1 dp_2$$

and by iterating this procedure whenever the differential term is a path of length ≥ 2 we can represent each class in $dR^1_{rel}\ \mathbb{M}$ as a combination from

$$\mathbb{M}e\ dx + \mathbb{M}e\ dy + \mathbb{M}e\ du + \mathbb{M}f\ dv$$

Now, $\mathbb{M}e = e\mathbb{M}e + f\mathbb{M}e$ and let $p \in f\mathbb{M}e$. Then, we have in $dR^1_{rel}\ \mathbb{M}$

$$d(xp) = p\ dx + x\ dp$$

but by our description of $\Omega^1\ \mathbb{M}$ the left hand term is zero as is the second term on the right, whence $p\ dx = 0$. A similar argument holds replacing x by y. As for u, let $p \in e\mathbb{M}e$, then we have in $dR^1_{rel}\ \mathbb{M}$

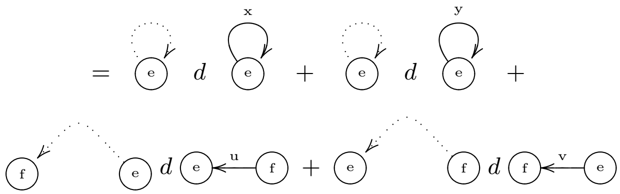
$$d(up) = p\ du + u\ dp$$

and again the left-hand and the second term on the right are zero whence $p\ du = 0$. An analogous result holds for v and $p \in f\mathbb{M}f$. Therefore, we have the description of noncommutative 1-forms on $\mathbb{M}$

$$dR^1_{rel}\ \mathbb{M} = e\mathbb{M}e\ dx + e\mathbb{M}e\ dy + f\mathbb{M}e\ du + e\mathbb{M}f\ dv$$

That is, in graphical terms

$dR^1_{rel}\ \mathbb{M} =$



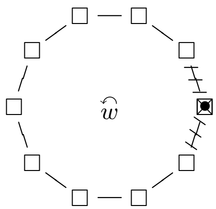
Using the above descriptions of $\mathtt{dR}^i_{rel}\ \mathbb{C}Q$ for $i = 0, 1$ and the differential $\mathtt{dR}^0_{rel}\ \mathbb{C}Q \xrightarrow{d} \mathtt{dR}^1_{rel}\ \mathbb{C}Q$ we can define *partial differential operators* associated to any arrow $j \xleftarrow{a} i$ in Q.

$$\frac{\partial}{\partial a} : \mathtt{dR}^0_{rel}\ \mathbb{C}Q \longrightarrow v_i \mathbb{C}Q v_j \qquad \text{by} \qquad df = \sum_{a \in Q_a} \frac{\partial f}{\partial a} da$$

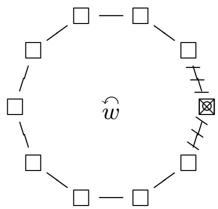
To take the partial derivative of a necklace word w with respect to an arrow a, we run through w and each time we encounter a we open the necklace by removing that occurrence of a and then take the sum of all the paths obtained.

Example 7.24 For the path algebra $\mathbb{M}$ we have the partial differential operators

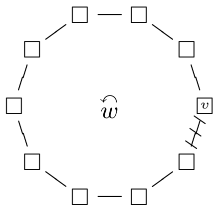
$$\frac{\partial w}{\partial x} = \sum_{\boxed{\bullet}}$$



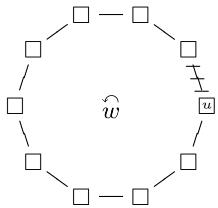
$$\frac{\partial w}{\partial y} = \sum_{\boxed{\circ}}$$



$$\frac{\partial w}{\partial u} = \sum_{\boxed{\blacktriangledown}}$$



$$\frac{\partial w}{\partial v} = \sum_{\boxed{\blacktriangledown}}$$



Recall that a *symplectic structure* on a (commutative) manifold M is given by a closed differential 2-form. The non-degenerate 2-form ω gives a canonical isomorphism

$$T\ M \simeq T^*\ M$$

that is, between vector fields on M and differential 1-forms. Further, there is a unique $\mathbb{C}$-linear map from functions f on M to vectorfields ξ_f by the requirement that $-df = i_{\xi_f}\omega$ where i_ξ is the contraction of n-forms to $n-1$-forms using the vectorfield ξ. We can make the functions on M into a *Poisson algebra* by defining

$$\{f, g\} = \omega(\xi_f, \xi_g)$$

and one verifies that this bracket satisfies the Jacobi and Leibnitz identities.

The *Lie derivative* L_ξ with respect to ξ is defined by the Cartan homotopy formula

$$L_\xi\ \varphi = i_\xi d\varphi + di_\xi\varphi$$

for any differential form φ. A vectorfield ξ is said to be *symplectic* if it preserves the symplectic form, that is, $L_\xi\omega = 0$. In particular, for any function f on M we have that ξ_f is symplectic. Moreover the assignment

$$f \longrightarrow \xi_f$$

defines a Lie algebra morphism from the functions $\mathcal{O}(M)$ on M equipped with the Poisson bracket to the Lie algebra of symplectic vectorfields, $Vect_\omega\ M$. Moreover, this map fits into the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(M) \longrightarrow Vect_\omega\ M \longrightarrow H^1_{dR}\ M \longrightarrow 0$$

Recall the definition of the double quiver Q^d of a quiver Q given in section 5.5 by assigning to every arrow $a \in Q_a$ an arrow a^* in Q^d in the opposite direction.

Definition 7.3 *The canonical* noncommutative symplectic structure *on the double quiver Q^d is given by the element*

$$\omega = \sum_{a \in Q_a} dada^* \in \mathtt{dR}^2_{rel}\ \mathbb{C}Q^d$$

We will use ω to define a correspondence between the noncommutative 1-forms $\mathtt{dR}^1_{rel}\ \mathbb{C}Q^d$ and the *noncommutative vectorfields* which are define to be $B = \mathbb{C}^{Q_v}$-derivations of the path algebra $\mathbb{C}Q^d$. Recall that if $\theta \in Der_B\ \mathbb{C}Q^d$ we define operators L_θ and i_θ on $\Omega\ \mathbb{C}Q^d$ and on $\mathtt{dR}\ \mathbb{C}Q^d$ by the rules

$$\begin{cases} L_\theta(a) = \theta(a) & L_\theta(da) = d\theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

and that the following identities are satisfied for all $\theta, \gamma \in Der_B\ \mathbb{C}Q^d$

$$[L_\theta, L_\gamma] = L_{[\theta,\gamma]} \qquad \text{and} \qquad [i_\theta, i_\gamma] = i_{[\theta,\gamma]}$$

These operators allow us to define a linear map

$$Der_B\ \mathbb{C}Q \xrightarrow{\tau} \mathtt{dR}^1_{rel}\ \mathbb{C}Q \qquad \text{by} \qquad \tau(\theta) = i_\theta(\omega)$$

We claim that this is an isomorphism. Indeed, every B-derivation θ on $\mathbb{C}Q^d$ is fully determined by its image on the arrows in Q^d which satisfy if $a = \textcircled{j} \xleftarrow{a} \textcircled{i}$

$$\theta(a) = \theta(v_j a v_i) = v_j \theta(a) v_i \in v_j \mathbb{C}Q^d v_i$$

so determines an element $\theta(a)da^* \in \mathtt{dR}^1_{rel}\ \mathbb{C}Q^d$. Further, we compute

$$\begin{aligned} i_\theta(\omega) &= \sum_{a \in Q_a} i_\theta(da)da^* - i_\theta(da^*)da \\ &= \sum_{a \in Q_a} \theta(a)da^* - \theta(a^*)da \end{aligned}$$

which lies in $\mathtt{dR}^1_{rel}\ \mathbb{C}Q^d$. As both B-derivations and 1-forms are determined by their coefficients, τ is indeed bijective.

Example 7.25 For the path algebra of the double quiver $\mathbb{M}$,the analog of the classical isomorphism $T\ M \simeq T^*\ M$ is the isomorphism

$$Der_{\mathbb{C}\times\mathbb{C}}\ \mathbb{M} \xrightarrow{i_{.}\omega} dR^1_{rel}\ \mathbb{M}$$

as for any $\mathbb{C} \times \mathbb{C}$-derivation θ we have

$$\begin{aligned} i_\theta\ \omega &= i_\theta(dx)dy - dx i_\theta(dy) + i_\theta(du)dv - du i_\theta(dv) \\ &= \theta(x)dy - dx\theta(y) + \theta(u)dv - du\theta(v) \\ &\equiv \theta(x)dy - \theta(y)dx + \theta(u)dv - \theta(v)du \end{aligned}$$

and using the relations in $\mathbb{M}$ we can easily prove that any $\mathbb{C} \times \mathbb{C}$ derivation on $\mathbb{M}$ must satisfy

$$\theta(x) \in e\mathbb{M}e \quad \theta(y) \in e\mathbb{M}e \quad \theta(u) \in e\mathbb{M}f \quad \theta(v) \in f\mathbb{M}e$$

so the above expression belongs to $dR^1_{rel}\ \mathbb{M}$. Conversely, any θ defined by its images on the generators x, y, u and v by

$$-\theta(y)dx + \theta(x)dy - \theta(v)du + \theta(u)dv \in dR^1_{rel}\ \mathbb{M}$$

induces a derivation on $\mathbb{M}$.

In analogy with the commutative case we define a derivation $\theta \in Der_B\ \mathbb{C}Q^d$ to be *symplectic* if and only if $L_\theta \omega = 0 \in \mathtt{dR}^2_{rel}\ \mathbb{C}Q^d$. We will denote the subspace of symplectic derivations by $Der_\omega\ \mathbb{C}Q$. It follows from the noncommutative analog of the Cartan homotopy equality

$$L_\theta = i_\theta \circ d + d \circ i_\theta$$

and the fact that ω is a closed form, that $\theta \in Der_\omega\ \mathbb{C}Q^d$ implies

$$L_\theta \omega = d i_\theta \omega = \tau(\theta) = 0$$

That is, $\tau(\theta)$ is a closed form which by the acyclicity of the Karoubi complex shows that it must be an exact form. That is we have an isomorphism of exact sequences of $\mathbb{C}$-vectorspaces

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "="'] & \mathtt{dR}^0_{rel}\ \mathbb{C}Q^d \arrow[r, "d"] \arrow[d, "\simeq"'] & (\mathtt{dR}^1_{rel}\ \mathbb{C}Q)_{exact} \arrow[r] \arrow[d, "\tau^{-1}"'] & 0 \\
0 \arrow[r] & B \arrow[r] & \frac{\mathbb{C}Q^d}{[\mathbb{C}Q^d, \mathbb{C}Q^d]} \arrow[r] & Der_\omega\ \mathbb{C}Q^d \arrow[r] & 0
\end{tikzcd}$$

n the next section we will show that this is in fact an exact sequence of Lie algebras.

7.8 Necklace Lie algebras

Let Q be a quiver on k vertices, Q^d its double and $\omega = \sum_{a \in Q_a} da da^*$ the canonical symplectic form on $\mathbb{C}Q^d$. Recall from last section the definition of the partial differential operators $\frac{\partial}{\partial a}$ for an arrow a in Q^d.

Definition 7.4 *The* Kontsevich bracket *on the necklace words in* Q^d, $\mathtt{dR}^0_{rel}\ \mathbb{C}Q^d$ *is defined to be*

$$\{w_1, w_2\}_K = \sum_{a \in Q_a} \left(\frac{\partial w_1}{\partial a}\frac{\partial w_2}{\partial a^*} - \frac{\partial w_1}{\partial a^*}\frac{\partial w_2}{\partial a}\right) \mathtt{mod}\ [\mathbb{C}Q^d, \mathbb{C}Q^d]$$

That is, to compute $\{w_1, w_2\}_K$ we consider for every arrow $a \in Q_a$ all occurrences of a in w_1 and a^* in w_2. We then open up the necklaces removing these factors and gluing the open ends together to form a new necklace word. We then replace the roles of a^* and a and redo this operation (with a minus sign), see figure 7.4. Finally, we add all the obtained necklace words.

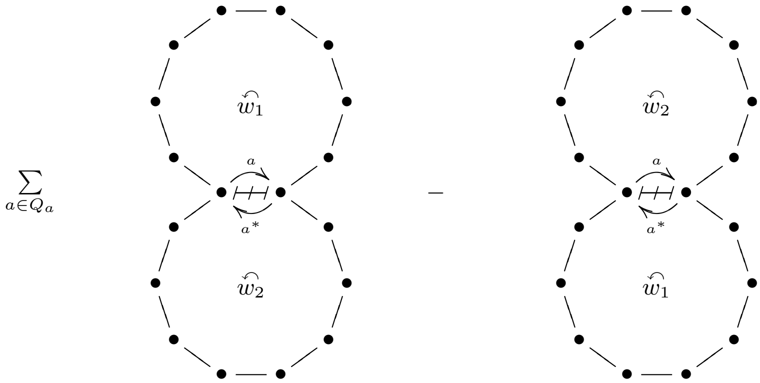
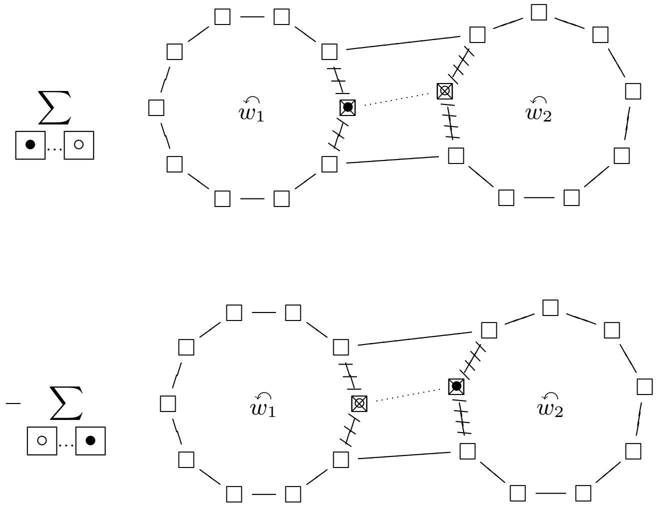
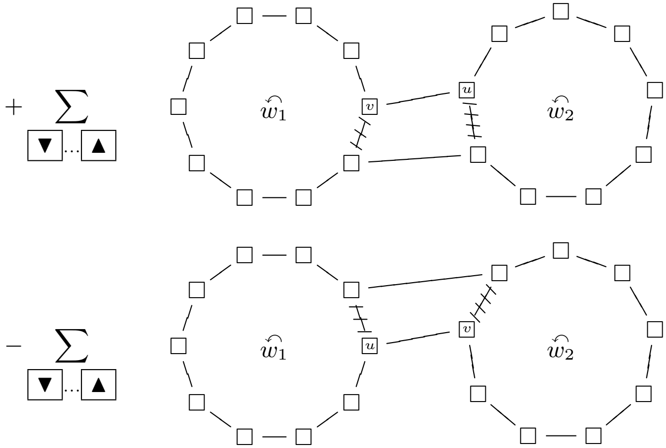


Figure 7.4: Kontsevich bracket $\{w_1, w_2\}_K$.

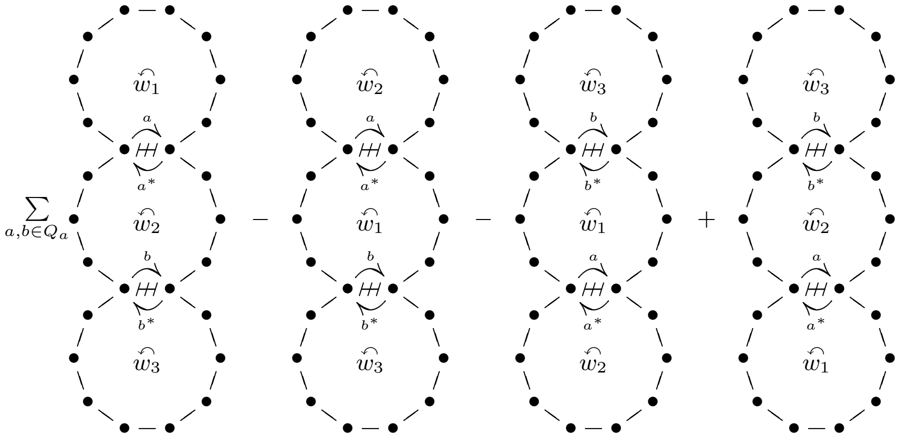
Example 7.26 For the path algebra $\mathbb{M}$ the canonical symplectic form is $\omega = dxdy + dudv$. Using the above graphical description we have that the Kontsevich bracket $\{w_1, w_2\}_K$ is equal to



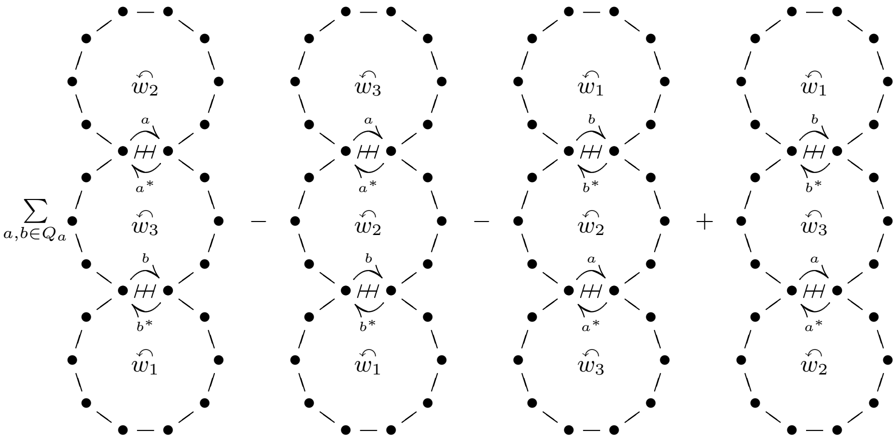


Using this graphical description of the Kontsevich bracket, it is an enjoyable exercise to verify that the bracket turns $\mathtt{dR}^0_{rel}\ \mathbb{C}Q^d$ into a Lie algebra. That is, for all necklace words w_i, the bracket satisfies the *Jacobi identity*

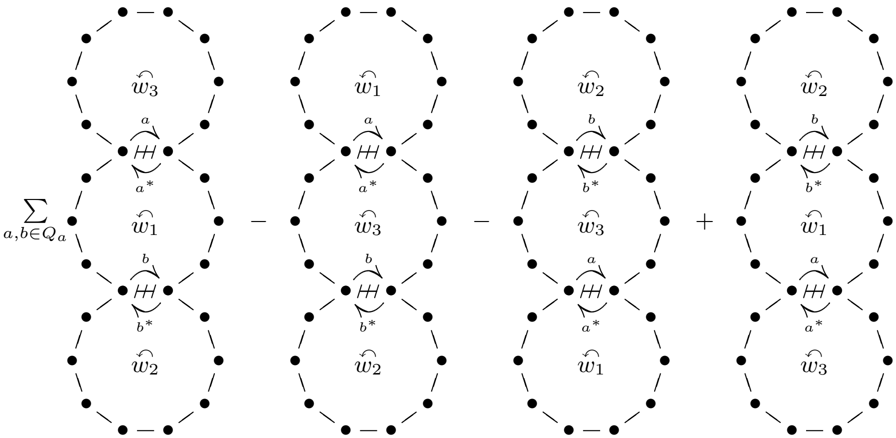
$$\{\{w_1,w_2\}_K,w_3\}_K+\{\{w_2,w_3\}_K,w_1\}_K+\{\{w_3,w_1\}_K,w_2\}_K=0$$



$\{\{w_1,w_2\}_K,w_3\}_K$



$$\{\{w_2, w_3\}_K, w_1\}_K$$



Term $1a$ vanishes against $2c$, term $1b$ against $3d$, $1c$ against $3a$, $1d$ against $2b$, $2a$ against $3c$ and $2d$ against $3b$.

Recall the exact commutative diagram from last section

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "="'] & \mathtt{dR}^0_{rel}\ \mathbb{C}Q^d \arrow[r, "d"] \arrow[d, "\simeq"'] & (\mathtt{dR}^1_{rel}\ \mathbb{C}Q)_{exact} \arrow[r] \arrow[d, "\tau^{-1}"'] & 0 \\ 0 \arrow[r] & B \arrow[r] & \dfrac{\mathbb{C}Q^d}{[\mathbb{C}Q^d, \mathbb{C}Q^d]} \arrow[r] & Der_\omega\ \mathbb{C}Q^d \arrow[r] & 0 \end{tikzcd}$$

Clearly, the symplectic derivations $Der_\omega\ \mathbb{C}Q^d$ are equipped with a Lie algebra structure via $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - \theta_2 \circ \theta_1$.

For every necklace word w we have a derivation $\theta_w = \tau^{-1} dw$ which is defined by

$$\begin{cases} \theta_w(a) & = \frac{\partial w}{\partial a^*} \\ \theta_w(a^*) & = -\frac{\partial w}{\partial a} \end{cases}$$

With this notation we get the following interpretations of the Kontsevich bracket

$$\{w_1, w_2\}_K = i_{\theta_{w_1}}(i_{\theta_{w_2}}\omega) = L_{\theta_{w_1}}(w_2) = -L_{\theta_{w_2}}(w_1)$$

where the next to last equality follows because $i_{\theta_{w_2}}\omega = dw_2$ and the fact that $i_{\theta_{w_1}}(dw) = L_{\theta_{w_1}}(w)$ for any w. More generally, for any B-derivation θ and any necklace word w we have the equation

$$i_\theta(i_{\theta_w}\omega) = L_\theta(w)$$

By the commutation relations for the operators L_θ and i_θ we have for all B-derivations θ_i the equalities

$$\begin{aligned} L_{\theta_1} i_{\theta_2} i_{\theta_3}\omega - i_{\theta_2} i_{\theta_3} L_{\theta_1}\omega &= [L_{\theta_1}, i_{\theta_2}] i_{\theta_3}\omega + i_{\theta_2} L_{\theta_1} i_{\theta_3}\omega \\ &\quad - i_{\theta_2} L_{\theta_1} i_{\theta_3}\omega + i_{\theta_2}[L_{\theta_1}, i_{\theta_3}]\omega \\ &= i_{[\theta_1,\theta_2]} i_{\theta_3}\omega + i_{\theta_2} i_{[\theta_1,\theta_3]}\omega \end{aligned}$$

By the homotopy formula we have $L_{\theta_w}\omega = 0$ for every necklace word w, whence we get

$$L_{\theta_{w_1}} i_{\theta_2} i_{\theta_3}\omega = i_{[\theta_{w_1},\theta_2]} i_{\theta_3}\omega + i_{\theta_2} i_{[\theta_{w_1},\theta_3]}\omega$$

Take $\theta_2 = \theta_{w_2}$, then the left hand side is equal to

$$\begin{aligned} L_{\theta_{w_1}} i_{\theta_{w_2}} i_{\theta_3}\omega &= -L_{\theta_{w_1}} i_{\theta_3} i_{\theta_{w_2}}\omega \\ &= -L_{\theta_{w_1}} L_{\theta_3} w_2 \end{aligned}$$

whereas the last term on the right equals

$$
\begin{aligned}
i_{\theta_{w_2}} i_{[\theta_{w_1},\theta_3]}\omega &= -i_{[\theta_{w_1},\theta_3]} i_{\theta_{w_2}}\omega \\
&= -L_{[\theta_{w_1},\theta_3]} w_2 = -L_{theta_{w_1}} L_{\theta_3} w_2 + L_{\theta_3} L_{\theta_{w_1}} w_2
\end{aligned}
$$

and substituting this we obtain that

$$
\begin{aligned}
i_{[\theta_{w_1},\theta_{w_2}]} i_{\theta_3}\omega &= -L_{\theta_{w_1}} L_{\theta_3} w_2 + L_{\theta_{w_1}} L_{\theta_3} w_2 - L_{\theta_3} L_{\theta_{w_1}} w_2 \\
&= -L_{\theta_3} L_{\theta_{w_1}} w_2 = -L_{\theta_3}\{w_1, w_2\}_K \\
&= -i_{\theta_3} i_{\theta_{\{w_1,w_2\}_K}}\omega = i_{\theta_{\{w_1,w_2\}_K}} i_{\theta_3}\omega
\end{aligned}
$$

Finally, if we take $\theta = [\theta_{w_1}, \theta_{w_2}] - \theta_{\{w_1,w_2\}_K}$ we have that $i_\theta\omega$ is a closed 1-form and that $i_\theta i_{\theta_3}\omega = -i_{\theta_3} i_\theta\omega = 0$ for all θ_3. But then by the homotopy formula $L_{\theta_3} i_\theta \omega = 0$ whence $i_\theta\omega = 0$, which finally implies that $\theta = 0$. This concludes the proof of :

Theorem 7.14 *With notations as before, the necklace words* $\mathtt{dR}^0_{rel}\ \mathbb{C}Q^d$ *is a Lie algebra for the Kontsevich bracket, and the sequence*

$$
0 \longrightarrow B \longrightarrow \mathtt{dR}^0_{rel}\ \mathbb{C}Q^d \xrightarrow{\tau^{-1}d} Der_\omega\ \mathbb{C}Q^d \longrightarrow 0
$$

is an exact sequence (hence a central extension) of Lie algebras.

This result will be crucial in the study of coadjoint orbits in the final chapter.

References

The formal structure on smooth affine varieties is due to M. Kapranov [39] and its extension to module varieties is due to L. Le Bruyn [62]. The semi-invariants of quivers have been obtained independently by various people, among which A. Schofield, M. Van den Bergh [75], H. Derksen, J. Weyman [24] and M. Domokos , A. Zubkov [25]. We follow here the approach of [75]. The results of section 7.3 are due to A. Schofield [72] or based on discussions with him. The results of section 7.4 are due to L. Le Bruyn [57] and is inspired by prior work of B. Westbury [83]. The results of section 7.5 are due to J. Cuntz and D. Quillen [23]. The acyclicity results of section 7.6 for the free algebra is due to M. Kontsevich [46] and in the quiver case to R. Bockland , L. Le Bruyn [11] and V. Ginzburg [30] independently as is the description of the necklace Lie algebra.

8 — Moduli Spaces

So far, the more interesting applications of the theory developed in the previous chapter have not been to noncommutative manifolds but to families $(Y_n)_n$ of varieties in which the role of Quillen-smooth algebras is replaced by Cayley-smooth algebras and where the sum-maps are replaced by gluing into a larger space. In this chapter we give the details of Ginzburg's coadjoint-orbit result for Calogero-Moser phase space which was the first instance of such a situation.

$$\begin{tikzcd} & Hilb_n\ \mathbb{C}^2 \arrow[dl, "\pi"'] \arrow[dr, "H"] & \\ S^n\ \mathbb{C}^2 \arrow[rr, dotted] & & Calo_n \end{tikzcd}$$

Here, $Hilb_n\ \mathbb{C}^2$ is the *Hilbert scheme* of n points in the complex plane $\mathbb{C}^2$ which is a desingularization of the symmetric power $S^n\ \mathbb{C}^2$. On the other hand, $S^n\ \mathbb{C}^2$ can be viewed as the special fiber of a family of which the general fiber is isomorphic to $Calo_n$, the phase space of Calogero-Moser particles. $Calo_n$ is a smooth affine variety and we will see that it is isomorphic to $\mathtt{triss}_n\ A_n$ for some Cayley-smooth order $A_n \in \mathtt{alg@n}$. Surprisingly, forgetting the complex structure, $Calo_n$ itself is diffeomorphic (as a $\mathbb{C}^\infty$-manifold) to $Hilb_n\ \mathbb{C}^2$ via rotations of hyper-Kähler structures.

George Wilson has shown that the varieties $Calo_n$ can be glued together to form an infinite dimensional manifold, the *adelic Grassmannian*

$$\bigsqcup_n Calo_n = Gr^{ad}$$

The adelic Grassmannian can be identified with the isomorphism classes of right ideals in the first Weyl algebra $A_1(\mathbb{C})$ and as the automorphism group of the Weyl algebra acts on this set with countably many orbits it was conjectured that every $Calo_n$ might be a coadjoint orbit. This fact was proved by Victor Ginzburg who showed that, indeed,

$$Calo_n \hookrightarrow \mathfrak{g}^*$$

for some infinite dimensional Lie algebra $\mathfrak{g}$ which is nothing but the necklace Lie algebra of the path algebra of a double quiver naturally associated to the situation. After reading through this chapter, the reader will have no problem to prove for herself that every quiver variety, in the sense of Nakajima, is diffeomorphic to a coadjoint orbit of a necklace Lie algebra.

8.1 Moment maps

In section 2.8 we have studied in some detail the *real moment map* of m-tuples of $n \times n$ matrices. In this section we will first describe the obvious extension to representation spaces of quivers and then to prove the properties of the *real moment map* for moduli spaces of θ-semistable representations.

We fix a quiver Q on k vertices $\{v_1, \ldots, v_k\}$ and a dimension vector $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}^k$. We take the standard Hermitian inproduct on each of the vertex spaces $\mathbb{C}^{\oplus a_i}$ and this induces the standard *operator inner product* on every arrow-component of $\mathtt{rep}_\alpha\ Q$. That is, for every arrow

$$\textcircled{j} \xleftarrow{\ a\ } \textcircled{i} \qquad \text{we define} \qquad (V_a, W_a) = tr(V_a W_a^*)$$

on the component $Hom_{\mathbb{C}}(\mathbb{C}^{\oplus a_i}, \mathbb{C}^{\oplus a_j})$ for all $V, W \in \mathtt{rep}_\alpha$ and where W_a^* is the *adjoint matrix* $(\overline{w_{ji}})_{i,j}$ of $W_a = (w_{ij})_{i,j}$. The *Hermitian inproduct* on $\mathtt{rep}_\alpha\ Q$ is defined to be

$$(V, W) = \sum_{a \in Q_a} tr(V_a W_a^*)$$

The *maximal compact subgroup* of the basechange group $GL(\alpha) = \prod_{i=1}^k GL_{a_i}$ is the *multiple unitary group*

$$U(\alpha) = \prod_{i=1}^k U_{a_i}$$

which preserves the Hermitian inproduct under the basechange action as subgroup of $GL(\alpha)$. The Lie algebra $Lie\ U(\alpha)$ is the algebra of multiple skew-Hermitian matrices

$$Lie\ U(\alpha) = \bigoplus_{j=1}^k iHerm_{a_j} = \{\ h = (h_1, \ldots, h_k)\ \mid\ h_j = -h_j^*\ \}$$

and the induced action of $Lie\ U(\alpha)$ on $\mathtt{rep}_\alpha\ Q$ is given by the rule

$$(h.V)_a = h_j V_a - V_a h_i \qquad \text{for} \qquad \textcircled{j} \xleftarrow{\ a\ } \textcircled{i}$$

for all $V \in \mathtt{rep}_\alpha\ Q$. This action allows us to define the *real moment map* μ for the action of $U(\alpha)$ on the representation space $\mathtt{rep}_\alpha\ Q$ by the assignment

$$\mathtt{rep}_\alpha\ Q \xrightarrow{\ \mu\ } (iLie\ U(\alpha))^* \qquad\qquad V \longrightarrow (h \mapsto i(h.V, V))$$

That is, the moment map is determined by

$$\begin{aligned}(h.V, V) &= \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} tr(h_j V_a V_a^* - V_a h_i V_a^*) \\ &= \sum_{v_i \in Q_v} tr(h_i(\sum_{\textcircled{i} \xleftarrow{a} \bigcirc} V_a V_a^* \ - \sum_{\bigcirc \xleftarrow{a} \textcircled{i}} V_a^* V_a\ ))\end{aligned}$$

Using the nondegeneracy of the Killing form on *Lie* $U(\alpha)$ we have the identification

$$\mu^{-1}(\underline{0}) = \{V \in \mathtt{rep}_\alpha\ Q \mid \sum_{\textcircled{\scriptsize i} \xleftarrow{a} \circ} V_a V_a^* = \sum_{\circ \xleftarrow{a} \textcircled{\scriptsize i}} V_a^* V_a \qquad \forall v_i \in Q_v\}$$

The *real moment map* $\mu_{\mathbb{R}}$ is then defined to be

$$\mathtt{rep}_\alpha\ Q \xrightarrow{\mu_{\mathbb{R}}} Lie\ U(\alpha) \qquad V \mapsto i[V, V^*] = i(\sum_{\textcircled{\scriptsize j} \xleftarrow{a} \circ} V_a V_a^* - \sum_{\circ \xleftarrow{a} \textcircled{\scriptsize j}} V_a^* V_a)_j$$

Reasoning as in section 2.8 we can prove the following moment map description of the isomorphism classes of semi-simple α-dimensional representations of Q.

Theorem 8.1 *There are natural one-to-one correspondences between*

1. *points of* $\mathtt{iss}_\alpha\ Q$*, and*
2. $U(\alpha)$*-orbits in* $\mu_{\mathbb{R}}^{-1}(\underline{0})$.

Next, we will prove a similar result to describe the points of $M_\alpha^{ss}(Q, \theta)$, the moduli space of θ-semistable α-dimensional representations of Q, introduced and studied in section 4.8. Fix, an integral k-tuple $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ with associated *character*

$$GL(\alpha) \xrightarrow{\chi_\theta} \mathbb{C}^* \qquad\qquad g = (g_1, \ldots, g_k) \mapsto \prod_{i=1}^k det(g_i)^{t_i}$$

We have seen in section 4.8 that in order to describe $M_\alpha^{ss}(Q, \theta)$ we consider the extended representation space $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$. We introduce a function N on this extended space replacing the norm in the above discussion.

$$\mathtt{rep}_\alpha\ Q \oplus \mathbb{C} \xrightarrow{N} \mathbb{R}_+ \qquad\qquad (V, z) \mapsto |z| e^{\frac{1}{2}\| V \|^2}$$

where $\| V \|$ is the norm coming from the Hermitian inproduct on $\mathtt{rep}_\alpha\ Q$. Sometimes, the function N is called the *Kähler potential* for the inproduct on $\mathtt{rep}_\alpha\ Q$. We will investigate the properties of N.

Lemma 8.1 *Let* X *be a closed subvariety of* $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ *disjoint from* $rep'_\alpha\ Q = \{(V, 0) \mid V \in \mathtt{rep}_\alpha\ Q\} \hookrightarrow \mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$*. Then, the restriction of* N *to* X *is* proper *and therefore achieves its minimum.*

Proof. Because X and $rep'_\alpha\ Q$ are disjoint closed subvarieties of $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$, there is a polynomial $f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}] = \mathbb{C}[\mathtt{rep}_\alpha\ Q][z]$ such that $f \mid X = 1$ and $f \mid rep'_\alpha\ Q = 0$. That is, X is contained in the hypersurface

$$\mathbb{V}(f-1) = \mathbb{V}(zP_1(V) + \ldots + z^n P_n(V) - 1) \hookrightarrow \mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$$

where the $P_i \in \mathbb{C}[\mathtt{rep}_\alpha\ Q]$.

Now, N is *proper* if the inverse images $N^{-1}([0,r])$ are compact for all $r \in \mathbb{R}_+$, that is, there exist constants r_1 and r_2 depending on X and r such that

$$N(z,V) \leq r \qquad \text{implies} \qquad |z| \leq r_1 \text{ and } \|V\| \leq r_2.$$

We can always take $r_1 = r$ so we only need to bound $\|V\|$. If $|z| \leq re^{-\frac{1}{2}\|V\|^2}$, then we have that

$$|zP_1(V) + \ldots + z^n P_n(V)| \leq r|P_1(V)|e^{-\frac{1}{2}\|V\|^2} + \ldots + r^n|P_n(V)|e^{-\frac{n}{2}\|V\|^2}$$

Choose r_2, depending on r and P_i such that the condition

$$\|V\| > r_2 \qquad \text{implies that} \qquad |P_i(V)| < \frac{1}{n} r^{-i} e^{\frac{i}{2}\|V\|^2} \qquad \forall 1 \leq i \leq n$$

But then if $\|V\| > r_2$, we have $|zP_1(V) + \ldots + z^n P_n(V)| < 1$ and so (V,z) does not belong to X. $\square$

Recall that $GL(\alpha)$ acts on the extended representation space $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ via

$$g.(V,z) = (g.V, \chi_\theta^{-1}(g)z)$$

Lemma 8.2 *Let $\mathcal{O}$ be a $GL(\alpha)$-orbit in the extended representation space $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ which is disjoint from $rep'_\alpha\ Q$. Then, if the restriction of N to $\mathcal{O}$ achieves its minimum, then $\mathcal{O}$ is a closed orbit.*

Proof. Assume that N achieves its minimum in the point $V_z = (V,z) \in \mathcal{O}$. If $\mathcal{O}$ is not a closed orbit we can by the Hilbert criterium find a one-parameter subgroup λ of $GL(\alpha)$ such that

$$\lim_{t \mapsto 0} \lambda(t).V_z \notin \mathcal{O}$$

and the limit exists in $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$. Decompose the representation $V = \sum_{n \in \mathbb{Z}} V_n$ into eigenspaces with respect to the one-parameter subgroup λ, that is,

$$\lambda(t).V = \sum_{n \in \mathbb{Z}} t^n V_n$$

Because the limit exists, we have that $V_n = 0$ whenever $n < 0$ and $\theta(\lambda) \leq 0$. Because the limit is not contained in $\mathcal{O}$ we have that $V_n \neq 0$ for some $n > 0$. Further, by conjugating λ if necessary we may assume that the weightspace decomposition $V = \sum_n V_n$ is orthogonal with respect to the inproduct in $\mathtt{rep}_\alpha\ Q$.

Using these properties we then have that

$$N(\lambda(t).(V,z)) = |z| e^{\frac{1}{2}|V_0|^2} |t|^{-\theta(\lambda)} e^{\frac{1}{2}\sum_{n>0}|t|^n \|V_n\|^2}$$

This expression will decrease when t approaches zero, contradicting the assumption that the minimum of $N \mid \mathcal{O}$ was achieved in (V,z). This contradiction implies that $\mathcal{O}$ must be a closed orbit. $\square$

Recall from section 4.8 that an orbit $\mathcal{O}(V,z)$ is closed and disjoint from $rep'_\alpha\ Q$ for some $z \in \mathbb{C}^*$ if and only if V is the direct sum of θ-stable representations of Q. Recall the real moment map

$$\mathtt{rep}_\alpha\ Q \xrightarrow{\mu} (iLie\ U(\alpha))^*$$

And consider the special real valued function $d\chi_\theta$ on $Lie\ U(\alpha)$ which is the restriction to $Lie\ U(\alpha)$ of the differential of $GL(\alpha) \xrightarrow{\chi_\theta} \mathbb{C}^*$ at the identity element (which takes real values). In fact, for any $m = (m_1, \ldots, m_k) \in Lie\ GL(\alpha) = M_\alpha(\mathbb{C})$ we have that

$$d\chi_\theta(m) = \sum_{v_j \in Q_v} t_j tr(m_j) = \sum_{v_j \in Q_v} tr(m_j t_j \mathbb{1}_{a_j})$$

With these notations we have the promised extension to moduli spaces of θ-semistable representations.

Theorem 8.2 *There are natural one-to-one correspondences between*

1. *points of $M_\alpha^{ss}(Q,\theta)$, and*
2. *$U(\alpha)$-orbits in $\mu^{-1}(d\chi_\theta)$.*

Proof. Let $V_z = (V,z) \in \mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ with $z \neq 0$. For any $h = (h_1, \ldots, h_k)$ in $iLie\ U(\alpha)$ we define the functions

$$\begin{aligned} m_V(h) &= \frac{d}{dt}\ |_{t=0}\ log\ N(e^{th}.V_z) \\ &= (h.V, V) - d\chi_\theta(h) \end{aligned}$$

$$\begin{aligned} m_V^{(2)}(h) &= \frac{d}{dt}\ |_{t=0}\ log\ N(e^{th}.V_z) \\ &= 2\|h.V\|^2 \end{aligned}$$

The function m_V is the zero map if and only if the restriction of N to the orbit $\mathcal{O}(V_z)$ has a critical point at V_z. As the basechange action of $U(\alpha)$ on the extended representation space $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ preserves the Kähler potential N, N induces a function on the quotient $\mathcal{O}(V_z)/U(\alpha)$. The formula for $m_V^{(2)}$ shows that this function is strictly convex (except in directions along the fibers $\{(V,c) \mid c \in \mathbb{C}\}$ where it is linear). Hence, a critical point is a minimum and there can be at most one such critical point. From the lemmas above we have that N has a minimum on $\mathcal{O}(V_z)$ if and only if $\mathcal{O}(V_z)$ is a closed orbit, which in its turn is equivalent to V being the direct sum of θ-stable representations, whence determining a point of $M_\alpha^{ss}(Q,\theta)$. $\square$

Finally, for any $h \in iLie\ U(\alpha)$ we have the formulas

$$\mu(V)(h) = i \sum_{v_i \in Q_v} tr(h_i(\sum_{\textcircled{i} \xleftarrow{a} \bigcirc} V_a V_a^* - \sum_{\bigcirc \xleftarrow{a} \textcircled{i}} V_a^* V_a))$$

$$d\chi_\theta(h) = \sum_{v_i \in Q_v} tr(h_i t_i \mathbb{1}_{a_i})$$

whence by nondegeneracy of the Killing form, the equality $\mu(V) = d\chi_\theta$ is equivalent to the conditions

$$\sum_{\textcircled{j} \xleftarrow{a} \bigcirc} V_a V_a^* - \sum_{\bigcirc \xleftarrow{a} \textcircled{j}} V_a^* V_a = it_j \mathbb{1}_{a_j} \qquad \forall v_j \in Q_v$$

We can assign to $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ the element $i\theta\mathbb{1}_\alpha = (it_1\mathbb{1}_{a_1}, \ldots, it_k\mathbb{1}_{a_k}) \in Lie\ U(\alpha)$. We then can rephrase the results of this section as

Theorem 8.3 *There are natural identification between the spaces*

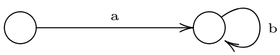
$$\mathtt{iss}_\alpha\ Q \longleftrightarrow \mu_{\mathbb{R}}^{-1}(\underline{0})/U(\alpha)$$

and between the spaces

$$M_\alpha^{ss}(Q,\theta) \longleftrightarrow \mu_{\mathbb{R}}^{-1}(i\theta\mathbb{1}_\alpha)/U(\alpha)$$

8.2 Dynamical systems

In this chapter we will illustrate what we have learned on the simplest *wild quiver* Q which is neither Dynkin nor extended Dynkin

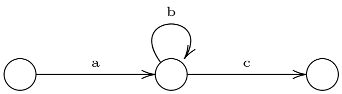


In this section we will show that the representation theory of this quiver is of importance in system theory.

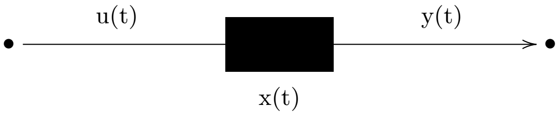
A *linear time invariant dynamical system* Σ is determined by the following system of differential equations

$$\begin{cases} \frac{dx}{dt} & = Bx + Au \\ y & = Cx. \end{cases} \tag{8.1}$$

Here, $u(t) \in \mathbb{C}^m$ is the *input* or *control* of the system at tome t, $x(t) \in \mathbb{C}^n$ the *state* of the system and $y(t) \in \mathbb{C}^p$ the *output* of the system Σ. *Time invariance* of Σ means that the matrices $A \in M_{n \times m}(\mathbb{C})$, $B \in M_n(\mathbb{C})$ and $C \in M_{p \times n}(\mathbb{C})$ are constant, that is $\Sigma = (A, B, C)$ is a representation of the quiver $\tilde{Q}$



of dimension vector $\alpha = (m, n, p)$. The system Σ can be represented as a *black box*



which is in a certain state $x(t)$ that we can try to change by using the input controls $u(t)$. By reading the output signals $y(t)$ we can try to determine the state of the system.

Recall that the *matrix exponential* e^B of any $n \times n$ matrix B is defined by the infinite series

$$e^B = \mathbb{1}_n + B + \frac{B^2}{2!} + \ldots + \frac{B^m}{m!} + \ldots$$

The importance of this construction is clear from the fact that e^{Bt} is the *fundamental matrix* for the homogeneous differential equation $\frac{dx}{dt} = Bx$. That is, the columns of e^{Bt} are a basis for the n-dimensional space of solutions of the equation $\frac{dx}{dt} = Bx$.

Motivated by this, we look for a solution to equation (8.1) as the form $x(t) = e^{Bt}g(t)$ for some function $g(t)$. Substitution gives the condition

$$\frac{dg}{dt} = e^{-Bt}Au \quad \text{whence} \quad g(\tau) = g(\tau_0) + \int_{\tau_0}^{\tau} e^{-Bt}Au(t)dt.$$

Observe that $x(\tau_0) = e^{B\tau_0}g(\tau_0)$ and we obtain the solution of the linear dynamical system $\Sigma = (A, B, C)$:

$$\begin{cases} x(\tau) & = e^{(\tau-\tau_0)B}x(\tau_0) + \int_{\tau_0}^{\tau} e^{(\tau-t)B}Au(t)dt \\ y(\tau) & = Ce^{B(\tau-\tau_0)}x(\tau_0) + \int_{\tau_0}^{\tau} Ce^{(\tau-t)B}Au(t)dt. \end{cases}$$

Differentiating we see that this is indeed a solution and it is the unique one having a prescribed starting state $x(\tau_0)$. Indeed, given another solution $x_1(\tau)$ we have that $x_1(\tau) - x(\tau)$ is a solution to the homogeneous system $\frac{dx}{dt} = Bt$, but then

$$x_1(\tau) = x(\tau) + e^{\tau B} e^{-\tau_0 B}(x_1(\tau_0) - x(\tau_0)).$$

We call the system Σ *completely controllable* if we can steer any starting state $x(\tau_0)$ to the zero state by some control function $u(t)$ in a finite time span $[\tau_0, \tau]$. That is, the equation

$$0 = x(\tau_0) + \int_{\tau_0}^{\tau} e^{(\tau_0 - t)B} Au(t)dt$$

has a solution in τ and $u(t)$. As the system is time-invariant we may always assume that $\tau_0 = 0$ and have to satisfy the equation

$$0 = x_0 + \int_0^{\tau} e^{tB} Au(t)dt \quad \text{for every} \quad x_0 \in \mathbb{C}^n \tag{8.2}$$

Consider the *control matrix* $c(\Sigma)$ which is the $n \times mn$ matrix

$$c(\Sigma) = \begin{bmatrix} \boxed{A} & \boxed{BA} & \boxed{B^2A} & \cdots & \boxed{B^{n-1}A} \end{bmatrix}$$

Assume that $rk\ c(\Sigma) < n$ then there is a non-zero state $s \in \mathbb{C}^n$ such that $s^{tr} c(\Sigma) = 0$, where s^{tr} denotes the transpose (row column) of s. Because B satisfies the characteristic polynomial $\chi_B(t)$, B^n and all higher powers B^m are linear combinations of $\{\mathbb{1}_n, B, B^2, \ldots, B^{n-1}\}$. Hence, $s^{tr} B^m A = 0$ for all m. Writing out the power series expansion of e^{tB} in equation (8.2) this leads to the contradiction that $0 = s^{tr} x_0$ for all $x_0 \in \mathbb{C}^n$. Hence, if $rk\ c(\Sigma) < n$, then Σ is not completely controllable.

Conversely, let $rk\ c(\Sigma) = n$ and assume that Σ is not completely controllable. That is, the space of all states

$$s(\tau, u) = \int_0^{\tau} e^{-tB} Au(t)dt$$

is a proper subspace of $\mathbb{C}^n$. But then, there is a non-zero state $s \in \mathbb{C}^n$ such that $s^{tr} s(\tau, u) = 0$ for all τ and all functions $u(t)$. Differentiating this with respect to τ we obtain

$$s^{tr} e^{-\tau B} Au(\tau) = 0 \quad \text{whence} \quad s^{tr} e^{-\tau B} A = 0 \tag{8.3}$$

for any τ as $u(\tau)$ can take on any vector. For $\tau = 0$ this gives $s^{tr} A = 0$. If we differentiate (8.3) with respect to τ we get $s^{tr} B e^{-\tau B} A = 0$ for all τ and for $\tau = 0$ this gives $s^{tr} BA = 0$. Iterating this process we show that $s^{tr} B^m A = 0$ for any m, whence

$$s^{tr} \begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A \end{bmatrix} = 0$$

contradicting the assumption that $rk\ c(\Sigma) = n$. That is, we have proved :

Proposition 8.1 *A linear time-invariant dynamical system Σ determined by the matrices (A,B,C) is completely controllable if and only if $rk\ c(\Sigma)$ is maximal.*

We say that a state $x(\tau)$ at time τ is *unobservable* if $Ce^{(\tau-t)B}x(\tau)=0$ for all t. Intuitively this means that the state $x(\tau)$ cannot be detected uniquely from the output of the system Σ. Again, if we differentiate this condition a number of times and evaluate at $t=\tau$ we obtain the conditions

$$Cx(\tau)=CBx(\tau)=\ldots=CB^{n-1}x(\tau)=0.$$

We say that Σ is *completely observable* if the zero state is the only unobservable state at any time τ. Consider the *observation matrix* $o(\Sigma)$ of the system Σ which is the $pn\times n$ matrix

$$o(\Sigma)=\begin{bmatrix}C^{tr} & (CB)^{tr} & \ldots & (CB^{n-1})^{tr}\end{bmatrix}^{tr}$$

An analogous argument as in the proof of proposition 8.1 gives us that a linear time-invariant dynamical system Σ determined by the matrices (A,B,C) is completely observable if and only if $rk\ o(\Sigma)$ is maximal.

Assume we have two systems Σ and Σ', determined by matrix triples from $\mathtt{rep}_\alpha\ Q=M_{n\times m}(\mathbb{C})\times M_n(\mathbb{C})\times M_{p\times n}(\mathbb{C})$ producing the same output $y(t)$ when given the same input $u(t)$, for all possible input functions $u(t)$. We recall that the output function y for a system $\Sigma=(A,B,C)$ is determined by

$$y(\tau)=Ce^{B(\tau-\tau_0)}x(\tau_0)+\int_{\tau_0}^{\tau}Ce^{(\tau-t)B}Au(t)dt.$$

Differentiating this a number of times and evaluating at $\tau=\tau_0$ as in the proof of proposition 8.1 equality of input/output for Σ and Σ' gives the conditions

$$CB^iA=C'{B'}^iA' \quad \text{for all} \quad i.$$

But then, we have for any $v\in\mathbb{C}^{mn}$ that $c(\Sigma)(v)=0\Leftrightarrow c(\Sigma')(v)=0$ and we can decompose $\mathbb{C}^{pn}=V\oplus W$ such that the restriction of $c(\Sigma)$ and $c(\Sigma')$ to V are the zero map and the restrictions to W give isomorphisms with $\mathbb{C}^n$. Hence, there is an invertible matrix $g\in GL_n$ such that $c(\Sigma')=gc(\Sigma)$ and from the commutative diagram

$$\begin{tikzcd}
\mathbb{C}^{mn} \arrow[r, two heads, "c(\Sigma)"] \arrow[d, equal] & \mathbb{C}^n \arrow[r, hook, "o(\Sigma)"] \arrow[d, "g"] & \mathbb{C}^{pn} \arrow[d, equal] \\
\mathbb{C}^{mn} \arrow[r, two heads, "c(\Sigma')"] & \mathbb{C}^n \arrow[r, hook, "o(\Sigma')"] & \mathbb{C}^{pn}
\end{tikzcd}$$

we obtain that also $o(\Sigma')=o(\Sigma)g^{-1}$.

Consider the system $\Sigma_1 = (A_1, B_1, C_1)$ *equivalent* with Σ under the base-change matrix g. That is, $\Sigma_1 = g.\Sigma = (gA, gBg^{-1}, Cg^{-1})$. Then,

$$\left[A_1, B_1A_1, \ldots, B_1^{n-1}A_1\right] = gc(\Sigma) = c(\Sigma') = \left[A', B'A', \ldots, {B'}^{n-1}A'\right]$$

and so $A_1 = A'$. Further, as $B_1^{i+1}A_1 = {B'}^{i+1}A'$ we have by induction on i that the restriction of B_1 on the subspace of ${B'}^{i}Im(A')$ is equal to the restriction of B' on this space. Moreover, as $\sum_{i=0}^{n-1}{B'}^{i}Im(A') = \mathbb{C}^n$ it follows that $B_1 = B'$. Because $o(\Sigma') = o(\Sigma)g^{-1}$ we also have $C_1 = C'$. This finishes the proof of :

Proposition 8.2 *Let $\Sigma = (A, B, C)$ and $\Sigma' = (A', B', C')$ be two completely controllable and completely observable dynamical systems. The following are equivalent*

1. *The input/output behavior of Σ and Σ' are equal.*
2. *The systems Σ and Σ' are* equivalent, *that is, there exists an invertible matrix $g \in GL_n$ such that*

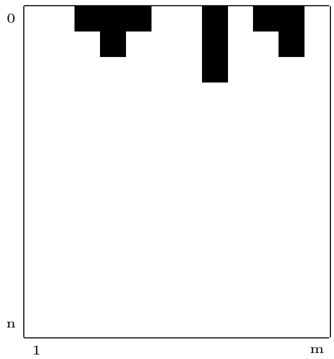
$$A' = gA, \quad B' = gBg^{-1} \quad \text{and} \quad C' = Cg^{-1}.$$

Hence, in *system identification* it is important to classify completely controllable and observable systems $\Sigma \in \mathtt{rep}_\alpha\ \tilde{Q}$ under this restricted basechange action. We will concentrate on the input part and consider *completely controllable minisystems*, that is, representations $\Sigma = (A, B) \in \mathtt{rep}_\alpha\ Q$ where $\alpha = (m, n)$ such that $c(\Sigma)$ is of maximal rank. First, we relate the system theoretic notion to that of θ-semistability for $\theta = (-n, m)$ (observe that $\theta(\alpha) = 0$).

Lemma 8.3 *If $\Sigma = (A, B) \in \mathtt{rep}_\alpha\ Q$ is θ-semistable, then Σ is completely controllable and $m \leq n$.*

Proof. If $m > n$ then $(Ker\ A, 0)$ is a proper subrepresentation of Σ of dimension vector $\beta = (dim\ Im\ A - m, 0)$ with $\theta(\beta) < 0$ so Σ cannot be θ-semistable. If Σ is not completely controllable then the subspace W of $\mathbb{C}^{\oplus n}$ spanned by the images of $A, BA, \ldots, B^{n-1}A$ has dimension $k < n$. But then, Σ has a proper subrepresentation of dimension vector $\beta = (m, k)$ with $\theta(\beta) < 0$, contradicting the θ-semistability assumption. □

We introduce a combinatorial gadget : the *Kalman code* . It is an array consisting of $(n+1) \times m$ boxes each having a position label (i, j) where $0 \leq i \leq n$ and $1 \leq j \leq m$. These boxes are ordered *lexicographically* that is $(i', j') < (i, j)$ if and only if either $i' < i$ or $i' = i$ and $j' < j$. Exactly n of these boxes are painted black subject to the rule that if box (i, j) is black, then so is box (i', j) for all $i' < i$. That is, a Kalman code looks like



We assign to a completely controllable couple $\Sigma = (A, B)$ its Kalman code $K(\Sigma)$ as follows : let $A = \begin{bmatrix} A_1 & A_2 & \ldots & A_m \end{bmatrix}$, that is A_i is the i-th column of A. Paint the box (i, j) black if and only if the column vector $B^i A_j$ is linearly independent of the column vectors $B^k A_l$ for all $(k, l) < (i, j)$.

The painted array $K(\Sigma)$ is indeed a Kalman code. Assume that box (i, j) is black but box (i', j) white for $i' < i$, then

$$B^{i'} A_j = \sum_{(k,l)<(i',j)} \alpha_{kl} B^k A_l \quad \text{but then,} \quad B^i A_j = \sum_{(k,l)<(i',j)} \alpha_{kl} B^{k+i-i'} A_l$$

and all $(k + i - i', l) < (i, l)$, a contradiction. Moreover, $K(\Sigma)$ has exactly n black boxes as there are n linearly independent columns of the control matrix $c(\Sigma)$ when Σ is completely controllable.

The Kalman code is a discrete invariant of the orbit $\mathcal{O}(\Sigma)$ under the restricted basechange action by GL_n. This follows from the fact that $B^i A_j$ is linearly independent of the $B^k A_l$ for all $(k, l) < (i, j)$ if and only if $gB^i A_j$ is linearly independent of the $gB^k A_l$ for any $g \in GL_n$ and the observation that $gB^k A_l = (gBg^{-1})^k (gA)_l$.

With $\mathtt{rep}^c_\alpha\ Q$ we will denote the open subset of $\mathtt{rep}_\alpha\ Q$ of all completely controllable couples (A, B). We consider the map

$$\begin{array}{ccc} \mathtt{rep}_\alpha\ Q & \xrightarrow{\quad\psi\quad} & M_{n\times(n+1)m}(\mathbb{C}) \\ \\ (A, B) & \mapsto & \begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A & B^nA \end{bmatrix} \end{array}$$

The matrix $\psi(A, B)$ determines a linear map $\psi_{(A,B)} : \mathbb{C}^{(n+1)m} \longrightarrow \mathbb{C}^n$ and (A, B) is a completely controllable couple if and only if the corresponding linear map $\psi_{(A,B)}$ is surjective. Moreover, there is a linear action of GL_n on $M_{n\times(n+1)m}(\mathbb{C})$ by left multiplication and the map ψ is GL_n-equivariant.

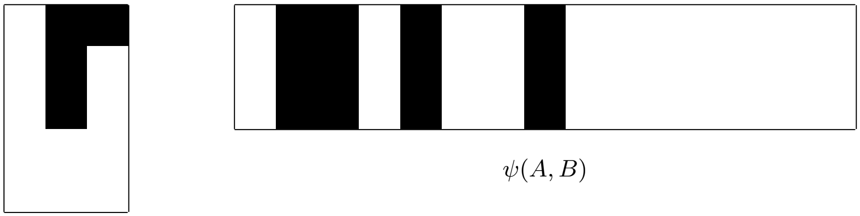


Figure 8.1: Kalman code and barcode.

The Kalman code induces a *barcode* on $\psi(A,B)$, that is the $n \times n$ minor of $\psi(A,B)$ determined by the columns corresponding to black boxes in the Kalman code, see figure 8.1 By construction this minor is an invertible matrix $g^{-1} \in GL_n$. We can choose a canonical point in the orbit $\mathcal{O}(\Sigma)$: $g.(A,B)$. It does have the characteristic property that the $n \times n$ minor of its image under ψ, determined by the Kalman code is the identity matrix $\mathbb{1}_n$. The matrix $\psi(g.(A,B))$ will be denoted by $b(A,B)$ and is called barcode of the completely controllable pair $\Sigma = (A,B)$. We claim that the barcode determines the orbit uniquely.

The map ψ is injective on the open set $\mathtt{rep}_\alpha^c\ Q$. Indeed, if

$$\begin{bmatrix} A & BA & \dots & B^n A \end{bmatrix} = \begin{bmatrix} A' & B'A' & \dots & {B'}^n A' \end{bmatrix}$$

then $A = A'$, $B \mid Im(A) = B' \mid Im(A)$ and hence by induction also

$$B \mid B^i Im(A) = B' \mid {B'}^i Im(A') \quad \text{for all } i \leq n-1.$$

But then, $B = B'$ as both couples (A,B) and (A',B') are completely controllable. Hence, the barcode $b(A,B)$ determines the orbit $\mathcal{O}(\Sigma)$ and is a point in the *Grassmannian* $Grass_n(m(n+1))$. We have

$$\begin{tikzcd} V_c \arrow[r, hook, "\psi"] \arrow[dr, "b(.)"'] & M^{max}_{n \times m(n+1)}(\mathbb{C}) \arrow[d, "\chi"] \\ & Grass_n(m(n+1)) \end{tikzcd}$$

where ψ is a GL_n-equivariant embedding and χ the orbit map. Observe that the barcode matrix $b(A,B)$ shows that the stabilizer of (A,B) is trivial. Indeed, the minor of $g.b(A,B)$ determined by the Kalman code is equal to g. Moreover, continuity of b implies that the orbit $\mathcal{O}(\Sigma)$ is closed in $\mathtt{rep}_\alpha^c\ Q$.

Consider the differential of ψ. For all $(A,B) \in \mathtt{rep}_\alpha\ Q$ and $(X,Y) \in T_{(A,B)}\ \mathtt{rep}_\alpha\ Q \simeq \mathtt{rep}_\alpha\ Q$ we have

$$(B+\epsilon Y)^j(A+\epsilon X) = B^nA + \epsilon\ (B^nX + \sum_{i=0}^{j-1} B^iYB^{n-1-i}A).$$

Therefore the differential of ψ in $(A,B) \in \mathtt{rep}_\alpha\ Q$, $d\psi_{(A,B)}(X,Y)$ is equal to

$$\begin{bmatrix} X & BX+YA & B^2X+BYA+YBA & \ldots & B^nX+\sum_{i=0}^{n-1} B^iYB^{n-1-i}A \end{bmatrix}.$$

Assume $d\psi_{(A,B)}(X,Y)$ is the zero matrix, then $X = 0$ and substituting in the next term also $YA = 0$. Substituting in the third gives $YBA = 0$, then in the fourth $YB^2A = 0$ and so on until $YB^{n-1}A = 0$. But then,

$$Y\begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A \end{bmatrix} = 0.$$

If (A,B) is a completely controllable pair, this implies that $Y = 0$ and hence shows that $d\psi_{(A,B)}$ is injective for all $(A,B) \in \mathtt{rep}^c_\alpha\ Q$. By the implicit function theorem, ψ induces a GL_n-equivariant diffeomorphism between $\mathtt{rep}^c_\alpha\ Q$ and a locally closed submanifold of $M^{max}_{n\times(n+1)m}(\mathbb{C})$. The image of this submanifold under the orbit map χ is again a manifold as all fibers are equal to GL_n. This concludes the difficult part of the *Kalman theorem* :

Theorem 8.4 *The orbit space $O_c = \mathtt{rep}^c_\alpha\ Q/GL_n$ of equivalence classes of completely controllable couples is a locally closed submanifold of dimension $m.n$ of the Grassmannian $Grass_n(m(n+1))$. In fact $\mathtt{rep}^c_\alpha\ Q \xrightarrow{b} O_c$ is a principal GL_n-bundle.*

To prove the dimension statement, consider $\mathtt{rep}^c_\alpha(K)$ the set of completely controllable pairs (A,B) having Kalman code K and let $O_c(K)$ be the image under the orbit map. After identifying $\mathtt{rep}^c_\alpha(K)$ with its image under ψ, the barcode matrix $b(A,B)$ gives a section $O_c(K) \xrightarrow{s} \mathtt{rep}^c_\alpha(K)$. In fact,

$$GL_n \times O_c(K) \longrightarrow V_c(K) \qquad (g,x) \mapsto g.s(x)$$

is a GL_n-equivariant diffeomorphism because the $n \times n$ minor determined by K of $g.b(A,B)$ is g. Consider the *generic* Kalman code K^g of figure 8.2 obtained by painting the top boxes black from left to right until one has n black boxes. Clearly $\mathtt{rep}^c_\alpha(K^g)$ is open in $\mathtt{rep}^c_\alpha$ and one deduces

$$dim\ O_c = dim\ O_c(K^g) = dim\ V_c(K^g) - dim\ GL_n = mn + n^2 - n^2 = mn.$$

The Kalman orbit space also naturally defines an order over the moduli space $M^{ss}_\alpha(Q,\theta)$. First, observe that whenever $m \leq n$ we have θ-stable representations of dimension vector $\alpha = (m,n)$ for $\theta = (-n,m)$. Then,

$$dim\ M^{ss}_\alpha(Q,\theta) = dim\ \mathtt{rep}_\alpha\ Q - dim\ GL(\alpha) + 1 = n^2 + mn - n^2 - m^2 + 1 = m(n-m)+1$$

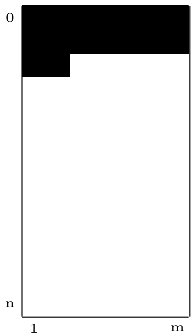


Figure 8.2: Generic Kalman code.

By the lemma we have that $\mathtt{rep}^{ss}_\alpha\ Q$ is an open subset of $\mathtt{rep}^{c}_\alpha\ Q$ and let O_{ss} be the open subset of O_c it determines. Then, the natural quotient map

$$O_{ss} \twoheadrightarrow M^{ss}_\alpha(Q,\theta)$$

is generically a principal PGL_m-fibration, so determines a central simple algebra over the function field of $M^{ss}_\alpha(Q,\theta)$.

In particular, if $m=1$ then $O_{ss} \simeq M^{ss}_\alpha(Q,\theta)$ and both are isomorphic to $\mathbb{A}^n$ and the orbits are parametrized by an old acquaintance, the *companion matrix* and its canonical *cyclic vector*

$$A = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & & & & x_n \\ -1 & 0 & & & x_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & x_2 \\ & & & -1 & x_1 \end{bmatrix}$$

Trivial as this case seems, we will see that it soon gets interesting if we consider its extension to the double quiver Q^d and to deformed preprojective algebras.

8.3 Deformed preprojective algebras

Recall the construction of *deformed preprojective algebras* given in section 5.5. Let Q be a quiver on k vertices and Q^d its *double quiver* , that is to each arrow $a \in Q_a$ we add an arrow a^* with the reverse orientation in Q^d_a and define the *commutator element* $c = \sum_{a \in Q_a} [a, a^*]$ in the path algebra

$\mathbb{C}Q^d$. For a weight $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ we define the *deformed preprojective algebra*

$$\Pi_\lambda = \frac{\mathbb{C}Q^d}{c - \lambda}$$

In this section we will give an outline of the determination of the dimension vectors of simple Π_λ-representations due to W. Crawley-Boevey [21].

We know already that a dimension vector $\alpha = (a_1, \ldots, a_k)$ can be the dimension vector of a Π_λ-representation only if $\lambda.\alpha = 0$, so we will denote this subset of $\mathbb{N}^k$ by $\mathbb{N}^k_\lambda$. With Δ^+_λ we will denote the subset of *positive roots* α of Q lying in $\mathbb{N}^k_\lambda$ and with $\mathbb{N}\Delta^+_\lambda$ the additive semigroup they generate.

If v_i is a loop-free vertex of Q we have defined the *reflexion* $\mathbb{Z}^k \xrightarrow{r_i} \mathbb{Z}^k$ by

$$r_i(\alpha) = \alpha - T_Q(\alpha, \epsilon_i)$$

and we define its *dual reflexion* $\mathbb{C}^k \xrightarrow{s_i} \mathbb{C}^k$ by the formula

$$s_i(\lambda)_j = \lambda_j - T_Q(\epsilon_i, \epsilon_j)\lambda_i$$

Clearly, we have $s_i(\lambda).\alpha = \lambda.r_i(\alpha)$. We say that a loop-free vertex v_i in Q is *admissible* for (λ, α) (or for λ) if $\lambda_i \neq 0$. We define an equivalence relation $\sim$ on pairs $(\lambda, \alpha) \in \mathbb{C}^k \times \mathbb{Z}^k$ induced by $(\lambda, \alpha) \sim (s_i(\lambda), r_i(\alpha))$ whenever v_i is an admissible vertex for (λ, α). We want to relate the representation theory of Π_λ to that of $\Pi_{s_i(\lambda)}$.

Theorem 8.5 *If v_i is an admissible vertex for λ, then there is an equivalence of categories*

$$\Pi_\lambda - \mathtt{rep} \xrightarrow{E_i} \Pi_{s_i(\lambda)} - \mathtt{rep}$$

that acts as the reflection r_i on the dimension vectors.

Proof. Because the definition of Π_λ does not depend on the orientation of the quiver Q we may assume that there are no arrows in Q having starting vertex v_i. Let $V \in \mathtt{rep}_\alpha\, \Pi_\lambda$ and consider V as a representation of the double quiver Q^d. Consider the vectorspace

$$V_\oplus = \bigoplus_{(i) \xleftarrow{a} (j)} V_j$$

where the sum is taken over all arrows $a \in Q_a$ terminating in v_i. Let μ_a and π_a be the inclusion and projection between V_j and $V_\oplus$ and define maps $V_i \xrightarrow{\mu} V_\oplus$ and $V_\oplus \xrightarrow{\pi} V_i$ by the formulas

$$\pi = \frac{1}{\lambda_i} \sum_{(i) \xleftarrow{a} ()} V_a \circ \pi_a \qquad \text{and} \qquad \mu = \sum_{(i) \xleftarrow{a} ()} \mu_a \circ V_a$$

then $\pi \circ \mu = \mathbb{1}_{V_i}$ whence $\mu \circ \pi$ is an idempotent endomorphism on $V_\oplus$.

We define the representation V' of Q^d by the following data : $V'_j = V_j$ for $j \neq i$, $V'_a = V_a$ and $V'_{a^*} = V_{a^*}$ whenever the terminating vertex of a is not v_i. Further,

$$V'_i = Im\ \mathbb{1} - \mu \circ \pi = Ker\ \pi$$

and for an arrow $(i) \xleftarrow{a} (j)$ in Q we define

$$\begin{cases} V'_a & = -\lambda_i(\mathbb{1} - \mu \circ \pi) \circ \mu_a \ : \ V'_j \longrightarrow V'_i \\ V'_{a^*} & = \pi_a \mid V'_i \ : \ V'_i \longrightarrow V'_j \end{cases}$$

We claim that V' is a representation of $\Pi_{s_i(\lambda)}$. Indeed, for a vertex v_i we have

$$\sum_{(i) \xleftarrow{a} ()} V'_a V'_{a^*} = \sum_{(i) \xleftarrow{a} ()} -\lambda_i(\mathbb{1} - \mu \circ \pi) \circ \mu_a \circ \pi_a \mid V'_i = -\lambda_i(\mathbb{1} - \mu \circ \pi) \mid V'_i = -\lambda_i \mathbb{1}_{V'_i}$$

and $(s_i\lambda)_i = -\lambda_i$. Further, for an arrow $(i) \xleftarrow{a} (j)$ in Q then

$$V'_{a^*} V'_a = \pi_a \circ (-\lambda_i(\mathbb{1} - \mu \circ \pi) \circ \mu_a) = -\lambda_i \pi_a \circ \mu_a + \lambda_i \pi_a \circ \mu \circ \pi \circ \mu_a = -\lambda_i \mathbb{1}_{V_j} + V_{a^*} V_a$$

but then, whenever $j \neq i$ we have the equality

$$\sum_{(j) \xleftarrow{a} ()} V'_a V'_{a^*} - \sum_{() \xleftarrow{a} (j)} V'_{a^*} V'_a = \sum_{(j) \xleftarrow{a} ()} V_a V_{a^*} - \sum_{() \xleftarrow{a} (j)} V_{a^*} V_a - T_Q(\epsilon_j, \epsilon_i) \lambda_i \mathbb{1}_{V_j}$$

because there are $-T_Q(\epsilon_j, \epsilon_i)$ arrows from v_j to v_i. Then, this reduces to

$$\lambda_j \mathbb{1}_{V_j} - T_Q(\epsilon_j, \epsilon_i) \lambda_i \mathbb{1}_{V_i} = (s_i\lambda)_j \mathbb{1}_{V_j}$$

The assignment $V \mapsto V'$ extends to a functor E_i and the exact sequence

$$0 \longrightarrow V'_i \longrightarrow V_\oplus \xrightarrow{\pi} V_i \longrightarrow 0$$

shows that it acts as r_i on the dimension vectors. Finally, the reflection also defines a functor $E'_i : \Pi_{s_i(\lambda)} - \texttt{rep} \longrightarrow \Pi_\lambda - \texttt{rep}$ and one shows that there is a natural equivalence $V \longrightarrow E'_i(E_i(V))$ finishing the proof. $\square$

Recall from section 5.5 that for a fixed dimension vector α we have the complex moment map

$$\texttt{rep}_\alpha\ Q^d \xrightarrow{\mu_\alpha} M_\alpha \qquad \mu_\alpha(V)_i = \sum_{() \xleftarrow{a} (i)} V_a V_{a^*} - \sum_{(i) \xleftarrow{a} ()} V_{a^*} V_a$$

and that we have the identification $\underline{rep}_\alpha\ \Pi_\lambda = \mu_\alpha^{-1}(\lambda)$. A geometric interpretation of the proof of the foregoing theorem tells us that the schemes $\mu_\alpha^{-1}(\lambda)$ and $\mu_{r_i(\alpha)}^{-1}(s_i(\lambda))$ have the same number of irreducible components and that

$$dim\ \mu_\alpha^{-1}(\lambda) - \alpha.\alpha = dim\ \mu_{r_i(\alpha)}^{-1}(s_i(\lambda)) - r_i(\alpha).r_i(\alpha)$$

see [21, lemma 1.2] for full details. The set of λ-Schur roots S_λ was defined to be the set of $\alpha \in \mathbb{N}^k$ such that

$$p_Q(\alpha) \geq p_Q(\beta_1) + \ldots + p_Q(\beta_r)$$

for all decompositions $\alpha = \beta_1 + \ldots + \beta_r$ with the $\beta_i \in \Delta_\lambda^+$. If we demand a proper inequality $>$ for all decompositions we get a subset Σ_λ and call it the *set of λ-simple roots* . Recall that S_λ and hence Σ_λ consists of Schur roots of Q.

As in the case of Kac's theorem where one obtains the set of all roots from the subsets $\Pi = \{\epsilon_i \mid v_i \text{ has no looops}\}$ and the *fundamental set of roots* $F_Q = \{\alpha \in \mathbb{N}^k - \underline{0} \mid T_Q(\alpha, \epsilon_i) \leq 0$ and $supp(\alpha)$ is connected $\}$, we can use the above reflection functors E_i to reduce pairs (λ, α) under the equivalence relation $\sim$ to a particularly nice form, see [21, Thm. 4.8].

Theorem 8.6 *If $\alpha \in \Sigma_\lambda$, then $(\lambda, \alpha) \sim (\lambda', \alpha')$ with*

$$\begin{cases} \alpha' \in \Pi & \text{if } \alpha \text{ is a real root,} \\ \alpha' \in F_Q & \text{if } \alpha \text{ is an imaginary root.} \end{cases}$$

Proposition 8.3 *If (λ, α) is such that $\alpha \in \Sigma_\lambda$, then $\underline{rep}_\alpha\ \Pi_\lambda = \mu_\alpha^{-1}(\lambda)$ is irreducible and*

$$dim\ \mu_\alpha^{-1}(\lambda) = \alpha.\alpha - 1 + 2p_Q(\alpha)$$

In particular, $\mu_\alpha^{-1}(\lambda)$ is a complete intersection.

Proof. If $\alpha \in \Sigma_\lambda$, then we know by theorem 5.18 that

$$dim\ \mu_\alpha^{-1}(\lambda) = \alpha.\alpha - \chi_Q(\alpha, \alpha) + p_Q(\alpha) = \alpha.\alpha - 1 + 2p_Q(\alpha)$$

as $p_Q(\alpha) = 1 - \chi_Q(\alpha, \alpha)$. Moreover, this number is also the relative dimension of the complex moment map μ_α. Therefore, $\mu_\alpha^{-1}(\lambda)$ is equidimensional and we only have to prove that it is irreducible.

By theorem 8.6 and the geometric interpretation of the reflexion functor equivalence we may reduce to the case where α is either a coordinate vector or lies in the fundamental region. The former case being trivial, we assume $\alpha \in F_Q$. Consider the projection map

$$\mu_\alpha^{-1}(\lambda) \xrightarrow{\pi} \mathtt{rep}_\alpha\ Q$$

then the image of π is described in theorem 5.17 and any non-empty fiber $\pi^{-1}(V) \simeq (Ext^1_{\mathbb{C}Q}(V,V))^*$ is irreducible. As in the proof of theorem 5.18 we can decompose $\mathtt{rep}_\alpha\ Q$ according to representation types in $\mathtt{rep}_\alpha(\tau)$. Because $\alpha \in \Sigma_\lambda$ we have that $dim\ \pi^{-1}(\mathtt{rep}_\alpha(\tau)) < d = \alpha.\alpha - 1 + 2p_Q(\alpha)$. for all $\tau \neq (1,\alpha)$.

Because α is a Schur root, $\mathtt{rep}_\alpha(1,\alpha)$ is an open set and $\pi^{-1}(\mathtt{rep}_\alpha\ Q - \mathtt{rep}_\alpha(1,\alpha))$ has dimension less than d, whence it is sufficient to prove that $\pi^{-1}(\mathtt{rep}_\alpha(1,\alpha))$ is irreducible. Because it is an open subset of $\mu_\alpha^{-1}(\lambda)$ it is equidimensional of dimension d and every fiber is irreducible. But, if $X \longrightarrow Y$ is a dominant map with Y irreducible and all fibers irreducible of the same dimension, then X is irreducible, finishing the proof. $\square$

The term λ-simple roots for Σ_λ is justified by the following result.

Theorem 8.7 *Let (λ,α) be such that $\alpha \in \Sigma_\lambda$. Then, $\underline{rep}_\alpha\ \Pi_\lambda = \mu_\alpha^{-1}(\lambda)$ is a reduced and irreducible complete intersection of dimension $d = \alpha.\alpha - 1 + 2p_Q(\alpha)$ and the general element of $\mu_\alpha^{-1}(\lambda)$ is a simple representation of Π_λ.*

In particular, $\mathtt{iss}_\alpha\ \Pi_\lambda$ s an irreducible variety of dimension $2p_Q(\alpha)$.

Proof. We know that $\mu^{-1}_{|alpha}(\lambda)$ is irreducible of dimension d. By the type stratification, it is enough to prove the existence of one simple representation of dimension vector α. The reflection functors being equivalences of categories, we may assume that α is either in Π or in F_Q. Clearly, for α a dimension vector, there is a simple representation, whence assume $\alpha \in F_Q$.

Assume there is no simple α-dimensional representation of Π_λ. Because $\underline{rep}_\alpha\ \Pi_\lambda$ is irreducible, there is a dimension vector $\beta < \alpha$ and an open subset of representations containing a subrepresentation of dimension vector β. As the latter condition is closed, every α-dimensional representation of Π_λ contains a β-dimensional subrepresentation.

Because α is a Schur root for Q, the general α-dimensional representation of Q extends to Π_λ and hence contains a subrepresentation of dimension vector β, that is $\beta \xhookrightarrow{Q} \alpha$. Applying the same argument to the quiver Q^o we also have $\beta \xhookrightarrow{Q^o} \alpha$.

If we now consider duals,this implies that the general α-dimensional representation of Q has a subrepresentation of dimension vector $\alpha - \beta$. But then, by the results of section 4.7 we have $ext(\beta, \alpha - \beta) = 0 = ext(\alpha - \beta, \beta)$ whence a general α-dimensional representation of Q decomposes as a direct sum of representations of dimension β and $\alpha - \beta$, contradicting the fact that α is a Schur root. Hence, there are α-dimensional simple representations of Π_λ.

Let V be a simple representation in $\mu_\alpha^{-1}(\lambda)$, then computing differentials it follows that μ_α is smooth at V, whence $\mu_\alpha^{-1}(\lambda)$ is generically reduced. But then, being a complete intersection, it is Cohen-Macaulay and therefore reduced. $\square$

This finishes the proof of the *easy* part of the characterization of simple roots for Π_λ due to W. Crawley-Boevey, [21].

Theorem 8.8 *The following are equivalent*

1. Π_λ *has* α*-dimensional simple representations.*

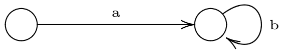
2. $\alpha \in \Sigma_\lambda$.

The proof of [21] involves a lengthy case-by-case study and awaits a more transparent argument, perhaps along the lines of hyper-Kähler reduction as in section 8.5.

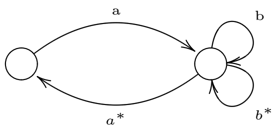
If $\alpha \in \Sigma_\lambda$, then $\Pi_\lambda(\alpha)$ is an order in a central simple algebra over the functionfield of $\mathtt{iss}_\alpha\ \Pi_\lambda$.

8.4 Hilbert schemes

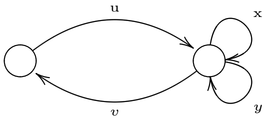
In this section we will illustrate some of the foregoing results in the special case of the quiver Q coming from the study of linear dynamical systems, and its double quiver Q^d



and



In order to avoid heavy use of stars, we denote as in the previous chapters, $a = u$, $a^* = v$, $b = x$ and $b^* = y$, so the path algebra of the double Q^d



is the algebra $\mathbb{M}$ considered before. We fix the dimension vector $\alpha = (1, n)$ and the character $\theta = (-n, 1)$ and recall from section 8.2 that the moduli space $M_\alpha^{ss}(Q, \theta) \simeq \mathbb{C}^n$.

We say that u is a *cyclic vector* for the matrix-couple $(X, Y) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ if there is no proper subspace of $\mathbb{C}^n$ containing u which is stable under left multiplication by X and Y.

Lemma 8.4 *A representation* $V = (X, Y, u, v) \in \mathtt{rep}_\alpha\ \mathbb{M}$ *is* θ*-semistable if and only if* u *is a cyclic vector for* (X, Y)*. Moreover, in this case* V *is even* θ*-stable.*

Proof. If there is a proper subspace of $\mathbb{C}^n$ of dimension k containing u and stable under the multiplication with X and Y then V contains a subrepresentation of dimension $\beta = (1, k)$ and $\theta(\beta) < 0$. If u is cyclic for (X, Y) then the only proper subrepresentations of V are of dimension $(0, k)$ for some k, but for those $\theta(\beta) > 0$ whence V is θ-stable. $\square$

The complex moment map $\mu = \mu_\alpha$ for this situation is

$$\begin{array}{ccc} \mathtt{rep}_\alpha\ Q^d = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*} & \xrightarrow{\quad\mu\quad} & \mathbb{C} \oplus M_n(\mathbb{C}) \\ (X,Y,u,v) & \mapsto & (-v.u, [Y,X] + u.v) \end{array}$$

Observe that the image is contained in $M^0_\alpha(\mathbb{C}) = \{(c,M) \mid c + tr(M) = 0\}$. The differential $d\mu$ in the point (X,Y,u,v) is equal to

$$d\mu_{(X,Y,u,v)}\ (A,B,c,d) = (-v.c - d.u, [B,X] + [Y,A] + u.d + c.v).$$

Lemma 8.5 *The second component of the differential $d\mu$ is surjective in (X,Y,u,v) if u is a cyclic vector for (X,Y).*

Proof. Consider the *nondegenerate* symmetric bilinear form $tr(MN)$ on $M_n(\mathbb{C})$ With respect to this inproduct on $M_n(\mathbb{C})$ the space orthogonal to the image of (the second component of) $d\mu_{(X,Y,u,v)}$ is equal to

$$\{M \in M_n(\mathbb{C}) \mid\ tr([B,X]M + [Y,A]M + u.dM + c.vM) = 0, \forall (A,B,c,d)\}$$

Because the trace does not change under cyclic permutations and is nondegenerate we see that this space is equal to

$$\{M \in M_n(\mathbb{C}) \mid\ [M,X] = 0 \quad [Y,M] = 0 \quad Mu = 0 \quad \text{and} \quad vM = 0\}$$

But then, the kernel $ker\ M$ is a subspace of $\mathbb{C}^n$ containing u and stable under left multiplication by X and Y. By the cyclicity assumption this implies that $ker\ M = \mathbb{C}^n$ or equivalently that $M = 0$. As $d\mu^\perp_{(X,Y,u,v)} = 0$ and tr is nondegenerate, this implies that the differential is surjective. □

Let $\mathtt{rep}^{ss}_\alpha\ Q^d = \mathtt{rep}^s_\alpha\ Q^d = \mathtt{rep}^s_\alpha\ \mathbb{M}$ be the open variety of θ-(semi)stable representations.

Proposition 8.4 *For every matrix $(c,M) \in M^0_\alpha(\mathbb{C})$ in the image of the map*

$$\mathtt{rep}^s_\alpha\ \mathbb{M} \xrightarrow{\mu} M^0_\alpha(\mathbb{C})$$

the inverse image $\mu^{-1}(M)$ is a submanifold of $\mathtt{rep}_\alpha\ \mathbb{M}$ of dimension $n^2 + 2n$.

This is a special case of theorem 5.19. Observe that for the quiver Q we have $p_Q(m,n) = mn + 1 - m^2$. As any decomposition of $\alpha = (1,n)$ is of the form

$$(1,n) = (1,a_1) + (0,a_2) + \ldots + (0,a_k) \qquad \text{with} \qquad \sum_i a_i = n$$

we have that $p_Q(\alpha) = n \geq \sum_i p_Q(\beta_i) = a_1 + 1 + \ldots + 1$ and equality only occurs for $(1,1)+(0,1)+\ldots+(0,1)$. Therefore $\alpha \in S_{\underline{0}}$.

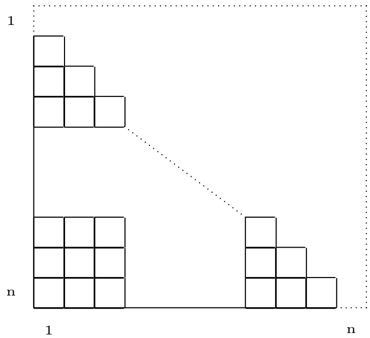
We now turn to the description of the moduli space $M^{ss}_\alpha(Q^d, \theta)$. In this particular case we clearly have.

Lemma 8.6 *For $\alpha = (1,n)$ and $\theta = (-n,1)$ there is a natural one-to-one correspondence between*

1. *$GL(\alpha)$-orbits in $\mathtt{rep}^s_\alpha\ \mathbb{M}$, and*

2. *GL_n-orbits in $\mathtt{rep}^s_\alpha\ \mathbb{M}$ under the induced action.*

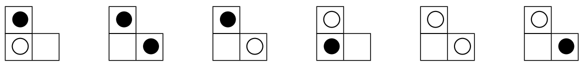
For the investigation of the $GL_n(\mathbb{C})$-orbits on $\mathtt{rep}^s_\alpha\ \mathbb{M}$ we introduce a combinatorial gadget : the *Hilbert n-stair*. This is the lower triangular part of a square $n \times n$ array of boxes



filled with *go-stones* according to the following two rules :

- each row contains exactly one stone, and
- each column contains at most one stone of each color.

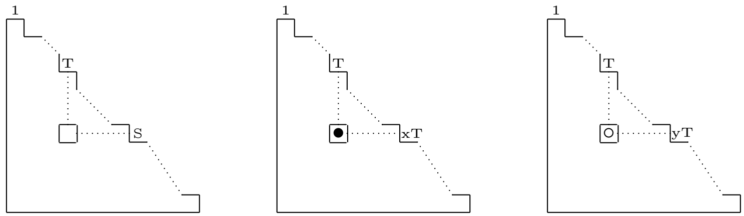
For example, the set of all possible Hilbert 3-stairs is given below.



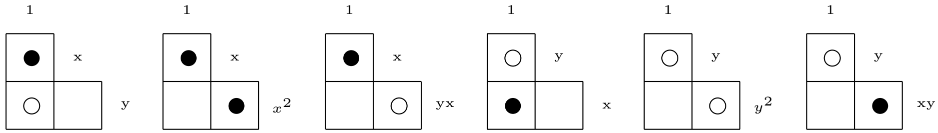
To every Hilbert stair σ we will associate a sequence of monomials $W(\sigma)$ in the free noncommutative algebra $\mathbb{C}\langle x, y\rangle$, that is $W(\sigma)$ is a sequence of words in x and y.

At the top of the stairs we place the identity element 1. Then, we descend the stairs according to the following rule.

- Every go-stone has a *top word* T which we may assume we have constructed before and a *side word* S and they are related as indicated below



For example, for the Hilbert 3-stairs we have the following sequences of non-commutative words



We will evaluate a Hilbert n-stair σ with associated sequence of non-commutative words $W(\sigma) = \{1, w_2(x,y), \ldots, w_n(x,y)\}$ on

$$\mathtt{rep}_\alpha\ \mathbb{M} = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$$

For a quadruple (X, Y, u, v) we replace every occurrence of x in the word $w_i(x,y)$ by X and every occurrence of y by Y to obtain an $n \times n$ matrix $w_i = w_i(X,Y) \in M_n(\mathbb{C})$ and by left multiplication on u a column vector $w_i.v$. The *evaluation of* σ *on* (X, Y, u, v) is the determinant of the $n \times n$ matrix

$$\sigma(X,Y,u,v) = det \begin{bmatrix} u & w_2.u & w_3.u & \cdots & w_n.u \end{bmatrix}$$

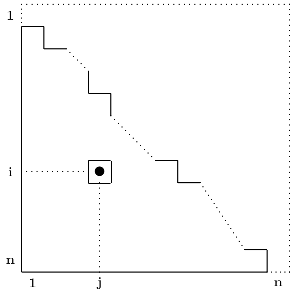
For a fixed Hilbert n-stair σ we denote with *rep*(σ) the subset of quadruples (X, Y, u, v) in $\mathtt{rep}_\alpha\ \mathbb{M}$ such that the evaluation $\sigma(v, X, Y) \neq 0$.

Theorem 8.9 *For every Hilbert n-stair, rep $(\sigma) \neq \emptyset$*

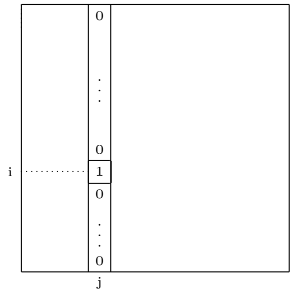
Proof. Let u be the basic column vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let every black stone in the Hilbert stair σ fix a column of X by the rule



$X =$



That is, one replaces every black stone in σ by 1 at the same spot in X and fills the remaining spots in the same column by zeroes. The same rule applies to Y for white stones. We say that such a quadruple (X, Y, u, v) is in *σ-standard form.*

With these conventions one easily verifies by induction that

$$w_i(X,Y)e_1 = e_i \quad \text{for all } 2 \leq i \leq n.$$

Hence, filling up the remaining spots in X and Y arbitrarily one has that $\sigma(X, Y, u, v) \neq 0$ proving the claim. $\square$

Hence, $rep\ (\sigma)$ is an open subset of $\mathtt{rep}_\alpha\ \mathbb{M}$ (and even of $\mathtt{rep}^s_\alpha\ \mathbb{M}$) for every Hilbert n-stair σ. Further, for every word (monomial) $w(x, y)$ and every $g \in GL_n(\mathbb{C})$ we have that

$$w(gXg^{-1}, gYg^{-1})gv = gw(X,Y)v$$

and therefore the open sets $rep\ (\sigma)$ are stable under the $GL_n(\mathbb{C})$-action on $\mathtt{rep}_\alpha\ \mathbb{M}$. We will give representatives of the orbits in $rep\ (\sigma)$.

Let $W_n = \{1, x, \ldots, x^n, xy, \ldots, y^n\}$ be the set of all words in the non-commuting variables x and y of length $\leq n$, ordered lexicographically.

For every quadruple $(X, Y, u, v) \in \mathtt{rep}_\alpha\ \mathbb{M}$ consider the $n \times m$ matrix

$$\psi(X,Y,u,v) = \begin{bmatrix} u & Xu & X^2u & \ldots & Y^nu \end{bmatrix}$$

where $m = 2^{n+1} - 1$ and the j-th column is the column vector $w(X,Y)v$ with $w(x,y)$ the j-th word in W_n.

Hence, $(X,Y,u,v) \in rep\ (\sigma)$ if and only if the $n \times n$ minor of $\psi(X,Y,u,v)$ determined by the word-sequence $\{1, w_2, \ldots, w_n\}$ of σ is invertible. Moreover, as

$$\psi(gXg^{-1}, gYg^{-1}, gu, vg^{-1}) = g\psi(v, X, Y)$$

we deduce that the $GL_n(\mathbb{C})$-orbit of $(X,Y,u,v) \in \mathtt{rep}_\alpha\ \mathbb{M}$ contains a *unique* quadruple (X_1, Y_1, u_1, v_1) such that the corresponding minor of $\psi(X_1, Y_1, u_1, v_1) = \mathbb{1}_n$.

Hence, each $GL_n(\mathbb{C})$-orbit in $rep\ (\sigma)$ contains a unique representant in σ-standard form. Therefore,

Proposition 8.5 *The action of $GL_n(\mathbb{C})$ on rep (σ) is free and the orbit space*

$$rep\ (\sigma)/GL_n(\mathbb{C})$$

is an affine space of dimension $n^2 + 2n$.

Proof. The dimension is equal to the number of non-forced entries in X, Y and v. As we fixed $n-1$ columns in X or Y this dimension is equal to

$$k = 2n^2 - (n-1)n + n = n^2 + 2n.$$

The argument above shows that every $GL_n(\mathbb{C})$-orbit contains a unique quadruple in σ-standard form so the orbit space is an affine space. □

Theorem 8.10 *For $\alpha = (1, n)$ and $\theta = (-n, 1)$, the moduli space*

$$M_\alpha^{ss}(Q^d, \theta) = M_\alpha^{ss}(\mathbb{M}, \theta)$$

is a complex manifold of dimension $n^2 + 2n$ and is covered by the affine spaces rep (σ).

Proof. Recall that $\mathtt{rep}_\alpha^s\ \mathbb{M}$ is the open submanifold consisting of quadruples (x, Y, u, v) such that u is a cyclic vector of (X, Y) or equivalently such that

$$\mathbb{C}\langle X, Y\rangle u = \mathbb{C}^n$$

where $\mathbb{C}\langle X, Y\rangle$ is the not necessarily commutative subalgebra of $M_n(\mathbb{C})$ generated by the matrices X and Y.

Hence, clearly $rep\ (\sigma) \subset \mathtt{rep}_n\ \mathbb{M}$ for any Hilbert n-stair σ. Conversely, we claim that a quadruple $(X, Y, u, v) \in \mathtt{rep}_\alpha^s\ \mathbb{M}$ belongs to at least one of the open subsets $rep\ (\sigma)$.

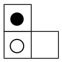
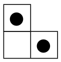
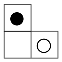
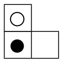
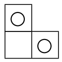
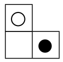
Indeed, either $Xu \notin \mathbb{C}u$ or $Yu \notin \mathbb{C}u$ as otherwise the subspace $W = \mathbb{C}u$ would contradict the cyclicity assumption. Fill the top box of the stairs with the corresponding stone and define the 2-dimensional subspace $V_2 = \mathbb{C}u_1 + \mathbb{C}u_2$ where $u_1 = u$ and $u_2 = w_2(X, Y)u$ with w_2 the corresponding word (either x or y).

Assume by induction we have been able to fill the first i rows of the stairs with stones leading to the sequence of words $\{1, w_2(x,y), \ldots, w_i(x,y)\}$ such that the subspace $V_i = \mathbb{C}u_1 + \ldots + \mathbb{C}u_i$ with $u_i = w_i(X,Y)v$ has dimension i.

Then, either $Xu_j \notin V_i$ for some j or $Yu_j \notin V_i$ (if not, V_i would contradict cyclicity). Then, fill the j-th box in the $i+1$-th row of the stairs with the corresponding stone. Then, the top $i+1$ rows of the stairs form a Hilbert $i+1$-stair as there can be no stone of the same color lying in the same column. Define $w_{i+1}(x,y) = xw_i(x,y)$ (or $yw_i(x,y)$) and $u_{i+1} = w_{i+1}(X,Y)u$. Then, $V_{i+1} = \mathbb{C}u_1 + \ldots + \mathbb{C}u_{i+1}$ has dimension $i+1$.

Continuing we end up with a Hilbert n-stair σ such that $(X,Y,u,v) \in rep\ (\sigma)$. This concludes the proof. $\square$

Example 8.1 (The moduli space $M_\alpha^{ss}(Q^d,\theta)$ when $n=3$) Representatives for the $GL_3(\mathbb{C})$-orbits in *rep* (σ) are given by the following quadruples for σ a Hilbert 3-stair :

						
X	$\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$	$\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}$	$\begin{bmatrix} 0 & a & b \\ 0 & c & d \\ 1 & e & f \end{bmatrix}$	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} a & 0 & b \\ c & 0 & d \\ e & 1 & f \end{bmatrix}$
Y	$\begin{bmatrix} 0 & g & h \\ 0 & i & j \\ 1 & k & l \end{bmatrix}$	$\begin{bmatrix} d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}$	$\begin{bmatrix} g & 0 & h \\ i & 0 & j \\ k & 1 & l \end{bmatrix}$	$\begin{bmatrix} 0 & g & h \\ 1 & i & j \\ 0 & k & l \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & j \\ 1 & 0 & k \\ 0 & 1 & l \end{bmatrix}$	$\begin{bmatrix} 0 & g & h \\ 1 & i & j \\ 0 & k & l \end{bmatrix}$
u	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
v	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$

We now turn to the deformed preprojective algebras. Let $\lambda = (-n\lambda, \lambda \mathbb{1}_n) \in M_\alpha^0(\mathbb{C})$ for $\lambda \in \mathbb{C}$. Then,

$$\Pi_\lambda = \frac{\mathbb{M}}{(v.u + \lambda v_1, [Y,X] + u.v - \lambda v_2)}$$

then if we denote by $M_\alpha^{ss}(\Pi_\lambda,\theta)$ the moduli space of θ-semistable representations of Π_λ, then we have the following situation

$$\begin{tikzcd} \mu^{-1}(\lambda) \cap \mathtt{rep}^s_\alpha\ \mathbb{M} \arrow[r, hook] \arrow[d] & \mathtt{rep}^s_\alpha\ \mathbb{M} \arrow[d] \\ M^{ss}_\alpha(\Pi_\lambda, \theta) \arrow[r, hook] & M^{ss}_\alpha(\mathbb{M}, \theta) \end{tikzcd}$$

and from the theorem above we obtain :

Theorem 8.11 *For a $\lambda \in M^0_\alpha(\mathbb{C})$, the orbit space of θ-semistable representations of the deformed preprojective algebra*

$$M^{ss}_\alpha(\Pi_\lambda, \theta)$$

is a submanifold of $M^{ss}_\alpha(\mathbb{M}, \theta)$ of dimension $2n$.

We will identify the special case of the preprojective algebra (that is $\lambda = \underline{0}$ with the *Hilbert scheme of n points in the plane* .

Consider a codimension n ideal $\mathfrak{i} \triangleleft \mathbb{C}[x, y]$ and fix a basis $\{v_1, \ldots, v_n\}$ of the quotient space

$$V_\mathfrak{i} = \frac{\mathbb{C}[x, y]}{\mathfrak{i}} = \mathbb{C}v_1 + \ldots + \mathbb{C}v_n.$$

Multiplication by x on $\mathbb{C}[x, y]$ induces a linear operator on the quotient $V_\mathfrak{i}$ and hence determines a matrix $X_\mathfrak{i} \in M_n(\mathbb{C})$ with respect to the chosen basis $\{v_1, \ldots, v_n\}$. Similarly, multiplication by y determines a matrix $Y_\mathfrak{i} \in M_n(\mathbb{C})$.

Moreover, the image of the unit element $1 \in \mathbb{C}[x, y]$ in $V_\mathfrak{i}$ determines with respect to the basis $\{v_1, \ldots, v_n\}$ a column vector $u \in \mathbb{C}^n = V_\mathfrak{i}$. Clearly, this vector and matrices satisfy :

$$[X_\mathfrak{i}, Y_\mathfrak{i}] = 0 \quad \text{and} \quad \mathbb{C}[X_\mathfrak{i}, Y_\mathfrak{i}]u = \mathbb{C}^n.$$

Here, $\mathbb{C}[X_\mathfrak{i}, Y_\mathfrak{i}]$ is the n-dimensional subalgebra of $M_n(\mathbb{C})$ generated by the two matrices $X_\mathfrak{i}$ and $Y_\mathfrak{i}$. In particular, u is a cyclic vector for the matrix-couple (X, Y).

Conversely, if $(X, Y, u) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n$ is a cyclic triple such that $[X, Y] = 0$, then $\mathbb{C}\langle X, Y\rangle = \mathbb{C}[X, Y]$ is an n-dimensional commutative subalgebra of $M_n(\mathbb{C})$, then the kernel of the natural epimorphism

$$\mathbb{C}[x, y] \twoheadrightarrow \mathbb{C}[X, Y] \qquad x \mapsto X \quad y \mapsto Y$$

is a codimension n ideal $\mathfrak{i}$ of $\mathbb{C}[x, y]$.

However, there is some redundancy in the assignment $\mathfrak{i} \longrightarrow (X_\mathfrak{i}, Y_\mathfrak{i}, u_\mathfrak{i})$ as it depends on the choice of basis of $V_\mathfrak{i}$. If we choose a different basis $\{v'_1, \ldots, v'_n\}$ with basechange matrix $g \in GL_n(\mathbb{C})$, then the corresponding triple is

$$(X'_\mathfrak{i}, Y'_\mathfrak{i}, u'_\mathfrak{i}) = (g.X_\mathfrak{i}.g^{-1}, g.Y_\mathfrak{i}.g^{-1}, gu_\mathfrak{i})$$

The above discussion shows that there is a one-to-one correspondence between

- codimension n ideals $\mathfrak{i}$ of $\mathbb{C}[x,y]$, and
- $GL_n(\mathbb{C})$-orbits of cyclic triples (X,Y,u) in $M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n$ such that $[X,Y]=0$.

Example 8.2 (The Hilbert scheme $Hilb_2$) Consider a triple $(X,Y,u) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C}^2$ and assume that either X or Y has distinct eigenvalues (type a). As

$$[\begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}] = \begin{bmatrix} 0 & (\nu_1 - \nu_2)b \\ (\nu_2 - \nu_1)c & 0 \end{bmatrix}$$

we have a representant in the orbit of the form

$$(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})$$

where cyclicity of the column vector implies that $u_1 u_2 \neq 0$.

The stabilizer subgroup of the matrix-pair is the group of diagonal matrices $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow GL_2(\mathbb{C})$, hence the orbit has a unique representant of the form

$$(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

The corresponding ideal $\mathfrak{i} \triangleleft \mathbb{C}[x,y]$ is then

$$\mathfrak{i} = \{ f(x,y) \in \mathbb{C}[x,y] \mid f(\lambda_1, \mu_1) = 0 = f(\lambda_2, \mu_2) \}$$

hence these orbits correspond to sets of two *distinct* points in $\mathbb{C}^2$.

The situation is slightly more complicated when X and Y have only one eigenvalue (type b). If (X,Y,u) is a cyclic commuting triple, then either X or Y is not diagonalizable. But then, as

$$[\begin{bmatrix} \nu & 1 \\ 0 & \nu \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}] = \begin{bmatrix} c & d-a \\ 0 & c \end{bmatrix}$$

we have a representant in the orbit of the form

$$(\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \mu & \beta \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})$$

with $[\alpha : \beta] \in \mathbb{P}^1$ and $u_2 \neq 0$. The stabilizer of the matrixpair is the subgroup

$$\{ \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \mid c \neq 0 \} \hookrightarrow GL_2(\mathbb{C})$$

and hence we have a unique representant of the form

$$\left(\begin{bmatrix}\lambda & \alpha\\ 0 & \lambda\end{bmatrix}, \begin{bmatrix}\mu & \beta\\ 0 & \mu\end{bmatrix}, \begin{bmatrix}0\\ 1\end{bmatrix}\right)$$

The corresponding ideal $\mathfrak{i} \lhd \mathbb{C}[x,y]$ is

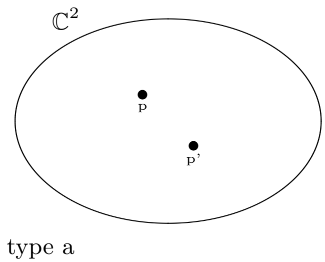
$$\mathfrak{i} = \{f(x,y) \in \mathbb{C}[x,y] \mid f(\lambda,\mu) = 0 \ \text{ and } \ \alpha\frac{\partial f}{\partial x}(\lambda,\mu) + \beta\frac{\partial f}{\partial y}(\lambda,\mu) = 0\}$$

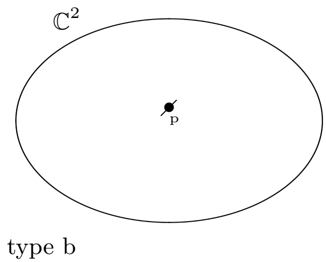
as one proves by verification on monomials because

$$\begin{bmatrix}\lambda & \alpha\\ 0 & \lambda\end{bmatrix}^k \begin{bmatrix}\mu & \beta\\ 0 & \mu\end{bmatrix}^l \begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}k\alpha\lambda^{k-1}\mu^l + l\beta\lambda^k\mu^{l-1}\\ \lambda^k\mu^l\end{bmatrix}$$

Therefore, $\mathfrak{i}$ corresponds to the set of two points at $(\lambda,\mu) \in \mathbb{C}^2$ infinitesimally attached to each other in the direction $\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$. For each point in $\mathbb{C}^2$ there is a $\mathbb{P}^1$ family of such *fat points*.

Thus, points of $Hilb_2$ correspond to either of the following two situations :





The Hilbert-Chow map $Hilb_2 \xrightarrow{\pi} S^2\ \mathbb{C}^2$ (where $S^2\ \mathbb{C}^2$ is the symmetric power of $\mathbb{C}^2$, that is $S_2 = \mathbb{Z}/2\mathbb{Z}$ orbits of couples of points from $\mathbb{C}^2$) sends a point of type a to the formal sum $[p] + [p']$ and a point of type b to $2[p]$. Over the complement of (the image of) the diagonal, this map is a one-to-one correspondence.

However, over points on the diagonal the fibers are $\mathbb{P}^1$ corresponding to the directions in which two points can approach each other in $\mathbb{C}^2$. As a matter of fact, the symmetric power $S^2\ \mathbb{C}^2$ has singularities and the Hilbert-Chow map $Hilb_2 \xrightarrow{\pi} S^2\ \mathbb{C}^2$ is a *resolution of singularities.*

Theorem 8.12 *Let* $\mathtt{rep}_\alpha\ \mathbb{M} \xrightarrow{\mu} M^0_\alpha(\mathbb{C})$ *be the complex moment map, then*

$$Hilb_n \simeq M^{ss}_\alpha(\Pi_0, \theta)$$

and is therefore a complex manifold of dimension $2n$.

Proof. We identify the triples $(X,Y,u) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n$ such that u is a cyclic vector of (X,Y) and $[X,Y]=0$ with the subspace

$$\{(X,Y,u,\underline{0}) \ | \ [X,Y]=0 \quad \text{and } u \text{ is cyclic } \} \hookrightarrow \mathtt{rep}^s_\alpha \ \mathbb{M}$$

which is clearly contained in $\mu^{-1}(0)$. To prove the converse inclusion assume that (X,Y,u,v) is a cyclic quadruple such that

$$[X,Y]+uv=0.$$

Let $m(x,y)$ be any word in the noncommuting variables x and y. We claim that

$$v.m(X,Y).u=0.$$

We will prove this by induction on the length $l(m)$ of the word $m(x,y)$. When $l(m)=0$ then $l(x,y)=1$ and we have

$$v.l(X,Y).u = v.u = tr(u.v) = tr([X,Y]) = 0.$$

Assume we proved the claim for all words of length $< l$ and take a word of the form $m(x,y) = m_1(x,y)yxm_2(x,y)$ with $l(m_1)+l(m_2)+2=l$. Then, we have

$$\begin{aligned}
wm(X,Y) = & \quad wm_1(X,Y)YXm_2(X,Y) \\
= & \quad wm_1(X,Y)([Y,X]+XY)m_2(X,Y) \\
= & \quad (wm_1(X,Y)v).wm_2(X,Y) + wm_1(X,Y)XYm_2(X,Y) \\
= & \quad wm_1(X,Y)XYm_2(X,Y)
\end{aligned}$$

where we used the induction hypotheses in the last equality (the bracketed term vanishes).

Hence we can reorder the terms in $m(x,y)$ if necessary and have that $wm(X,Y) = wX^{l_1}Y^{l_2}$ with $l_1+l_2=l$ and l_1 the number of occurrences of x in $m(x,y)$. Hence, we have to prove the claim for $X^{l_1}Y^{l_2}$.

$$\begin{aligned}
wX^{l_1}Y^{l_2}v = & \quad tr(X^{l_1}Y^{l_2}vw) \\
= & \quad -tr(X^{l_1}Y^{l_2}[X,Y]) \\
= & \quad -tr([X^{l_1}Y^{l_2},X]Y) \\
= & \quad -tr(X^{l_1}[Y^{l_2},X]Y) \\
= & \quad -\textstyle\sum_{i=0}^{l_2-1} tr(X^{l_1}Y^i[Y,X]Y^{l_2-i}) \\
= & \quad -\textstyle\sum_{i=0}^{l_2-1} tr(Y^{l_2-i}X^{l_1}Y^i[Y,X] \\
= & \quad -\textstyle\sum_{i=0}^{l_2-1} tr(Y^{l_2-i}X^{l_1}Y^iv.w \\
= & \quad -\textstyle\sum_{i=0}^{l_2-1} wY^{m_2-i}X^{l_1}Y^iv
\end{aligned}$$

But we have seen that $wY^{l_2-i}X^{l_1}Y^i = wX^{l_1}Y^{l_2}$ hence the above implies that $wX^{l_1}Y^{l_2}v = -l_2 wX^{l_1}Y^{l_2}v$. But then $wX^{l_1}Y^{l_2}v = 0$, proving the claim.

Consequently, $w.\mathbb{C}\langle X,Y\rangle.v = 0$ and by the cyclicity condition we have $w.\mathbb{C}^n = 0$ hence $w = 0$. Finally, as $v.w + [X,Y] = 0$ this implies that $[X,Y] = 0$ and we can identify the fiber $\mu^{-1}(0)$ with the indicated subspace. From this the result follows. $\square$

We can use the affine covering of $M_\alpha^{ss}(\mathbb{M},\theta)$ by Hilbert stairs, to cover the Hilbert scheme $Hilb_n$ by the intersections $Hilb(\sigma) = rep(\sigma) \cap Hilb_n$.

Example 8.3 (The Hilbert scheme $Hilb_2$) Consider $Hilb_2$ (). Because

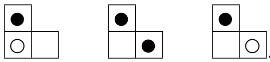
$$[\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}, \begin{bmatrix} c & d \\ e & f \end{bmatrix}] = \begin{bmatrix} ae-d & af-ac-bd \\ c+be-f & d-ae \end{bmatrix}$$

this subset can be identified with $\mathbb{C}^4$ using the equalities

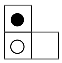
$$d = ar \quad \text{and} \quad f = c + be.$$

Similarly, $Hilb_2$ () $\simeq \mathbb{C}^4$.

Example 8.4 (The Hilbert scheme $Hilb_3$) Up to change of colors there are three 3-stairs to consider



We claim that

$Hilb_3$ () $\simeq \mathbb{C}^6$.

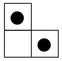
For consider the commutator

$$[\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}, \begin{bmatrix} 0 & g & h \\ 0 & i & j \\ 1 & k & k \end{bmatrix}] = \begin{bmatrix} b-g & ai+bk-cg-eh & aj+bl-dg-fh \\ d-i & g+dk-ej & h+cj+dl-di-fj \\ f-k & -a-ck-el+ei+fk & -b-dk+ej \end{bmatrix}$$

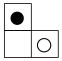
Taking the Groebner basis of these relations one finds the following relations

$$\begin{cases} f &= k \\ g &= ej - ik \\ d &= i \\ h &= i^2 - cj + jk - il \\ b &= g \\ a &= ei - ck + k^2 - el \end{cases}$$

from which the claim follows. In a similar manner one proves that

$Hilb_3$ () $\simeq \mathbb{C}^6$.

However, the situation for

$Hilb_3$ ()

is more complicated.

Theorem 8.13 *The Hilbert scheme $Hilb_n$ of n points in $\mathbb{C}^2$ is a complex connected manifold of dimension $2n$.*

Proof. The symmetric power $S^n\ \mathbb{C}^1$ parametrizes *sets* of n-points on the line $\mathbb{C}^1$ and can be identified with $\mathbb{C}^n$. Consider the map

$$Hilb_n \xrightarrow{\pi} S^n\ \mathbb{C}^1$$

defined by mapping a cyclic triple (X, Y, u) with $[X, Y] = 0$ in the orbit corresponding to the point of $Hilb_n$ to the set $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues of X. Observe that this map does not depend on the point chosen in the orbit.

Let Δ be the *big diagonal* in $S^n\ \mathbb{C}^1$, that is, $S^n\ \mathbb{C}^1 - \Delta$ is the space of all sets of n distinct points from $\mathbb{C}^1$. Clearly, $S^n\ \mathbb{C}^1 - \Delta$ is a connected n-dimensional manifold. We claim that

$$\pi^{-1}(S^n\ \mathbb{C}^1 - \Delta) \simeq (S^n\ \mathbb{C}^1 - \Delta) \times \mathbb{C}^n$$

and hence is connected.

Indeed, take a matrix X with n distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. We may diagonalize X. But then, as

$$\left[\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \begin{bmatrix} y_{11} & \ldots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \ldots & y_{nn} \end{bmatrix}\right] = \begin{bmatrix} (\lambda_1 - \lambda_1)y_{11} & \ldots & (\lambda_1 - \lambda_n)y_{1n} \\ \vdots & & \vdots \\ (\lambda_n - \lambda_1)y_{n1} & \ldots & (\lambda_n - \lambda_n)y_{nn} \end{bmatrix}$$

we see that also Y must be a diagonal matrix with entries $(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ where $\mu_i = y_{ii}$. But then the cyclicity condition implies that all coordinates of v must be non-zero.

Now, the stabilizer subgroup of the commuting (diagonal) matrix-pair (X, Y) is the *maximal torus* $T_n = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ of diagonal invertible $n \times n$ matrices. Using its action we may assume that all coordinates of v are equal to 1. That is, the points in $\pi^{-1}(\{\lambda_1,\ldots,\lambda_n\})$ with $\lambda_i \neq \lambda_j$ have unique (up to permutation as before) representatives of the form

$$(\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix})$$

that is $\pi^{-1}(\{\lambda_1,\ldots,\lambda_n\}$ can be identified with $\mathbb{C}^n$, proving the claim.

Next, we claim that *all* the fibers of π have dimension at most n. Let $\{\lambda_1,\ldots,\lambda_n\} \in S^n\ \mathbb{C}^1$ then there are only finitely many X in Jordan normalform with eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$. Fix such an X, then the subset $T(X)$ of cyclic triples (X,Y,u) with $[X,Y] = 0$ has dimension at most $n + dim\ C(X)$ where $C(X)$ is the centralizer of X in $M_n(\mathbb{C})$, that is,

$$C(X) = \{Y \in M_n(\mathbb{C}) \mid XY = YX\}.$$

The stabilizer subgroup $Stab(X) = \{g \in GL_n(\mathbb{C}) \mid gXg^{-1} = X\}$ is an open subset of the vectorspace $C(X)$ and acts freely on the subset $T(X)$ because the action of $GL_n(\mathbb{C})$ on $\mu^{-1}(0) \cap \mathtt{rep}^s_\alpha\ \mathbb{M}$ has trivial stabilizers.

But then, the orbitspace for the $Stab(X)$-action on $T(X)$ has dimension at most

$$n + dim\ C(X) - dim\ Stab(X) = n.$$

As we only have to consider finitely many X this proves the claim. The diagonal Δ has dimension $n-1$ in $S^n\ \mathbb{C}^1$ and hence by the foregoing we know that the dimension of $\pi^{-1}(\Delta)$ is at most $2n-1$. Let H be the connected component of $Hilb_n$ containing the connected subset $\pi^{-1}(S^n\ \mathbb{C}^1 - \Delta)$. If $\pi^{-1}(\Delta)$ were not entirely contained in H, then $Hilb_n$ would have a component of dimension less than $2n$, which we proved not to be the case. This finishes the proof. $\square$

We can give a representation theoretic interpretation of the *resolution of singularities Hilbert-Chow morphism*

$$Hilb_n \xrightarrow{\pi} S^n\ \mathbb{C}^2$$

$\Sigma_0 = \{(1,0),(0,1)\}$, that is the only simple Π_0-representations are one-dimensional. Any semi-simple representation of Π_0 of dimension vector $\alpha = (1,n)$ therefore decomposes as $T_0 \oplus S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$ with T_0 the unique simple $(1,0)$-dimensional representation and the S_i in the two-dimensional family of $(0,1)$-simple representations of Π_0 (corresponding to couples $(\lambda_i,\mu_i) \in \mathbb{C}^2$). Therefore we have the projective bundle morphism

$$Hilb_n = M^{ss}_\alpha(\Pi_0,\theta) \xrightarrow{\pi'} \mathtt{iss}_\alpha\ \Pi_0 = S^n\ \mathbb{C}^2$$

where the mapping sends a point of $Hilb_n$ determined by a cyclic triple (X,Y,u) to the n-tuple of eigenvalues (λ_i,μ_i) of X and Y.

8.5 Hyper Kähler structure

Again, Q is a quiver on k vertices and Q^d its double. We fix a dimension vector $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}^k$ and a character $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ and a weight $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ such that the numerical conditions

$$\theta(\alpha) = \sum_{i=1}^{k} t_i a_i = 0 \qquad \text{and} \qquad \lambda(\alpha) = \sum_{i=1}^{n} \lambda_i a_i = 0$$

are satisfied. The first is required to have θ-semistable representations, the second for λ to lie in the image of the complex moment map

$$\mathtt{rep}_\alpha\, Q^d \xrightarrow{\mu_{\mathbb{C}}} M^0_\alpha(\mathbb{C}) \qquad V \mapsto \sum_{\substack{\circ \xleftarrow{a} \circ \\ a \in Q_a}} [V_a, V_{a^*}]$$

where a^* is the arrow in Q^d_a corresponding to $a \in Q_a$ (that is with the opposite direction).

Recall that the *quaternion algebra* $\mathbb{H}$ is the 4-dimensional division algebra over $\mathbb{R}$ defined by

$$\mathbb{H} = \mathbb{R}.1 \oplus \mathbb{R}.i \oplus \mathbb{R}.j \oplus \mathbb{R}.k \qquad i^2 = j^2 = k^2 = -1 \quad k = ij = -ji$$

Definition 8.1 *A C^∞ (real) manifold M is said to be a hyper-Kähler manifold if $\mathbb{H}$ acts on H by diffeomorphisms.*

Lemma 8.7 *For any quiver Q, the representation space $\mathtt{rep}_\alpha\, Q^d$ is a hyper-Kähler manifold.*

Proof. We have to specify the actions. They are defined as follows, for $V \in \mathtt{rep}_\alpha\, Q^d$ for all arrows $b \in Q^d_a$ and all arrows $a \in Q_a$

$$\begin{aligned} (i.V)_b &= iV_b \\ (j.V)_a &= -V^\dagger_{a^*} \quad (j.V)_{a^*} = V^\dagger_a \\ (k.V)_a &= -iV^\dagger_{a^*} \quad (k.V)_{a^*} = iV^\dagger_a \end{aligned}$$

where this time we denote the *Hermitian adjoint* of a matrix M by $M^\dagger$ to distinguish it from the star-operation on the arrows of Q^d. A calculation shows that these operations satisfy the required relations. $\square$

In section 8.1 we introduced the *real moment map* for quiver representations. If we apply this to the double quiver Q^d we can take

$$\mathtt{rep}_\alpha\, Q^d \xrightarrow{\mu_{\mathbb{R}}} \mathit{Lie}\ U(\alpha) \qquad V \mapsto \sum_{\substack{\circ \xleftarrow{b} \circ \\ b \in Q^d_a}} \frac{i}{2}[V_b, V_b^\dagger]$$

We will use the action by non-zero elements of $\mathbb{H}$ to obtain C^∞-diffeomorphisms between certain subsets of $\mathtt{rep}_\alpha\ Q^d$. Let $h = \frac{i-k}{\sqrt{2}}$ then we have

$$\begin{aligned}
\mu_{\mathbb{C}}(h.V) &= \frac{1}{2} \sum_{a \in Q_a} [iV_a + iV_{a^*}^\dagger, iV_{a^*} - iV_a^\dagger] \\
&= \frac{1}{2} \sum_{a \in Q_a} (\ -[V_a, V_{a^*}] + [V_a, V_a^\dagger] - [V_{a^*}^\dagger, V_{a^*}] + [V_{a^*}^\dagger, V_a^\dagger]\ ) \\
&= \frac{1}{2} \sum_{a \in Q_a} (\ [V_a, V_{a^*}]^\dagger - [V_a, V_{a^*}]\ ) + \frac{1}{2} \sum_{a \in Q_a} (\ [V_a, V_a^\dagger] + [V_{a^*}, V_{a^*}^\dagger]\ ) \\
&= \frac{1}{2}(\mu_{\mathbb{C}}(V)^\dagger - \mu_{\mathbb{C}}(V)) - i\mu_{\mathbb{R}}(V)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\mathbb{R}}(h.V) &= \frac{i}{4}(\sum_{a \in Q_a} [iV_a + iV_{a^*}^\dagger, -iV_a^\dagger - iV_{a^*}] + \sum_{a \in Q_a} [iV_{a^*} - iV_a^\dagger, -iV_{a^*}^\dagger + iV_a]\ ) \\
&= \frac{i}{4} \sum_{a \in Q_a} (\ [V_a, V_a^\dagger] + [V_a, V_{a^*}] + [V_{a^*}^\dagger, V_a^\dagger] + [V_{a^*}^\dagger, V_{a^*}] \\
&\quad + [V_{a^*}, V_{a^*}^\dagger] - [V_{a^*}, V_a] - [V_a^\dagger, V_{a^*}^\dagger] + [V_a^\dagger, V_a]\ ) \\
&= \frac{i}{4}(2\mu_{\mathbb{C}}(V) + 2\mu_{\mathbb{C}}(V)^\dagger)
\end{aligned}$$

In particular we have

Proposition 8.6 *If $\lambda \in \mathbb{R}^k$, then we have a homeomorphism between the real varieties*

$$\mu_{\mathbb{C}}^{-1}(\lambda \mathbb{1}_\alpha) \cap \mu_{\mathbb{R}}^{-1}(\underline{0}) \xrightarrow{h.} \mu_{\mathbb{C}}^{-1}(\underline{0}) \cap \mu_{\mathbb{R}}^{-1}(i\lambda \mathbb{1}_\alpha)$$

Moreover, the hyper-Kähler structure commutes with the base-change action of $U(\alpha)$, whence we have a natural one-to-one correspondence between the quotient spaces

$$(\mu_{\mathbb{C}}^{-1}(\lambda \mathbb{1}_\alpha) \cap \mu_{\mathbb{R}}^{-1}(\underline{0}))/U(\alpha) \xrightarrow{h.} (\mu_{\mathbb{C}}^{-1}(\underline{0}) \cap \mu_{\mathbb{R}}^{-1}(i\lambda \mathbb{1}_\alpha))/U(\alpha)$$

By the results of section 8.1 we can identify both sides. To begin, by definition of the complex moment map $\mu_{\mathbb{C}}$ we have that

$$\mu_{\mathbb{C}}^{-1}(\underline{0}) = \underline{rep}_\alpha\ \Pi_0 \qquad \text{and} \qquad \mu_{\mathbb{C}}^{-1}(\lambda \mathbb{1}_\alpha) = \underline{rep}_\alpha\ \Pi_\lambda$$

Moreover, applying theorem 8.3 to the double quiver Q^d we have

$$\mathtt{iss}_\alpha\ Q^d \simeq \mu_{\mathbb{R}}^{-1}(\underline{0})/U(\alpha) \qquad \text{and} \qquad M_\alpha^{ss}(Q^d, \lambda) \simeq \mu_{\mathbb{R}}^{-1}(i\lambda \mathbb{1}_\alpha)/U(\alpha)$$

when $\lambda \in \mathbb{Z}^k$. This concludes the proof of

Theorem 8.14 *For a character $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ such that $\theta(\alpha) = 0$, there is a natural one-to-one correspondence between*

$$\mathtt{iss}_\alpha\ \Pi_\theta \xrightarrow{h.} M_\alpha^{ss}(\Pi_0, \theta)$$

which is an homeomorphism in the (induced) real topology.

Note however that this bijection does *not* respect the complex structures of these varieties. This is already clear from the fact that $\mathtt{iss}_\alpha\ \Pi_\theta$ is an affine complex variety and $M_\alpha^{ss}(\Pi_0, \theta)$ is a projective bundle over $\mathtt{iss}_\alpha\ \Pi_0$.

If $V \in \mathtt{rep}_\alpha\ Q^d$ belongs to $\mu_{\mathbb{R}}^{-1}(\underline{0})$ we know that V is a semisimple representation, that is,

$$V = S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$$

with the S_i simple representations of dimension vector β_i. Further, if $W \in \mathtt{rep}_\alpha^{-1}(i\theta \mathbb{1}_\alpha)$, then W is a direct sum of θ-stable representations, that is,

$$W = T_1^{\oplus f_1} \oplus \ldots \oplus T_s^{\oplus f_s}$$

with the T_i θ-stable representations of dimension vector γ_i. By the explicit form of the map, we have that if $W = h.V$ that $r = s$, $e_i = f_i$ and $\beta_i = \gamma_i$. That is,

Proposition 8.7 *Let θ be a character such that $\theta(\alpha) = 0$, then the deformed preprojective algebra Π_θ has semi-simple representations of dimension vector α of representation type $\tau = (e_1, \beta_1; \ldots; e_r, \beta_r)$ if and only if the preprojective algebra Π_0 has θ-stable representations of dimension vectors β_i for all $1 \leq i \leq r$.*

In particular, Π_θ has a simple representation of dimension vector α if and only if Π_0 has a θ-stable representation of dimension vector α.

The variety $M_\alpha^{ss}(\Pi_0, \theta)$ is locally controlled by noncommutative algebras. Indeed, as in the case of moduli spaces of θ-semistable quiver-representations, it is locally isomorphic to $\mathtt{iss}_\alpha\ (\Pi_0)_\Sigma$ for some universal localization of Π_0. We can determine the α-smooth locus of the corresponding sheaf of Cayley-Hamilton algebras.

Proposition 8.8 *Let $\alpha \in \Sigma_\theta$, then the α-smooth locus of $M_\alpha^{ss}(\Pi_0, \theta)$ is the open subvariety $M_\alpha^s(\Pi_0, \theta)$ of θ-stable representations of Π_0.*

In particular, if the sheaf of Cayley-Hamilton algebras over $M_\alpha^{ss}(\Pi_0, \theta)$ is a sheaf of α-smooth algebras if and only if α is a minimal dimension vector in Σ_θ.

Proof. As $\alpha \in \Sigma_\theta$ we know that $\mathtt{iss}_\alpha\ \Pi_\theta$ has dimension $2p_Q(\alpha) = 2 - T_Q(\alpha, \alpha)$. By the hyper-Kähler correspondence so is the dimension of $M_\alpha^{ss}(\Pi_0, \theta)$, whence the open subset of $\mu_{\mathbb{C}}^{-1}(\underline{0})$ consisting of θ-semistable representations has dimension

$$\alpha.\alpha - 1 + 2p_Q(\alpha)$$

as there are θ-stable representations in it. Take a $GL(\alpha)$-closed orbit $\mathcal{O}(V)$ in this open set. That is, V is the direct sum of θ-stable subrepresentations

$$V = S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$$

with S_i a θ-stable representation of Π_0 of dimension vector β_i occurring in V with multiplicity e_i whence $\alpha = \sum_i e_i \beta_i$.

As all S_i are Π_0-representations we can determine the local quiver Q_V by the knowledge of all $Ext^1_{\Pi_0}(S_i, S_j)$ from proposition 5.12

$$Ext^1_{\Pi_0}(S_i, S_j) = 2\delta_{ij} - T_Q(\beta_i, \beta_j)$$

But then the dimension of the normal space to the orbit is

$$dim\ Ext^1_{\Pi_0}(V, V) = 2\sum_{i=1}^r e_i - T_Q(\alpha, \alpha)$$

whence the étale local structure in an n-smooth point is of the form $GL(\alpha) \times^G L(\tau) Ext^1(V, V)$ where $\tau = (e_1, \ldots, e_r)$ and is therefore of dimension

$$\alpha.\alpha + \sum_{i=1}^2 e_i^2 - T_Q(\alpha, \alpha)$$

This number is equal to the dimension of the subvariety of θ-semistable representations of Π_0 which has dimension $\alpha.\alpha - 1 + 2 - T_Q(\alpha, \alpha)$ if and only if $r = 1$ and $e_1 = 1$, that is if V is θ-stable. $\square$

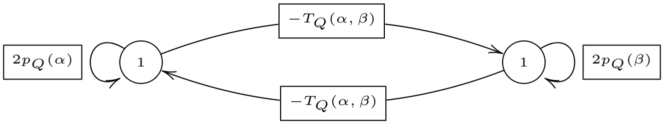
Even in points of $M^{ss}_\alpha(\Pi_0, \theta)$ which are not in the α-smooth locus we can use the local quiver to deduce combinatorial properties of the set of dimension vectors Σ_θ of simple representations of Π_θ.

Proposition 8.9 *Let $\alpha, \beta \in \Sigma_\theta$, then*

1. *If $T(\alpha, \beta) \leq -2$ then $\alpha + \beta \in \Sigma_\theta$,*
2. *If $T(\alpha, \beta) \geq -1$ then $\alpha + \beta \notin \Sigma_\theta$.*

Proof. The property that α and β are Schur roots of Q such that $T_Q(\alpha, \beta) \leq -2$ ensures that $\gamma = \alpha + \beta$ is a Schur root of Q and hence that $\mu_{\mathbb{C}}^{-1}(\theta \mathbb{1}_\gamma$ has dimension $\gamma.\gamma - 1 + 2p_Q(\gamma)$, whence so is the subvariety of θ-semistable γ-dimensional representations of Π_0. We have to prove that Π_0 has a θ-stable γ-dimensional representation.

Let $V = S \oplus T$ with S resp. T a θ-stable representation of Π_0 of dimension vector α resp. β (they exist by the hyper-Kähler correspondence). But then the local quiver Q_V has the following form



and by a calculation similar to the one in the foregoing proof we see that the image of the slice morphism in the space $GL(\gamma) \times^{\mathbb{C}^* \times \mathbb{C}^*} \mathtt{rep}_{(1,1)}\ Q_V$ has codimension 1. However, as $T_Q(\alpha,\beta) \leq -2$ there are at least 3 algebraically independent new invariants coming from the non-loop cycles in Q_V, so they cannot all vanish on the image. This means that $(1,\alpha;1,\beta)$ cannot be the generic type for θ-semistables of dimension γ, so by the stratification result, there must exist θ-stables of dimension γ.

For the second assertion, assume that $\gamma = \alpha+\beta$ is the dimension vector of a simple representation of Π_θ, then $\mathtt{iss}_\gamma\ \Pi_\theta$ has dimension $2p_Q(\gamma) = 2 - T_Q(\alpha,\beta,\alpha+\beta) = 2p_Q(\alpha) + 2p_Q(\beta)$ whence so is the dimension of $M^{ss}_\gamma(\Pi_0,\theta)$. By assumption $(1,\alpha;1,\beta)$ cannot be the generic type for θ-semistable representations, but the stratum consisting of direct sums $S \oplus T$ with $S \in M^s_\alpha(\Pi_0,\theta)$ and $T \in M^s_\beta(Q,\theta)$ has the same dimension as the total space, a contradiction. $\square$

The first part of the foregoing proof can also be used to show that usually the moduli spaces $M^{ss}_\alpha(\Pi_),\theta)$ and the quotient varieties $\mathtt{iss}_\alpha\ \Pi_\theta$ have lots of singularities.

Proposition 8.10 *Let $\alpha \in |sigma_\theta$ such that $\alpha = \beta+\gamma$ with $\beta,\gamma \in \Sigma_\theta$. Then,*

$$M^{ss}_\alpha(\Pi_0,\theta) \qquad \textit{and} \qquad \mathtt{iss}_\alpha\ \Pi_\theta$$

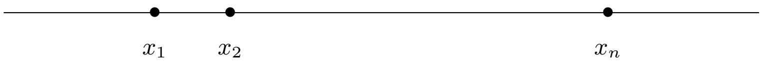
is singular along the stratum of points of type $(1,\beta;1,\gamma)$.

Proof. The quotient space of the local quiver situation (as in the foregoing proof) contains singularities at the trivial representations which remain singularities in any codimension one subvariety. $\square$

Still, if α is a minimal dimension vector in Σ_θ, the varieties $M^{ss}_\alpha(\Pi_0,\theta)$ and $\mathtt{iss}_\alpha\ \Pi_\theta$ are smooth. In fact, we will show in section 8.7 that the affine smooth variety $\mathtt{iss}_\alpha\ \Pi_\theta$ is in fact a coadjoint orbit.

8.6 Calogero particles

The *Calogero system* is a classical particle system of n particles on the real line with inverse square potential.



That is, if the i-th particle has position x_i and velocity (momentum) y_i, then the Hamiltonian is equal to

$$H = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

The Hamiltonian *equations of motions* is the system of $2n$ differential equations

$$\begin{cases} \dfrac{dx_i}{dt} & = \quad \dfrac{\partial H}{\partial y_i} \\[2ex] \dfrac{dy_i}{dt} & = \quad -\dfrac{\partial H}{\partial x_i} \end{cases}$$

This defines a dynamical system which is *integrable* .

A convenient way to study this system is as follows. Assign to a position defined by the $2n$ vector $(x_1, y_1; \dots, x_n, y_n)$ the couple of *Hermitian* $n \times n$ matrices

$$X = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 & \frac{i}{x_1 - x_2} & \cdots & \cdots & \frac{i}{x_1 - x_n} \\ \frac{i}{x_2 - x_1} & y_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{i}{x_{n-1} - x_n} \\ \frac{i}{x_n - x_1} & \cdots & \cdots & \frac{i}{x_n - x_{n-1}} & y_n \end{bmatrix}$$

Physical quantities are given by invariant polynomial functions under the action of the unitary group $U_n(\mathbb{C})$ under simultaneous conjugation. In particular one considers the functions

$$F_j = tr\ \frac{Y^j}{j}$$

For example,

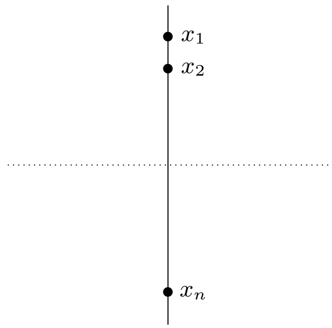
$$\begin{cases} tr(Y) = \sum y_i & \text{the total momentum} \\ \frac{1}{2} tr(Y^2) = \frac{1}{2}\sum y_i^2 - \sum_{i<j} \frac{1}{(x_i - x_j)^2} & \text{the Hamiltonian} \end{cases}$$

We can now consider the $U_n(\mathbb{C})$-translates of these matrix couples. This is shown to be a manifold with a free action of $U_n(\mathbb{C})$ such that the orbits are in one-to-one correspondence with points $(x_1, y_1; \ldots; x_n, y_n)$ in the phase space (that is, we agree that two such $2n$ tuples are determined only up to permuting the couples (x_i, y_i). The n-functions F_j give a completely integrable system on the phase space via *Liouville's theorem* , see for example [1].

In the classical case, all points are assumed to lie on the real axis and the potential is repulsive so that collisions do not appear. G. Wilson [85] considered an alternative where the points are assumed to lie in the complex numbers and such that the potential is attractive (to allow for collisions), that is, the Hamiltonian is of the form

$$H = \frac{1}{2}\sum_i y_i^2 - \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

giving again rise to a dynamical system via the equations of motion. One recovers the classical situation back if the particles are assumed only to move on the imaginary axis.



In general, we want to extend the phase space of n distinct points analytically to allow for collisions.

When all the points are distinct, that is, if all eigenvalues of X are distinct we will see in a moment that there is a unique $GL_n(\mathbb{C})$-orbit of couples of $n \times n$ matrices (up to permuting the n couples (x_i, y_i)).

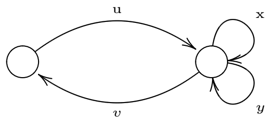
$$X = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 & \frac{1}{x_1 - x_2} & \cdots & \cdots & \frac{1}{x_1 - x_n} \\ \frac{1}{x_2 - x_1} & y_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{x_{n-1} - x_n} \\ \frac{1}{x_n - x_1} & \cdots & \cdots & \frac{1}{x_n - x_{n-1}} & y_n \end{bmatrix}$$

For matrix couples in this standard form one verifies that

$$[Y, X] + \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} = \mathbb{1}_n$$

This equality suggests an approach to extend the phase space of n distinct complex Calogero particles to allow for collisions.

Assign the representation $(X, Y, u, v) \in \mathtt{rep}_\alpha\ \mathbb{M}$ where $\alpha = (1, n)$ and $\mathbb{M}$ is the path algebra of the quiver Q^d is



where X and Y are the matrices above and where

$$u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$$

Recall that the complex moment map for this quiver-setting is defined to be

$$\begin{array}{ccc} \mathtt{rep}_\alpha\ Q^d = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*} & \xrightarrow{\quad\mu\quad} & \mathbb{C} \oplus M_n(\mathbb{C}) \\ (X, Y, u, v) & \mapsto & (-v.u, [Y, X] + u.v) \end{array}$$

Therefore, the above equation entails that $(X, Y, u, v) \in \mu_{\mathbb{C}}^{-1}(\theta\mathbb{1}_\alpha)$ where $\theta = (-n, 1)$, that is $(X, Y, u, v) \in \mathtt{rep}_\alpha\ \Pi_\theta$. Observe that $\alpha = (1, n) \in \Sigma_\theta$ (in fact, α is a minimal element in Σ_θ), whence theorem 8.7, $\mathtt{rep}_\alpha\ \Pi_\theta$ is an irreducible complete intersection of dimension $d = n^2 + 2n$ and there are α-dimensional simple representations of Π_θ. In particular, $\mathtt{iss}_\alpha\ \Pi_\theta$ is an irreducible variety of dimension $2n$.

We can define the *phase space for Calogero collisions* of n particles to be the quotient space

$$Calo_n = \mathtt{iss}_\alpha\ \Pi_\theta$$

In a moment we will show that this is actually an orbit-space and :

Theorem 8.15 *The phase space $Calo_n$ of Calogero collisions of n-particles is a connected complex manifold of dimension $2n$.*

Theorem 8.16 *Let* $\mathtt{rep}_\alpha\ \mathbb{M} \xrightarrow{\mu_{\mathbb{C}}} M^0_\alpha(\mathbb{C})$ *be the complex moment map, then any* $V = (X, Y, u, v) \in \mathtt{rep}_\alpha\ \Pi_\theta$ *is a* θ*-stable representation. Therefore,*

$$Calo_n = \mu_{\mathbb{C}}^{-1}(\theta \mathbb{1}_\alpha)/GL(\alpha) = \mathtt{iss}_\alpha\ \Pi_\theta \simeq (\mu_{\mathbb{C}}^{-1}(\theta \mathbb{1}_\alpha) \cap \mathtt{rep}^s_\alpha\ \mathbb{M})/GL(\alpha) = M^s_\alpha(\Pi_\theta, \theta)$$

and is therefore a complex manifold of dimension $2n$*, which is connected by theorem 8.7.*

Proof. The result will follow if we can prove that any Calogero quadruple (X, Y, u, v) has the property that u is a cyclic vector, that is, lies in $\mathtt{rep}^s_\alpha\ \mathbb{M}$.

Assume that U is a subspace of $\mathbb{C}^n$ stable under X and Y and containing u. U is then also stable under left multiplication with the matrix

$$A = [X, Y] + \mathbb{1}_n$$

and we have that $tr(A \mid U) = tr(\mathbb{1}_n \mid U) = dim\ U$. On the other hand, $A = u.v$ and therefore

$$A. \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} . \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} . \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = (\sum_{i=1}^n v_i c_i) \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Hence, if we take a basis for U containing u, then we have that

$$tr(A \mid U) = a$$

where $A.u = au$, that is $a = \sum u_i v_i$.

But then, $tr(A \mid U) = dim\ U$ is independent of the choice of U. Now, $\mathbb{C}^n$ is clearly a subspace stable under X and Y and containing u, so we must have that $a = n$ and so the only subspace U possible is $\mathbb{C}^n$ proving cyclicity of u with respect to the matrix-couple (X, Y). □

Again, it follows that we can cover the phase space $Calo_n$ by open subsets

$$Calo_n\ (\sigma) = \{(X, Y, u, v) \quad \text{in } \sigma\text{-standard form such that } [Y, X] + u.v = \mathbb{1}_n\ \}$$

where σ runs over the Hilbert n-stairs.

Example 8.5 (The phase space $Calo_2$) Consider $Calo_2$ ($\boxed{\circ}$). Because

$$[\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}, \begin{bmatrix} c & d \\ e & f \end{bmatrix}] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \begin{bmatrix} g & h \end{bmatrix} - \mathbb{1}_2 = \begin{bmatrix} g - d + ae - 1 & h + af - ac - bd \\ c - f + be & d - ae - 1 \end{bmatrix}$$

We obtain after taking Groebner bases that the defining equations are

$$\begin{cases} g &= 2 \\ h &= b \\ f &= c + eh \\ d &= 1 + ae \end{cases}$$

In particular we find

$$Calo_2\ (\boxed{\circ}) = \{(\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}, \begin{bmatrix} c & 1+ae \\ e & c+be \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & b \end{bmatrix}) \mid a,b,c,e \in \mathbb{C}\} \simeq \mathbb{C}^4$$

and a similar description holds for $Calo_2\ (\boxed{\bullet})$.

Example 8.6 (The phase space $Calo_3$) We claim that

$$Calo_3\ (\begin{matrix} \boxed{\circ} & \\ \boxed{\bullet} & \boxed{} \end{matrix}) \simeq \mathbb{C}^6$$

For, if we compute the 3×3 matrix

$$[\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}, \begin{bmatrix} 0 & g & h \\ 0 & i & j \\ 1 & k & l \end{bmatrix}] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} . \begin{bmatrix} m & n & o \end{bmatrix} - \mathbb{1}_3$$

then the Groebner basis for its entries gives the following defining equations

$$\begin{cases} m &= 3 \\ n &= c+k \\ o &= i+l \\ f &= k \\ d &= o-l \\ g &= 2+b \\ l &= g-ej-kl+ko \\ h &= 2jk+2l^2-jn-3lo+o^2 \\ a &= 2k^2-2el-kn+eo \end{cases}$$

In a similar manner one can show that

$$Calo_3\ (\begin{matrix} \boxed{\circ} & \\ \boxed{} & \boxed{\circ} \end{matrix}) \simeq \mathbb{C}^6 \qquad \text{but} \qquad Calo_3\ (\begin{matrix} \boxed{\circ} & \\ \boxed{} & \boxed{\bullet} \end{matrix})$$

is again more difficult to describe.

We can identify the classical Calogero situations as an open subset of $Calo_n$.

Proposition 8.11 *Let $(X, Y, u, v) \in \mathtt{rep}_\alpha\ \Pi_\theta$ and suppose that X is diagonalizable. Then*

1. all eigenvalues of X are distinct, and

2. the $GL(\alpha)$-orbit contains a representative such that

$$X = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad Y = \begin{bmatrix} \alpha_1 & \frac{1}{\lambda_1-\lambda_2} & \dots & \dots & \frac{1}{\lambda_1-\lambda_n} \\ \frac{1}{\lambda_2-\lambda_1} & \alpha_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{\lambda_{n-1}-\lambda_n} \\ \frac{1}{\lambda_n-\lambda_1} & \dots & \dots & \frac{1}{\lambda_n-\lambda_{n-1}} & \alpha_n \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$$

and this representative is unique up to permutation of the n couples (λ_i, α_i).

Proof. Choose a representative with X a diagonal matrix as indicated. Equating the diagonal entries in $[Y,X] + u.v = \mathbb{1}_n$ we obtain that for all $1 \leq i \leq n$ we have $u_i v_i = 1$. Hence, none of the entries of

$$[X,Y] + \mathbb{1}_n = u.v$$

is zero. Consequently, by equating the (i,j)-entry it follows that $\lambda_i \neq \lambda_j$ for $i \neq j$.

The representative with X a diagonal matrix is therefore unique up to the action of a diagonal matrix D and of a permutation. The freedom in D allows us to normalize u and v as indicated, the effect of the permutation is described in the last sentence.

Finally, the precise form of Y can be calculated from the normalized forms of X, u and v and the equation $[Y,X] + u.v = \mathbb{1}_n$. □

Invoking the hyper-Kähler structure on $\mathtt{rep}_\alpha\ \mathbb{M}$ we have by theorem 8.14an homeomorphism, in fact in this case a C^∞-diffeomorphism between the Calogero phase-space and the Hilbert scheme

$$Calo_n = \mathtt{iss}_\alpha\ \Pi_\theta \xrightarrow{h.} M^{ss}_\alpha(\Pi_0, \theta) = Hilb_n$$

8.7 Coadjoint orbits

In this section we will give an important application of `noncommutative geometry@n` developed in the foregoing chapter. If α is a minimal dimension vector in Σ_θ we will prove that the quotient variety $\mathtt{iss}_\alpha\ \Pi_\theta$ is smooth and a coadjoint orbit for the dual of the necklace algebra. In particular, the phase space of Calogero particles is a coadjoint orbit.

We fix a quiver Q on k vertices, a dimension vector $\alpha \in \mathbb{N}^k$ and a character $\theta \in \mathbb{Z}^k$ such that $\theta(\alpha) = 0$ with corresponding weight $\theta\mathbb{1}_\alpha$. Recall that Σ_θ is the subset of dimension vectors α such that

$$p_Q(\alpha) > p_Q(\beta_1) + \ldots + p_Q(\beta_r)$$

for all decompositions $\alpha = \beta_1 + \ldots + \beta_r$ with the $\beta_i \in \Delta^+\theta$, that is, β_i is a positive root for the quiver Q and $\theta(\beta_i) = 0$. With Σ_θ^{min} we will denote the subset of *minimal dimension vectors* in Σ_θ, that is, such that for all $\beta < \alpha$ we have $\beta \notin \Sigma_\theta$.

Proposition 8.12 *If $\alpha \in \Sigma_\theta^{min}$, then the deformed preprojective algebra Π_θ is α-smooth, that is $\mathtt{rep}_\alpha\ \Pi_\theta$ is a smooth $GL(\alpha)$-variety of dimension $d = \alpha.\alpha - 1 + 2p_Q(\alpha)$.*

Moreover, the quotient variety $\mathtt{iss}_\alpha\ \Pi_\theta$ is a smooth variety of dimension $2p_Q(\alpha)$, and the quotient map

$$\mathtt{rep}_\alpha\ \Pi_\theta \twoheadrightarrow \mathtt{iss}_\alpha\ \Pi_\theta$$

is a principal $PGL(\alpha)$-fibration, so determines a central simple algebra.

Proof. Let $V \in \mathtt{rep}_\alpha\ \Pi_\theta$ and let V^{ss} be its semisimplification. As Σ_θ is the set of simple dimension vectors of Π_θ by theorem 8.8 and α is a minimal dimension vector in this set, V^{ss} must be simple. As V^{ss} is the direct sum of the Jordan-Hölder components of V, it follows that $V \simeq V^{ss}$ is simple and hence its orbit $\mathcal{O}(V)$ is closed. As the stabilizer subgroup of V is $\mathbb{C}^*\mathbb{1}_\alpha$ computing the differential of the complex moment map shows that V is a smooth point of $\mu_{\mathbb{C}}^{-1}(\theta\mathbb{1}_\alpha = \mathtt{rep}_\alpha\ \Pi_\theta$.

Therefore, $\mathtt{rep}_\alpha\ \Pi_\theta$ is a smooth $GL(\alpha)$-variety whence Π_θ is α-smooth. Because each α-dimensional representation is simple, the quotient map

$$\mathtt{rep}_\alpha\ \Pi_\theta \xrightarrow{\pi} \mathtt{iss}_\alpha\ \Pi_\theta$$

is a principal $PGL(\alpha)$-fibration in the étale topology. The total space being smooth, so is the basespace $\mathtt{iss}_\alpha\ \Pi_\theta$. □

The trace pairing identifies $\mathtt{rep}_\alpha\ Q^d$ with the *cotangent bundle* $T^*\ \mathtt{rep}_\alpha\ Q$ and as such it comes equipped with a *canonical symplectic structure* . More explicit, for every arrow $(j) \xleftarrow{a} (i)$ in Q we have an $a_j \times a_i$ matrix of coordinate functions A_{uv} with $1 \leq u \leq a_j$ and $1 \leq v \leq a_i$ and for the adjoined arrow $(j) \xrightarrow[a^*]{} (i)$ in Q^d an $a_i \times a_j$ matrix of coordinate functions A^*_{vu}. The canonical symplectic structure on $\mathtt{rep}_\alpha\ Q^d$ is then induced by the closed 2-form

$$\omega = \sum_{(j) \xleftarrow{a} (i)}^{\substack{1\leq v\leq a_i \\ 1\leq u\leq a_j}} dA_{uv} \wedge dA^*_{vu}$$

This symplectic structure induces a *Poisson bracket* on the coordinate ring $\mathbb{C}[\mathtt{rep}_\alpha\ Q^d]$ by the formula

$$\{f,g\} = \sum_{j \xleftarrow{a} i}^{\substack{1 \leq v \leq a_i \\ 1 \leq u \leq a_j}} \left(\frac{\partial f}{\partial A_{uv}} \frac{\partial g}{\partial A^*_{vu}} - \frac{\partial f}{\partial A^*_{vu}} \frac{\partial g}{\partial A_{uv}} \right)$$

The basechange action of $GL(\alpha)$ on the representation space $\mathtt{rep}_\alpha\ Q^d$ is *symplectic* which means that for all *tangentvectors* $t, t' \in T\ \mathtt{rep}_\alpha\ Q^d$ we have for the induced $GL(\alpha)$ action that

$$\omega(t,t') = \omega(g.t, g.t')$$

for all $g \in GL(\alpha)$.

The infinitesimal $GL(\alpha)$ action gives a Lie algebra homomorphism

$$Lie\ PGL(\alpha) \longrightarrow Vect_\omega\ \mathtt{rep}_\alpha\ Q^d$$

which factorizes through a Lie algebra morphism H to the coordinate ring making the diagram below commute

$$\begin{tikzcd}
& Lie\ PGL(\alpha) \arrow[dl, "H=\mu^*_{\mathbb{C}}"'] \arrow[dr] & \\
\mathbb{C}[\mathtt{rep}_\alpha\ Q^d] \arrow[rr, "f \mapsto \xi_f"'] & & Vect_\omega\ \mathtt{rep}_\alpha\ Q^d
\end{tikzcd}$$

where $\mu_{\mathbb{C}}$ is the complex moment map introduced before. We say that the action of $GL(\alpha)$ on $\mathtt{rep}_\alpha\ \mathbb{M}$ is *Hamiltonian* .

This makes the ring of polynomial invariants $\mathbb{C}[\mathtt{rep}_\alpha\ Q^d]^{GL(\alpha)}$ into a *Poisson algebra* and we will write

$$\mathfrak{lie} = (\mathbb{C}[\mathtt{rep}_\alpha\ Q^d]^{GL(\alpha)}, \{-,-\})$$

for the corresponding abstract *infinite dimensional Lie algebra.*

The dual space of this Lie algebra $\mathfrak{lie}^*$ is then a *Poisson manifold* equipped with the *Kirillov-Kostant bracket* .

Evaluation at a point in the quotient variety $\mathtt{iss}_\alpha\ Q^d$ defines a linear function on $\mathfrak{lie}$ and therefore evaluation gives an embedding

$$\mathtt{iss}_\alpha\ Q^d \hookrightarrow \mathfrak{lie}^*$$

as Poisson varieties. That is, the induced map on the polynomial functions is a morphism of Poisson algebras.

Let us return to the setting of deformed preprojective algebras. So let θ be a character with $\theta(\alpha) = 0$ and corresponding weight $\theta \mathbb{1}_\alpha \in Lie\ PGL(\alpha)$.

Theorem 8.17 *Let $\alpha \in \Sigma_\theta^{min}$, then $\mathtt{iss}_\alpha\ \Pi_\theta$ is an affine symplectic manifold and the Poisson embeddings*

$$\mathtt{iss}_\alpha\ \Pi_\theta \hookrightarrow \mathtt{iss}_\alpha\ Q^d \hookrightarrow \mathfrak{lie}^*$$

make $\mathtt{iss}_\alpha\ \Pi_\theta$ into a closed coadjoint orbit of the infinite dimensional Lie algebra $\mathfrak{lie}^*$.

Proof. We know from proposition 8.12 that $\mathtt{iss}_\alpha\ \Pi_\theta$ is a smooth affine variety and that $PGL(\alpha)$ acts freely on $\mu_{\mathbb{C}}^{-1}(\theta \mathbb{1}_\alpha) = \mathtt{rep}_\alpha\ \Pi_\theta$. Moreover, the infinitesimal coadjoint action of $\mathfrak{lie}$ on $\mathfrak{lie}^*$ preserves $\mathtt{iss}_\alpha\ \Pi_\theta$ and therefore $\mathbb{C}[\mathtt{iss}_\alpha\ \Pi_\theta]$ is a quotient Lie $\overline{\mathfrak{lie}}$ algebra (for the induced bracket) of $\mathfrak{lie}$.

In general, if X is a smooth affine variety, then the differentials of polynomial functions on X span the tangent spaces at all points x of X. Therefore, if X is in addition symplectic, the infinitesimal Hamiltonian action of the Lie algebra $\mathbb{C}[X]$ (with the natural Poisson bracket) on X is infinitesimally transitive. But then, the evaluation map makes X a coadjoint orbit of the dual Lie algebra $\mathbb{C}[X]^*$.

Hence, the quotient variety $oss_\alpha\ \Pi_\theta$ is a coadjoint orbit in $\overline{\mathfrak{lie}}^*$. Therefore, the infinite dimensional group Ham generated by all *Hamiltonian flows* on $\mathtt{iss}_\alpha\ \Pi_\theta$ acts with *open* orbits.

By proposition 8.12 $\mathtt{iss}_\alpha\ \Pi_\theta$ is an irreducible variety, whence is a single Ham-orbit, finishing the proof. □

The Lie algebra $\mathfrak{lie}$ depends on the dimension vector α. By the general principle of **`noncommutative geometry@n`** we would like to construct a noncommutative variety from a family of coadjoint quotient spaces of deformed preprojective algebras. For this reason we need a larger Lie algebra, the *necklace Lie algebra*.

Recall that the necklace Lie algebra introduced in section 7.8

$$\mathfrak{neck} = \mathtt{dR}^0_{rel}\ \mathbb{C}Q^d = \frac{\mathbb{C}Q^d}{[\mathbb{C}Q^d, \mathbb{C}Q^d]}$$

is the vectorspace with basis all the necklace words w in the quiver Q^d, that is, all equivalence classes of oriented cycles in the quiver Q^d, equipped with the Kontsevich bracket

$$\{w_1, w_2\}_K = \sum_{a \in Q_a} \left(\frac{\partial w_1}{\partial a}\frac{\partial w_2}{\partial a^*} - \frac{\partial w_1}{\partial a^*}\frac{\partial w_2}{\partial a} \right) \quad \mathtt{mod}\ [\mathbb{C}Q^d, \mathbb{C}Q^d]$$

We recall that the algebra of polynomial quiver invariants $\mathbb{C}[\mathtt{iss}_\alpha\ Q^d] = \mathbb{C}[\mathtt{rep}_\alpha\ Q^d]^{GL(\alpha)}$ is generated by traces of necklace words. That is, we have a map

$$\mathfrak{neck} = \frac{\mathbb{C}Q^d}{[\mathbb{C}Q^d, \mathbb{C}Q^d]} \xrightarrow{tr} \mathfrak{lie} = \mathbb{C}[\mathtt{iss}_\alpha\ Q^d]$$

Recalling the definition of the Lie bracket on $\mathfrak{lie}$ we see that this map is actually a Lie algebra map, that is, for all necklace words w_1 and w_2 in Q^d we have the identity

$$tr\ \{w_1, w_2\}_K = \{tr(w_1), tr(w_2)\}$$

Now, the image of tr contains a set of *algebra* generators of $\mathbb{C}[\mathtt{iss}_\alpha\ Q^d]$, so the elements $tr\ \mathfrak{neck}$ are enough to separate points in $\mathtt{iss}_\alpha\ Q^d$ and in the closed subvariety $\mathtt{iss}_\alpha\ \Pi_\theta$. That is, the composition

$$\mathtt{iss}_\alpha\ \Pi_\theta \hookrightarrow \mathtt{iss}_\alpha\ Q^d \xrightarrow{tr^*} \mathfrak{neck}^*$$

is injective. Again, the differentials of functions on $\mathtt{iss}_\alpha\ \Pi_\theta$ obtained by restricting traces of necklace words span the tangent spaces at all points if the affine variety $\mathtt{iss}_\alpha\ \Pi_\theta$ is smooth. That is, we have :

Theorem 8.18 *Let $\alpha \in \Sigma_\theta^{min}$. Then, the quotient variety of the preprojective algebra $\mathtt{iss}_\alpha\ \Pi_\theta$ is an affine smooth manifold and the embeddings*

$$\mathtt{iss}_\alpha\ \Pi_\theta \hookrightarrow \mathtt{iss}_\alpha\ Q^d \hookrightarrow \mathfrak{lie}^* \hookrightarrow \mathfrak{neck}^*$$

make $\mathtt{iss}_\alpha\ \Pi_\theta$ into a closed coadjoint orbit in the dual of the necklace Lie algebra $\mathfrak{neck}$.

We have proved in section 7.8 that there is an exact sequence of Lie algebras

$$0 \longrightarrow \underbrace{\mathbb{C} \oplus \ldots \oplus C}_{k} \longrightarrow \mathfrak{neck} \longrightarrow Der_\omega\ \mathbb{C}Q^d \longrightarrow 0$$

That is, the necklace Lie algebra $\mathfrak{neck}$ is a central extension of the Lie algebra of symplectic derivations of $\mathbb{C}Q^d$. This Lie algebra corresponds to the automorphism group of all $B = \mathbb{C} \times \ldots \times \mathbb{C}$-automorphisms of the path algebra $\mathbb{C}Q^d$ preserving the *moment map element*, the commutator

$$c = \sum_{a \in Q_a} [a, a^*]$$

That is, we expect a transitive action of an extension of this automorphism group on the quotient varieties of deformed preprojective algebras $\mathtt{iss}_\alpha\ \Pi_\theta$ when $\alpha \in \Sigma_\theta^{min}$. Further, it should be observed that these coadjoint cases are precisely the situations were the preprojective algebra Π_θ is α-smooth. That is, whereas the Lie algebra of vectorfields of the smooth noncommutative variety corresponding to $\mathbb{C}Q^d$ has rather unpredictable behavior on the *singular* noncommutative closed subvariety corresponding to the quotient algebra Π_θ, it behaves as expected on those α-dimensional components where Π_θ is α-smooth.

8.8 Adelic Grassmannian

At the moment of writing it is unclear which coadjoint orbits $\mathtt{iss}_\alpha\ \Pi_\theta$ should be taken together to form an object in `noncommutative geometry@n`, for a general quiver Q. In this section we will briefly recall how the phase spaces $Calo_n$ of Calogero particles can be assembled together to form an infinite dimensional cellular complex, the *adelic Grassmannian* Gr^{ad}.

Let $\lambda \in \mathbb{C}$, a subset $V \subset \mathbb{C}[x]$ is said to be λ*-primary* if there is some power $r \in \mathbb{N}_+$ such that

$$(x-\lambda)^r\mathbb{C}[x] \subset V \subset \mathbb{C}[x]$$

A subset $V \subset \mathbb{C}[x]$ is said to be *primary decomposable* if it is the finite intersection

$$V = V_{\lambda_1} \cap \ldots \cap V_{\lambda_r}$$

with $\lambda_i \neq \lambda_j$ if $i \neq j$ and V_{λ_i} is a λ_i-primary subset. Let k_{λ_i} be the codimension of V_{λ_i} in $\mathbb{C}[x]$ and consider the polynomial

$$p_V(x) = \prod_{i=1}^{r}(x-\lambda_i)^{k_{\lambda_i}}$$

Finally, take $W = p_V(x)^{-1}V$, then W is a vectorsubspace of the rational functionfield $\mathbb{C}(x)$ in one variable.

Definition 8.2 *The* adelic Grassmannian Gr^{ad} *is the se of subspaces* $W \subset \mathbb{C}(x)$ *that arise in this way.*

We can decompose Gr^{ad} in affine cells as follows. For a fixed $\lambda \in \mathbb{C}$ we define

$$Gr_\lambda = \{W \in Gr^{ad} \ \mid \ \exists k,l \in \mathbb{N} \ : \ (x-\lambda)^k\mathbb{C}[x] \subset W \subset (x-\lambda)^{-l}\mathbb{C}[x]\}$$

Then, we can write every element $w \in W$ as a *Laurent series*

$$w = \alpha_s(x-\lambda)^s + \text{ higher terms}$$

Consider the increasing set of integers $S = \{s_0 < s_1 < \ldots\}$ consisting of all *degrees* s of elements $w \in W$. Now, define natural numbers

$$v_i = i - s_i \qquad \text{then} \qquad v_0 \geq v_1 \geq \ldots \geq v_z = 0 = v_{z+1} = \ldots$$

That is, to $W \in Gr_\lambda$ we can associate a *partition*

$$p(W) = (v_0, v_1, \ldots, v_{z-1})$$

Conversely, if p is a partition of some n, then the set of all $W \in Gr_\lambda$ with associated partition $p_W = p$ form an affine space $\mathbb{A}^n$ of dimension n. Hence, Gr_λ has a cellular structure indexed by the set of all partitions.

As $Gr^{ad} = \prod'_{\lambda \in \mathbb{C}} Gr_\lambda$ because for any $W \in Gr^{ad}$ there are uniquely determined $W(\lambda_i) \in Gr_{\lambda_i}$ such that $W = W(\lambda_1) \cap \ldots \cap W(\lambda_r)$, there is a natural number n associated to W where $n = |p_i|$ where $p_i = p(W(\lambda_i))$ is the partition determined by $W(\lambda_i)$. Again, all $W \in Gr^{ad}$ with corresponding $(\lambda_1, p_1; \ldots; \lambda_r, p_r)$ for an affine cell $\mathbb{A}^n$ of dimension n. In his way, the adelic Grassmannian Gr^{ad} becomes an infinite cellular space with the cells indexed by r-tuples of complex numbers and partitions for all $r \geq 0$. The adelic Grassmannian is an important object in the theory of dynamical systems as it parametrizes rational solutions of the *KP hierarchy* . A surprising connection between Gr^{ad} and the Calogero system was discovered by G. Wilson in [85].

Theorem 8.19 *Let $Gr^{ad}(n)$ be the collection of all cells of dimension n in gr^{ad}, then there is a set-theoretic bijection*

$$Gr^{ad}(n) \longleftrightarrow Calo_n$$

between $Gr^{ad}(n)$ and the phase space of n Calogero particles.

The adelic Grassmannian also appears in the study of right ideals of the first *Weyl algebra*

$$A_1(\mathbb{C}) = \frac{\mathbb{C}\langle x, y\rangle}{(xy - yx - 1)}$$

which is an infinite dimensional simple $\mathbb{C}$-algebra, having no finite dimensional representations. Consider right ideals of $A_1(\mathbb{C})$ under isomorphism, that is

$$\mathfrak{p} \simeq \mathfrak{p}' \qquad \text{iff} \qquad f.\mathfrak{p} = g.\mathfrak{p}' \quad \text{for some } f, g \in A_1(\mathbb{C}).$$

If we denote with $D_1(\mathbb{C})$ the *Weyl skewfield* , that is, the field of fractions of $A_1(\mathbb{C})$, then the foregoing can also be expressed as

$$\mathfrak{p} \simeq \mathfrak{p}' \qquad \text{iff} \qquad \mathfrak{p} = h.\mathfrak{p}' \quad \text{for some } h \in D_1(\mathbb{C}).$$

The set of isomorphism classes will be denoted by $Weyl$.

The connection between right ideals of $A_1(\mathbb{C})$ and gr^{ad} is contained (in disguise) in the paper of R. Cannings and M. Holland [16]. $A_1(\mathbb{C})$ acts as differential operators on $\mathbb{C}[x]$ and for every right ideal I of $A_1(\mathbb{C})$ they show that $I.\mathbb{C}[x]$ is primary decomposable. Conversely, if $V \subset \mathbb{C}[x]$ is primary decomposable, they associate the right ideal

$$I_V = \{\theta \in A_1(\mathbb{C}) \mid \theta.\mathbb{C}[x] \subset V\}$$

of $A_1(\mathbb{C})$ to it. Moreover, isomorphism classes of right ideals correspond to studying primary decomposable subspaces under multiplication with polynomials. Hence,

$$Gr^{ad} \simeq Weyl$$

The group $Aut\ A_1(\mathbb{C})$ of $\mathbb{C}$-algebra automorphisms of $A_1(\mathbb{C})$ acts on the set of right ideals of $A_1(\mathbb{C})$ and respects the notion of isomorphism whence acts on $Weyl$. The group $Aut\ A_1(\mathbb{C})$ is generated by automorphisms σ_i^f defined by

$$\begin{cases} \sigma_1^f(x) & = x + f(y) \\ \sigma_1^f(y) & = y \end{cases} \quad \text{with } f \in \mathbb{C}[y], \quad \begin{cases} \sigma_2^f(x) & = x \\ \sigma_2^f(y) & = y + f(x) \end{cases} \quad \text{with } f \in \mathbb{C}[x]$$

We claim that for any polynomial in one variable $f(z) \in \mathbb{C}[z]$ we have that

$$f(xy).x^n = x^n.f(xy-n) \qquad \text{and} \qquad f(xy).y^n = y^n.f(xy+n)$$

Indeed, we have $(xy).x = x.(yx) = x.(xy-1)$ and therefore

$$f(xy).x = x.f(xy-1)$$

from which the claim follows by recursion. In particular, as $x^n.y^n = x^{n-1}(xy)y^{n-1} = x^{n-1}y^{n-1}(xy+n-1)$ we get by recurrence that

$$x^n y^n = xy(xy+1)(xy+2)\dots(xy+n-1)$$

In calculations with the Weyl algebra it is often useful to decompose $A_1(\mathbb{C})$ in weight spaces. For $t \in \mathbb{Z}$ let us define

$$A_1(\mathbb{C})(t) = \{ \ f \in A_1(\mathbb{C}) \ \mid \ [xy, f] = tf \ \}$$

then the foregoing asserts that $A_1(\mathbb{C}) = \oplus_{t\in\mathbb{Z}} A_1(\mathbb{C})(t)$ where $A_1(\mathbb{C})(t)$ is equal to

$$\begin{cases} y^t\mathbb{C}[xy] = \mathbb{C}[xy]y^t & \text{for } t \geq 0 \\ x^{-t}\mathbb{C}[xy] = \mathbb{C}[xy]x^{-t} & \text{for } t < 0. \end{cases}$$

For a natural number $n \geq 1$ we define the *n-th canonical right ideal* of $A_1(\mathbb{C})$ to be

$$\mathfrak{p}_n = x^{n+1}A_1(\mathbb{C}) + (xy+n)A_1(\mathbb{C}).$$

Lemma 8.8 *The weight space decomposition of $\mathfrak{p}_n$ is given for $t \in \mathbb{Z}$*

$$\mathfrak{p}_n(t) = x^{n+1}A_1(\mathbb{C})(t+n+1) + (xy+n)A_1(\mathbb{C})(t)$$

which is equal to

$$\begin{cases} (xy+n)\mathbb{C}[xy]y^t & \text{for } t \geq 0, \\ (xy+n)\mathbb{C}[xy]x^{-t} & \text{for } -n \leq t < 0, \\ \mathbb{C}[xy]x^{-t} & \text{for } t < -n. \end{cases}$$

Proof. Let $t = -1$, then $\mathfrak{p}_n(-1)$ is equal to

$$x^{n+1}\mathbb{C}[xy]y^n + (xy+n)\mathbb{C}[xy]x$$

Using $x^{n+1}y^{n+1} = xy(xy+1)\dots(xy+n)$ this is equal to

$$xy(xy+1)\dots(xy+n)\mathbb{C}[xy]y^{-1} + (xy+n)\mathbb{C}[xy]x$$

The first factor is $(xy+1)\dots(xy+n)\mathbb{C}[xy]x$ from which the claim follows. For all other t the calculations are similar. $\square$

One can show that $\mathfrak{p}_n \not\simeq \mathfrak{p}_m$ whenever $n \neq m$ so the isomorphism classes $[\mathfrak{p}_n]$ are distinct points in $Weyl$ for all n. We define

$$Weyl_n = Aut\ A_1(\mathbb{C}).[\mathfrak{p}_n] = \{\ [\sigma(\mathfrak{p}_n)] \quad \forall \sigma \in Aut\ A_1(\mathbb{C})\}$$

the orbit in $Weyl$ of the point $[\mathfrak{p}_n]$ under the action of the automorphism group.

Example 8.7 (The Weyl right ideals $Weyl_1$) For a point $(a,b) \in \mathbb{C}^2$ we define a right ideal of $A_1(\mathbb{C})$ by

$$\mathfrak{p}_{a,b} = (x+a)^2 A_1(\mathbb{C}) + ((x+a)(y+b)+1)A_1(\mathbb{C}).$$

Observe that $\mathfrak{p}_1 = \mathfrak{p}_{0,0}$. Consider the action of the automorphism σ_2^f on these right ideals. As $f \in \mathbb{C}[x]$ we can write

$$f = f(-a) + (x+a)f_1 \qquad \text{with } f_1 \in \mathbb{C}[x].$$

Then, recalling the definition of σ_2^f we have

$$\begin{aligned}\sigma_2^f(\mathfrak{p}_{a,b}) &= (x+a)^2 A_1(\mathbb{C}) + ((x+a)(y+b+f(-a)+(x+a)f_1)+1)A_1(\mathbb{C}) \\ &= (x+a)^2 A_1(\mathbb{C}) + ((x+a)(y+b+f(-a))+1)A_1(\mathbb{C}) = \mathfrak{p}_{a,b+f(-a)}\end{aligned}$$

Now, consider the action of an automorphism σ_1^f. We claim that

$$\mathfrak{p}_{a,b} = A_1(\mathbb{C}) \cap (y+b)^{-1}(x+a)A_1(\mathbb{C})$$

This is easily verified on the special case $\mathfrak{p}_1$ using the above lemma, the arbitrary case follows by changing variables. We have

$$\begin{aligned}\mathfrak{p}_{a,b} &= A_1(\mathbb{C}) \cap (y+b)^{-1}(x+a)A_1(\mathbb{C}) \\ &\simeq (x+a)^{-1}(y+b)A_1(\mathbb{C}) \cap A_1(\mathbb{C}) \quad (\text{multiply with } h = (x+a)^{-1}(y+b)) \\ &= (y+b)^2 A_1(\mathbb{C}) + ((y+b)(x+a)-1)A_1(\mathbb{C}) \stackrel{def}{=} \mathfrak{q}_{b,a}\end{aligned}$$

Writing $f = f(-b) + (y+b)f_1$ with $f_1 \in \mathbb{C}[y]$ we then obtain by mimicking the foregoing

$$\begin{aligned} \sigma_1^f(\mathfrak{p}_{a,b}) & \simeq \sigma_1^f(\mathfrak{q}_{b,a}) \\ & = \mathfrak{q}_{b,a+f(-b)} \\ & \simeq \mathfrak{p}_{a+f(-b),b} \end{aligned}$$

and therefore there is an $h \in D_1(\mathbb{C})$ such that $\sigma_1^f(\mathfrak{p}_{a,b}) = h\mathfrak{p}_{a+f(-b),b}$.

As the group $Aut\ A_1(\mathbb{C})$ is generated by the automorphisms σ_1^f and σ_2^f we see that

$$Weyl_1 = Aut\ A_1(\mathbb{C}).[\mathfrak{p}_1] \hookrightarrow \{ [\mathfrak{p}_{a,b} \mid a,b \in \mathbb{C} \}$$

Moreover, this inclusion is clearly surjective by the above arguments. Finally, we claim that $Weyl_1 \simeq \mathbb{C}^2$. That is we have to prove that if

$$\mathfrak{p}_{a,b} = h.\mathfrak{p}_{a',b'} \quad \Rightarrow \quad (a,b) = (a',b').$$

Observe that $A_1(\mathbb{C}) \hookrightarrow \mathbb{C}(x)[y,\delta]$ where this algebra is the differential polynomial algebra over the field $\mathbb{C}(x)$ and is hence a right principal ideal domain. That is, we may assume that the element $h \in D_1(\mathbb{C})$ actually lies in $\mathbb{C}(x)[y,\delta]$. Now, induce the filtration by y-degree on $\mathbb{C}(x)[y,\delta]$ to the subalgebra $A_1(\mathbb{C})$. This is usually called the *Bernstein filtration* . Because $A_1(\mathbb{C})$ and $\mathbb{C}(x)[y,\delta]$ are domains we have for all $f \in A_1(\mathbb{C})$ that

$$deg(h.f) = deg(h) + deg(f).$$

Now, as both $\mathfrak{p}_{a,b}$ and $\mathfrak{p}_{a'.b'}$ contain elements of degree zero x^2+a resp. x^2+a' we must have that $h \in \mathbb{C}(x)$.

View y as the differential operator $-\frac{\partial}{\partial x}$ on $\mathbb{C}[x]$ and define for every right ideal $\mathfrak{p}$ of $A_1(\mathbb{C})$ its *evaluation* to be the *subspace* of polynomials

$$ev(\mathfrak{p}) = \{ D.f \mid D \in \mathfrak{p} \ , \ f \in \mathbb{C}[x] \}$$

where $D.f$ is the evaluation of the differential operator on f. One calculates that

$$ev(\mathfrak{p}_{a,b}) = \mathbb{C}(1 + b(x+a)) + (x+a)^2\mathbb{C}[x]$$

and as from $\mathfrak{p}_{a,b} = h.\mathfrak{p}_{a',b'}$ and $h \in \mathbb{C}(x)$ follows that

$$ev(\mathfrak{p}_{a,b}) = hev(\mathfrak{p}_{a',b'})$$

we deduce that $h \in \mathbb{C}^*$ and hence that $\mathfrak{p}_{a,b} = \mathfrak{p}_{a',b'}$ and $(a,b) = (a',b')$.

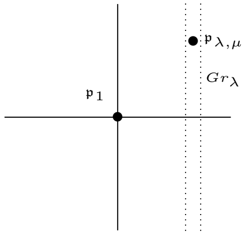
Yu. Berest and G. Wilson proved in [7] that the Cannings-Holland correspondence respects the automorphism orbit decomposition.

Theorem 8.20 *We have $Weyl = \bigsqcup_n Weyl_n$ and there are set-theoretic bijections*

$$Weyl_n \longleftrightarrow Gr^{ad}(n)$$

whence also with $Calo_n$.

Example 8.8 Consider the special case $n = 1$. As $\square$ is the only partition of 1, for every $\lambda \in \mathbb{C}$, gr_λ is a one-dimensional cell $\mathbb{A}^1$, whence $Gr^{ad}(1) \simeq \mathbb{A}^2$. In fact we have



where the origin corresponds to the canonical right ideal $\mathfrak{p}_1$ and the right ideal corresponding to (λ, μ) is $\mathfrak{p}_{\lambda,\mu} = (x-\lambda)^2 A_1(\mathbb{C}) + ((x-\lambda)(y-\mu)+1)A_1(\mathbb{C})$.

Finally, let us verify that $\mathfrak{p}_n$ should correspond to a point in $Gr^{ad}(n)$. As $\mathfrak{p}_{\mathfrak{n}} = x^{n+1}A_1(\mathbb{C}) + (xy+n)A_1(\mathbb{C})$ we have that

$$\mathfrak{p}_n.\mathbb{C}[x] = \mathbb{C} + \mathbb{C}x + \ldots + \mathbb{C}x^{n-1} + (x^{n+1})\mathbb{C}[x]$$

whence $(x^{n+1})\mathbb{C}[x] \subset \mathfrak{p}_n.\mathbb{C}[x] \subset \mathbb{C}[x]$ and converting this to Gr^{ad} the corresponding subspace is

$$(x^n)\mathbb{C}[x] \subset x^{-1}\mathfrak{p}_n.\mathbb{C}[x] \subset x^{-1}\mathbb{C}[x]$$

The associated sequence of degrees is $(-1, 0, 1, \ldots, n-2, n, \ldots)$ giving rise to the partition $p = \underbrace{(1, 1, \ldots, 1)}_{n}$ proving the claim.

If we trace the action of $Aut\ A_1(\mathbb{C})$ on $Weyl_n$ through all the identifications, we get a transitive action of $Aut\ A_1(\mathbb{C})$ on $Calo_n$. However, this action is non-differentiable hence highly non-algebraic. Berest and Wilson asked whether it is possible to identify $Calo_n$ with a coadjoint orbit in some infinite dimensional Lie algebra. We have seen before that this is indeed the case if we consider the necklace Lie algebra.

It is our hope that similar results are true for more general quivers and certain families of coadjoint orbits coming from quotient varieties of deformed preprojective algebras.

References

The results of section 8.1 are due to A. King [43]. Section 8.2 is based on the lecture notes of H-P. Kraft [50] and those of A. Tannenbaum [80]. Theorem 8.5 is due to W. Crawley-Boevey and M. Holland [22], the other results of section 8.3 are due to W. Crawley-Boevey [21]. The description of Hilbert schemes via Hilbert stairs is my interpretation of a result of M. Van den Bergh [81]. Example 8.2 is taken from the lecture notes of H. Nakajima [65]. The hyper-Kähler correspondence is classical, see for example the notes of Nakajima, [65], here we follow W. Crawley-Boevey [20]. The applications to the hyper-Kähler correspondence are due to R. Bockland and L. Le Bruyn [11]. The treatment of Calogero particles is taken from G. Wilson's paper [85]. The fact that the Calogero phase space is a coadjoint orbit is due to V. Ginzburg [29]. The extension to deformed preprojective algebras is independently due to V. Ginzburg [30] and R. Bockland and L. Le Bruyn [11]. The results on adelic Grassmannians are due to G. Wilson [85] and Y. Berest and G. Wilson [7]. More details on the connection with right ideals of the Weyl algebra can be found in the papers of R. C. Cannings and M. Holland [16] and L. Le Bruyn [54].

Bibliography

[1] V. I. Arnol′d, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 199?, Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition. MR MR1345386 (96c:70001)

[2] M. Artin, *On Azumaya algebras and finite dimensional representations of rings.*, J. Algebra **11** (1969), 532–563. MR MR0242890 (39 #4217)

[3] ———, *Brauer-Severi varieties*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin, 1982, pp. 194–210. MR MR657430 (83j:14015)

[4] ———, *Local structure of maximal orders on surfaces*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin, 1982, pp. 146–181. MR MR657428 (83k:14003)

[5] ———, *Maximal orders of global dimension and Krull dimension two*, Invent. Math. **84** (1986), no. 1, 195–222. MR MR830045 (88e:16008a)

[6] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75–95. MR MR0321934 (48 #299)

[7] Yuri Berest and George Wilson, *Automorphisms and ideals of the Weyl algebra*, Math. Ann. **318** (2000), no. 1, 127–147. MR MR1785579 (2001m:16036)

[8] David Birkes, *Orbits of linear algebraic groups*, Ann. of Math. (2) **93** (1971), 459–475. MR MR0296077 (45 #5138)

[9] Raf Bocklandt, *Smooth quiver representation spaces*, J. Algebra **253** (2002), no. 2, 296–313. MR MR1929191 (2003i:16020)

[10] ———, *Symmetric quiver settings with a regular ring of invariants*, Linear Algebra Appl. **365** (2003), 25–43, Special issue on linear algebra methods in representation theory. MR MR1987326 (2004k:16031)

[11] Raf Bocklandt and Lieven Le Bruyn, *Necklace Lie algebras and noncommutative symplectic geometry*, Math. Z. **240** (2002), no. 1, 141–167. MR MR1906711 (2003g:16013)

[12] Raf Bocklandt, Lieven Le Bruyn, and Stijn Symens, *Isolated singularities, smooth orders, and Auslander regularity*, Comm. Algebra **31** (2003), no. 12, 6019–6036. MR MR2014913 (2004j:16019)

[13] Raf Bocklandt, Lieven Le Bruyn, and Geert Van de Weyer, *Smooth order singularities*, J. Algebra Appl. **2** (2003), no. 4, 365–395. MR MR2020934 (2005e:16029)

[14] Nicolas Bourbaki, *Éléments de mathématique: groupes et algèbres de Lie*, Masson, Paris, 1982, Chapitre 9. Groupes de Lie réels compacts. [Chapter 9. Compact real Lie groups]. MR MR682756 (84i:22001)

[15] Lieven Le Bruyn, *Etale cohomology in non-commutative geometry*, TMR-Lecture Notes, 1997.

[16] R. C. Cannings and M. P. Holland, *Right ideals of rings of differential operators*, J. Algebra **167** (1994), no. 1, 116–141. MR MR1282820 (95f:16028)

[17] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR MR1433132 (98i:22021)

[18] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, and Bruno Kahn, *The Bloch-Ogus-Gabber theorem*, Algebraic K-theory (Toronto, ON, 1996), Fields Inst. Commun., vol. 16, Amer. Math. Soc., Providence, RI, 1997, pp. 31–94. MR MR1466971 (98j:14021)

[19] William Crawley-Boevey, *Geometry of representations of algebras*, Lecture Notes, 1993.

[20] William Crawley-Boevey, *On the exceptional fibres of Kleinian singularities*, Amer. J. Math. **122** (2000), no. 5, 1027–1037. MR MR1781930 (2001f:14009)

[21] ______, *Geometry of the moment map for representations of quivers*, Compositio Math. **126** (2001), no. 3, 257–293. MR MR1834739 (2002g:16021)

[22] William Crawley-Boevey and Martin P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635. MR MR1620538 (99f:14003)

[23] Joachim Cuntz and Daniel Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289. MR MR1303029 (96c:19002)

[24] Harm Derksen and Jerzy Weyman, *On the canonical decomposition of quiver representations*, Compositio Math. **133** (2002), no. 3, 245–265. MR MR1930979 (2003h:16017)

[25] M. Domokos and A. N. Zubkov, *Semi-invariants of quivers as determinants*, Transform. Groups **6** (2001), no. 1, 9–24. MR MR1825166 (2002d:16015)

[26] Edward Formanek, *The polynomial identities and invariants of $n \times n$ matrices*, CBMS Regional Conference Series in Mathematics, vol. 78, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1991. MR MR1088481 (92d:16031)

[27] Edward Formanek, Patrick Halpin, and Wen Ch'ing Winnie Li, *The Poincaré series of the ring of* 2×2 *generic matrices*, J. Algebra **69** (1981), no. 1, 105–112. MR MR613860 (82i:16020)

[28] P. Gabriel and A. V. Roiter, *Representations of finite-dimensional algebras*, Springer-Verlag, Berlin, 1997, Translated from the Russian, With a chapter by B. Keller, Reprint of the 1992 English translation. MR MR1475926 (98e:16014)

[29] Victor Ginzburg, *Non-commutative symplectic geometry and Calogero-Moser space*, preprint Chicago, preliminary version, 1999.

[30] Victor Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett. **8** (2001), no. 3, 377–400. MR MR1839485 (2002m:17020)

[31] G. B. Gurevich, *Foundations of the theory of algebraic invariants*, Translated by J. R. M. Radok and A. J. M. Spencer, P. Noordhoff Ltd., Groningen, 1964. MR MR0183733 (32 #1211)

[32] Dieter Happel and Claus Michael Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443. MR MR675063 (84d:16027)

[33] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, Berlin Heidelberg New York, 1977.

[34] Michiel Hazewinkel, *On representations of the symmetric groups, nilpotent matrices, systems, vectorbundles and Schubert cells*, Representations of algebras (Puebla, 1980), Lecture Notes in Math., vol. 903, Springer, Berlin, 1981, pp. 145–167. MR MR654708 (83g:14028)

[35] Wim H. Hesselink, *Desingularizations of varieties of nullforms*, Invent. Math. **55** (1979), no. 2, 141–163. MR MR553706 (81b:14025)

[36] Klaus Hulek, *On the classification of stable rank-r vector bundles over the projective plane*, Vector bundles and differential equations (Proc. Conf., Nice, 1979), Progr. Math., vol. 7, Birkhäuser Boston, Mass., 1980, pp. 113–144. MR MR589223 (82b:14012)

[37] V. G. Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), no. 1, 57–92. MR MR557581 (82j:16050)

[38] ______, *Infinite root systems, representations of graphs and invariant theory. II*, J. Algebra **78** (1982), no. 1, 141–162. MR MR677715 (85b:17003)

[39] M. Kapranov, *Noncommutative geometry based on commutator expansions*, J. Reine Angew. Math. **505** (1998), 73–118. MR MR1662244 (2000b:14003)

[40] Bernhard Keller, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl. **3** (2001), no. 1, 1–35 (electronic). MR MR1854636 (2004a:18008a)

[41] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Springer-Verlag, Berlin, 1973, Lecture Notes in Mathematics, Vol. 339. MR MR0335518 (49 #299)

[42] George Kempf and Linda Ness, *The length of vectors in representation spaces*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 233–243. MR MR555701 (81i:14032)

[43] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530. MR MR1315461 (96a:16009)

[44] Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, 1984. MR MR766741 (86i:58050)

[45] Friedrich Knop, *Der Scheibensatz für algebraische Transformationsgruppen*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 100–113. MR MR1044587

[46] Maxim Kontsevich, *Formal (non)commutative symplectic geometry*, The Gel′fand Mathematical Seminars, 1990–1992, Birkhäuser Boston, Boston, MA, 1993, pp. 173–187. MR MR1247289 (94i:58212)

[47] Maxim Kontsevich, *Triangulated categories and geometry*, Course at the Ecole Normale Sup., Paris, 1998.

[48] Maxim Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys. **48** (1999), no. 1, 35–72, Moshé Flato (1937–1998). MR MR1718044 (2000j:53119)

[49] H. Kraft and Ch. Riedtmann, *Geometry of representations of quivers*, Representations of algebras (Durham, 1985), London Math. Soc. Lecture Note Ser., vol. 116, Cambridge Univ. Press, Cambridge, 1986, pp. 109–145. MR MR897322 (88k:16028)

[50] Hanspeter Kraft, *Geometric methods in representation theory*, Representations of algebras (Puebla, 1980), Lecture Notes in Math., vol. 944, Springer, Berlin, 1982, pp. 180–258. MR MR672117 (84c:14007)

[51] ———, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics, D1, Friedr. Vieweg & Sohn, Braunschweig, 1984. MR MR768181 (86j:14006)

[52] ———, *Klassische Invariantentheorie. Eine Einführung*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 41–62. MR MR1044584

[53] Lieven Le Bruyn, *Trace rings of generic* 2 *by* 2 *matrices*, Mem. Amer. Math. Soc. **66** (1987), no. 363, vi+100. MR MR878906 (88b:16032)

[54] ———, *Moduli spaces for right ideals of the Weyl algebra*, J. Algebra **172** (1995), no. 1, 32–48. MR MR1320617 (96b:16025)

[55] ———, *Nilpotent representations*, J. Algebra **197** (1997), no. 1, 153–177. MR MR1480781 (98k:14070)

[56] ———, *Orbits of matrix tuples*, Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), Sémin. Congr., vol. 2, Soc. Math. France, Paris, 1997, pp. 245–261. MR MR1601151 (99a:16019)

[57] ———, *Noncommutative compact manifolds constructed from quivers*, AMA Algebra Montp. Announc. (1999), Paper 1, 5 pp. (electronic). MR MR1734896 (2001m:14004)

[58] ———, *Local structure of Schelter-Procesi smooth orders*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4815–4841. MR MR1695028 (2001b:16018)

[59] ———, *Optimal filtrations on representations of finite-dimensional algebras*, Trans. Amer. Math. Soc. **353** (2001), no. 1, 411–426. MR MR1707199 (2001c:16025)

[60] Lieven Le Bruyn and Claudio Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 585–598. MR MR958897 (90e:16048)

[61] Lieven Le Bruyn and Aidan Schofield, *Rational invariants of quivers and the ring of matrixinvariants*, Perspectives in ring theory (Antwerp, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 233, Kluwer Acad. Publ., Dordrecht, 1988, pp. 21–29. MR MR1048393 (91f:14047)

[62] Lieven LeBruyn, *Noncommutative geometry @n*, math.AG/9904171, 1999.

[63] Domingo Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33. MR MR0342523 (49 #7269)

[64] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR MR559531 (81j:14002)

[65] Hiraku Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999. MR MR1711344 (2001b:14007)

[66] Richard S. Pierce, *Associative algebras*, Graduate Texts in Mathematics, vol. 88, Springer-Verlag, New York, 1982, Studies in the History of Modern Science, 9. MR MR674652 (84c:16001)

[67] C. Procesi, *The invariant theory of $n \times n$ matrices*, Advances in Math. **19** (1976), no. 3, 306–381. MR MR0419491 (54 #7512)

[68] Claudio Procesi, *A formal inverse to the Cayley-Hamilton theorem*, J. Algebra **107** (1987), no. 1, 63–74. MR MR883869 (88b:16033)

[69] Y. P. Razmyslov, *Trace identities of full matrix algebras over a field of characteristic zero*, Math. USSR-Izv. **8** (1974), 727–760.

[70] I. Reiner, *Maximal orders*, London Mathematical Society Monographs. New Series, vol. 28, The Clarendon Press Oxford University Press, Oxford, 2003, Corrected reprint of the 1975 original, With a foreword by M. J. Taylor. MR MR1972204 (2004c:16026)

[71] William F. Schelter, *Smooth algebras*, J. Algebra **103** (1986), no. 2, 677–685. MR MR864437 (88a:16034)

[72] A. H. Schofield, *Representation of rings over skew fields*, London Mathematical Society Lecture Note Series, vol. 92, Cambridge University Press, Cambridge, 1985. MR MR800853 (87c:16001)

[73] Aidan Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 46–64. MR MR1162487 (93d:16014)

[74] ______, *Birational classification of moduli spaces of representations of quivers*, Indag. Math. (N.S.) **12** (2001), no. 3, 407–432. MR MR1914089 (2003k:16028)

[75] Aidan Schofield and Michel van den Bergh, *Semi-invariants of quivers for arbitrary dimension vectors*, Indag. Math. (N.S.) **12** (2001), no. 1, 125–138. MR MR1908144 (2003e:16016)

[76] Gerald W. Schwarz, *Lifting smooth homotopies of orbit spaces*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 51, 37–135. MR MR573821 (81h:57024)

[77] Stephen S. Shatz, *Profinite groups, arithmetic, and geometry*, Princeton University Press, Princeton, N.J., 1972, Annals of Mathematics Studies, No. 67. MR MR0347778 (50 #279)

[78] K. S. Siberskii, *Algebraic invariants for a set of matrices*, Siberian Math. J. **9** (1968), 115–124.

[79] Peter Slodowy, *Der Scheibensatz für algebraische Transformationsgruppen*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 89–113. MR MR1044587

[80] Allen Tannenbaum, *Invariance and system theory: algebraic and geometric aspects*, Lecture Notes in Mathematics, vol. 845, Springer-Verlag, Berlin, 1981. MR MR611155 (83g:93009)

[81] Michel Van den Bergh, *The Brauer-Severi scheme of the trace ring of generic matrices*, Perspectives in ring theory (Antwerp, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 233, Kluwer Acad. Publ., Dordrecht, 1988, pp. 333–338. MR MR1048420 (91g:13001)

[82] Freddy Van Oystaeyen, *Algebraic geometry for associative algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 232, Marcel Dekker Inc., New York, 2000. MR MR1807463 (2002i:14002)

[83] Bruce W. Westbury, *On the character varieties of the modular group*, preprint, Nottingham, 1995.

[84] Hermann Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, N.J., 1939. MR MR0000255 (1,42c)

[85] George Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), no. 1, 1–41, With an appendix by I. G. Macdonald. MR MR1626461 (99f:58107)

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