lecture 1

MANIFOLDS

1.1 Some examples of algebras

Throughout, ℓ will be an arbitrary field with algebraic closure $\bar{\ell}$. In order to get a feeling for definitions to come it is good to have an arsenal of examples.

matrix-algebra : $M_n(\ell)$ is the algebra of all $n \times n$ matrices with entries in ℓ.

group-algebra : ℓG for a finite group G is the ℓ-vectorspace with basis $\{e_g \; : \; g \in G\}$ and multiplication induced from the rule that $e_g \, e_h = e_{gh}$.

polynomial algebra : $\ell[x_1, \ldots, x_n]$ the algebra of commutative polynomials in the variables $x_1, \ldots, x_n$.

free algebra : $\ell\langle x_1, \ldots, x_n \rangle$ the algebra of non-commutative polynomials in the variables $x_1, \ldots, x_n$ (or, equivalently, the tensor algebra of an n-dimensional ℓ-vectorspace). An ℓ-basis consists of all words in the alphabet $\{x_1, \ldots, x_n\}$ and multiplication is induced by concatenation of words.

first Weyl algebra : $A_1(\ell)$ is generated by two elements x and y satisfying the so-called canonical commutation relation $[x, y] = xy - yx = 1$. It is also the ring of differential operators on the affine line.

path algebra : ℓQ for a finite quiver (an oriented graph) is the ℓ-vectorspace with basis the oriented paths in Q of length ≥ 0 and multiplication induced by concatenation of paths.

coordinate ring : $\ell[X]$ for an affine ℓ-variety X is the algebra of polynomial functions on X. For example, if C is a smooth elliptic curve with equation $y^2 = x^3 + ax + b$ then $\ell[C] = \ell[x, y]/(x^3 + ax + b - y^2)$.

1.2 Projective A-modules

We quickly run through some basic homological algebra. For more details we refer to J. Rotman 'Introduction to homological algebra'. If A is an ℓ-algebra we denote by $A - mod$ the category of all left A-modules, that is, ℓ-vectorspaces M with a linear action on the left $. : A \times M \longrightarrow M$ satisfying $1.m = m$ and $a.(a'.m) = (aa').m$. Morphisms in $A - mod$ are linear maps $f \; : \; M \longrightarrow N$ such that $f(a.m) = a.f(m)$. Two left A-modules are *isomorphic* $M \simeq N$ iff there are A-module morphisms $f : M \longrightarrow N$ and $g : N \longrightarrow M$ such that $f \circ g = id_N$ and $g \circ f = id_M$.

A *free* left module F is isomorphic to the direct sum A^I for some indexset I. A left A-module P is a *projective* module if $P \lhd F$ that is, is a direct summand of a free module F.

Exercise 1.1 (By no means easy) Classify all (or construct examples of) projective A-modules for A in the list of seven examples in the previous section. Failing this, look up the literature about this problem.

Recall the notion of *exactness* of sequences of left A-modules. Consider a sequence of left A-modules and left A-module morphisms

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \ldots$$

This is called a *complex* whenever $f_i \circ f_{i-1} = 0$ for all i, that is, if for all i we have that $Im(f_{i-1}) \subset Ker(f_i)$. It is called *exact in* M_i if $Ker(f_i) = Im(f_{i-1})$ and *exact* if it is exact in all M_i. In general, the *cohomology* vector spaces

$$H_i = \frac{Ker(f_i)}{Im(f_{i-1})}$$

measures the obstruction to exactness in M_i.

Exercise 1.2 (Easy) Prove that a left A-module P is projective if and only if every diagram of left A-module morphisms

$$
\begin{array}{ccc}
 & & P \\
 & \overset{f'}{\swarrow} & \downarrow {\scriptstyle f} \\
M & \longrightarrow N & \longrightarrow 0
\end{array}
$$

with the lower sequence exact (that is, a surjection) can be completed with a left A-module morphism f'. Hint : use the fact that every left A-module M has a short exact sequence

$$F \longrightarrow M \longrightarrow 0$$

with F a free A-module (choose an ℓ-basis in M) and use that the lifting property is trivial for free modules.

A *covariant functor* $F : A - mod \longrightarrow \ell - vect$ is said to be *left exact* iff for every exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M"$$

also the sequence

$$0 \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M")$$

is exact (so, F preserves monomorphisms). Similarly, one defines a *right exact functor* (preserves epimorphisms).

Lemma 1.3 *For every* $N \in A - mod$, *the functor* $F(-) = Hom_A(N, -)$ *is left exact.*

Proof. Take an exact sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M"$ then the sequence

$$0 \longrightarrow Hom_A(N, M') \xrightarrow{F\alpha} Hom_A(N, M) \xrightarrow{F\beta} Hom_A(N, M")$$

is defined by $F\alpha(\phi) = \alpha \circ \phi$ and $F\beta(\psi) = \beta \circ \psi$. Suppose $F\alpha$ is not injective, then there is a morphism $\phi : N \longrightarrow M'$ such that $F\alpha(\phi) = 0$, that is, $\alpha(\phi(m)) = 0 \; \forall m \in M$. Because α is injective, $\phi(m) = 0$ for all m whence $\phi = 0$. As an exercise, check that the sequence is also exact in the middle term. $\qquad \square$

In general, this functor is *not* right exact. However, we have the following characterization of projective left A-modules.

Lemma 1.4 *P is a projective left A-module if and only if $Hom_A(P, -)$ is an exact functor (that is, is both a left- and a right-exact functor).*

Proof. Remains to prove that for an epimorphism $M \longrightarrow M" \longrightarrow 0$ also the induced map

$$Hom_A(P, M) \longrightarrow Hom_A(P, M") \longrightarrow 0$$

is epi. But this is just the lifting property for projectives. $\square$

1.3 Ext-spaces

Just as $Hom_A(M, -)$ is a covariant functor, $Hom_A(-, N)$ is a *contravariant functor* (that i, reverses the direction of the arrows. So, if $M \xrightarrow{f} M'$ is a morphism of left A-modules, then there is a map

$$Hom_A(M', N) \xrightarrow{Hf} Hom_A(M, N)$$

defined by sending ϕ to $\phi \circ f$. Now, take a *projective resolution* of M that is a long exact sequence

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0$$

with all P_i projective left A-modules and consider the *complex*

$$0 \longrightarrow Hom_A(P_0, N) \xrightarrow{Hom(d_1, N)} Hom_A(P_1, N) \xrightarrow{Hom(d_2, N)} Hom_A(P_2, N) \longrightarrow \cdots$$

and we define the *n-th Ext-space*

$$Ext_A^n(M, N) = \frac{Ker\ Hom(d_{n+1}, N)}{Im\ Hom(d_n, N)}$$

to be the obstructions to exactness of this complex. Clearly we have that

$$Ext_A^0(M, N) = Hom_A(M, N)$$

The importance of these Ext-spaces (as is the case for all so called *derived functors*) is that it turns short exact sequences into long exact sequences (in either entry). So if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

is a short exact sequence of left A-modules, then we have a long exact sequence of ℓ-vectorspaces

$$0 \longrightarrow Hom(M", N) \longrightarrow Hom(M, N) \longrightarrow Hom(M', N) \longrightarrow Ext^1(M", N) \longrightarrow$$

$$Ext^1(M, N) \longrightarrow Ext^1(M', N) \longrightarrow Ext^2(M", N) \longrightarrow \cdots$$

and, similarly, for a short exact sequence of left A-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N" \longrightarrow 0$$

there is a long exact of ℓ-spaces

$$0 \longrightarrow Hom(M, N') \longrightarrow Hom(M, N) \longrightarrow Hom(M", N) \longrightarrow Ext^1(M, N') \longrightarrow$$

$$Ext^1(M, N) \longrightarrow Ext^1(M, N") \longrightarrow Ext^2(M', N) \longrightarrow \cdots$$

For our applications in non-commutative geometry it is very important to have a concrete interpretation of $Ext_A^1(M, N)$ as equivalence classes of *extensions*. An *extension of N by M* is a short exact sequence of left A-modules

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$$

and two such sequences are called equivalent if there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
& & \downarrow{id_N} & & \downarrow{\phi} & & \downarrow{id_M} & & \\
0 & \longrightarrow & N & \longrightarrow & X' & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

(exercise : verify that this is indeed an equivalence relation!). Here is the procedure to define a map from such extensions to $Ext_A^1(M, N)$. Consider the diagram

$$
\begin{array}{ccccccccccc}
\longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
& \vdots & & \vdots\,{\alpha} & & \vdots & & \downarrow{id_M} & & \\
& 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

(where the dotted arrows exist by the lifting property for projectives). By commutativity of the diagram, $\alpha \circ d_2 = 0$ so α determines an element of $Ker\ Hom_A(d_2, N)$ and hence a class in $Ext_A^1(M, N)$. As an exercise, verify that this is independent of the choice of α making the diagram commute and of the equivalence class of extension.

Another use of Ext-spaces is that it gives still another property of projective modules.

Lemma 1.5 *If P is a projective left A-module, then $Ext_A^n(P, M) = 0$ for all $n \geq 1$ and all $M \in A - mod$.*

Proof. Take as a projective resolution of P the sequence $\ldots 0 \longrightarrow 0 \longrightarrow P \longrightarrow P \longrightarrow 0$ and apply the definitions. $\square$

Note that the zero-vector in $Ext_A^1(M, N)$ corresponds with the extension with middle term $M \oplus N$ and obvious maps. Hence, if $Ext_A^1(M, N) = 0$ then every extension of N by M *splits*. This allows us to give yet another characterization of projectives (Proof as an easy exercise to the above)

Lemma 1.6 *M is a projective left A-module if and only if $Ext_A^1(M, N) = 0$ for all $N \in A - mod$.*

Finally, there is one more result we will need later on

Lemma 1.7 *If P is a projective left A-module and if there is a short exact sequence*

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

then for every $X \in A - mod$ and all $n \geq 1$ there is an isomorphism

$$Ext_A^{n+1}(M, X) \simeq Ext_A^n(N, X)$$

Proof. Applying $Hom_A(-, X)$ to the short exact sequence gives us filaments

$$Ext^i(P, X) \longrightarrow Ext^i(N, X) \longrightarrow Ext^{i+1}(M, X) \longrightarrow Ext^{i+1}(P, X)$$

Because P is projective the first and last terms are zero. $\square$

1.4 Bimodules and Hochschild cohomology

M is called an *A-bimodule* if M is both a left- and a right A-module satisfying

$$(a_1 m)a_2 = a_1(m a_2) \qquad \forall a_i \in A, m \in M$$

and bimodule morphisms are defined in the obvious way. We need to define the notions of projective modules, cohomology etc. in the category $A - bimod$ of all A-bimodules rather than in $A - mod$. Fortunately, we need no extra work.

The *enveloping algebra* A^e of an ℓ-algebra A is the tensor product algebra

$$A^e = -A \otimes_\ell A^{op}$$

where A^{op} (the *opposite algebra*) is A as an ℓ-space with multiplication $a.a' = a'a$ (that is, the reversed multiplication of A). There is an *equivalence of categories*

$$A - bimod \quad \Leftrightarrow \quad A^e - mod$$

using the definition of a left A^e-module structure on any A-bimodule M via

$$(a \otimes a').m = ama'$$

(Exercise : verify that this all fits).

Exercise 1.8 Compute the enveloping algebra A^e for

$$A = M_{n_1}(\ell) \oplus \cdots \oplus M_{n_k}(\ell)$$

and deduce from this the classification of all A-bimodules.

We define the *Hochschild cohomology* $H^i(M)$ for $M \in A - bimod$ to be

$$H^i(M) = Ext^i_{A^e}(A, M)$$

where we give A the natural A-bimodule structure. As before, these spaces can be calculated if we have a projective A^e-resolution of A, that is a projective resolution of A as bimodules. We can easily define even a free resolution. Observe that $A \otimes A$ (all tensors are taken over ℓ) is the free A-bimodule of rank one. Denote

$$\overline{A} = A/\ell.1 \qquad \text{as } \ell\text{-vectorspace}$$

then $A \otimes \overline{A}^{\otimes n} \otimes A$ is the free A-bimodule on a basis of $\overline{A}^{\otimes n}$. Now, define a sequence of A-bimodules

$$\cdots \xrightarrow{b} A \otimes \overline{A}^{\otimes 2} \otimes A \xrightarrow{b} A \otimes \overline{A} \otimes A \xrightarrow{b} A \otimes A \xrightarrow{b} A \longrightarrow 0$$

where we define

$$b(a_0, \underbrace{a_1, \cdots, a_n}_{\in \overline{A}^{\otimes n}}, a_{n+1}) = \sum_{i=0}^{n} (-1)^i (a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1})$$

Exercise 1.9 Verify that $b : A \otimes A \longrightarrow A$ is just multiplication and that

$$A \otimes \overline{AA} \xrightarrow{b} A \otimes A$$

sends (a_0, a_1, a_2) to $(a_0 a_1, a_2) - (a_0, a_1 a_2)$. Observe that $b \circ b = 0$ and verify that this also holds for higher terms, so the above sequence is a complex.

1.5 (Non-commutative) differential forms

There is another interpretation of the free bimodules appearing in this resolution

$$A \otimes \overline{A}^{\otimes n} \otimes A = \Omega^n A \otimes A$$

where $\Omega^n A = A \otimes \overline{A}^{\otimes n}$ are the *non-commutative differential n-forms* using the dictionary

$$(a_0, a_1, \cdots, a_n) = a_0 da_1 d_2 \cdots da_n$$

We can put a graded algebra structure on the space of all differential forms

$$\Omega A = \oplus_{i=0}^{\infty} \Omega^n A$$

by the rule

$$(a_0, a_1, \cdots, a_n) \cdot (a_{n+1}, \cdots, a_{n+k}) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+k})$$

which determines a map $\Omega^n A \otimes \Omega^{k-1} A \longrightarrow \Omega^{n+k-1} A$. Because $\Omega^0 A = A$ this multiplication defines an A-bimodule structure on all of the $\Omega^n A$. Important for us will be the differential 1-forms $\Omega^1 A$.

Recall that for an A-bimodule M, an ℓ-*derivation* is an ℓ-linear map $D : A \longrightarrow M$ satisfying

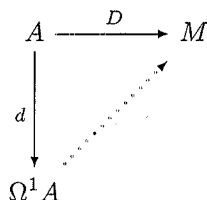$$D(\ell) = 0 \qquad \text{and} \qquad D(ab) = D(a)b + aD(b)$$

Example 1.10 Consider the map

$$A \xrightarrow{d} \Omega^1 A = A \otimes \overline{A} \qquad a \mapsto (1, a)$$

then this is an ℓ-derivation as

$$d(a)b + ad(b) = (1, a) \cdot b + a \cdot (1, b) = -(a, b) + (1, ab) + (a, b) = (1, ab) = d(ab)$$

In fact, it is the *universal ℓ-derivation* meaning that for any ℓ-derivation D there exists a unique A-bimodule morphism making the diagram below commute

$$
\begin{array}{ccc}
A & \xrightarrow{\quad D \quad} & M \\
\Big\downarrow{\scriptstyle d} & \nearrow & \\
\Omega^1 A & &
\end{array}
$$

Observe that if M is an A-bimodule, we can define the *tensor algebra*

$$T_A(M) = A \oplus M \oplus (M \otimes_A M) \oplus (M \otimes_A M \otimes_A M) \oplus \cdots$$

with the natural multiplication. This definition allows us to control ΩA.

Lemma 1.11 $\Omega A = T_A(\Omega^1 A)$

Proof.   Have algebra map (inclusion) $A \longrightarrow \Omega A$ and a bimodule map (inclusion) $\Omega^1 A \longrightarrow \Omega A$ so from the universal property of tensor-algebras we have an algebra map

$$T_A(\Omega^1 A) \longrightarrow \Omega A$$

which is an isomorphism as by induction we have

$$\Omega^n \otimes_A \Omega^1 = \Omega^n A \otimes_A (A \otimes \overline{A}) = \Omega^n \otimes \overline{A} = \Omega^{n+1} A$$

$\square$

We have an exact sequence of A-bimodules

$$0 \longrightarrow \Omega^1 A \overset{j}{\longrightarrow} A \otimes A \overset{m}{\longrightarrow} A \longrightarrow 0$$

with m multiplication and $j(a_0 da_1) = a_0 a_1 \otimes 1 - a_0 \otimes a_1$. As A is a free left (resp. right) A-module (*NOT* necessarily a projective A bimodule!) we know that the sequence splits as a sequence of left A-modules (or right A-modules) whence we can tensor with $\Omega^n \otimes_A -$ and obtain an exact sequence of A-bimodules

$$0 \longrightarrow \Omega^n A \otimes_A \Omega^1 \longrightarrow \Omega^n A \otimes_A A \otimes \otimes A \longrightarrow \Omega^n A \otimes_A A \longrightarrow 0$$

whence an exact sequence of A-bimodules

$$0 \longrightarrow \Omega^{n+1} A \overset{j}{\longrightarrow} \Omega^n \otimes A \overset{m}{\longrightarrow} \Omega^n A \longrightarrow 0$$

where the maps are defined by

$$\begin{cases} j(\omega da) = \omega a \otimes 1 - \omega \otimes a \\ m(\omega \otimes a) = \omega a \end{cases}$$

1.6   (Non-commutative) points

When M is an A-bimodule, its Hochschild cohomoloy $H^i(M) = Ext^i_{A^e}(A, M)$ can be calculated as the cohomology of the complex

$$C^n(A, M) \quad = Hom_{A^e}(A \otimes \overline{A}^{\otimes n} \otimes A.M) = Hom_\ell(\overline{A}^{\otimes n}, M)$$

$$\Big\downarrow \delta$$

$$C^{n+1}(A, M) = Hom_{A^e}(A \otimes \overline{A}^{\otimes n+1} \otimes A, M) = Hom_\ell(\overline{A}^{\otimes n+1}, M)$$

where for $f \in Hom_\ell(\overline{A}^{\otimes n}, M)$, $(\delta f)(a_1, \ldots, a_{n+1})$ is defined to be

$$a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}$$

This allows us to compute low dimensional cohomology groups from investigation of

$$C^0(A, M) \overset{\delta_0}{\longrightarrow} C^1(A, M) \overset{\delta_1}{\longrightarrow} C^2(A, M)$$

$$\Big\downarrow = \qquad\qquad \Big\downarrow = \qquad\qquad \Big\downarrow =$$

$$M \longrightarrow Hom_\ell(\overline{A}, M) \longrightarrow Hom_\ell(\overline{A} \otimes \overline{A}, M)$$

where $\delta_0(m)$ is the map sending $\overline{a}$ to $am - ma$ (check that this is well-defined). Therefore,

$$H^0(M) = Ker\ \delta_0 = \{m \in M | am = ma\} = M^A$$

the *center of M*. Likewise, if $D \in Hom_\ell(\overline{A}, M)$ then

$$\delta_1 D(a_1, a_2) = a_1 D(a_2) - D(a_1 a_2) + D(a_1) a_2$$

whence $Ker\ \delta_1$ is precisely the space of ℓ-derivations and the image of δ_0 are by the above the *inner derivations*, whence

$$H^1(M) = \frac{\ell\text{-derivations on } M}{\text{inner derivations}}$$

This brings us to our first theorem (definition)

Theorem 1.12 *The following are equivalent*

1. *A has cohomological dimension 0 with respect to Hochschild cohomology, that is, $H^i(M) = 0$ for all $i \geq 1$ and all $M \in A - bimod$.*

2. *A is a projective A-bimodule.*

3. *Every ℓ-derivation of A in a A-bimodule is inner.*

Proof. $(1) \Rightarrow (3)$: $H^1(M) = 0$ and so the statement follows from the above description of H^1.

$(3) \Rightarrow (2)$: $H^1(M) = Ext^1_{A^e}(A, M) = 0$ for all $M \in A^e - mod$ whence A is a projective left A^e-module.

$(2) \Rightarrow (1)$: If A is a projective A^e-module, then $Ext^i_{A^e}(A, M) = 0$ for all $i \geq 1$ and all $M \in A - bimod$, whence the statement follows. $\square$

An ℓ-algebra satisfying these equivalent conditions is called a *(non-commutative) point*. We will now characterize such points.

Because A is a projective A-bimodule, we have a bimodule splitting $A \xrightarrow{\ s\ } A \otimes A$ of the multiplication sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \xrightarrow{\ m\ } A \longrightarrow 0$$

$s \in Hom_{A^e}(A, A \otimes A) = (A \otimes A)^A$ and is fully determined by $z = s(1)$. If $z = \sum x_i \otimes y_i$ then we have that

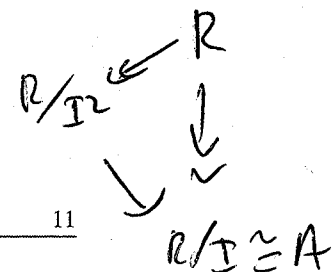$$\begin{cases} m(z) = 1 \quad \text{so} \quad \sum_i x_i y_i = 1 \\ \forall a \in A\ :\ \sum a x_i \otimes y_i = \sum x_i \otimes y_i a \\ z.(a \otimes 1 - 1 \otimes a) = 0 \end{cases}$$

such elements are usually called *separability idempotents*. Indeed we have that $z^2 = z$ as

$$z^2 = (\sum x_i \otimes y_i) z = \sum (x_i \otimes 1)(1 \otimes y_i) z = \sum (x_i \otimes 1)(y_i \otimes 1) z = (\sum x_i y_i) \otimes z = z$$

Therefore,

Theorem 1.13 *Non-commutative points are exactly the separable ℓ-algebras.*

Exercise 1.14 Prove that $M_n(\ell)$ is a point with separability idempotent

$$z = \frac{1}{n} \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}$$

What happens if $char(\ell)|n$?

Exercise 1.15 If $char(\ell)$ does not divide the order $\#G$ of a finite group G, then the group algebra lG is a non-commutative point with separability idempotent

$$z = \frac{1}{\#G} \sum_{g \in G} g \otimes g^{-1}$$

1.7 (Non-commutative) curves

If M is an A-bimodule, we can define an ℓ-algebra structure on $A \oplus M$ by

$$(a,m) \, (a',m') = (aa', am' + ma')$$

Note that $I = 0 \oplus M$ is a twosided ideal of this algebra and that $I^2 = 0$ and $(A \oplus M)/I \simeq A$. We call $A \oplus M$ the *trivial square-zero extension* of A by M.

More generally, an ℓ-algebra B is said to be a *square-zero extension* if there is a twosided ideal $I \triangleleft B$ with $I^2 = 0$ and $B/I \simeq A$.

Exercise 1.16 Show that the kernel I of the projection $B \longrightarrow A$ from a square-zero extension has a well-defined A-bimodule structure. Hint : for $\pi(b) = a$ and $i \in I$ define $a.i = bi$ and $i.a = ib$ and prove that this does not depend on the choices made (because $I^2 = 0$).

Hence, any square-zero extension B determines an A-bimodule and we call two square-zero extensions (with the same bimodule M) equivalent iff there is an ℓ-algebra map ϕ making the diagram commute

$$
\begin{array}{ccccc}
M & \hookrightarrow & B & \longrightarrow\!\!\!\!\rightarrow & A \\
\downarrow{\scriptstyle id_M} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id_A} \\
M & \hookrightarrow & B' & \longrightarrow\!\!\!\!\rightarrow & A
\end{array}
$$

Precisely as we calculated H^0 and H^1 one can show that $H^2(M)$ classifies equivalence classes of square-zero extensions of A determined by the A-bimodule M and that the zero-vector in this space corresponds to the trivial square-zero extension $A \oplus M$.

Theorem 1.17 *The following are equivalent*

  1. A has cohomological dimension 1 with respect to Hochschild cohomology, that is, $H^j(M) = 0$ for all $j \geq 2$ and all $M \in A - bimod.$

  2. $\Omega^1 A$ is a projective A-bimodule.

3. For every square-zero extension (B, I) of A there is a lifting algebra map

$$A \dashrightarrow B$$

with diagonal map $=$ to B/I and vertical map $B \to B/I$.

Proof. $(3) \Rightarrow (2)$: Lifting means that every square-zero extension is equivalent to a trivial square-zero extension, whence $H^2(M) = 0$ for every $M \in A - bimod.$ Hence, for all $M \in A - bimod$

$$0 = Ext^2_{A^e}(A, M) = Ext^1_{A^e}(\Omega^1 A, M)$$

(because $0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$ with middle term a free A-bimodule). Hence, $\Omega^1 A$ is indeed a projective A-bimodule.

$(2) \Rightarrow (1)$: We know that $Ext^i_{A^e}(\Omega^1 A, M) = 0$ for all $i \geq 1$ and all $M \in A - bimod$, But then, $Ext^{i+1}_{A^e}(A, M) = H^{i+1}(M) = 0.$

$(1) \Rightarrow (3)$: trivial from the above remarks. $\square$

An ℓ-algebra A satisfying these equivalent conditions is called a *(non-commutative) curve.* Other terminology for this class of algebras is : *quasi-free algebras* (Cuntz-Quillen) and *formally smooth algebras* (Kontsevich).

Recall that an ℓ-algebra A is said to be *hereditary* if every left A-module M has a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of length $\leq 1.$

Theorem 1.18 *A (non-commutative) curve is an hereditary algebra.*

Proof. Because $A \otimes A$ is a free onesided A-module, the sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

splits as a sequence of right A-modules. Therefore, tensoring this sequence with a left A-module M we till have an exact sequence

$$0 \longrightarrow \Omega^1 A \otimes_A M \longrightarrow A \otimes A \otimes_A M \longrightarrow A \otimes_A M \longrightarrow 0$$
$$\downarrow = \qquad\qquad \downarrow = \qquad\qquad \downarrow =$$
$$0 \longrightarrow \Omega^1 \otimes_A M \longrightarrow A \otimes M \longrightarrow M \longrightarrow 0$$

Here, the middle term is a free left A-module and as $\Omega^1 A$ is a projective A-bimodule it is a direct summand of some $A \otimes V \otimes A$ whence

$$\Omega^1 A \otimes_A M \lhd A \otimes V \otimes A \otimes_A M = A \otimes V \otimes M$$

and the right hand side is a free left A-module. $\square$

1.8   Back to the examples

Let us go back to the classes of ℓ-algebras we gave at the beginning and determine in which class they fall.

$M_n(\ell)$ is a simple algebra and a non-commutative point. If $\ell \subset L$ is a finite inseperable field extension of ℓ (which then must have prime characteristic) then $M_n(L)$ is a simple algebra but is not a point.

ℓG is a semi-simple algebra and a non-commutative point provided $char(\ell)$ does not divide the order of the group G.

$l[x_1, \ldots, x_n]$ is a non-commutative point iff $n = 0$ and a non-commutative curve iff $n = 1$ (see next time).

$l\langle x_1, \ldots, x_n \rangle$ is a non-commutative point iff $n = 0$ and for all n it is a non-commutative curve (see next time).

$A_1(\ell)$ is hereditary iff $char(\ell) = 0$ but is never a non-commutative curve.

ℓQ is a non-commutative point iff there are no arrows in Q and is always a non-commutative curve.

$\ell[X]$ is a non-commutative point (resp. curve) iff X is the disjoint union of a finite number of points (resp. of smooth affine curves).

lecture 2

MACHINES

2.1 Grothendieck's characterization

Recall from last time that an ℓ-algebra A is said to be a

- *(non-commutative) point* iff A is a projective A-bimodule iff A is a separable ℓ-algebra

- *(non-commutative) curve* iff $\Omega^1 A$ is a projective A-bimodule iff A has the lifting property for square-zero extensions

From now on we will call such algebras just *(non-commutative) manifolds* for we will show that if one extends one of the many characterizations of smooth affine commutative algebras to the non-commutative world we find exactly the above mentioned algebras. This is the first surprise of non-commutative geometry : non-commutative manifolds are morally either points or curves!

Recall that a (commutative) *affine variety* is a subset of common zeroes

$$X = \mathbb{V}(f_1, \ldots, f_l) \subset \mathbb{A}^N(\ell)$$

for a finite set of polynomials $f_i \in \ell[x_1, \ldots, x_N]$. For more information on varieties and schemes, read the first chapter of R. Hartshorne 'Algebraic Geometry'.

The *coordinate ring* of the affine variety X is defined to be the quotient

$$\ell[X] = \frac{\ell[x_1, \ldots, x_N]}{(f_1, \ldots, f_l)}$$

that is, it is the ℓ-algebra of polynomial functions defined on X.

We say that $\ell[X]$ has the *nilpotent lifting property* if for every *commutative* ℓ-algebra C, every ideal $I \lhd C$ with $I^2 = 0$ and every algebra map ϕ, there exist an ℓ-algebra lift $\tilde{\phi}$

$$
\begin{array}{ccc}
\ell[X] & \xrightarrow{\ \ \tilde{\phi}\ \ } & C \\
& \phi \searrow & \downarrow \\
& & C/I
\end{array}
$$

making the diagram commute.

J.P. Serre proved that an affine variety X is *smooth* of dimension d if and only if $\ell[X]$ has finite global dimension d (that is, every $\ell[X]$-module has a projective resolution of length $\leq d$). On the other hand, A. Grothendieck proved the following facts :

- If $\ell[X]$ has the nilpotent lifting property then $\ell[X]$ has finite global dimension.

- If ℓ is a perfect field (e.g. if ℓ is finite or algebraically closed) then any $\ell[X]$ having finite global dimension satisfies the nilpotent lifting property.

Therefore, a natural generalization of smooth varieties to the non-commutative world is to take ℓ-algebras satisfying the *nilpotent lifting property* in alg the category of *all* ℓ-algebras, that is, where we allow C in the above diagram to be non-commutative. Observe that this condition is a lot stronger than the commutative lifting property so it is not true in general that a commutative ring satisfying the commutative lifting property also satisfies the non-commutative one.

Example 2.1 Take $X = \mathbb{A}^n(\ell)$ then $\ell[X] = \ell[x_1, \ldots, x_n]$ clearly satisfies the commutative nilpotent lifting property (just left the images of $\phi(x_i)$ to elements of C, say c_i and define ϕ by sending x_i to c_i. However, this argument does no longer work in alg. Take for example the 4-dimensional non-commutative ℓ-algebra

$$B = \frac{\ell\langle x, y\rangle}{(x^2, y^2, xy + yx)} = \ell.1 \oplus \ell.x \oplus \ell.y \oplus \ell.xy$$

and I the one-dimensional nilpotent twosided ideal in B

$$I = (xy - yx) = \ell.xy \qquad \text{with} \qquad I^2 = 0$$

then the quotient is a commutative ℓ-algebra

$$B/I = \frac{\ell[x, y]}{(x^2, y^2)} = \ell.1 \oplus \ell.x \oplus \ell.y$$

so there is an algebra map $\phi : \ell[X] \longrightarrow B/I$ sending $x_1 \mapsto x$, $x_2 \mapsto y$ and $x_i \mapsto 0$ for all $i \geq 3$. A potential lift $\tilde{\phi} : \ell[X] \longrightarrow B$ must be of the form

$$\tilde{\phi}(x_1) = x + \alpha xy \qquad \text{and} \qquad \tilde{\phi}(x_2) = y + \beta xy$$

But then $[\tilde{\phi}(x_1), \tilde{\phi}(x_2)] \neq 0$ in B contradicting the fact $[x_1, x_2] = 0$ in $\ell[X]$. So, whenever $n \geq 2$ then $\ell[x_1, \ldots, x_n]$ does *not* have the nilpotent lifting property in alg.

We will show that A has the nilpotent lifting property in alg if and only if A is a non-commutative curve.

2.2   The generic square-zero extension

Consider the (usually extremely large) tensor algebra over ℓ

$$T(A) = \ell \oplus A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \ldots$$

For any ℓ-algebra R a *based linear map* is an ℓ-linear map

$$A \xrightarrow{\ \rho\ } R \qquad \text{s.t.} \qquad \rho(1_A) = 1_R$$

By the universal property of tensor algebras, any based linear map ρ extends to an algebra map

$$T(A) \xrightarrow{\ \tilde{\rho}\ } R \qquad a_1 \otimes \ldots \otimes a_k \mapsto \rho(a_1)\rho(a_2) \ldots \rho(a_k)$$

and as $\rho(1_A) = 1_R$ this algebra map factors through the quotient

$$B(A) = \frac{T(A)}{(1_A - 1)}$$

where 1_A is the element in $T(A)$ of degree one and 1 is the unit element of $T(A)$. That is, $B(A)$ has the universal property that every based linear map ρ extends to an ℓ-algebra map making the diagram
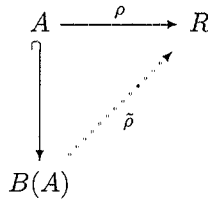
commute. In particular, apply this to the identity map $id : A \longrightarrow A$ then we get a twosided ideal

$$I_A = Ker\ \tilde{id} \qquad \text{such that} \qquad B(A)/I_A \simeq A$$

Therefore, $B(A)/I_A^2$ is a square-zero extension of A which we call the *generic square-zero extension*.

2.3 Characterizing manifolds

Let us call a ℓ-algebra A a *(non-commutative) manifold* if it satisfies the nilpotent lifting property in alg.

Theorem 2.2 *The following are equivalent*

1. *A has the nilpotent lifting property in* alg.

2. *A is a non-commutative curve.*

Proof. (1) $\Rightarrow$ (2) : Immediate from the characterization of curves via lifting of square-zero extensions.

(2) $\Rightarrow$ (1) : Let B be a ℓ-algebra having a twosided ideal I with $I^2 = 0$ and an algebra map $\phi : A \longrightarrow B/I$. We can always lift ϕ to a based linear map ρ (not necessarily an algebra map!) making the diagram commute
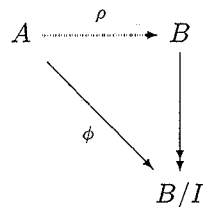
This ρ extends to an algebra map $\tilde{\rho}$ by the universal property of $B(A)$ and we obtain the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & B/I \\
\uparrow & \nwarrow (3) & \uparrow \\
B(A) & \xrightarrow{\tilde{\rho}} & B \\
\downarrow & \nearrow (1) & \\
B(A)/I_A^2 &
\end{array}
$$

By commutativity of the upper square, $\tilde{\rho}(I_A) \subset I$ whence $\tilde{\rho}(I_A^2) \subset I^2 = 0$ and therefore $\tilde{\rho}$ factors through an algebra map (1). Now, $B(A)/I_A^2$ is a square-zero extension of A hence there is a lifting algebra map

$$(2) \; : \; A \longrightarrow B(A)/I_A^2$$

But then we have found the required algebra lift $(3) = (1) \circ (2)$. $\square$

2.4 Representation schemes

Our next aim will be to show that (non-commutative) manifolds (or curves if you want) are really *machines* (as M. Kontsevich described it) producing a *family* of commutative affine manifolds. The first family to consider are the affine schemes $\mathrm{rep}_n\, A$ of *n-dimensional representations* of A. Recall that an n-dimensional representation is an algebra map

$$A \xrightarrow{\phi} M_n(\ell)$$

We will now describe the coordinate ring $\ell[\mathrm{rep}_n\, A]$. Assume A is an affine ℓ-algebra with presentation

$$A = \frac{\ell\langle x_1, \ldots, x_N \rangle}{(f_i(x_1, \ldots, x_N) \; : \; i \in I)}$$

where the index set I is not necessarily finite. Consider the affine space $\mathbb{A}^{Nn^2}(\ell)$ with coordinate functions $x_{ij}(k)$ (where $1 \le i,j \le n$ and $1 \le k \le N$) and define the *generic $n \times n$ matrices*

$$X_k = \begin{bmatrix} x_{11}(k) & \cdots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \cdots & x_{nn}(k) \end{bmatrix}$$

We can evaluate each of the identities $f_i(x_1, \ldots, x_N)$ in the non-commuting variables x_k in the matrices X_k and obtain an $n \times n$ matrix

$$f_i(X_1, \ldots, X_N) \in M_n(\ell[\mathbb{A}^{Nn^2}])$$

and we define the ideal $I_n(A) \lhd \ell[\mathbb{A}^{Nn^2}]$ to be generated by all entries of all these $n \times n$ matrices $f_i(X_1, \ldots, X_N)$. Observe that whereas I may be infinite, the ideal $I_n(A)$ is clearly finitely generated by Noetherianity of $\ell[\mathbb{A}^{Nn^2}]$. We define the *n-th representation scheme* of A via its coordinate ring

$$\ell[\mathrm{rep}_n\, A] = \frac{\ell[\mathbb{A}^{Nn^2}]}{I_n(A)}$$

The whole point of this construction is that an ℓ-point of $\mathrm{rep}_n\ A$ corresponds to an n-dimensional representation of A. One should be extremely careful though not to confuse the *geometric points* of $\mathrm{rep}_n\ A$ with n-dimensional ℓ-representations. A *geometric point* of $\mathrm{rep}_n\ A$ corresponds to an ℓ-algebra map

$$A \longrightarrow M_n(\overline{\ell})$$

and for a subfield $\ell \subset L \subset \overline{\ell}$ this is said to be an L-point is the image is contained in $M_n(L)$.

Example 2.3 Let $\ell \subset L$ be a finite separable field extension of ℓ (in particular, L is a non-commutative manifold, even a point), then we can write

$$L = \frac{\ell[x]}{(f(x))}$$

for some irreducible polynomial $f(x) \in \ell[x]$ and L is the field generated by the roots of $f(x)$. Then

$$\mathrm{rep}_1\ L(\ell) = \emptyset \qquad \text{whereas} \qquad \mathrm{rep}_1\ L(\overline{\ell}) = \{ \text{ roots of } f(x) \}$$

As an exercise, if $deg(f) = n$ find an ℓ-point in $\mathrm{rep}_n\ L$ (Hint : think of companion matrices...).

For a general ℓ-algebra A one can also characterize the representation scheme $\mathrm{rep}_n\ A$ (or, equivalently, its coordinate ring $\ell[\mathrm{rep}_n\ A]$) by the following universal property. There is a natural algebra map

$$A \xrightarrow{\ j_n\ } M_n(\ell[\mathrm{rep}_n\ A]) \quad \text{defined by} \quad x_k \mapsto X_k$$

Let C be a *commutative* ℓ-algebra, then for every ℓ-algebra map ϕ, there is a *unique* algebra map $\psi\ :\ \ell[\mathrm{rep}_n\ A] \longrightarrow C$ making the diagram

commutative. Clearly, ψ is defined by sending the class of $x_{ij}(k)$ to the (i,j)-entry of $\phi(x_k)$. This allows us to prove a fundamental result

Theorem 2.4 *If A is a non-commutative manifold, then for all n the representation scheme* $\mathrm{rep}_n\ A$ *is a smooth (commutative) variety.*

Proof. By Grothendieck's characterization we have to show that $\ell[\mathrm{rep}_n\ A]$ satisfies the nilpotent lifting property with respect to commutative algebras. So, take a commutative ℓ-algebra C an ideal $I \lhd C$ satisfying $I^2 = 0$ and an ℓ-algebra map

$$\ell[\mathrm{rep}_n\ A] \xrightarrow{\ \phi\ } C/I$$

which we want to lift to C. Consider the following diagram

Recall that the product of two irreducible varieties is again irreducible and that the image of an irreducible variety is irreducible. Therefore, there is a *unique* component $\text{rep}_\gamma A$ in $\text{rep}_{m+n} A$ containing the image of the irreducible variety $\text{rep}_\alpha A \times \text{rep}_\beta A$ under the sum-map. We denote this relationship by

$$\alpha + \beta = \gamma$$

and in this way we define a *commutative semigroup* structure on $\text{comp}(A)$ the set of all irreducible components of representation varieties of the manifold A. (For general algebras A one can do a similar construction but then one has to restrict to *connected* components; for manifolds these two notions coincide).

Example 2.7 The algebra $A = \ell \times \ldots \times \ell$ (k factors) is semi-simple with simple representations M_i ($1 \le i \le k$) hence every finite dimensional representation M of A is isomorphic to

$$M \simeq M_1^{\oplus a_1} \oplus \ldots \oplus M_k^{\oplus a_k}$$

for some $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}^k$ and with $|\alpha| = \sum_i a_i$. Therefore, for each n we have that $\text{rep}_n A$ is the disjoint union of a finite number of GL_n-orbits

$$\text{rep}_n A = \bigsqcup_{|\alpha|=n} \text{rep}_\alpha A$$

and hence $\text{comp}(A) = \mathbb{N}^k$ with generators the components of $\text{rep}_1 A$ corresponding to the simple representations M_i.

Exercise 2.8 For the manifolds below, prove that their *component semigroups* are as indicated (in the case $\ell = \bar{\ell}$).

1. $A = M_n(\ell)$ then $\text{comp}(A) = \mathbb{N}n \subset \mathbb{N}$.

2. $A = \ell G$, then $\text{comp}(A) \simeq \mathbb{N}^k$ where k is the number of characters of G.

3. $A = \ell\langle x_1, \ldots, x_N \rangle$, then $\text{comp}(A) \simeq \mathbb{N}$.

4. $A = \ell[C]$, then $\text{comp}(A) \simeq \mathbb{N}$.

5. $A = \ell Q$, then $\text{comp}(A) \simeq \mathbb{N}^k$ where k is the number of vertices of Q.

2.6   The component semigroup, ℓ arbitrary

An n-dimensional representation $A \xrightarrow{\phi} M_n(\ell)$ is the same thing as an n-dimensional left A-module $M_\phi = \ell^n$ (considered as column vectors) with action

$$a.v = \phi(a)v \in \ell^n$$

Clearly, this depends on the choice of basis in M_ϕ and two different bases lead to isomorphic left A-modules. That is, there is an action of the base-change group GL_n on the affine variety $\text{rep}_n A$ such that its orbits are precisely the isomorphism classes of n-dimensional A-representations.

Let us start with an example showing some of the problems which occur when trying to extend the definition of a component semigroup to manifolds A over an arbitrary basefield ℓ.

Where $\psi = M_n(\phi) \circ j_n$. As $M_n(I)$ is a square-zero ideal in $M_n(C)$ we know from the manifold condition on A the existence of a lifted algebra map $\tilde{\psi}$. But then, using the universal property of the map j_n there exists a unique algebra map

$$\ell[\text{rep}_n A] \longrightarrow C$$

making the diagram commute. This map is the required algebra lift.   □

Exercise 2.5 To see that this result allows us to prove smoothness of varieties that would be intractable otherwise, take $a, b \in \ell$ such that

$$C = \mathbb{V}(x^3 + ax + b - y^2) \subset \mathbb{A}^2$$

is a smooth affine elliptic curve. Determine explicitly the 8 (4 from the above equation and 4 from $[x, y] = 0$) equations defining $\text{rep}_2 \ \ell[C]$ in $\mathbb{A}^8$ and try to show that this variety is smooth. Attempt a similar approach to $\text{rep}_3 \ \ell[C]$ and higher n...

2.5 The component semigroup, $\ell = \overline{\ell}$

In this section we assume for the first time that $\ell = \overline{\ell}$ is algebraically closed.In the next section we will show how one can extend the results to an arbitrary field. In the next lecture we will give a more elegant solution replacing the semigroup to be introduced here by a (braided) coalgebra.

If A is a manifold we know that each of the representation varieties $\text{rep}_n A$ is a smooth affine manifold but it may have different irreducible components (which must be disjoint by smoothness).

Example 2.6 Let $A = \ell \times \ldots \times \ell$ (k factors). Then,

$$\text{rep}_1 A = \{M_1, \ldots, M_k \mid M_i = 0 \times \ldots \times \ell \times 0 \ldots \times 0\}$$

has k distinct points.

Let us denote this decomposition

$$\text{rep}_n A = \bigsqcup_\alpha \text{rep}_\alpha A$$

where α is just a label of the irreducible component $\text{rep}_\alpha A$. We call α a *dimension vector* of *total dimension* n and denote this by $|\alpha| = n$. We will now impose relations among these α's. Assume that $M \in \text{rep}_n A$ is determined by the matrices $m_i \in M_n(\ell)$ and $N \in \text{rep}_m A$ determined by matrices $n_i \in M_m(\ell)$ (where $i \in I$ a set of algebra generators of A), then we can form the *direct sum* representation

$$M \oplus N \in \text{rep}_{m+n} A \quad \text{determined by} \quad \begin{bmatrix} m_i & 0 \\ 0 & n_i \end{bmatrix}$$

These morphisms are called the *sum maps*

$$\text{rep}_n A \times \text{rep}_m A \longrightarrow \text{rep}_{m+n} A$$

And decomposing all varieties involved in their disjoint irreducible components we have a map

$$\bigsqcup_{|\alpha|=n} \text{rep}_\alpha A \times \bigsqcup_{|\beta|=m} \text{rep}_\beta A \longrightarrow \bigsqcup_{|\gamma|=m+n} \text{rep}_\gamma A$$

Example 2.9 Let $\ell \subset L$ be a finite separable field extension of dimension k (observe that L is a manifold, even a non-commutative point). As L is (semi)simple, every finite dimensional L-module is isomorphic to $L^{\oplus a}$ for some $a \in \mathbb{N}$. Therefore, there can only be ℓ-points in $\operatorname{rep}_n L$ if n is a multiple of k. Still, it is perfectly possible to define the ℓ-scheme $\operatorname{rep}_n L$ for arbitrary n. Assume that

$$L = \frac{\ell[t]}{f(t)}$$

where $f(t)$ is an irreducible polynomial of degree k. Then, $\ell[\operatorname{rep}_n L]$ is the quotient of the polynomial algebra $\ell[x_{11}, x_{12}, \ldots, x_{nn}]$ in the entries of a generic $n \times n$ matrix X by the ideal generated by the n^2 entries of $f(X)$. For example,

$$\ell[\operatorname{rep}_1 L] = \frac{\bar{\ell}[x]}{f(x)} \simeq L$$

An ℓ-point of $\operatorname{rep}_n L$ corresponds to a morphism $\ell[\operatorname{rep}_n L] \longrightarrow \ell$ whence $\operatorname{rep}_1 L$ has *no* ℓ-rational points. Moreover, *irreducible components* of the ℓ-scheme $\operatorname{rep}_n L$ correspond to *minimal prime ideals* of $\ell[\operatorname{rep}_n L]$, so for example, as an ℓ-scheme $\operatorname{rep}_1 L$ has just one component. Still, $L \otimes_\ell \bar{\ell} \simeq \bar{\ell} \times \ldots \times \bar{\ell}$ and we have seen that $\operatorname{rep}_1 L \otimes_\ell \bar{\ell}$ has exactly k irreducible components!

Therefore, we define the *component semigroup* $\operatorname{comp} A$ to be the sub semigroup of $\operatorname{comp} A \otimes_\ell \bar{\ell}$ consisting of those components α such that

$$\operatorname{rep}_\alpha A \otimes_\ell \bar{\ell} \quad \text{contains an } \ell\text{-point}$$

An equivalent description is as follows : consider the *Galois group* $G = \operatorname{Gal}(\bar{\ell}/\ell)$, then this group acts on all the representation varieties $\operatorname{rep}_n A \otimes_\ell \bar{\ell}$ and hence induces an action by automorphisms on $\operatorname{comp} A \otimes_\ell \bar{\ell}$. For this action we have that

$$\operatorname{comp} A = (\operatorname{comp} A \otimes_\ell \bar{\ell})^G$$

the sub semigroup consisting of elements fixed by G.

Exercise 2.10 Show that these two definitions are really the same!

Example 2.11 In the above example we have that the Galois group $\operatorname{Gal}(\bar{\ell}/\ell)$ acts on $\operatorname{comp}(L \otimes_\ell \bar{\ell}) = \mathbb{N}^k$ by permuting the entries, whence

$$\operatorname{comp}(L) = (\mathbb{N}^k)^G = \{(a, a, \ldots, a) \mid a \in \mathbb{N}\}$$

consistent with the observation that all finite dimensional L-modules are of the form $L^{\oplus a}$.

2.7 The Euler-form on $\operatorname{rep} A$

As we are only doing homological algebra in this section, ℓ can be arbitrary. We have seen before that if A is a manifold, then A is a *hereditary algebra* meaning that every left A-module M has a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of length ≤ 1. If we apply this fact to the method of computation of $\operatorname{Ext}_A^1(M, N)$ we get

Lemma 2.12 *If A is a manifold, then for any $M, N \in \text{rep } A$ we have*

$$Ext_A^i(M, N) = 0 \qquad i \geq 2$$

where $\text{rep } A$ is the set of all finite dimensional left A-modules.

An alternative method to prove this is to observe that

$$Ext_A^i(M, N) = Ext_{A^e}^i(A, Hom_\ell(M, N)) = 0$$

for $i \geq 2$ as $\Omega^1 A$ is a projective A-bimodule. As M and N are finite dimensional spaces, so is $Hom_A(M, N) = Ext_A^0(M, N)$ being a subspace of the finite dimensional space $Hom_\ell(M, N)$ and also from our description of $Ext_A^1(M, N)$ as classifying equivalence classes of extensions of N by M we deduce that $Ext_A^1(M, N)$ is a finite dimensional ℓ-space. This allows us to define the *Euler-form* for any $M, N \in \text{rep } A$

$$\chi(M, N) = dim_\ell Hom_A(M, N) - dim_\ell Ext_A^1(M, N)$$

Lemma 2.13 *Both $\chi(-, N)$ and $\chi(M, -)$ are additive on short exact sequences.*

Proof. A short exact sequence of finite dimensional left A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

leads to a long exact sequence of ℓ-vector spaces

$$0 \longrightarrow Hom_A(M', N) \longrightarrow Hom_A(M, N) \longrightarrow Hom_A(M", N) \longrightarrow$$

$$Ext_A^1(M', N) \longrightarrow Ext_A^1(M, N) \longrightarrow Ext_A^1(M", N) \longrightarrow Ext_A^2(M', N) = 0$$

and hence the alternating sum of their dimensions is equal to zero, whence

$$\chi_A(M, N) = \chi_A(M', N) + \chi_A(M", N)$$

The same argument (reversing the arrows) applies to $\chi_A(M, -)$. $\qquad \square$

Lemma 2.14 *If we know $\chi_A(S, S')$ on all simple left A-modules $S, S' \in \text{simp } A$, then we know $\chi_A(M, N)$ for all $M, N \in \text{rep } A$.*

Proof. Recall that any $M \in \text{rep } A$ has a finite *Jordan-Hölder filtration*

$$0 = M_{l+1} \subset M_l \subset M_{l-1} \subset \ldots \subset M_1 \subset M_0 = M$$

with all successive quotients

$$S_i = \frac{M_i}{M_{i+1}} \quad \text{a simple } A\text{-module}$$

This filtration gives us a collection of short exact sequences

$$0 \longrightarrow S_l \longrightarrow M_{l-1} \longrightarrow S_{l-1} \longrightarrow 0$$

$$0 \longrightarrow M_{l-1} \longrightarrow M_{l-2} \longrightarrow S_{l-2} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow S_1 \longrightarrow 0$$

and applying the foregoing to these sequences from top to bottom we see that we can compute $\chi_A(M, N)$ in terms of the $\chi_A(S_i, N)$. Applying a similar argument to a Jordan-Hölder filtration of N we can compute also all $\chi_A(S_i, N)$ in terms of $\chi_A(S_i, T_j)$ where the T_j are the simple components of the filtration for N. $\qquad \square$

2.8 The Euler form on $\text{comp}(A)$

We now come to the first major result stating that the Euler-form is constant along modules lying in the same irreducible components. We will give only a sketch of the proof, as it involves some *geometric invariant theory (GIT)*.

Theorem 2.15 $(\ell = \bar{\ell})$ χ_A *defines a bilinear form on the commutative semigroup* $\text{comp}(A)$. *That is, if* $M, M' \in \text{rep}_\alpha A$ *and* $N, N' \in \text{rep}_\beta A$, *then*

$$\chi_A(M, N) = \chi_A(M', N')$$

and this common value we denote with $\chi_A(\alpha, \beta)$.

Proof. (Sketch) If M has simple Jordan-Hölder factors $S_1, \ldots, S_u$ and N has Jordan-Hölder factors $T_1, \ldots, T_v$ then it follows from the previous section that

$$\chi_A(M, N) = \chi_A(S_1 \oplus \cdots \oplus S_u, T_1 \oplus \cdots \oplus T_v)$$

By (GIT) the semi-simple module $M^{ss} = \oplus S_i$ lies in the same component as M (and $N^{ss} = \oplus_j T_j$ in the same component as N). Next, we use that the function

$$\chi_A(-, -) : \text{rep}_\alpha A \times \text{rep}_\beta A \longrightarrow \mathbb{Z}$$

is *upper-semicontinuous*, that is, there is a Zariski open subset consisting of couples (M, N) where $\chi_A(M, N)$ is minimal. By irreducibility, we can therefore find a couple (M, N) such that both $\chi_A(M, N)$ and $\chi_A(N, M)$ are minimal and by the above we may assume that M and N are both semi-simple (just replace them by M^{ss} and N^{ss}).

Another consequence of (GIT) is that for every semi-simple module $X \in \text{rep}_\gamma A$ we have that

$$|\gamma|^2 - \chi_A(X, X) = dim_\ell \, T_X(\text{rep}_\gamma A)$$

where the right hand side is the tangent space to the representation component $\text{rep}_\gamma A$. This follows from the *étale slice theorem* stating that locally in X the representation component resembles the fiber bundle

$$GL_{|\gamma|} \times^{GL(X)} Ext^1_A(X, X)$$

where $GL(X)$ is the stabilizer subgroup of X which is an open piece in $Hom_A(X, X)$. Because all $\text{rep}_\gamma A$ are smooth varieties, it follows that

$$\chi_A(X, X) = \chi_A(X', X')$$

whenever X and X' are semi-simple representations in $\text{rep}_\gamma A$.

Now, take M' a semi-simple module in $\text{rep}_\alpha A$ and N' a semi-simple in $\text{rep}_\beta A$, then we have from bilinearity and the previous remark that

$$\chi_A(M', N') + \chi_A(N', M') = \chi_A(M, N) + \chi_A(N, M)$$

and by the choice of (M, N) before also $\chi_A(M', N') \geq \chi_A(M, N)$ and $\chi_A(N', M') \geq \chi_A(N, M)$ from which equality of both inequalities follows and we are done! □

Now, take ℓ again arbitrary and M, N two finite dimensional A-modules where A is a manifold. If $\bar{A} = A \otimes_\ell \bar{\ell}$, then we have

$$Hom_A(M, N) = Hom_{\bar{A}}(M \otimes_\ell \bar{\ell}, N \otimes_\ell \bar{\ell}) \quad \text{and} \quad Ext^1_A(M, N) = Ext^1_{\bar{A}}(M \otimes_\ell \bar{\ell}, N \otimes_\ell \bar{\ell})$$

from which we deduce from the previous result :

Theorem 2.16 *(ℓ arbitrary) χ_A defines a bilinear form on the commutative semigroup* $\text{comp}(A)$. *That is, if* $M, M' \in \text{rep}_\alpha A(\ell)$ *and* $N, N' \in \text{rep}_\beta A(\ell)$, *then*

$$\chi_A(M, N) = \chi_A(M', N')$$

and this common value we denote with $\chi_A(\alpha, \beta)$.

lecture 3

MODULARS

3.1 Variety machines

Recall that we call an affine ℓ-algebra A a *manifold* (aka quasi-free, formally smooth or non-commutative curve) iff $\Omega^1 A$ is a projective A-bimodule iff A has the nilpotent lifting property. The main motivation for studying this class of non-commutative algebras is that such an A provides us with *families* of (commutative) manifolds. We consider three examples :

representation varieties : Let $\operatorname{rep}_n A = \{\phi : A \longrightarrow M_n(\ell)\}$ be the affine scheme of n-dimensional representations of A, then $\{\operatorname{rep}_n A : n \in \mathbb{N}\}$ is a family of smooth affine varieties whenever A s a manifold.

Hilbert schemes : Consider the open subset of $\operatorname{rep}_n A \times \ell^n$ consisting of couples (ϕ, v) such that $\phi(A).v = \ell^n$. The base-change action of GL_n on this set is given by

$$g.(\phi, v) = (g\phi g^{-1}, gv)$$

and is a *free* action whence we can form the orbit-space which we denote by $\operatorname{Hilb}_n A$ and call the (non-commutative) *Hilbert scheme* of A. It classifies all codimension n left-ideals of A. If A is a manifold then $\{\operatorname{Hilb}_n A : n \in \mathbb{N}\}$ is a family of smooth varieties because the smooth variety $\operatorname{rep}_n A \times \ell^n$ is a principal GL_n-bundle over $\operatorname{Hilb}_n A$. One can generalize this to *higher Hilbert schemes* $\operatorname{HiHilb}_n^m A$ starting from the open subset of

$$\{(\phi, v_1, \ldots, v_k) \in \operatorname{rep}_n A \times \ell^n \times \ldots \times \ell^n \} \quad \text{such that} \quad \phi(A)v_1 + \ldots + \phi(A)v_k = \ell^n$$

with GL_n-action $g.(\phi, v_1, \ldots, v_k) = (g\phi g^{-1}, gv_1, \ldots, gv_k)$. Again, if A is a manifold then $\{\operatorname{HiHilb}_n^m A : m, n \in \mathbb{N}\}$ is a family of smooth varieties.

Simple representations : Consider the open subset of $\operatorname{rep}_n A$

$$\{\phi : \phi \otimes \bar{\ell} : A \otimes \bar{\ell} \longrightarrow M_n(\bar{\ell}) \}$$

of absolutely simple n-dimensional representations of A, then its orbit space $\operatorname{simp}_n A$ is a smooth variety whenever A is a manifold because the smooth open subset defined above is a principal PGL_n-bundle over it. One should observe that principal GL_n-bundles are *a lot* easier to study than principal PGL_n-bundles. In fact one can even argue that non-commutative algebra owns its existence from the fact that there are non-trivial principal PGL_n-bundles.

3.2 Counting points

WE want to understand the geometry of these families. One way to get at conjectural descriptions is to *count points* over finite fields. So, assume that $\ell = \mathbb{F}_q$ and let A be a

manifold defined over $\mathbb{F}_q$, then it follows from the constructions that the sets of $\mathbb{F}_q$-rational points

$$\# \, \mathtt{rep}_n \, A(\mathbb{F}_q) \qquad \# \, \mathtt{Hilb}_n \, A(\mathbb{F}_q) \qquad \# \, \mathtt{simp}_n \, A(\mathbb{F}_q)$$

are finite numbers. In his mini-course, Markus Reineke proved the following result

Theorem 3.1 (Reineke) *If we know* $\# \, \mathtt{rep}_n \, A(\mathbb{F}_q)$ *for all* $n \in \mathbb{N}$ *and some extra information, then we can compute* $\# \, \mathtt{Hilb}_n \, A(\mathbb{F}_q)$ *and* $\# \, \mathtt{simp}_n \, A(\mathbb{F}_q)$.

To compute $\# \, \mathtt{Hilb}_n \, A(\mathbb{F}_q)$ the extra information consists of knowledge of the component semigroup $\mathrm{comp}(A)$ together with the Euler form on it as well as the numbers $\# \, \mathtt{rep}_\alpha \, A(\mathbb{F}_q)$ for all components $\alpha \in \mathrm{comp}(A)$.

To compute $\# \, \mathtt{simp}_n \, A$ we need all this extra information not only for A but for all extensions $A \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$ together with the action of the Galois group on the representation spaces.

The main problem is that there are virtually no examples known of manifolds A where we do have all this precise information, except in the case when A is the path algebra of a quiver.

3.3 The case of path algebras

Let Q be a finite quiver on k vertices $\{v_1, \ldots, v_k\}$ and let a_{ij} be the number of directed arrows in Q from v_i to v_j. Then we know all required information for the path algebra $A = \ell Q$ from representation theory, see for example the lecture notes by Harm Derksen for more details.

To begin, the component semigroup $\mathrm{comp}(A) \simeq \mathbb{N}^k$ with generators $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ corresponding to the *vertex-simples*. Further, we can describe the components $\mathtt{rep}_\alpha \, A$. To start, take $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}^k$ and define the *quiver representation space*

$$\mathtt{rep}_\alpha \, Q = \oplus_{1 \le i,j \le k} M_{a_i \times a_j}(\ell)^{\oplus a_{ij}}$$

be the affine space of all representations of Q of dimension vector α, that is, all assignments of square matrices of the appropriate dimensions to all arrows in Q. On this space there is a base-change action by the *base-change group*

$$GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k}$$

which acts via : $g = (g_1, \ldots, g_k)$ and $V = (M_a \; : \; a \text{ arrow})$ then

$$g.V = (g_j M_a g_i^{-1} \; : \; a \text{ arrow from } v_i \text{ to } v_j)$$

Representations belong to the same $GL(\alpha)$-orbits in $\mathtt{rep}_\alpha \, Q$ if and only if they are *isomorphic* as Q-representations.

Let the total dimension of α be $n = |\alpha| = \sum_i a_i$ then there is a natural embedding of $GL(\alpha) \hookrightarrow GL_n$ along the diagonal. In this way, $GL(\alpha)$ acts on the product space

$$GL_n \times \mathtt{rep}_\alpha \, Q \qquad \text{via} \qquad g.(h, V) = (hg^{-1}, g.M)$$

which one verifies to be a *free action* whence we can form the orbit space which is called the *principal fiber bundle*

$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha \, Q$$

This fiber bundle is the component $\text{rep}_\alpha A$ of $\text{rep}_n A$ corresponding to $\alpha \in \text{comp}(A)$. But then we can compute the number of points in such a component over a finite field (use freeness of the action)

$$\# \text{rep}_\alpha A(\mathbb{F}_q) = \frac{\# GL_n(\mathbb{F}_q) . \# \text{rep}_\alpha Q(\mathbb{F}_q)}{\# GL(\alpha)(\mathbb{F}_q)} = \frac{\# GL_n(\mathbb{F}_q) . q^{\sum_{i,j} a_i a_j a_{ij}}}{\prod_{i=1}^k GL_{a_i}(\mathbb{F}_q)}$$

a number which we can compute using the standard fact that

$$\# GL_m(\mathbb{F}_q) = q^{\binom{m}{2}} (q-1)(q^2-1) \cdots (q^m - 1)$$

Remains to know the *Euler-form* on $\text{com } A = \mathbb{N}^k$. Define the integral matrix $E(Q) \in M_k(\mathbb{Z})$ whose (i,j)-th entry is equal to $\delta_{ij} - a_{ij}$, then one can prove that the Euler form

$$\chi_A(\alpha, \beta) = \alpha^\tau E(Q) \beta$$

and is therefore computable from the information on the quiver Q. Finally, observe that

$$A \otimes \mathbb{F}_{q^\tau} = \mathbb{F}_{q^\tau} Q$$

and the action of the Galois group is given by its action on the scalars, so we know all extra information we need in this special (but important) case.

3.4 Free products

We want to find a large class of new examples for which we can compute everything explicitly. Free algebras of path algebras (or even the simplest case of free products of semisimple algebras) provide such a class.

Recall that if A and B are ℓ-algebras, then the algebra free product $A *_\ell B$ (or simply $A * B$) i the ℓ-algebra determined by the universal property that for every ℓ-algebra D there is a natural isomorphism

$$Hom_\ell(A, D) \times Hom_\ell(B, D) \simeq Hom_\ell(A * B, D)$$

Here is how to construct $A * B$: choose an ℓ-basis a_i : $i \in I$ of A and a ℓ-basis b_j : $j \in J$ of B (both including 1), then $A * B$ has an ℓ-basis consisting of all alternating words

$$w = a_{i_1} b_{j_1} a_{i_2} \cdots a_{i_l} b_{j_l}$$

and where multiplication is induced by concatenation. Even for the simplest of algebras A and B (say both ℓ-semisimple) not much is known about these algebra free products.

Lemma 3.2 *If A and B are manifolds, then so is $A * B$.*

Proof. Follows from the universal property by using the nilpotent lift characterization of manifolds. $\square$

Lemma 3.3 $\text{rep}_n A * B = \text{rep}_n A * \text{rep}_n B$.

Proof. A representation is an ℓ-algebra map $A * B \longrightarrow M_n(\ell)$ whence determines (and is determined by) algebra maps $A \longrightarrow M_n(\ell)$ and $B \longrightarrow M_n(\ell)$. $\square$

The connected component semigroup $\text{comp}(A)$ comes equipped with a semigroup map

$$\text{comp}(A) \xrightarrow{deg} \mathbb{N}$$

assigning to a dimension vector α its total dimension $deg(\alpha) = |\alpha|$. This allows us to define the *fibered product*

$$\text{comp}(A) \times_\mathbb{N} \text{comp}(B) = \{(\alpha, \beta) \in \text{comp}(A) \times \text{comp}(B) \mid |\alpha| = |\beta|\}$$

Lemma 3.4 $\text{comp}(A * B) = \text{comp}(A) \times_\mathbb{N} \text{comp}(B)$.

Proof. Follows from the previous lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

So, we only have to produce the Euler-form on the component semigroup of a free product algebra. We will do this in the special case when A and B are both semisimple ℓ-algebras by reducing to a certain path algebra for which we know the Euler-form. We start with an example :

Example 3.5 Let $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ be the (projective) modular group (important in number-theory as well as in knot-theory). Its group algebra is of the above type

$$\ell PSL_2(\mathbb{Z}) \simeq \ell\mathbb{Z}_2 * \ell\mathbb{Z}_3 \simeq (\ell \times \ell) * (\ell \times \ell \times \ell)$$

(at least if $char(\ell)$ is not 2 or 3 and ℓ contains a primitive 3-rd root of unity). Let us clarify what we know already about this algebra.

$$\text{comp}(\ell PSL_2(\mathbb{Z})) \simeq \mathbb{N}^2 \times_\mathbb{N} \mathbb{N}^3 = \{(a-1, a_2; b_1, b_2, b_3) \mid a_1 + a_2 = b_1 + b_2 + b_3\}$$

and for $\alpha = (a_1, a_2; b_1, b_2, b_3)$ in this semigroup with $n = a_1 + a_2$

$$\text{rep}_\alpha \ell PSL_2(\mathbb{Z}) = (GL_n/GL_{a_1} \times GL_{a_2}) \times GL_n/(GL_{b_1} \times GL_{b_2} \times GL_{b_3})$$

from which it follows that we can calculate the number of $\mathbb{F}_q$-points in such a component for all dimension vectors α. Let $M \in \text{rep}_n \ell PSL_2(\mathbb{Z})$ and consider the restrictions

$$M \downarrow_{\mathbb{Z}_2} \simeq V_+^{\oplus a_1} \oplus V_-^{\oplus a_2} \quad \text{and} \quad M \downarrow_{\mathbb{Z}_3} \simeq V_1^{\oplus b_1} \oplus V_\rho^{\oplus b_2} \oplus V_{\rho^2}^{\oplus b_3}$$

with obvious notations. Choosing bases in these eigenspaces, we can relate M to a representation of a bipartite quiver Q having two left and three right vertices with one arrow connecting each left-vertex to each right-vertex and where the maps corresponding to these arrows make up the blocks of an $n \times n$ matrix

$$A_M = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

describing the base-change matrix between our two sets of bases for M. That is, to M we can associate a representation in the open subset of $\text{rep}_\alpha Q$ consisting of those representations such that the $n \times n$ matrix on their arrow-matrices is invertible. A moments thought shows that we can also reverse this process. But then we have for all finite dimensional $A = \ell PSL_2(\mathbb{Z})$-representations M and N that

$$\chi_A(M, N) = \chi_{\ell Q}(A_M, A_N)$$

where the right hand side can be computed using the Euler-matrix of the quiver Q.

Clearly, this idea can be extended to algebra free products of (split) semi simples.

Theorem 3.6 *The Euler-form of the algebra free product*

$$A = (M_{k_1}(\ell) \oplus \ldots \oplus M_{k_u}(\ell)) * (M_{l_1}(\ell) \oplus \ldots \oplus M_{l_v}(\ell))$$

can be computed from that of the path algebra ℓQ where Q is the bipartite quiver on u left vertices and v right vertices having precisely $k_i l_j$ directed arrows from the i-th left vertex to the j-th right vertex.

3.5 Project 1

By the results of the last section we have all the required information to compute (using the results of Markus Reineke) the generating sequences for the number of points of Hilbert schemes and simples of any algebra free product of two semisimple ℓ-algebras. But why should we invest time in trying to perform these tedious calculations?

Well, there are a number of conjectures in non-commutative geometry that need some extra testing. One such conjectural description of the *quasi-free world* is that whereas *commutative manifolds* correspond to *quasi-free algebras*, their *tangent spaces* should correspond to *path algebras of quivers*. More precisely,

Conjecture 3.7 *To any quasi-free algebra A one can associate a quiver Q_A such that A is étale Morita equivalent to teh path algebra ℓQ_A whatever that means.*

If this conjecture is true then one would expect a *nice relation* between the generating sequences of A and those of the algebra Morita equivalent to ℓQ_A and we can compute all this in teh case when A is the free product of two split semisimple ℓ-algebras. By a *nice relation* one might mean that their quotient is a rational function or that there is a simple algebraic relation between these generating functions. I would advice to perform the calculations first in some easy cases such as

$$A = \ell PSL_2(\mathbb{Z}) \qquad \text{or} \qquad A = M_n(\ell) * M_m(\ell)$$

But, what might this quiver Q_A be associated to these algebras? In general, if A is a manifold, consider teh semigroup $\text{comp}(A)$ and assume taht it is finitely generated as a semigroup. In fact, here is another

Conjecture 3.8 *If A is an affine quasi-free ℓ-algebra, then $\text{comp}(A)$ is finitely generated as semigroup.*

Let $\{\alpha_1, \ldots, \alpha_k\}$ be a minimal set of semigroup generators (that is, such that none of the α_i can be written as a sum of the others) then we define the quiver Q_A to be the one on k vertices $\{v_1, \ldots, v_k\}$ where v_i corresponds to the generator component α_i and such that there are precisely

$$\delta_{ij} - \chi_A(\alpha_i, \alpha_j)$$

directeda rrows from v_i to v_j in Q_A. In the *'One quiver to rule them all'* paper I gave some evidence to support the following conjectural truth

Theorem 3.9 *If A is a manifold, then its tangent space is the ring Morita equivalent to ℓQ_A where the Morita equivalence is determined by teh integers $|\alpha_1|, \ldots, |\alpha_k|$.*

Using our knowledge to compute the Euler form χ_A for A a free product of semisimples it is easy to work out the following in our two prime cases :

Lemma 3.10 *The quiver Q_A corresponding to $A = M_n(\ell) * M_m(\ell)$ is the* bouquet *quiver with one vertex and with excatly*

$$N = a_0^2 + b_0^2 - nma_0b_0 \qquad loops$$

where $a_0 = lcm(m,n)/n$ and $b_0 = lcm(m,n)/m$. We expect A to be étale equivalent to

$$M_{lcm(m,n)}(\ell\langle x_1, \cdots, x_N\rangle)$$

For $A = \ell PSL_2(\mathbb{Z})$ the associated quiver Q_A is teh double quiver of the extended Dynkin quiver $\tilde{A}_5$ (a cycle on 6 vertices). In this case we expect $\ell PSL_2(\mathbb{Z})$ to be étale equivalent to ℓQ_A as all generators have total dimension one.

Hence, teh first project is to work out all generating sequences for those two quasi-free algebras and their associated path algebras or more generally for any free algebra product of two semisimple ℓ-algebras (in fact, a next step may be to do all this for the free product of two path algebras with obvious modifications to compute their Euler product).

3.6 Project 2

If $\ell = \bar{\ell}$ is algebraically closed, then the semigroup comp(A) contains all information required, but over an arbitrary field ℓ this semigroup can be ridiculously small compared to the semigroup of $A \otimes \bar{\ell}$. For example, if $\ell \subset L$ is a k-dimensional separable field extension, then comp(L) $= k\mathbb{N} \subset \mathbb{N}$ whereas comp($L \otimes \bar{\ell}$) $\simeq \mathbb{N}^k$. For this project we want to construct *something* associated to a quasi-free algebra A over an arbitrary basefield ℓ having the property that it has the same size as the corresponding object for $A \otimes \bar{\ell}$ and that this obkect over the algebraic closure enables us to reconstruct the component semigroup comp($A \otimes \bar{\ell}$).

We need to recall first some facts from *commutative* algebra. If C is a commutative affine ℓ-algebra then we call C *unramified* if and only if

$$C \otimes_\ell \bar{\ell} \simeq \bar{\ell} \times \ldots \bar{\ell}$$

a finite number of times. It is well known that the only unramified algebras over a field ℓ are of the form

$$C = L_1 \times \ldots \times L_k$$

where each L_i is a finite dimensional separable fields extension of ℓ. It follows from this that any subalgebra of an unramified algebra is unramified, that the tensor-product of two unramified algebras is unramified and that an epimorphic image of an unramified algebra remains unramified.

From this it follows that if C is an affine ℓ-algebra then there exists a *unique maximal unramified ℓ-subalgebra* of C which we denote with $\pi_0(C)$. Here are some useful facts about this subalgebra.

If $\ell \subset L$ is a field extension, then

$$\pi_0(C) \otimes_\ell L \simeq \pi_0(C \otimes L)$$

so everything defined in terms of π_0's will be of teh same size over ℓ as over its algebraic closure $\bar{\ell}$. Moreover, if $C = \ell[X]$ is the coordinate ring of an affine variety defined over ℓ, then we call X (or $\ell[X]$) *connected* if and only if $\ell[X]$ contains no non-trivial idempotents, that is, we cannot write $\ell[X]$ as the direct sum $\ell[X_1] \oplus \ell[X_2]$. We call X (or $\ell[X]$) *geometrically connected* iff $\ell[X] \otimes \bar{\ell}$ is connected. Here is teh geometrical interpretation of te subalgebra $\pi_0(C)$.

Theorem 3.11 *X or $\ell[X]$ is connected if and only if $\pi_0(\ell[X])$ is a field.*

X or $\ell[X]$ is geometrically connected if and only if $\pi_0(\ell[X]) = \ell$.

If X is connected and has an ℓ-rational point, then X is geometrically connected.

If X has several connected componments, say $\ell[X] = \ell[X_1] \oplus \ldots \oplus \ell[X_k]$, then we have that

$$\pi_0(\ell[X]) \simeq L_1 \times \ldots \times L_k$$

with each component L_i a finite dimensional separable field extension of ℓ.

That is, $\pi_0(\ell[X])$ contains all information about the connected components of X and how they might decompose further over field extensions $\ell \subset L \subset \bar{\ell}$. Moreover, π_0 behaves nicely with respect to morphisms and products of ℓ-varieties.

Theorem 3.12 *A morphism $X_1 \longrightarrow X_2$ of affine ℓ-varieties induces an ℓ-algebra map*

$$\pi_0(\ell[X_2]) \longrightarrow \pi_0(\ell[X_1])$$

If X_1 and X_2 are affine ℓ-varieties, then the natural map

$$\pi_0(\ell[X_1]) \otimes_\ell \pi_0(\ell[X_2]) \longrightarrow \pi_0(\ell[X_1] \otimes_\ell \ell[X_2]) = \pi_0(\ell[X_1 \times X_2])$$

is an ℓ-algebra isomorphism.

Of course we will apply all of this to our *sum morphisms*

$$\mathrm{rep}_n A \times \mathrm{rep}_m A \longrightarrow \mathrm{rep}_{n+m} A$$

which then give us algebra maps

$$\pi_0(\ell[\mathrm{rep}_{n+m} A]) \xrightarrow{\Delta_{m,n}} \pi_0(\ell[\mathrm{rep}_n A]) \otimes \pi_0(\ell[\mathrm{rep}_m A])$$

which allow us to define on the graded vector-space

$$\pi_0(A) = \oplus_{n=0}^\infty \pi_0(n) = \oplus_{n=0}^\infty \pi_0(\ell[\mathrm{rep}_n A])$$

a gradation preserving map (the *comultiplication*)

$$\pi_0(A) \xrightarrow{\Delta} \pi_0(A) \otimes \pi_0(A) \qquad \pi_0(N) \xrightarrow{\sum_{n+m=N} \Delta_{m,n}} \sum_{n+m=N} \pi_0(n) \otimes \pi_0(m)$$

as well as a *counit* $\epsilon : \pi_0(A) \longrightarrow l = \pi_0(0)$ projecting to the degree zero component. As all the $\pi_0(n)$ are l-algebras (hence so is $\pi_0(A)$) it is an exercise to show that

Lemma 3.13 *$\pi_0(A)$ with the above structures is a commutative and co-commutative bialgebra.*

remains to clarify teh connection with the component semigroup $\mathrm{comp}(A \otimes \bar{\ell})$.

Theorem 3.14 *Recall that the function bialgebra* Fun $\mathrm{comp}(A \otimes \bar{\ell})$ *is teh algebra of $\bar{\ell}$-valued functions on teh semigroup* $\mathrm{comp}(A \otimes \bar{\ell})$ *and has a comultiplication induced by*

$$\Delta(\alpha) = \sum_{\beta + \gamma = \alpha} \beta \otimes \gamma$$

Then, tehre is a natural bialgebra isomorphism

$$\pi_0(A) \otimes_\ell \bar{\ell} = \pi_0(A \otimes_\ell \bar{\ell}) \simeq \mathrm{Fun}\, \mathrm{comp}(A \otimes \bar{\ell})$$

Note that the function bialgebra is sort of dual to the usual semigroup-algebra $\bar{\ell}\,\text{comp}(A\otimes\bar{\ell})$. So we cannot descent the semigroup $\text{comp}(A\otimes\bar{\ell})$ to a reasonable semigroup over ℓ but we *can* descent the function bialgebra to obtain an ℓ-bia;gebra $\pi_0(A)$ cintaining enough information. In fact there is another description of this bialgebra.

Lemma 3.15 *The Galois group* $\text{Gal}(\bar{\ell}/\ell)$ *acts on* $A\otimes\bar{\ell}$ *(and hence on* $\text{comp}(A\otimes\bar{\ell})$ *and its function bialgebra) and we have*

$$\pi_0(A) \simeq \text{Func comp}(A\otimes\bar{\ell})^{\text{Gal}(\bar{\ell}/\ell)}$$

It would be interesting to compute these bialgebras as explicitly as possible for some easy classes of quasi-free algebras A.

So we have a big enough object defined over ℓ but are still missing one essential ingredient : the Euler-form. Here is a way to encode this information too. The Euler-form defines a bi-cocharacter (teh dual of a bi-character on teh level of the semigroup algebra) and we can use it to *twist* teh coalgebra structure resulting in a *braided co- or bi-algebra*. In this way, we get a braided cocommutative bialgebra which not only encodes the Euler-form but is also related to the *self dual* nature of the *Hall algebra* (this last became clear after talking to Markus Reineke). Projects to work out might be to check the validity of the commuting diagram

$$\text{Hall}((A)) \xrightarrow{\ \ j\ \ } \mathbb{Q}_{tw}[[\text{comp}(A\otimes\bar{\mathbb{F}}_q)]]$$

$$\|^* \qquad\qquad\qquad \|^*$$

$$\text{Hall}((A))^* \xleftarrow{\ \ f^*\ \ } \pi_0(A)_{tw}$$

(here the top map is teh algebra map defined by Markus Reineke in his mini-course) in particular, whether one can lift the (twisted) bialgebra structure over $\mathbb{F}_q$ to one over $\mathbb{Z}$ (or ℓ). Again, it would be interesting to work all this out in greater detail.