noncommutative geometry@n

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CONTENTS

lecture 1

QUOTIENT SINGULARITIES

1.1 Orbifold constructions

In this section we will run through the construction of the *orbifold* $\mathbb{C}^n/G$ which is the orbit-space of the action of a finite group G on an n-dimensional representation $V = \mathbb{C}^n$ of G. We start by recalling some standard facts on the representation theory of finite groups. For more details on this we refer to [**?**, Chp 1-2]. Some of this results we will generalize in later chapters to other reductive groups and mor egeneral non-commutative algebras.

Definition 1.1 An *n-dimensional representation* of a finite group G is a group-morphism

$$G \xrightarrow{\phi} GL_n(\mathbb{C})$$

Equivalently, ϕ defines a G-action on the n-dimensional space $V \simeq \mathbb{C}^n$ via the rule $g.v = \phi(g)v$ where elements of V are viewed as column-vectors.

As we will work throughout this book with finite dimensional representations of algebras, let us bring in the finite dimensional (usually non-commutative) *group algebra*

$$\mathbb{C}G = \oplus_{g \in G} \mathbb{C}e_g \qquad \text{with multiplication induced by} \qquad e_g.e_h = e_{gh}$$

and observe that an n-dimensional G-representation ϕ determines an n-dimensional *left $\mathbb{C}G$-module* $M_\phi \simeq \mathbb{C}^n$ with module structure induced by

$$e_g.v = \phi(g)v \qquad \text{for all } g \in G$$

Alternatively, we say that M_ϕ is an *n-dimensional representation of* $\mathbb{C}G$. Conversely, an n-dimensional left $\mathbb{C}G$-module $M = \mathbb{C}G$ defines an n-dimensional G-representation ϕ_M with $\phi_M(g)$ the $n \times n$ matrix expressing the left action by e_g on M. Hence, both approaches are equivalent.

A *G-linear map* (sometimes called a *G-equivariant map*) between G-representations V and W is a *linear map* $\psi \; : \; V \longrightarrow W$ such that for all $g \in G$ the diagram below commutes

$$\begin{array}{ccc} V & \xrightarrow{\psi} & W \\ {\scriptstyle g.}\downarrow & & \downarrow{\scriptstyle g.} \\ V & \xrightarrow{\psi} & W \end{array}$$

In particular, two G-representations V and W are *isomorphic* iff they have the same dimension n and if their actions are *conjugate*, that is, there is an invertible $n \times n$ matrix A such that

$$\phi_W(g) = A^{-1}\phi_V(g)A \qquad \text{for all } g \in G.$$

The vectorspace of all G-linear maps from V to W will be denoted by $Hom_G(V,W)$. If $\psi \in Hom_G(V,W)$ is injective, V is called a *subrepresentation* of W. A representation W without *proper* subrepresentations is called *irreducible* (or *simple*).

There are standard procedures to construct new representations from known representations :

- *the direct sum* $V \oplus W$ with action $g.(v+w) = \phi_V(g)v + \phi_W(g)w$.
- *the tensor product* $V \otimes W$ with action $g.(v \otimes w) = \phi_V(g)v \otimes \phi_W(g)w$.
- *the dual* $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$ with action $g.v^* = \phi_V(g^{-1})^\tau v^*$. (Here, A^τ denotes the *transpose* of A)

The *regular representation* R is the underlying space of the group algebra $\mathbb{C}G$ with action $g.e_h = e_{gh}$.

A crucial property of finite groups is that they are *reductive*. That is, every finite dimensional representation is isomorphic to a direct sum of simple representations. This fact follows by induction on the dimension of the representation from the *averaging argument* below. A similar argument replacing sums by integrals on unitary groups can be used to prove that GL_n is reductive, a fact we will use later on.

Lemma 1.2 *If W is a G-subrepresentation of V, then there is a G-subrepresentation W' of V such that (as G-representations)*

$$V = W \oplus W'$$

Proof. Take a $\mathbb{C}$-vectorspace complement $V = W \oplus U$ of W and consider the $\mathbb{C}$-linear projection map $\pi_1 : V \twoheadrightarrow W$ on the first component. Average this map over the finite group G, that is, define

$$V \overset{\pi}{\twoheadrightarrow} W \qquad \text{via} \qquad \pi(v) = \frac{1}{\# G} \sum_{g \in G} g.(\pi_1(g^{-1}.v))$$

(where $\# G$ is the *order* of G) and verify that this is a G-linear map on W. The *kernel* of π, $\mathtt{ker}\ \pi = \{v \in V \mid \pi(v) = 0\}$ will be a G-subrepresentation of V and $V = W \oplus \mathtt{ker}\ \pi$ as G-representations. □

A second important ingredient is *Schur's lemma* which can be extended verbatim to simple finite dimensional representations of algebras.

Lemma 1.3 (Schur's lemma) *If $V \xrightarrow{\psi} W$ is a G-linear map between simple G-representations, then either $\phi = 0$ or ϕ is an isomorphism which is given by scalar multiplication by $\lambda \in \mathbb{C}$.*

Proof. Because $\mathtt{ker}\ \psi$ and $\mathtt{im}\ \psi$ are G-subrepresentations of V resp. W it follows from irreducibility that ψ is either the zero-map or an isomorphism. If $\psi \neq 0$, take $V = W$ and let λ be an eigenvalue of the matrix describing ψ. Then, $\psi - \lambda 1_V$ is a G-linear map with non-zero kernel and so $V = \mathtt{ker}\ \psi - \lambda 1_V$ or, equivalently, $\psi = \lambda 1_V$ on V. □

Proposition 1.4 (Complete Reducibility) *Any finite dimensional G-representation V can be decomposed as*

$$V = V_1^{\oplus e_1} \oplus \ldots \oplus V_k^{\oplus e_k}$$

with the V_i non-isomorphic irreducible representations. In this decomposition, the irreducible factors and their multiplicities are uniquely determined.

Proof. If W is a G-representation with decomposition $\oplus_j W_j^{\oplus f_j}$ and if $\phi : V \longrightarrow W$ is a G-linear map, then by Schur's lemma, ϕ must map the factor $V_i^{\oplus e_i}$ into that factor $W_j^{\oplus f_j}$ for which $W_j \simeq V_i$. Applying this to the identity map of V, the uniqueness statement follows. □

The principal problem in the representation theory of algebras is to determine methods to test whether two finite dimension representations are isomorphic. Later on, we will see that *traces* can be used to distinguish non-isomorphic *semi-simple modules*. The archetypical instance of this result is the classical notion of *group characters*.

Definition 1.5 The *character* χ of an n-dimensional representation $\phi : G \longrightarrow GL_n(\mathbb{C})$ of a finite group G is the map

$$\chi : G \longrightarrow \mathbb{C} \qquad \text{defined by} \qquad \chi(g) = tr(\phi(g))$$

where tr denotes the trace of the square matrix.

Clearly, as traces of conjugate matrices are equal, the character is an isomorphism invariant of a representation. Moreover, one easily verifies (see [**?**, Chp 2]) that $\chi(1)$ equals teh dimension of the representation, that $\chi(g)$ is a *class function*, that is, is constant along conjugacy classes and that the character of a direct sum of representations is the sum of the characters. One defines an *inproduct* on the characters of G by the rule

$$\langle \chi, \chi' \rangle = \frac{1}{\# G} \sum_{g \in G} \overline{\chi(g)} \chi'(g)$$

With respect to this inproduct, the main results on characters are summarized in :

Proposition 1.6 *Let G be a finite group with $\phi_1, \phi_2, \ldots$ the set of distinct simple representations of G having characters $\chi_1, \chi_2, \ldots$.*

1. *The characters χ_i are orthogonal, that is,*

$$\langle \chi_i, \chi_j \rangle = \delta_{ij} \qquad \textit{for all } i, j$$

2. *The number of isomorphism classes of irreducible G-representations is finite and equals the number of conjugacy classes in G.*
3. *If d_i is the dimension of the irreducible G-representation ϕ_i, then $d_i | \# G$ and*

$$\# G = d_1^2 + \ldots + d_r^2$$

where r is the number of conjugacy classes of G.

Proof. See [**?**, Chp. 2] for details. □

It follows that the representation theory of G is fully encoded in its *character table* which is a square matrix with the rows corresponding to the distinct irreducible G-representations, the columns to the conjugacy classes of G and the (i, j)-entry is $\chi_i(g)$ with g an element of the j-th conjugacy class.

After this brief recap, let us address the topic of this section. Let V be an n-dimensional G-representation and consider the linear action of the finite group G on the n-dimensional affine space $\mathbb{C}^n = V$. The corresponding *orbifold* will be the *orbit space* $\mathbb{C}^n/G$, that is the set of equivalence classes

$$(a_1,\ldots,a_n) \sim (b_1,\ldots,b_n) \qquad \text{iff} \qquad g.\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad \text{for some } g \in G$$

We will prove that $\mathbb{C}^n/G$ is an affine variety (usually with singularities) with coordinate ring $\mathcal{O}(\mathbb{C}^n/G) = \mathbb{C}[x_1,\ldots,x_n]^G$, the ring of *polynomial G-invariants*. Again, this is an archetypical case of a more general result : if a reductive group acts on a vectorspace, then the best algebraic approximation to the orbit-space (which does not have to exist in general, due to the existence of non-closed orbits) is the affine variety associated to the affine algebra of polynomial invariants. We will encounter many instances of this result later on in this book.

An n-dimensional G-representation V determines a group-morphism

$$G \xrightarrow{\phi} GL_n(\mathbb{C})$$

which also determines an action of G by *automorphisms* on the polynomial algebra $\mathbb{C}[x_1,\ldots,x_n]$ via the rule

$$\begin{bmatrix} \phi_g(x_1) \\ \vdots \\ \phi_g(x_n) \end{bmatrix} = \phi(g).\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

That is, every $g \in G$ sends x_i to a linear combination $\phi_g(x_i)$ and induces therefore a *degree preserving* automorphism on $\mathbb{C}[x_1,\ldots,x_n]$. Under this automorphism, an arbitrary polynomial of degree d, $f(x_1,\ldots,x_n)$ is send to the polynomial of degree d

$$\phi_g(f) = f(\phi_g(x_1),\ldots,\phi_g(x_n)) \in \mathbb{C}[x_1,\ldots,x_n]$$

Definition 1.7 Let $G \xrightarrow{\phi} GL_n(\mathbb{C})$ be determined by an n-dimensional G-representation V and let $\{\phi_g \,:\, g \in G\}$ be the induced algebra automorphisms on the polynomial algebra $\mathbb{C}[x_1,\ldots,x_n]$. A polynomial $f(x_1,\ldots,x_n) \in \mathbb{C}[x_1,\ldots,x_n]$ is said to be an *invariant* under G if

$$\phi_g(f) = f \quad \text{for all } g \in G$$

The subalgebra (verify!) of all invariant polynomials is denoted $\mathbb{C}[x_1,\ldots,x_n]^G$ and is called the *ring of polynomial invariants*.

Example 1.8 Let V_4 be the *Klein Vierergruppe* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and consider the 2-dimensional representation determined by

$$V_4 \longrightarrow \{\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}\} \subset GL_2(\mathbb{C})$$

then V_4 is generated by the two matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and a polynomial $f(x,y) = \sum_{ij} a_{ij}x^iy^j \in \mathbb{C}[x,y]$ is invariant under V_4 if and only if

$$f(x,y) = f(-x,y) = f(x,-y)$$

that is, if and only if, $a_{ij} = 0$ whenever i or j is odd. Therefore, the ring of polynomial V_4-invariants is

$$\mathbb{C}[x,y]^{V_4} = \mathbb{C}[x^2,y^2]$$

The fact that this ring of invariants is again a polynomial ring is rather special. On the other hand, the fact that it is generated by finitely many elements is general as we will now prove. If $f_1, \ldots, f_m$ are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ we will denote with $\mathbb{C}[f_1, \ldots, f_m]$ the *subalgebra* of $\mathbb{C}[x_1, \ldots, x_n]$ generated by the f_i. Observe that we do *not* mean by this that this subalgebra is again a polynomial ring (there may be algebraic relations among the f_i). The averaging trick we used before also has its use here.

Definition 1.9 Let $G \longrightarrow GL_n(\mathbb{C})$ determine an n-dimensional G-representation V. The corresponding *Reynolds operator* is the map

$$\mathbb{C}[x_1, \ldots, x_n] \xrightarrow{R_G} \mathbb{C}[x_1, \ldots, x_n]$$

defined by

$$R_G(f)(x_1, \ldots, x_n) = \frac{1}{\# G} \sum_{g \in G} f(\phi_g(x_1), \ldots, \phi_g(x_n)) = \frac{1}{\# G} \sum_{g \in G} \phi_g(f)$$

Lemma 1.10 *Let R_G be the Reynolds operator corresponding to an action $G \longrightarrow GL_n(\mathbb{C})$. Then,*

1. *R_G is a $\mathbb{C}$-linear map.*
2. *For all $f \in \mathbb{C}[x_1, \ldots, x_n]$ we have that $R_G(f) \in \mathbb{C}[x_1, \ldots, x_n]^G$.*
3. *If $f \in \mathbb{C}[x_1, \ldots, x_n]^G$, then $R_G(f) = f$.*

Proof. R_G is a linear combination of the algebra morphisms ϕ_g whence $\mathbb{C}$-linear proving (1). To prove (2) let $h \in G$ then

$$\begin{aligned}\phi_h(R_G(f)) =& \phi_h(\frac{1}{\# G} \sum_{g \in G} \phi_g(f)) = \frac{1}{\# G} \sum_{g \in G} \phi_h(\phi_g(f)) \\ =& \frac{1}{\# G} \sum_{hg \in G} \phi_{hg}(f) = R_G(f)\end{aligned}$$

As for (3) if $f \in \mathbb{C}[x_1, \ldots, x_n]^G$ then

$$R_G(f) = \frac{1}{\# G} \sum_{g \in G} \phi_g(f) = \frac{1}{\# G} \sum_{g \in G} f = f$$

and we are done! □

We therefore see that R_G is a surjective map which gives us a method to produce lots of polynomial G-invariants. We say that a *monomial* $m = x_1^{a_1} \ldots x_n^{a_n}$ has degree d and denote $\mathtt{deg} m = d$ if $d = a_1 + \ldots + a_n$.

Proposition 1.11 *The ring of polynomial G-invariants*

$$\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[R_G(m) \ : \ \mathtt{deg} m \leq N]$$

is generated by the images under the Reynolds operator of all monomials of degree at most the order of G and hence is finitely generated.

Proof. For any $k \in \mathbb{N}$ we can expand

$$(x_1 + \ldots + x_n)^k = \sum_{\mathtt{deg} m = k} a_m m(x_1, \ldots, x_n)$$

as a linear combination of all monomials of degree m. Now, take another set of variables $u_1, \ldots, u_n$ and consider in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_n]$ for every $g \in G$

$$(u_1\phi_g(x_1) + \ldots + u_n\phi_g(x_n))^k = \sum_{\mathtt{deg} m = k} a_m \phi_g(m(x_1, \ldots, x_n)) m(u_1, \ldots, u_n)$$

But then, we have the equality

$$\begin{aligned} \sum_{g \in G} (u_1\phi_g(x_1) + \ldots + u_n\phi_g(x_n))^k &= \textstyle\sum_{g \in G} \sum_{\mathtt{deg} m = k} a_m \phi_g(m) m(u_1, \ldots, u_n) \\ &= \textstyle\sum_{\mathtt{deg} m = k} (N a_m) R_G(m) m(u_1, \ldots, u_n) \end{aligned}$$

Denote $v_g = u_1\phi_g(x_1) + \ldots + u_n\phi_g(x_n)$, then the left-hand expression is $S_k(v_1, \ldots, v_N)$ the S_N-invariant where S_N is the symmetric group on N letters acting by permuting the variables of $\mathbb{C}[v_1, \ldots, v_N]$. In the exercises we will see that for any $k \in \mathbb{N}$, S_k is a polynomial in the invariants $S_1, \ldots, S_N$. That is, we can write

$$S_k(v_1, \ldots, v_N) = P(S_1(v_1, \ldots, v_N), \ldots, S_N(v_1, \ldots, v_N))$$

and resubstituting in this expression $v_g = u_1\phi_g(x_1) + \ldots + u_n\phi_g(x_n)$ and working out both sides of the equality above and comparing x_i-terms belonging to the same monomial in the u_i we deduce that

$$(N a_m) R_G(m(x_1, \ldots, x_n)) = Q(R_G(m'(x_1, \ldots, x_n)) \ : \ \mathtt{deg} m' \leq N)$$

for some polynomial Q. As a consequence (using linearity of the Reynolds operator) we see that the image $R_G(f)$ for any $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a certain polynomial in the images $R_G(m)$ where the degree of the monomial m is at most N. $\square$

If V is an n-dimensional G-representation, then the groupmorphism $G \longrightarrow GL_n(\mathbb{C})$ also determines an action of G on $\mathbb{C}^n$ $p \mapsto g.p$ via

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mapsto \phi(g). \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

and we want to describe the *orbit space* $\mathbb{C}^n/G$, that is, we want to describe the G-orbits

$$\mathcal{O}(p) = \{q \in \mathbb{C}^n \mid q = g.p \text{ for some } g \in G\}$$

Clearly, any polynomial invariant $f \in \mathbb{C}[x_1, \ldots, x_n]^G$ remains constant over the G-orbit of a point $p = (a_1, \ldots, a_n)$ as

$$f(g.(a_1, \ldots, a_n)) = \phi_g(f)(a_1, \ldots, a_n) = f(a_1, \ldots, a_n)$$

and as affine varieties are determined by their polynomial functions we hope that the coordinate ring of the orbit-space (or the *quotient variety*) is given buy the invariants, that is

$$\mathbb{C}[\mathbb{C}^n/G] = \mathbb{C}[x_1, \ldots, x_n]^G$$

We begin by describing the affine variety determined by the invariant ring. In the previous section we have seen that the invariant ring is finitely generated, say

$$\mathbb{C}[x_1,\ldots,x_n]^G = \mathbb{C}[f_1,\ldots,f_m] \subset \mathbb{C}[x_1,\ldots,x_n]$$

for certain invariant polynomials f_i. However, there may be relations among these f_i, that is, we can write

$$\mathbb{C}[x_1,\ldots,x_n]^G = \frac{\mathbb{C}[y_1,\ldots,y_m]}{I}$$

where I is the kernel of the surjective algebra morphism $\mathbb{C}[y_1,\ldots,y_m] \twoheadrightarrow \mathbb{C}[f_1,\ldots,f_m]$ defined by $y_i \mapsto f_i$. In this way we can associate to the ring of invariants an affine variety

$$\mathbb{V}(I) \subset \mathbb{C}^m = \mathbb{A}^m$$

and we have a mapping

$$\mathbb{C}^n \xrightarrow{\pi} \mathbb{V}(I_V) \qquad \text{by} \qquad p \mapsto (f_1(p),\ldots,f_m(p))$$

and as the f_i are constant along G-orbits, this map factors over the orbit-space $\mathbb{C}^n/G$.

Lemma 1.12 *The factored map*

$$\mathbb{C}^n/G \longrightarrow \mathbb{V}(I) \subset \mathbb{C}^m$$

is injective.

Proof. Assume $p = (a_1,\ldots,a_n)$ and $q = (b_1,\ldots,b_n)$ are two points in $\mathbb{C}^n$ such that $\mathcal{O}(p) \neq \mathcal{O}(q)$ (that is, $\mathcal{O}(p) \cap \mathcal{O}(q) = \emptyset$) then we have to show that $f_i(p) \neq f_i(q)$ for some $1 \leq i \leq m$.

Consider $S = \mathcal{O}(q) \cup (\mathcal{O}(p) - \{p\})$, then S is a finite subset of $\mathbb{C}^n$ and hence is an affine variety. Because $p \notin S$ there is a polynomial $f \in I(S) \triangleleft \mathbb{C}[x_1,\ldots,x_n]$ such that $f(s) = 0$ for all $s \in S$ but $f(p) \neq 0$. Consider the Reynolds image of f : $R_G(f) = (1/N)\sum_{g\in G}\phi_g(f)$ then it follows that

$$R_G(f)(q) = \frac{1}{N}\sum_{g\in G}\phi_g(f)(q) = \frac{1}{N}\sum_{g\in G} f(g.q) = 0$$

whereas

$$R_G(f)(p) = \frac{1}{N}\sum_{g\in G} f(g.p) = \frac{M}{N} f(p) \neq 0$$

where M is the number of $g \in G$ such that $g.p = p$. Because $R_G(f) \in \mathbb{C}[x_1,\ldots,x_n]^G$, there is a polynomial $P(f_1,\ldots,f_m) = R_G(f)$. But then as

$$P(f_1(q),\ldots,f_m(q)) = R_G(f)(q) \neq R_G(f)(p) = P(f_1(p),\ldots,f_m(p))$$

it follows that for at least one i we have $f_i(p) \neq f_i(q)$ and hence the image of p and q under the quotient mapπ are different. □

Before we can prove surjectivity of the factored map, we need to give another description of the ring of invariants. By giving each variable x_i degree one, the polynomial ring becomes a *graded algebra*

$$\mathbb{C}[x_1,\ldots,x_n] = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$$

where R_d is the finite dimensional vectorspace spanned by all monomials m such that $\mathtt{deg} m = d$. As the G-action on polynomials preserves the degree, each of the R_d is a finite dimensional G-representation and hence decomposes into its irreducible factors

$$R_d = V_1^{\oplus e_1(d)} \oplus V_2^{\oplus e_2(d)} \oplus \ldots \oplus V_r^{\oplus e_r(d)}$$

where V_1 is the trivial G-representation. With $R_d(i)$ we will denote the subspace $V_i^{\oplus e_i(d)}$ of R_d.

Also the subring of invariant polynomials is graded (the induced gradation)

$$\mathbb{C}[x_1, \ldots, x_n]^G = S_0 \oplus S_1 \oplus S_2 \oplus \ldots$$

and, by definition $g.f = f$ for all $f \in S_d$. That is, we can identify

$$S_d = R_d(1) = V_1^{\oplus e_1(d)}$$

the space spanned by all factors of R_d isomorphic to the trivial representation. Moreover note that for any $1 \leq i \leq r$ we have that

$$S_d R_{d'}(i) \subset R_{d+d'}(i)$$

If $M \subset \mathbb{C}[x_1, \ldots, x_n]$ is G-stable, that is, if $g.f \in M$ for all $f \in M$ and all $g \in G$, then M can be decomposed into irreducible G-representations and we denote by $M(i)$ the collection of all subspaces of type V_i. Using these facts we can now finish the proof of

Proposition 1.13 *The ring of polynomial invariants $\mathbb{C}[x_1, \ldots, x_n]^G$ is the coordinate ring of the orbit space $\mathbb{C}^n/G$, that is, the factored map*

$$\mathbb{C}^n/G \longrightarrow \mathbb{V}(I) \qquad \textit{induced by} \qquad \mathbb{C}^n \xrightarrow{\pi} \mathbb{V}(I) \subset \mathbb{C}^m$$

is a bijection.

Proof. It remains to prove that the map π is surjective. Let $p \in \mathbb{V}(I)$ and let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]^G$ corresponding to p. We claim that the *extended ideal*

$$\mathfrak{m}\mathbb{C}[x_1, \ldots, x_n] \neq \mathbb{C}[x_1, \ldots, x_n]$$

In fact, we claim the stronger property that

$$\mathfrak{m}\mathbb{C}[x_1, \ldots, x_n] \cap \mathbb{C}[x_1, \ldots, x_n]^G = \mathfrak{m}$$

Indeed, $M = \mathfrak{m}\mathbb{C}[x_1, \ldots, x_n]$ is G-stable and therefore

$$M = \oplus_{i=1}^r M(r) = \oplus_{i=1}^r (\mathfrak{m}\mathbb{C}[x_1, \ldots, x_n])(i) = \oplus_{i=1}^r \mathfrak{m}(\mathbb{C}[x_1, \ldots, x_n](i))$$

the last equality holding because $\mathfrak{m} \subset \mathbb{C}[x_1, \ldots, x_n](1)$. Restricting to the component of the trivial representation we get

$$\mathfrak{m}\mathbb{C}[x_1, \ldots, x_n] \cap \mathbb{C}[x_1, \ldots, x_n]^G = (\mathfrak{m}\mathbb{C}[x_1, \ldots, x_n])(1) = \mathfrak{m}(\mathbb{C}[x_1, \ldots, x_n](1)) = \mathfrak{m}$$

because $\mathbb{C}[x_1, \ldots, x_n](1) = \mathbb{C}[x_1, \ldots, x_n]^G$, proving the claim.

Let $\mathfrak{p}$ be any maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$ containing the proper ideal $\mathfrak{m}\mathbb{C}[x_1, \ldots, x_n]$ and $q \in \mathbb{C}^n$ be the corresponding point, then $\pi(q) = p$. $\square$

1.2 McKay quivers

Let V be an n-dimensional representation of G, we will associate to V a *quiver* (that is, a finite directed graph) $Q_V = \mathtt{mck}_G(V)$, the *McKay quiver* of V.

Definition 1.14 The *vertices* $\{v_1, \ldots, v_r\}$ of $Q_V = \mathtt{mck}_G(V)$ are in one-to-one correspondence with the distinct irreducible representations V_i (for $1 \le i \le r$) of G where we let v_1 correspond to the *trivial representation*.

Let V_j be the irreducible G-representation corresponding to vertex v_j, then we have that

$$V \otimes V_j = V_1^{\oplus a_{1j}} \oplus \ldots \oplus V_r^{\oplus a_{rj}}$$

for certain $a_{ij} \in \mathbb{N}$. In the McKay quiver we use these integers to draw a_{ij} directed *arrows* from vertex v_i to vertex v_j. Repeating this procedure for all vertices, we obtain the McKay quiver $Q_V = \mathtt{mck}_G(V)$.

Example 1.15 (The cyclic group $C_3 = \mathbb{Z}/3\mathbb{Z}$) Write the Abelian group $\mathbb{Z}/3\mathbb{Z}$ multiplicatively, that is, $C_3 = \{1, \rho, \rho^2\}$ where ρ is a primitive third root of unity. Then, C_3 has three distinct conjugacy classes $\{1\}$, $\{\rho\}$ and $\{\rho^2\}$ whence C_3 must have three distinct irreducible representation which must be necessarily of dimension one. These representations $V_i : C_3 \longrightarrow \mathbb{C}^*$ are determined by the image of ρ and are

$$V_1 = \{\rho \mapsto 1\} \qquad V_2 = \{\rho \mapsto \rho\} \qquad V_3 = \{\rho \mapsto \rho^2\}$$

and therefore the character table of C_3 is given by the matrix

	$\{1\}$	$\{\rho\}$	$\{\rho^2\}$
V_1	1	1	1
V_2	1	ρ	ρ^2
V_3	1	ρ^2	ρ

Consider the two-dimensional C_3-representation

$$V = V_2 \oplus V_3 \qquad \text{then} \qquad \chi_V = \chi_{V_2} + \chi_{V_3}$$

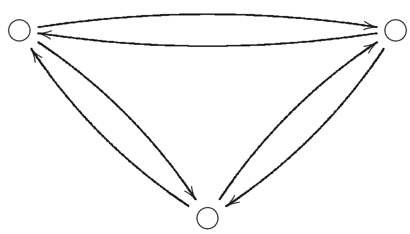
and therefore $\chi_{V \otimes V_i} = \chi_{V_2}\chi_{V_i} + \chi_{V_3}\chi_{V_i}$ giving

	$\{1\}$	$\{\rho\}$	$\{\rho^2\}$
$V \otimes V_1$	2	-1	-1
$V \otimes V_2$	2	$-\rho$	$3-\rho^2$
$V \otimes V_3$	2	$-\rho^2$	$-\rho$

Therefore, we have the following decompositions

$$\begin{cases} V \otimes V_1 \simeq V_2 \oplus V_3 \\ V \otimes V_2 \simeq V_3 \oplus V_1 \\ V \otimes V_3 \simeq V_1 \oplus V_2 \end{cases}$$

whence the McKay quiver $Q_V = \mathtt{mck}_{C_3}(V)$ has the form



On the other hand, for the three-dimensional C_3-representation

$$V = V_2 \oplus V_2 \oplus V_2$$

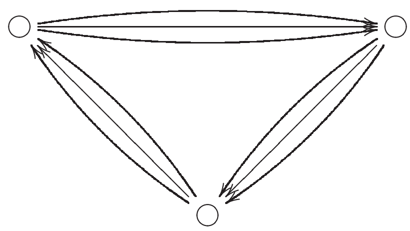
then $\chi_V = 3\chi_2$ and hence $\chi_{V \otimes V_i} = 3\chi_2\chi_i$ giving

	$\{1\}$	$\{\rho\}$	$\{\rho^2\}$
$V \otimes V_1$	3	3ρ	$3\rho^2$
$V \otimes V_2$	3	$3\rho^2$	3ρ
$V \otimes V_3$	3	3	3

Hence we have the decompositions

$$\begin{cases} V \otimes V_1 = & V_2 \oplus V_2 \oplus V_2 \\ V \otimes V_2 = & V_3 \oplus V_3 \oplus V_3 \\ V \otimes V_3 = & V_1 \oplus V_1 \oplus V_1 \end{cases}$$

which gives us that the McKay quiver $Q_V = \mathtt{mck}_{C_3}(V)$ has the following form



Historically, the McKay quiver was assigned to a *Kleinian singularity* linking them to *tame quivers* and their *isotropic roots*.

Definition 1.16 Let $G \subset SL_2(\mathbb{C})$ a finite subgroup of the 3-dimensional complex Lie group

$$SL_2(\mathbb{C}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc = 1\}$$

then G has a natural 2-dimensional representation via the embedding $SL_2(\mathbb{C}) \subset GL_2(\mathbb{C})$ and we can consider the quotient varieties $\mathbb{C}^2/G$ which in this case are called *Kleinian singularities*.

One has a complete classification of all finite subgroups of $SL_2(\mathbb{C})$. There are two infinite families and three exceptional cases.

The cyclic groups C_n : Let ρ be a primitive n-th root of unity and consider the subgroup of $SL_2(\mathbb{C})$ generated by the matrix

$$\begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix}$$

which is clearly cyclic of order n. It acts on $\mathbb{C}[x, y]$ via the automorphism

$$\begin{cases} x \mapsto \rho x \\ y \mapsto \rho^{-1} y \end{cases}$$

and one verifies immediately that the ring of polynomial invariants is

$$\mathbb{C}[x, y]^{C_n} = \mathbb{C}[x^n, y^n, xy] = \frac{\mathbb{C}[X, Y, Z]}{(X^n - YZ)}$$

which has an isolated singularity at the origin. Btw. the frontispiece gives the real picture of the Kleinian singularity $\mathbb{C}^3/C_3$.

The dihedral groups D_n : If ρ is a primitive $2n$-th root of unity, consider the subgroup of $SL_2(\mathbb{C})$ generated by the two matrices

$$a = \begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

which is a group of order $4n$ and it acts on $\mathbb{C}[x,y]$ via the two automorphisms

$$\begin{cases} x \mapsto \rho x \\ y \mapsto \rho^{-1} y \end{cases} \qquad \text{and} \qquad \begin{cases} x \mapsto iy \\ y \mapsto ix \end{cases}$$

With some difficulty one can prove that the ring of invariants

$$\mathbb{C}[x,y]^{D_n} = \mathbb{C}[x^2y^2, x^{2n} + (-1)^n y^{2n}, xy(x^{2n} - (-1)^n y^{2n})] = \frac{\mathbb{C}[X,Y,Z]}{(Z^2 + X(Y^2 + X^n))}$$

which again has an isolated singularity at the origin.

The exceptional groups : Recall that $SU_2(\mathbb{C})$ (the group of special unitary 2×2 matrices) is the universal 2-fold cover of the rotation group $SO_3(\mathbb{R})$. Thus, we can lift any finite subgroup of the rotation group to a finite subgroup of $SU_2(\mathbb{C}) \subset SL_2(\mathbb{C})$. In particular one can do this for the group of rotations leaving the *tetrahedron, octahedron* and *icosahedron* fixed, giving us the finite subgroups

$$T, \quad O, \quad I \quad \subset \quad SL_2(\mathbb{Z}) \quad \text{of order, resp. } 24, 48 \text{ and } 120$$

We will not give precise matrix representations of these groups but only state the result on the rings of invariants in these three cases :

$$\mathbb{C}[x,y]^T = \frac{\mathbb{C}[X,Y,Z]}{(X^4 + Y^3 + Z^2)}$$

$$\mathbb{C}[x,y]^O = \frac{\mathbb{C}[X,Y,Z]}{(X^3 + XY^3 + Z^2)}$$

$$\mathbb{C}[x,y]^I = \frac{\mathbb{C}[X,Y,Z]}{(X^5 + Y^3 + Z^2)}$$

all of which have an isolated singularity at the origin.

The *tame quivers* are the directed graphs obtained from a so called *extended Dynkin diagrams* given in figure 1.2 by putting some orientation on each of the edges. Tame quivers come equipped with an *isotropic root* α which is a certain dimension vector depicted on the right hand side of figure 1.2.

Definition 1.17 A *dimension vector* for a quiver on r vertices is an integral vector $\alpha \in \mathbb{N}^r$.For a McKay quiver $Q_V = \mathtt{mck}_G(V)$ there is a *distinguished dimension vector*

$$\alpha_G = (d_1, \ldots, d_r) \qquad \text{with} \qquad d_i = \mathtt{dim} V_i$$

Definition 1.18 If Q is an arbitrary quiver, for example an extended Dynkin graph with some orientation on the edges, then one can define its *double* $\mathbb{D}(Q)$ to be the extended quiver obtained by adjoining for each arrow a in Q an arrow a^* with the reverse orientation

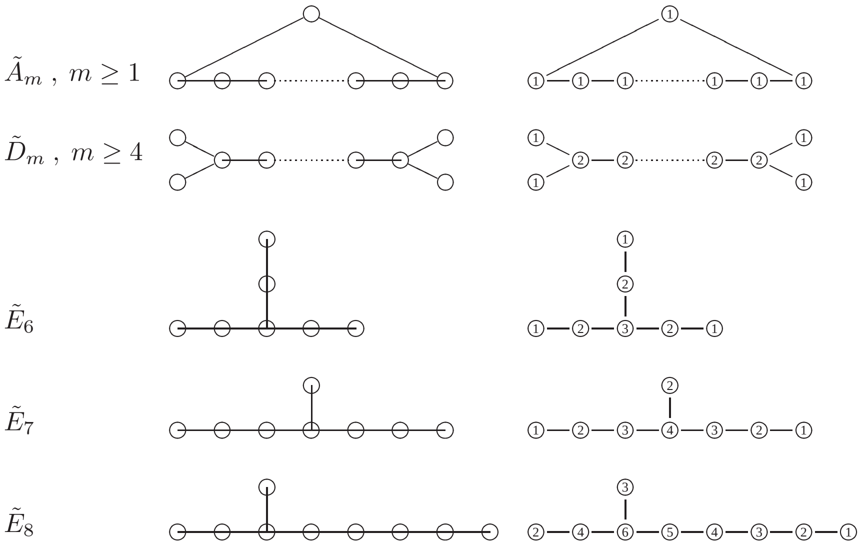


Fig. 1.1: The extended Dynkin diagrams.

With this notation *McKay's observation* can be stated as :

Proposition 1.19 *The McKay quiver settings corresponding to a two-dimensional Kleinian singularity are precisely the quiver settings*

$$(Q_V, \alpha_G) = (\mathbb{D}(Q), \alpha)$$

where Q is an extended Dynkin quiver and α its corresponding isotropic root. In this correspondence the cyclic subgroups C_n correspond to $\tilde{A}_{n-1}$, the dihedral subgroups D_n to $\tilde{D}_n$ and the exceptional subgroups T, O and I respectively to $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

To illustrate this so called *McKay correspondence* we consider the special case of the cyclic subgroup $C_6 = \mathbb{Z}_6$ in detail. If ρ is a primitive 6-th root of unity, then a complete list of irreducible $\mathbb{Z}_6$-representation is given by

$$\{V_0, V_1, V_2, V_3, V_4, V_5\} \qquad \text{with} \qquad V_i = \mathbb{C}v_i \quad g.v_i = \rho^i v_i$$

for the generator $g = \overline{1}$ of $\mathbb{Z}_6$. The action of $\mathbb{Z}_6$ on $V = \mathbb{C}^2$ is given by the matrix

$$\begin{bmatrix} \rho & 0 \\ 0 & \rho^5 \end{bmatrix} \qquad \text{whence} \qquad V \simeq V_1 \oplus V_5$$

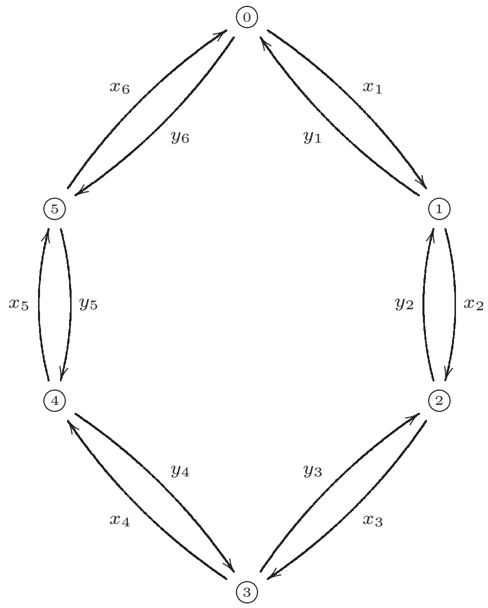
The tensor-products of simple representations are easily worked out in this case

$$V_i \otimes V_j \simeq V_{i+j \ mod \ 6}$$

from which it follows that

$$V_i \otimes V = V_{i+1} \oplus V_{i-1}$$

where all indices are taken modulo 6. Therefore, the corresponding McKay quiver has the form if we denote the vertex (i) to correspond to the simple representation V_i



and as the corresponding dimension vector $\alpha = (1,1,1,1,1,1)$ is the isotropic root of $\tilde{A}_5$ and as the subquiver of the x_i arrows is $\tilde{A}_5$ we get McKay's observation in this case by taking $x_i^* = y_i$.

1.3 Hilbert schemes

If $\alpha = (n_1, \ldots, n_r)$ is a dimension vector, the *representation space*

$$\texttt{rep}_\alpha \, Q_V = \oplus_{1 \leq i,j \leq r} M_{n_j \times n_i}(\mathbb{C})^{\oplus a_{ij}}$$

is the vectorspace such that every arrow from vertex v_i to vertex v_j determines a linear map (a matrix) from $\mathbb{C}^{n_i}$ to $\mathbb{C}^{n_j}$.

Lemma 1.20 *For the distinguished dimension vector $\alpha_G = (d_1, \ldots, d_r)$ of the McKay quiver $Q_V = \texttt{mck}_G(V)$, there is a natural identification*

$$\texttt{rep}_{\alpha_G} \, Q_V = Hom_G(R, V \otimes R)$$

where R is the regular representation of G.

Proof. The number a_{ij} of directed arrows from v_i to v_j is by definition the number of V_i-components in the G-tensor product $V \otimes V_j$, that is, by Schur's lemma

$$a_{ij} = Hom_G(V_i, V \otimes V_j)$$

But then, by definition of the representation space we have that

$$\begin{aligned}
\texttt{rep}_{\alpha_G} \, Q_V &= \oplus_{1 \leq i,j \leq r} Hom_G(V_i, V \otimes_{\mathbb{C}} V_j) \otimes Hom_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) \\
&= Hom_G(\oplus_{1 \leq i \leq r} V_i \otimes_{\mathbb{C}} \mathbb{C}^{d_i}, \oplus_{1 \leq j \leq r} V \otimes (V_j \otimes_{\mathbb{C}} \mathbb{C}^{d_j})) \\
&= Hom_G(\oplus_{1 \leq i \leq r} V_i^{\oplus d_i}, V \otimes (\oplus_{1 \leq j \leq r} V_j^{\oplus d_j})) \\
&= Hom_G(R, V \otimes R)
\end{aligned}$$

finishing the proof. □

There is a natural action of the *base change group*

$$GL(\alpha_G) = GL_{a_1} \times \ldots \times GL_{a_k}$$

on $\mathtt{rep}_{\alpha_G} Q_V$ by base-change. That is, if $g = (g_1, \ldots, g_k) \in GL(\alpha)$ and M_a is the $a_j \times a_i$ matrix corresponding to the arrow $(i) \longrightarrow (j)$, then

$$g.M_a = g_j.M_a.g_i^{-1}$$

The $GL(\alpha)$-orbits under this action are precisely the isomorphism classes of quiver-representations. We have seen before that if G is a finite group acting on a vectorspace, then we can parametrize all orbits by the maximal ideals of the ring of polynomial invariants. For more general reductive groups such as $GL(\alpha_G)$ we will see later that it is not always possible to construct an orbit space due to the existence of non-closed orbits (in the case of finite groups, each orbit is a finite number of points whence closed). Still, as before, the variety determined by the ring of polynomial invariants

$$\mathbb{C}[\mathtt{rep}_{\alpha_G} Q_V]^{GL(\alpha_G)}$$

is the best algebraic approximation to this non-existent orbit space. Later we will see methods to determine such invariant rings but in the special case of G being Abelian there is a direct route because in this case $GL(\alpha_G) \simeq \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ is a torus and one can find polynomial invariants by determining integer solutions to a linear system of equations.

We will illustrate this in the special case of $G \simeq \mathbb{Z}_6$ studied above. So let (Q, α) be the $\mathbb{Z}_6$-quiver setting, then

$$\mathbb{C}[\mathtt{rep}_\alpha Q] = \mathbb{C}[x_1, \ldots, x_6, y_1, \ldots, y_6] \qquad \text{and} \qquad GL(\alpha) = \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{6}$$

and $\lambda = (\lambda_1, \ldots, \lambda_6) \in GL(\alpha)$ acts via

$$\begin{cases} \lambda.x_i &= \lambda_i \lambda_{i-1}^{-1} x_i \\ \lambda.y_i &= \lambda_{i-1} \lambda_i^{-1} y_i \end{cases}$$

and consequently the action of λ on any monomial multiplies this monomial with some scalar. As a consequence the ring of polynomial invariants is generated by the monomials where this scalar factor is 1. We can represent any monomial in the x_i and the y_j by an integral vector

$$x_1^{a_1} \ldots x_6^{a_6} y_1^{b_1} \ldots y_6^{b_l} \quad \mapsto \quad (a_1, \ldots, a_6, b_1, \ldots, b_6) \in \mathbb{N}^{12}$$

and the monomials left invariant by the torus-action are precisely the solutions to the linear set of relations determined by the 12×6 matrix, where the columns correspond to the variables and the rows to the action of the different components of $GL(\alpha)$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and one verifies that the *fundamental solutions* are given by the 6 short cycles x_iy_i, that is

$$c_1 = (1,0,0,0,0,0,1,0,0,0,0,0), \ldots, c_6 = (0,0,0,0,0,1,0,0,0,0,0,1)$$

and the two long cycles

$$c_x = (1,1,1,1,1,1,0,0,0,0,0,0) \quad \text{and} \quad c_y = (0,0,0,0,0,0,1,1,1,1,1,1)$$

From this it follows that the ring of invariants

$$\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)} \simeq \frac{\mathbb{C}[c_1,c_2,c_3,c_4,c_5,c_6,c_x,c_y]}{(c_1c_2c_3c_4c_5c_6 - c_xc_y)}$$

which is a singular hypersurface in $\mathbb{C}^8$ so is 7-dimensional. Clearly this does not yet give us the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_6$ but we still have to divide out the *commuting matrices relations*.

Observe that $V = \mathbb{C}x + \mathbb{C}y = V_0 \oplus V_5$ and we have the identification

$$Hom_{\mathbb{Z}_6}(\mathbb{C}\mathbb{Z}_6, V \otimes \mathbb{C}\mathbb{Z}_6) = \mathtt{rep}_\alpha\ Q$$

Any $B \in Hom_{\mathbb{Z}_6}(\mathbb{C}\mathbb{Z}_6, V \otimes \mathbb{C}\mathbb{Z}_6)$ is determined by two 6×6 matrices B_x and B_y defined by the rule that

$$B(c) = x \otimes B_x(c) + y \otimes B_y(c) \qquad \text{for all } c \in \mathbb{C}\mathbb{Z}_6$$

Identifying $\mathbb{C}\mathbb{Z}_6$ with the space spanned by the vertex-idempotents $\mathbb{C}v_0 + \ldots + \mathbb{C}v_5$ any $B = (x_1, \ldots, x_6, y_1, \ldots, y_6) \in \mathtt{rep}_\alpha\ Q$ determines the matrices

$$B_x = \begin{bmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_5 \\ x_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & y_6 \\ y_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_5 & 0 \end{bmatrix}$$

and the *commuting matrix relations* are given by setting the entries of the commutator $[B_x, B_y]$ equal to zero. Now, $[B_x, B_y] =$

$$\begin{bmatrix} x_1y_1 - y_6x_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2y_2 - y_1x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3y_3 - y_4x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4y_4 - y_5x_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5y_5 - y_6x_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6y_6 - y_1x_1 \end{bmatrix}$$

Consider the affine subvariety of $\mathtt{rep}_\alpha\ Q$ consisting of the representations satisfying these conditions

$$X = \mathbb{V}(x_1y_1 = x_2y_2 = \ldots = x_6y_6) \subset \mathtt{rep}_\alpha\ Q$$

then the corresponding quotient variety $X//GL(\alpha)$ has as coordinate ring the quotient of the invariant ring of $\mathtt{rep}_\alpha\ Q$ on which we imposed these relations, that is,

$$\mathbb{C}[X]^{GL(\alpha)} = \frac{\mathbb{C}[c_1,c_2,c_3,c_4,c_5,c_6,c_x,c_y]}{(c_1c_2c_3c_4c_5c_6 - c_xc_y, c_1 - c_2, c_2 - c_3, \ldots, c_6 - c_1)} \simeq \frac{\mathbb{C}[x,y,z]}{(x^6 - yz)}$$

which is the coordinate ring of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_6$. This is no accident. There is a general result recovering quotient singularities from the McKay quiver setting by restricting to the variety of commuting matrices.

Proposition 1.21 *Let $G \subset SL_d(\mathbb{C})$ act freely outside the origin of $\mathbb{C}^d$. Let (Q, α) be the McKay quiver setting corresponding to the quotient singularity $\mathbb{C}^d/G$. Let $X \subset \mathtt{rep}_\alpha\, Q$ denote the affine subvariety consisting of representations for which the corresponding G-equivariant map $B \in Hom_G(R, V \otimes R)$ satisfies the equation*

$$B \wedge B = 0 \in Hom_G(R, \wedge^2 V \otimes R)$$

then the corresponding quotient variety

$$X//GL(\alpha) \simeq \mathbb{C}^d/G$$

is isomorphic to the quotient singularity.

The variety $S^n\, \mathbb{C}^2$ parametrizing unordered n-tuples in the complex plane $\mathbb{C}^2$ is singular and there is a natural *desingularization*

$$\mathtt{Hilb}_n\, \mathbb{C}^2 \twoheadrightarrow S^n\, \mathbb{C}^2$$

where $\mathtt{Hilb}_n\, \mathbb{C}^2$ is the *Hilbert scheme* of n points in the plane. That is, the Hilbert scheme parametrizes ideals I of $\mathbb{C}[x, y]$ such that

$$dim_{\mathbb{C}}\, \frac{\mathbb{C}[x, y]}{I} = n$$

We want to describe the points in $\mathtt{Hilb}_6\, \mathbb{C}^2$ which are isomorphic as $\mathbb{Z}_6$-representation to $\mathbb{C}\mathbb{Z}_6$, that is we want to classify the codimension 6 ideals I of $\mathbb{C}[x, y]$ which are stable under the $\mathbb{Z}_6$-action on $\mathbb{C}[x, y]$ and such that the corresponding quotient representation

$$\frac{\mathbb{C}[x, y]}{I} \simeq \mathbb{C}\mathbb{Z}_6$$

such a point in $\mathtt{Hilb}_6\, \mathbb{C}^2$ is called a $\mathbb{Z}_6$-*constellation* and the classifying variety will be denoted by $\mathbb{Z}_6 - \mathtt{Hilb}\, \mathbb{C}^2$.

To start off, there is a natural onto mapping

$$\mathbb{Z}_6 - \mathtt{Hilb}\, \mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_6$$

defined as follows : take a codimension 6-ideal I of $\mathbb{C}[x, y]$ and let $\mathfrak{m}_P$ be the maximal ideal of a point $P \in \mathbb{C}^2$ such that $I \subset \mathfrak{m}_P$, then map the point of $\mathbb{Z}_6 - \mathtt{Hilb}\, \mathbb{C}^2$ determined by I to the point $[P] \in \mathbb{C}^2/\mathbb{Z}_6$. As I is stable under the $\mathbb{Z}_6$-action, the choice of the particular P is irrelevant. This map is surjective for take a point $[0] \neq P \in \mathbb{C}^2/\mathbb{Z}_6$, then we can consider the ideal

$$I = \cap_{g \in \mathbb{Z}_6} \mathfrak{m}_{g.P}$$

and for $[0]$ take for example the ideal $I = (x^6, y)$. As before, identify $\mathbb{C}\mathbb{Z}_6$ with $\mathbb{C}v_0 + \ldots + \mathbb{C}v_5$ the space spanned by the vertex-idempotents then under a $\mathbb{Z}_6$-isomorphism

$$\frac{\mathbb{C}[x, y]}{I} \simeq \mathbb{C}\mathbb{Z}_6 = \mathbb{C}v_0 + \ldots + \mathbb{C}v_5$$

the image of 1 corresponds to v_0. Moreover, multiplication by x, resp. by y, in the quotient $\mathbb{C}[x, y]/I$ determines two commuting 6×6 matrices B_x and B_y in $End_{\mathbb{C}}(\mathbb{C}\mathbb{Z}_6)$ and by assumption the induced linear map

$$\mathbb{C}\mathbb{Z}_6 \longrightarrow (\mathbb{C}x + \mathbb{C}y) \otimes \mathbb{C}\mathbb{Z}_6 \quad \text{defined by} \quad v \mapsto x \otimes B_x.v + y \otimes B_y.v$$

is $\mathbb{Z}_6$-equivariant, that is determines an element of

$$Hom_{\mathbb{Z}_6}(\mathbb{C}\mathbb{Z}_6, V \otimes \mathbb{C}\mathbb{Z}_6) = \mathtt{rep}_\alpha\, Q$$

and hence describes a point in the commuting matrix subvariety $X \subset \mathtt{rep}_\alpha\, Q$. However, conversely, it is not true that an arbitrary point of X determines a point of $\mathbb{Z}_6 - \mathtt{Hilb}\,\mathbb{C}^2$ as for such an ideal we have the extra condition that the image of 1 under the identification

$$\frac{\mathbb{C}[x,y]}{I} = \mathbb{C}\mathbb{Z}_6 = \mathbb{C}v_0 + \ldots + \mathbb{C}v_5$$

that is v_0, must generate the whole 6-dimensional representation when acted upon by the matrices $B_x^k B_y^l$. So, for example, a point in X with $x_1 = y_6 = 0$ does not satisfy this extra requirement.

Still, if $x = (x_1, \ldots, x_6, y_1, \ldots, x_6) \in X$ such that for every vertex $v_i \neq v_0$ we have a path P in the quiver Q such that $P(x) \neq 0$, then the generating condition *is* satisfied and x determines a point of the Hilbert scheme $\mathbb{Z}_6 - \mathtt{Hilb}\,\mathbb{C}^2$. This gives us a way to calculate the Hilbert scheme. Let X^s be the set of points satisfying this path-condition, then clearly X^s is a Zariski open subset of X. However, it is not true that X^s is *affine*. Still, we can cover X^s by affine open pieces U_i and take the corresponding $GL(\alpha)$-quotients $U_i//GL(\alpha)$. Gluing these affine quotients together gives a covering of $\mathbb{Z}_6 - \mathtt{Hilb}\,\mathbb{C}^2$. We claim that $\mathbb{Z}_6 - \mathtt{Hilb}\,\mathbb{C}^2$ is a smooth variety.

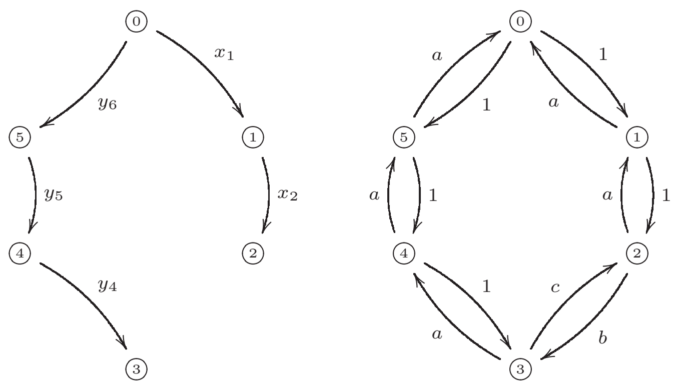
Let us consider one of these affine open pieces (the calculations for the others are similar). Take

$$U = \{x \in X^s \mid x_2 x_1(x) \neq 0 \neq y_4 y_5 y_6(x)\}$$

then this is an affine open piece with coordinate ring

$$\mathbb{C}[U] = \mathbb{C}[X]_f \qquad \text{with} \qquad f = x_2 x_1 y_4 y_5 y_6$$

Now, we have to divide out the $GL(\alpha)$-action. In every orbit of U we can find a representant of the form



Here, we used the $GL(\alpha)$-action to normalize the certain non-zero arrows to 1 and then the commuting matrix relations imply that most of the remaining arrows must be equal to $a \in \mathbb{C}$. As the ring of invariants is generated by (traces of) cycles in Q we have

$$\mathbb{C}[U]^{GL(\alpha)} = \frac{\mathbb{C}[a,b,c]}{(bc-a)} \simeq \mathbb{C}[b,c]$$

whence $U//GL(\alpha) \simeq \mathbb{A}^2$ and is smooth. As a consequence, we have that the natural map

$$\mathbb{Z}_6 - \mathtt{Hilb}\,\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}^2/\mathbb{Z}_6$$

is a *resolution of singularities*. Again, this is no accident and there is a general result.

Definition 1.22 If G is a finite group acting on $\mathbb{C}^d$ freely outside the origin we define the *G-equivariant Hilbert scheme* $G - \texttt{Hilb}\,\mathbb{C}^d$ to be the variety parametrizing codimension $\#G$-ideals I of $\mathbb{C}[x_1, \ldots, x_d]$ stable under the action of G and such that the corresponding quotient

$$\frac{\mathbb{C}[x_1, \ldots, x_d]}{I} \simeq \mathbb{C}G$$

as G-representations.

Proposition 1.23 *If G is a finite group acting on $\mathbb{C}^d$ freely outside the origin, then there is a natural map*

$$G - \texttt{Hilb}\,\mathbb{C}^d \twoheadrightarrow \mathbb{C}^d/G$$

which is a (partial) resolution of the quotient singularity. If $d = 2$ (and for many cases, including all Abelian G, in $d = 3$) this is a resolution of singularities.

1.4 Non-commutative algebras

lecture 2

THE CONIFOLD ALGEBRA

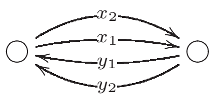
2.1 The algebra

In this section we give two different characterizations of the *conifold algebra*, a non-commutative algebra of current interest in stringtheory : as a skew-group algebra and as a Clifford algebra.

Quiver-diagrams play an important role in stringtheory as they encode intersection information of so called *wrapped D-branes* (higher dimensional strings) in *Calabi-Yau manifolds* (some special three dimensional manifolds over the complex numbers $\mathbb{C}$). One of the earliest models, studied by I. R. Klebanov and E. Witten [**?**], was based on the *conifold singularity*. Recall that the conifold singularity is the singularity of the *affine cone* (see [**?**,]) over the image of the *Segre embedding* (see [**?**,]) $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. That is, an affine presentation of the conifold singularity is given by the affine commutative $\mathbb{C}$-algebra

$$C_{con} = \frac{\mathbb{C}[a,b,c,d]}{(ab-cd)}$$

A $D3$-*brane* is a three-dimensional (over the real numbers $\mathbb{R}$) submanifold of a Calabi-Yau manifold and as this is a six-dimensional (again over the real numbers) manifold it follows that two $D3$-branes in sufficiently general position intersect each other in a finite number of points. If one wraps two sufficiently general $D3$-branes around a conifold singularity, their intersection data will be encoded in the quiver-diagram



Without going into details (for more information see [**?**]) one can associate to such a quiver-diagram a non-commutative algebra describing the vacua with respect to a certain *super-potential* which is a suitable linear combination of oriented cycles in the quiver-diagram. In the case of two $D3$-branes wrapped around a conifold singularity one obtains :

Definition 2.1 The *conifold algebra* A_{con} is the non-commutative affine $\mathbb{C}$-algebra generated by three non-commuting variables X, Y and Z and satisfying the following relations

$$\begin{cases} XZ &= -ZX \\ YZ &= -ZY \\ X^2Y &= YX^2 \\ Y^2X &= XY^2 \\ Z^2 &= 1 \end{cases}$$

That is, A_{con} has a presentation

$$A_{con} = \frac{\mathbb{C}\langle X,Y,Z\rangle}{(Z^2-1, XZ+ZX, YZ+ZY, [X^2,Y],[Y^2,X])}$$

where $\mathbb{C}\langle X,Y,Z\rangle$ is the free associative algebra on three non-commuting variables and where $[A,B]=AB-BA$ denotes the commutator.

Actually, one sometimes sees another presentation of A_{con} as

$$\frac{\mathbb{C}\langle X,Y,Z\rangle}{(Z^2-1, XZ+ZX, YZ+ZY, [Z[X,Y],X],[Z[X,Y],Y])}$$

but as Z is a unit, it is easily seen that both presentations give isomorphic $\mathbb{C}$-algebras.

In general, the structure of a non-commutative affine $\mathbb{C}$-algebra can be quite complicated but here we are in luck as there is a large *central subalgebra* in the conifold algebra. Recall that an element r of a non-commutative algebra R is said to be *central* if it commutes with all elements, that is, $[r,s]=rs-sr=0$ for all $s \in R$.

Lemma 2.2 *In the conifold algebra A_{con} the following elements*

$$x = X^2, \qquad y = Y^2 \qquad \textit{and} \qquad z = \frac{1}{2}(XY+YX)$$

are algebraically independent central elements and A_{con} is a free module over the central subalgebra $C=\mathbb{C}[x,y,z]$ with basis

$$A_{con} = C.1 \oplus C.X \oplus C.Y \oplus C.Z \oplus C.XY \oplus C.XZ \oplus C.YZ \oplus C.XYZ$$

In fact, the conifold algebra is a skew group algebra

$$A_{con} \simeq \mathbb{C}[z,X][Y,\sigma,\delta]\#\mathbb{Z}/2\mathbb{Z}$$

for some automorphism σ and σ-derivation δ.

Proof. Consider the subalgebra S of A_{con} generated by X and Y, that is

$$S = \frac{\mathbb{C}\langle X,Y\rangle}{([X^2,Y],[Y^2,X])}$$

Then clearly x and y are central elements of S as is $z=\frac{1}{2}(XY+YX)$ because

$$(XY+YX)X = XYX+YX^2 = YXY+X^2Y = X(YX+XY)$$

Now, consider the *Öre extension* (see [**?**,])

$$S' = \mathbb{C}[z,X][Y,\sigma,\delta] \quad \text{with} \quad \sigma(z)=z, \sigma(X)=-X \quad \text{and} \quad \delta(z)=0, \delta(X)=2z$$

This means that z is a central element of S' and that $YX=\sigma(X)Y+\delta(X)=-XY+2z$ whence the map

$$S \longrightarrow S' \qquad \text{defined by} \qquad X \mapsto X \quad \text{and} \quad Y \mapsto Y$$

is an isomorphism. By standard results, the *center* of S' is equal to

$$Z(S') = \mathbb{C}[x,y,z]$$

whence the three elements are algebraically independent. Consider the automorphism defined by $\phi(X) = -X$ and $\phi(Y) = -Y$ on S, then the conifold algebra can be written as the *skew group ring* (see [**?**,])

$$A_{con} \simeq S\#\mathbb{Z}/2\mathbb{Z}$$

This is the ring on all elements $s\#g$ with $g \in \mathbb{Z}/2\mathbb{Z}$ and $(s\#g)(s'\#g') = s\phi_g(s')\%gg'$. As $Z(S) = \mathbb{C}[x,y,z]$ is fixed under ϕ the elements $x = x\#1$, $y = y\#1$ and $z = z\#1$ are central in A_{con} and as S' is free over $Z(S')$ with basis

$$S' = Z(S').1 \oplus Z(S').X \oplus Z(S').Y \oplus Z(S').XY$$

the result on freeness of A_{con} over $\mathbb{C}[x,y,z]$ follows. □

If C is a commutative $\mathbb{C}$-algebra and if M_q is a *symmetric* $m \times m$ matrix with entries in C, then we have a *bilinear form* on the free C-module $V = C \oplus \ldots \oplus C$ of rank m defined by

$$B_q(v,w) = \begin{bmatrix} v_1 & v_2 & \ldots & v_m \end{bmatrix} . \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{12} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & \ldots & b_{nn} \end{bmatrix} . \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

The associated *Clifford algebra* $Cl_q(V)$ is then the quotient of the *tensor algebra* $T_C(V) = C\langle v_1, \ldots, v_m\rangle$ where $\{v_1, \ldots, v_m\}$ is a basis of the free C-module V and the defining relations are

$$Cl_q(V) = \frac{T_C(V)}{(v \otimes w + w \otimes v - 2B_q(v,w) \ : \ v,w \in V)}$$

As an example, the algebra $S \simeq S'$ constructed in the above proof is the Clifford algebra of the binary quadratic form over $C = \mathbb{C}[x,y,z]$

$$B_q = \begin{bmatrix} x & z \\ z & y \end{bmatrix} \qquad \text{on} \qquad V = C.X \oplus C.Y$$

as $B_q(X,X) = x, B_q(Y,Y) = y$ and $B_q(X,Y) = z$. As the entries of the symmetric variable are independent variables, we call this algebra the *generic binary Clifford algebra*, see [**?**] for more details and the structure of higher generic Clifford algebras.

Lemma 2.3 *The conifold algebra A_{con} is the* Clifford algebra *of a non-degenerate ternary quadratic form over* $\mathbb{C}[x,y,z]$.

Proof. Consider the free $C = \mathbb{C}[x,y,z]$-module of rank three $V = C.X \oplus C.Y \oplus C.Z$ and the symmetric 3×3 matrix

$$B_q = \begin{bmatrix} x & z & 0 \\ z & y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then it follows that $A_{con} \simeq Cl_q(V)$ as $B_q(X,Z) = 0, B_q(Y,Z) = 0$, $B_q(Z,Z) = 0$ and the remaining inproducts are those of $S \simeq S'$ above. □

What is the connection between A_{con} and the conifold singularity? Whereas $C = \mathbb{C}[x,y,z]$ is a central *subalgebra* of A_{con}, the center itself is strictly larger. Take $D = XYZ - YXZ$

and verify that

$$\begin{aligned}(XYZ - YXZ)X &= -X(2z - XY)Z + xYZ \\ &= -2zXZ + 2xYZ \\ &= xYZ - (2zXZ - YX^2Z) \\ &= X(XYZ - YXZ)\end{aligned}$$

and a similar calculation shows that $DY = YD$ and $DZ = ZD$. Moreover, $D \notin \mathbb{C}[x,y,z]$. Indeed, in the description $A_{con} \simeq S\#\mathbb{Z}/2\mathbb{Z}$ we have that

$$\mathbb{C}[x,y,z] \subset S\#1 \qquad \text{whereas} \qquad D = XYZ - YXZ = (XY - YX)\#Z \in S\#Z$$

Moreover, we have that $D^2 \in \mathbb{C}[x,y,z]$ because

$$D^2 = (XYZ - YXZ)^2 = 2z(XY + YX) - 4xy = 4(z^2 - xy) \in \mathbb{C}[x,y,z]$$

Lemma 2.4 *The center Z_{con} of the conifold algebra A_{con} is isomorphic to the conifold singularity*

$$Z_{con} \simeq \frac{\mathbb{C}[a,b,c,d]}{(ab - cd)}$$

Proof. Let Z be the central subalgebra generated by x, y, z and D, then a representation of Z is

$$Z = \frac{\mathbb{C}[x,y,z,D]}{(D^2 - 4(z^2 - xy))} \simeq \frac{\mathbb{C}[a,b,c,d]}{(ab - cd)}$$

where the second isomorphism comes from the following change of coordinates

$$a = D + 2z, \quad b = D - 2z, \quad c = 2x \quad \text{and} \quad d = 2y$$

As a consequence Z is the coordinate ring of the conifold singularity and is in particular integrally closed. As A_{con} is a finite module over Z it follows that if $Z \neq Z_{con}$ then the field of fractions L of Z_{con} would be a proper extension of the field of fractions K of Z. This can be contradicted using classical results on Clifford algebras over fields. To begin, note that as the ternary form

$$B_q = \begin{bmatrix} x & z & 0 \\ z & y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has square-free determinant $xy - z^2 \notin \mathbb{C}(x,y,z)^{*2}$ the Clifford algebra over the rational field $\mathbb{C}(x,y,z)$

$$A_{con} \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}(x,y,z)$$

is a central simple algebra of dimension 4 over its center K' which is a quadratic field extension of $\mathbb{C}(x,y,z)$ determined by adjoining the square root of the determinant. As $[K : \mathbb{C}(x,y,z)] = 2$ it follows that $K = K'$ and hence also that $K = L$ whence $Z = Z_{con}$. $\square$

2.2 The space

In this section we will determine the non-commutative affine variety corresponding to the conifold algebra.

Definition 2.5 The *non-commutative affine variety* $\mathtt{max}\ A$ of a non-commutative algebra A is the set of all maximal two-sided ideals of A equipped with the *Zariski topology*. The Zariski topology is the topology determined by taking as its closed sets the subsets

$$\mathbb{V}(I) = \{\mathfrak{m} \in \mathtt{max}\ A \mid I \subset \mathfrak{m}\}$$

for all two-sided ideals I of A.

Observe that the Zariski topology on $\mathtt{max}\ A$ is indeed a topology as $\mathbb{V}(0) = \mathtt{max}\ A$, $\mathbb{V}(A) = \emptyset$, $\mathbb{V}(\sum_i I_i) = \cap_i \mathbb{V}(I_i)$ and $\mathbb{V}(I.J) = \mathbb{V}(I) \cup \mathbb{V}(J)$. The last equality follows from the fact that maximal two-sided ideals of A are *two-sided prime ideals*, that is ideals P of A satisfying that if $I.J \subset P$ then either $I \subset P$ or $J \subset P$.

We want to relate these non-commutative affine varieties to ordinary (that is, commutative) varieties for example those determined by central subalgebras.

Definition 2.6 A $\mathbb{C}$-algebra morphism $A \xrightarrow{f} B$ is said to be a *central extension* provided

$$B = Z_B(A)f(A) \qquad \text{where} \qquad Z_B(A) = \{b \in B \mid bf(a) = f(a)b\ \forall a \in A\}$$

In particular, any epimorphism $A \twoheadrightarrow B$ is a central extension as is any monomorphism $A \subset B$ where A is a central subalgebra of B as in this case $Z_B(A) = B$.

Lemma 2.7 *If $A \xrightarrow{f} B$ is a central extension and B is a finite A-module, then the map*

$$f^* \ :\ \mathtt{max}\ B \longrightarrow \mathtt{max}\ A \qquad \textit{defined by} \qquad \mathfrak{m} \mapsto f^{-1}(\mathfrak{m})$$

is continuous for the Zariski topologies on $\mathtt{max}\ A$ and $\mathtt{max}\ B$.

Proof. If $I \triangleleft A$ is a two-sided ideal, then $Z_B(A)f(I)$ is a tow-sided ideal of B as for any $b = \sum_i z_i f(a_i) \in B$ (with $z_i \in Z_B(A)$) we have that

$$b.f(I) = \sum_i z_i f(a_i) f(I) \subset \sum_i z_i f(I) \subset Z_B(A)f(I)$$

an similarly for multiplication by b on the right-hand side. As a consequence, $f^{-1}(P)$ is a two-sided prime ideal of A whenever P is a two-sided prime ideal of B for if $I.J \subset f^{-1}(P)$ then

$$Z_B(A)f(I)f(J) = (Z_B(A)f(I))(Z_B(A)f(J)) \subset P$$

whence $Z_B(A)f(I) \subset P$ or $Z_B(A)f(J) \subset P$. If $\mathfrak{m}$ is a maximal two-sided ideal of B, then $f^{-1}(\mathfrak{m})$ is a prime ideal of A which must be maximal as B is a finite A-module under f. Hence, the map is well-defined and verification that the map is continuous, that is that $f^*(\mathbb{V}(I)) = \mathbb{V}(f^{-1}(I))$ is obvious. □

Observe that we do not need the condition that B is a finite A-module if we consider the *prime ideal spectrum* $\mathtt{spec}$ (instead of $\mathtt{max}$) in the statement where $\mathtt{spec}\ A$ is the set of all two-sided prime ideals of A, again equipped with the Zariski topology.

After these generalities, let us go back to the study of the conifold algebra A_{con} and relate the non-commutative affine variety $\mathtt{max}\ A_{con}$ with that of the central subalgebra $\mathtt{max}\ \mathbb{C}[x,y,z] = \mathbb{A}^3$.

Lemma 2.8 *Intersecting maximal ideals $\mathfrak{m}$ of A_{con} with the central subalgebra $\mathbb{C}[x,y,z]$ determines a continuous map*

$$\mathtt{max}\ A_{con} \xrightarrow{\phi} \mathbb{A}^3$$

with the following fiber information :

1. *If* $\mathfrak{n} \notin \mathbb{V}(xy - z^2)$*, then* $\phi^{-1}(\mathfrak{n})$ *consists of two points.*
2. *If* $(x,y,z) \neq \mathfrak{n} \in \mathbb{V}(xy - z^2)$*, then* $\phi^{-1}(\mathfrak{n})$ *consists of one point.*
3. *If* $(x,y,z) = \mathfrak{n}$*, then* $\phi^{-1}(\mathfrak{n})$ *consists of two points.*

Proof. For $P = (a,b,c) \in \mathbb{A}^3$ the corresponding maximal ideal of $\mathbb{C}[x,y,z]$ is $\mathfrak{n}_P = (x-a, y-b, z-c)$ and therefore the quotient of A_{con} by the extended two-sided ideal $A_{con}.\mathfrak{n}_P$ is the Clifford algebra Cl_P over $\mathbb{C}$ of the ternary quadratic form

$$B_P = \begin{bmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the elements of $\phi^{-1}(\mathfrak{n}_P)$ are the two-sided maximal ideals of Cl_P. We can diagonalize the symmetric matrix, that is there is a base-change matrix $M \in GL_3$ such that

$$M^\tau . \begin{bmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & 1 \end{bmatrix} .M = \begin{bmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{bmatrix} = B_Q$$

(with $uv = ab - c^2$) and hence $Cl_P \simeq Cl_Q$. The Clifford algebra Cl_Q is the 8-dimensional $\mathbb{C}$-algebra generated by x_1, x_2 and x_3 satisfying the defining relations

$$x_1^2 = u,\ x_2^2 = v,\ x_3^2 = 1 \qquad \text{and} \qquad x_i x_j + x_j x_i = 0 \text{ for } i \neq j$$

If $uv \neq 0$ then B_Q is a non-degenerate ternary quadratic form with determinant a square in $\mathbb{C}^*$ whence Cl_Q is the direct sum of two copies of $M_2(\mathbb{C})$. If $uv = 0$, say $u = 0$ and $v \neq 0$, then x_1 generates a nilpotent two-sided ideal of Cl_Q and the quotient is the Clifford algebra of the non-degenerate binary quadratic form

$$B_R = \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{whence} \qquad Cl_R \simeq M_2(\mathbb{C})$$

as any such algebra is a quaternion algebra. Finally, if both $u = 0 = v$ then the two-sided ideal I generated by x_1 and x_2 is nilpotent and the quotient

$$Cl_R/I = \mathbb{C}[x_3]/(x_3^2 - 1) \simeq \mathbb{C} \oplus \mathbb{C}$$

As the maximal ideals of a non-commutative algebra R and of a quotient R/I by a nilpotent ideal coincide, the statements follow. □

This allows us to relate the non-commutative affine variety $\texttt{max}\ A_{con}$ with the affine variety $\texttt{max}\ Z_{con}$ of its center, that is with the conifold singularity.

Lemma 2.9 *Intersecting with the center* Z_{con} *determines a continuous map*

$$\texttt{max}\ A_{con} \xrightarrow{\psi} \texttt{max}\ Z_{con} \qquad \mathfrak{m} \mapsto \mathfrak{m} \cap Z_{con}$$

which is a one-to-one correspondence away from the unique singularity of $\texttt{max}\ Z_{con}$ *whose fiber consists of two points.*

Proof. The inclusion $\mathbb{C}[x,y,z] \subset Z_{con}$ of commutative $\mathbb{C}$-algebras determines a two-fold cover

$$\texttt{max}\ Z_{con} = \mathbb{V}(D^2 - 4(z^2 - xy)) \subset \mathbb{A}^4 \xrightarrow{c} \mathbb{A}^3 \qquad (x,y,z,D) \mapsto (x,y,z)$$

which is *ramified* over $\mathbb{V}(z^2 - xy)$. That is, if $P = (a,b,c) \notin \mathbb{V}(z^2 - xy)$ then there are exactly two point lying over it

$$P_1 = (a,b,c,+\sqrt{c^2 - ab}) \qquad \text{and} \qquad P_2 = (a,b,c,-\sqrt{c^2 - ab})$$

On the other hand, if $P = (a,b,c) \in \mathbb{V}(z^2 - xy)$, then there is just one point lying over it : $(a,b,c,0)$. The statement then follows from combining this covering information with the composition map

$$\mathtt{max}\ A_{con} \xrightarrow{\psi} \mathtt{max}\ Z_{con} \xrightarrow{c} \mathbb{A}^3$$

which is ϕ and the foregoing lemma. □

Observe that ψ is a homeomorphism on $\mathtt{max}\ A_{con} - \mathbb{V}(x,y,z)$ and hence can be seen as a non-commutative birational map. If $\mathfrak{m}$ lies in this open set then

$$A_{con}/\mathfrak{m} \simeq M_2(\mathbb{C})$$

whereas for the two maximal ideals $\mathfrak{m}_+ = (X,Y,Z-1)$ and $\mathfrak{m}_- = (X,Y,Z+1)$ lying over the conifold singularity we have

$$A_{con}/\mathfrak{m}_+ \simeq \mathbb{C} \simeq A_{con}/\mathfrak{m}_-$$

We call the open set $\mathtt{max}\ Z_{con} - \mathbb{V}(z^2 - xy)$ the *Azumaya locus* of the conifold algebra A_{con} and its complement the *ramification locus*.

2.3 The representations

In this section we will clarify why we say that $\mathtt{max}\ A_{con} \longrightarrow \mathtt{max}\ Z_{con}$ is a non-commutative desingularization by proving that the representation variety corresponding to the conifold algebra is smooth.

Definition 2.10 An *n-dimensional representation* of a non-commutative $\mathbb{C}$-algebra A is an algebra morphism

$$A \xrightarrow{\phi} M_n(\mathbb{C})$$

A representation ϕ determines an *n-dimensional right A-module* M_ϕ by identifying $M_\phi = \mathbb{C}^{\oplus n}$ and defining the A-action on M_ϕ via

$$m.a = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}.\phi(a)$$

Two n-dimensional right A-modules M_ϕ and M_ψ are said to be *isomorphic* if there is a linear isomorphism $M_\phi \xrightarrow{g} M_\psi$ such that $g(m._\phi a) = g(m)._\psi a$. Using the above identifications, this means there is a $g \in GL_n$ such that for all $m = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}$

$$m.\phi(a).g = g(m._\phi a) = g(m)._\psi a = m.g.\psi(a)$$

or. equivalently, that the n-dimensional representations ϕ and ψ are *conjugated*, that is

$$\exists g \in GL_n, \forall a \in A \ : \quad \phi(a) = g.\psi(a).g^{-1}$$

Lemma 2.11 *If A is a non-commutative affine $\mathbb{C}$-algebra, then for each n there exists an ideal $I_n(A)$ in some polynomial ring $\mathbb{C}[z_1,\dots,z_N]$ such that its* geometric points, *that is*

$$\mathbb{V}(I_n(A)) \subset \mathbb{A}^N \qquad \longleftrightarrow \qquad \mathtt{rep}_n\ A$$

are in one-to-one correspondence with n-dimensional representations $\mathtt{rep}_n\ A$ of A. Moreover, there is an action of GL_n on $\mathbb{V}(I_n(A))$ such that orbits *under this action correspond to isomorphism classes of n-dimensional right A-modules.*

Proof. Take an affine presentation of A, that is

$$A \simeq \frac{\mathbb{C}\langle x_1, \ldots, x_k \rangle}{I}$$

for some two-sided ideal $I \triangleleft \mathbb{C}\langle x_1, \ldots, x_k \rangle$. For each $1 \leq i \leq k$ consider the *generic* $n \times n$ *matrix*

$$X_i = \begin{bmatrix} x_{11}(i) & \ldots & x_{1n}(i) \\ \vdots & & \vdots \\ x_{n1}(i) & \ldots & x_{nn}(i) \end{bmatrix}$$

where all the $x_{uv}(i)$ are commuting variables. Let $N = k.n^2$ and consider the polynomial ring R on the N variables $x_{uv}(i)$. For each $i = f(x_1, \ldots, x_k) \in I \subset \mathbb{C}\langle x_1, \ldots, x_k \rangle$ we can consider the $n \times n$ matrix with all its entries contained in R by substituting each occurrence of x_i in i by the generic matrix X_i

$$i_n = f(X_1, \ldots, X_k) = \begin{bmatrix} i_n(1,1) & \ldots & i_n(1,n) \\ \vdots & & \vdots \\ i_n(n,1) & \ldots & i_n(n,n) \end{bmatrix} \in M_n(R)$$

and define $I_n(A)$ to be the ideal of $R \simeq \mathbb{C}[z_1, \ldots, z_N]$ generated by all entries $i_n(u,v)$ for all $i \in I$ (observe that even when I is not finitely generated, the ideal $I_n(A)$ will be as R is Noetherian). Hence, there is an algebra morphism

$$A \xrightarrow{j_n} M_n(\frac{\mathbb{C}[z_1, \ldots, z_N]}{I_n(A)})$$

Every point $P \in \mathbb{V}(I_n(A)) \subset \mathbb{A}^N$ determines a maximal ideal of $\mathbb{C}[z_1, \ldots, z_N]/I_n(A)$ and hence an algebra morphism $\pi_P : \mathbb{C}[z_1, \ldots, z_N]/I_n(A) \twoheadrightarrow \mathbb{C}$. Therefore, $P \in \mathbb{V}(I_n(A))$ defines the n-dimensional representation

$$A \xrightarrow{j_n} M_n(\frac{\mathbb{C}[z_1, \ldots, z_N]}{I_n(A)}) \xrightarrow{M_n(\pi_P)} M_n(\mathbb{C})$$

Conversely, if $A \xrightarrow{\phi} M_n(\mathbb{C})$ is an n-dimensional representation with

$$\phi(x_i) = \begin{bmatrix} a_{11}(i) & \ldots & a_{1n}(i) \\ \vdots & & \vdots \\ a_{n1}(i) & \ldots & a_{nn}(i) \end{bmatrix}$$

then the point P_ϕ with entries all $a_{uv}(i)$ lies in $\mathbb{V}(I_n(A))$. Finally, there is a natural GL_n-action by automorphisms on $\mathbb{C}[z_1, \ldots, z_N]$ by sending for $g \in GL_n$ the (u,v)-entry of X_i to the (u,v)-entry of $g^{-1}X_i g$ and the ideal $I_n(A)$ is invariant under this action, that is, $g.p \in I_n(A)$ for all $p \in I_n(A)$. This follows from the fact that for any $f \in \mathbb{C}\langle x_1, \ldots, x_k \rangle$

$$f(g^{-1}X_1 g, \ldots, g^{-1}X_k g) = g^{-1} f(X_1, \ldots, X_k) g$$

$\square$

Lemma 2.12 *For the conifold algebra A_{con}, the representation variety $\texttt{rep}_1\ A_{con}$ consists of two points corresponding to the one-dimensional representations*

$$\phi_+ = \begin{cases} X \mapsto 0 \\ Y \mapsto 0 \\ Z \mapsto +1 \end{cases} \qquad \textit{and} \qquad \phi_- = \begin{cases} X \mapsto 0 \\ Y \mapsto 0 \\ Z \mapsto -1 \end{cases}$$

Observe that these are the two points of $\texttt{max}\ A_{con}$ lying over the conifold singularity.

Proof. An algebra map $A_{con} \xrightarrow{\phi} \mathbb{C}$ must satisfy $\phi(Z)^2 = 1$ whence $\phi(Z) = \pm 1$. Moreover, the images $\phi(X), \phi(Y)$ and $\phi(Z)$ commute, so

$$0 = \phi(XZ + ZX) = 2\phi(X)\phi(Z) = \pm 2\phi(X) \quad \text{whence} \quad \phi(X) = 0$$

and similarly $\phi(Y) = 0$. □

For $n > 1$ it is more complicated to determine $I_n(A_{con})$ and the associated representation variety $\mathtt{rep}_n\ A_{con}$. For example, to determine $I_2(A_{con})$ we consider the generic 2×2 matrices

$$X \mapsto \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \quad Y \mapsto \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \quad Z \mapsto \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$$

and have to work out the matrix-identities induced by the defining relations of A_{con}. For example

$$XZ + ZX \mapsto \begin{bmatrix} 2x_1z_1 + x_2z_3 + x_3z_2 & x_1z_2 + x_2z_4 + x_2z_1 + x_4z_2 \\ x_1z_3 + x_3z_1 + x_3z_4 + x_4z_3 & 2x_4z_4 + x_2z_3 + x_3z_2 \end{bmatrix}$$

$$YZ + ZY \mapsto \begin{bmatrix} 2y_1z_1 + y_2z_3 + y_3z_2 & y_1z_2 + y_2z_4 + y_2z_1 + y_4z_2 \\ y_1z_3 + y_3z_1 + y_3z_4 + y_4z_3 & 2y_4z_4 + y_2z_3 + y_3z_2 \end{bmatrix}$$

So, even in this case we do not get much insight into simple geometric questions about $\mathtt{rep}_2\ A_{con}$ such as smoothness, dimension, orbit structure etc. For larger n the situation becomes even more complicated.

This is where non-commutative geometry enters. We will use ringtheoretic properties of the non-commutative algebra A to get some grip on the representation varieties $\mathtt{rep}_n\ A$. In the special case of $\mathtt{rep}_2\ A_{con}$ we can use some ad-hoc arguments.

Lemma 2.13 *For the conifold algebra A_{con}, the representation variety $\mathtt{rep}_2\ A_{con}$ is a smooth affine variety having three disjoint irreducible components. Two of these components are a point, the third component $\mathtt{trep}_2\ A$ has dimension 6.*

Proof. From the defining relation $Z^2 = 1$ it follows that the image of Z in any finite dimensional representation has eigenvalues ± 1. Hence, after simultaneous conjugation of the images of X, Y and Z we may assume that Z has one of the following three forms

$$Z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad Z \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad Z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The first two possibilities are easily dealt with. Here, the image of Z is a central unit so it follows from the relations $XZ + ZX = 0 = YZ + ZY$ as in the previous lemma that $X \mapsto 0$ and $Y \mapsto 0$. That is, these two components consist of just one point (the action of GL_2 by simultaneous conjugation fixes these matrices) corresponding to the 2-dimensional *semi-simple* representations

$$M_+ = \phi_+ \oplus \phi_+ \qquad \text{and} \qquad M_- = \phi_- \oplus \phi_-$$

The interesting case is the third one. Because X^2 and Y^2 are central elements it follows (for example using the characteristic polynomial of 2×2 matrices) that in any 2-dimensional representation $A_{con} \xrightarrow{\phi} M_2(\mathbb{C})$ we have that $tr(\phi(X)) = 0$ and $tr(\phi(Y)) = 0$. Hence, the third component of $\mathtt{rep}_2\ A_{con}$ consists of those 2-dimensional representations ϕ such that

$$tr(\phi(X)) = 0 \qquad tr(\phi(Y)) = 0 \qquad \text{and} \qquad tr(\phi(Z)) = 0$$

For this reason we denote this component by $\texttt{trep}_2\ A_{con}$ and call it the variety of *trace preserving* 2*-dimensional representations*. To describe the coordinate ring of this component we can use *trace zero* generic 2×2 matrices

$$X \mapsto \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \quad Y \mapsto \begin{bmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{bmatrix} \quad Z \mapsto \begin{bmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{bmatrix}$$

which drastically reduces the defining equations as T^2 and $TS + ST$ are both scalar matrices for any trace zero 2×2 matrices. More precisely, we have

$$XZ + ZX \mapsto \begin{bmatrix} 2x_1z_1 + x_2z_3 + x_3z_2 & 0 \\ 0 & 2x_1z_1 + x_2z_3 + x_3z_2 \end{bmatrix}$$

$$YZ + ZY \mapsto \begin{bmatrix} 2y_1z_1 + y_2z_3 + y_3z_2 & 0 \\ 0 & 2y_1z_1 + y_2z_3 + y_3z_2 \end{bmatrix}$$

$$Z^2 \mapsto \begin{bmatrix} z_1^2 + z_2z_3 & 0 \\ 0 & z_1^2 + z_2z_3 \end{bmatrix}$$

and therefore the coordinate ring of $\texttt{trep}_2\ A_{con}$

$$\mathbb{C}[\texttt{trep}_2\ A_{con}] = \frac{\mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3]}{(2x_1z_1 + x_2z_3 + x_3z_2, 2y_1z_1 + y_2z_3 + y_3z_2, z_1^2 + z_2z_3 - 1)}$$

To verify that $\texttt{trep}_2\ A_{con}$ is a smooth 6-dimensional affine variety we therefore have to show that the *Jacobian matrix*

$$\begin{bmatrix} 2z_1 & z_3 & z_2 & 0 & 0 & 0 & 2x_1 & x_3 & x_2 \\ 0 & 0 & 0 & 2z_1 & z_3 & z_2 & 2y_1 & y_3 & y_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2z_1 & z_3 & z_2 \end{bmatrix}$$

has constant rank 3 on $\texttt{trep}_2\ A_{con}$. This is forced by the submatrices $\begin{bmatrix} 2z_1 & z_3 & z_2 \end{bmatrix}$ along the 'diagonal' of the Jacobian unless $z_1 = z_2 = z_3 = 0$ but this cannot hold for a point in $\texttt{trep}_2\ A_{con}$ by the equation $z_1^2 + z_2z_3 = 1$. $\square$

2.4 The quotient

Because $\texttt{trep}_2\ A_{con}$ is a smooth affine variety, we call the conifold algebra A_{con} a smooth@2-algebra and say that $\texttt{max}\ A_{con} \twoheadrightarrow \texttt{max}\ Z_{con}$ is a *non-commutative desingularization* of the conifold singularity. In this section we give the connection between $\texttt{trep}_2\ A_{con}$ and the conifold singularity by showing that the latter is the *quotient variety* of the former under the base-change action by GL_2.

We will give an alternative proof of the fact that the trace preserving representation variety $\texttt{trep}_2\ A_{con}$ is a smooth variety. Recall that up to simultaneous basechange we could bring the image of Z in the form

$$Z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Taking the generic 2×2 matrices

$$X \mapsto \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \qquad Y \mapsto \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$$

it follows from the relations $XZ + ZX = 0 = YZ + ZY$ that $x_1 = x_4 = 0 = y_1 = y_4$. Therefore, such a 2-dimensional representation of A_{con} can be simultaneously conjugated to one of the form

$$X \mapsto \begin{bmatrix} 0 & x_2 \\ x_2 & 0 \end{bmatrix} \quad Y \mapsto \begin{bmatrix} 0 & y_2 \\ y_3 & 0 \end{bmatrix} \quad Z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and as the images of X^2 and Y^2 are scalar matrices the remaining defining relations $[X^2, Y] = 0 = [Y^2, X]$ are automatically satisfied. 2-dimensional representations of A_{con} in this canonical form hence form a smooth 4-dimensional affine space

$$\mathbb{A}^4 = \mathbb{V}(x_1, x_4, y_1, y_4, z_1 - 1, z_2, z_3, z_4 + 1) \subset \mathbb{A}^{12}$$

To recover $\mathtt{trep}_2\ A_{con}$ from this affine space we have to let GL_2 act on it. The subgroup of GL_2 fixing the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{is} \quad T = \{ \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \mid \lambda, \mu \in \mathbb{C}^* \},$$

the two-dimensional *torus*. There is an action of T on the product $GL_2 \times \mathbb{A}^4$ via

$$t.(g, P) = (gt^{-1}, t.P) \qquad \text{for all } t \in T, g \in GL_2 \text{ and } P \in \mathbb{A}^4$$

and where $t.P$ means the action by simultaneous conjugation by the 2×2 matrix $t \in T \subset GL_2$ on the three 2×2 matrix-components of P.

Lemma 2.14 *Under the action-map*

$$GL_2 \times \mathbb{A}^4 \longrightarrow \mathtt{trep}_2\ A_{con} \qquad (g, P) \mapsto g.P$$

two points (g, P) and (g', P') are mapped to the same point if and only if they belong to the same T-orbit in $GL_2 \times \mathbb{A}^4$. That is, we can identify $\mathtt{trep}_2\ A_{con}$ with the principal fiber bundle *(or orbit-space)*

$$\mathtt{trep}_2\ A_{con} \simeq GL_2 \times^T \mathbb{A}^4 = (GL_2 \times \mathbb{A}^4)/T$$

In particular, there is a natural one-to-one correspondence between GL_2-orbits in $\mathtt{trep}_2\ A_{con}$ and T-orbits in $\mathbb{A}^4$.

Proof. If $g.P = g'.P'$, then $P = g^{-1}g'.P'$ and as both P and P' have as their third 2×2 matrix component

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

it follows that $g^{-1}g'$ is in the stabilizer subgroup of this matrix so $g^{-1}g' = t^{-1}$ for some $t \in T$ whence $g' = gt^{-1}$ and as $(g^{-1}g')^{-1}.P = P'$ also $t.P = P'$ whence

$$t.(g, P) = (gt^{-1}, t.P) = (g', P')$$

Hence we can identify $\mathtt{trep}_2\ A_{con} = GL_2.\mathbb{A}^4$ with the orbit-space of the T-action which is usually denoted by $GL_2 \times^T \mathbb{A}^4$ and called the principal (or associated) fiber bundle. Incidentally, this gives another proof for smoothness of $\mathtt{trep}_2\ A_{con}$ as it is the base of a fibration with smooth fibers of the smooth top space $GL_2 \times \mathbb{A}^4$.

GL_2 acts on $GL_2 \times \mathbb{A}^4$ by $g.(g', P') = (gg', P')$ and this action commutes with the T-action so induces a GL_2-action on the orbit-space

$$GL_2 \times (GL_2 \times^T \mathbb{A}^4) \longrightarrow GL_2 \times^T \mathbb{A}^4 \qquad g.\overline{(g', P')} = \overline{(gg', P')}$$

As we have identified $GL_2 \times^T \mathbb{A}^4$ with $\mathtt{trep}_2\ A_{con}$ via the action map, that is $\overline{(g, P)} = g.P$ the remaining statement follows. $\square$

We would like to construct an orbit space for the GL_2-action on $\mathtt{trep}_2\ A_{con}$ as its points are the isomorphism classes of 2-dimensional representations. However, such an orbit space only exists when all orbits are closed and $\texttt{trep}_2\ A_{con}$ has non-closed orbits, for example

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix}.(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = (\begin{bmatrix} 0 & \epsilon x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})$$

all belong to the same orbit for every $\epsilon \neq 0$, but its limiting point is the representation sending X and Y to the zero matrix which is a non-isomorphic representation.

As we will see later, the best algebraic approximation to the non-existent orbit space is the affine variety corresponding to the ring of *polynomial invariants* $\mathbb{C}[\texttt{trep}_2\ A_{con}]^{GL_2}$ which in this case is isomorphic to the ring of *polynomial torus invariants* $\mathbb{C}[\mathbb{A}^4]^T$ by the foregoing lemma.

Lemma 2.15 *The ring of polynomial invariants*

$$\mathbb{C}[\texttt{trep}_2\ A_{con}]^{GL_2} \simeq \mathbb{C}[\mathbb{A}^4]^T$$

are isomorphic to the coordinate ring of the conifold singularity Z_{con}. As a consequence, the quotient map

$$\texttt{trep}_2\ A_{con} \longrightarrow\!\!\!\!\!\rightarrow \texttt{spec}\ Z_{con}$$

maps a two-dimensional representation to the direct sum of its Jordan-Hölder components as the quotient variety $\texttt{spec}\ Z_{con}$ *parametrizes isomorphism classes of two-dimensional semi-simple representations of A_{con}.*

Proof. The action of the two-dimensional torus T on $\mathbb{A}^4 = \{(x_2, x_3, y_2, y_3)\}$ is given by

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.(\begin{bmatrix} 0 & x_2 \\ x_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y_2 \\ y_3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) =$$

$$(\begin{bmatrix} 0 & \lambda\mu^{-1}x_2 \\ \lambda^{-1}\mu x_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \lambda\mu^{-1}y_2 \\ \lambda^{-1}\mu y_3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})$$

Hence, the action of $(\lambda, \mu) \in T$ on $\mathbb{C}[\mathbb{A}^4] = \mathbb{C}[X_2, X_3, Y_2, Y_3]$ is defined by

$$X_2 \mapsto \lambda^{-1}\mu X_2 \quad X_3 \mapsto \lambda\mu^{-1}X_3 \quad Y_2 \mapsto \lambda^{-1}\mu Y_2 \quad Y_3 \mapsto \lambda\mu^{-1}Y_3$$

and this action sends any monomial in the variables to a scalar multiple of that monomial. So, in order to determine the ring of polynomial invariants

$$\mathbb{C}[X_2, X_3, Y_2, Y_3]^T = \{f = f(X_2, X_3, Y_2, Y_3) \mid (\lambda, \mu).f = f\ \forall(\lambda, \mu) \in T\}$$

it sufficers to determine all invariant monomials, or equivalently, all positive integer quadruplets (a, b, c, d) such that $a - b + c - d = 0$ as

$$(\lambda, \mu).X_2^a X_3^b Y_2^c Y_3^d = \lambda^{-a+b-c+d}\mu^{a-b+c-d} X_2^a X_3^b Y_2^c Y_3^d$$

Clearly, such quadruplets are all generated (as Abelian group under addition) by the four basic oones

$$(1,1,0,0) \mapsto X_2X_3 \quad (1,0,0,1) \mapsto X_2Y_3 \quad (0,1,1,0) \mapsto X_3Y_2 \quad (0,0,1,1) \mapsto Y_2Y_3$$

and therefore

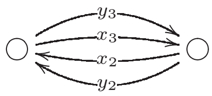
$$\mathbb{C}[\texttt{trep}_2\ A_{con}]^{GL_2} \simeq \mathbb{C}[X_2, X_3, Y_2, Y_3]^T = \mathbb{C}[X_2X_3, X_2Y_3, X_3Y_2, Y_2, Y_3] \simeq \frac{\mathbb{C}[p, q, r, s]}{(ps - qr)}$$

is the conifold singularity Z_{con}. We know already that $\texttt{spec}\ Z_{con}$ has as its points the isomorphism classes of 2-dimensional semi-simple representations with $\phi_+ \oplus \phi_-$ as the semi-simple representation corresponding to the singularity and all other points classify a unique simple 2-dimensional representation. □

2.5 The desingularization

It is all well to call the map $\mathtt{spec}\ A_{con} \longrightarrow\!\!\!\!\!\rightarrow \mathtt{spec}\ Z_{con}$ a non-commutative desingularization of the conifold singularity, but sceptical people would like to construct an ordinary (that is, commutative) desingularization of $\mathtt{spec}\ Z_{con}$ from the conifold algebra A_{con}. In this section we will achieve this by constructing *moduli spaces* of certain open sets of 2-dimensional representations of A_{con}.

First, we need to clarify the connection with the quiver-diagram Q_{con} :



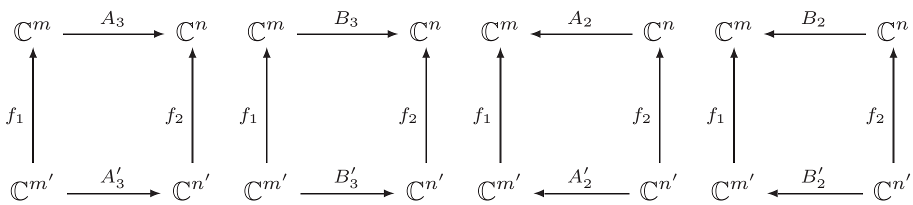
A *representation* of Q_{con} of *dimension vector* $\alpha = (m, n)$ is by definition the assignment of a vectorspace of the appropriate dimension to each vertex of Q_{con} and a linear map between the vertex-spaces to every arrow in Q_{con}, that is a quadruple of matrices

$$(A_3, B_3, A_2, B_2) \in M_{m\times n}(\mathbb{C}) \oplus M_{m\times n}(\mathbb{C}) \oplus M_{n\times m}(\mathbb{C}) \oplus M_{n\times m}(\mathbb{C}) = \mathtt{rep}_\alpha\ Q_{con}$$

Base change in the vertex-spaces induces an action of the *basechange group* $GL(\alpha) = GL_m \times GL_n$ on the space of all representations $\mathtt{rep}_\alpha\ A_{con}$ via

$$(g, h).(A_3, B_3, A_2, B_2) = (g^{-1}A_3h, g^{-1}B_3h, h^{-1}A_2g, h^{-1}B_2g)$$

and two α-dimensional representations are said to be *isomorphic* if they belong to the same orbit. For $\beta = (m', n')$ and $V' = (A'_3, B'_3, A'_2, B'_2) \in \mathtt{rep}_\beta\ Q_{con}$ and $V = (A_3, B_3, A_2, B_2) \in \mathtt{rep}_\alpha\ Q_{con}$ a *morphism* $F : V' \longrightarrow V$ consists of linear maps $(f_1, f_2) \in M_{m'\times m}(\mathbb{C}) \times M_{n'\times n}(\mathbb{C})$ between the vertex spaces such that all the corresponding arrow-diagrams are commutative, that is, the diagrams



all all commuting. If both f_i are monomorphisms we say that F is a monomorphism or that V' is a *subrepresentation* of V and if both f_i are epimorphisms we say that F is an epimorphism or that V is a quotient representation of V'.

With these definitions we can identify the T-action on $\mathbb{A}^4$ above with the action of the base-change group $T = \mathbb{C}^* \times \mathbb{C}^* = GL(\alpha)$ for $\alpha = (1, 1)$ on the space of all α-dimensional representations $\mathtt{rep}_\alpha\ Q_{con} = \{(x_3, y_3, x_2, y_2)\} = \mathbb{A}^4$. Next, we bring in a *stability structure* $\theta = (-1, 1)$ (observe that $\theta.\alpha = 0$). We call a representation $V = (x_3, y_3, x_2, y_2) \in \mathtt{rep}_\alpha\ Q_{con}$ *θ-stable* if for all proper subrepresentations $V' \subset V$ of dimension vector β we have that $\theta.\beta > 0$. In our case, for $\alpha = (1, 1)$ this condition just says that V has no subrepresentations of dimension vector $\beta = (1, 0)$. That is,

$$V = (x_3, y_3, x_2, y_2) \quad \text{is } \theta\text{-stable} \quad \Leftrightarrow \quad x_3 \neq 0 \text{ or } y_3 \neq 0$$

The subset $\mathtt{rep}_\alpha^{|theta}\ Q_{con}$ is a Zariski open (though not affine) subset of $\mathtt{rep}_\alpha\ Q_{con}$ and the *stabilizer subgroup* of any point $V \in \mathtt{rep}_\alpha^\theta\ Q_{con}$ is

$$\mathtt{stab}_T\ V = \{(\lambda, \lambda) \mid \lambda \in \mathbb{C}^*\} = T_c$$

and hence the group $PGL(\alpha) = T/T_c$ acts freely on $\mathtt{rep}^{\theta}_{\alpha}\ Q_{con}$ and therefore we can construct the orbit space classifying the isomorphism classes of θ-stable α-dimensional representations of Q_{con}

$$\mathtt{moduli}^{\theta}_{\alpha}\ Q_{con} = \mathtt{rep}^{\theta}_{\alpha}\ Q_{con}/T$$

which is called the *moduli space* of θ-stable representations.

For the action of the torus T on $\mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}] = \mathbb{C}[x_3, y_3, x_2, y_2]$ a polynomial $f = f(x_3, y_3, x_2, y_2)$ is said to be a *θ-semi-invariant of weight k* provided

$$(\lambda, \mu).f = \lambda^{-k}\mu^k f$$

In particular, θ-semi-invariants of weight 0 are just the polynomial invariants and the product of θ-semi-invariants of weight k resp. l is a semi-invariant of weight $k + l$. Therefore we have a *graded* subalgebra of $\mathbb{C}[x_3, y_3, x_2, y_2]$ of all θ-semi-invariants

$$\mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta} = \mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta}_0 \oplus \mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta}_1 \oplus \dots$$

where $\mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta}_k$ is the space of all θ-semi-invariants of weight k.

Recall from [**?**, p.76] that $\mathtt{proj}\ R$ of any positively graded commutative algebra $R = R_0 \oplus R_1 \oplus \dots$ is the set of all *graded prime ideals* which do not contain the positive part $R_+ = R_1 \oplus R_2 \oplus \dots$. One defines on $\mathtt{proj}\ R$ the Zariski topology by taking as the closed subsets

$$\mathbb{V}(I) = \{P \in \mathtt{proj}\ R \mid I \subset P\}$$

for any graded ideal I of R. Intersecting a graded prime ideal with the part of degree zero R_0 defines a continuous map

$$\mathtt{proj}\ R \overset{\pi}{\twoheadrightarrow} \mathtt{spec}\ R_0$$

which is surjective and *projective*, that is, all fibers $\pi^{-1}(\mathfrak{p})$ are projective varieties.

Lemma 2.16 *The moduli space of all θ-stable α-dimensional representations*

$$\mathtt{moduli}^{\theta}_{\alpha}\ Q_{con} \simeq \mathtt{proj}\ \mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta}$$

is the $\mathtt{proj}$ of the ring of θ-semi-invariants and as the semi-invariants of weight zero are the polynomial invariants we get a projective morphism

$$\mathtt{proj}\ \mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta} \twoheadrightarrow \mathtt{spec}\ Z_{con}$$

which is a desingularization of the conifold singularity.

Proof. As in the case of polynomial invariants, the space $\mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta}_k$ is spanned by monomials

$$x_2^a x_3^b y_2^c y_3^d \qquad \text{satisfying} \qquad -a + b - c + d = k$$

and one verifies that this space is the module over the ring of polynomial invariants generated by all monomials of degree k in x_3 and y_3. That is

$$\mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta} = \mathbb{C}[x_2x_3, x_2y_3, x_3y_2, y_2y_3][x_3, y_3] \subset \mathbb{C}[x_2, y_2, x_3, y_3]$$

with the generators $a = x_2x_3, b = x_2y_3, c = x_3y_2$ and $d = y_2y_3$ of degree zero and $e = x_3$ and $f = y_3$ of degree one. As a consequence, we can identify $\mathtt{proj}\ \mathbb{C}[\mathtt{rep}_{\alpha}\ Q_{con}]^{\theta}$ with the closed subvariety

$$\mathbb{V}(ad - bc, af - be, cf - de) \subset \mathbb{A}^4 \times \mathbb{P}^1$$

with (a, b, c, d) the affine coordinates of $\mathbb{A}^4$ and $[e : f]$ projective coordinates of $\mathbb{P}^1$. The projection $\texttt{proj}\ \mathbb{C}[\texttt{rep}_\alpha\ Q_{con}]^\theta \twoheadrightarrow \texttt{spec}\ Z_{con}$ is projection onto the $\mathbb{A}^4$-component of $\mathbb{A}^4 \times \mathbb{P}^1$.

To prove smoothness we cover $\mathbb{P}^1$ with the two affine opens $e \neq 0$ (with affine coordinate $x = f/e$ and $f \neq 0$ with affine coordinate $y = e/f$. In the affine coordinates (a, b, c, d, x) the relations become

$$ad = bc \qquad ax = b \qquad \text{and} \qquad cx = d$$

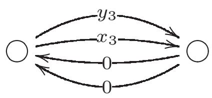
whence the coordinate ring is $\mathbb{C}[a, c, x]$ and so the variety is smooth on this affine open. Similarly, the coordinate ring on the other affine open is $\mathbb{C}[b, d, y]$ and smoothness follows. Moreover, π is *birational* over the complement of the singularity. This follows from the relations

$$ax = b, \quad cx = d, \quad by = a, \quad dy = c$$

which determine x (or y and hence the point in $\texttt{proj}$) lying over any $(a, b, c, d) \neq (0, 0, 0, 0)$ in $\texttt{spec}\ Z_{con}$. Therefore, the map π is a desingularization and the *exceptional fiber*

$$E = \pi^{-1}(0, 0, 0, 0) \simeq \mathbb{P}^1$$

which classifies the θ-stable representations which lie over $(0, 0, 0, 0)$ (that is, those such that $x_2x_3 = x_2y_3 = x_3y_2 = y_2y_3 = 0$) as they are all of the form

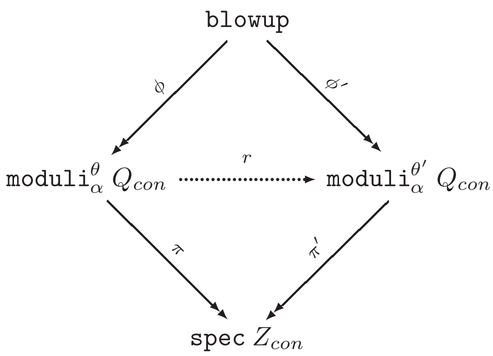


with either $x_3 \neq 0$ or $y_3 \neq 0$ and the different T-orbits of those are parametrized by the points of $\mathbb{P}^1$. As the smooth points of $\texttt{spec}\ Z_{con}$ are known to correspond to isomorphism classes of simple (hence certainly θ-stable) representations we have proved that

$$\texttt{proj}\ \mathbb{C}[\texttt{rep}_\alpha\ Q_{con}]^\theta \simeq \texttt{moduli}^\theta_\alpha\ Q_{con}$$

is the moduli space of all θ-stable α-dimensional representations of Q_{con}. □

Clearly, we could have done the same calculations starting with another stability structure $\theta' = (1, -1)$ and obtained another desingularization replacing the roles of x_2, y_2 and x_3, y_3. This gives us the situation



Here, $\texttt{blowup}$ denotes the desingularization of $\texttt{spec}\ Z_{con}$ one obtains by blowing-up the point $(0, 0, 0, 0) \in \mathbb{A}^4$ and which has exceptional fiber $\mathbb{P}^1 \times \mathbb{P}^1$. Blowing down either of these lines (the maps ϕ and ϕ') one obtains the 'minimal' resolutions given by the moduli spaces. These spaces are related by the *rational map* r which is called the *Atiyah flop* in string theory-literature.