k alg closed char 0 (2)

$$\begin{tikzcd}
{}^{(2)}A \arrow[r, dash] \arrow[d, "tr(2)"] & \Sigma \arrow[r, dash] \arrow[d, "tr(2)"] & M_n(\bar{K}) \arrow[d, "tr(2)"] \\
{}^{(2)}R \arrow[r, dash] & K \arrow[r, dash] & \bar{K}
\end{tikzcd}$$

$BS(\Sigma)(L) = \{$ left ideals in $\Sigma \otimes L$ of dimension $n \}$

$BS(\Sigma)(\bar{K}) = \mathbb{P}^{n-1}_{\bar{K}}$

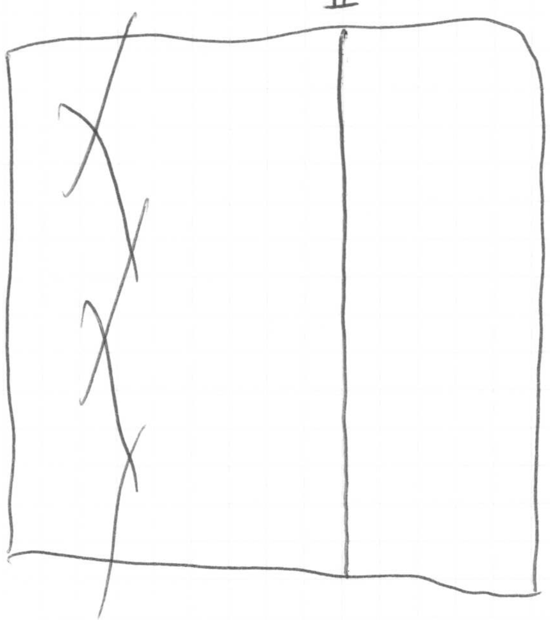
central simple alg $\overset{1-1}{\longleftrightarrow} H^1(Gal(\bar{K}/K), PGL_n) \overset{1-1}{\longleftrightarrow}$ Brauer-Severi

$$\Sigma = \left(\begin{matrix} a, b \\ K \end{matrix} \right)$$

$$BS(\Sigma) = \mathbb{V}(x_0^2 - a x_1^2 - b x_2^2) \subset \mathbb{P}^2_K$$

$$BS(\Sigma)(L) \neq \emptyset \iff \Sigma \otimes L \cong M_n(L)$$

$\mathbb{P}^{n-1}$



$$BS(A) \left\{ \begin{array}{l} \text{Grothendieck (Azumaya)} \\ \text{Artin-Mumford (cartan orders)} \end{array} \right.$$

$$\downarrow \pi$$

(2)  max R

Ramif. Azumaya

$$\downarrow$$

$$(Q_m, \alpha_m) \longrightarrow \hat{A}_m, \hat{R}_m$$

laten staan op 1 klein hoofd.

M. Van den Bergh (1986)

$$GL_n \circlearrowright \mathrm{trep}_n A = \{ A \xrightarrow{\varphi} M_n(k) \; : \; \varphi(tr(a)) = Tr\, \varphi(a) \}$$

$$GL_n \circlearrowright k^n = \text{column vectors}$$

Artin + Procesi

$$\begin{tikzcd}
\{(\varphi, v) : \varphi(A) v = k^n\} \arrow[d, two heads, "\text{orbits}", "\substack{\text{principal} \\ GL_n\text{-fibr.}}"'] & \subset & (\mathrm{trep}_n A) \times k^n \arrow[d, two heads, "\text{alg quotient} /\!/ GL_n"] \\
BS(A) \arrow[rr, two heads, "\pi"] & & \max R
\end{tikzcd}$$

B. Schelter – C. Procesi ($\pm$ 1985)

<u>dfn</u> A smooth order $\Leftrightarrow$ $\mathrm{trep}_n\, A$ is smooth variety

<u>ex</u>: – hereditary orders

– trace rings $\mathbb{T}_{m,n}$ of m generic $n \times n$ matrices

– size $n \times n$ approximations of nonc. manifolds

– every order has open set where it is smooth

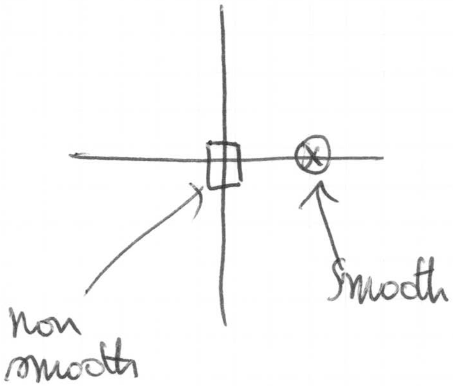
<u>"Thm"</u>: A smooth order $\Longrightarrow$ $BS(A)$ smooth variety

$\Leftarrow$ – – "
MORALLY

<u>Example</u> $\quad A = k_q[X,Y] \qquad q^n = 1$

$R = k[X^n, Y^n] \qquad \max R = \mathbb{A}^2$

$\max(R)$



singularities of $BS(A) = \left\{ \begin{array}{l} \varphi \text{ nilpotent rep} \\ \text{having cyclic vector} \end{array} \right. +$ singular pt in $\operatorname{trep}_n A$

(n=2) $\quad X \mapsto \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad Y \mapsto \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$

$XY + YX = 0$

$\mathbb{V}(2ad + bf + ec) \subset \mathbb{A}^6 \qquad \exists!$ sing $\overline{\mathcal{O}} = \mathcal{O}_{trep}$

$\Rightarrow BS(A)$ smooth

op 2e klei bord

(n=3) $\quad$ C. Ingalls smooth

(n=4) $\quad$ " $\quad \exists$ singular pt.

~~Illustration~~

30 min

idea

1) what are central singularities

2) can we describe fibers (as they might contain parts of except f(1))

3) flatness of π should be related to singularities

35 min

Thm 1: A smooth order over R, then singularities of $\max R$ are known

dim	1	2	3	4	5	6	7	...
# types	0	0	1	3	10	53	...	
	hereditary smooth curve	!	conifold					

+ Raf Bocklandt (quotient van smooth)

Geert Van de Weyer (Popov quiver reps)

$\max R$

m

(Q_m, α_m)

$\Downarrow$

$\hat{R}_m \;,\; \hat{A}_m$

q. houd met theorij

$$M_m = S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$$

$$\dim(S_i) = d_i$$

Q quiver on k vertices $\quad v_i \leftrightarrow S_i$

$$\#\{ v_i \longrightarrow v_j \} = \dim_k Ext^1_A(S_i, S_j)$$

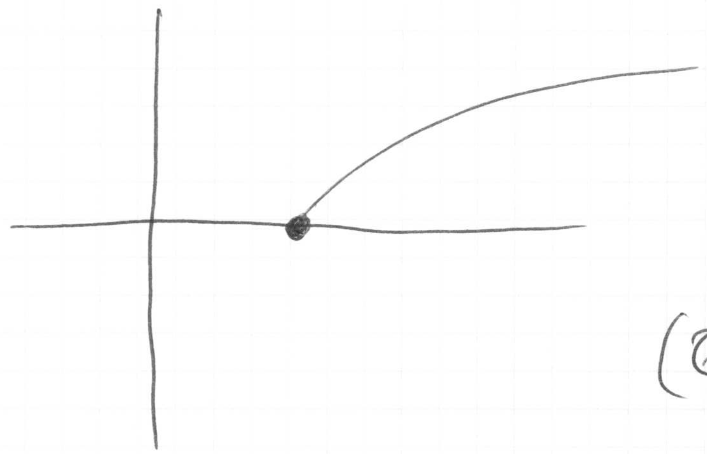
$$\alpha_m = (e_1, \dots, e_k)$$

$$rep_{\alpha_m}(Q) \cong Ext^1_A(M_m, M_m)$$

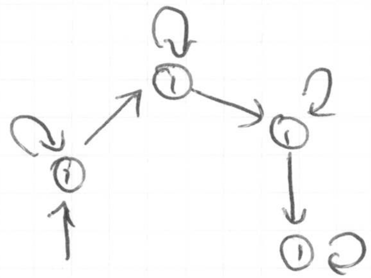
normal space to GL_n-orbit in rep_n

need subquiver Q_m for $trep_n$

Example knot $k_q[X,Y]$

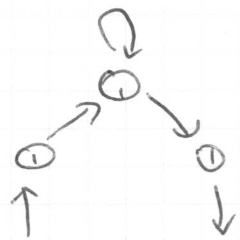


$$M_m = S_1 \oplus S_2 \oplus \cdots \oplus S_n$$



(Q, α_m)

(Q_m, α_m)



$$X \mapsto \begin{pmatrix} a & & & \\ & qa & & \\ & & q^2a & \\ & & & \ddots & \\ & & & & q^{n-1}a \end{pmatrix}$$

$$Y \mapsto \begin{pmatrix} \ddots & \\ & 0 & \ddots \end{pmatrix}$$

$\mathrm{Tr}(X) = \mathrm{Tr}(X^2) = \cdots = \mathrm{Tr}(X^{n-1})$

$= 0$

gives $n-1$ linear relat

↑ least volpundo blz ↑

proof of thm 1

3 Reduction steps

$\exists$ fin. many reduced quiver setting for given di

make list

separate m^i/m^{i+1}

fingerprint: quiver setting also allows to determine nearby singular types + dimension of strata.

few exceptional cases are easily proved to be isomorphic

How does (Q_m, α_m) determine $\hat{R}_m$, A_m

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left, "a"] \arrow[r, "b"'] & \textcircled{1} \arrow[l, bend left, "c"'] \arrow[l, bend left=60, "d"] \end{tikzcd}$$

reduced quiver setting for conifold sing

R are invariants = traces along oriented cycles

$$R = k[ac, bc, ad, bd] = \frac{k[x,y,z,t]}{(xt - yz)}$$

A are (Morita equiv.) to equivariant maps.

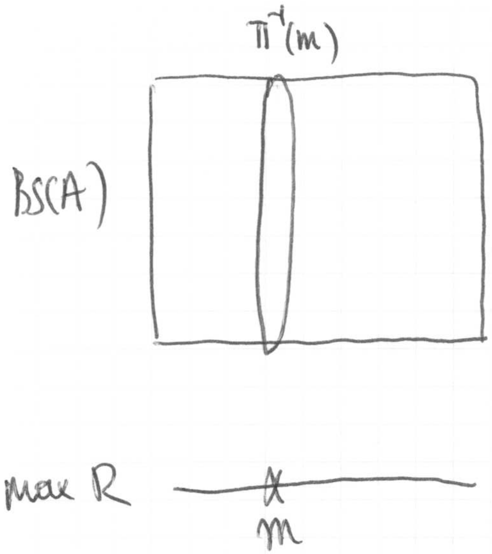
$$A = \begin{bmatrix} R & Ra + Rb \\ Rc + Rd & R \end{bmatrix}$$

R module gen. by ~~traces~~ paths from vertex i to vertex j

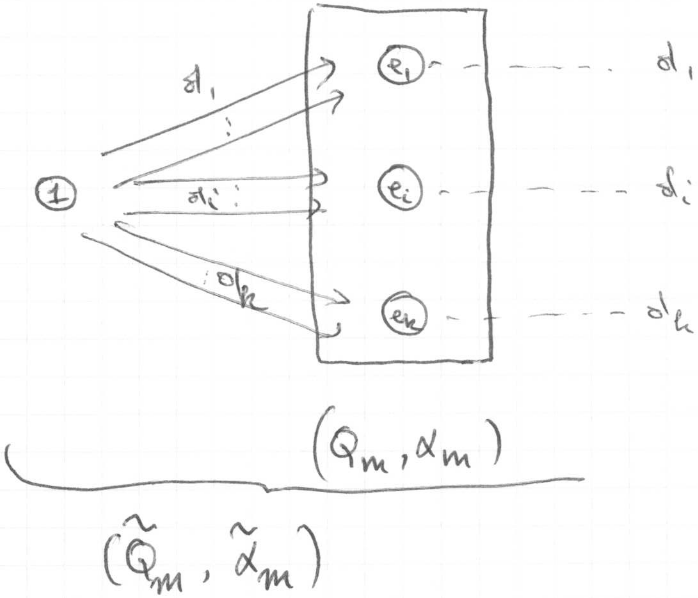
determine by dim of simples.

is sing which appeared yesterday in Ian's talk.

$$\boxed{\text{q-plane ob uitvegen en vervangen door dit.}}$$



10 à 15 minute



<u>Thm 2</u> : $$\pi^{-1}(m) = \Big[\underbrace{\text{Nullcone } \mathrm{rep}_{\tilde{\alpha}_m} \tilde{Q}_m}_{\text{exist combinatorial description of them}} \cap \ *\text{-generated reps}\Big] \underset{\text{orbits}}{/\!\!/} GL(\tilde{\alpha}_m)$$

$*$-generated reps: subrepresentation generated by 1 is the whole rep.

Examples

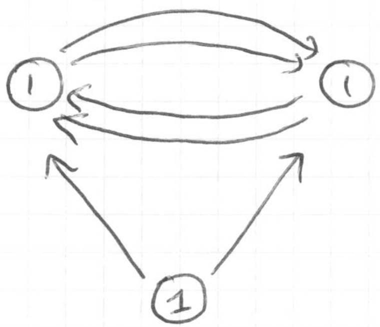
① Azumaya pt $\qquad M = S$ (dim n) $\qquad ① \overset{n}{\rightrightarrows} ①$: dim max R

nullcone ⇒ all loops must be zero

$① \rightrightarrows ①$ $*$-generated   one of the arrows must be non-zero

$/\!/ GL(\alpha) = \mathbb{P}^{n-1}$

② conifold singularity

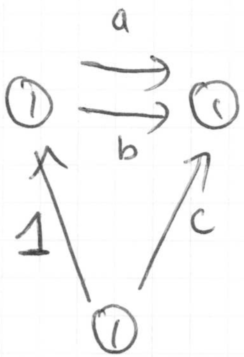


nullcone ⇒ if one of 2 top is $\neq 0$

⇒ two bottom are zero

two cones

$\mathbb{P}^2 \vee \mathbb{P}^2$ intersecting in one point.



$\neq 0$

$*$-generated

$(a,b,c) \neq (0,0,0)$

$/\!/ GL(\alpha) = \mathbb{P}^2$

③ $\mathbb{T}_{m,n}$ trace ring of m generic $n \times n$-matrices

$$M_{m} = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k} \qquad \dim S_i = d_i$$

Thm (+ G. Seelinger) $\pi^{-1}(m)$ has $\frac{(e_1+\dots+e_k)!}{e_1! \dots e_k!}$ irreducible comp. each of dimension

$$n + (m-1)\sum_{i<j} e_i e_j d_i d_j + (m-1)\sum_i \frac{e_i(e_i-1)}{2} - \sum_i e_i$$

Thm 3: ① dim 1 or 2 $\Rightarrow$ π is flat morphism for maximal smooth orders ($\rightsquigarrow$ Artin-Mumford ex.)

② other extreme : $\mathbb{T}_{m,n}$ π is flat $\Leftrightarrow$ $(m,n) = (2,2)$

(only case when center is smooth)

and flat-locus(π) = smooth-locus(center)

③ in general : flat-locus(π) $\subset$ smooth-locus(center)

(Popov)

So Brown-Severi might contain useful info about good desingularizations, remains to find out how much!