Representations of virtually free groups

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Abstract

We use non-commutative geometry to study finite dimensional representations of virtually free groups. In particular, we determine the dimensions of the varieties classifying isomorphism classes of n-dimensional representations by reducing the representation theory of virtually free groups to that of quivers. As the arguments apply as well to the larger class of fundamental algebras of graphs of separable algebras, we state them in that generality.

1 Graphs of separable algebras

Fix a commutative basefield ℓ and recall that a finite dimensional ℓ-algebra S is said to be *separable* if and only if S is the direct sum of simple algebras each of which has a center which is a separable field extension of ℓ. For example, the group algebra ℓG of a finite group G is separable if and only if the order of G is a unit in ℓ.

A finitely generated ℓ-algebra A is said to be *quasi-free* [1] (or *formally smooth* [4]) if either of the following two equivalent conditions is satisfied

- The universal bimodule $\Omega^1_\ell(A)$ of derivations is a projective A-bimodule.
- A satisfies the lifting property modulo nilpotent ideals in $\ell - \mathtt{alg}$, the category of ℓ-algebras.

For example, the free algebra $\ell\langle x_1, \ldots, x_m\rangle$ is quasi-free as is the path algebra ℓQ of a finite quiver Q. Moreover, any separable ℓ-algebra S is quasi-free by [1, §4]. Remark that the standard assumption of [1] is that $\ell = \mathbb{C}$ the field of complex numbers. However, with minor modifications most results remain valid over an arbitrary basefield and we will refer to statements in [1] whenever the argument can be repeated verbatim.

We will imitate the Bass-Serre theory of fundamental groups of graph of groups, see [12] or [2], to construct a class of quasi-free algebras. Let $G = (V, E)$ be a finite graph with vertex-set V and edges E. A *G-graph of separable algebras* $\mathcal{S}_G$ is the assignment of separable ℓ-algebras S_v and S_e to any vertex $v \in V$ and to any edge $e \in E$ of G such that there are embeddings as ℓ-algebras

$$S_e \xhookrightarrow{i_{e,v}} S_v \quad \text{and} \quad S_e \xhookrightarrow{i_{e,w}} S_w \quad \text{whenever} \quad \textcircled{v} \overset{e}{\text{———}} \textcircled{w}$$

In order to define the *fundamental algebra* $\pi_1(\mathcal{S}_G)$ of a graph of separable algebras $\mathcal{S}_G$ we need to have algebra equivalents for *amalgamated groups* [12, §1.2] and of the *HNN construction* [12, §1.4]. If S is a separable ℓ-algebra and if A and A' are S-algebras, then the *coproduct* $A *_S A'$ is the algebra representing the functor

$$Hom_{S-\mathtt{alg}}(A,-) \times Hom_{S-\mathtt{alg}}(A',-)$$

in the category $S - \mathtt{alg}$ of S-algebras, see for example [10, Chp. 2] for its construction and properties. As for the HNN-counterpart, let $\alpha, \beta : S \hookrightarrow A$ be two ℓ-algebra embeddings of S in A and consider the algebra

$$A*_S^{\alpha,\beta} = \frac{A * \ell[t,t^{-1}]}{(\beta(s) - t^{-1}\alpha(s)t \ : \ \forall s \in S)}$$

Lemma 1 *Let S be a separable ℓ-algebra, A and A' quasi-free ℓ-algebras and embeddings $\alpha,\beta : S \hookrightarrow A$ and $S \hookrightarrow A'$. Then, the ℓ-algebras*

$$A *_S A' \qquad \textit{and} \qquad A*_S^{\alpha,\beta}$$

are again quasi-free ℓ-algebras.

Proof. The crucial property to use can be deduced from [1, Prop. 6.1.2] and asserts that any two ℓ-algebra morphisms $\phi, \psi : S \longrightarrow B$ such that $\overline{\phi}$ and $\overline{\psi} : S \longrightarrow B/I$ (where I is a nilpotent ideal of B) are conjugate via a unit $\overline{b}$ in B/I then there is a unit $b \in B$ mapping to $\overline{b}$ and such that b conjugates ϕ to ψ.

A morphism $A *_S A' \xrightarrow{g} B/I$ is fully determined by morphisms $A \xrightarrow{f} B/I$ and $A' \xrightarrow{f'} B/I$ such that $f|S = f'|S$. As A and A' are quasi-free one has ℓ-algebra lifts $\tilde{f} : A \longrightarrow B$ and $\tilde{f'} : A' \longrightarrow B$ whence two morphisms on S which have to be conjugated by an $b \in B^*$ such that $\overline{b} = 1_{B/I}$, that is $f'(s) = b^{-1}f(s)b$ for all $s \in S$. But then, we have a lift $A *_S A' \longrightarrow B$ determined by the morphisms $b^{-1}fb$ and f'.

A morphism $A *_S^{\alpha,\beta} \xrightarrow{g} B/I$ determines (and is determined by) a morphism $A \xrightarrow{f} B/I$ and a unit $\overline{b} = g(t)$ such that $f \circ \alpha$ and $f \circ \beta : S \longrightarrow B/I$ are conjugated via $\overline{b}$. Because A is quasi-free we have a lift $\tilde{f} : A \longrightarrow B$ and algebra maps $\tilde{f} \circ \alpha$ and $\tilde{f} \circ \beta : S \longrightarrow B$ which reduce to $\overline{b}$ conjugate morphisms. But then there is a unit $b \in B^*$ conjugating $\tilde{f} \circ \alpha$ to $\tilde{f} \circ \beta$ and mapping t to b produces the required lift $A *_S^{\alpha,\beta} \longrightarrow B$. □

The construction of the fundamental algebra $\pi_1(\mathcal{S}_G)$ can now be continued as in the case for graphs of groups. Let T be a maximal subtree of G and construct $\pi_1(\mathcal{S}_T)$ by induction on the number of edges in T while extending the formalism of graphs of seperable algebras to graphs of quasi-free algebras with edge-algebras separable. If v is a leaf vertex with edge e and other vertex w, we go to a smaller tree T' by dropping v and e, keep the same algebras on all remaining edges and all vertices different from w and defining as the new w-vertex algebra the coproduct $S_v *_{S_e} S_w$. Observe that for every vertex $u \in V$ we have a canonical embedding $i_u : S_u \longrightarrow \pi_1(\mathcal{S}_T)$.

We will use these embeddings to go from $\pi_1(\mathcal{S}_T)$ to $\pi_1(\mathcal{S}_G)$ by induction on the number of edges in G which are not in the maximal subtree T. At each step, we will have constructed an algebra A with embeddings $i_u : S_u \hookrightarrow A$. For the next edge e connecting vertices v and w we have two embeddings

$$\alpha_e : S_e \hookrightarrow S_v \xhookrightarrow{i_v} A \qquad \text{and} \qquad \beta_e : S_e \hookrightarrow S_w \xhookrightarrow{i_w} A$$

and will delete the edge e and replace A by the HNN-construction $A*_{S_e}^{\alpha_e,\beta_e}$. From this construction and the previous lemma we deduce

Theorem 1 *For any graph of separable ℓ-algebras $\mathcal{S}_G$ the associated fundamental algebra $\pi_1(\mathcal{S}_G)$ is a quasi-free ℓ-algebra.*

2 Quasi-free group algebras

The classification of quasi-free ℓ-algebras is way out of reach at the moment so it is important to have partial classifications. In [1, §6] the finite dimensional ℓ-algebras were shown to be the hereditary finite dimensional ℓ-algebras. In this section we will classify the quasi-free group algebras ℓH for H a finitely generated group. The desired answer is that these are precisely the ℓH with H a *virtually free group* (that is, H has a free subgroup of finite index) but we have to take precautions depending on the characteristic of ℓ.

If $\mathcal{G}_G$ is a graph of *finite groups* as in [12] such that all orders are invertible in ℓ, then we can associate to it a graph of separable ℓ-algebras $\mathcal{S}_G$ by taking

$$S_v = \ell G_v \quad \forall v \in V \qquad \text{and} \qquad S_e = \ell G_e \quad \forall e \in E$$

with embeddings determined by the group-embeddings. If $\pi_1(\mathcal{G}_G)$ is the *fundamental group* of $\mathcal{G}_G$ as in [12, §5.1] then the point of the construction in the previous section is that

$$\ell\pi_1(\mathcal{G}_G) \simeq \pi_1(\mathcal{S}_G)$$

and hence these group algebras are quasi-free ℓ-algebras. The connection with virtually free groups is provided by a result of Karrass, see for example [14, Thm. 3.5]. The following statements are equivalent for a finitely generated group H

- $H = \pi_1(\mathcal{G}_G)$ for a graph of finite groups.
- H is a virtually free group.

For example, all congruence subgroups in the modular group $SL_2(\mathbb{Z})$ are virtually free. On the other hand, the third braid group $B_3 = \langle s,t \mid s^2 = t^3 \rangle$ is not virtually free. Note that very little is known about simple representations of congruence subgroups. For some low dimensional classifications of $SL_2(\mathbb{Z})$-representations see [13]. Our approach to this problem uses non-commutative geometry and therefore the next result is crucial.

Theorem 2 *The following statements are equivalent :*

1. *The group algebra ℓH is a quasi-free ℓ-algebra.*
2. *H is a virtually free group such that in a description $H = \pi_1(\mathcal{G}_G)$ all orders of the vertex groups G_v are finite and invertible in ℓ.*

Proof. If ℓH is a quasi-free ℓ-algebra, it has to be hereditary by [1, Prop. 5.1] and hence, in particular, its augmentation ideal ω_H mast be a projective left ℓH-module. By a result of Dunwoody, see [2, Thm. IV.2.12] this is equivalent to H being the fundamental group of a graph of finite groups $\mathcal{G}_G$ such that all vertex-group orders are invertible in ℓ, whence $\mathbf{2}$ follows. The converse implication follows from the discussion preceding the statement and the last section. □

3 The zero quiver $Q_0(\mathcal{S}_G)$ of $\pi_1(\mathcal{S}_G)$

In this section we will assume that $\ell = \overline{\ell}$ is algebraically closed and fix a graph $\mathcal{S}_G$ of separable ℓ-algebras. One of the guiding principles of non-commutative geometry as in [1], [4] or [5] is that quasi-free algebras are the coordinate rings of affine smooth non-commutative varieties and that path algebras of quivers correspond to tangent spaces to these non-commutative manifolds. In [7] a procedure was given to assign to any quasi-free $\overline{\ell}$-algebra A a quiver-setting (Q_A, α_A), consisting of a quiver Q_A and a dimension-vector α_A, encoding all the relevant information to study finite dimensional representations of A. In this section we will calculate the quiver-setting corresponding to the quasi-free algebra $\pi_1(\mathcal{S}_G)$. As an intermediary step we will construct a finite quiver $Q_0(\mathcal{S}_G)$ such that finite dimensional representations of $\pi_1(\mathcal{S}_G)$ correspond to certain finite dimensional representations of the path algebra $\overline{\ell}Q_0(\mathcal{S}_G)$.

As $\overline{\ell}$ is algebraically closed we have decomposition of the vertex- and edge-algebras

$$S_v = M_{d_v(1)}(\overline{\ell}) \oplus \ldots \oplus M_{d_v(n_v)}(\overline{\ell}) \quad \text{resp.} \quad S_e = M_{d_e(1)}(\overline{\ell}) \oplus \ldots \oplus M_{d_e(n_e)}(\overline{\ell})$$

The embeddings $S_e \hookrightarrow S_v$ are depicted via their Bratelli-diagrams or, equivalently, by natural numbers $a_{ij}^{(ev)}$ for $1 \leq i \leq n_e$ and $1 \leq j \leq n_v$ satisfying the numerical restrictions

$$d_v(j) = \sum_{i=1}^{n_e} a_{ij}^{(ev)} d_e(i) \qquad \text{for all } 1 \leq j \leq n_v \text{ and all } v \in V \text{ and } e \in E$$

remark that these numbers give the *restriction data*, that is, the multiplicities of the simple components of S_e occurring in the restriction $V_j^{(v)} \downarrow_{S_e}$ for the simple components V_j of S_v. From these decompositions and Schur's lemma it follows that for any edge $v \overset{e}{\text{———}} w$ in the graph G we define the numbers

$$Hom_{S_e}(V_i^{(v)}, V_j^{(w)}) = \sum_{k=1}^{n_e} a_{ki}^{(ev)} a_{kj}^{(ew)} = n_{ij}^{(e)}$$

We are now in a position to construct the quiver $Q_0(\mathcal{S}_G)$. As its vertices, for any vertex $v \in V$ of G take n_v vertices $\{\mu_1^{(v)}, \ldots, \mu_{n_v}^v\}$. Fix an orientation $\vec{G}$ on all of the edges of G. For any edge $v \overset{e}{\text{———}} w$ in G we add for each $1 \leq i \leq n_v$ and each $1 \leq j \leq n_w$ precisely $n_{ij}^{(e)}$ arrows between the vertices $\mu_i^{(v)}$ and $\mu_j^{(w)}$ oriented in the same way as the edge e in $\vec{G}$.

An *n-dimension vector* for $Q_0(\mathcal{S}_G)$ is a vector $\alpha = (\alpha_i^{(v)} : v \in V, 1 \leq i \leq n_v)$ of natural numbers satisfying the following numerical conditions

$$\sum_{i=1}^{n_v} d_v(i) \alpha_i^{(v)} = n \qquad \text{for all } v \in V$$

Recall that the *representation space* $rep_\alpha\, Q_0(\mathcal{S}_G)$ is the affine $\overline{\ell}$-space

$$rep_\alpha\, Q_0(\mathcal{S}_G) = \bigoplus_{v \xrightarrow{e} w} \oplus_{i=1}^{n_v} \oplus_{j=1}^{n_w} M_{\alpha_j^{(w)} \times \alpha_i^{(v)}}(\overline{\ell})$$

and two α-dimensional representations are said to be *isomorphic* if they are conjugated via the natural base-change action of $GL(\alpha) = \times_{v \in V} \times_{i=1}^{n} GL(\alpha_i^{(v)})$.

For any edge $ⓥ \xrightarrow{e} ⓦ$ we denote by Q_e the *bipartite* subquiver of $Q_0(\mathcal{S}_G)$ on the vertices $\{\mu_1^{(v)}, \ldots, \mu_{n_v}^{(v)}\}$, $\{\mu_1^{(w)}, \ldots, \mu_{n_w}^{(w)}\}$ and the $n_{ij}^{(e)}$ arrows between $\mu_i^{(v)}$ and $\mu_j^{(w)}$ determined by the embeddings $S_e \hookrightarrow S_v$ and $S_e \hookrightarrow S_w$.

We say that a representation $M \in rep_\alpha\ Q_0(\mathcal{S}_G)$ (for some n-dimension vector α) is e-(semi)stable if the restriction $M|Q_e$ is θ-(semi)stable (in the terminology of [3]) for $\theta = (-d_v(1), \ldots, -d_v(n_v), d_w(1), \ldots, d_w(n_w))$. That is, there is no proper Q_e-subrepresentation N of $M|Q_e$ of dimension vector $(n_1, \ldots, n_{n_v}, n'_1, \ldots, n'_{n_w})$ such that $\sum_{i=1}^{n_w} n'_i d_w(i) < \sum_{i=1}^{n_v} n_i d_v(i)$ for e-semistable and such that $\sum_{i=1}^{n_w} n'_i d_w(i) \leq \sum_{i=1}^{n_v} n_i d_v(i)$ for e-stable. We say that a representation $M \in rep_\alpha\ Q_0(\mathcal{S}_G)$ (for some n-dimension vector α) is $\mathcal{S}_G$-(semi)stable if M is e-(semi)stable for all edges $e \in E$. The relevance of the quiver $Q_0(\mathcal{S}_G)$ and the introduced terminology is contained in the following result.

Theorem 3 *Every n-dimensional representation $\pi_1(\mathcal{S}_G) \xrightarrow{\phi} M_n(\bar{\ell})$ determines (and is determined by) an $\mathcal{S}_G$-semistable representation $M_\phi \in rep_\alpha\ Q_0(\mathcal{S}_G)$ for some n-dimension vector α. Moreover, if ϕ and ϕ' are isomorphic representations of $\pi_1(\mathcal{S}_G)$, then M_ϕ and $M_{\phi'}$ are isomorphic as quiver representations.*

Proof. Let $N = \bar{\ell}_\phi^n$ be the n-dimensional module determined by ϕ. For each vertex $v \in V$ we have a decomposition by restricting N to the separable subalgebra S_v

$$N \downarrow_{S_v} \simeq V_{1,v}^{\oplus \alpha_1^{(v)}} \oplus \ldots \oplus V_{n_v,v}^{\oplus \alpha_{n_v}^{(v)}}$$

where the $V_{i,v}$ are the distinct simple modules of S_v of dimension $d_v(i)$. Choose an $\bar{\ell}$-basis $\mathcal{B}_v$ of $N \downarrow_{S_v}$ compatible with this decomposition. These decompositions determine an n-dimension vector α. For any edge $ⓥ \xrightarrow{e} ⓦ$ the embeddings $S_e \overset{\alpha}{\hookrightarrow} S_v$ and $S_e \overset{\beta}{\hookrightarrow} S_w$ determine two n-dimensional S_e-representations

$$(N \downarrow_{S_v}) \downarrow_{S_e}^{\alpha} \qquad \text{and} \qquad (N \downarrow_{S_w}) \downarrow_{S_e}^{\beta}$$

which, by construction of $\pi_1(\mathcal{S}_G)$ are isomorphic. That is, the basechange map $\mathcal{B}_v \xrightarrow{\psi_{vw}} \mathcal{B}_w$ is an invertible element of

$$Hom_{S_e}(N \downarrow_{S_v}, N \downarrow_{S_w}) = \oplus_{i=1}^{n_v} \oplus_{j=1}^{n_w} M_{\alpha_j^{(w)} \times \alpha_i^{(v)}}(Hom_{S_e}(V_{i,v}, V_{j,w}))$$

and hence ψ_{vw} determines a representation of the bipartite quiver Q_e of dimension vector $\alpha|Q_e$. Repeating this for all edges $e \in E$ we obtain a representation $M_\phi \in rep_\alpha\ Q_0(\mathcal{S}_G)$. Invertibility of the mapψ_{vw} is equivalent to M_ϕ being e-semistable, so M_ϕ is $\mathcal{S}_G$-semistable. Isomorphic representations ϕ and ϕ' determine isomorphic vertex-decompositions whence, by Schur's lemma, bases which are transferred into each other via an element of $GL(\alpha)$ and hence the quiver representations M_ϕ and $M_{\phi'}$ are isomorphic. From the construction of the fundamental algebra $\pi_1(\mathcal{S}_G)$ it follows that one can reverse this procedure to construct on n-dimensional representation of $\pi_1(\mathcal{S}_G)$ from a $\mathcal{S}_G$-stable representation $M \in rep_\alpha\ Q_0(\mathcal{S}_G)$ for some n-dimension vector α. $\square$

Under this correspondence simple $\pi_1(\mathcal{S}_G)$-representations correspond to $\mathcal{S}_G$-stable representations. If α is an n-dimension vector such that $rep_\alpha\, Q_0(\mathcal{S}_G)$ contains $\mathcal{S}_G$-stable representations (which then form a Zariski open subset), then α is a *Schur root* of $Q_0(\mathcal{S}_G)$ and consequently the dimension of the classifying variety is equal to $1-\chi(\alpha,\alpha)$ where χ is the *Euler form* of the quiver $Q_0(\mathcal{S}_G$, that is, the bilinear form on $\mathbb{Z}^l$ (where l is the number of vertices in $Q_0(\mathcal{S}_G)$ defined by

$$\chi(\epsilon_i,\epsilon_j) = \delta_{ij} - \#\{ (i) \longrightarrow (j) \}$$

for the vertex-dimensions $\epsilon_i = (\delta_{ij})_j$. For this result and related material on Schur roots we refer to [11]. We deduce from this

Theorem 4 *Isomorphism classes of simple n-dimensional representations of $\pi_1(\mathcal{S}_G)$ are parametrized by the points of a smooth quasi-affine variety (possibly with several irreducible components)*

$$simp_n\, \pi_1(\mathcal{S}_G) = \bigsqcup_\alpha simp_\alpha\, \pi_1(\mathcal{S}_G)$$

where α runs over all n-dimension vectors such that $rep_\alpha\, Q_0(\mathcal{S}_G)$ contains $\mathcal{S}_G$-stable representations. These components have dimensions

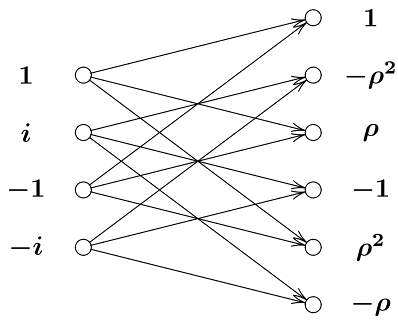
$$dim\ simp_\alpha\, \pi_1(\mathcal{S}_G) = 1-\chi(\alpha,\alpha)$$

where χ is the Euler form of the quiver $Q_0(\mathcal{S}_G)$.

As an example consider the modular group $SL_2(\mathbb{Z})$ which is the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, see for example [2, I §7]. If $char(\overline{\ell}) \neq 2,3$ the group-algebra $\overline{\ell}SL_2(\mathbb{Z})$ is the fundamental algebra of the graph of separable $\overline{\ell}$-algebras

$$(v) \xrightarrow{e} (w) \qquad \text{with} \qquad S_v = \overline{\ell}\mathbb{Z}_4 \qquad S_w = \overline{\ell}\mathbb{Z}_6 \qquad S_e = \overline{\ell}\mathbb{Z}_2$$

As all simples are one-dimensional (determined by their eigenvalue), it is easy to verify that the zero quiver $Q_0(\overline{\ell}SL_2(\mathbb{Z}))$ has the following form



(ρ is a primitive 3rd root of unity) which is the disjoint union of two copies of the quiver associated to $PSL_2(\mathbb{Z})$ in [15].

4 The one quiver $Q_1(\mathcal{S}_G)$ of $\pi_1(\mathcal{S}_G)$

The étale GL_n-local structure of the representation schemes $rep_n\, \pi_1(\mathcal{S}_G)$ of n-dimensional representations near the orbit of a semi-simple representation is isomorphic to that of some semi-simple orbit in $rep_n\, A(\mathcal{S}_G)$ where $A(\mathcal{S}_G)$ is a quasi-free

$\overline{\ell}$-algebra Morita equivalent to the path algebra $\overline{\ell}Q_1(\mathcal{S}_G)$ of the so-called *one quiver* $Q_1(\mathcal{S}_G)$ which we will now describe.

By the foregoing theorem, the $rep_n \pi_1(\mathcal{S}_G)$ are smooth varieties having as many irreducible components as there are n-dimension vectors α of $Q_0(\mathcal{S}_G)$ such that there exists $\mathcal{S}_G$-semistable representations in $rep_\alpha\ Q_0(\mathcal{S}_G)$. All such dimension vectors α form a sub-semigroup of the semigroup of all dimension vectors of $Q_0(\mathcal{S}_G)$ which we denote with $comp\ \pi_1(\mathcal{S}_G)$ and call the *component semigroup* (see [9]) of $\pi_1(\mathcal{S}_G)$. Let $\{\alpha_1, \ldots, \alpha_k\}$ be the minimal set of semigroup generators of $comp\ \pi_1(\mathcal{S}_G)$ (observe that this is precisely the set of irreducible components $rep_\alpha\ \pi_1(\mathcal{S}_G)$ of some $rep_n\ \pi_1(\mathcal{S}_G)$ on which the basechange action by GL_n is a free PGL_n-action).

The *one quiver* $Q_1(\mathcal{S}_G)$ has k vertices $\{\nu_1, \ldots, \nu_k\}$ where ν_i corresponds to the semigroup generator α_i. In $Q_1(\mathcal{S}_G)$ the number of directed arrows between μ_i and μ_j is given by

$$\# \{ (i) \longrightarrow (j) \} = \delta_{ij} - \chi(\alpha_i, \alpha_j)$$

where χ is (as before) the Euler-form for the zero quiver $Q_0(\mathcal{S}_G)$. A first application of $Q_1(\mathcal{S}_G)$ to the representation theory of $\pi_1(\mathcal{S}_G)$ is that it allows us to compute the components $rep_\alpha\ \pi_1(\mathcal{S}_G)$ which contain simple representations.

Theorem 5 *If $\alpha = c_1\alpha_1 + \ldots + c_k\alpha_k \in comp\ \pi_1(\mathcal{S}_G)$ then the component $rep_\alpha\ \pi_1(\mathcal{S}_G)$ contains simple representations if and only if*

$$\chi_1(\gamma, \epsilon_i) \leq 0 \qquad \textit{and} \qquad \chi_1(\epsilon_i, \gamma) \leq 0$$

for all $1 \leq i \leq k$ where $\gamma = (c_1, \ldots, c_k)$ and $\epsilon_i = (\delta_{1i}, \ldots, \delta_{ki})$ and where χ_1 is the Euler form of the one quiver $Q_1(\mathcal{S}_G)$.

Proof. This is [7, Thm. 2] adapted to the situation at hand. □

If $char(\overline{\ell}) = 0$ one can apply Luna slice machinery to construct a Zariski open subset of all simple representations in $rep_\alpha\ \pi_1(\mathcal{S}_G)$ from the knowledge of low-dimensional simples. For example, suppose we have found simple representations

$$S_i \in rep_{\alpha_i}\ \pi_1(\mathcal{S}_G) \qquad \text{for all } 1 \leq i \leq k$$

and consider the point M in the affine space $rep_\alpha\ Q_0(\mathcal{S}_G)$ determined by the semi-simple representation of $\pi_1(\mathcal{S}_G)$

$$M = S_1^{\oplus c_1} \oplus \ldots \oplus S_k^{\oplus c_k}$$

then the normal space to the $GL(\alpha)$-orbit $\mathcal{O}(M)$ is isomorphic to $Ext^1_{\pi_1(\mathcal{S}_G)}(M, M)$ which can be identified to $rep_\gamma\ Q_1(\mathcal{S}_G)$ (again by the results from [7]).

Theorem 6 *Let $\alpha = c_1\alpha_1 + \ldots + c_k\alpha_k$ be a component such that $rep_\alpha\ \pi_1(\mathcal{S}_G)$ contains simple representations. In the affine space $rep_\alpha Q_0(\mathcal{S}_G)$ identify the normal space to the orbit $\mathcal{O}(M)$ of the semi-simple representation M (as above) with*

$$N_M = \{ M + V \mid V \in rep_\gamma\ Q_1(\mathcal{S}_G)\ \}$$

where $\gamma = (c_1, \ldots, c_k)$. Then, $GL(\alpha).N_M$ contains a Zariski open subset of all α-dimensional simple representations of $\pi_1(\mathcal{S}_G)$.

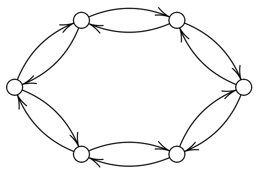
Proof. This is a special case of the Luna slice theorem, see for example [6] for more details. □

In fact, one can generalize this result to other known semi-simple representations N of $\pi_1(\mathcal{S}_G)$ but then one has to replace $Q_1(\mathcal{S}_G)$ by the *local quiver* Q_N of N, the structure of which can be entirely deduced from the one quiver $Q_1(\mathcal{S}_G)$, see [7] for more details.

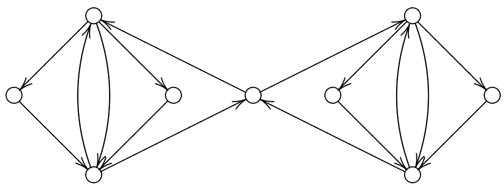
In the $SL_2(\mathbb{Z})$ example, $comp\ \overline{\ell}SL_2(\mathbb{Z})$ is generated by the 12 components of two-dimensional representations of $Q_0(\overline{\ell}SL_2(\mathbb{Z}))$

$$\nu_{ij} = (\delta_{1i}, \ldots, \delta_{4i}; \delta_{1j}, \ldots, \delta_{6j}) \qquad 1 \leq i \leq 4, 1 \leq j \leq 6$$

From this the structure of the one quiver $Q_1(\overline{\ell}SL_2(\mathbb{Z}))$ (corresponding to the 12 one-dimensional simples of $\overline{\ell}SL_2(\mathbb{Z})$) can be verified to be



Here, the vertices of the first component correspond (in cyclic order) to $\nu_{11}, \nu_{33}, \nu_{15}, \nu_{31}, \nu_{13}, \nu_{35}$ and those of the second component (in cyclic order) to $\nu_{22}, \nu_{44}, \nu_{26}, \nu_{42}, \nu_{24}, \nu_{46}$. Applications to the representation theory of the modular group $SL_2(\mathbb{Z})$ and its central extension B_3 (the third braid group) will be given elsewhere.



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