Qurves and Quivers

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Abstract

In this paper we associate to an $\overline{\ell}$-*qurve* A (formerly known as a quasi-free algebra [3] or formally smooth algebra [7]) the *one-quiver* $Q_1(A)$ and dimension vector $\alpha_1(A)$. This pair contains enough information to reconstruct for all $n \in \mathbb{N}$ the GL_n-étale local structure of the representation scheme $rep_n\ A$. In an appendix we indicate how one might extend this to qurves over non-algebraically closed fields. Further, we classify all finitely generated groups G such that the group algebra ℓG is an ℓ-qurve. If $char(\ell) = 0$ these are exactly the virtually free groups. We determine the one-quiver setting in this case and indicate how it can be used to study the finite dimensional representations of virtually free groups. As this approach also applies to *fundamental algebras* of *graphs of separable ℓ-algebras*, we state the results in this more general setting.

1 Qurves

In this paper, ℓ is a commutative field with algebraic closure $\overline{\ell}$. Algebras will be associative ℓ-algebras with unit and (usually) finitely generated over ℓ. For an ℓ-algebra A let A' be the ℓ-vectorspace $A/\ell.1$ and define (following [3, §1]) the graded algebra of *non-commutative differential forms*

$$\Omega A = \oplus_{i=o}^{\infty} \Omega^i A \qquad \text{with} \qquad \Omega^i\ A = A \otimes A'^{\otimes i}$$

with multiplication defined by the maps $\Omega^n A \otimes \Omega^{k-1} A \longrightarrow \Omega^{n+k-1} A$ where

$$(a_0, \ldots, a_n).(a_{n+1}, \ldots, a_{n+k}) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+k})$$

As $\Omega^0 A = A$ this multiplication defines an A-bimodule structure on each $\Omega^n A$ and one proves [3, Prop. 2.3] that $\Omega A = T_A(\Omega^1 A)$ the tensor algebra of the A-bimodule $\Omega^1 A$. Remark that the standard assumption of [3] is that $\ell = \mathbb{C}$ the field of complex numbers. However, with minor modifications most results remain valid over an arbitrary basefield and we will refer to statements in [3] whenever the argument can be repeated verbatim.

Definition 1 *A finitely generated ℓ-algebra A is said to be an* ℓ-qurve *(or* quasi-free *[3] or* formally smooth *[7]) if either of the following two equivalent conditions is satisfied*

- *The universal bimodule $\Omega^1_\ell(A)$ of derivations is a projective A-bimodule.*

- *A satisfies the lifting property modulo nilpotent ideals in $\ell - alg$, the category of ℓ-algebras.*

Whereas the lifting property extends Grothendieck's characterization of commutative regular algebras (see for example [6]) to the non-commutative setting, such algebras are known to be hereditary *by [3, Prop. 5.1] and hence they behave* quite *like* curves*.*

Recall that a finite dimensional ℓ-algebra S is said to be *separable* if and only if S is the direct sum of simple algebras each of which has a center which is a separable field extension of ℓ. For example, the group algebra ℓG of a finite group G is separable if and only if the order of G is a unit in ℓ. Separable ℓ-algebras are known to be ℓ-qurves by [3, §4] but should be thought of as corresponding to *points*. In fact, they are characterized by either of the following two equivalent conditions

- A is a projective A-bimodule.

- A satisfies the *conjugate* lifting property modulo nilpotent ideals in $\ell - alg$.

That is, if $I \triangleleft B$ is a nilpotent ideal and if $\overline{\phi}, \overline{\psi} : S \rightrightarrows B/I$ are two ℓ-algebra morphisms which are conjugated by a unit $\overline{b} \in B/I$ then there exist algebra lifts $\phi, \psi : S \rightrightarrows B$ and a unit $b \in B$ (mapping to $\overline{b}$) conjugating ϕ and ψ, see [3, Prop. 6.1.2].

Genuine examples of ℓ-qurves are : the free algebra $\ell\langle x_1, \ldots, x_m\rangle$, the path algebra ℓQ of a finite quiver Q and the coordinate ring $\ell[C]$ of a smooth affine commutative curve C. From these more complicated examples are construed by universal constructions such as taking algebra free products $A * A'$ or universal localizations A_Σ. In the next section we will introduce a new class of ℓ-qurve examples.

For an ℓ-algebra A recall that the *representation scheme* $rep_n\ A$ is the affine ℓ-scheme representing the functor

$$\ell - commalg \longrightarrow sets \qquad \text{defined by} \qquad C \mapsto Hom_{\ell-alg}(A, M_n(C))$$

where $\ell - commalg$ is the category of commutative ℓ-algebras. A major motivation for studying ℓ-qurves comes from the result mentioned in [8], [7] and proved in [11, (2.2)].

Proposition 1 *If A is a ℓ-qurve, then all representation schemes $rep_n\ A$ are smooth affine varieties (possibly having several connected components).*

2 Qurves from graphs

In this section we will imitate the Bass-Serre theory of the fundamental group of a graph of groups, see [19] or [4], to construct a large class of examples of ℓ-qurves.

Definition 2 *Let $G = (V, E)$ be a finite graph with vertex-set V and edges E. A* G-graph of ℓ-qurves $\mathcal{Q}_G$ *is the assignment of*

- *An ℓ-qurve A_v to every vertex $v \in V$.*

- *A separable ℓ-algebra S_e to every edge $e \in E$.*

- *Inclusions of ℓ-algebras*

$$S_e \overset{i_{e,v}}{\hookrightarrow} A_v \quad \text{and} \quad S_e \overset{i_{e,w}}{\hookrightarrow} A_w \quad \text{for every edge} \quad \textcircled{v} \overset{e}{\text{———}} \textcircled{w}$$

*If, moreover, all vertex-algebras are separable algebras S_v we call this data a G-*graph of separable algebras *and denote it by $\mathcal{S}_G$.*

In order to construct the *fundamental algebra* $\pi_1(\mathcal{Q}_G)$ of a G-graph of qurves $\mathcal{Q}_G$ we need to have ℓ-algebra equivalents for the notions of *amalgamated group products* [19, §1.2] and of the *HNN construction* [19, §1.4]. If S is a separable ℓ-algebra and if A and A' are S-algebras, then the *coproduct* $A *_S A'$ is the algebra representing the functor

$$Hom_{S-\mathtt{alg}}(A,-) \times Hom_{S-\mathtt{alg}}(A',-)$$

in the category $S-alg$ of S-algebras, see for example [17, Chp. 2] for its construction and properties. As for the HNN-construction, let $\alpha, \beta : S \hookrightarrow A$ be two ℓ-algebra embeddings of S in A, consider the algebra

$$A *_S^{\alpha,\beta} = \frac{A * \ell[t,t^{-1}]}{(\beta(s) - t^{-1}\alpha(s)t \ : \ \forall s \in S)}$$

Lemma 1 *Let S be a separable ℓ-algebra, A and A' ℓ-qurves and ℓ-embeddings $\alpha, \beta : S \hookrightarrow A$ and $S \hookrightarrow A'$. Then, the ℓ-algebras*

$$A *_S A' \qquad \text{and} \qquad A *_S^{\alpha,\beta}$$

are again ℓ-qurves.

Proof. Our edge-algebras need to be separable ℓ-algebras because we will need the conjugate lifting property modulo nilpotent ideals.

A morphism $A *_S A' \xrightarrow{g} B/I$ is fully determined by morphisms $A \xrightarrow{f} B/I$ and $A' \xrightarrow{f'} B/I$ such that $f|S = f'|S$. As A and A' are quasi-free one has ℓ-algebra lifts $\tilde{f} : A \longrightarrow B$ and $\tilde{f}' : A' \longrightarrow B$ whence two morphisms on S which have to be conjugated by an $b \in B^*$ such that $\overline{b} = 1_{B/I}$, that is $f'(s) = b^{-1}f(s)b$ for all $s \in S$. But then, we have a lift $A *_S A' \longrightarrow B$ determined by the morphisms $b^{-1}fb$ and f'.

A morphism $A *_S^{\alpha,\beta} \xrightarrow{g} B/I$ determines (and is determined by) a morphism $A \xrightarrow{f} B/I$ and a unit $\overline{b} = g(t)$ such that $f \circ \alpha$ and $f \circ \beta : S \longrightarrow B/I$ are conjugated via $\overline{b}$. Because A is quasi-free we have a lift $\tilde{f} : A \longrightarrow B$ and algebra maps $\tilde{f} \circ \alpha$ and $\tilde{f} \circ \beta : S \longrightarrow B$ which reduce to $\overline{b}$ conjugate morphisms. But then there is a unit $b \in B^*$ conjugating $\tilde{f} \circ \alpha$ to $\tilde{f} \circ \beta$ and mapping t to b produces the required lift $A *_S^{\alpha,\beta} \longrightarrow B$. $\square$

However, as often with universal constructions, we have to take care not to end up with the trivial algebra! Because S is semi-simple and A and A' are faithful S-algebras it follows from [17, Chp. 2] that there are inclusions $A \hookrightarrow A *_S A'$ and $A' \hookrightarrow A *_S A'$. To prove that $A \hookrightarrow A *_S^{\alpha,\beta}$ we give another description of the HNN-construction mimicking [19, §1.4]. For any $n \in \mathbb{Z}$ take $A[n] \simeq A$ and construct the following amalgamated products

$$A_0 = A, \quad A_1 = A[-1] *_S A_0 *_S A[1], \ \dots \ A_k = A[-k] *_S A_{k-1} *_S A[k]$$

with respect to the following embeddings

$$\begin{tikzcd}[column sep=small]
A[-1] & & A[0] & & A[1] & & A[2] \\
& S \arrow[ul, "\beta"] \arrow[ur, "\alpha"'] & & S \arrow[ul, "\beta"] \arrow[ur, "\alpha"'] & & S \arrow[ul, "\beta"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

As S is semi-simple we have by [17, Chp. 2] embeddings $A_0 \subset A_1 \subset A_2 \subset \dots$ and hence A embeds in the limit $\tilde{A} = \underrightarrow{lim} A_n$. The shift-identity

$$\dots \longrightarrow A[k-1] \xrightarrow{id} A[k] \xrightarrow{id} A[k+1] \longrightarrow \dots$$

induces an automorphism ϕ on $\tilde{A}$ and as the two algebras below have the same universal property they are isomorphic

$$A *_S^{\alpha,\beta} \simeq \tilde{A}[t, t^{-1}, \phi] \qquad \text{whence} \qquad A \hookrightarrow A *_S^{\alpha,\beta}$$

Definition 3 *Let $\mathcal{Q}_G$ be a graph of ℓ-qurves and let T be a maximal subtree of G. We construct the ℓ-algebra A_T by induction on the number t of edges in T. If $t = 0$ so $V = \{v\}$ then $A_T = A_v$. If $t > 0$, consider a leaf vertex v with connecting edge $v \stackrel{e}{-\!\!-\!\!-\!\!-} w$ in T. Construct a new tree T' on $t-1$ edges by dropping the vertex v and edge e and construct a new graph of ℓ-qurves $\mathcal{Q'}_{T'}$ by*

$$A'_w = A_v *_{A_e} A_w, \quad A'_u = A_u \quad \text{for } v \neq u \in V, \quad A'_f = A_f \text{ for } e \neq f \in E$$

then $A_T \simeq A_{T'}$. Observe that there are embeddings $S_u \overset{i_u}{\hookrightarrow} A_T$ for every $u \in V$.

Let $G - T = \{e_1, \dots, e_r\}$ and take $A_0 \simeq A_T$. For every edge $v \stackrel{e_i}{-\!\!-\!\!-\!\!-} w$ in $G - T$ there are two embeddings

$$\alpha_i \ : \ S_e \overset{i_{e_i,v}}{\hookrightarrow} S_v \overset{i_v}{\hookrightarrow} A_{i-1} \quad \text{and} \quad \beta_i \ : \ S_e \overset{i_{e_i,w}}{\hookrightarrow} S_w \overset{i_w}{\hookrightarrow} A_{i-1}$$

and we define

$$A_i \simeq A_{i-1} *_{S_e}^{\alpha_i,\beta_i}$$

The algebra A_r is then called the fundamental algebra of the graph of ℓ-qurves $\mathcal{Q}_G$ *and is denoted by $\pi_1(\mathcal{Q}_G)$.*

Theorem 1 *If $\mathcal{Q}_G$ is a graph of ℓ-qurves, the fundamental algebra $\pi_1(\mathcal{Q}_G)$ is again an ℓ-qurve.*

Proof. Immediate from the construction and lemma 1. □

3 Qurve group algebras

The classification of ℓ-qurves is way out of reach at the moment so it is important to have partial classifications. In [3, §6] the finite dimensional ℓ-qurves were shown to be the hereditary finite dimensional ℓ-algebras (and hence Morita equivalent to path algebras ℓQ of a finite quiver Q without oriented cycles). In this section we will classify the group algebras ℓH for H a finitely generated group which are ℓ-qurves. The desired answer is that these are precisely the ℓH with H a *virtually free group* (that is, H has a free subgroup of finite index) but we have to take the characteristic of ℓ into account (observe that finite groups are virtually free).

If $\mathcal{G}_G$ is a graph of *finite groups* as in [19] such that all orders are invertible in ℓ, then we can associate to it a graph of separable ℓ-algebras $\mathcal{S}_G$ by taking

$$S_v = \ell G_v \quad \forall v \in V \qquad \text{and} \qquad S_e = \ell G_e \quad \forall e \in E$$

with embeddings determined by the group-embeddings. If $\pi_1(\mathcal{G}_G)$ is the *fundamental group* of $\mathcal{G}_G$ as in [19, §5.1] then the point of the construction in the previous section is that

$$\ell\pi_1(\mathcal{G}_G) \simeq \pi_1(\mathcal{S}_G)$$

and hence these group algebras are ℓ-qurves. The connection with virtually free groups is provided by a result of Karrass, see for example [21, Thm. 3.5]. The following statements are equivalent for a finitely generated group H

- $H = \pi_1(\mathcal{G}_G)$ for a graph of finite groups.
- H is a virtually free group.

For example, all congruence subgroups in the modular group $SL_2(\mathbb{Z})$ are virtually free. On the other hand, the third braid group $B_3 = \langle s, t \mid s^2 = t^3 \rangle$ is not virtually free. Note that very little is known about simple representations of congruence subgroups. For some low dimensional classifications of $SL_2(\mathbb{Z})$-representations see [20].

Theorem 2 *The following statements are equivalent for a finitely generated group H:*

1. *The group algebra ℓH is an ℓ-qurve.*
2. *H is a virtually free group such that in a description $H = \pi_1(\mathcal{G}_G)$ all orders of the vertex groups G_v are finite and invertible in ℓ.*

Proof. If ℓH is a quasi-free ℓ-algebra, it has to be hereditary by [3, Prop. 5.1] and hence, in particular, its augmentation ideal ω_H mast be a projective left ℓH-module. By a result of Dunwoody, see [4, Thm. IV.2.12] this is equivalent to H being the fundamental group of a graph of finite groups $\mathcal{G}_G$ such that all vertex-group orders are invertible in ℓ, whence (2) follows. The converse implication follows from the discussion preceding the statement and the last section. $\square$

If $char(\ell) = 0$ it follows from this and proposition 1 that all representation schemes $rep_n\ \ell H$ are smooth affine varieties whenever H is a finitely generated virtually free group.

4 The component semigroup

From now on we will assume that $\ell = \overline{\ell}$ is algebraically closed. In the appendix we will replace the component semigroup by a component co-algebra over an arbitrary basefield ℓ. If A is an $\overline{\ell}$-qurve we know from proposition 1 that all representation schemes are smooth affine varieties.

Definition 4 *For an $\overline{\ell}$-qurve A the smooth variety $rep_n\ A$ decomposes into connected (equivalently, irreducible) components*

$$rep_n\ A = \bigsqcup_{|\alpha|=n} rep_\alpha\ A$$

where α is a label. We call α a dimension vector *of* total dimension $|\alpha| = n$.

An $\overline{\ell}$-point of $rep_n\ A$ is an n-dimensional left A-module and the direct sum of modules defines the *sum maps*

$$rep_n\ A \times rep_m\ A \longrightarrow rep_{n+m}\ A$$

If we decompose these varieties into their connected components and use the fact that the image of two connected varieties is again connected, we can define a semigroup.

Definition 5 *The* component semigroup $comp(A)$ *is the set of all dimension vectors α equipped with the addition $\alpha + \beta = \gamma$ where γ determines the unique component $rep_\gamma\ A$ of $rep_{n+m}\ A$ containing the image of $rep_\alpha\ A \times rep_\beta\ A$ under the sum map*

$$\bigsqcup_{|\alpha|=n} rep_\alpha\ A \times \bigsqcup_{|\beta|=m} rep_\beta\ A \longrightarrow \bigsqcup_{|\gamma|=n+m} rep_\gamma\ A$$

$comp(A)$ is a commutative semigroup with an augmentation map $comp(A) \longrightarrow \mathbb{N}$ sending a dimension vector α to its total dimension $|\alpha|$.

Here are some examples :

- For $A = M_{n_1}(\overline{\ell}) \oplus \ldots \oplus M_{n_k}(\overline{\ell})$ semi-simple, $comp(A) = (\mathbb{N}n_1, \ldots, \mathbb{N}n_k) \subset \mathbb{N}^k$.
- For $A = \overline{\ell}Q$ a path algebra we have $comp(A) = \mathbb{N}^k$ where k is the number of vertices of the quiver Q.
- For a direct sum $A = A_1 \oplus A_2$ we have $comp(A) = comp(A_1) \oplus comp(A_2)$.
- For a free algebra product $A = A_1 * A_2$ we have that $comp(A_1)$ is the fibered product (using the augmentation) $comp(A_1) \times_{\mathbb{N}} comp(A_2)$, see [14, Prop. 1].

In [14, Question 2] K. Morrison asked whether $comp(A)$ is always a free Abelian semigroup (as in the examples above). However, even for A an $\overline{\ell}$-qurve, reality is more complex as one can remove components by the process of universal localization (see for example [17] for definition and properties of universal localization).

Proposition 2 *For every sub semigroup $S \subset \mathbb{N}$, there is an $\overline{\ell}$-qurve A with*

$$comp(A) = S$$

as augmented semigroups.

Proof. Suppose first that $gcd(S) = 1$, that is the elements of S are coprime. By using results on polynomial- and rational identities of matrices (see for example [16]) it was proved in [10] that there is an affine ℓ-algebra with presentation

$$A = \frac{\overline{\ell}\langle x_1, \ldots, x_a, y_1, \ldots, y_b \rangle}{(1 - y_i p_i(x_1, \ldots, x_a, y_1, \ldots, y_{i-1}) \ : \ 1 \leq i \leq b)}$$

(with each of the $p_i \in \ell\langle x_1, \ldots, x_a, y_1, \ldots, y_{i-1}\rangle$) having the property that A has finite dimensional representations of dimensions exactly the elements of S. A is a universal localization of $\overline{\ell}\langle x_1, \ldots, x_a\rangle$ and hence is an $\overline{\ell}$-qurve (for example use [17, Thm. 10.6] to prove that $\Omega^1(A)$ is a projective A-bimodule). As such, for every n, $rep_n\ A$ is a Zariski open subset (possibly empty) of $rep_n\ \overline{\ell}\langle x_1, \ldots, x_a\rangle = M_n(\overline{\ell})^{\times a}$ and is therefore irreducible (when non-empty). Therefore, $comp(A) = S \subset \mathbb{N}$ and consists precisely of those $n \in \mathbb{N}$ for which none of the p_i (when expressed as a rational non-commutative function in $x_1, \ldots, x_a$) is a rational identity for $n \times n$ matrices.

For the general case, assume that $gcd(S) = m$ and take $S' = S/m$ with associated algebra (as above) A' for which $comp(A') = S' \subset \mathbb{N}$. But then,

$$comp(A' * M_m(\overline{\ell})) = S' \times_{\mathbb{N}} \mathbb{N}m = S$$

and $A = A' * M_m(\overline{\ell})$ is again an $\overline{\ell}$-qurve. □

5 Tits and Euler forms

In this section we will define bilinear forms on $comp(A)$ (when A is an $\overline{\ell}$-qurve) generalizing the Tits- and Euler-forms on the dimension vectors of a quiver. Let $rep\ A$ be the Abelian category of all finite dimensional representations of A. If A is an affine $\overline{\ell}$-algebra, then $Hom_A(M, N)$ and $Ext^1_A(M, N)$ are finite dimensional $\overline{\ell}$-spaces for all $M, N \in rep\ A$.

If A is hereditary (for example, if A is an $\overline{\ell}$-qurve) we have that $\chi_A(M, -)$ and $\chi_A(-, N)$ are additive on short exact sequences in $rep\ A$ where

$$\chi_A(M, N) = dim_{\overline{\ell}} Hom_A(M, N) - dim_{\overline{\ell}} Ext^1_A(M, N)$$

For $M \in rep\ A$ define its *semi-simplification* M^{ss} to be the semi-simple A-module obtained by taking the direct sum of the Jordan-Hölder components of M. From additivity on short exact sequences it follows for all $M, N \in rep\ A$ that

$$\chi_A(M, N) = \chi_A(M^{ss}, N^{ss})$$

For $\alpha, \beta \in comp(A)$ it follows from [9] and [2, lemma 4.3] that the functions

$$rep_\alpha\ A \times rep_\beta\ A \longrightarrow \mathbb{Z} \qquad (M, N) \mapsto \begin{cases} dim_{\overline{\ell}} Hom_A(M, N) \\ dim_{\overline{\ell}} Ext^1_A(M, N) \end{cases}$$

are upper semicontinuous. In particular, there are Zariski open subsets (whence dense by irreducibility) of $rep_\alpha\ A \times rep_\beta\ A$ where these functions attain a minimum. Following [18] we will denote these minimal values by $hom(\alpha, \beta)$ resp. $ext(\alpha, \beta)$.

The group GL_n acts on $rep_n\ A$ by base-change and orbits $\mathcal{O}(M)$ under this action are precisely the isomorphism classes of n-dimensional left A-modules. From

[5] we recall that the semi-simplification M^{ss} belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of the orbit and that $Ext^1_A(M,M)$ can be identified to the *normal space* to the orbit $\mathcal{O}(M)$ with respect to the scheme structure on $rep_n\ A$.

Proposition 3 *Let A be an affine $\overline{\ell}$-algebra.*

1. *If $rep_\gamma\ A$ is a smooth variety, then for all $M \in rep_\gamma\ A$ we have*

$$|\gamma|^2 - \chi_A(M,M) = dim\, rep_\gamma\ A$$

and hence $\chi_A(M,M)$ is constant on $rep_\gamma\ A$.

2. *If $rep_\alpha\ A$, $rep_\beta\ A$ and $rep_{\alpha+\beta}\ A$ are smooth varieties, then*

$$\chi_A(M,N) + \chi_A(N,M)$$

is a constant function on $rep_\alpha\ A \times rep_\beta\ A$.

Proof. If $rep_\gamma\ A$ is smooth in M, it follows from the above remarks that

$$T_M rep_\gamma\ A = Ext^1_A(M,N) \oplus T_M\mathcal{O}(M), \qquad \mathcal{O}(M) = GL_{|\gamma|}/Stab(M)$$

where $Stab(M)$ is the stabilizer subgroup which by [9] has the same dimension as $Hom_A(M,M)$. Therefore,

$$\begin{aligned} dim\, rep_\gamma\ A &= dim_{\overline{\ell}} T_M rep_\gamma\ A \\ &= dim_{\overline{\ell}} Ext^1_A(M,M) + |\gamma|^2 - dim_{\overline{\ell}} Hom_A(M,M) \end{aligned}$$

whence (1). (2) follows from this by considering the point $M \oplus N \in rep_{\alpha+\beta}\ A$ and using bi-additivity of χ_A. □

Definition 6 *If A is an $\overline{\ell}$-qurve, then for all $\alpha \in comp(A)$ the representation variety $rep_\alpha\ A$ is smooth. Therefore, the constant value*

$$(\alpha,\beta)_A = \chi_A(M,N) + \chi_A(N,M)$$

on $rep_\alpha\ A \times rep_\beta\ A$ defines a symmetric bilinear form

$$(-,-)_A\ :\ comp(A) \times comp(A) \longrightarrow \mathbb{Z}$$

which we call the Tits-form *of the $\overline{\ell}$-qurve A.*

For general affine $\overline{\ell}$-algebras $\chi_A(M,N) + \chi_A(N,M)$ does not have to be constant and the foregoing result can be used to deduce singularity of specific representation varieties.

Example 1 *Let $A = \overline{\ell}B_3$ be the group-algebra of the* third braid group $B_3 = \langle s,t \mid s^2 = t^3 \rangle$*. The one dimensional representation variety is the cusp minus the singular origin*

$$rep_1\ A = \{(x,y) \in \overline{\ell}^2 \mid x^3 = y^2\} - \{(0,0)\}$$

and hence is a smooth affine variety. As all points are simple A-modules we have that $dim_{\overline{\ell}}\ Hom_A(-,-)$ is equal to zero on the open set $rep_1\ A \times rep_1\ A - \Delta$

and is equal to one on the diagonal Δ. *As for* $dim_{\overline{\ell}}\ Ext^1_A(-,-)$ *this is zero on* $rep_1\ A \times rep_1\ A - (\Delta \sqcup \Delta_1 \sqcup \Delta_2)$ *where*

$$\begin{cases} \Delta_1 & = \{((x,y),(\rho x,-y))\ :\ x^3 = y^2\} \\ \Delta_2 & = \{((x,y),(\rho^2 x,-y))\ :\ x^3 = y^2\} \end{cases}$$

for ρ *a primitive third root of unity. As a consequence,* $\chi_A(M,N)$ *is zero on the Zariski open subset* $rep_1\ A \times rep_1\ A - (\Delta_1 \sqcup \Delta_2)$ *and is equal to* -1 *on* $\Delta_1 \sqcup \Delta_2$*. Therefore,* $\overline{\ell}B_3$ *is not an* $\overline{\ell}$*-qurve. In fact,* $rep_2\ \overline{\ell}B_3$ *is not smooth.*

If α is the dimension vector of a simple representation of A, then there is a Zariski open subset $simp_\alpha\ A$ of simple representations in $rep_\alpha\ A$.

Proposition 4 *If* A *is an* $\overline{\ell}$*-qurve and* α, β *are dimension vectors of simple representations, then the function*

$$\chi_A(S,T)$$

is constant on $simp_\alpha\ A \times simp_\beta\ A$.

Proof. There is a Zariski open subset $U \subset simp_\alpha\ A \times simp_\beta\ A$ consisting of couples (S',T') such that

$$dim_{\overline{\ell}} Ext^1_A(S',T') = ext(\alpha,\beta) \quad \text{and} \quad dim_{\overline{\ell}} Ext^1_A(T',S') = ext(\beta,\alpha)$$

Hence, for all $(S,T) \in simp_\alpha\ A \times simp_\beta\ A$

$$\begin{cases} dim_{\overline{\ell}} Ext^1_A(S,T) \geq dim_{\overline{\ell}} Ext^1_A(S',T') \\ dim_{\overline{\ell}} Ext^1_A(T,S) \geq dim_{\overline{\ell}} Ext^1_A(T',S') \end{cases}$$

If $\alpha \neq \beta$ (or if $\alpha = \beta$ and $S \not\simeq T$) $\chi_A(S,T) = -dim_{\overline{\ell}} Ext^1_A(S,T)$ and hence the above inequalities must be equalities by proposition 3. Remains to prove for $S,T \in simp_\alpha\ A$ with $S \not\simeq T$ that $\chi_A(S,S) = \chi_A(S,T)$. Consider the two semi-simple representations $M = S \oplus S$ and $N = S \oplus T$ in $rep_{2\alpha}\ A$. From proposition 3 (1) we get

$$\begin{aligned} 4\chi_A(S,S) &= \chi_A(S,S) + \chi_A(T,T) + \chi_A(S,T) + \chi_A(T,S) \\ &= 2\chi_A(S,S) + 2\chi_A(S,T) \end{aligned}$$

(using proposition 3 (1) and the above fact that $\chi_A(S,T) = \chi_A(T,S)$) whence $\chi_A(S,S) = \chi_A(S,T)$. $\square$

If $M \in rep\ A$, its semi-simplification has as isotypical decomposition

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

with all S_i non-isomorphic. If $S_i \in rep_{\beta_i}\ A$ we say that the *representation type* of M (which is determined upto permutation of the (e_i, β_i) terms).

$$\tau(M) = (e_1, \beta_1; \ldots ; e_k, \beta_k)$$

Proposition 5 *If* A *is an* $\overline{\ell}$*-qurve, the* Euler-form

$$\chi_A(M,N) = dim_{\overline{\ell}} Hom_A(M,N) - dim_{\overline{\ell}} Ext^1_A(M,N)$$

depends only on the representation types $\tau(M)$ *and* $\tau(N)$.

Proof. Follows from the foregoing result by observing that $\chi_A(M,N) = \chi_A(M^{ss},N^{ss})$. $\square$

In particular, there is a Zariski open subset in $rep_\alpha\ A \times rep_\beta\ A$ of couples (M,N) on which the value of $\chi_A(M,N)$ is constant and equal to the *Euler form*

$$\chi_A(\alpha,\beta) = hom(\alpha,\beta) - ext(\alpha,\beta)$$

Clearly, this open set contains all representations of *generic representation type* τ_{gen}, see for example [13]. In fact, if $char(\overline{\ell}) = 0$ the proof of proposition 7 implies that $\chi_A(M,N)$ is constant on $rep_\alpha\ A \times rep_\beta\ A$.

6 One quiver to rule them all

If A is an $\overline{\ell}$-qurve, we will denote with Σ_A the minimal set of semigroup-generators of the component semigroup $comp(A)$. Observe that Σ_A is well-defined as it follows from the Jordan-Hölder decomposition that

$$\Sigma_A = \{\alpha \in comp(A) \mid simp_\alpha\ A = rep_\alpha\ A\}$$

In particular, it follows from proposition 5 that $\chi_A(S,T) = \chi_S(\alpha,\beta)$ for all representations $S \in rep_\alpha\ A$ and $T \in rep_\beta\ A$ if $\alpha,\beta \in \Sigma_A$. In all examples known to us, Σ_A is a finite set.

Definition 7 *If A is an $\overline{\ell}$-qurve, we define its* one-quiver $Q_1(A)$ *to be the quiver on the (possibly infinite) vertex set $\{v_\alpha \mid \alpha \in \Sigma_A\}$ such that the number of directed arrows (loops) from v_α to v_β is given by*

$$\#\{\ \textcircled{\scriptsize α} \longrightarrow \textcircled{\scriptsize β}\ \} = \delta_{\alpha\beta} - \chi_A(\alpha,\beta)$$

The one-dimension vector $\alpha_1(A)$ *for A is the dimension vector for $Q_1(A)$ having as its v_α-component the total dimension $|\alpha|$.*

If $Q_1(A)$ is a quiver on finitely many vertices $\{v_1,\ldots,v_k\}$ and $\alpha_1(A) = (n_1,\ldots,n_k)$, we can define the $\overline{\ell}$-algebra

$$B(Q_1(A),\alpha_1(A)) = \begin{bmatrix} B_{11} & \ldots & B_{1k} \\ \vdots & & \vdots \\ B_{k1} & \ldots & B_{kk} \end{bmatrix}$$

where B_{ij} is the $n_i \times n_j$ block matrix having all its components equal to the sub vectorspace of the path algebra $\overline{\ell}Q_1(A)$ spanned by all oriented paths in $Q_1(A)$ starting at vertex v_i and ending in v_j. Observe, that $B(Q_1(A),\alpha_1(A))$ is Morita equivalent to the path algebra $\overline{\ell}Q_1(A)$ and as such is again an $\overline{\ell}$-qurve.

Example 2 (Deligne-Mumford curves) *Recall from [1, Coroll. 7.8] that a* smooth Deligne-Mumford curve *which is generically a scheme, determines (and is determine by) a smooth affine curve X and an* hereditary order A *over $\overline{\ell}[X]$. As such, A is an $\overline{\ell}$-qurve with center $\overline{\ell}[X]$ and is a subalgebra of $M_n(\overline{\ell}(X))$ for some n called* *the p.i.-degree of A. If $\mathfrak{m}_x$ is the maximal ideal of $\overline{\ell}[X]$ corresponding to the point $x \in X$ then for all but finitely many exceptions $\{x_1, \ldots, x_l\}$ we have that*

$$A/\mathfrak{m}_x A \simeq M_n(\overline{\ell})$$

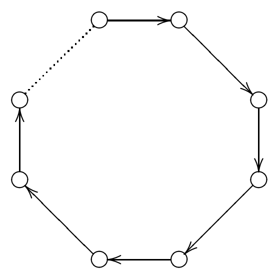
For the exceptional points (the ramification locus *of A) there are finitely many maximal ideals $\{P_1(i), \ldots, P_{k_i}(i)\}$ of A lying over $\mathfrak{m}_{x_i}$ and*

$$A/P_j(i) \simeq M_{n_j(i)}(\overline{\ell}) \qquad \textit{with} \qquad n_1(i) + \ldots + n_{k_i}(i) = n$$

As a consequence, $rep_l\ A$ for all $l < n$ consists of finitely many closed orbits each corresponding to a maximal ideal $P_j(i)$ such that $A/P_j(i) \simeq M_l(\overline{\ell})$. Hence, the component semigroup $comp(A)$ has generators $\alpha_j(i)$ for all $1 \leq i \leq l$ and $1 \leq j \leq k_i$ and relations for all $1 \leq i, j \leq l$

$$\alpha_1(i) + \ldots + \alpha_{k_i}(i) = \alpha_1(j) + \ldots + \alpha_{k_j}(j)$$

From direct calculation or using [12, Prop. 6.1] it follows that the one quiver $Q_1(A)$ is the disjoint union of l quivers of type $\tilde{A}_{k_i}$, that is the i-th component is $Q_1(A)(i)$ and is the quiver on k_i vertices



and the corresponding components for the one dimension vector $\alpha_1(A)$ are $\alpha_1(A)(i) = (n_1(i), \ldots, n_{k_i}(i))$. Therefore, the associated algebra

$$B(Q_1(A), \alpha_1(A)) = B_1 \oplus \ldots \oplus B_l$$

where B_i is the block-matrix algebra

$$\begin{bmatrix} M_{n_1(i)\times n_1(i)}(\overline{\ell}[x]) & M_{n_1(i)\times n_2(i)}(\overline{\ell}[x]) & \ldots & M_{n_1(i)\times n_{k_i}(i)}(\overline{\ell}[x]) \\ M_{n_2(i)\times n_1(i)}(x\overline{\ell}[x]) & M_{n_2(i)\times n_2(i)}(\overline{\ell}[x]) & \ldots & M_{n_2(i)\times n_{k_i}(i)}(\overline{\ell}[x]) \\ \vdots & \vdots & & \vdots \\ M_{n_{k_i}(i)\times n_1(i)}(x\overline{\ell}[x]) & M_{n_{k_i}(i)\times n_2(i)}(x\overline{\ell}[x]) & \ldots & M_{n_{k_i}(i)\times n_{k_i}(i)}(\overline{\ell}[x]) \end{bmatrix}$$

It follows from [15, Chp. 9] or [12, Prop. 6.1] that in a neighborhood of x_i the $\overline{\ell}$-qurve A is étale isomorphic to B_i.

Elsewhere, we will generalize this example by relating the $\overline{\ell}$-qurve A with the algebra $B(Q_1(A), \alpha_1(A))$ using the formal tubular neighborhood theorem [3, §6]. Here, we will use the *one-quiver-setting* $(Q_1(A), \alpha_1(A))$ to describe the GL_n-étale local structure of $rep_n\ A$ in the neighborhood of a semi-simple representation. As this description uses the Luna slice result, we will assume that $char(\overline{\ell}) = 0$ in the remainder of this section. We recall the construction of the *local quiver* and refer to [11] and [12] for details and proofs.

Definition 8 *Let $M \in rep_\alpha\ A$ be a semi-simple A-module of representation type $\tau_M = (e_1, \gamma_1; \ldots ; e_l, \beta_l)$, that is*

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_l^{\oplus e_l}$$

with all S_i non-isomorphic and of dimension vector γ_i.

The local quiver *Q_M is the quiver on l vertices (corresponding to the distinct simple components of M) such that the number of directed arrows from v_i to v_j is equal to $dim_{\overline{\ell}}\, Ext^1_A(S_i, S_j)$.*

The local dimension vector *$\alpha_M = (e_1, \ldots, e_l)$ determined by the multiplicities e_i of the simple components of M.*

Observe that we know already that the quiver Q_M only depends on the representation type τ_M of M and not on the choice of the simple components S_i. The relevance of this *local quiver setting* (Q_M, α_M) is that it determines the GL_n-equivariant étale structure of $rep_\alpha\ A$ in a neighborhood of the closed orbit $\mathcal{O}(M)$ by the results from [11].

As $n = \sum_i e_i |\gamma_i|$ there is an embedding of $GL(\alpha_M)$ into GL_n and with respect to this embedding there is a GL_n-equivariant étale isomorphism between

- $rep_\alpha\ A$ in a neighborhood of $\mathcal{O}(M)$, and
- $GL_n \times^{GL(\alpha_M)} rep_{\alpha_M}\ Q_M$ is a neighborhood of $\mathcal{O}(1_n, 0)$

where 0 is the zero representation. We will show that the one-quiver setting $(Q_1(A), \alpha_1(A))$ contains enough information to describe all these local quiver settings (Q_M, α_M) whenever A is an $\overline{\ell}$-qurve.

$\Sigma_A = \{\beta_i \mid i \in I\}$ is the set of semigroup generators of $comp(A)$ (possibly infinite). For any $\alpha \in comp(A)$ we can write

$$\alpha = a_1\beta_1 + \ldots + a_k\beta_k \qquad a_i \in \mathbb{N}$$

(possibly in many several ways) with the $\beta_i \in \Sigma_A$. If the set of vertices $V \leftrightarrow \Sigma_A$ is infinite, we can always replace the infinite one-quiver setting $(Q_1(A), \alpha_1(A))$ by a finite quiver setting $(supp(\alpha), \alpha_1(A)|supp(\alpha))$ where $supp(\alpha)$ is the *support* of α, that is those vertices $\beta_i \in V \leftrightarrow \Sigma_A$ such that $a_i \in \mathbb{N}_+$ in a fixed description of α in terms of the semigroup generators. For notational reasons, we denote this finite quiver setting again by $(Q_1(A), \alpha_1(A))$.

Proposition 6 *The one-quiver setting $(Q_1(A), \alpha_1(A))$ contains enough information to determine $simp(A)$ the set of all dimension vectors of simple finite dimensional representations of A.*

Proof. If $\alpha \in comp(A)$, fix a description

$$\alpha = a_1\beta_1 + \ldots + a_k\beta_k$$

with $a_i \in \mathbb{N}_+$ and $\{\beta_1, \ldots, \beta_k\}$ among the semigroup generators of $comp(A)$. This implies that there are points in $rep_\alpha\ A$ corresponding to semi-simple representations

$$M = S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k}$$

where the S_i are distinct simple representations in $rep_{\beta_i}\ A$. But then the local quiver setting of M in $rep_\alpha\ A$, (Q_M, α_M) is just $(Q_1(A), \epsilon)$ where $\epsilon = (a_1, \ldots, a_k)$.

Because $rep_\alpha\ A$ is irreducible, it follows that $\alpha \in simp(A)$ if and only if ϵ is the dimension vector of a simple representation of $Q_1(A)$. These dimension vectors have been classified in [13] and we recall the result.

Let χ be the Euler-form of $Q_1(A)$, that is $\chi = (c_{ij})_{i,j} \in M_k(\mathbb{Z})$ with $c_{ij} = \delta_{ij} - \#\{ (i) \longrightarrow (j) \}$ and let δ_i be the dimension vector of a vertex-simple concentrated in vertex v_i. Then, ϵ is the dimension vector of a simple representation of Q_A if and only if the following conditions are satisfied :

1. the support $supp(\epsilon)$ is a strongly connected subquiver of Q_A (there is an oriented cycle in $supp(\epsilon)$ containing each pair (i,j) such that $\{v_i, v_j\} \subset supp(\epsilon)$)
2. for all $v_i \in supp(\epsilon)$ we have the numerical conditions

$$\chi(\epsilon, \delta_i) \leq 0 \qquad \text{and} \qquad \chi(\delta_i, \epsilon) \leq 0$$

unless $supp(\epsilon)$ in an oriented cycle of type $\tilde{A}_l$ for some l in which case all components of ϵ must be equal to one.

The statement follows from this. $\square$

Proposition 7 *The one-quiver setting $(Q_1(A), \alpha_1(A))$ contains enough information to compute the $\overline{\ell}$-dimension of $Ext^1_A(S,T)$ for all finite dimensional simple representations S and T of A.*

If $S \in rep_\alpha\ A$ where $\alpha = \sum_i a_i \beta_i$ and $T \in rep_\beta\ A$ where $\beta = \sum_i b_i \beta_i$, then

$$dim_{\overline{\ell}}\ Ext^1_A(S,T) = -\chi_{Q_1(A)}(\epsilon, \eta)$$

for $\epsilon = (a_1, \ldots, a_k)$ and $\eta = (b_1, \ldots, b_k)$.

Proof. Let S_i and T_i be distinct simples in $rep_{\beta_i}\ A$ and consider the semi-simple representations M resp. N in $rep_\alpha\ A$ resp. $rep_\beta\ A$

$$M = S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k} \quad \text{and} \quad N = T_1^{\oplus b_1} \oplus \ldots \oplus T_k^{\oplus b_k}$$

By the foregoing proposition, we have complete information on the local quiver setting of $M \oplus N$ in $rep_{\alpha+\beta}\ A$ from $(Q_1(A), \alpha_1(A))$. By assumption on α and β there is a Zariski open subset of simples $S' \in rep_\alpha\ A$ and simples $T' \in rep_\beta\ A$ such that $S' \oplus T'$ lies in a neighborhood of $M \oplus N$.

It follows from [13] that one can reconstruct the local quiver setting of $S' \oplus T'$ from that of $M \oplus N$. This local quiver has two vertices $\{v_1, v_2\}$ with $-\chi_Q(\eta, \epsilon)$ arrows from v_1 to v_2 and $-\chi_Q(\epsilon, \eta)$ arrows from v_2 to v_1. In v_1 there are $1 - \chi_Q(\epsilon, \epsilon)$ loops and in v_2 there are $1 - \chi_Q(\eta, \eta)$ loops. The dimension vector is $(1,1)$. From this we deduce that

$$dim_{\overline{\ell}}\ Ext^1_A(S', T') = -\chi(\epsilon, \eta)$$

but we have seen before that the extension-dimension only depends on the representation type and not on the choice of simples, hence this number is also equal to $dim_{\overline{\ell}}\ Ext^1_A(S,T)$. $\square$

Theorem 3 *The one-quiver setting $(Q_1(A), \alpha_1(A))$ contains enough information to construct the local quiver setting (Q_M, α_M) for every semi-simple representation*

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_l^{\oplus e_l}$$

of A.

Proof. This is a direct consequence of the foregoing two propositions. To begin, we can determine the possible dimension vectors α_i of the simple components S_i. Write $\alpha_i = \sum_{j=1}^{k} a_j(i)\beta_j$ then $\epsilon_i = (a_1(i), \ldots, a_k(i))$ must be the dimension vector of a simple representation of the associated quiver $Q_1(A)$. Moreover, by the previous theorem we know that

$$dim_{\overline{\ell}}\, Ext^1_A(S_i, S_j) = \delta_{ij} - \chi(\epsilon_i, \epsilon_j)$$

and hence we have full knowledge of the local quiver Q_M. □

7 The one-quiver for $\pi_1(\mathcal{S}_G)$

In this section we will construct the one-quiver setting for the fundamental algebra $\pi_1(\mathcal{S}_G)$ of a graph $\mathcal{S}_G$ of separable (that is, semi-simple) $\overline{\ell}$-algebras. As an intermediary step we will construct a finite quiver $Q_0(\mathcal{S}_G)$ such that finite dimensional representations of $\pi_1(\mathcal{S}_G)$ correspond to certain finite dimensional representations of the path algebra $\overline{\ell}Q_0(\mathcal{S}_G)$.

We have decomposition of the vertex- and edge-algebras

$$S_v = M_{d_v(1)}(\overline{\ell}) \oplus \ldots \oplus M_{d_v(n_v)}(\overline{\ell}) \quad \text{resp.} \quad S_e = M_{d_e(1)}(\overline{\ell}) \oplus \ldots \oplus M_{d_e(n_e)}(\overline{\ell})$$

The embeddings $S_e \hookrightarrow S_v$ are depicted via Bratelli-diagrams or, equivalently, by natural numbers $a_{ij}^{(ev)}$ for $1 \leq i \leq n_e$ and $1 \leq j \leq n_v$ satisfying the numerical restrictions

$$d_v(j) = \sum_{i=1}^{n_e} a_{ij}^{(ev)} d_e(i) \qquad \text{for all } 1 \leq j \leq n_v \text{ and all } v \in V \text{ and } e \in E$$

Remark that these numbers give the *restriction data*, that is, the multiplicities of the simple components of S_e occurring in the restriction $V_j^{(v)} \downarrow_{S_e}$ for the simple components $V_j^{(v)}$ of S_v. From these decompositions and Schur's lemma it follows that for any edge $\textcircled{v} \overset{e}{\text{———}} \textcircled{w}$ in the graph G we have

$$Hom_{S_e}(V_i^{(v)}, V_j^{(w)}) = \sum_{k=1}^{n_e} a_{ki}^{(ev)} a_{kj}^{(ew)} = n_{ij}^{(e)}$$

Definition 9 *For a graph $\mathcal{S}_G$ of separable $\overline{\ell}$-algebras we define a quiver $Q_0(\mathcal{S}_G)$ as follows*

- *Vertices : for any vertex $v \in V$ of G take n_v vertices $\{\mu_1^{(v)}, \ldots, \mu_{n_v}^{v}\}$.*
- *Arrows : fix an orientation $\vec{G}$ on all of the edges of G. For any edge $\textcircled{v} \overset{e}{\text{———}} \textcircled{w}$ in G we add for each $1 \leq i \leq n_v$ and each $1 \leq j \leq n_w$ precisely $n_{ij}^{(e)}$ arrows between the vertices $\mu_i^{(v)}$ and $\mu_j^{(w)}$ oriented in the same way as the edge e in $\vec{G}$.*

We call $Q_0(\mathcal{S}_G)$ the Zariski quiver *of the graph of separable algebras $\mathcal{S}_G$.*

The *representation space* $\boldsymbol{rep_\alpha\, Q_0(\mathcal{S}_G)}$ is the affine $\overline{\ell}$-space

$$rep_\alpha\, Q_0(\mathcal{S}_G) = \bigoplus_{\textcircled{v} \xrightarrow{e} \textcircled{w}} \oplus_{i=1}^{n_v} \oplus_{j=1}^{n_w} M_{\alpha_j^{(w)} \times \alpha_i^{(v)}}(\overline{\ell})$$

and two α-dimensional representations are said to be *isomorphic* if they are conjugated via the natural base-change action of $GL(\alpha) = \times_{v \in V} \times_{i=1}^{n} GL(\alpha_i^{(v)})$.

A dimension vector $\alpha = (\alpha_i^{(v)} \; : \; v \in V, 1 \leq i \leq n_v)$ for $Q_0(\mathcal{S}_G)$ is said to be an *n-dimension vector* if the following numerical conditions are satisfied

$$\sum_{i=1}^{n_v} d_v(i) \alpha_i^{(v)} = n$$

for all $v \in V$.

For any edge $\textcircled{v} \xrightarrow{e} \textcircled{w}$ we denote by Q_e the *bipartite* subquiver of $Q_0(\mathcal{S}_G)$ on the vertices $\{\mu_1^{(v)}, \ldots, \mu_{n_v}^{(v)}\}$, $\{\mu_1^{(w)}, \ldots, \mu_{n_w}^{(w)}\}$ and the $n_{ij}^{(e)}$ arrows between $\mu_i^{(v)}$ and $\mu_j^{(w)}$ determined by the embeddings $S_e \hookrightarrow S_v$ and $S_e \hookrightarrow S_w$.

Definition 10 *Let α be an n-dimension vector, $M \in rep_\alpha\, Q_0(\mathcal{S}_G)$ and $e \in E$:*

- *M is said to be e-semistable iff for all Q_e- subrepresentations N of $M|Q_e$ of dimension vector $(n_1, \ldots, n_{n_v}, n'_1, \ldots, n'_{n_w})$ we have*

$$\sum_{i=1}^{n_w} n'_i d_w(i) \geq \sum_{i=1}^{n_v} n_i d_v(i)$$

- *M is said to be e-stable iff for all proper Q_e-subrepresentations N of $M|Q_e$ of dimension vector $(n_1, \ldots, n_{n_v}, n'_1, \ldots, n'_{n_w})$ we have*

$$\sum_{i=1}^{n_w} n'_i d_w(i) > \sum_{i=1}^{n_v} n_i d_v(i)$$

- *M is said to be $\mathcal{S}_G$-semistable (resp. $\mathcal{S}_G$-stable) iff M is e-semistable (resp. e-stable) for all edges $e \in E$.*

The relevance of the quiver $Q_0(\mathcal{S}_G)$ and the introduced terminology is contained in the next result.

Proposition 8 *Every n-dimensional representation $\pi_1(\mathcal{S}_G) \xrightarrow{\phi} M_n(\overline{\ell})$ determines (and is determined by) an $\mathcal{S}_G$-semistable representation $M_\phi \in rep_\alpha\, Q_0(\mathcal{S}_G)$ for some n-dimension vector α. Moreover, if ϕ and ϕ' are isomorphic representations of $\pi_1(\mathcal{S}_G)$, then M_ϕ and $M_{\phi'}$ are isomorphic as quiver representations.*

Proof. Let $N = \overline{\ell}_\phi^n$ be the n-dimensional module determined by ϕ. For each vertex $v \in V$ we have a decomposition by restricting N to the separable subalgebra S_v

$$N \downarrow_{S_v} \simeq V_{1,v}^{\oplus \alpha_1^{(v)}} \oplus \ldots \oplus V_{n_v,v}^{\oplus \alpha_{n_v}^{(v)}}$$

where the $V_{i,v}$ are the distinct simple modules of S_v of dimension $d_v(i)$. Choose an $\overline{\ell}$-basis $\mathcal{B}_v$ of $N \downarrow_{S_v}$ compatible with this decomposition. These decompositions determine an n-dimension vector α. For any edge $\textcircled{v} \xrightarrow{e} \textcircled{w}$ the embeddings $S_e \overset{\alpha}{\hookrightarrow} S_v$ and $S_e \overset{\beta}{\hookrightarrow} S_w$ determine two n-dimensional S_e-representations

$$(N \downarrow_{S_v}) \downarrow^{\alpha}_{S_e} \qquad \text{and} \qquad (N \downarrow_{S_w}) \downarrow^{\beta}_{S_e}$$

which, by construction of $\pi_1(\mathcal{S}_G)$ are isomorphic. That is, the basechange map $\mathcal{B}_v \xrightarrow{\psi_{vw}} \mathcal{B}_w$ is an invertible element of

$$Hom_{S_e}(N \downarrow_{S_v}, N \downarrow_{S_w}) = \oplus_{i=1}^{n_v} \oplus_{j=1}^{n_w} M_{\alpha_j^{(w)} \times \alpha_i^{(v)}}(Hom_{S_e}(V_{i,v}, V_{j,w}))$$

and hence ψ_{vw} determines a representation of the bipartite quiver Q_e of dimension vector $\alpha | Q_e$. Repeating this for all edges $e \in E$ we obtain a representation $M_\phi \in rep_\alpha\ Q_0(\mathcal{S}_G)$. Invertibility of the map ψ_{vw} is equivalent to M_ϕ being e-semistable, so M_ϕ is $\mathcal{S}_G$-semistable. Isomorphic representations ϕ and ϕ' determine isomorphic vertex-decompositions whence, by Schur's lemma, bases which are transferred into each other via an element of $GL(\alpha)$ and hence the quiver representations M_ϕ and $M_{\phi'}$ are isomorphic. From the construction of the fundamental algebra $\pi_1(\mathcal{S}_G)$ it follows that one can reverse this procedure to construct on n-dimensional representation of $\pi_1(\mathcal{S}_G)$ from a $\mathcal{S}_G$-stable representation $M \in rep_\alpha\ Q_0(\mathcal{S}_G)$ for some n-dimension vector α. □

Under this correspondence simple $\pi_1(\mathcal{S}_G)$-representations correspond to $\mathcal{S}_G$-stable representations. If α is an n-dimension vector such that $rep_\alpha\ Q_0(\mathcal{S}_G)$ contains $\mathcal{S}_G$-stable representations (which then form a Zariski open subset), then α is a *Schur root* of $Q_0(\mathcal{S}_G)$ and consequently the dimension of the classifying variety is equal to $1 - \chi_0(\alpha, \alpha)$ where χ is the *Euler form* of the quiver $Q_0(\mathcal{S}_G)$. For this result and related material on Schur roots we refer to [18].

Proposition 9 *Isomorphism classes of simple n-dimensional representations of $\pi_1(\mathcal{S}_G)$ are parametrized by the points of a smooth quasi-affine variety (possibly with several irreducible components)*

$$isosimp_n\ \pi_1(\mathcal{S}_G) = \bigsqcup_{\alpha} isosimp_\alpha\ \pi_1(\mathcal{S}_G)$$

where α runs over all n-dimension vectors such that $rep_\alpha\ Q_0(\mathcal{S}_G)$ contains $\mathcal{S}_G$-stable representations. These components have dimensions

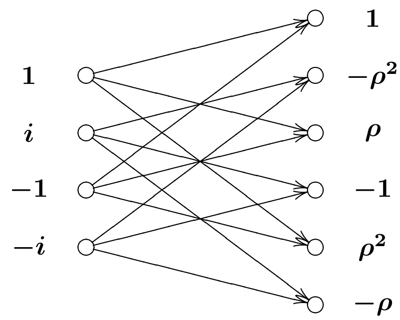
$$dim\ isosimp_\alpha\ \pi_1(\mathcal{S}_G) = 1 - \chi_0(\alpha, \alpha)$$

where χ_0 is the Euler form of the quiver $Q_0(\mathcal{S}_G)$.

As an example consider the modular group $SL_2(\mathbb{Z})$ which is the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, see for example [4, I §7]. If $char(\overline{\ell}) \neq 2, 3$ the group-algebra $\overline{\ell} SL_2(\mathbb{Z})$ is the fundamental algebra of the graph of separable $\overline{\ell}$-algebras

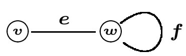
$$\textcircled{v} \xrightarrow{e} \textcircled{w} \qquad \text{with} \qquad S_v = \overline{\ell}\mathbb{Z}_4 \quad S_w = \overline{\ell}\mathbb{Z}_6 \quad S_e = \overline{\ell}\mathbb{Z}_2$$

As all simples are one-dimensional (determined by their eigenvalue), it is easy to verify that the zero quiver $Q_0(\overline{\ell}SL_2(\mathbb{Z}))$ has the following form



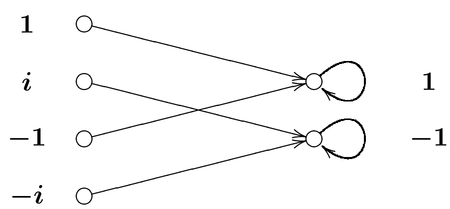
(ρ is a primitive 3rd root of unity) which is the disjoint union of two copies of the quiver associated to $PSL_2(\mathbb{Z})$ in [22].

The congruence subgroup $\Gamma_0(2) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ with c even $\}$ is the fundamental group of the graph of finite groups



$$G_w = G_e = G_f = \mathbb{Z}_2,\ G_v = \mathbb{Z}_4$$

If $char(\overline{\ell}) \neq 2$, the group algebra $\overline{\ell}\Gamma_0(2)$ is the fundamental algebra of a graph of separable $\overline{\ell}$-algebras and the zero quiver $Q_0(\overline{\ell}\Gamma_0(2))$ has the following form



Definition 11 *For a graph $\mathcal{S}_G$ of separable $\overline{\ell}$-algebras we define a quiver $Q_1(\mathcal{S}_G)$ as follows*

- *Vertices : Let $\{\alpha_1, \ldots, \alpha_k\}$ be the minimal set of generators for the sub-semigroup of dimension vectors α of $Q_0(\mathcal{S}_G)$ which are n-dimension vectors for some $n \in \mathbb{N}$ and such that $rep_\alpha\ Q_0(\mathcal{S}_G)$ contains $\mathcal{S}_G$-semistable representations. The vertices $\{\nu_1, \ldots, \nu_k\}$ are in one-to-one correspondence with these generators $\{\alpha_1, \ldots, \alpha_k\}$.*

- *Arrows : The number of directed arrows in $Q_1(\mathcal{S}_G)$ from ν_i to ν_j*

$$\#\{ \textcircled{i} \longrightarrow \textcircled{j} \} = \delta_{ij} - \chi_0(\alpha_i, \alpha_j)$$

where χ_0 is the Euler-form of the Zariski quiver $Q_0(\mathcal{S}_G)$.

We call $Q_1(\mathcal{S}_G)$ the one-quiver of the graph of separable algebras $\mathcal{S}_G$.

The one-quiver $Q_1(\mathcal{S}_G)$ allows us to determine the components $rep_\alpha\ \pi_1(\mathcal{S}_G)$ which contain (a Zariski open subset of) simple representations. Remark that the description of Schur roots is a lot harder than that of dimension vectors of simple representations.

Proposition 10 *If $\alpha = c_1\alpha_1 + \ldots + c_k\alpha_k \in comp\, \pi_1(\mathcal{S}_G)$ then the component $rep_\alpha\, \pi_1(\mathcal{S}_G)$ contains simple representations if and only if*

- $$\chi_1(\gamma, \epsilon_i) \leq 0 \qquad \textit{and} \qquad \chi_1(\epsilon_i, \gamma) \leq 0$$
  for all $1 \leq i \leq k$ where $\gamma = (c_1, \ldots, c_k)$ and $\epsilon_i = (\delta_{1i}, \ldots, \delta_{ki})$ and where χ_1 is the Euler form of the one quiver $Q_1(\mathcal{S}_G)$

- *$supp(\gamma)$ is a strongly connected subquiver of $\pi_1(\mathcal{S}_G)$ and if $supp(\gamma)$ is of extended Dynkin type $\tilde{A}_l$ then all non-zero components of γ must be equal to one.*

Proof. Follows from the proof of proposition 6. □

If $char(\overline{\ell}) = 0$ one can apply Luna slice machinery to construct a Zariski open subset of all simple representations in $rep_\alpha\, \pi_1(\mathcal{S}_G)$ from the knowledge of low-dimensional simples. For example, suppose we have found simple representations

$$S_i \in rep_{\alpha_i}\, \pi_1(\mathcal{S}_G) \qquad \text{for all } 1 \leq i \leq k$$

and consider the point M in the affine space $rep_\alpha\, Q_0(\mathcal{S}_G)$ determined by the semi-simple representation of $\pi_1(\mathcal{S}_G)$

$$M = S_1^{\oplus c_1} \oplus \ldots \oplus S_k^{\oplus c_k}$$

then the normal space to the $GL(\alpha)$-orbit $\mathcal{O}(M)$ is isomorphic to $Ext^1_{\pi_1(\mathcal{S}_G)}(M, M)$ which we have seen can be identified to $rep_\gamma\, Q_1(\mathcal{S}_G)$.

Proposition 11 *Let $\alpha = c_1\alpha_1 + \ldots + c_k\alpha_k$ be a component such that $rep_\alpha\, \pi_1(\mathcal{S}_G)$ contains simple representations. In the affine space $rep_\alpha Q_0(\mathcal{S}_G)$ identify the normal space to the orbit $\mathcal{O}(M)$ of the semi-simple representation M (as above) with*

$$N_M = \{ M + V \mid V \in rep_\gamma\, Q_1(\mathcal{S}_G) \}$$

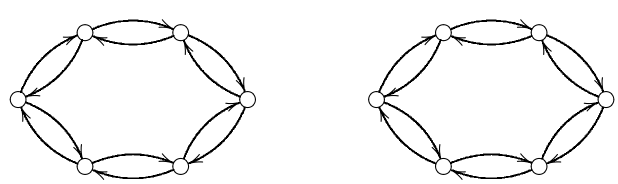
where $\gamma = (c_1, \ldots, c_k)$. Then, $GL(\alpha).N_M$ contains a Zariski open subset of all α-dimensional simple representations of $\pi_1(\mathcal{S}_G)$.

Proof. This is a special case of the Luna slice result applied to the local quiver setting. In fact, one can generalize this result to other known semi-simple representations N of $\pi_1(\mathcal{S}_G)$ but then one has to replace $Q_1(\mathcal{S}_G)$ by the *local quiver* Q_N of N. □

In the $SL_2(\mathbb{Z})$ example, $comp(\overline{\ell}SL_2(\mathbb{Z}))$ is generated by the 12 components of two-dimensional representations of $Q_0(\overline{\ell}SL_2(\mathbb{Z}))$

$$\nu_{ij} = (\delta_{1i}, \ldots, \delta_{4i}; \delta_{1j}, \ldots, \delta_{6j}) \qquad 1 \leq i \leq 4, 1 \leq j \leq 6$$

for which i and j are both even or both odd. From this the structure of the one quiver $Q_1(\overline{\ell}SL_2(\mathbb{Z}))$ (corresponding to the 12 one-dimensional simples of $\overline{\ell}SL_2(\mathbb{Z})$) can be verified to be



Here, the vertices of the first component correspond (in cyclic order) to $\nu_{11}, \nu_{33}, \nu_{15}, \nu_{31}, \nu_{13}, \nu_{35}$ and those of the second component (in cyclic order) to $\nu_{22}, \nu_{44}, \nu_{26}, \nu_{42}, \nu_{24}, \nu_{46}$. Applications to the representation theory of the modular group $SL_2(\mathbb{Z})$ and its central extension B_3 (the third braid group) will be given elsewhere.

In the $\Gamma_0(2)$ example, $comp(\overline{\ell}\Gamma_0(2))$ is generated by the 4 dimension vectors

$$(1,0,0,0;1,0),\ (0,0,1,0;1,0),\ (0,1,0,0;0,1),\ (0,0,0,1;0,1)$$

and one verifies that the one-quiver $Q_1(\overline{\ell}\Gamma_0(2))$ has the following form

Appendix : The component coalgebra $coco(A)$

Over an algebraically closed field $\overline{\ell}$ we have seen that the component semigroup and Euler form contain useful information on the finite dimensional representations of an $\overline{\ell}$-qurve. Clearly, one can repeat all arguments verbatim for an arbitrary ℓ by restricting at those components which contain ℓ-rational points. However, this sub-semigroup $\mathtt{comp}(A)$ of $\mathtt{comp}(A \otimes \overline{\ell})$ is usually too small to be of interest.

Example 3 *Let $\ell \subset L$ be a finite separable field extension of dimension k. As L is a simple algebra, all its finite dimensional representations are of the form $L^{\oplus a}$ and hence only components of $rep_n\ L$ containing ℓ-rational points exist when $k|n$. Over the algebraic closure we have*

$$L \otimes \overline{\ell} = \underbrace{\overline{\ell} \times \ldots \times \overline{\ell}}_{k}$$

whence $comp(L \otimes \overline{\ell}) \simeq \mathbb{N}^k$ generated by the factors of $L \otimes \overline{\ell}$. We have $comp(A) \subset comp(A \otimes \overline{\ell})$ sending the generator k to $(1,\ldots,1)$.

We recall some standard facts from [6, Chp. 1] on unramified commutative algebras over an arbitrary basefield ℓ. A commutative affine ℓ-algebra C is said to be *unramified* whenever

$$C \otimes \overline{\ell} \simeq \overline{\ell} \times \ldots \times \overline{\ell}$$

It is well known that all unramified ℓ-algebras are of the form

$$C \simeq L_1 \times \ldots \times L_k$$

where each L_i is a finite dimensional separable field extension of ℓ. From this it follows that subalgebras, tensorproducts and epimorphic images of unramified ℓ-algebras are again unramified. As a consequence, an affine commutative ℓ-algebra C has a unique *maximal unramified ℓ-subalgebra* $\pi_0(C)$. In case $C = \ell[X]$ is the coordinate algebra of an affine ℓ-scheme X, the algebra $\pi_0(C)$ contains all information about the connected components of X. Recall that an affine ℓ-scheme X (or its coordinate algebra $\ell[X]$) is said to be *connected* if $\ell[X]$ contains no non-trivial idempotents and is called *geometrically connected* if $\ell[X] \otimes \overline{\ell}$ is connected. We summarize [6, I.7] in

Proposition 12 *For an affine ℓ-scheme X we have*

1. *X is connected iff $\pi_0(\ell[X])$ is a field.*
2. *X is geometrically connected iff $\pi_0(\ell[X]) = \ell$.*
3. *If X is connected and has an ℓ-rational point, then X is geometrically connected.*
4. *If $\pi_0(\ell[X]) = L_1 \times \ldots \times L_k$ with all L_i separable field extensions of ℓ, then X has exactly k connected components.*
5. *If Y is an affine ℓ-scheme and $X \longrightarrow Y$ a morphism, then $\pi_0(\ell[Y]) \longrightarrow \pi_0(\ell[X])$ is an ℓ-algebra morphism.*
6. *If Y is an affine ℓ-scheme, then the natural map*
$$\pi_0(\ell[X]) \otimes \pi_0(\ell[Y]) \longrightarrow \pi_0(\ell[X] \otimes \ell[Y]) = \pi_0(\ell[X \times Y])$$
is an ℓ-algebra isomorphism.

Definition 12 *For A an ℓ-qurve consider the sum-maps*
$$rep_n\ A \times rep_m\ A \longrightarrow rep_{m+n}\ A$$
which determine ℓ-algebra morphisms
$$\Delta_{m,n}\ :\ \pi_0(\ell[rep_{m+n}\ A]) \longrightarrow \pi_0(\ell[rep_n\ A]) \otimes \pi_0(\ell[rep_m\ A])$$
Denote $\pi_0(n) = \pi_0(\ell[rep_n\ A])$ and consider the graded ℓ-vectorspace
$$coco(A) = \pi_0(0) \oplus \pi_0(1) \oplus \pi_0(2) \oplus \ldots$$
Define a coalgebra structure by taking as the comultiplication *map*
$$coco(A) \xrightarrow{\Delta} coco(A) \otimes coco(A)$$
$$\sum_{m+n=N} \Delta_{m,n}\ :\ \pi_0(N) \longrightarrow \sum_{n+m=N} \pi_0(n) \otimes \pi_0(m)$$
and as the counit $\mathtt{coco}(A) \xrightarrow{\epsilon} \pi_0(0) = \ell$*. We call $(coco(A), \Delta, \epsilon)$ the* component coalgebra *of the ℓ-qurve A.*

In fact, it follows from the foregoing proposition that $coco(A)$ is in fact a *mock bialgebra*, that is a bialgebra without a unit-map. Recall that if G is a finite group, its *function bialgebra* $func(G)$ is the space of all ℓ-valued functions on G with pointwise multiplication and co-multiplication induced by
$$\Delta(x_g) = \sum_{g'.g"=g} x_{g'} \otimes x_{g"}$$
where x_h is the function mapping $h \mapsto 1$ and all other $h' \in G$ to zero. If G is no longer finite, $func(G)$ is still a mock bialgebra.

Proposition 13 *If A is an ℓ-qurve, then there is an isomorphism of mock bialgebras*
$$coco(A) \otimes \overline{\ell} \simeq func(comp(A \otimes \overline{\ell}))$$
and hence $coco(A)$ contains enough information to reconstruct the component semigroup $comp(A \otimes \overline{\ell})$. Alternatively, the Galois group $Gal(\overline{\ell}/\ell)$ acts on $A \otimes \overline{\ell}$ and hence on $comp(A \otimes \overline{\ell})$ and the function coalgebra. The component coalgebra $coco(A)$ can be obtained by Galois descent
$$coco(A) = func(comp(A \otimes \overline{\ell}))^{Gal(\overline{\ell}/\ell)}$$

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