# A non-commutative topology on $\mathtt{rep}$ A

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**Abstract**

We extend the Zariski topology on $\mathtt{simp}\ A$, the set of all simple finite dimensional representations of $A$, to a non-commutative topology (in the sense of Fred Van Oystaeyen) on $\mathtt{rep}\ A$, the set of all finite dimensional representations of $A$, using Jordan-Hölder filtrations. The non-commutativity of the topology is enforced by the order of the composition factors.

All algebras will be affine associative $\mathbb{k}$-algebras with unit over an algebraically closed field $\mathbb{k}$. The *non-commutative affine 'scheme'* associated to an algebra $A$ is, as a set, the disjoint union

$$\mathtt{rep}\ A = \bigsqcup_n \mathtt{rep}_n\ A$$

where $\mathtt{rep}_n\ A$ is the (commutative) affine scheme of $n$-dimensional representations of $A$. In this note we will equip $\mathtt{rep}\ A$ with a non-commutative topology in the sense of Fred Van Oystaeyen [5, §7.2] (or, more precisely, a slight generalization of it).

Here is the main idea. The twosided prime ideal spectrum $\mathtt{spec}\ A$ is an (ordinary) topological space via the Zariski topology, see for example [4] or [1, §II.6]. Hence, the subset $\mathtt{simp}\ A$ of all simple finite dimensional $A$-representations can be equipped with the induced topology. This topology can then be extended to a non-commutative topology on $\mathtt{rep}\ A$ using Jordan-Hölder filtrations. The non-commutative nature of the topology is enforced by the order of the composition factors.

We give a few examples, connect this notion with that of Reineke's composition monoid and remark on the difference between quotient varieties and moduli spaces from the perspective of non-commutative topology. Finally, we note that this construction can be generalized verbatim to any Artinian Abelian category as soon as we have a topology on the set of simple objects.

# 1 The Zariski topology on $\texttt{simp}\ A$.

Recall that a prime ideal $P$ of $A$ is a twosided ideal satisfying the property that if $I.J \subset P$ then $I \subset P$ or $J \subset P$ for any pair of twosided ideals $I, J$ of $A$. The *prime spectrum* $\texttt{spec}\ A$ is the set of all twosided prime ideals of $A$. The *Zariski topology* on $\texttt{spec}\ A$ has as its closed subsets

$$\mathbb{V}(S) = \{P \in \texttt{spec}\ A \mid S \subset P\}$$

where $S$ varies over all subsets of $A$, see for example [1, Prop. II.6.2]. Note that an algebra morphism $\phi : A \longrightarrow B$ does *not* necessarily induce a continuous map $\phi^* : \texttt{spec}\ B \longrightarrow \texttt{spec}\ A$ but is does so in the case $\phi$ is a *central extension* in the sense of [1, §II.6].

If $M \in \texttt{rep}_n\ A$ is a simple $n$-dimensional representation, there is a defining epimorphism $\psi_M : A \twoheadrightarrow M_n(\Bbbk)$ and the kernel of this morphism $\texttt{ker}\ \psi_M$ is a twosided maximal (hence prime) ideal of $A$. We define the Zariski topology on the set of all simple finite dimensional representations $\texttt{simp}\ A$ by taking as its closed subsets

$$\mathbb{V}(S) = \{M \in \texttt{simp}\ A \mid S \subset \texttt{ker}\ \psi_M\}$$

Again, one should be careful that whereas an algebra map $\phi : A \longrightarrow B$ induces a map $\phi^* : \texttt{rep}\ B \longrightarrow \texttt{rep}\ A$ it does *not* in general map $\texttt{simp}\ B$ to $\texttt{simp}\ A$ (unless $\phi$ is a central extension).

With $\mathcal{L}_A$ we will denote the set of all open subsets of $\texttt{simp}\ A$. $\mathcal{L}_A$ will be the set of *letters* on which to base our non-commutative topology.

# 2 Non-commutative topologies (and generalizations).

In [5, Chp. 7] Fred Van Oystaeyen defined *non-commutative topologies* which are generalizations of usual topologies in which it is no longer true that $A \cap A$ is equal to $A$ for an open set $A$. In order to keep dichotomies of possible definitions to a minimum he imposed left-right symmetric conditions on the definition. However, for applications to representation theory it seems that the most natural non-commutative topologies are truly one-sided. For this reason we take some time to generalize some definitions and results of [5, Chp. 7].

We fix a partially ordered set $(\Lambda, \leq)$ with a unique minimal element $0$ and a unique maximal element $1$, equipped with two operations $\wedge$ and $\vee$. With $i_\Lambda$ we will denote the set of all *idempotent elements* of $\Lambda$, that is, those $x \in \Lambda$ such that $x \wedge x = x$. A *finite global cover* is a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ such that $1 = \lambda_1 \vee \ldots \vee \lambda_n$. In the table below we have listed the conditions for a (one-sided) non-commutative topology. Note that some requirements are less essential than others. For example, the covering condition (A10) is only needed if we want to fit non-commutative topologies in the framework of non-commutative Grothendieck topologies [5] and the weak modularity condition (A9) is not required if every basic open is $\vee$-idempotent (as is the case in most examples).

| | | | |
|---|---|---|---|
| (A1) | $x \wedge y \leq x$ | | $x \wedge y \leq y$ |
| (A2) | $x \wedge 1 = x$<br>$x \wedge 0 = 0$ | | $1 \wedge x = x$<br>$0 \wedge x = 0$ |
| (A3) | | $(x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y \wedge z$ | |
| (A4) | $x \leq y \Rightarrow z \wedge x \leq z \wedge y$ | | $x \leq y \Rightarrow x \wedge z \leq y \wedge z$ |
| (A5) | $x \leq x \vee y$ | | $y \leq x \vee y$ |
| (A6) | $x \vee 1 = 1$<br>$x \vee 0 = x$ | | $1 \vee x = 1$<br>$0 \vee x = x$ |
| (A7) | | $(x \vee y) \vee z = x \vee (y \vee z) = x \vee y \vee z$ | |
| (A8) | $x \leq y \Rightarrow x \vee z \leq y \vee z$ | | $x \leq y \Rightarrow z \vee x \leq z \vee y$ |
| (A9) | $a \vee (a \wedge b) \leq (a \vee a) \wedge b$ | | $a \vee (b \wedge a) \leq (a \vee b) \wedge a$ |
| (A10) | $x = (x \wedge \lambda_1) \vee \ldots \vee (x \wedge \lambda_n)$ | | $x = (\lambda_1 \wedge x) \vee \ldots \vee (\lambda_n \wedge x)$ |

**Definition 1** Let $(\Lambda, \leq)$ be a partially ordered set with minimal and maximal element $0$ and $1$ and operations $\wedge$ and $\vee$. Then,

$\Lambda$ is said to be a *left non-commutative topology* if and only if the left and middle column conditions of (A1)-(A10) are valid for all $x, y, z \in \Lambda$, all $a, b \in i_\Lambda$ with $a \leq b$ and all finite global covers $\{\lambda_1, \ldots, \lambda_n\}$.

$\Lambda$ is said to be a *right non-commutative topology* if and only if the middle and right column conditions of (A1)-(A10) are valid for all $x, y, z \in \Lambda$, all $a, b \in i_\Lambda$ with $a \leq b$ and all finite global covers $\{\lambda_1, \ldots, \lambda_n\}$.

$\Lambda$ is said to be a *non-commutative topology* if and only if the conditions (A1)-(A10) are valid for all $x, y, z \in \Lambda$, all $a, b \in i_\Lambda$ with $a \leq b$ and all finite global covers $\{\lambda_1, \ldots, \lambda_n\}$.

There are at least two ways of building a genuine non-commutative topology out of these sets of basic opens. We briefly sketch the procedures here and refer to the forthcoming monograph [6] for details in the symmetric case (the one-sided versions present no real problems).

Let $T(\Lambda)$ be the set of all finite $(\wedge, \vee)$-words in the *contractible* idempotent elements $i_\Lambda$ (that is, $\lambda \in i_\Lambda$ such that for all $\lambda_1, \lambda_2$ with $\lambda \leq \lambda_1 \vee \lambda_2$ we have that $\lambda = (\lambda \wedge \lambda_1) \vee (\lambda \wedge \lambda_2)$). If $\Lambda$ is a (left,right) non-commutative topology, then so is $T(\Lambda)$. The $\vee$-*complete topology of virtual opens* $T'(\Lambda)$ is then the set of all $(\wedge, \vee)$-words in the contractible idempotents of finite length in $\wedge$ (but not necessarily of finite length in $\vee$). This non-commutative topology has properties very similar to that of an ordinary topology and, in fact, has associated to it a *commutative shadow*.

The second construction, leading to the *pattern topology*, starts with the equivalence classes of *directed systems* $S \subset \Lambda$ (that is, if for all $x, y \in S$ there is a $z \in S$ such that $z \leq x$ and $z \leq y$) and where the equivalence relation $S \sim S'$ is defined by

$$\begin{cases} \forall a \in S, \exists a' \in S, a' \leq a \text{ and } b \leq a' \leq b' \text{ for some } b, b' \in S' \\ \forall b \in S', \exists b' \in S', b' \leq b \text{ and } a \leq b' \leq a' \text{ for some } a, a' \in S \end{cases}$$

One can extend the $\wedge, \vee$ operations on $\Lambda$ to the equivalence classes $C(\Lambda) = \{[S] \mid S \text{ directed } \}$ in the obvious way such that also $C(\Lambda)$ is a (left,right) non-commutative topology. A directed set $S \subset \Lambda$ is said to be *idempotent* if for all $a \in S$, there is an $a' \in S \cap i_\Lambda$ such that $a' \leq a$. If $S$ is idempotent then $[S] \in i_{C(\Lambda)}$ and those idempotents will be called *strong idempotents*. The pattern topology $\Pi(\Lambda)$ is the (left,right) non-commutative topology of finite $(\wedge, \vee)$-words in the strong idempotents of $C(\Lambda)$. A directed system $[S]$ is called a *point* iff $[S] \leq \vee [S_\alpha]$ implies that $[S] \leq [S_\alpha]$ for some $\alpha$.

## 3 The basic opens.

For an $n$-dimensional representation $M$ of $A$ we call a finite filtration of length $u$

$$\mathcal{F}^u \ : \ 0 = M_0 \subset M_1 \subset \ldots \subset M_u = M$$

of $A$-representations a *Jordan-Hölder filtration* if the successive quotients

$$\mathcal{F}_i = \frac{M_i}{M_{i-1}}$$

are simple $A$-representations. Recall that $\mathcal{L}_A$ is the set of all open subsets $V$ of $\mathtt{simp}\ A$. With $\mathbb{W}_A$ we denote the non-commutative words in these letters

$$\mathbb{W}_A = \{V_1 \dots V_k \mid V_i \in \mathcal{L}_A, k \in \mathbb{N}\}$$

For a given word $w = V_1 V_2 \dots V_k \in \mathbb{W}_A$ we define the *left basic open set*

$$\mathcal{O}^l_w = \{M \in \mathtt{rep}\ A \mid \exists \mathcal{F}^u \text{ Jordan-Hölder filtration on } M \text{ such that } \mathcal{F}_i \in V_i\}$$

and the *right basic open set*

$$\mathcal{O}^r_w = \{M \in \mathtt{rep}\ A \mid \exists \mathcal{F}^u \text{ Jordan-Hölder filtration on } M \text{ such that } \mathcal{F}_{u-i} \in V_{k-i}\}$$

Finally, to make these definitions symmetric we define the *basic open set*

$$\mathcal{O}_w = \{M \in \mathtt{rep}\ A \mid \exists \mathcal{F}^u \text{ Jordan-Hölder filtration on } M \text{ such that } \mathcal{F}_{i_j} \in V_j$$
$$\text{for some } 1 \leq i_1 < i_2 < \dots < i_k \leq u\ \}$$

Clearly, $\mathcal{O}^l_w$ consists of those representations having prescribed bottom structure, whereas $\mathcal{O}^r_w$ consists of those with prescribed top structure. In order to avoid three sets of definitions we will denote from now on $\mathcal{O}^\bullet_w$ whenever we mean $\bullet \in \{l, r, \emptyset\}$.

If $w = L_1 \dots L_k$ and $w' = M_1 \dots M_l$, we will denote with $w \cup w'$ the *multi-set* $\{N_1, \dots, N_m\}$ where each $N_i$ is one of $L_j, M_j$ and $N_i$ occurs in $w \cup w'$ as many times as its maximum number of factors in $w$ or $w'$. With $\mathtt{rep}(w \cup w')$ we denote the subset of $\mathtt{rep}\ A$ consisting of the representations of $M$ having a Jordan-Hölder filtration having factor-multi-set containing $w \cup w'$. For any triple of words $w, w'$ and $w$" we denote $\mathcal{O}^\bullet_{w"}(w \cup w') = \mathcal{O}^\bullet_{w"} \cap \mathtt{rep}(w \cup w')$.

We define an equivalence relation on the basic open sets by

$$\mathcal{O}^\bullet_w \approx \mathcal{O}^\bullet_{w'} \qquad \Leftrightarrow \qquad \mathcal{O}^\bullet_w(w \cup w') = \mathcal{O}^\bullet_{w'}(w \cup w')$$

The reason for this definition is that the condition of $M \in \mathcal{O}^\bullet_w$ is void if $M$ does not have enough Jordan-Hölder components to get all factors of $w$ which makes it impossible to define equality of basic open sets defined by different words.

We can now define the partially ordered sets $\Lambda^\bullet_A$ as consisting of all basic open subsets $\mathcal{O}^\bullet_w$ of $\mathtt{rep}\ A$. The partial ordering $\leq$ is induced by set-theoretic inclusion modulo equivalence, that is,

$$\mathcal{O}^\bullet_w \leq \mathcal{O}^\bullet_{w'} \qquad \Leftrightarrow \qquad \mathcal{O}^\bullet_w(w \cup w') \subseteq \mathcal{O}^\bullet_{w'}(w \cup w')$$

As a consequence, equality $=$ in the set $\Lambda^\bullet_A$ coincides with equivalence $\approx$. Observe that these partially ordered sets have a unique minimal and a unique maximal element (upto equivalence)

$$0 = \emptyset = \mathcal{O}^\bullet_\emptyset \qquad \text{and} \qquad 1 = \mathtt{rep}\ A = \mathcal{O}^\bullet_{\mathtt{simp}\ A}$$

The operations $\vee$ and $\wedge$ are defined as follows : $\vee$ is induced by ordinary set-theoretic union and $\wedge$ is induced by concatenation of words, that is

$$\mathcal{O}^\bullet_w \wedge \mathcal{O}^\bullet_{w'} \approx \mathcal{O}^\bullet_{ww'}$$

**Theorem 1** *With notations as before :*

- $(\Lambda^l_A, \leq, \approx, 0, 1, \vee, \wedge)$ *is a left non-commutative topology on* $\texttt{rep}\ A$*.*
- $(\Lambda^r_A, \leq, \approx, 0, 1, \vee, \wedge)$ *is a right non-commutative topology on* $\texttt{rep}\ A$*.*

**Proof.** The tedious verification is left to the reader. Here, we only stress the importance of the equivalence relation for example in verifying $x \wedge 1 = x$. So, let $w = L_1 \ldots L_k$ then

$$\mathcal{O}^l_w \wedge 1 = \mathcal{O}^l_{L_1 \ldots L_k \texttt{simp} A} \subset \mathcal{O}^l_w$$

and this inclusion is proper (look at elements in $\mathcal{O}^l_w$ having exactly $k$ composition factors). However, as soon as the representation has $k+1$ composition factors, it is contained in the left hand side whence $\mathcal{O}^l_w \wedge 1 \approx \mathcal{O}^l_w$. A similar argument is needed in the covering condition. $\square$

Note however that $(\Lambda_A, \leq, \approx, 0, 1, \vee, \wedge)$ is not necessarily a non-commutative topology : the problematic conditions are $\mathcal{O}_w \wedge 1 = \mathcal{O}_w = 1 \wedge \mathcal{O}_w$ and the covering condition. The reason is that for $w = L_1 \ldots L_k$ as before and $M \in \mathcal{O}_w$ having $> k$ factors, it may happen that the last factor is the one in $L_k$ leaving no room for a successive factor in $\texttt{simp}\ A$ (whence $\mathcal{O}_w \cap 1$ is not equivalent to $\mathcal{O}_w$).

**Example 1** Let $A$ be a finite dimensional algebra, then $A$ has a finite number of simple representations $\texttt{simp}\ A = \{S_1, \ldots, S_n\}$ and the Zariski topology is the discrete topology. If for some $1 \leq i, j \leq n$ we have that

$$Ext^1_A(S_i, S_j) = 0 \qquad \text{and} \qquad Ext^1_A(S_j, S_i) \neq 0$$

then $\Lambda^l_A$ is a genuinely non-commutative topology, for example

$$\mathcal{O}^l_{S_i} \wedge \mathcal{O}^l_{S_j} = \mathcal{O}^l_{S_i S_j} \neq \mathcal{O}^l_{S_j S_i} = \mathcal{O}^l_{S_j} \wedge \mathcal{O}^l_{S_i}$$

as a non-trivial extension $0 \longrightarrow S_i \longrightarrow X \longrightarrow S_j \longrightarrow 0$ belongs to $\mathcal{O}^l_{S_i S_j}(S_i S_j \cup S_j S_i)$ but not to $\mathcal{O}^l_{S_j S_i}(S_i S_j \cup S_j S_i)$.

## 4 Reineke's mon(str)oid.

When $A$ is the path algebra of a quiver without oriented cycles we can generalize the foregoing example and connect the previous definitions to the *composition monoid* introduced and studied by Markus Reineke in [2].

Let $Q$ be a quiver without oriented cycles, then its path algebra $A = \Bbbk Q$ is finite dimensional hereditary with all simple representations one-dimensional and in one-to-one correspondence with the vertices of $Q$. For every dimension $n$ we have that

$$\texttt{rep}_n\ A = \bigsqcup_{|\alpha| = n} GL_n \times^{GL(\alpha)} \texttt{rep}_\alpha\ Q$$

where $\alpha$ runs over all dimension vectors of total dimension $n$ and where $\mathtt{rep}_\alpha\, Q$ is the affine space of all $\alpha$-dimensional representations of the quiver $Q$ with base-change group action by $GL(\alpha)$.

The *Reineke monstroid* $\mathcal{M}(Q)$ has as its elements the set of all irreducible closed $GL(\alpha)$-stable subvarieties of $\mathtt{rep}_\alpha\, Q$ for all dimension vectors $\alpha$, equipped with a product

$$\mathcal{A} * \mathcal{B} = \{X \in \mathtt{rep}_{\alpha+\beta}\, Q \mid \text{there is an exact sequence}$$
$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0 \quad M \in \mathcal{A}, N \in \mathcal{B}\,\}$$

if $\mathcal{A}$ (resp. $\mathcal{B}$) is an element of $\mathcal{M}(Q)$ contained in $\mathtt{rep}_\alpha\, Q$ (resp. in $\mathtt{rep}_\beta\, Q$). It is proved in [2, lemma 2.2] that $\mathcal{A} * \mathcal{B}$ is again an element of $\mathcal{M}(Q)$. This defines a monoid structure on $\mathcal{M}(Q)$ which is too unwieldy to study directly. Observe that we changed the order of the terms wrt. the definition given in [2]. That is, we will work with the *opposite* monoid of [2].

On the other hand, the *Reineke composition monoid* is very tractable. It is the submonoid $\mathcal{C}(Q)$ of $\mathcal{M}(Q)$ generated by the vertex-representation spaces $R_i = \mathtt{rep}_{\delta_i}\, Q$. These generators satisfy specific commutation relations which can be read off from the quiver structure, see [2, §5]. For example, if there are no arrows between $v_i$ and $v_j$ then

$$R_i * R_j = R_j * R_i$$

and if there are no arrows from $v_i$ to $v_j$ but $n$ arrows from $v_j$ to $v_i$, then

$$\begin{cases} R_i^{*(n+1)} * R_j = R_i^{*n} * R_j * R_i \\ R_i * R_j^{*(n+1)} = R_j * R_i * R_j^{*n} \end{cases}$$

For more details on the structure of $\mathcal{C}(Q)$ we refer to [2, §5].

There is a relation between $\mathcal{C}(Q)$ and the left- and right- non-commutative topologies $\Lambda^l_A$ and $\Lambda^r_A$. Because the Zariski topology on $\mathtt{simp}\, A$ is the discrete topology on the set $\{S_1, \ldots, S_k\}$ of vertex simples, it is important to understand $\mathcal{O}^r_w$ where $w$ is a word in the $S_i$, say $w = S_{i_1} S_{i_2} \ldots S_{i_u}$. In fact, we could have based our definition of a one-sided non-commutative topology on the set $\mathcal{L}_A$ of *irreducible* open subsets of $\mathtt{simp}\, A$ and then these basic opens would be all. If $\mathcal{C}$ is a $GL(\alpha)$-stable subset of $\mathtt{rep}_\alpha\, Q$ with $|\alpha| = n$, we will denote the subset $GL_n \times^{GL(\alpha)} \mathcal{C}$ of $\mathtt{rep}_n\, A$ by $\tilde{\mathcal{C}}$.

**Proposition 1**

$$\mathcal{O}^l_w = \bigcup_{w'} \tilde{\mathcal{A}}_{w'} \qquad \textit{resp.} \qquad \mathcal{O}^r_w = \bigcup_{w'} \tilde{\mathcal{A}}_{w'}$$

*where $\mathcal{A}_{w'}$ is a $*$-word in the generators $R_i$ of the composition monoid such that $w'$ can be rewritten (using the relations in $\mathcal{C}(Q)$) in the form*

$$w' = R_{i_1} * R_{i_2} * \ldots * R_{i_u} * w" \quad \textit{resp.} \quad w' = w" * R_{i_1} * R_{i_2} * \ldots * R_{i_u}$$

*for another $*$-word $w"$.*

Also, the equivalence relation introduced before can be expressed in terms of $\mathcal{C}(Q)$. If $w = S_{i_1}S_{i_2}\dots S_{i_u}$ and $w' = S_{j_1}S_{j_2}\dots S_{j_v}$ such that $w \cup w' = \{S_{k_1},\dots,S_{k_w}\}$, then

**Proposition 2** *$\mathcal{O}^l_w \approx \mathcal{O}^l_{w'}$ if and only if every $*$-word $v = R_{a_1} * \dots * R_{a_z}$ containing in it distinct factors $R_{k_1},\dots,R_{k_w}$ which can be brought in $\mathcal{C}(Q)$ in the form*

$$v = R_{i_1} * \dots * R_{i_u} * v'$$

*can also be written in the form*

$$v = R_{j_1} * \dots * R_{j_v} * v"$$

*(and conversely). A similar result describes $\mathcal{O}^r_w \approx \mathcal{O}^r_{w'}$.*

In particular, in this setting there will be hardly any *idempotent* basic opens (that is, satisfying $\mathcal{O}^r_w \wedge \mathcal{O}^r_w \approx \mathcal{O}^r_w$). Clearly, if $\{S_{e_1},\dots,S_{e_a}\}$ are simples such that the quiver restricted to $\{v_{e_1},\dots,v_{e_a}\}$ has no arrows, then any word $w$ in the $S_{e_j}$ gives an idempotent $\mathcal{O}^r_w$. In the following section we will give an example where *every* basic open is idempotent and hence we get a commutative topology.

# 5 The commutative case.

If $A$ is a commutative affine $\mathbb{k}$-algebra, then any simple representation is one-dimensional, $\texttt{simp}\ A = X_A$ the affine (commutative) variety corresponding to $A$ and the Zariski topologies on both sets coincide. Still, one can define the non-commutative topologies on $\texttt{rep}\ A$. However,

**Proposition 3** *If $A$ is a commutative affine $\mathbb{k}$-algebra, then both $\Lambda^l_A$ and $\Lambda^r_A$ are commutative topologies. That is, for all words $w$ and $w'$ in $\mathcal{L}_A$ we have*

$$\mathcal{O}^l_w \wedge \mathcal{O}^l_{w'} \approx \mathcal{O}^l_{w'} \wedge \mathcal{O}^l_w \qquad \textit{and} \qquad \mathcal{O}^r_w \wedge \mathcal{O}^r_{w'} \approx \mathcal{O}^r_{w'} \wedge \mathcal{O}^r_w$$

**Proof.** We claim that every basic open $\mathcal{O}^l_w$ is idempotent. Observe that all simple $A$-representations are one-dimensional and that there are only self-extensions of those, that is, if $S$ and $T$ are non-isomorphic simples, then $Ext^1_A(S,T) = 0 = Ext^1_A(T,S)$. However, there are self-extensions with the dimension of $Ext^1_A(S,S)$ being equal to the dimension of the tangent space at $X_A$ in the point corresponding to $S$. As a consequence we have for any Zariski open subsets $U$ and $V$ of $X_A$ that

$$\mathcal{O}^l_{UV} = \mathcal{O}^l_{VU}$$

as we can change the order of the filtration factors (a representation $M$ is the direct sum of submodules $M_1 \oplus \dots \oplus M_s$ with each $M_i$ concentrated in a single simple $S_i$ and we can add the successive $S_i$ factors of $M$ at any wanted place in the filtration sequence). Hence, for every word $w$ we have that

$$\mathcal{O}^l_w \approx \mathcal{O}^l_w \wedge \mathcal{O}^l_w$$

and also for any pair of words $w$ and $w'$ we have that

$$\mathcal{O}^l_w \wedge \mathcal{O}^l_{w'} = \mathcal{O}^l_{ww'} = \mathcal{O}^l_{w'w} = \mathcal{O}^l_{w'} \wedge \mathcal{O}^l_w$$

Observe that in [5] it is proved that a non-commutative topology in which every basic open is idempotent is commutative. We cannot use this here as the proof of that result uses both the left- and right- conditions. However, we are dealing here with a very simple example. $\square$

# 6 Quotient varieties versus moduli spaces.

Having defined a one-sided non-commutative topology on $\mathtt{rep}\ A$ we can ask about the induced topology on the quotient variety $\mathtt{iss}\ A$ of all isomorphism classes of semi-simple $A$-representations or on the moduli space $\mathtt{moduli}_\theta\ A$ with respect to a certain stability structure $\theta$, cfr. [3]. Experience tells us that it is a lot easier to work with quotient varieties than with moduli spaces and non-commutative topology may give a partial explanation for this.

Indeed, as the points of $\mathtt{iss}\ A$ are semi-simple representations, it is clear that the induced non-commutative topology on $\mathtt{iss}\ A$ is in fact commutative. However, as the points of $\mathtt{moduli}_\theta\ A$ correspond to isomorphism classes of direct sums of stable representations (not simples!), the induced non-commutative topology on $\mathtt{moduli}_\theta\ A$ will in general remain non-commutative. Still, in nice examples, such as representations of quivers, one can define another non-commutative topology on $\mathtt{moduli}_\theta\ A$ which does become commutative. Use universal localization to cover $\mathtt{moduli}_\theta\ A$ by opens isomorphic to $\mathtt{iss}\ A_\Sigma$ for some families $\Sigma$ of maps between projectives and equip $\mathtt{moduli}_\theta\ A$ with a non-commutative topology (which then will be commutative!) obtained by gluing the induced non-commutative topologies on the $\mathtt{rep}\ A_\Sigma$.

# 7 Generalizations.

It should be evident that our construction can be carried out verbatim in the setting of any Artinian Abelian category (that is, an Abelian category having Jordan-Hölder sequences) as soon as we have a natural topology on the set of simple objects. In fact, the same procedure can be applied when we have a left (or right) non-commutative topology on the simples.

In fact, the construction may even be useful in Abelian categories in which every object is filtered by special objects on which we can define a (one-sided) (non-commutative) topology.

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