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One quiver to rule them all

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ABSTRACT: In math.AG/9904171 it was shown that the étale local structure of finite dimensional representations for a formally smooth algebra is determined by (varying) local quiver settings. In this note we prove that there is one quiver setting (Q_A, α_A) depending only on the formally smooth algebra A which contains enough information to reconstruct all these local quiver settings. Conjecturally, the formally smooth algebra A is locally isomorphic in a (yet to be developed) non-commutative étale topology to an algebra B Morita equivalent (determined by the dimension vector α_A) to the path algebra $\mathbb{C}Q_A$.

Contents

In memory of International Law † *20-03-2003*

Following [7] and [6] one defines a non-commutative smooth affine variety to correspond to a *formally smooth algebra* A. Such an algebra A has the lifting property with respect to nilpotent ideals in `alg`, the category of all associative $\mathbb{C}$-algebras with unit. That is, for every $B \in$ `alg`, every nilpotent ideal $I \triangleleft B$ and every $\mathbb{C}$-algebra morphism $\phi : A \longrightarrow B/I$, there exists a lifted algebra morphism $\tilde{\phi}$ making the diagram below commutative

$$\begin{tikzcd} A \arrow[r, dotted, "\tilde{\phi}"] \arrow[dr, "\phi"'] & B \arrow[d] \\ & \frac{B}{I} \end{tikzcd}$$

This notion generalizes Grothendieck's characterization of commutative regular algebras (replacing `alg` by `commalg`, the category of all commutative $\mathbb{C}$-algebras) but as the lifting property in `alg` is stronger not all commutative regular algebras will be formally smooth. In fact, by [6] any commutative formally smooth affine algebra is the coordinate ring of a disjoint union of points and smooth affine curves.

Typical non-commutative examples of formally smooth algebras are path algebras $\mathbb{C}Q$ of finite quivers Q (see [9]), in particular free associative algebras $\mathbb{C}\langle x_1, \ldots, x_n \rangle$. Following [7] (or [9]) one assigns to a formally smooth affine algebra A the family of finite dimensional representation schemes $\{\mathtt{rep}_n\ A : n \in \mathbb{N}\}$ each element of which is a smooth affine (commutative) variety (possibly containing several connected components). There is a natural base-change action by GL_n on $\mathtt{rep}_n\ A$ with quotient variety $\mathtt{iss}_n\ A$ parametrizing isomorphism classes of semi-simple n-dimensional representations. The étale local structure of the quotient varieties $\mathtt{iss}_n\ A$ is described in terms of local quiver-settings which we will recall briefly.

1. Local quiver settings

A point ξ in $\mathtt{iss}_n\ A$ corresponds to the isomorphism class of a semi-simple n-dimensional representation of A

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i are the distinct simple components (say of dimension d_i) occurring in A with multiplicity e_i. Construct a quiver Q_ξ on k-vertices $\{v_1, \ldots, v_k\}$ (corresponding to the distinct simple components of M_ξ) with the property that the number of directed arrows from v_i to v_j is equal to the dimension of the extension group $Ext^1_A(S_i, S_j)$. Consider the dimension vector $\alpha_\xi = (e_1, \ldots, e_k)$ (corresponding to the multiplicities of the simple components in M_ξ) and recall that $\mathtt{rep}_{\alpha_{xi}}\ Q_\xi$ is the affine space of all α_ξ-dimensional representations of the quiver Q_ξ. On this space there is a base-change action by the group $GL(\alpha_\xi) = GL_{e_1} \times \ldots \times GL_{e_k}$ and the corresponding affine quotient variety $\mathtt{iss}_{\alpha_\xi}\ Q_\xi$ parametrizes semi-simple α_ξ-dimensional representations of Q_ξ, see [11]. As $n = \sum_i e_i d_i$ there is a natural embedding of $GL(\alpha_\xi)$ in GL_n. With these notations, the étale local structure of the quotient variety $\mathtt{iss}_n\ A$ near ξ is described by the following result of [9].

Theorem 1 *If A is a formally smooth affine algebra, there is a GL_n-equivariant étale isomorphism between*

$$\mathtt{rep}_n\ A \qquad \textit{and} \qquad GL_n \times^{GL(\alpha_\xi)} \mathtt{rep}_{\alpha_\xi}\ Q_\xi$$

in a neighborhood of the orbit of M_ξ (resp. the orbit of the zero representation). As a consequence, there is an étale isomorphism between

$$\mathtt{iss}_n\ A \qquad \textit{and} \qquad \mathtt{iss}_{\alpha_\xi}\ Q_\xi$$

in a neighborhood of ξ (resp. a neighborhood of the point $\overline{0}$ corresponding to the zero representation).

The main result of this note asserts that there is one quiver setting (Q_A, α_A) depending only on the formally smooth algebra A that contains enough information to reconstruct all these local quiver settings (Q_ξ, α_ξ) for $\xi \in \mathtt{iss}_n\ A$ for any $n \in \mathbb{N}$.

2. The quiver setting (Q_A, α_A)

If A is a formally smooth affine algebra we can decompose the affine smooth variety

$$\mathtt{rep}_n\ A = \bigsqcup_{|\alpha|=n} \mathtt{rep}_\alpha\ A$$

into its connected components $\mathtt{rep}_\alpha\ A$ and we will call the label α a dimension vector of A of total dimension n. The set $\mathtt{comp}\ A$ of all dimension vectors of A can be equipped with an Abelian semigroup structure by defining $\alpha + \beta = \gamma$ whenever $\mathtt{rep}_\gamma\ A$ is the connected component of $\mathtt{rep}_{m+n}\ A$ containing the image of $\mathtt{rep}_\alpha\ A \times \mathtt{rep}_\beta\ A$ under the direct sum map

$$\mathtt{rep}_n\ A \times \mathtt{rep}_m\ A \xrightarrow{\oplus} \mathtt{rep}_{m+n}\ A$$

for α a dimension vector of total dimension n and β of total dimension m. The semigroup $\texttt{comp}\ A$ is augmented as there is a natural map $\texttt{comp}\ A \longrightarrow \mathbb{N}$ sending α to its total dimension $|\alpha|$. In fact, for a formally smooth algebra A this definition of $\texttt{comp}\ A$ coincides with the component semigroup introduced and studied by K. Morrison in [12].

Clearly, $\texttt{comp}\ \mathbb{C}Q \simeq \mathbb{N}^k$ if Q has k vertices as the semigroup generators correspond to the vertex-simples. If X is a smooth affine curve, then $\texttt{comp}\ \mathbb{C}[X] \simeq \mathbb{N}$ as all $\texttt{rep}_n\ \mathbb{C}[X]$ are irreducible smooth affine varieties. In general though the structure of $\texttt{comp}\ A$ can be quite complicated.

Example 1 For every sub-semigroup $S \subset \mathbb{N}$ there exists an affine formally smooth algebra A such that

$$\texttt{comp}\ A \simeq S$$

as augmented semigroups.

Proof. A special affine algebra of inversion depth b A has a presentation

$$A = \frac{\mathbb{C}\langle x_1, \ldots, x_a, y_1, \ldots, y_b\rangle}{(1 - y_i p_i(x_1, \ldots, x_a, y_1, \ldots, y_{i-1})\ ,\ 1 \leq i \leq b)}$$

where the p_i are polynomials in the noncommuting variables $\{x_1, \ldots, x_a, y_1, \ldots, y_{i-1}\}$. Therefore, A is a universal localization (see e.g. [15] for definition and properties) of the free associative algebra $\mathbb{C}\langle x_1, \ldots, x_a\rangle$ and as such A is formally smooth. Because $\texttt{rep}_n\ \mathbb{C}\langle x_1, \ldots, x_a\rangle = M_n(\mathbb{C})^{\oplus a}$ is irreducible, $\texttt{comp}\ A \subset \mathbb{N}$ and consists of those $n \in \mathbb{N}$ such that none of the p_i is a rational identity for $n \times n$ matrices (see [8]).

In the special case when $gcd(S) = 1$ it was proved in [8, Coroll. 1] that there is a special affine algebra A such that $\texttt{comp}\ A = S$. In general, if $gcd(S) = n$ let $S' = S/n$ and A' a special affine algebra such that $\texttt{comp}\ A' = S'$. Then, the free algebra product $A = M_n(\mathbb{C}) * A'$ is again a formally smooth algebra and satisfies $\texttt{comp}\ A = S$.

In all these examples, the component semigroup is finitely generated but we do not know whether this is always the case for an affine formally smooth algebra. From now on, we will impose :

Assumption 1 : A is an affine formally smooth algebra such that $\texttt{comp}\ A$ has a finite number of semigroup generators.

There exists a ringtheoretic characterization of these semigroup generators :

Lemma 1 *The following are equivalent*

1. *α is a semigroup generator of $\texttt{comp}\ A$ of total dimension n.*
2. *The quotient map $\texttt{rep}_\alpha\ A \twoheadrightarrow \texttt{iss}_\alpha\ A$ is a principal PGL_n-fibration in the étale topology.*
3. *The ring $\int_\alpha A$ of all GL_n-equivariant maps from $\texttt{rep}_\alpha\ A$ to $M_n(\mathbb{C})$ is an Azumaya algebra with center $\mathbb{C}[\texttt{iss}_\alpha\ A]$.*

Proof. If $M \in \mathtt{rep}_\alpha\ A$ is not a simple representation, then the orbit closure $\mathcal{O}(M)$ contains a semi-simple representation corresponding to the Jordan-Hölder decomposition of M into distinct simple components

$$N = S_1^{\oplus e_1} \oplus \ldots \oplus S_l^{\oplus e_l}$$

where S_i is simple of dimension vector β_i. But then, $\alpha = \sum_i e_i \beta_i$. As the stabilizer subgroup of a simple representation is $\mathbb{C}^*$ (by Schur's lemma) this proves that 1 and 2 are equivalent. The equivalence of 2 and 3 follows from [13] and the fact that both are classified by the étale cohomology group $H^1_{et}(\mathtt{iss}_\alpha\ A, PGL_n)$.

For $\alpha, \beta \in \mathtt{comp}\ A$ and $N \in \mathtt{rep}_\alpha\ A$. $M \in \mathtt{rep}_\beta\ A$ we know from [5, Lemma 4.3] that the dimension of the extension space

$$dim_{\mathbb{C}}\ Ext^1_A(N, M)$$

is an upper semi-continuous function on $\mathtt{rep}_\alpha\ A \times \mathtt{rep}_\beta\ A$. In particular, there is a Zariski open subset $E(\alpha, \beta)$ of this product where this dimension attains its minimal value which we will denote by $ext(\alpha, \beta)$.

Definition 1 Let A be an affine formally smooth algebra having a finite set of semigroup generators $\{\beta_1, \ldots, \beta_k\}$ of $\mathtt{comp}\ A$. Let Q_A be the quiver on k vertices $\{v_1, \ldots, v_k\}$ such that the number of directed arrows from v_i to v_j is equal to $ext(\beta_i, \beta_j)$ and the number of loops in v_i is equal to $dim\ \mathtt{iss}_{\beta_i}\ A$. Let α_A be the dimension vector $(n_1, \ldots, n_k)$ where $n_i = |\beta_i|$. We say that A is a formally smooth algebra of type (Q_A, α_A).

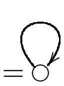
3. Some examples

Let A be an affine formally smooth algebra of type (Q_A, α_A) with $\alpha_A = (n_1, \ldots, n_k)$ and construct the algebra

$$B = \begin{bmatrix} B_{11} & \ldots & B_{1k} \\ \vdots & & \vdots \\ B_{k1} & \ldots & B_{kk} \end{bmatrix}$$

where B_{ij} is the $n_i \times n_j$ block matrix with all its components equal to the subspace of $\mathbb{C}Q_A$ spanned by all oriented paths in Q_A starting at v_i and ending at v_j. That is, B is an affine formally smooth algebra which is Morita equivalent to the path algebra $\mathbb{C}Q_A$ with the Morita equivalence determined by the dimension vector α_A. The hope is that in a (yet to be developed) non-commutative étale topology the algebras A and B are locally isomorphic. The examples below may add some weight to this conjecture.

Example 2 Let X be an affine smooth curve with coordinate ring $A = \mathbb{C}[X]$, then $\mathtt{comp}\ A = \mathbb{N}$ with generating component $\mathtt{rep}_1\ A = \mathtt{iss}_1\ A = X$. Therefore, A is of type

$Q_A =$  and $\alpha_A = (1)$

The associated algebra B is in this case isomorphic to the polynomial ring $\mathbb{C}[x]$ which is indeed locally isomorphic (in the étale topology) to A.

Example 3 If A is the path algebra $\mathbb{C}Q$ then $\texttt{comp}\ A = \mathbb{N}^k$ with semigroup generators the vertex dimension vectors $\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Clearly,

$$\texttt{rep}_{\delta_i}\ A = \texttt{iss}_{\delta_i}\ A = \mathbb{A}^{l_i}$$

where l_i is the number of loops in the i-th vertex of Q. If $\delta_i \neq \delta_j$ then we obtain from the Euler-form formula that $dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j)$ is equal to the number of arrows from the i-th vertex to the j-th vertex of Q for every $S_i \in \texttt{rep}_{\beta_i}\ A$ and $S_j \in \texttt{rep}_{\beta_j}\ A$. As a consequence,

$$Q_A = Q \qquad \text{and} \qquad \alpha_A = (1, \ldots, 1)$$

In this case the associated algebra B is isomorphic to $\mathbb{C}Q_A = \mathbb{C}Q$.

Example 4 Let A be a hereditary order over a smooth affine curve X (or, if you prefer : a smooth Deligne-Mumford stack which is generically a curve see [4, Coroll. 7.8]). Then, A is an affine formally smooth algebra with center $\mathbb{C}[X]$ and is a subalgebra of $M_n(\mathbb{C}(X))$ for some n (the p.i.-degree of A). If $\mathfrak{m}_x$ is the maximal ideal corresponding to x then for all but finitely many x we have that

$$A/\mathfrak{m}_x A \simeq M_n(\mathbb{C})$$

For the exceptional points $\{x_1, \ldots, x_l\}$ (the so called ramified points) there are finitely many maximal ideals of A

$$\{P_1(i), \ldots, P_{k_i}(i)\} \quad \text{lying over} \quad \mathfrak{m}_i$$

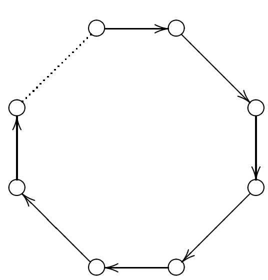
These ideals correspond to simple representations of A of dimension $\{n_1(i), \ldots, n_{k_i}(i)\}$ such that $n_1(i) + \ldots + n_{k_i}(i) = n$.

Therefore, the representation schemes $\texttt{rep}_l\ A$ for $l < n$ consist of finitely many closed orbits and the semigroup generators of $\texttt{comp}\ A$ are given by $\alpha_j(i)$ for all $1 \leq i \leq l$ and $1 \leq j \leq k_i$ where

$$\texttt{rep}_{\alpha_j(i)}\ A = \mathcal{O}(A/P_j(i)) \hookrightarrow \texttt{rep}_{n_j(i)}\ A$$

It follows from [10, Prop. 6.1] that in this case the quiver Q_A is the disjoint union of l quivers $Q_A(i)$ of type $\tilde{A}_{k_i}$ (with circular orientation), that is,

$Q_A(i) =$



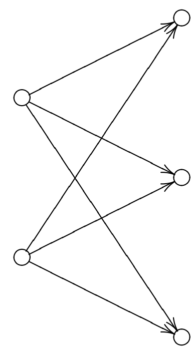
and $\qquad \alpha_A(i) = (n_1(i), \ldots, n_{k_i}(i))$

The associated algebra $B = B_1 \oplus \ldots \oplus B_l$ where

$$B_i = \begin{bmatrix} M_{n_1(i)}(\mathbb{C}[x]) & \ldots & M_{n_1(i) \times n_{k_i}(i)}(\mathbb{C}[x]) \\ \vdots & & \vdots \\ M_{n_{k_i}(i) \times n_1}(x\mathbb{C}[x]) & \ldots & M_{n_{k_i}(i)}(\mathbb{C}[x]) \end{bmatrix}$$

(ideals below the main diagonal) and it follows from [14, Chpt. 9] or [10, Prop. 6.1] that A in a neighborhood of x_i is étale isomorphic to B_i.

Example 5 The modular groupalgebra $A = \mathbb{C}PSL_2(\mathbb{Z})$ can be identified with the free algebra product $\mathbb{C}\mathbb{Z}/2\mathbb{Z} * \mathbb{C}\mathbb{Z}/3\mathbb{Z}$ and therefore is a formally smooth affine algebra and every finite dimensional representation of it is isomorphic to a representation of the bipartite quiver



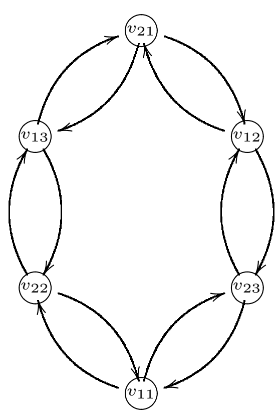
where the left-hand vertex-spaces are the eigenspaces for the $\mathbb{Z}/2\mathbb{Z}$-action and those on the right-hand the eigenspaces for the $\mathbb{Z}/3\mathbb{Z}$-action. In particular, an n-dimensional $\mathbb{C}PSL_2(\mathbb{Z})$ representation has dimension vector $(a_1, a_2; b_1, b_2, b_3)$ such that $a_1 + a_2 = b_1 + b_2 + b_3 = n$ and such that the matrix determined by the 6 arrows is invertible (it gives a base-change on the n-dimensional representation), see [1] or [16] for more details. As a consequence $\mathtt{comp}\ A$ has semigroup generators

$$v_{11} = (1,0;1,0,0),\ v_{12} = (1,0;0,1,0),\ v_{13} = (1,0;0,0,1)$$

$$v_{21} = (0,1;1,0,0),\ v_{22} = (0,1;0,1,0),\ v_{23} = (0,1;0,0,1)$$

and each component $\mathtt{rep}_{v_{ij}}\ A$ consists of a single simple 1-dimensional module S_{ij}. Because A is a universal localization of the path algebra of this bipartite quiver we can compute the dimensions of $Ext^1_A(S_{ij}, S_{kl})$ from the corresponding dimension vectors of the quiver. As a consequence, the associated quiver setting (Q_A, α_A) is

$Q_A =$



and $\quad \alpha_A = (1,1,1,1,1,1)$

and the associated algebra B is the path algebra $\mathbb{C}Q_A$.

4. Simple dimension vectors

Definition 2 $\alpha \in \mathtt{comp}\ A$ is said to be a simple dimension vector provided there is a non-empty Zariski open subset of $\mathtt{rep}_\alpha\ A$ consisting of simple representations. The set of all simple dimension vectors of A will be denoted by $\mathtt{simp}\ A$.

Theorem 2 *If A is a formally smooth algebra then Q_A contains enough information to determine* $\mathtt{simp}\ A$.

Proof. If $\alpha \in \mathtt{comp}\ A$ we can write (possibly in several ways)

$$\alpha = a_1\beta_1 + \ldots + a_k\beta_k$$

with $a_i \in \mathbb{N}$ and $\{\beta_1, \ldots, \beta_k\}$ the semigroup generators of $\mathtt{comp}\ A$. This implies that there are points in $\mathtt{rep}_\alpha\ A$ corresponding to semi-simple representations

$$M = S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k}$$

where the S_i are distinct simple representations in $\mathtt{rep}_{\beta_i}\ A$ and we can choose the S_i such that for all $1 \leq i, j \leq k$ we have that $S_i \oplus S_j \in E(\beta_i, \beta_j)$. But then the local quiver setting of M in $\mathtt{rep}_\alpha\ A$ is determined by (Q_A, ϵ) where $\epsilon = (a_1, \ldots, a_k)$. Because $\mathtt{rep}_\alpha\ A$ is irreducible, it follows from section 1 that $\alpha \in \mathtt{simp}\ A$ if and only if ϵ is the dimension vector of a simple representation of Q_A. These dimension vectors have been classified in [11] and we recall the result.

Let χ be the Euler-form of Q_A, that is $\chi = (c_{ij})_{i,j} \in M_k(\mathbb{Z})$ with $c_{ij} = \delta_{ij} - \#\{$ arrows from v_i to v_j $\}$ and let δ_i be the dimension vector of a vertex-simple concentrated in vertex v_i. Then, ϵ is the dimension vector of a simple representation of Q_A if and only if the following conditions are satisfied : (1) the support $\mathtt{supp}(\epsilon)$ is a strongly connected subquiver of Q_A (there is an oriented cycle in $\mathtt{supp}(\epsilon)$ containing each pair (i,j) such that $\{v_i, v_j\} \subset \mathtt{supp}(\epsilon)$) and (2) for all $v_i \in \mathtt{supp}(\epsilon)$ we have the numerical conditions

$$\chi(\epsilon, \delta_i) \leq 0 \qquad \text{and} \qquad \chi(\delta_i, \epsilon) \leq 0$$

unless $\mathtt{supp}(\epsilon)$ in an oriented cycle of type $\tilde{A}_l$ for some l in which case all components of ϵ must be equal to one.

Example 6 If we apply this result to the setting (Q_A, α_A) for the modular groupalgebra $A = \mathbb{C}PSL_2(\mathbb{Z})$ we find that a dimension vector $(a_1, a_2; b_1, b_2, b_3)$ has an open subset of simple $PSL_2(\mathbb{Z})$-representations if and only if

$$b_j \leq a_i \qquad \text{for all } 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 3$$

which is the criterium found by Bruce Westbury in [16].

5. $ext(\alpha, \beta)$ on simples

In the foregoing section we were careful to take $S_i \oplus S_j \in E(\beta_i, \beta_j)$ but this is not really necessary.

Lemma 2 *For every simple S_i in $\mathtt{rep}_{\beta_i}\ A$ and simple S_j in $\mathtt{rep}_{\beta_j}\ A$ we have that*

$$dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j) = ext(\beta_i, \beta_j)$$

Proof. For $\alpha = \beta_i + \beta_j$ the local structure of $\texttt{rep}_\alpha\ A$ near the orbit of $M = S_i \oplus S_j$ is determined by the local quiver setting (Q_M, α_M)

$$\begin{tikzcd} \textcircled{1} \arrow[loop left, "d_i"] \arrow[r, bend left, "e_{ij}"] & \textcircled{1} \arrow[l, bend left, "e_{ji}"] \arrow[loop right, "d_j"] \end{tikzcd} .$$

where $d_i = dim\ \texttt{iss}_{\beta_i}\ A$, $d_j = dim\ \texttt{iss}_{\beta_j}\ A$, $e_{ij} = dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j)$ and $e_{ji} = dim_{\mathbb{C}}\ Ext^1_A(S_j, S_i)$ and $\alpha_M = (1,1)$. Now, $\texttt{rep}_{\alpha_M}\ Q_M$ can be identified with $Ext^1_A(M,M)$ which is the normal space N_M to the orbit of M in $\texttt{rep}_\alpha\ A$ which has therefore dimension

$$dim\ N_M = d_i + d_j + e_{ij} + e_{ji}$$

On the other hand, there is a point $N = S'_i \oplus S'_j$ in $\texttt{rep}_\alpha\ A$ with $N \in E(\beta_i, \beta_j)$ and $N \in E(\beta_j, \beta_i)$ and we have

$$e_{ij} \geq ext(\beta_i, \beta_j) \qquad e_{ji} \geq ext(\beta_j, \beta_i)$$

and as for the normal space N_N to the orbit of N in $\texttt{rep}_\alpha\ A$ we have that

$$dim\ N_N = d_i + d_j + ext(\beta_i, \beta_j) + ext(\beta_j, \beta_i)$$

As the stabilizer subgroup of N and M is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and $\texttt{rep}_\alpha\ A$ is smooth it follows that $dim\ N_M = dim\ N_N$ from which the claim follows.

Theorem 3 *If A is a formally smooth algebra, then Q_A contains enough information to compute the dimension of $Ext^1_A(S,T)$ for all simple representations S and T. More precisely,*

$$dim\ Ext^1_A(S,T) = -\chi(\epsilon, \eta)$$

if S is a simple representation in $\texttt{rep}_\alpha\ A$ where $\alpha = \sum_i a_i \beta_i$ and $\epsilon = (a_1, \ldots, a_k)$ and if T is a simple representation of $\texttt{rep}_\beta\ A$ where $\beta = \sum_i b_i \beta_i$ and $\eta = (b_1, \ldots .b_k)$.

Proof. Let S_i and T_i be distinct simples in $\texttt{rep}_{\beta_i}\ A$ and consider the semi-simple representations M resp. N in $\texttt{rep}_\alpha\ A$ resp. $\texttt{rep}_\beta\ A$

$$M = S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k} \quad \text{and} \quad N = T_1^{\oplus b_1} \oplus \ldots \oplus T_k^{\oplus b_k}$$

By the foregoing lemma we have complete information on the local quiver setting of $M \oplus N$ in $\texttt{rep}_{\alpha+\beta}\ A$ from Q_A. By assumption on α and β there is a Zariski open subset of simples $S' \in \texttt{rep}_\alpha\ A$ and simples $T' \in \texttt{rep}_\beta\ A$ such that $S' \oplus T'$ lies in a neighborhood of $M \oplus N$. By the result of [11] we can therefore reconstruct the local quiver setting of $S' \oplus T'$ from that of $M \oplus N$. This quiver setting has the following form

$$\begin{tikzcd} \textcircled{1} \arrow[loop left, "1-\chi(\epsilon,\epsilon)"] \arrow[r, bend left, "-\chi(\epsilon,\eta)"] & \textcircled{1} \arrow[l, bend left, "-\chi(\eta,\epsilon)"] \arrow[loop right, "1-\chi(\eta,\eta)"] \end{tikzcd} .$$

from which we deduce that

$$ext(\alpha.\beta) = -\chi(\epsilon, \eta)$$

But then comparing the local quiver settings of $S' \oplus T'$ with $S \oplus T$ and repeating the argument of the foregoing lemma, the result follows.

6. The main result

Theorem 4 *If A is a formally smooth algebra, the associated quiver Q_A contains enough information to reconstruct the local quiver settings (Q_ξ, α_ξ) for any semi-simple representation*

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_l^{\oplus e_l}$$

of A.

Proof. This is a direct consequence of the foregoing two sections. To begin, we can determine the possible dimension vectors α_i of the simple components S_i. Write $\alpha_i = \sum_{j=1}^k a_j(i)\beta_j$ then $\epsilon_i = (a_1(i), \ldots, a_k(i))$ must be the dimension vector of a simple representation of the associated quiver Q_A. Moreover, by the previous theorem we know that

$$dim\ Ext^1_A(S_i, S_j) = \delta_{ij} - \chi(\epsilon_i, \epsilon_j)$$

and hence have full knowledge of the local quiver Q_ξ.

Recall that the results of [2] and [3] allow us to classify the singularities of the quotient varieties $\mathtt{iss}_\alpha\ A$ upto smooth equivalence and, in particular, to determine their singular loci.

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