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Canonical systems and non-commutative geometry

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ABSTRACT: Inspired by ideas from non-commutative geometry, unions of moduli spaces of linear control systems are identified as open subsets of infinite Grassmannians.

Contents

For Michiel Hazewinkel on his 60th birthday.

1. Introduction

A *linear control system* Σ of type $(m,n,p) \in \mathbb{N}^3$ is determined by the system of linear differential equations

$$\begin{cases} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx, \end{cases}$$

where $u(t) \in \mathbb{C}^m$ is the *input or control* at time t, $x(t) \in \mathbb{C}^n$ is the *state* of the system and $y(t) \in \mathbb{C}^p$ is its *output*. That is, Σ is described by a triple of matrices

$$\Sigma = (A,B,C) \in M_n(\mathbb{C}) \times M_{n\times m}(\mathbb{C}) \times M_{p \times n}(\mathbb{C}) = V_{m,n,p}$$

and is said to be equivalent to a system $\Sigma' = (A',B',C') \in V_{m,n,p}$ if and only if there is a basechange matrix $g \in GL_n = GL_n(\mathbb{C})$ in the state-space such that

$$\Sigma \sim \Sigma' \quad \Leftrightarrow \quad A' = gAg^{-1}, \quad B' = gB \quad \text{and} \quad C' = Cg^{-1}.$$

A system $\Sigma = (A,B,C) \in V_{m,n,p}$ is said to be *completely controllable* (resp. *completely observable*) if and only if the matrix

$$c(\Sigma) = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \quad \text{(resp.} \quad o(\Sigma) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{)}$$

is of maximal rank. These conditions define GL_n-open subsets $V^{cc}_{m,n,p}$, resp. $V^{co}_{m,n,p}$, consisting of systems with trivial GL_n-stabilizer, whence we have corresponding orbit spaces

$$\mathtt{sys}^{cc}_{m,n,p} = V^{cc}_{m,n,p}/GL_n \qquad \text{and} \qquad \mathtt{sys}^{co}_{m,n,p} = V^{co}_{m,n,p}/GL_n,$$

which are known to be smooth quasi-projective varieties of dimension $(m+p)n$, see for example [13, Part IV]. A system $\Sigma = (A, B, C) \in V_{m,n,p}$ is said to be *canonical* if it is both completely controllable and completely observable. The corresponding moduli space

$$\mathtt{sys}^c_{m,n,p} = (V^{cc}_{m,n,p} \cap V^{co}_{m,n,p})/GL_n$$

classifies canonical systems having the same input-output behavior, that is, such that all the $p \times m$ matrices CA^iB for $i \in \mathbb{N}$ are equal [13, Part VI - VII]. Conversely, if $F = \{F_j \ : \ j \in \mathbb{N}_+\}$ is a sequence of $p \times m$ matrices such that the corresponding *Hankel matrices*

$$H_{ij}(F) = \begin{bmatrix} F_1 & F_2 & \dots & F_j \\ F_2 & F_3 & \dots & F_{j+1} \\ \vdots & \vdots & & \vdots \\ F_i & F_{i+1} & \dots & F_{i+j-1} \end{bmatrix}$$

are such that there exist integers r and s such that $rk\ H_{rs}(F) = rk\ H_{r+1,s+j}(F)$ for all $j \in \mathbb{N}_+$, then F is *realizable* by a canonical system $\Sigma = (A, B, C) \in V^c_{m,n,p}$ (for some n which is equal to $rk\ H_{rs}(F)$), that is,

$$F_j = CA^{j-1}B \qquad \text{for all } j \in \mathbb{N}_+,$$

see for example [13, Part VI - VII] for connections between this realization problem and classical problems in analysis. These problems would be facilitated if there was an infinite dimensional manifold X together with a natural stratification

$$X = \bigsqcup_n \mathtt{sys}^c_{m,n,p}$$

by the moduli spaces of canonical systems (for fixed m and p and varying n).

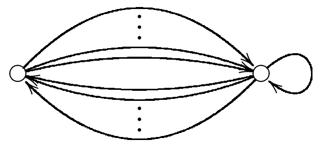
Non-commutative geometry, as outlined by M. Kontsevich in [9], offers a possibility to glue together closely related moduli spaces into an infinite dimensional variety controlled by a non-commutative algebra. The individual moduli spaces are then recovered as moduli spaces of simple representations (of specific dimension vectors) of the non-commutative algebra. An illustrative example is contained in the recent work by G. Wilson and Yu. Berest [15] [1] relating Calogero-Moser spaces to the adelic Grassmannian (see also [3] and [6] for the connection with non-commutative geometry). The main aim of the present paper is to offer another (and more elementary) example :

Theorem 1 *The equivalence classes of canonical systems with fixed input- and output-dimensions m and p form a specific open submanifold*

$$\bigsqcup_n \mathtt{sys}^c_{m,n,p} \hookrightarrow \mathtt{Gras}_{m+p}(\infty)$$

of the infinite Grassmannian of $m+p$-dimensional subspaces.

This paper is organized as follows. In section two we show that Kontsevich's approach is applicable to moduli spaces of canonical systems by proving that there is a natural one-to-one correspondence between equivalence classes of canonical systems with n-dimensional state space and isomorphism classes of simple representations of dimension vector $(1, n)$ of the formally smooth path algebra of the quiver



with m arrows pointing right and p arrows pointing left. This observation gives a short proof of the following result, due to M. Hazewinkel ([13, thm. VI.2.5] or [8, (2.5.7)]):

Theorem 2 (Hazewinkel) *The moduli space $\mathtt{sys}^c_{m,n,p}$ of canonical systems is a quasi-affine variety.*

In section 3 we prove that the moduli spaces $\mathtt{sys}^{cc}_{m,n,p}$ (resp. $\mathtt{sys}^{co}_{m,n,p}$) of completely controllable (resp. completely observable) systems are isomorphic to moduli spaces (in the sense of A. King [10]) of θ-stable representations of dimension vector $(1, n)$ for this quiver, where $\theta = (-n, 1)$ (resp. $\theta = (n, -1)$). By computing the cohomology of these moduli spaces, as in [14], we were then led to

Theorem 3 *The moduli space $\mathtt{sys}^{cc}_{m,n,p}$ of completely controllable systems has a cell decomposition identical to the natural cell decomposition of a vectorbundle of rank $(p+1)n$ on the Grassmannian $\mathtt{Gras}_n(m+n-1)$ with respect to the Schubert cells on the Grassmannian.*

In an earlier version of this note we claimed that the moduli space itself is a vectorbundle over the Grassmannian. However, this cannot be the case when $m = n$ as the referee kindly pointed out.

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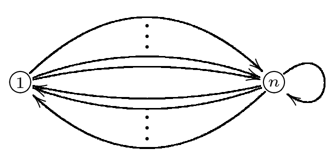
2. Proof of theorem 2

Consider the quiver setting (Q, α) where the dimension vector is $\alpha = (1, n)$ and the quiver Q



has m arrows $\{b_1, \ldots, b_m\}$ from left to right and p arrows $\{c_1, \ldots, c_p\}$ from right to left. We can identify $V_{m,n,p}$ with $\mathtt{rep}_\alpha\, Q$, where we associate to a system $\Sigma = (A, B, C)$ the representation V_Σ which assigns to the arrow b_i (resp. c_j) the i-th column B_i of B (resp. the j-th row C^j of C) and the matrix A to the loop. The basechange action of $(\lambda, g) \in GL(\alpha) = \mathbb{C}^* \times GL_n$ acts on the representation $V_\Sigma = (A, B_1, \ldots, B_m, C^1, \ldots, C^p)$ as follows:

$$(\lambda.g).V_\Sigma = (gAg^{-1}, gB_1\lambda^{-1}, \ldots, gB_m\lambda^{-1}, \lambda C^1 g^{-1}, \ldots, \lambda C^p g^{-1}),$$

and as the central subgroup $\mathbb{C}^*(1, \mathbf{1}_n)$ acts trivially on $\mathtt{rep}_\alpha\, Q$, there is a natural one-to-one correspondence between equivalence classes of systems in $V_{m,n,p}$ and isomorphism classes of α-dimensional representations in $\mathtt{rep}_\alpha\, Q$. If $\mathbb{C}Q$ denotes the *path algebra* of the quiver Q, then it is well known that $\mathbb{C}Q$ is a formally smooth algebra in the sense of [4], and that there is an equivalence of categories between finite dimensional right $\mathbb{C}Q$-modules and representations of Q. It is perhaps surprising that the system theoretic notion of canonical system corresponds under these identifications to the algebraic notion of simple module.

Lemma 1 *The following are equivalent:*

1. $\Sigma = (A, B, C) \in V_{m,n,p}$ *is a canonical system,*
2. $V_\Sigma = (A, B_1, \dots, B_m, C^1, \dots, C^p) \in \mathtt{rep}_\alpha\, Q$ *is a simple representation.*

Proof. $1 \Rightarrow 2$: If V_Σ has a proper subrepresentation of dimension vector $\beta = (1, l)$ for some $l < n$, then the rank of the control-matrix $c(\Sigma)$ is at most l, contradicting complete controllability. If V_Σ has a proper subrepresentation of dimension vector $\beta' = (0, l)$ with $l \neq 0$, then the observation-matrix $o(\Sigma)$ has rank at most $n - l$, contradicting complete observability. $2 \Rightarrow 1$: If $rk\ c(\Sigma) = l < n$ then there is a proper subrepresentation of dimension vector $(1, l)$ of V_Σ. If $rk\ o(\Sigma) = n - l$ with $l > 0$, then there is a proper subrepresentation of dimension vector $(0, l)$ of V_Σ.

From [12] we recall that for a general quiver setting (Q, α) the isomorphism classes of α-dimensional semi-simple representations are classified by the *affine* algebraic quotient variety

$$\mathtt{rep}_\alpha\, Q // GL(\alpha) = \mathtt{iss}_\alpha\, Q$$

whose coordinate ring is generated by all traces along oriented cycles in the quiver Q. If α is the dimension vector of a simple representation, this affine quotient has dimension $1 - \chi_Q(\alpha, \alpha)$ where χ_Q is the *Euler form* of Q. Moreover, the isomorphism classes of *simple* representations form a Zariski open *smooth subvariety* of $\mathtt{iss}_\alpha\, Q$. Specializing these general results from [12] to the case of interest, we recover Hazewinkels theorem.

Theorem 4 (Hazewinkel) *The moduli space* $\mathtt{sys}^c_{m,n,p}$ *of canonical systems is a smooth quasi-affine variety of dimension* $(m + p)n$.

In fact, combining the theory of local quivers (see for example [11]) with the classification of all quiver settings having a smooth quotient variety due to Raf Bocklandt [2], it follows that (unless $m = p = 1$) $\mathtt{sys}^c_{m,n,p}$ is precisely the smooth locus of the affine quotient variety $\mathtt{iss}_\alpha\, Q$.

3. Proof of theorem 3

For (Q, α) a quiver setting on k vertices and if $\theta \in \mathbb{Z}^k$, a representation $V \in \mathtt{rep}_\alpha\, Q$ is said to be *θ-semistable* (resp. *θ-stable*) if and only if for every *proper* non-zero subrepresentation W of V we have that $\theta.\beta \geq 0$ (resp. $\theta.\beta > 0$), where β is the dimension vector of W. In the special case when $\alpha = (1, n)$ and Q is the quiver introduced before, there are essentially two different *stability structures* on $\mathtt{rep}_\alpha\, Q$ determined by the integral vectors

$$\theta_+ = (-n, 1) \qquad \text{and} \qquad \theta_- = (n, -1)$$

By the identification of $\mathtt{rep}_\alpha\, Q$ with $V_{m,n,p}$ and the proof of lemma 1 we have

Lemma 2 *For $\theta_+ = (-n, 1)$ the following are equivalent:*

1. *$\Sigma \in V_{m,n,p}$ is completely controllable,*
2. *$V_\Sigma \in \mathtt{rep}_\alpha\, Q$ is θ_+-stable.*

For $\theta_- = (n, -1)$ the following are equivalent:

1. *$\Sigma \in V_{m,n,p}$ is completely controllable,*
2. *$V_\Sigma \in \mathtt{rep}_\alpha\, Q$ is θ_--stable.*

For a general stability structure θ and quiver setting (Q, α), A. King [10] introduced and studied the *moduli space* $\mathtt{moduli}^\theta_\alpha\, Q$ of θ-semistable representations, the points of which classify isomorphism classes of direct sums of θ-stable representations. In the case of interest to us we have

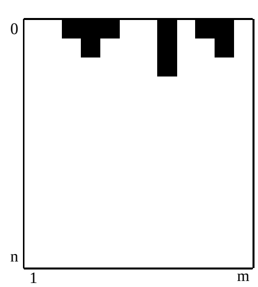
$$\mathtt{sys}^{cc}_{m,n,p} = \mathtt{moduli}^{\theta_+}_\alpha\, Q \qquad \text{and} \qquad \mathtt{sys}^{co}_{m,n,p} = \mathtt{moduli}^{\theta_-}_\alpha\, Q.$$

In [14] the Harder-Narasinham filtration associated to a stability structure was used to compute the cohomology of the moduli spaces $\mathtt{moduli}^\theta_\alpha\, Q$ (at least if the quiver Q has no oriented cycles). For general quivers the same methods can be applied to compute the number of $\mathbb{F}_q$-points of these moduli spaces, where $\mathbb{F}_q$ is the finite field of $q = p^l$ elements. In the case of interest to us, we get the rational functions

$$\begin{cases} \#\, \mathtt{moduli}^{\theta_+}_\alpha\, Q\, (\mathbb{F}_q) & = q^{n(p+1)} \prod_{i=1}^n \frac{q^{m+i-1}-1}{q^i-1} \\ \\ \#\, \mathtt{moduli}^{\theta_-}_\alpha\, Q\, (\mathbb{F}_q) & = q^{n(m+1)} \prod_{i=1}^n \frac{q^{p+i-1}-1}{q^i-1}, \end{cases}$$

which suggests that the moduli space $\mathtt{sys}^{cc}_{m,n,p}$ is a vectorbundle of rank $n(p+1)$ over the Grassmannian $\mathtt{Gras}_n(m+n-1)$, and that the moduli space $\mathtt{sys}^{co}_{m,n,p}$ is a vectorbundle of rank $n(m+1)$ over $\mathtt{Gras}_n(p+n-1)$.

To a completely controllable $\Sigma = (A, B, C)$ one associates its *Kalman code* K_Σ, which is an array of $n \times m$ boxes $\{(i,j) \mid 0 \leq i < n, 1 \leq j \leq\}$, ordered lexicographically, with exactly n boxes painted black. If the column $A^i B_j$ is linearly independent of all column vectors $A^k B_l$ with $(k,l) < (i,j)$ we paint box (i,j) black. From this rule it is clear that if (i,j) is a black box so are (i', j) for all $i' \leq i$. That is, the Kalman code K_Σ (which only depends on the GL_n-orbit of Σ) looks like



Assume $\kappa = K_\Sigma$ has k black boxes on its first row at places $(0, i_1), \ldots, (0, i_k)$. Then we assign to κ the strictly increasing sequence

$$1 \leq j_\kappa(1) = i_1 < j_\kappa(2) = i_2 < \ldots < j_\kappa(k) = i_k \leq m$$

and another sequence $p_\kappa(1), \ldots, p_\kappa(k)$, where $p_\kappa(j)$ is the total number of black boxes in the i_j-th column of κ, that is,

$$p_\kappa(1) + p_\kappa(2) + \ldots + p_\kappa(k) = n.$$

It is clear that there is a one-to-one correspondence between Kalman codes and pairs of functions satisfying these conditions. Further, define the strictly increasing sequence

$$h_\kappa(0) = 0 < h_\kappa(1) = p_\kappa(1) < \ldots < h_\kappa(j) = \sum_{i=1}^{j} p_\kappa(i) < \ldots < h_\kappa(k) = n.$$

With these notations we have the following canonical form for $\Sigma = (A, B, C) \in V^{cc}_{m,n,p}$ which is essentially [5, lemma 3.2]:

Lemma 3 *For a completely reachable system $\Sigma = (A, B, C)$ with Kalman code $\kappa = K_\Sigma$, there is a unique $g \in GL_n$ such that $g.(A, B, C) = (A', B', C')$ with*

- $B'_{j_\kappa(i)} = \mathbf{1}_{h_\kappa(i-1)+1}$ *for all* $1 \leq i \leq k$.
- $A'_i = \mathbf{1}_{i+1}$ *for all* $i \notin \{h_\kappa(1), h_\kappa(2), \ldots, h_\kappa(k)\}$.
- *All entries in the remaining columns of A' and B' are determined as the quotient of two specific $n \times n$ minors of $c(\Sigma)$.*
- $C' = Cg^{-1}$.

This allows us to prove theorem 3 :

Theorem 5 *The moduli space $\mathtt{sys}^{cc}_{m,n,p}$ of completely controllable systems has a cell decomposition identical to the natural cell decomposition of a vectorbundle of rank $n(p+1)$ over the Grassmann manifold $\mathtt{Gras}_n(m+n-1)$.*

Proof. Define a map $V^{cc}_{m,n,p} \xrightarrow{\phi} \mathtt{Gras}_n(m+n-1)$ by sending a completely reachable system $\Sigma = (A, B, C)$ to the point in $\mathtt{Gras}_n(m+n-1)$ determined by the $n \times (m+n-1)$ matrix

$$M_\Sigma = \left[B'_1 \; \ldots \; B'_m \; A'_1 \; \ldots \; A'_{n-1} \right],$$

where (A', B', C') is the canonical form of Σ given by the previous lemma. By construction, M_Σ has rank n with invertible $n \times n$ matrix determined by the columns

$$I_\kappa = \{j_\kappa(1) < \ldots < j_\kappa(k) < m + c_1 < \ldots < m + c_{n-k}\} \subset \{1, \ldots, m+n-1\},$$

where $\{c_1, \ldots, c_{n-k}\} = \{1, \ldots, n\} - \{h_\kappa(1), \ldots, h_\kappa(k)\}$. As all remaining entries of (A', B') are determined by $c(\Sigma)$ it follows that $\phi(\Sigma)$ depends only on the GL_n-orbit of Σ, whence the map factorizes through

$$\mathtt{sys}^{cc}_{m,n,p} \xrightarrow{\psi} \mathtt{Gras}_n(m+n-1),$$

and we claim that ψ is surjective. To begin, all multi-indices $I = \{1 \le d_1 < d_2 < \ldots < d_n \le n+m-1\}$ are of the form I_κ for some Kalman code κ. Define

$$\{d_1, \ldots, d_n\} = \{i_1, \ldots, i_k\} \cup \{m + c_1, \ldots, m + c_{n-k}\}$$

with $i_j \le m$ and $1 \le c_j < n$, and let $\{e_1 < \ldots < e_k\} = \{1, \ldots, n\} - \{c_1, \ldots, c_{n-k}\}$, and set $e_0 = 0$. Construct the Kalman code κ having $e_j - e_{j-1}$ black boxes in the i_j-th column and verify that I is indeed I_κ.

$\mathtt{Gras}_n(m+n-1)$ is covered by modified Schubert cells S_I (isomorphic to some affine space) consisting of points such that the I-minor is invertible, where I is a multi-index $\{d_1, \ldots, d_n\}$, and the dimension of the subspace spanned by the first k columns is i iff $k < d_{i+1}$. A point in S_I can be taken such that the d_i-th column is equal to

$$\begin{cases} \mathbf{1}_{h_\kappa(i-1)+1} & \text{for } d_i \le m \\ \mathbf{1}_{j+1} & \text{for } d_i = m + i, \end{cases}$$

where $I = I_\kappa$. This determines a $n \times (n+m-1)$ matrix $\begin{bmatrix} B_1 & \ldots & B_m & A_1 & \ldots & A_{n-1} \end{bmatrix}$, and choosing any last column A_n and any $p \times n$ matrix C we obtain a system $\Sigma = (A, B, C)$ which is completely controllable, and which is mapped to the given point under ψ. This finishes the proof.

Because the map $(A, B, C) \longrightarrow (A^{tr}, C^{tr}, B^{tr})$ defines a duality between $V^{co}_{m,n,p}$ and $V^{cc}_{p,n,m}$, we have a similar result for the moduli spaces of completely observable systems.

Theorem 6 *The moduli space of completely observable systems* $\mathtt{sys}^{co}_{m,n,p}$ *has a cell decomposition identical to that of a vectorbundle of rank* $n(p+1)$ *over the Grassmann manifold* $\mathtt{Gras}_n(p+n-1)$.

4. Proof of theorem 1

The counting argument of the previous section gives us also a conjectural description of the infinite dimensional variety admitting a stratification by the moduli spaces $\mathtt{sys}^{cc}_{m,n,p}$. It follows from the explicit rational form of $\# \, \mathtt{sys}^{cc}_{m,n,p} \, (\mathbb{F}_q)$ and the *q-binomial theorem* that

$$\sum_{n=0}^{\infty} \# \, \mathtt{sys}^{cc}_{m,n,p} \, (\mathbb{F}_q) \, t^n = \prod_{i=1}^{m} \frac{1}{1 - q^{p+i} t}$$

In the special case when $p = 0$ we recover the cohomology of the infinite Grassmannian $\mathtt{Gras}_m(\infty)$ of m-dimensional subspaces of a countably infinite dimensional vectorspace. For $p \ge 1$ we only get a factor of the cohomology of $\mathtt{Gras}_{m+p}(\infty)$, which led to the following result.

Theorem 7 *The disjoint union* $\bigsqcup_n \, \mathtt{sys}^{cc}_{m,n,p}$ *is the open subset of the infinite dimensional Grassmann manifold* $\mathtt{Gras}_{m+p}(\infty)$ *which is the union of all standard affine open sets corresponding to a multi-index set* $I = \{1 \le d_1 < d_2 < \ldots < d_{m+p}\}$ *such that*

$$\{m+1, m+2, \ldots, m+p, m+p+n\} \subset I.$$

Proof. Let $\Sigma = (A, B, C)$ be a completely controllable system in canonical form represented by the point $p_\Sigma \in \mathtt{sys}^{cc}_{m,n,p}$. Consider the $n \times (m+p+n)$ matrix

$$L_\Sigma = \begin{bmatrix} B \ C^{tr} \ A \end{bmatrix}.$$

The submatrix $M_\Sigma = \begin{bmatrix} B_1 \ \dots \ B_m \ A_1 \ \dots \ A_{n-1} \end{bmatrix}$ has rank n, whence so has L_Σ, and p_Σ determines a point in $\mathtt{Gras}_n(n+m+p)$. Under the natural duality

$$\mathtt{Gras}_n(m+p+n) \xrightarrow{D} \mathtt{Gras}_{m+p}(m+p+n),$$

the point p_Σ is mapped to the point determined by the $(m+p) \times (m+p+n)$ matrix N_Σ whose rows give a basis for the linear relations holding among the columns of L_Σ. Because M_Σ has rank n it follows that the columns of C^{tr} and the last column A_n of A are linearly dependent of those of M_Σ. As a consequence the matrix

$$N_\Sigma = \begin{bmatrix} U_1 \ \dots \ U_m \ V_1 \ \dots \ V_p \ W_1 \ \dots \ W_n \end{bmatrix}$$

has the property that the submatrix $\begin{bmatrix} V_1 \ \dots \ V_p \ W_n \end{bmatrix}$ has rank $p+1$. This procedure defines a morphism

$$\mathtt{sys}^{cc}_{m,n,p} \xrightarrow{\gamma_n} \mathtt{Gras}_{m+p}(m+p+n),$$

the image of which is the open union of all standard affine opens determined by a multi-index set $I = \{1 \leq d_1 < d_2 < \ldots < d_{m+p} \leq m+p+n\}$ satisfying

$$\{m+1, m+2, \ldots, m+p, m+p+n\} \subset I.$$

Therefore, the image of the morphism

$$\bigsqcup_n \mathtt{sys}^{cc}_{m,n,p} \xrightarrow{\sqcup\gamma_n} \mathtt{Gras}_{m+p}(\infty)$$

is the one of the statement of the theorem. The dimension n of the system corresponding to a point in this open set of $\mathtt{Gras}_{m+p}(\infty)$ is determined by $d_{m+p} = m+p+n$.

By the duality between $V^{cc}_{m,n,p}$ and $V^{co}_{p,n,m}$ used in the previous section we deduce:

Theorem 8 *The disjoint union $\bigsqcup_n \mathtt{sys}^{co}_{m,n,p}$ is the open subset of $\mathtt{Gras}_{m+p}(\infty)$ which is the union of all standard affine opens corresponding to a multi-index set $I = \{1 \leq d_1 < d_2 < \ldots < d_{m+p}\}$ such that*

$$\{1, 2, \ldots, m, m+p+n\} \subset I.$$

This, in turn, proves theorem 1 :

Theorem 9 *The disjoint union $\bigsqcup_n \mathtt{sys}^{c}_{m,n,p}$ of all moduli spaces of canonical systems with fixed input- and output-dimension m and p is the open subset of the infinite Grassmannian $\mathtt{Gras}_{m+p}(\infty)$ of $m+p$-dimensional subspaces of a countably infinite dimensional vectorspace which is the intersection of all possible standard open subsets X_I and X_J, where I and J are multi-index sets satisfying the conditions*

$$\{m+1, m+2, \ldots, m+p, m+p+n\} \subset I \qquad \textit{and} \qquad \{1, 2, \ldots, m, m+p+n\} \subset J.$$

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