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Lieven Le Bruyn

Departement Wiskunde en Informatica, Universiteit Antwerpen (UIA)
B-2610 Antwerp (Belgium)
E-mail : lebruyn@uia.ua.ac.be

ABSTRACT: We give examples of classification problems from a variety of topics (conjugacy classes of matrices, linear control systems, Hilbert schemes, Calogero phase spaces, (gravitational) instantons and quotient singularities) which are best understood using noncommutative geometry. That is, each of these problems is in essence a classification upto isomorphism of certain finite dimensional representations of an affine (noncommutative) algebra. In this first part, we study the first four examples. More details can be found in the text 'noncommutative geometry@n' (ng@n) which can be obtained from the Courses-page.

Contents

1. Noncommutative geometry

Over the last couple of decades it has become clear that several important classification problems in geometry and physics are best understood in terms of finite dimensional representations of a specific affine $\mathbb{C}$-algebra.

Let A be an affine $\mathbb{C}$-algebra, that is, A has a presentation

$$A \simeq \mathbb{C}\langle x_1, \ldots, x_m \rangle / I_A$$

where $\mathbb{C}\langle x_1, \ldots, x_m \rangle$ is the free algebra on m noncommuting variables x_i and I_A is the ideal of relations holding in A. An n-dimensional representation of A is an n-dimensional left A-module $M = \mathbb{C}^n$, that is there is a bilinear map

$$A \otimes_{\mathbb{C}} M \longrightarrow M$$

such that $1.m = m$ and $a.(b.m) = (ab).m$. Fixing a basis in M we can view the action of $a \in A$ as multiplication by the $n \times n$ matrix $\phi(a)$, that is, there is a $\mathbb{C}$-algebra morphism

$$A \xrightarrow{\phi} M_n(\mathbb{C})$$

giving an alternative definition of a finite dimensional representation. The set of all n-dimensional representations can be given the structure of an affine variety $\mathtt{rep}_n\, A$, the *representation variety* of A. Because every $\mathbb{C}$-algebra morphism

$$\mathbb{C}\langle x_1, \ldots, x_m \rangle \xrightarrow{\phi} M_n(\mathbb{C})$$

is fully determined by the images of the free variables $\phi(x_i) \in M_n(\mathbb{C})$ it follows that the representation variety is the affine space

$$\mathtt{rep}_n\ \mathbb{C}\langle x_1, \ldots, x_m\rangle = \underbrace{M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})}_{m}$$

If a relation $f(x_1, \ldots, x_m) \in \mathbb{C}\langle x_1, \ldots, x_m\rangle$ holds among the generators of A, then it must also hold among the $a_i = \phi(x_i)$ for any n-dimensional representation ϕ of A, that is,

$$\mathtt{rep}_n\ A = \{(a_1, \ldots, a_m) \in M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C}) \mid f(a_1, \ldots, a_m) = 0 \text{ for all } f \in I_A\}$$

There is a natural action of the group of invertible $n \times n$ matrices GL_n on $\mathtt{rep}_n\ A$ by *simultaneous conjugation*, that is,

$$g.(a_1, \ldots, a_m) = (ga_1g^{-1}, \ldots, ga_mg^{-1})$$

and the orbits $\mathcal{O}(\phi) = \{g.\phi \mid g \in GL_n\}$ for this action are precisely the *isomorphism classes* of n-dimensional left A-modules. The geometric classification of these orbits is the essence of noncommutative geometry.

Definition 1 A classification problem, expressed in terms of the study of G-orbits (for some algebraic group G) on a variety X, is said to be a *noncommutative variety* if there is an affine (noncommutative) $\mathbb{C}$-algebra A, a dimension n and a Zariski open subset

$$U \hookrightarrow \mathtt{rep}_n\ A$$

such that there is a natural one-to-one correspondence between G-orbits in X (the classified objects) and GL_n-orbits in U (isomorphism classes of n-dimensional A-representations in U.

2. Conjugacy classes of matrices

More information on this section can be found in ag@n §1.1. From linear algebra we recall that two $n \times n$ matrices $A, B \in M_n(\mathbb{C})$ are said to be *equivalent*

$$A \sim B \quad \Leftrightarrow \quad \exists g \in GL_n\ :\ B = gAg^{-1}$$

Equivalence classes of matrices, that is the action of GL_n on the variety $M_n(\mathbb{C})$ by conjugation is a noncommutative variety as

$$M_n(\mathbb{C}) = \mathtt{rep}_n\ \mathbb{C}[x]$$

and the action of GL_n on both sides is the same. A *set-theoretic* solution of this classification problem is given by the *Jordan normalform* of a matrix. From it some important lessons can be learned.

It is *not* always possible to have an *orbit map*

$$X \xrightarrow{\pi} X/G$$

with X/G an affine variety such that its points classify the G-orbits in X. Indeed, this can only be achieved if *all* G-orbits in X are *closed* (as they must be all of the form $\pi^{-1}(\xi)$ for some $\xi \in X/G$). Consider the matrices

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

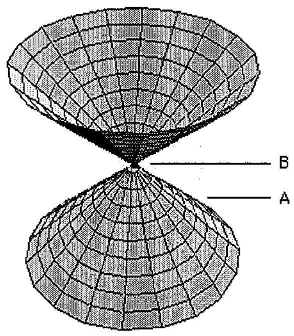
which belong to distinct orbits as they are different Jordan forms. For any $\epsilon \neq 0$ we have that

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & \epsilon \\ 0 & \lambda \end{bmatrix}$$

belongs to the orbit of A. Hence if $\epsilon \mapsto 0$, we see that B lies in the closure of $\mathcal{O}_A$ whence $\mathcal{O}_A$ cannot be a closed orbit in M_2. As any matrix in $\mathcal{O}_A$ has trace 2λ, the orbit is contained in the 3-dimensional subspace

$$\begin{bmatrix} \lambda + x & y \\ z & \lambda - x \end{bmatrix} \hookrightarrow M_2$$

In this space, the orbit-closure $\overline{\mathcal{O}_A}$ is the set of points satisfying $x^2 + yz = 0$ (the determinant has to be λ^2), which is a cone having the origin as its top :



The orbit $\mathcal{O}_B$ is the top of the cone and the orbit $\mathcal{O}_A$ is the complement.

Therefore, the best we can hope for is a *quotient map* $X \xrightarrow{\pi} X//G$ classifying the *closed* G-orbits in X. We will see later that if G is a *reductive* algebraic group (such as GL_n) and if X is an affine variety, then such a quotient variety always exists and its coordinate ring

$$\mathbb{C}[X//G] = \mathbb{C}[X]^G$$

is the ring of G-invariant polynomials on X. In the example of the GL_n-action by conjugation on $M_n(\mathbb{C})$ a supply of GL_n-invariant function is given by the *elementary symmetric* *functions* σ_l in the *eigenvalues* $\{\lambda_1, \ldots, \lambda_n\}$

$$\sigma_l(\lambda_1, \ldots, \lambda_l) = \sum_{i_1 < i_2 < \ldots < i_l} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_l}.$$

If $A \in M_n$ has eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ then

$$\prod_{j=1}^{n}(t - \lambda_j) = \chi_A(t) = det(t\mathbb{1}_n - A) = t^n + \sum_{i=1}^{n}(-1)^i\sigma_i(A)t^{n-i}$$

Developing the determinant $det(t\mathbb{1}_n - A)$ we see that each of the coefficients $\sigma_i(A)$ is a polynomial function in the entries of A and is clearly GL_n-invariant. We can define a map

$$M_n(\mathbb{C}) \xrightarrow{\pi} \mathbb{C}^n \qquad A \mapsto (\sigma_1(A), \ldots, \sigma_n(A))$$

which is onto (apply it to the *companion matrix*), As the orbit $\mathcal{O}_A$ is closed if and only if A is *diagonalizable* (apply an argument as in the 2×2 case above to any off diagonal entry in a Jordan normal form of A) we see that

$$M_n(\mathbb{C})//GL_n \simeq \mathbb{C}^n$$

as the point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ determines (and is determined by) the closed orbit of

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ are the roots of $t^n + \sum_{i=1}^{n}(-1)^i a_i t^{n-i}$.

There is an open subset of $\mathbb{C}^n$ (determined by the non-zero locus of the discriminant of the above polynomial) consisting of points $\xi = (a_1, \ldots, a_n)$ such that the *fiber* $\pi^{-1}(\xi)$ consists of a *unique* closed orbit. However, for all remaining points, the fiber is a union of several orbits (determined by the different types of possible Jordan normal forms), the 'worst case' occurring for $\xi = (0, \ldots, 0) = \vec{0}$. The fiber

$$\pi^{-1}(\vec{0}) = \mathtt{null} = \{A \in M_n(\mathbb{C}) \mid A^n = 0\}$$

is called the *nullcone* and consists of all *nilpotent matrices*. The orbits of such matrices correspond one-to-one to the different *partitions* of n. If A resp. B are nilpotent matrices with corresponding partitions $p_A = (a_1 \geq a_2 \geq \ldots \geq a_n \geq 0)$ resp. $p_B = (b_1 \geq b_2 \ldots \geq b_n \geq 0)$ then the *Gerstenhaber-Hesselink theorem* solves the orbit-closure problem for this nullcone :

$$\mathcal{O}_B \subset \overline{\mathcal{O}_A} \quad \text{if and only if} \quad \sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i \text{ for all } 1 \leq r \leq n$$

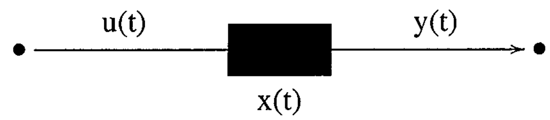
In general, the nullcone of a linear group G acting on a vectorspace V is the union of all orbits $\mathcal{O}_v$ such that $\vec{0} \in \overline{\mathcal{O}_v}$. It can contain infinitely many orbits and its precise structure is usually rather hard to determine.

3. Linear control systems

More information on this section can be found in ag@n §8.2. A linear time invariant control system Σ is governed by the following system of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= Bx + Au \\ y &= Cx. \end{cases} \tag{3.1}$$

Here, $u(t) \in \mathbb{C}^m$ is the *input* or *control* of the system at time t, $x(t) \in \mathbb{C}^n$ the *state* of the system and $y(t) \in \mathbb{C}^p$ the *output* of the system Σ. *Time invariance* of Σ means that the matrices $A \in M_{n \times m}(\mathbb{C})$, $B \in M_n(\mathbb{C})$ and $C \in M_{p \times n}(\mathbb{C})$ are constant. The system Σ can be represented as a *black box*



which is in a certain state $x(t)$ that we can try to change by using the input controls $u(t)$. By reading the output signals $y(t)$ we can try to determine the state of the system. A system $\Sigma = (A, B, C)$ is *completely controllable* if we can steer an arbitrary starting state to the zero state by some control function $u(t)$ in a finite time span. This control-theoretic notion is equivalent to the condition that the *control matrix*

$$c(\Sigma) = \begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A \end{bmatrix}$$

has maximal rank n. Dually, a system is said to be completely observable if we can determine the state of a system by observing the output function. This notion is equivalent to the condition that the *observation matrix*

$$o(\Sigma) = \begin{bmatrix} C \\ CB \\ CB^2 \\ \vdots \\ CB^{n-1} \end{bmatrix}$$

has maximal rank n.

An important control-theoretic problem is to determine when two completely controllable and observable systems $\Sigma = (A, B, C)$ and $\Sigma' = (A', B', C')$ are *equivalent*, that is, have the same input-output behavior. Somewhat surprisingly this condition can be rephrased into an orbit problem : Σ and Σ' are equivalent if and only if there is an invertible matrix $g \in GL_n$ such that

$$A' = gA \qquad B = gAg^{-1} \qquad C' = Cg^{-1}$$

That is, we can define a GL_n action on the vectorspace $M_{n\times m}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{p\times m}(\mathbb{C})$ as above and determine the Zariski open subset Sys of it consisting of all completely controllable and observable systems (both conditions are open as they are determined by non-vanishing of an $n \times n$ minor of the matrices $o(\Sigma)$ and $c(\Sigma)$.

Here we encounter another strategy to classify G-orbits : even when it is not possible to construct an orbit space for the whole variety X (because there are non-closed orbits) it may be possible to construct an orbit-space for the orbits in a specific open subset U of X (because all orbits are closed in U, for example because they all have the same dimension). One can then hope to reiterate this process starting with the G-closed subset $X - U$.

All GL_n-orbits of completely controllable systems are closed in the Zariski open subset Sys_c they determine in $M_{n\times m}(\mathbb{C}) \times M_n(\mathbb{C})$ because the *stabilizer subgroup*

$$Stab(\Sigma) = \{g \in GL_n \mid g.\Sigma = \Sigma\} = \{\mathbb{1}_n\}$$

and hence all such orbits have dimension n^2. To prove this, observe that $g \in GL_n$ acts on the control matrix $c(\Sigma)$ by multiplication on the left and because $c(\Sigma)$ contains an invertible $n \times n$ minor we have that $gc(\Sigma) = c(\Sigma)$ implies that $g = \mathbb{1}_n$. Recall that the *Grasmannian* $\mathtt{Gras}_k(l)$ of k-dimensional subspaces of $\mathbb{C}^l$ is the orbit space of the open subset

$$M_{k\times l}(\mathbb{C})^{max} \subset M_{k\times l}(\mathbb{C})$$

of matrices of maximal rank under the action by left multiplication of GL_k. This allows us to construct an orbit-space Sys_c/GL_n via

$$\begin{array}{ccc} Sys_c & \xhookrightarrow{\phi} & M_{n\times(n+1)m}(\mathbb{C})^{max} \\ \vdots & & \downarrow \\ Sys_c/GL_n & \hookrightarrow & \mathtt{Gras}_n((n+1)m) \end{array}$$

where $\phi(A, B) = \begin{bmatrix} A & AB & AB^2 & \ldots & AB^n \end{bmatrix}$. One calculates that the dimension of Sys_c/GL_n is mn. This can then be used to prove that the orbit-space of all completely controllable *and* observable systems Sys/GL_n is a *vectorbundle* of rank np over Sys_c/GL_n.

We will now prove that the classification problem of completely controllable systems (Sys_c, GL_n) is a noncommutative variety. A finite *quiver* Q is a directed graph determined by

- a finite set $Q_v = \{v_1, \ldots, v_k\}$ of *vertices*, and
- a finite set $Q_a = \{a_1, \ldots, a_l\}$ of *arrows* where we allow multiple arrows between vertices and loops in vertices.

Every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ has a *starting vertex* $s(a) = i$ and a *terminating vertex* $t(a) = j$. The description of the quiver Q is encoded in the integral $k \times k$ matrix

$$\chi_Q = \begin{bmatrix} \chi_{11} & \cdots & \chi_{1k} \\ \vdots & & \vdots \\ \chi_{k1} & \cdots & \chi_{kk} \end{bmatrix} \qquad \text{where} \quad \chi_{ij} = \delta_{ij} - \# \{ \textcircled{j} \longleftarrow \textcircled{i} \}$$

The corresponding bilinear form on $\mathbb{Z}^k$ is called the *Euler form* of the quiver Q.

The underlying vectorspace of the *path algebra* $\mathbb{C}Q$ of the quiver Q has as basis the directed paths in Q. Multiplication is induced by (left) concatenation of paths. More precisely, $1 = v_1 + \ldots + v_k$ is a decomposition of 1 into mutually orthogonal vertex-idempotents and we define

- $v_j.a$ is always zero unless $\textcircled{j} \xleftarrow{a} \circ$ in which case it is the path a,
- $a.v_i$ is always zero unless $\circ \xleftarrow{a} \textcircled{i}$ in which case it is the path a,
- $a_i.a_j$ is always zero unless $\circ \xleftarrow{a_i} \circ \xleftarrow{a_j} \circ$ in which case it is the path $a_i a_j$.

Example 1 Consider the quiver with Euler form

$$\begin{bmatrix} 1 & -m \\ 0 & 0 \end{bmatrix}$$

That is, the directed graph

$$\begin{tikzcd} \circ \arrow[r, Rightarrow, "m"] & \circ \arrow[loop right] \end{tikzcd}$$

then the path algebra of this quiver is the noncommutative algebra

$$\mathbb{C}Q = A = \begin{bmatrix} \mathbb{C} & \mathbb{C}[x]a_1 + \ldots + \mathbb{C}[x]a_m \\ 0 & \mathbb{C}[x] \end{bmatrix}$$

where x denotes the loop in the second vertex and $a_1, \ldots, a_m$ are the m arrows from the first to the second vertex. If M is an n-dimensional left A-module then we can use the idempotents

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

to decompose (as a $\mathbb{C}$-vectorspace) $M = M_1 \oplus M_2$ where $M_i = e_i M$. If $a = dim\ M_1$ and $b = dim\ M_2$ we say that M has *dimension vector* $\alpha = (a, b)$ and clearly $n = a + b$. Choosing a basis of M (that is, going to a different point in the orbit $\mathcal{O}(M)$) relative to this decomposition we may assume that the corresponding representation

$$A \xrightarrow{\phi} M_n(\mathbb{C})$$

is such that

$$\phi(e_1) = \begin{bmatrix} \mathbb{1}_a & 0 \\ 0 & 0 \end{bmatrix} \qquad \phi(e_2) = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1}_b \end{bmatrix}$$

But then, using that $e_1 a_i = a_i = a_i e_2$ and $e_2 a_i = 0 = a_i e_1$ we have for the remaining algebra generators, denoting

$$a_i = \begin{bmatrix} 0 & a_i \\ 0 & 0 \end{bmatrix} \qquad \text{then} \qquad \phi(a_i) = \begin{bmatrix} 0 & A_i \\ 0 & 0 \end{bmatrix}$$

for some matrix $A_i \in M_{a \times b}(\mathbb{C})$ and for

$$x = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \qquad \text{we have} \qquad \phi(x) = \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}$$

for some square matrix $X \in M_b(\mathbb{C})$. That is, by going to a different point in the GL_n-orbit of M we have associated to M a *representation* of the quiver Q.

In general, a *representation* V of a finite quiver Q is given by

- a finite dimensional $\mathbb{C}$-vector space V_i for each vertex $v_i \in Q_v$, and
- a linear map $V_j \xleftarrow{V_a} V_i$ for every arrow $(j) \xleftarrow{a} (i)$ in Q_a.

If $dim\ V_i = d_i$ we call the integral vector $\alpha = (d_1, \ldots, d_k) \in \mathbb{N}^k$ the *dimension vector* of V and denote it with $dim\ V$.

The set $\mathtt{rep}_\alpha\ Q$ of all representations V of Q such that $dim(V) = \alpha$ is an affine space

$$\mathtt{rep}_\alpha\ Q = \bigoplus_{(j) \xleftarrow{a} (i)} M_{d_j \times d_i}(\mathbb{C}) \simeq \mathbb{C}^r$$

where $r = \sum_{a \in Q_a} d_{s(a)} d_{t(a)}$.

A *morphism* $V \xrightarrow{\phi} W$ between two representations V and W of Q is determined by a set of linear maps

$$V_i \xrightarrow{\phi_i} W_i \qquad \text{for all vertices } v_i \in Q_v$$

satisfying the following compatibility conditions. For every arrow there is a commuting diagram $(j) \xleftarrow{a} (i)$ in Q_a

$$\begin{tikzcd} V_i \arrow[r, "V_a"] \arrow[d, "\phi_i"] & V_j \arrow[d, "\phi_j"] \\ W_i \arrow[r, "W_a"] & W_j \end{tikzcd}$$

Basechange in all the vertex spaces induces an action of the algebraic group $GL(\alpha) = GL_{d_1} \times \ldots \times GL_{d_k}$ on the affine space $\mathtt{rep}_\alpha Q$. That is, if $g = (g_1, \ldots, g_k) \in GL(\alpha)$ and if $V = (V_a)_{a \in Q_a}$ then $g.V$ is determined by the matrices

$$(g.V)_a = g_{t(a)} V_a g_{s(a)}^{-1}.$$

If V and W in $\mathtt{rep}_\alpha\, Q$ are isomorphic as representations of Q, such an isomorphism is determined by invertible matrices $g_i : V_i \longrightarrow W_i \in GL_{d_i}$ and therefore they belong to the same orbit under $GL(\alpha)$. Again, one can use the vertex-idempotents to decompose any finite dimensional representation of the path algebra as a vectorspace direct sum. Choosing the vertex bases accordingly (which amounts to going to an isomorphic representation) we may assume that the associated representation $\phi : \mathbb{C}Q \longrightarrow M_n(\mathbb{C})$ is such that

$$\phi(v_i) = \begin{bmatrix} \ddots & & \\ & \mathbb{1}_{d_i} & \\ & & \ddots \end{bmatrix}$$

inducing an embedding of $GL(\alpha) = GL_{d_1} \times \ldots \times GL_{d_k} \hookrightarrow GL_n$ with $n = |\alpha| = \sum_i d_i$. As a consequence the variety of all n-dimensional representation of the path algebra decomposes

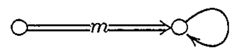
$$\mathtt{rep}_n\, \mathbb{C}Q = \bigsqcup_{|\alpha| = n} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q$$

into a disjoint union of *associated fiber bundles*. They are defined as follows : using the embedding $GL(\alpha) \hookrightarrow GL_n$ there is a $GL(\alpha)$-action on $GL_n \times \mathtt{rep}_\alpha\, Q$ via

$$g.(h, V) = (hg^{-1}, g.V) \qquad \forall h \in GL_n, V \in \mathtt{rep}_\alpha\, Q$$

and $GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q$ is the orbit space for this action. The GL_n-action on $GL_n \times \mathtt{rep}_\alpha\, Q$ by left multiplication on the first factor, factorizes through the $GL(\alpha)$-orbit space and there is a natural one-to-one correspondence between (1) GL_n-orbits in $GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q$ and (2) $GL(\alpha)$-orbits in $\mathtt{rep}_\alpha\, Q$.

Example 2 These facts allow us to finish the proof that the classification problem of equivalence classes of completely controllable systems is a noncommutative variety. For the quiver Q



take the dimension vector $\alpha = (1, n)$. There is a canonical identification

$$M_{n \times m}(\mathbb{C}) \times M_n(\mathbb{C}) \longrightarrow \mathtt{rep}_\alpha\, Q$$

sending a system (A, B) to the representation V of Q with

$$V_x = B, \qquad V_{a_i} = A_i$$

the i-th column of A. This map induces a natural one-to-one correspondence between

- Equivalence classes of systems under GL_n,

$$A' = gA, \qquad B' = gBg^{-1}$$

- $GL(\alpha)$-orbits of quiver representations in $\mathtt{rep}_\alpha\, Q$

$$A_i' = (cg)A_ic^{-1}, \qquad B = (cg)B(cg)^{-1}$$

for $(c,g) \in GL(\alpha) = \mathbb{C}^* \times GL_n$.

Having an identification between equivalence classes of linear systems and isomorphism classes of quiver representations, we want to identify the completely controllable systems in quiver terms. Let $\theta = (-n, 1)$, then $\theta.\alpha = 0$ and θ determines a *character* (that is, a group morphism)

$$GL(\alpha) = \mathbb{C}^* \times GL_n \longrightarrow \mathbb{C}^* \qquad (c,g) \mapsto c^{-n}det(g)$$

A representation $V \in \mathtt{rep}_\alpha\, Q$ is said to be θ-stable if and only if for every proper subrepresentation $W \subset V$ we have that

$$\theta.\beta > 0$$

where β is the dimension vector of W. Under the above identification we have a one-to-one correspondence between

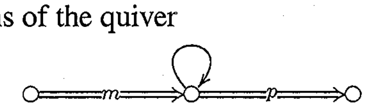
- Completely controllable systems $(A,B) \in Sys_c$, and
- θ-stable representations $(A_1, \ldots, A_m, B) \in \mathtt{rep}_\alpha\, Q$

Indeed, a representation $(A_1, \ldots, A_m, B)$ has a proper subrepresentation of dimension vector $(1,k)$ for some $k < n$ if and only if all vectors

$$A_iB^j \qquad 1 \leq i \leq m, 0 \leq j < n$$

span a vectorspace of dimension $\leq k$ but this is equivalent to the control matrix having rank at most k. using the natural one-to-one correspondence between $GL(\alpha)$-orbits in $\mathtt{rep}_\alpha\, Q$ and GL_n-orbits in the Zariski open component $GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q$ of $\mathtt{rep}_{n+1}\, \mathbb{C}Q$ we are done.

Exercise 1 Prove that the control-theoretic problem of classifying completely controllable and observable systems is a noncommutative variety. Identify $M_{n \times m}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{p \times n}(\mathbb{C})$ with representations of the quiver



of dimension vector $(1, n, 1)$ and show that Sys corresponds under this map to representations which are both $\theta = (-n, 1, 0)$ and $\theta' = (0, 1, -n)$-stable. Prove that the corresponding noncommutative algebra is

$$\mathbb{C}Q \simeq \begin{bmatrix} \mathbb{C} & \mathbb{C}[x] \otimes V & \mathbb{C}[x] \otimes V \otimes W \\ 0 & \mathbb{C}[x] & \mathbb{C}[x] \otimes W \\ 0 & 0 & \mathbb{C}[x] \end{bmatrix}$$

where V (resp. W) is an m (resp. p) dimensional vectorspace.

4. Hilbert schemes

In this section we will show that the *Hilbert scheme*, $\mathtt{Hilb}_n\ \mathbb{C}^2$ classifying all codimension n ideals $I \triangleleft \mathbb{C}[x, y]$ is a noncommutative variety. We will first reduce this classification problem to a GL_n-orbit problem and then identify the relevant noncommutative algebra. More information can be found in ng@n §8.4.

Let $I \triangleleft \mathbb{C}[x, y]$ be such that $V = \mathbb{C}[x, y]/I$ is an n-dimensional vectorspace and fix a basis $\{v_1, \ldots, v_n\}$ of V. Multiplication by x (resp. y) on $\mathbb{C}[x, y]$ induces a linear operator on V and hence determines a matrix $X \in M_n(\mathbb{C})$ (resp. $Y \in M_n(\mathbb{C})$). Clearly, $[X, Y] = 0$ and they generate an n-dimensional subalgebra $\mathbb{C}[X, Y] \simeq \mathbb{C}[x, y]/I$ of $M_n(\mathbb{C})$. Further, the image of the unit element $1 \in \mathbb{C}[x, y]$ determines a column vector $v \in V = \mathbb{C}^n$ with the property that

$$\mathbb{C}[X, Y]v = \mathbb{C}^n.$$

Note however that the triple $(v, X, Y) \in \mathbb{C}^n \oplus M_n \oplus M_n$ is not uniquely determined by the ideal I as it depends on the choice of basis of V. If we choose a different basis $\{v'_1, \ldots, v'_n\}$ with basechange matrix $g \in GL_n$, then the corresponding triple is

$$(v', X', Y') = (gv, gXg^{-1}, gYg^{-1}).$$

Consider the vectorspace of all triples

$$H_n = \mathbb{C}^n \oplus M_n \oplus M_n \quad \text{with action} \quad g.(v, X, Y) = (gv, gXg^{-1}, gYg^{-1})$$

for all $g \in GL_n$. The above discussion shows that the ideal $I \triangleleft \mathbb{C}[x, y]$ of codimension n determines an orbit $\mathcal{O}_I$ in H_n. Conversely, let C_n^c be the subset of triples $(v, X, Y) \in H_n$ satisfying the additional conditions :

1. The matrix pair commutes : $[X, Y] = 0$, and
2. v is a cyclic vector for this pair, that is : $\mathbb{C}[X, Y]v = \mathbb{C}^n$.

For $(v, X, Y) \in C_n^c$ we can define a map $\mathbb{C}[x, y] \xrightarrow{\phi} \mathbb{C}^n$ by sending a polynomial $f = f(x, y)$ to the vector $\phi(f) = f(X, Y)v$. By the second condition, ϕ is surjective and therefore, its kernel $I = \{f \in \mathbb{C}[x, y] \mid \phi(f) = 0\}$ is an ideal of codimension n. That is, the Hilbert scheme $\mathtt{Hilb}_n\ \mathbb{C}^2$ of n points in the plane $\mathbb{C}^2$ is the orbit space for the GL_n-action on the subset C_n^c.

Example 3 ($\mathtt{Hilb}_2\ \mathbb{C}^2$) Let us first consider the Hilbert scheme $\mathtt{Hilb}_1\ C^2$ of one point in $\mathbb{C}^2$ which we expect to be $\mathbb{C}^2$. Indeed, $H_1 = \{(v, X, Y) \mid v, X, Y \in \mathbb{C}\}$ and any pair (X, Y) is commuting. Moreover, v is cyclic for (X, Y) if and only if $v \neq 0$. That is, $C_1^c = \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$. The group $GL_1 = \mathbb{C}^*$ acts via $c.(v, X, Y) = (cv, X, Y)$ and hence the triples $\{(1, X, Y)\} = \mathbb{C}^2$ parametrize the orbits of C_1^c, that is, $\mathtt{Hilb}_1\ \mathbb{C}^2 = \mathbb{C}^2$ and the ideal I of codimension one corresponding to the point $p = (X, Y) \in \mathbb{C}^2$ is the ideal of polynomials $f \in \mathbb{C}[x, y]$ vanishing in p, $f(X, Y) = 0$.

Next, we consider the Hilbert scheme $\mathtt{Hilb}_2\ \mathbb{C}^2$ of two points in $\mathbb{C}^2$. Let $(v, X, Y) \in C_2^c$ and assume that either X or Y has distinct eigenvalues (type a). As

$$[\begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}] = \begin{bmatrix} 0 & (\nu_1 - \nu_2)b \\ (\nu_2 - \nu_1)a & 0 \end{bmatrix}$$

we have a representant in the orbit of the form

$$(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix})$$

where cyclicity of the column vector implies that $v_1 v_2 \neq 0$. The stabilizer subgroup of the matrix-pair is the group of diagonal matrices $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow GL_2$, hence the orbit has a unique representant with $v_1 = v_2 = 1$. The corresponding ideal $I \triangleleft \mathbb{C}[x, y]$ is then

$$I = \{f(x, y) \in \mathbb{C}[x, y] \mid f(\lambda_1, \mu_1) = 0 = f(\lambda_2, \mu_2)\}$$

hence these orbits in C_2^c correspond to sets of two distinct points in $\mathbb{C}^2$.

The situation is slightly more complicated when X and Y have only one eigenvalue (type b). If $(v, X, Y) \in C_2^c$ then either X or Y is not diagonalizable. But then, as

$$[\begin{bmatrix} \nu & 1 \\ 0 & \nu \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}] = \begin{bmatrix} c & d - a \\ 0 & c \end{bmatrix}$$

we have a representant in the orbit of the form

$$(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \mu & \beta \\ 0 & \mu \end{bmatrix})$$

with $[\alpha : \beta] \in \mathbb{P}^1$ and $v_2 \neq 0$. The stabilizer of the matrixpair is the subgroup

$$\{\begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \mid c \neq 0\} \hookrightarrow GL_2$$

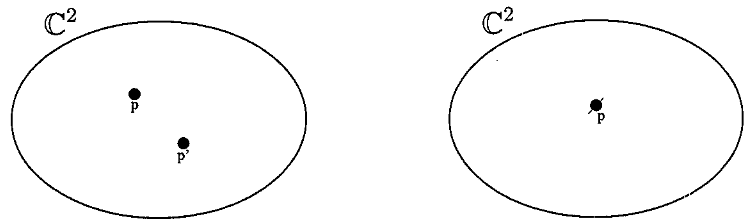
and hence we have a unique representant with $v_1 = 0$ and $v_2 = 1$. The corresponding ideal $I \triangleleft \mathbb{C}[x, y]$ is

$$I = \{f(x, y) \in \mathbb{C}[x, y] \mid f(\lambda, \mu) = 0 \text{ and } \alpha \frac{\partial f}{\partial x}(\lambda, \mu) + \beta \frac{\partial f}{\partial y}(\lambda, \mu) = 0\}$$

as one proves by verification on monomials because

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}^k \begin{bmatrix} \mu & \beta \\ 0 & \mu \end{bmatrix}^l \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k\alpha\lambda^{k-1}\mu^l + l\beta\lambda^k\mu^{l-1} \\ \lambda^k\mu^l \end{bmatrix}$$

Therefore, I corresponds to the set of two points at $(\lambda, \mu) \in \mathbb{C}^2$ infinitesimally attached to each other in the direction $\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$. For each point in $\mathbb{C}^2$ there is a $\mathbb{P}^1$ family of such *fat points*. Thus, points of $\mathtt{Hilb}_2\, \mathbb{C}^2$ correspond to either of the following two situations :



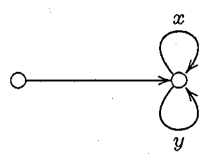
type a type b

The Hilbert-Chow map $\mathtt{Hilb}_2\, \mathbb{C}^2 \xrightarrow{\pi} S^2\, \mathbb{C}^2$ sends a point of type a to the formal sum $[p] + [p']$ and a point of type b to $2[p]$. Over the complement of (the image of) the diagonal, this map is a one-to-one correspondence. However, over points on the diagonal the fibers are $\mathbb{P}^1$, so π is not a one-to-one correspondence as in the case of $\mathtt{Hilb}_n\, \mathbb{C}^1$. The situation is nicer for $\mathbb{C}^1$ because there points can only collide along one direction, whereas in $\mathbb{C}^2$ they can approach each other along a $\mathbb{P}^1$ family of lines leading to different ideals. In fact, the symmetric power $S^2\, \mathbb{C}^2$ has singularities and the Hilbert-Chow map $\mathtt{Hilb}_2\, \mathbb{C}^2 \xrightarrow{\pi} S^2\, \mathbb{C}^2$ is a *resolution of singularities.*

Consider the noncommutative algebra

$$A = \begin{bmatrix} \mathbb{C} & \mathbb{C}[x,y] \\ 0 & \mathbb{C}[x,y] \end{bmatrix}$$

which is the path algebra of a quiver *with relations*. Consider the quiver Q



then $A \simeq \mathbb{C}Q/(xy - yx)$. The matrix idempotents can be used to decompose every finite dimensional representation, whence

$$\mathtt{rep}_n\, A = \bigsqcup_{|\alpha|=n} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, (Q|R)$$

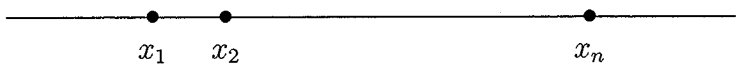
where the representation space of this quiver with relations is defined to be

$$\mathtt{rep}_\alpha\, (Q|R) = \{V \in \mathtt{rep}_\alpha\, Q \mid V_x V_y = V_y V_x\}$$

Exercise 2 Show that the Hilbert scheme $\mathtt{Hilb}_n\ \mathbb{C}^2$ is a noncommutative variety by relating it to the $\theta = (-n, 1)$-stable representations of $\alpha = (1, n)$-dimensional representations of A and hence to the orbits of a Zariski open subset of $\mathtt{rep}_{n+1}\ A$.

5. Calogero particles

More information on this section can be found in ag@n §8.6. The *Calogero system* is a classical particle system of n particles on the real line with inverse square potential.



That is, if the i-th particle has position x_i and velocity (momentum) y_i, then the Hamiltonian is equal to

$$H = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

The Hamiltonian *equations of motions* is the system of $2n$ differential equations

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial y_i} \\ \\ \dfrac{dy_i}{dt} = -\dfrac{\partial H}{\partial x_i} \end{cases}$$

This defines a dynamical system which is *integrable.*

A convenient way to study this system is as follows. Assign to a position defined by the $2n$ vector $(x_1, y_1; \dots, x_n, y_n)$ the couple of *Hermitian or self-adjoint* $n \times n$ matrices

$$X = \begin{bmatrix} y_1 & \frac{i}{x_1-x_2} & \cdots & \cdots & \frac{i}{x_1-x_n} \\ \frac{i}{x_2-x_1} & y_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{i}{x_{n-1}-x_n} \\ \frac{i}{x_n-x_1} & \cdots & \cdots & \frac{i}{x_n-x_{n-1}} & y_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}.$$

Physical quantities are given by invariant polynomial functions under the action of the unitary group $U_n(\mathbb{C})$ under simultaneous conjugation. In particular one considers the functions

$$F_j = tr\ \frac{X^j}{j}$$

For example,

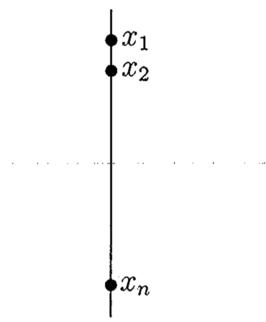
$$\begin{cases} tr(X) = \sum y_i & \text{the total momentum} \\ \frac{1}{2}tr(X^2) = \frac{1}{2}\sum y_i^2 - \sum_{i<j} \frac{1}{(x_i-x_j)^2} & \text{the Hamiltonian} \end{cases}$$

We can now consider the $U_n(\mathbb{C})$-translates of these matrix couples. This is shown to be a manifold with a free action of $U_n(\mathbb{C})$ such that the orbits are in one-to-one correspondence with points $(x_1, y_1; \ldots; x_n, y_n)$ in the phase space (that is, we agree that two such $2n$ tuples are determined only up to permuting the couples (x_i, y_i). The n-functions F_j give a completely integrable system on the phase space via Liouville's theorem.

In the classical case, all points are assumed to lie on the real axis and the potential is repulsive so that collisions do not appear. G. Wilson considered an alternative where the points are assumed to lie in the complex numbers and such that the potential is attractive (to allow for collisions), that is, the Hamiltonian is of the form

$$H = \frac{1}{2}\sum_i y_i^2 - \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

giving again rise to a dynamical system via the equations of motion. One recovers the classical situation back if the particles are assumed only to move on the imaginary axis.



In general, we want to extend the phase space of n distinct points analytically to allow for collisions. The strategy to do this is to assign to the n couples (x_i, y_i) the matrix-pair

$$X = \begin{bmatrix} y_1 & \frac{1}{x_1-x_2} & \cdots & \cdots & \frac{1}{x_1-x_n} \\ \frac{1}{x_2-x_1} & y_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{x_{n-1}-x_n} \\ \frac{1}{x_n-x_1} & \cdots & \cdots & \frac{1}{x_n-x_{n-1}} & y_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}.$$

and to observe that

$$[X, Y] + \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix} = \mathbb{1}_n$$

This equality suggests an approach to extend the phase space of n distinct complex Calogero particles to allow for collisions. Consider the following GL_n-orbit problem : consider the vectorspace

$$V_n = M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times \mathbb{C}^n \times (\mathbb{C}^n)^*$$

where $(\mathbb{C}^n)^*$ is the space of row-vectors. Define a GL_n-action on this space by

$$g.(X, Y, u, v) = (gXg^{-1}, gYg^{-1}, gu, vg^{-1})$$

Consider the GL_n-invariant closed subset of V_n

$$C_n = \{(X, Y, u, v) \mid [X, Y] + uv = \mathbb{1}_n\}$$

If Y is in diagonal form $diag(x_1, \ldots, x_n)$ then computing the diagonal entries of $[X, Y] + u.v = \mathbb{1}_n$ we find that $u_i v_i = 1$ for all $1 \leq i \leq n$, whence none of the entries of $[Y, X] + \mathbb{1}_n$ is zero and computing the (i, j)-entry we find that $x_i \neq x_j$ for $i \neq j$. The representant of Y is unique upto the action of a diagonal matrix D and a permutation matrix. The freedom in D allows us to normalize u and v such that

$$u = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}$$

and finally computing the equation $[X, Y] + u.v = \mathbb{1}_n$ with these normalized forms we find that X is the matrix described before. That is, every fourtuple $(X, Y, u, v) \in C_n$ with Y diagonalizable can be brought to the above standard form which is unique upto permuting the n points (x_i, y_i) hence the orbit corresponds to a Calogero state. Moreover, all these orbits are closed in C_n because u is a *cyclic vector* for any $(X, Y, u, v) \in C_n)$, that is, $\mathbb{C}^n$ is the smallest subspace containing u and stable under multiplication with X and Y.

Indeed, if U is a subspace of $\mathbb{C}^n$ stable under X and Y containing u, U is also stable under left multiplication with the matrix

$$A = [Y, X] + \mathbb{1}_n$$

and we have that $tr(A \mid U) = tr(\mathbb{1}_n \mid U) = dim\ U$. On the other hand, $A = u.v$ and therefore

$$A. \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} . \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} . \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = (\sum_{i=1}^{n} v_i c_i) \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Hence, if we take a basis for U containing u, then we have that

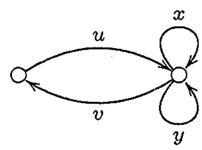
$$tr(A \mid U) = a$$

where $A.u = au$, that is $a = \sum u_i v_i$. But then, $tr(A \mid U) = dim\ U$ is independent of the choice of U. Now, $\mathbb{C}^n$ is clearly a subspace stable under X and Y and containing u, so we must have that $a = n$ and so the only subspace U possible is $\mathbb{C}^n$ proving cyclicity of u with respect to the matrix-couple (X, Y). Therefore, all GL_n-orbits in C_n are closed and we can construct an orbit-space

$$\mathtt{calo}_n = C_n / GL_n$$

which is called the *Calogero state space*, an open set of which describes the states such that all points are distinct (Y diagonalizable).

Exercise 3 Prove that C_n is a noncommutative variety. Hint : use the quiver Q



and consider the noncommutative algebra

$$A = \frac{\mathbb{C}Q}{(xy - yx + uv - e_2)}$$

where e_2 is the second vertexidempotent. Construct a one-to-one correspondence between C_n and $\mathtt{rep}_\alpha\ (Q|R)$ for $\alpha = (1, n)$ and show that there is a correspondence between GL_n-orbits in C_n and $GL(\alpha)$-orbits in $\mathtt{rep}_\alpha\ (Q|R)$. Finally, use this to identify the classification problem of Calogero particles with the isomorphism problem in a Zariski open subset of $\mathtt{rep}_{n+1}\ A$.

6. Research problem 1

Taken from Allen Tannenbaum "Invariance and System Theory : Algebraic and Geometric Aspects", Springer Lecture Notes in mathematics 845 (1981) p. 63-64.

Let $\mathtt{sys}_{m,n}$ be the orbitspace Sys_c/GL_n of completely controllable systems, then we have the diagram

$$\begin{tikzcd} Sys_c \arrow[r, hook] \arrow[d] & M_{n\times m}(\mathbb{C}) \times M_n(\mathbb{C}) \arrow[r, two heads, "pr_2"] & M_n(\mathbb{C}) \arrow[d, "\pi_n"] \\ \mathtt{sys}_{m,n} \arrow[rr, "\phi"'] & & \mathbb{C}^n \end{tikzcd}$$

where π_n is the quotient map of the conjugation action on $M_n(\mathbb{C})$. Describe explicitly the fibers of ϕ.

If U is the open subset of $\mathbb{C}^n$ determined by matrices with distinct non-zero eigenvalues, then for $c \in U$ one can prove that

$$\phi^{-1}(c) \simeq \underbrace{\mathbb{P}^{m-1} \times \ldots \times \mathbb{P}^{m-1}}_{n}$$

(verify this!). Therefore, the morphism $\phi\ :\ \mathtt{sys}_{m,n} \longrightarrow \mathbb{C}^n$ is a situation in which products of projective spaces are degenerating to varieties with rational singularities $\phi^{-1}(c)$ for $c \notin U$. "These degenerations should be very interesting from both a mathematical and system theoretic point of view", loc.cit. p. 64.

Use noncommutative geometry to prove that the situation is even nicer : prove that all fibers $\phi^{-1}(c)$ are smooth varieties and describe their structure explicitly.

the essential git&pi 2

Lieven Le Bruyn

Departement Wiskunde en Informatica, Universiteit Antwerpen (UIA)
B-2610 Antwerp (Belgium)
E-mail : lebruyn@uia.ua.ac.be

ABSTRACT: We introduce the representation varieties of noncommutative algebras, prove that the closed orbits correspond to semi-simple representations and describe the coordinate ring of the quotient variety. We illustrate these results with explicit computations for the conifold algebra.

Contents

1. Representation varieties

Let A be an affine $\mathbb{C}$-algebra with presentation

$$A = \frac{\mathbb{C}\langle x_1, \ldots, x_m \rangle}{I_A}$$

where I_A is the twosided ideal of relations holding in A. An n-dimensional *representation* is a $\mathbb{C}$-algebra morphism

$$A \xrightarrow{\phi} M_n(\mathbb{C})$$

which also determine a left A-module structure on $\mathbb{C}^n$ say M_ϕ by the rule that $a.v = \phi(a)v$ for all $a \in A$ and all $v \in \mathbb{C}^n$. We want to construct an affine variety (actually an affine *scheme*) $\mathtt{rep}_n\ A$ such that its $\mathbb{C}$-points are in one-to-one correspondence with the n-dimensional representations.

If A is the *free algebra* $A = \mathbb{C}\langle x_1, \ldots, x_m \rangle$, then any $\mathbb{C}$-algebra morphism $A \longrightarrow M_n(\mathbb{C})$ is fully determined by the images $a_i = \phi(x_i) \in M_n(\mathbb{C})$ of the variables. That is, the representation variety is just the affine space

$$\mathtt{rep}_n\ \mathbb{C}\langle x_1, \ldots, x_m \rangle = \underbrace{M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})}_{m} = \mathbb{C}^{mn^2}$$

the correspondence being given by $\phi \leftrightarrow (a_1, \ldots, a_m)$. Let $x_{ij}(k)$ be the n^2 coordinate functions on $\mathtt{rep}_n\ \mathbb{C}\langle x_1, \ldots, x_m \rangle$ corresponding to the k-th component $M_n(\mathbb{C})$ and define the *generic matrix*

$$X_k = \begin{bmatrix} x_{11}(k) & \ldots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \ldots & x_{nn}(k) \end{bmatrix}$$

Clearly, for an affine algebra A with defining ideal I_A as above, $\mathtt{rep}_n\, A$ is a closed subscheme of $\mathtt{rep}_n\, \mathbb{C}\langle x_1, \ldots, x_m\rangle$ and its coordinate ring is the quotient

$$\mathbb{C}[\mathtt{rep}_n\, A] = \frac{\mathbb{C}[x_{ij}(k)\ :\ 1 \leq i, j \leq n, 1 \leq k \leq m]}{(f_{ij}(X_1, \ldots, X_m)\ :\ f \in I_A)}$$

where for any $f(x_1, \ldots, x_m) \in I_A$ we can substitute a generic matrix for every variable and obtain an $n \times n$ matrix

$$f(X_1, \ldots, X_m) = \begin{bmatrix} f_{11}(X_1, \ldots, X_m) & \ldots & f_{1n}(X_1, \ldots, X_m) \\ \vdots & & \vdots \\ f_{n1}(X_1, \ldots, X_m) & \ldots & f_{nn}(X_1, \ldots, X_m) \end{bmatrix}$$

where each of the entries $f_{uv}(X_1, \ldots, X_m)$ is a polynomial in the variables $x_{ij}(k)$. Even if A is *not* finitely presented, that is I_A not finitely generated, then the ideal $I_n(A) = (f_{ij}(X_1, \ldots, X_m)\, \forall f \in I_A)$ is finitely generated because the polynomial algebra is Noetherian, so $\mathtt{rep}_n\, A$ is an *affine scheme*. Observe that it is not necessarily an *affine variety* because it may happen that $I_n(A)$ is not a radical ideal.

Still, the $\mathbb{C}$-points of $\mathtt{rep}_n\, A$, that is the maximal ideals of $\mathbb{C}[\mathtt{rep}_n\, A]$ or equivalently the algebra maps

$$\mathbb{C}[\mathtt{rep}_n\, A] \xrightarrow{\phi} \mathbb{C}$$

determine n-dimensional representations by sending the generator

$$\overline{x_i} \mapsto \begin{bmatrix} \phi(\overline{x_{11}(i)}) & \ldots & \phi(\overline{x_{1n}(i)}) \\ \vdots & & \vdots \\ \phi(\overline{x_{n1}(i)}) & \ldots & \phi(\overline{x_{nn}(i)}) \end{bmatrix}$$

There is an action by *simultaneous conjugation* of GL_n on $\mathtt{rep}_n\, \mathbb{C}\langle x_1, \ldots, x_m\rangle = M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})$

$$g.(a_1, \ldots, a_m) = (ga_1g^{-1}, \ldots, ga_mg^{-1})$$

which induces a GL_n-action by $\mathbb{C}$-algebra automorphisms on the polynomial algebra $\mathbb{C}[x_{ij}(k)\ : i, j, k]$ by sending $x_{ij}(k)$ to the (i, j)-entry of the $n \times n$ matrix of linear forms gX_kg^{-1}. Under this action it is clear that $I_n(A)$ is GL_n-stable, that is,

$$g.f \in I_n(A) \quad \text{for all } f \in I_n(A)$$

Therefore, there is an induced GL_n-action on the closed subscheme $\mathtt{rep}_n\, A$ and also on its associated reduced variety. The GL_n-orbits on the $\mathbb{C}$-points are precisely the isomorphism classes of n-dimensional left A-modules M_ϕ. Our main objective will be to develop tools to study this orbit space problem.

2. The conifold algebra

The conifold algebra is of current interest in stringtheory, see for example `hep-th/0110184`. This algebra is defined to be

$$A_c = \frac{\mathbb{C}\langle x, y, z\rangle}{(xz + zx, yz + zy, z^2 - 1, [z[x, y], x], [z[x, y], y])}$$

and we would like to understand the orbit-structure of the GL_n-schemes $\mathtt{rep}_n\ A_c$ for all n. The case $n = 1$ is pretty trivial : we assign to each variable a 1×1 matrix X, Y, Z and we obtain that

$$I_1(A_c) = (2XZ, 2YZ, Z^2 - 1)$$

whence $\mathbb{C}[\mathtt{rep}_1\ A_c] = \mathbb{C}[z]/(z^2 - 1) \simeq \mathbb{C} \times \mathbb{C}$ so $\mathtt{rep}_1\ A_c$ consists of two points corresponding to the one-dimensional representations

$$M_+ \begin{cases} x \mapsto 0 \\ y \mapsto 0 \\ z \mapsto 1 \end{cases} \qquad M_- \begin{cases} x \mapsto 0 \\ y \mapsto 0 \\ z \mapsto -1 \end{cases}$$

For $n > 1$ the situation is more complicated and can hardly be handled with the brute force methods of (commutative) algebraic geometry. Let us work out the case of two-dimensional representations. To define the ideal $I_2(A_c)$ we consider the generic matrices

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \qquad Z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$$

and we have to work out the matrix-identities induced by $I(A)$. For example,

$$XY + YX = \begin{bmatrix} 2x_1y_1 + x_2y_3 + x_3y_2 & x_1y_2 + x_2y_4 + x_2y_1 + x_4y_2 \\ x_1y_3 + x_3y_1 + x_3y_4 + x_4y_3 & 2x_4y_4 + x_2y_3 + x_3y_2 \end{bmatrix}$$

whence these four entries belong to $I_2(A_c)$. If we repeat this for the other defining relations of A_c we get a huge set of ideal generators of $I_2(A_c)$ which do not give us much insight into simple geometric questions about $\mathtt{rep}_2\ A_c$ such as : is it smooth ? what is the dimension ? let alone the orbit structure. This is were noncommutative geometry enters : we have to use ringtheoretic properties of the algebra A_c to get a grip on $\mathtt{rep}_n\ A_c$!

3. Semi-simple representations

Still, the general philosophy of *geometric invariant theory (git)* is quite helpful but we need to find a non-commutative algebraic interpretation of all the crucial concepts. In this section we will give a first example. We have seen that the best algebraic approximation to the (non-existent) orbit space classifies the *closed orbits*. In the case of the GL_n-action on $\mathtt{rep}_n\ A$ we will prove that the closed orbits are the isomorphism classes of semi-simple representations. In the proof we will need another big gun from invariant theory : the *Hilbert criterium*. More details can be found in ng@n §2.2, §2.3 and §2.4.

For $M \in \mathtt{rep}_n\ A$ we define its orbit

$$\mathcal{O}(M) = \{g.M \mid g \in GL_n\}$$

There are at least two 'natural' topologies on $\mathtt{rep}_n\ A$: the *Zariski topology* and the *analytic topology* (the induced topology from the embedding $\mathtt{rep}_n\ A \subset \mathbb{C}^{mn^2}$) so in principle we have to consider two closures of $\mathcal{O}(M)$. However, as $\mathcal{O}(M)$ is a *constructible set* it follows that

$$\overline{\mathcal{O}(M)}^{\mathbb{C}} = \overline{\mathcal{O}(M)}$$

Further, the complement $\overline{\mathcal{O}(M)} - \mathcal{O}(M)$ is a disjoint union of other orbits, all of which have *strictly smaller* dimension than $\mathcal{O}(M)$. This allows us to find a closed orbit in $\overline{\mathcal{O}(M)}$: take an orbit of minimal dimension! We will see in a moment that there is a *unique* closed orbit in $\overline{\mathcal{O}(M)}$.

A *one parameter subgroup* of GL_n is a morphism of algebraic groups

$$\lambda \, : \, \mathbb{C}^* \longrightarrow GL_n$$

any such morphism is fully determined by a $g \in GL_n$ and an n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ and is defined to be

$$\lambda(t) = g^{-1} \begin{bmatrix} t^{r_1} & & \\ & \ddots & \\ & & t^{r_n} \end{bmatrix} g$$

For $M \in \mathtt{rep}_n\ A$ we can hope to reach points in the orbit-closure $\overline{\mathcal{O}(M)}$ via the limit of one-parameter subgroup actions

$$N = \lim_{t \to 0} \lambda(t).M$$

This geometric construction has the following ringtheoretical interpretation : a limit point exists if and only if there is a finite *filtration*

$$0 = M_{k+1} \subset M_k \subset \ldots \subset M_1 \subset M_0 = M$$

of left A-submodules of M such that the *associated graded* module

$$gr(M) = \oplus_{i=0}^k M_i/M_{i+1} \simeq N$$

Suppose we have a limit, then we can decompose the underlying vectorspace $\mathbb{C}^n$ of M into *weight spaces*

$$\mathbb{C}^n = \oplus_i V_{\lambda,i} \qquad V_{\lambda,i} = \{v \in \mathbb{C}^n \mid \lambda(t)v = t^i v \ \forall t \in \mathbb{C}^*\}$$

dus andere $\oplus \Rightarrow V(k) = \bigoplus_{i \geq k} V_i$ A-deelmodules

$$\begin{tikzcd} V_i \arrow[r, "\varphi(a)."] & V_j \\ V_i \arrow[u, "\lambda(t)^{-1}"] \arrow[r, "\varphi'(a)"] & V_j \arrow[u, "\lambda(t)"'] \end{tikzcd}$$

$\varphi'(a) = t^{i-j}\varphi(a)$ limits only for $j \leq i$

and then one shows that $M_j = \oplus_{i>j} V_{\lambda,i}$ is a left A-submodule of M and these different submodules do define a filtration such that the associated graded corresponds to the limit point. Conversely, if we have a finite filtration, then decompose $\mathbb{C}^n$ into subspaces V_i for $0 \leq i \leq k$ such that $M_j = \oplus_{i=j}^k V_i$ and take λ to be the one-parameter subgroup defined by $\lambda(t) = t^i \mathbb{1}_{V_i}$. The relevance of this limit construction is clear from the Hilbert criterium, which specializes to the following form in the case of interest to us.

Theorem 1 (Hilbert criterium) *Let $M, N \in \mathtt{rep}_n\ A$ and assume that $\mathcal{O}(N)$ is a closed orbit contained in $\overline{\mathcal{O}(M)}$. Then, there is a one-parameter subgroup λ of GL_n such that*

$$\lim_{t \to 0} \lambda(t).M \in \mathcal{O}(N)$$

That is, there is a finite filtration by left A-submodules of M with associated graded module isomorphic to N.

An example showing that the condition of $\mathcal{O}(N)$ being a *closed orbit* is really necessary is given in ng@n §2.4. An immediate consequence of the Hilbert criterium is the ringtheoretic characterization of closed orbits in representation varieties.

Theorem 2 *For $M \in \mathtt{rep}_n\ A$ the orbit $\mathcal{O}(M)$ is closed if and only if M is a semi-simple left A-module (that is, a direct sum of simple A-modules).*

Proof. Assume that M is semi-simple, there is always a closed orbit $\mathcal{O}(N)$ contained in the orbit closure $\overline{\mathcal{O}(M)}$ whence there is a filtration by submodules on M having associated graded isomorphic to N. By semi-simplicity the associated graded N must be isomorphic to M so $\mathcal{O}(M)$ is closed.

Conversely, if $\mathcal{O}(M)$ is closed, consider a Jordan-Hölder filtration on M (such that the associated graded module is the direct sum of the composition factors of M), then $gr(M)$ is semi-simple and contained in $\overline{\mathcal{O}(M)} = \mathcal{O}(M)$ whence $gr(M) \simeq M$.

4. The conifold simples

The (geometric) description of all finite dimensional simple representations of an affine $\mathbb{C}$-algebra is difficult, in general. In the special case of the conifold algebra we will classify all simple representations by means of some classical ringtheory : the theory of *quadratic forms* and their associated *Clifford algebras*.

For R a commutative $\mathbb{C}$-algebra, an *m-ary quadratic form* is a quadratic polynomial in $R[X_1, \ldots, X_m]$

$$f(X_1, \ldots, X_m) = \begin{bmatrix} X_1 & \ldots & X_m \end{bmatrix} \cdot \begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{1m} & \ldots & a_{mm} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix}$$

where the matrix $A_f = (a_{ij})_{i,j} \in M_m(R)$ is *symmetric*. The quadratic form f induces a quadratic map Q_f resp. a symmetric bilinear pairing B_f on the free R-module $R^{\oplus m} = R \oplus \ldots \oplus R$ of rank m

$$Q_f \, : \, R^{\oplus m} \longrightarrow R \qquad B_f \, : \, R^{\oplus m} \times R^{\oplus m} \longrightarrow R$$

where for any two column vectors $\vec{x} = (x_1, \ldots, x_m)^{tr}$ and $\vec{y} = (y_1, \ldots, y_m)^{tr}$ in $R^{\oplus m}$ we define

$$Q_f(\vec{x}) = \vec{x}^{tr}.A_f.\vec{x} \qquad B_f(\vec{x}, \vec{y}) = \frac{1}{2}(Q_f(\vec{x}+\vec{y}) - Q_f(\vec{x}) - Q_f(\vec{y}))$$

The *radical* of the quadratic form f is the submodule

$$rad(f) = \{\vec{x} \in R^{\oplus m} \mid \forall \vec{y} \in R^{\oplus m} \; : \; B_f(\vec{x}, \vec{y}) = 0\}$$

The quadratic form f is said to be *regular* if $rad(f) = \{\vec{0}\}$ and is said to be *non-singular* if $A_f \in GL_m(R)$, that is, $det(A_f)$ is a unit in R.

The *Clifford algebra* Cl_f of the quadratic form f is the noncommutative R-algebra defined to be the quotient of the of the free R-algebra $R\langle X_1, \ldots, X_m\rangle$ (where the variables X_i are a basis of the free R-module $R^{\oplus m}$ by the twosided ideal generated by all elements of the form

$$\vec{x} \otimes \vec{x} - Q_f(\vec{x})$$

with $\vec{x} \in R^{\oplus m}$. If we give the free R-algebra the natural *gradation*, that is $deg(X_i) = 1$ and $deg(r) = 0$ for all $r \in R$, then $\vec{x} \otimes \vec{x}$ is *homogeneous* of degree two whereas $Q_f(\vec{x}) \in R$ has degree zero. Therefore, Cl_f has a natural $\mathbb{Z}/2\mathbb{Z}$-gradation

$$Cl_f = C_0 \oplus C_1 \qquad C_i.C_j \subset C_k \qquad k = i + j \; mod \; 2$$

Generators of C_0 are said to be *bosonic* whereas generators in C_1 are called *fermionic*.

Example 1 (Conifold algebra) Let R be the polynomial algebra $\mathbb{C}[x, y, z]$ and consider the Clifford algebra Cl over R associated to the symmetric 3×3 matrix

$$\begin{bmatrix} x & z & 0 \\ z & y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let X, Y and Z be the standard basis for $R^{\oplus 3}$, then the defining relations of Cl as an R-algebra are

$$\begin{cases} X^2 & = x \\ Y^2 & = y \\ Z^2 & = 1 \\ XY + YX & = 2z \\ XZ + ZX & = 0 \\ YZ + ZY & = 0 \end{cases}$$

$(XY+YX)X \overset{?}{=} X(XY+YX)$

$XYX + \underline{YX^2} = \underline{X^2Y} + XYX$

Alternatively, we can describe Cl as the $\mathbb{C}$-algebra, generated by X, Y and Z and such that x, y and z defined as above are central elements, that is

$$Cl = \frac{\mathbb{C}\langle X, Y, Z\rangle}{(Z^2 - 1, XZ + ZX, YZ + ZY, [X^2, Y], [Y^2, X])}$$

Observe that it follows from these equations that X^2, Y^2 and $XY + YX$ are central elements of Cl. Now, recall that the conifold algebra was described to be the algebra

$$A_c = \frac{\mathbb{C}\langle X, Y, Z\rangle}{(Z^2 - 1, XZ + ZX, YZ + ZY, [Z[X, Y], X], [Z[X, Y], Y])}$$

Because Z is a unit and $ZX = -XZ$, $ZY = -YZ$ it is clear that the two last identities are equivalent to

$$[Z[X, Y], X] = 0 \Leftrightarrow [X^2, Y] = 0 \quad \text{and} \quad [Z[X, Y], Y] = 0 \Leftrightarrow [Y^2, X] = 0$$

whence $A_c \simeq Cl$.

A *simple* n-dimensional representation of $A_c = Cl$ determines a $\mathbb{C}$-algebra epimorphism

$$A_c \xrightarrow{\phi} M_n(\mathbb{C})$$

and $I_\phi = ker(\phi)$ is a twosided maximal ideal of A_c. Because $R = \mathbb{C}[x, y, z]$ is a central subalgebra of A_c and A_c is a finitely generated R-module (actually, it is a free R-module of rank 8), the intersection $I_\phi \cap R$ is a maximal ideal of R, hence determines a point $p = (a, b, c) \in \mathbb{C}^3$ and we obtain an epimorphism $Cl_{f(p)} \longrightarrow M_n(\mathbb{C})$ from the Clifford algebra $Cl_{f(p)}$ over $\mathbb{C}$ associated to the ternary form

$$f(p) = \begin{bmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The algebraic structure of Clifford algebras over $\mathbb{C}$ is well-known allowing us to determine its simple factors. Let f be an m-ary quadratic form over $\mathbb{C}$ having a k-dimensional radical, then

$$Cl_f \simeq Cl_g \otimes'_{\mathbb{C}} \Lambda(\mathbb{C}^k)$$

for some *non-singular* $m - k$-ary quadratic form g over $\mathbb{C}$ and where $\Lambda(\mathbb{C}^k)$ is the *exterior algebra* on k letters, that is,

$$\Lambda(\mathbb{C}^k) = \mathbb{C}\langle y_1, \ldots, y_k\rangle / (y_i^2, y_i y_j + y_j y_i \; \forall 1 \leq i, j \leq k)$$

which is a finite dimensional *graded* $\mathbb{C}$-algebra of dimension 2^k having a unique simple representation (of dimension one, dividing out $(y_1, \ldots, y_k)$). In the isomorphism we used the modified tensorproduct $\otimes'$ of $\mathbb{Z}/2\mathbb{Z}$-algebras, that is

$$(c \otimes' e)(c' \otimes' e') = (-1)^{deg(e)deg(c')} cc' \otimes' ee' \qquad \forall c \in Cl_g, e \in \Lambda(\mathbb{C}^k)$$

Anyway, the simple factors of Cl_f are entirely determined by those of Cl_g so we have to recall the structure of Clifford algebras over $\mathbb{C}$ of $n = m - k$-ary *non-singular* quadratic forms :

$$Cl_g \simeq \begin{cases} M_{2^l}(\mathbb{C}) \times M_{2^l}(\mathbb{C}) & \text{if } n = 2l+1 \text{ is odd.} \\ M_{2^l}(\mathbb{C}) & \text{if } n = 2l \text{ is even.} \end{cases}$$

Theorem 3 *Every simple representation of the conifold algebra $A_c \simeq Cl_f$ has dimension ≤ 2 and determines a point $p = (a, b, c) \in \mathbb{C}^3$.*

1. *If $p \notin \mathbb{V}(xy - z^2)$, then there are two simple A_c-representations of dimension two lying over p.*

2. *If $p \in \mathbb{V}(xy - z^2) - \{(0,0,0)\}$, then there is a unique simple A_c-representation of dimension two lying over p.*

3. *If $p = (0,0,0)$, then there are two simple A_c-representations of dimension one lying over p.*

5. The quotient variety

In this section we will give generators of the ring of *polynomial invariants* $\mathbb{C}[\mathtt{rep}_n\ A]^{GL_n}$ and prove that the associated variety classifies the closed GL_n-orbits, that is the isomorphism classes of semi-simple n-dimensional representations of A. For this reason we will denote the algebraic quotient variety in this case

$$\mathtt{rep}_n\ A//GL_n = \mathtt{iss}_n\ A$$

for *i*somorphisms of *s*emi-*s*imple representations. We start with the ring of invariants, more details and proofs can be found in ng@n §1.3 and §2.5.

Consider the special case when $A = \mathbb{C}\langle x_1, \ldots, x_m \rangle$, then the action of GL_n on $\mathtt{rep}_n\ A = M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})$ is by simultaneous conjugation and we like to determine polynomial functions in the matrix-entries which are invariant under this action. Recall that the k-th *generic matrix* is defined to be the $n \times n$ matrix of coordinate functions

$$X_k = \begin{bmatrix} x_{11}(k) & \ldots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \ldots & x_{nn}(k) \end{bmatrix}$$

Clearly, as $g.X_k = gX_kg^{-1}$ we have $g.X_{i_1} \ldots X_{i_l} = gX_{i_1} \ldots X_{i_l}g^{-1}$ whence the polynomial function $tr(X_{i_1} \ldots X_{i_l})$ is GL_n-invariant.

Theorem 4 (Procesi-Razmyslov) *The ring of polynomial invariants*

$$\mathbb{C}[M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})]^{GL_n}$$

is generated by traces of monomials in the generic matrices

$$tr(X_{i_1} \ldots X_{i_l})$$

for all $1 \leq i_u \leq m$ *and* $l \leq n^2 + 1$.

Proof. (sketch, more details in na@g §1.3-1.6) First determine the *multi-linear* GL_n-invariants, that is, the linear maps

$$M_n(\mathbb{C})^{\otimes m} = M_n(\mathbb{C}) \otimes \ldots \otimes M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

invariant under the diagonal action of GL_n. Write $M_n(\mathbb{C}) = V_n \otimes V_n^*$ for $V_n = \mathbb{C}^n$ the standard GL_n-representation, then this is the problem of determining all GL_n-invariants of m vectors and m covectors, that is GL_n-invariant linear maps

$$(V_n^*)^{\otimes m} \otimes (V_n)^{\otimes m} \longrightarrow \mathbb{C}$$

which by *classical invariant theory* are linear combination of the invariants

$$\mu_\sigma(f_1 \otimes \ldots \otimes f_m \otimes v_1 \otimes \ldots \otimes v_m) = \prod_i f_i(v_{\sigma(i)})$$

for $\sigma \in S_m$. If we write $A_i = v_i \otimes f_i$ and if we decompose the partition into cycles

$$\sigma = (i_1 i_2 \ldots i_\alpha)(j_1 j_2 \ldots j_\beta) \ldots (z_1 z_2 \ldots z_\zeta)$$

then one calculates that

$$\mu_\sigma(A_1 \otimes \ldots \otimes A_m) = tr(A_{i_1} A_{i_2} \ldots A_{i_\alpha}) tr(A_{j_1} A_{j_2} \ldots A_{j_\beta}) \ldots tr(A_{z_1} A_{z_2} \ldots A_{z_\zeta})$$

For an arbitrary GL_n-invariant polynomial functions, one can reduce to its (multi) homogeneous components of degree $(d_1, \ldots, d_m)$ apply the *polarization process*, that is reduce to multi-linear polynomials by increasing the number of matrices, use the multi-linear result above and recover the polynomial back by the *restitution process* which proves that any polynomial invariant is a polynomial in the invariants

$$tr(X_{i_1} X_{i_2} \ldots X_{i_l})$$

for some $1 \leq i_u \leq m$ and some l. Because GL_n is a *reductive group* (that is, all its finite dimensional representations are completely reducible) we know from general theory that the ring of invariants is finitely generated (see ng@n §2.5 for a proof), so there is a bound on the l needed. Alternatively, one can use a non-commutative argument (based on the Nagata-Higman result) to get the bound $l \leq 2^m$, see ng@n §1.4. Using a lot more on the trace relations (that is, the identities holding among these generators) one can then reduce the bound to $l \leq n^2 + 1$, see ng@n §1.5-1.7.

Next, consider an arbitrary affine $\mathbb{C}$-algebra with presentation

$$A = \frac{\mathbb{C}\langle x_1, \ldots, x_m \rangle}{I_A}$$

then $\mathtt{rep}_n\, A \hookrightarrow M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})$ is a closed GL_n-subscheme and the epimorphism $\mathbb{C}[M_n \times \ldots \times M_n] \twoheadrightarrow \mathbb{C}[\mathtt{rep}_n\, A]$ induces an epimorphism on the level of invariants (use reductivity of GL_n and the fact that the GL_n action on $\mathtt{rep}_n\, A$ is *locally finite*, see ng@n §2.5)

$$\mathbb{C}[M_n(|C) \times \ldots \times M_n(\mathbb{C})]^{GL_n} \twoheadrightarrow \mathbb{C}[\mathtt{rep}_n\, A]^{GL_n}$$

we deduce that the ring of polynomial invariants $\mathbb{C}[\mathtt{rep}_n\, A]^{GL_n}$ is generated by the images of the traces in monomials in the generic matrices of length $l \leq n^2 + 1$.

Because by ng@n §2.5, the ring of polynomial invariants separates disjoint closed GL_n-stable subschemes of $\mathtt{rep}_n\, A$ and so in particular its points classify the closed orbits (that is, the isoclasses of n-dimensional semi-simple representations of A we will denote

$$\mathbb{C}[\mathtt{rep}_n\, A]^{GL_n} = \mathbb{C}[\mathtt{iss}_n\, A]$$

and denote the corresponding quotient scheme with $\mathtt{iss}_n\, A$. the inclusion $\mathbb{C}[\mathtt{iss}_n\, A] \subset \mathbb{C}[\mathtt{rep}_n\, A]$ induces a projection (the quotient map)

$$\mathtt{rep}_n\, A \xrightarrow{\pi} \mathtt{iss}_n\, A$$

which sends an n-dimensional representation V of A to the direct sum of its Jordan-Hölder components. In particular it follows that two n-dimensional representations V and V' have the same Jordan-Hölder decomposition if and only if all traces of monomials in the algebra generators of A evaluate to the same complex number when evaluated in V and V'.

6. The conifold quotient singularity

Next week we will be able to study the local geometry of $\mathtt{rep}_2\, A_c$ and prove that it is a smooth variety. Today we will determine the quotient variety $\mathtt{iss}_2\, A$ and relate it to our previous classification of the simple representations of A_c.

We have to determine the traces of monomials in the images of the 2×2 generic matrices determined by the generators X, Y and Z of A_c. Because $Z^2 = 1$ we have to separate three possible diagonal forms for Z

$$(a)\ Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (b)\ Z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (c)\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In the cases (a) and (b), Z is a central unit whence the relations $XZ + ZX = 0 = YZ + ZY$ imply that $X = Y = 0$, so these components of $\mathtt{iss}_2\, A_c$ correspond to just one point, resp.

$$(a)\ M_1^{\oplus 2} \qquad \text{and} \qquad (b)\ M_{-1}^{\oplus 2}$$

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where $M_{\pm 1}$ is the one-dimensional simple representation of A_c determined by $X \mapsto 0, Y \mapsto 0$ and $Z \mapsto \pm 1$. Th remaining component (c) is the interesting one corresponding to the Clifford algebra. Here, we have (because the characteristic polynomials of the images of the generic matrices in are $X^2 - x\mathbb{1}_2 = 0,\ Y^2 - y\mathbb{1}_2 = 0$ and $Z^2 - \mathbb{1}_2 = 0$)

$$tr(Z) = 0,\ tr(Z^2) = 2,\ tr(X) = 0,\ tr(X^2) = 2x,\ tr(Y) = 0,\ tr(Y^2) = 2y$$

Moreover, the equations $XZ + ZX = 0 = YZ + ZY$ imply that

$$tr(XZ) = 0 = tr(YZ) \qquad \text{whereas} \qquad tr(XY) = 2z$$

These relations allow us to prove that the trace of any monomial in just two of the generators $\{X, Y, Z\}$ is a polynomial in x, y and z. But there is another generator of the ring of polynomial invariants,

$$w = tr(XYZ)$$

To determine its value, we start from the characteristic polynomial

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$$(XYZ)^2 - tr(XYZ)XYZ + det(X)det(Y)det(Z) = 0$$

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work out the first term using the defining relations in A_c to get

$$(XYZ)^2 = -X^2Y^2 + 2zXY = -xy\mathbb{1}_2 + 2zXY$$

and take the trace of the characteristic polynomial. It follows that w satisfies the quadratic relation

$$-2xy + 4z^2 - w^2 - 2xy = 0 \quad \text{that is} \quad w^2 = 4(z^2 - xy)$$

By the defining relations in A_c (or equivalently the characteristic polynomials of X, Y and Z) it follows that every trace of a monomial in X, Y and Z is a polynomial in x, y, z and w. That is, the coordinate ring of the quotient variety is

$$\mathbb{C}[\mathtt{iss}_2\ A_c] = \frac{\mathbb{C}[x, y, z, w]}{w^2 - 4z^2 - 4xy} \simeq \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{(x_1x_2 - x_3x_4)}$$

which is the coordinate ring of the *conifold quotient singularity*.

By the foregoing section we know that the points in $\mathtt{iss}_2\ A_c$ classify the semi-simple 2-dimensional representations of A_c. We have seen that all of them are simple except for $M_1 \oplus M_{-1}$ determined by the origin. The embedding

$$\mathbb{C}[x, y, z] \subset \mathbb{C}[\mathtt{iss}_2\ A_c]$$

determines a projection

$$\mathtt{iss}_2\ A_c \xrightarrow{f} \mathbb{C}^3$$

which is *ramified* along $\mathbb{V}(z^2 - xy)$. That is, if $p \notin \mathbb{V}(z^2 - xy)$ there are two points of $\mathtt{iss}_2\ A_c$ lying over p whereas if $p \in \mathbb{V}(z^2 - xy)$ there is just one. This agrees with the fact that there is just one semi-simple (actually simple) representation for every point in $\mathtt{iss}_2\ A_c$ which is not the origin.

$\lambda(aa') = \rho_M(a)\lambda(a') + \lambda(a)\rho_M(a')$

$\lambda : A \to Hom_{\mathbb{C}}(M, M)$

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Lieven Le Bruyn

Departement Wiskunde en Informatica, Universiteit Antwerpen (UIA)
B-2610 Antwerp (Belgium)
E-mail : lebruyn@uia.ua.ac.be

ABSTRACT: In this part we show how one can compute the dimensions of components of $\mathtt{rep}_n\ A$ and give two methods to verify whether a component is smooth. the second method, using self-extensions defines a local quiver setting which (via the Luna slice theorem) describes the GL_n-étale local structure of $\mathtt{rep}_n\ A$. We illustrate these results with explicit calculations for the conifold algebra.

Contents

Last time we have seen how to approach the quotient variety (or scheme)

$$\mathtt{iss}_n\, A = \mathtt{rep}_n\, A//GL_n$$

classifying n-dimensional semi-simple A-representations. Today, we will use this to study the GL_n-geometry of the representation variety $\mathtt{rep}_n\, A$ itself.

1. Components and their dimensions

Representation varieties behave functorially, that is, if $B \xrightarrow{f} A$ is a $\mathbb{C}$-algebra morphism, there is an induced morphism between the representation schemes

$$\mathtt{rep}_n\, A \xrightarrow{f^*} \mathtt{rep}_n\, B$$

obtained by *restriction of scalars.* If $p \in \mathtt{rep}_n\, A$ determines the algebra morphism $A \xrightarrow{\phi_p} M_n(\mathbb{C})$ then $f^*(p)$ is the n-dimensional representation of B defined by the composition

$$B \xrightarrow{f} A \xrightarrow{\phi_p} M_n(\mathbb{C})$$

Assume that A has a set $\{e_1, \ldots, e_k\}$ of *orthogonal idempotents*, that is, elements satisfying

$$e_i^2 = e_i \qquad e_ie_j = 0 \text{ when } i \neq j \qquad e_1 + \ldots + e_k = 1$$

then these elements generate a semi-simple commutative subalgebra of A

$$B = \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{k} \hookrightarrow A$$

Because B is semi-simple, all its n-dimensional representations are semi-simple and B has precisely k simple (one-dimensional) representations determined by its components, that is,

$$M_i \,:\, e_1 \mapsto 0, \quad \ldots\; e_{i-1} \mapsto 0,\; e_i \mapsto 1,\; e_{i+1} \mapsto 0, \;\ldots \quad e_k \mapsto 0$$

Because all B-representations are semi-simple, their orbits are closed and the representation variety decomposes as a finite disjoint union

$$\mathtt{rep}_n\, B = \bigsqcup_{\substack{\sum_i a_i = n \\ \alpha=(a_1,\ldots,a_k)}} \mathcal{O}(M_1^{\oplus a_1} \oplus \ldots \oplus M_k^{\oplus a_k})$$

Hence, $\mathtt{rep}_n\, A$ also decomposes

$$\mathtt{rep}_n\, A = \bigsqcup_{\substack{\sum_i a_i = n \\ \alpha=(a_1,\ldots,a_k)}} \mathtt{rep}_\alpha\, A \quad \text{with} \quad \mathtt{rep}_\alpha\, A = (f^*)^{-1}(\mathcal{O}(M_1^{\oplus a_1} \oplus \ldots \oplus M_k^{\oplus a_k})$$

We call $\mathtt{rep}_\alpha\, A$ the *representation scheme of α-dimensional representations of A*. It consists of those representations $A \xrightarrow{\phi} M_n(\mathbb{C})$ such that

$$tr(\phi(e_1)) = a_1, tr(\phi(e_2)) = a_2, \ldots, tr(\phi(e_k)) = a_k$$

Example 1 The conifold algebra A_c contains the subalgebra

$$B = \frac{\mathbb{C}[z]}{(z^2-1)} \simeq \mathbb{C} \times \mathbb{C}$$

the idempotents being given by the elements $e_1 = \frac{1}{2}(1-z)$ and $e_2 = \frac{1}{2}(1+z)$. Therefore, $\mathtt{rep}_2\, A_c$ decomposes into three components

$$\mathtt{rep}_{(2,0)}\, A_c \sqcup \mathtt{rep}_{(1,1)}\, A_c \sqcup \mathtt{rep}_{(0,2)}\, A_c$$

and contain resp. the representations for which $tr(z) = -2$ $(2,0)$, $tr(z) = 0$ $(1,1)$ and $tr(z) = 2$ $(0,2)$. Another way to say the same thing is : because $\phi(z)$ is a *semi-simple* element in $M_2(\mathbb{C})$ it is a diagonalizable matrix, so is conjugated to one of the following three possible diagonal matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If ϕ lies in the first or the last component, $\phi(z)$ is (conjugated to) a central unit, but then

$$0 = \phi(x)\phi(z) + \phi(z)\phi(x) = 2\phi(x)\phi(z)$$

whence $\phi(x) = 0$ (and similarly $\phi(y) = 0$. That is,

$$\mathtt{rep}_{(2,0)}\, A_c = \mathcal{O}(M_- \oplus M_-) \qquad \mathtt{rep}_{(0,2)}\, A_c = \mathcal{O}(M_+ \oplus M_+)$$

Last time we calculated that the component $\mathtt{rep}_{(1,1)}\, A_c$ corresponds to the Clifford algebra and has quotient variety the three-dimensional conifold singularity.

In all relevant examples one is usually interested in those components $\mathtt{rep}_\alpha\ A$ which have a Zariski open subset of *simple* n-dimensional representations. Equivalently, there is a Zariski open subset $Az \subset \mathtt{iss}_\alpha\ A = \mathtt{rep}_\alpha\ A//GL_n$, called the *Azumaya locus* such that under the quotient map

$$\mathtt{rep}_\alpha\ A \xrightarrow{\pi} \mathtt{iss}_\alpha\ A$$

the *fiber* $\pi^{-1}(p)$ is a single orbit $\mathcal{O}(M)$ with M a simple A-module. This often allows us to compute the dimension of such components.

By *Schur's lemma* we know that $End_A(M)$, the A-module morphisms $M \longrightarrow M$ of a simple A-module M are reduced to scalar multiples $\mathbb{C}id_M$. Therefore, if M is a simple n-dimensional representation, its *stabilizer subgroup*

$$Stab(M) = \{g \in GL_n \mid g.M = M\} = \mathbb{C}^*\mathbb{1}_n$$

and therefore the dimension of the orbit is

$$dim\ \mathcal{O}(M) = dim\ GL_n - dim\ Stab(M) = n^2 - 1$$

Alternatively, the orbit $\mathcal{O}(M)$ is isomorphic to the *projective linear group* $PGL_n = GL_n/(\mathbb{C}^*\mathbb{1}_n)$. In fact, we will see in a moment that the orbit-map $\pi^{-1}(Az) \xrightarrow{\pi} Az$ over the Azumaya locus is a *principal* PGL_n*-fibration.*

Theorem 1 *Assume* $\mathtt{iss}_\alpha\ A$ *is an irreducible component of dimension* d *of* $\mathtt{iss}_n\ A$ *containing a Zariski open subset of simple* n*-dimensional representations. Then,*

$$dim\ \mathtt{rep}_\alpha\ A = d + n^2 - 1$$

Similar tricks can often be used to compute the dimension of components if we have information on the stabilizer subgroup of a *generic representation* (that is, for a Zariski open subset of its representations).

Example 2 The components $\mathtt{rep}_{(2,0)}\ A_c$ and $\mathtt{rep}_{(0,2)}\ A_c$ consist of a unique orbit $\mathcal{O}(M)$ so its dimension is $4 - dim\ Stab(M)$. As M is twice the sum of a simple one-dimensional representation, $End(M) = M_2(\mathbb{C})$ whence its stabilizer subgroup is GL_2 (or verify directly), whence these components reduce to a single fix-point.

We have seen last time that $\mathtt{iss}_{(1,1)}\ A_c$ is irreducible of dimension three. Moreover, all of its points but the origin determine a simple 2-dimensional representation, that is,

$$Az = \mathtt{iss}_{(1,1)}\ A_c - \{t\}$$

the Azumaya locus is the complement of the isolated singularity t. Therefore,

$$dim\ \mathtt{rep}_{(1,1)}\ A_c = 3 + (4-1) = 6$$

which would be hard to determine just starting from the defining equations of $\mathtt{rep}_2\ A_c$.

2. The tangent space

Having determined the good components and their dimensions, we would like to determine whether $\mathtt{rep}_\alpha\ A$ is a *smooth variety* and, if not, to locate its singularities.

If $A = \mathbb{C}\langle x_1, \ldots, x_m\rangle / I_A$ we have a closed embedding

$$\mathtt{rep}_\alpha\ A \hookrightarrow \mathbb{C}^{mn^2} = M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})$$

and every n-dimensional representation $M \in \mathtt{rep}_\alpha\ A$ is determined by an m-tuple of matrices $m = (m_1, \ldots, m_m)$ where $m_i = \phi(\overline{x_i})$ if $\phi : A \longrightarrow M_n(\mathbb{C})$ is the algebra map determined by M. The *tangent space* at m in the big space

$$T_m(M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C})) = M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})$$

and the element $(n_1, \ldots, n_m)$ in it can be represented by the m-tuple of $n \times n$ matrices over the *dual numbers* $\mathbb{C}[\epsilon] \simeq \mathbb{C}[x]/(x^2)$

$$(m_1 + \epsilon n_1, \ldots, m_m + \epsilon n_m) \in M_n(\mathbb{C}[\epsilon]) \times \ldots \times M_n(\mathbb{C}[\epsilon])$$

The *tangent space in* m *at the subscheme* $\mathtt{rep}_\alpha\ A$, $T_m(\mathtt{rep}_\alpha\ A)$ is the *subspace* consisting of those $(n_1, \ldots, n_m)$ such that

$$f(m_1 + \epsilon n_1, \ldots, m_m + \epsilon n_m) = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in M_n(\mathbb{C}[\epsilon])$$

for all $f(x_1, \ldots, x_m) \in I_A$. Even if we do not know explicitly the full ideal of relations of $\mathtt{rep}_n\ A$ or $\mathtt{rep}_\alpha\ A$ this is usually easy to work out as it is a system of linear equations in the entries if $(n_1, \ldots, n_m)$.

If the dimension of $\mathtt{rep}_m\ A$ in m is d, then m is a smooth point of $\mathtt{rep}_n\ A$ if and only if

$$dim_{\mathbb{C}}\ T_m(\mathtt{rep}_\alpha\ A) = d$$

To verify that $\mathtt{rep}_\alpha\ A$ is smooth we do not have to verify this condition in *all* points. For, the *singular locus* of $\mathtt{rep}_\alpha\ A$ is closed and GL_n-stable, so there is a singularity if and only if there is a closed orbit (a semi-simple representation) of singularities.

Theorem 2 $\mathtt{rep}_\alpha\ A$ *is smooth if and only if for every* semi-simple $M \in \mathtt{rep}_\alpha\ A$ *we have that*

$$dim_{\mathbb{C}}\ T_m(\mathtt{rep}_\alpha\ A) = dim_m\ \mathtt{rep}_\alpha\ A$$

Often, one only has to consider the 'worst' possible semi-simples in $\mathtt{rep}_\alpha\ A$.

Example 3 We will show that $\mathtt{rep}_{(1,1)}\ A_c$ is a smooth variety. We know already that the quotient variety $\mathtt{iss}_{(1,1)}\ A_c$ has an isolated singularity t and that the complement is the Azumaya locus of A_c. Therefore,

$$\pi^{-1}(\mathtt{iss}_{(1,1)}\ A_c - \{t\}) \xrightarrow{\pi} \mathtt{iss}_{(1,1)}\ A_c$$

is a principal PGL_n-fibration over a smooth variety and hence the total space is smooth. (This holds more generally, the part of $\mathtt{rep}_\alpha\ A$ lying over the intersection of the Azumaya locus and the smooth locus of $\mathtt{iss}_\alpha\ A$ contains only smooth points).Hence, possible singularities must lie in $\pi^{-1}(t)$ and there is a unique semi-simple representation in this closed GL_n-stable set, $m = M_+ \oplus M_-$ or in matrix-terms

$$x \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

That is, we have to solve which $x_1, \ldots, x_4, y_1, \ldots, y_4, z_1, \ldots, z_4$ satisfy all defining relations of A_c when we replace x, resp. y and z with the matrices over the dual numbers

$$x \mapsto \epsilon \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \qquad y \mapsto \epsilon \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}, \qquad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \epsilon \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$$

the equation $z^2 = 1$ becomes

$$(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \epsilon \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix})^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \epsilon \begin{bmatrix} 2z_1 & 0 \\ 0 & -2z_4 \end{bmatrix}$$

whence $z_1 = z_4 = 0$. The equation $xz + zx = 0$ becomes

$$\epsilon \begin{bmatrix} 2x_1 & 0 \\ 0 & -2x_4 \end{bmatrix}$$

whence $x_1 = x_4 = 0$ and similarly also $y_1 = y_4 = 0$ from the equation $yz + zy = 0$. All other relations are automatically satisfied. Therefore,

$$T_m(\mathtt{rep}_{(1,1)}\ A_c) = \mathbb{C}x_2 + \mathbb{C}x_3 + \mathbb{C}y_2 + \mathbb{C}y_3 + \mathbb{C}z_2 + \mathbb{C}z_3$$

has dimension 6 which coincides with the dimension of $\mathtt{rep}_{(1,1)}\ A_c$ whence m is also a smooth point.

Because $\mathtt{rep}_{(1,1)}\ A_c$ is a smooth variety with quotient variety the conifold singularity we say that the conifold algebra is a *noncommutative desingularization* (or more precisely, an $(1,1)$-dimensional noncommutative desingularization) of the conifold singularity. Next time we will see how one can use noncommutative desingularizations to obtain *commutative* desingularizations.

3. The normal space

In this section we will give another method to compute the dimension of the tangent space $T_m(\mathtt{rep}_\alpha\, A)$ by describing the *normalspace* N_m to the orbit $\mathcal{O}(M)$

$$T_m(\mathtt{rep}_\alpha\, A) = N_m \oplus T_m(\mathcal{O}(M))$$

There are two advantages : the description of N_m only uses module-theoretic facts on M (self-extensions) and N_m is not just a vectorspace but a $Stab(M)$-representation. In the important case when M is a semi-simple module this will give us a *local quiver setting* describing the étale local GL_n-structure of $\mathtt{rep}_\alpha\, A$ in a neighborhood of $\mathcal{O}(M)$. More details can be found in ng@n §3.3 and §4.2.

Let M and N be two A-representations of dimension m resp. n. An *extension* of N by M is an $n+m$-dimensional representation P such that M is a submodule of P with quotient N. Two extensions are *equivalent* if there is an A-module isomorphism $P \xrightarrow{\phi} P'$ such that the diagram describing the extensions

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\phi"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & P' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

is commutative. We can also express this is matrix-terms. Let $\rho_M : A \longrightarrow M_m(\mathbb{C})$ and $\sigma_N : A \longrightarrow M_n(\mathbb{C})$ be the representations associated to M and N, then any extension P defines a representation $\mu_P : A \longrightarrow M_{m+n}(\mathbb{C})$ such that

$$\mu_P(a) = \begin{bmatrix} \rho_M(a) & \lambda(a) \\ 0 & \sigma_N(a) \end{bmatrix} \qquad \text{for all } a \in A$$

where $\lambda : A \longrightarrow Hom_{\mathbb{C}}(N, M)$ is a linear map satisfying the *cycle condition*

$$\lambda(aa') = \rho_M(a)\lambda(a') + \lambda(a)\sigma_N(a')$$

The set of all such cycle linear maps form a vectorspace $Z(N, M)$ and two extensions are equivalent if and only if there difference is a *boundary*, that is there is a linear map $\beta \in Hom_{\mathbb{C}}(N, M)$ such that

$$\lambda(a) - \lambda(a') = \rho_M(a) \circ \beta - \beta \circ \sigma_N(a) \qquad \text{for all } a \in A$$

The set of all boundaries is a subspace $B(N, M)$ of $Z(N, M)$ and the quotient

$$\frac{Z(N, M)}{B(N, M)} = Ext^1_A(N, M)$$

is the space of all equivalence classes of extensions.

Example 4 Let us compute $Ext^1_{A_c}(M_\pm, M_\pm)$ for the one-dimensional simple A_c-modules

$$M_+ = \begin{cases} x \mapsto 0 \\ y \mapsto 0 \\ z \mapsto 1 \end{cases} \qquad M_- = \begin{cases} x \mapsto 0 \\ y \mapsto 0 \\ z \mapsto -1 \end{cases}$$

Any cycle $A \xrightarrow{\lambda} Hom_{\mathbb{C}}(M_\pm, M_\pm)$ is fully determined by the images of the generators and hence determines a triple $(a, b, c) \in \mathbb{C}^3$ such that the matrices

$$x \mapsto \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} \pm 1 & c \\ 0 & \pm 1 \end{bmatrix}$$

determine a two dimensional representation of A_c. The identity $z^2 = 1$ implies that $c = 0$ unless both terms are different. If they are both say M_+ then the identity $xz + zx = 0$ becomes

$$\begin{bmatrix} 0 & 2a \\ 0 & 0 \end{bmatrix} \qquad \text{whence } a = 0$$

and similarly $b = 0$. Therefore, $Z(M_+, M_+) = 0 = Z(M_-, M_-)$ and

$$Ext^1_{A_c}(M_+, M_+) = 0 \qquad \text{and} \qquad Ext^1_{A_c}(M_-, M_-) = 0$$

If the two terms are different, the equations $z^2 = 1$ and $xz + zx = 0$,$zy + yz = 0$ impose no conditions on (a, b, c) and the other defining equations of A_c are automatically satisfied. So,

$$Z(M_+, M_-) = \mathbb{C}^3 = Z(M_-, M_+)$$

The boundaries $B(M_+, M_-)$ are given by those $A_c \xrightarrow{\delta} Hom_{\mathbb{C}}(M_+, M_-) = \mathbb{C}$ such that for some $d \in Hom_{\mathbb{C}}(M_+, M_-)$ we have

$$\delta(x) = 0.d - d.0 = 0 \quad \delta(y) = 0.d - 0.d = 0 \quad \delta(z) = -1.d - d.1 = -2d$$

whence $Z(M_+, Z_-) = \{(0, 0, c)\}$ and therefore

$$Ext^1_{A_c}(M_+, M_-) = \mathbb{C}^2 = \{(a, b, 0)\} \quad \text{and similarly} \quad Ext^1_{A_c}(M_-, M_+) = \mathbb{C}^2$$

Fix $M \in \mathtt{rep}_\alpha\ A$, then the cycles $Z(M, M)$ can be identified to the tangent space $T_m(\mathtt{rep}_\alpha\ A)$. Let $\phi\ :\ A \longrightarrow M_n(\mathbb{C})$ be the algebra map determined by M and $\lambda\ :\ A \longrightarrow Hom_{\mathbb{C}}(M, M) = M_n(\mathbb{C})$ a cycle, that is, satisfying

$$\lambda(aa') = \phi(a)\lambda(a') + \lambda(a)\phi(a') \qquad \forall a, a' \in A$$

then we have an algebra map (and hence a tangent vector)

$$A \longrightarrow M_n(\mathbb{C}[\epsilon]) \qquad a \mapsto \phi(a) + \epsilon\lambda(a)$$

Indeed, for all $a, a' \in A$ we have

$$(\phi(a)+\epsilon\lambda(a))(\phi(a')+\epsilon\lambda(a')) = \phi(a)\phi(a')+\epsilon(\phi(a)\lambda(a')+\lambda(a)\phi(a')) = \phi(aa')+\epsilon\lambda(aa')$$

Conversely, every tangent vector $\chi : A \longrightarrow M_n(\mathbb{C}[\epsilon])$ determines a cycle λ by setting $\chi(a) = \phi(a) + \epsilon\lambda(a)$. On the other hand, the tangent space $T_{\mathbb{1}_n}(GL_n) \simeq M_n(\mathbb{C})$ and the *differential* of the *action map* of GL_n on $\mathtt{rep}_\alpha A$ sending g to $g.M$ is

$$M_n(C) \mapsto T_m(\mathtt{rep}_\alpha A) \qquad g \mapsto \lambda \text{ where } \lambda(a) = g\phi(a) - \phi(a)g$$

because $(\mathbb{1}_n + \epsilon g)\phi(1 - \epsilon g) = \phi + \epsilon(g.\phi - \phi.g)$. That is, the tangent space to the orbit $\mathcal{O}(M)$ in $\mathtt{rep}_\alpha A$ coincides with the boundaries $B(M, M)$.

Theorem 3 *The normal space N_m to the orbit $\mathcal{O}(M)$ in $\mathtt{rep}_\alpha A$ is isomorphic to the space of* self-extensions $Ext^1_A(M, M)$.

This gives an alternative method to compute the dimension of the tangent space as

$$\begin{aligned} dim_{\mathbb{C}}\, T_m(\mathtt{rep}_\alpha A) = & \quad dim_{\mathbb{C}}\, N_m + dim\, \mathcal{O}(M) \\ & = dim_{\mathbb{C}}\, Ext^1_A(M, M) + n^2 - dim\, Stab(M) \end{aligned}$$

and another test on smoothness of M.

4. The local quiver setting

The stabilizer subgroup $Stab(M)$ acts on the normal space $N_m = Ext^1_A(M, M)$ by conjugation. In the important case when M is a semi-simple n-dimensional representation, we will show that this action is isomorphic to the basechange action of a certain quiver setting : the *local quiver setting*. For more details we refer to ng@n §4.5.

Consider the decomposition of the semi-simple representation M in its simple components

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i are the distinct simple components, say of dimension d_i and occurring in M with *multiplicity* e_i. To begin, $n = d_1 e_1 + \ldots + d_k e_k$ and as for $i \neq j$ S_i and S_j are non-isomorphic we have that the stabilizer subgroup of M, that is, the invertible A-endomorphisms of M is (use $Hom_A(S_i, S_j) = 0$ if $i \neq j$ and $Hom_A(S_i, S_i) = \mathbb{C}$ by Schur's lemma)

$$Stab(M) \simeq GL_{e_1} \times \ldots \times GL_{e_k}$$

and if we choose a basis of M (that is, go to another point in the orbit if necessary) adapted to this decomposition we see that $Stab(M)$ is embedded in GL_n via

$$\begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{d_1}) & & \\ & \ddots & \\ & & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{d_k}) \end{bmatrix}$$

The Ext-functor is additive in both factors, whence we have a description of $Ext^1_A(M,M)$ as

$$\begin{bmatrix} M_{e_1}(Ext^1_A(S_1,S_1)) & \dots & M_{e_1\times e_k}(Ext^1_A(S_1,S_k)) \\ \vdots & & \vdots \\ M_{e_k\times e_1}(Ext^1_A(S_k,S_1)) & \dots & M_{e_k}(Ext^1_A(S_k,S_k)) \end{bmatrix}$$

which can be identified to the representation space $\mathtt{rep}_\beta\ Q$ of the *local quiver setting* (Q,β) where the dimension vector $\beta = (e_1,\dots,e_k)$ is given by the multiplicities of the simple components in M and where Q is a quiver on k vertices $\{v_1,\dots,v_k\}$ (corresponding to the distinct simple components of M) such that there are

$$a_{ij} = dim_{\mathbb{C}}\ Ext^1_A(S_i,S_j)$$

directed arrows from v_i to v_j. Observe that $Stab(M) \simeq GL(\beta)$ the base change group of $\mathtt{rep}_\beta\ Q$.

Theorem 4 *If M is a semi-simple representation with isotypical decomposition*

$$M = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{e_k}$$

then as a $Stab(M) = GL_{e_1} \times \dots \times GL_{e_k}$-representation the normal space N_m *to the orbit $\mathcal{O}(M)$ is isomorphic to the representation space*

$$\mathtt{rep}_\beta\ Q$$

where $\beta = (e_1,\dots,e_k)$ and Q has $dim_{\mathbb{C}}\ Ext^1_A(S_i,S_j)$ directed arrows from vertex v_i to vertex v_j. We call (Q,β) the local quiver setting of the semi-simple representation M.

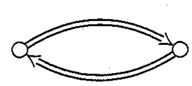
Example 5 Consider the semi-simple two-dimensional representation of A_c

$$M = M_+ \oplus M_-$$

The stabilizer subgroup is

$$Stab(M) \simeq \mathbb{C}^* \times \mathbb{C}^* = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\} \hookrightarrow GL_2$$

and by our previous calculations, the local quiver setting (Q,β) has quiver



and dimension vector $(1,1)$. A representation in $\mathtt{rep}_{(1,1)}\ Q$ is determined by matrices

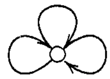
$$\begin{bmatrix} 0 & x_1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & x_2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ y_1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ y_2 & 0 \end{bmatrix}$$

on which $Stab(M)$ acts by simultaneous conjugation which is also the action by basechange of the group $GL(\beta)$.

If M is a two-dimensional simple representation in $\mathtt{rep}_{(1,1)}\ A_c$, then the stabilizer subgroup $Stab(M) \simeq \mathbb{C}^* = \mathbb{C}^*\mathbb{1}_2 \hookrightarrow GL_2$ by Schur's lemma and one verifies that

$$Ext^1_{A_c}(M, M) \simeq \mathbb{C}^3$$

That is, the local quiver setting (Q, β) for such a representation has quiver



and dimension vector $\beta = (1)$.

5. The Luna slice theorem

In this section we will apply the *Luna slice theorem* to get the GL_n-local structure of $\mathtt{rep}_\alpha\ A$ near the orbit $\mathcal{O}(M)$ of a semi-simple representation from the local quiver setting (Q, β). Proofs and more details can be found in ng@n §4.1, §4.3 and §4.5.

Recall that a morphism of commutative rings $R \xrightarrow{f} S$ is said to be *étale* if S is a finite R-module and for a presentation of S as

$$S = \frac{R[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)}$$

we have that the *Jacobian matrix*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

is an invertible $n \times n$ matrix over S. Many ringtheoretic properties (such as smoothness, complete intersection etc.) are preserved under étale morphisms. A morphism of affine varieties $X \longrightarrow Y$ is said to be *étale* if the induced morphism on the coordinate rings $\mathbb{C}[Y] \longrightarrow \mathbb{C}[X]$ is an étale morphism. One should think of étale maps as finite covers, they are locally isomorphic in the analytic topology (but *not* necessarily in the Zariski topology). For more details see ng@n Chp. 3.

A naive attempt to describe the GL_n-structure of $\mathtt{rep}_\alpha\ A$ near an orbit $\mathcal{O}(M)$ would be to say that it should look like the *product* $G/Stab(M) \times N_m$ of the orbit with the normal spaces, see figure 1. Surprisingly, this is 'almost' true provided the orbit $\mathcal{O}(M)$ is closed (that is, M is a semi-simple representation), m is a smooth point of $\mathtt{rep}_\alpha\ A$ and we work in the étale topology, see ag@n §4.3 for a proof :

Theorem 5 (Luna slice theorem) *Let M be an n-dimensional semi-simple representation of A with local quiver setting (Q, β) such that the corresponding point $m \in \mathtt{rep}_\alpha\ A$*

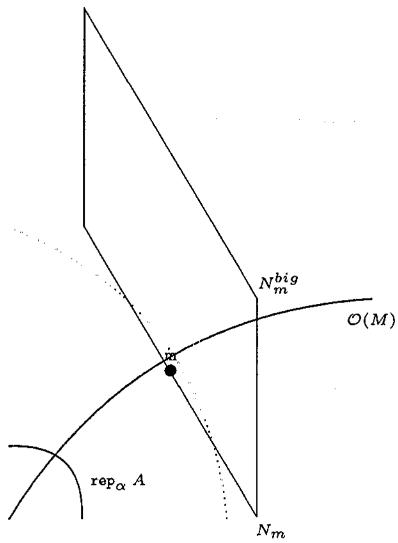


Figure 1: Normal spaces to the orbit.

is smooth. Then, there is a smooth $GL(\beta)$-stable subvariety $S_m \hookrightarrow \mathtt{rep}_\alpha\, A$ through m (the slice) and a diagram of varieties

$$\begin{tikzcd} GL_n \times^{GL(\beta)} \mathtt{rep}_\beta\, Q \arrow[d] & GL_n \times^{GL(\beta)} S_m \arrow[l, "GL_n \times^{GL(\beta)} \phi"'] \arrow[r, "\psi"] \arrow[d] & \mathtt{rep}_\alpha\, A \arrow[d] \\ \mathtt{iss}_\beta\, Q & S_m//GL(\beta) \arrow[l, "\overline{\phi}"'] \arrow[r, "\overline{\psi}"] & \mathtt{iss}_\alpha\, A \end{tikzcd}$$

such that the upper maps are GL_n-equivariant étale maps and the lower maps étale maps between the corresponding quotient varieties. The map ψ is the action-map and ϕ takes m to the zero-representation in $\mathtt{rep}_\beta\, Q$.

Example 6 All conditions are satisfied for the semi-simple module $M = M_+ \oplus M_-$ in $\mathtt{rep}_{(1,1)}\, A_c$. By the above result, the GL_2-structure of $\mathtt{rep}_{(1,1)}\, A_c$ near $\mathcal{O}(M)$ should be *étale isomorphic* to that of the associated fiber bundle

$$GL_2 \times^{\mathbb{C}^* \times C^*} \mathtt{rep}_{(1,1)}\, Q$$

(for the quiver Q described before) near the zero-representation and that there is an étale isomorphism between the quotient variety $\mathtt{iss}_{(1,1)}\, A_c$ near the image of m and the quiver-quotient variety $\mathtt{iss}_{(1,1)}\, Q$ near the image if the zero-representation.

In this case, these morphisms are actually isomorphisms in the Zariski topology. Indeed, do a basechange such that the action of z on a representation $N \in \mathtt{rep}_{(1,1)}\, A_c$ is given by the matrix

$$z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It follows from the equations $xz + zx = 0 = yz + zy$ that the action of x resp. y is given by

$$x \mapsto \begin{bmatrix} 0 & x_2 \\ x_3 & 0 \end{bmatrix} \qquad y \mapsto \begin{bmatrix} 0 & y_2 \\ y_3 & 0 \end{bmatrix}$$

and hence these matrices determine a representation in $\mathtt{rep}_{(1,1)}\ Q$. As we needed to perform a basechange, this gives a morphism

$$\mathtt{rep}_{(1,1)}\ A_c \longrightarrow GL_2 \times^{\mathbb{C}^* \times \mathbb{C}^*} \mathtt{rep}_{(1,1)}\ Q$$

Another application of the slice result is the étale local structure of $\mathtt{rep}_{(1,1)}\ A_c$ near a *simple* representation M. In this case we have that the local quiver setting is $(Q, (1))$ where Q is the three loop quiver. Therefore, a GL_2-neighborhood of the orbit is étale isomorphic to

$$GL_2 \times^{\mathbb{C}^*} \mathbb{C}[x, y, z] = PGL_2 \times \mathbb{C}[x, y, z]$$

as the action of $\mathbb{C}^*$ on $\mathtt{rep}_{(1)}\ Q$ is trivial. That is, over the Azumaya locus we have that the quotient map

$$\mathtt{rep}_{(1,1)}\ A_c \overset{\pi}{\twoheadrightarrow} \mathtt{iss}_{(1,1)}\ A_c$$

is a principal PGL_2-fibration in the étale topology.

When M is semi-simple but m is *not* smooth, one can still describe the GL_n-étale local structure of $\mathtt{rep}_\alpha\ A$ in terms of representations of quivers *with relations*. The idea is to look for an epimorphism $A' \twoheadrightarrow A$ such that the image of m is smooth in $\mathtt{rep}_\alpha\ A'$ (a drastic choice of A' would be the free algebra $\mathbb{C}\langle x_1, \ldots, x_m\rangle$). Use the previous result using the local quiver setting (Q', β') for m with respect to A' (this gives the 'big' normal space of figure 1), then one can define a *slice* for m in $\mathtt{rep}_\alpha\ A$ by taking the inverse image of the inclusion $\mathtt{rep}_\alpha\ A \hookrightarrow \mathtt{rep}_\alpha\ A'$ under the action map. Finally, take the image of this closed subset in $GL_n \times^{GL(\beta')} \mathtt{rep}_{\beta'}\ Q'$ which gives us the relations which must hold for the quiver-representations.

the essential git&pi 4

Lieven Le Bruyn

Departement Wiskunde en Informatica, Universiteit Antwerpen (UIA)
B-2610 Antwerp (Belgium)
E-mail : lebruyn@uia.ua.ac.be

ABSTRACT: In this part we show how to work with quotient varieties of quiver representations, introduce stability structures and define the corresponding moduli spaces. We apply noncommutative geometry to desingularize singular varieties and present a second research problem.

Contents

Last time we have seen that the local study of representation varieties reduces to that of representations of quivers (possibly with relations). This week we will see what is known in this case and introduce the *moduli spaces* of quiver-representations.

1. Quiver quotient varieties

Let Q be a quiver with k vertices $\{v_1, \ldots, v_k\}$ and such that there are a_{ij} arrows with starting vertex v_i and terminating vertex v_j. The *Euler form* of Q is given by the matrix

$$\chi_Q = (\delta_{ij} - a_{ij})_{i,j} \in M_k(\mathbb{Z})$$

which determines a bilinear form on $\mathbb{Z}^k$. The vertex-idempotents e_i form a complete set of orthogonal idempotents in the path algebra $\mathbb{C}Q$ whence $\mathbb{C}Q$ has the semi-simple algebra $\mathbb{C} \times \ldots \times \mathbb{C}$ (k factors) as a subalgebra. Therefore,

$$\mathtt{rep}_n\, \mathbb{C}Q = \bigsqcup_{\substack{\alpha=(a_1,\ldots,a_k)\\ \sum_i a_i = n}} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q$$

where $\mathtt{rep}_\alpha\, Q$ is the affine space of all α-dimensional representations of $Q : V = (V_a)_a$ where for each arrow a in Q with $s(a) = v_i$ and $t(a) = v_j$ we have that $V_a \in M_{a_j \times a_i}(\mathbb{C})$. On this space the *basechange group* $GL(\alpha) = \times_i GL_{a_i}$ acts via

$$(g_1, \ldots, g_k).V_a = g_j V_a g_i^{-1}$$

The to $V = (V_a)_a \in \mathtt{rep}_\alpha\, Q$ corresponding n-dimensional representations of $\mathbb{C}Q$ is given by sending the vertex idempotent e_i to the diagonal matrix

$$\phi_V(e_i) = \begin{bmatrix} \ddots & & 0 \\ & \mathbb{1}_{a_i} & \\ 0 & & \ddots \end{bmatrix}$$

with 1's at places $\sum_{j=1}^{i-1} a_j + 1 \leq l \leq \sum_{j=1}^{i} a_j$. The image of the arrow-generator a is an $a_j \times a_i$ block-matrix

$$\phi_V(a) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V_a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

at spot (i,j). As there is a natural one-to-one correspondence between GL_n-orbits in the associated fiber bundle $GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q$ and $GL(\alpha)$-orbits in $\mathtt{rep}_\alpha\, Q$ we have that

$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q // GL_n \simeq \mathtt{rep}_\alpha\, Q // GL(\alpha) = \mathtt{iss}_\alpha\, Q$$

the variety parametrizing isomorphism classes of *semi-simple* α-dimensional representations of Q. Its coordinate ring

$$\mathbb{C}[\mathtt{iss}_\alpha\, Q] = \mathbb{C}[\mathtt{rep}_\alpha\, Q]^{GL(\alpha)} = \mathbb{C}[GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q]^{GL_n}$$

can be computed via the Procesi-Razmyslov theorem from the traces of monomial of the images of the generic matrices under the relations of $\mathbb{C}Q$. This means that the generic matrix corresponding to the vertex idempotent e_i is send to the above diagonal matrix and that corresponding to a to a block-matrix as above with all its entries variables. Calculating multiplication of such block-matrices we see that a monomial in these matrices has non-zero trace if and only if the monomial corresponds to an oriented circuit in the quiver Q. This proves (for more details see ng@n §3.3)

Theorem 1 *The ring of polynomial quiver-invariants*

$$\mathbb{C}[\mathtt{rep}_\alpha\, Q]^{GL(\alpha)}$$

is generated by taking traces of oriented cycles of total length $\leq n^2 + 1$ in the quiver Q.

Example 1 For the conifold algebra we have seen that

$$\mathtt{rep}_{(1,1)}\, A_c \simeq GL_2 \times^{\mathbb{C}^* \times \mathbb{C}^*} \mathtt{rep}_{(1,1)}\, Q$$

for the quiver setting

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left=40, "x_2"] \arrow[r, bend left=10, "x_1"'] & \textcircled{1} \arrow[l, bend left=10, "y_1"'] \arrow[l, bend left=40, "y_2"] \end{tikzcd}$$

then, the images of the generic matrices are respectively

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & x_1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & x_2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ y_1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ y_2 & 0 \end{bmatrix}$$

and the traces in monomials of these matrices are generated by the polynomials

$$\{x = x_1y_1, y = x_2y_2, u = x_1y_2, v = x_2y_1\}$$

but they satisfy the relation $xy = uv$, whence $\mathbb{C}[\mathtt{iss}_{(1,1)}\ Q]$ is the conifold singularity

$$\frac{\mathbb{C}[x,y,u,v]}{(xy-uv)}$$

2. Quiver local quivers

We are interested in those dimension vectors α such that

$$\mathtt{rep}_\alpha\ \mathbb{C}Q = GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ Q$$

contains a Zariski open subset consisting of *simple* representations. We have a full answer to this question in terms of the Euler form of the quiver Q

Theorem 2 *There is a Zariski open subset of simple representations in* $\mathtt{rep}_\alpha\ \mathbb{C}Q$ *if and only if*

$$\chi_Q(\alpha, \delta_i) \leq 0 \quad \textit{and} \quad \chi_Q(\delta_i, \alpha) \leq 0$$

for all vertex-simples $\delta_i = (\delta_{ij})_j$. + supp($\alpha$) strongly connected

One direction is easy : if say $\chi_Q(\alpha, \delta_i) > 0$ for some vertex v_i, this means that the total number of *incoming* dimensions (multiplied with the number of arrows) is strictly smaller that the vertex-dimension at v_i. But then we can produce a proper subrepresentation by taking in vertex v_i the subspace spanned by all incoming matrices. The harder part is to prove the converse implication, see ng@n §3.4 for more details.

If α is such a *simple root*, then we also know the dimension of the quotient variety $\mathtt{iss}_\alpha\ Q = \mathtt{rep}_\alpha\ Q//GL(\alpha)$ as we know the total dimension of $\mathtt{rep}_\alpha\ Q$ and because there is a Zariski open subset where the quotient map is a principal $PGL(\alpha) = GL(\alpha)/C^*$-fibration. Therefore

$$dim\ \mathtt{iss}_\alpha\ \mathbb{C}Q = dim_{\mathbb{C}}\ \mathtt{iss}_\alpha\ Q = 1 - \chi_Q(\alpha,\alpha)$$

Thus, for a given dimension vector α we can determine all possible *representation types*

$$\tau = (\beta_1, e_1; \ldots; \beta_z, e_z) \qquad \sum_{i=1}^{z} e_i\beta_i = \alpha$$

where the β_i are dimension vectors of simple representations of Q and where the e_i are the multiplicities. With $\mathtt{iss}_\alpha\, Q(\tau)$ we denote the subset of all α-dimensional semi-simple representations of type τ. It follows that this locally closed subset is of dimension

$$dim\ \mathtt{iss}_\alpha\, Q(\tau) = \sum_{i=1}^{z} (1 - \chi_Q(\beta_i, \beta_i))$$

Next, let $V \in \mathtt{rep}_\alpha\, Q$ be a semi-simple representation of type τ, then we would like to determine the *local quiver setting* (Q_τ, α_τ) of V entirely in terms of the type τ. We have seen that to do so it is important to be able to compute the dimension of Ext^1-spaces. For representations of quivers we have the following result

Theorem 3 *Let V resp. W be representations of a quiver Q of dimension vector α resp. β, then*

$$dim\ Hom_{\mathbb{C}Q}(V, W) - dim\ Ext^1_{\mathbb{C}Q}(V, W) = \chi_Q(\alpha, \beta)$$

For a proof we refer to ng@n §3.3. Specializing to the case when V and W are *simple* representations we know by Schur's lemma that $dim\ Hom_{\mathbb{C}Q}(V, W)$ is one or zero depending on whether (or not) $V \simeq W$. Therefore

Theorem 4 *Let V be an α-dimensional semi-simple representation of Q of type $\tau = (\beta_1, e_1; \ldots; \beta_z, e_z)$, that is,*

$$V = W_1^{\oplus e_1} \oplus \ldots \oplus W_z^{\oplus e_z}$$

where $dim\ W_i = \beta_i$ and W_i is a simple representation. The local quiver setting (Q_τ, α_τ) only depends on τ. The quiver Q_τ has z vertices $\{w_1, \ldots, w_z\}$ such that the number of arrows from w_i to w_j is

$$\delta_{ij} - \chi_Q(\beta_i, \beta_j)$$

and the dimension vector $\alpha_\tau = (e_1, \ldots, e_z)$.

Exercise 1 Consider the quiver setting (Q, α) coming from linear control systems

$$\begin{tikzcd} \textcircled{1} \arrow[r, Rightarrow, "m"] & \textcircled{n} \arrow[loop right] \end{tikzcd}$$

Determine $\mathbb{C}[\mathtt{iss}_\alpha\, Q]$ and deduce that $\mathtt{iss}_\alpha\, Q \simeq \mathbb{C}^n$. Determine all dimension vectors of simple representations of Q and deduce all possible representation types occurring in $\mathtt{rep}_\alpha\, Q$. For each type τ determine the local quiver setting (Q_τ, α_τ). Determine for which types τ we have that

$$\mathtt{iss}_\alpha\, Q(\tau') \subset \overline{\mathtt{iss}_\alpha\, Q(\tau)}$$

How would you extend this to arbitrary quiver settings ?

$0 \to Hom_Q(V, W) \to \bigoplus Hom(V_{s_i}, W_{s_i}) \to \bigoplus Hom_{\mathbb{C}}(V_a, W_a) \to Ext^1_Q(V, W) \to 0$

$(f_1, \ldots, f_k) \mapsto (f_j \circ V_a - W_a \circ f_i)$

3. Stability structures

If the quiver Q has no oriented cycles, there are no non-trivial polynomial invariants whence the quotient variety $\mathtt{iss}_\alpha\ Q$ is reduced to one point. Still, it may be that there are perfectly good orbit-spaces for a Zariski open subset of $\mathtt{rep}_\alpha\ Q$.

Example 2 Consider the quiver setting

$$① \overset{n+1}{\Longrightarrow} ①$$

with $n+1$ arrows $\{x_0, x_1, \ldots, x_n\}$ from v_1 to v_2. The basechange group acts via

$$(\lambda, mu).(x_0, x_1, \ldots, x_n) = \mu\lambda^{-1}(x_0, x_1, \ldots, x_n)$$

If we consider the Zariski open subset $\mathtt{rep}_{(1,1)}\ Q - \{(0, \ldots, 0)\}$ then the $GL(\alpha)$-orbits in this open set are all closed (they have the same dimension) and they are classified by the points in $\mathbb{P}^n$, the projective n-space.

Let Q be a quiver on k vertices and let $\theta = (p_1, \ldots, p_k) \in \mathbb{Z}^k$ such that $\theta.\alpha = \sum_{i=1}^k p_i a_i = 0$. A representation $V \in \mathtt{rep}_\alpha\ Q$ is said to be θ-*semistable* if and only if for every proper subrepresentation $0 \neq W \hookrightarrow V$ we have that

$$\theta.dim\ W \geq 0$$

If this is a strict inequality for all proper subrepresentations, we call V θ-stable. One can show (see $\mathtt{ng@n}$ §7.3 for details) that the θ-semistable representations in $\mathtt{rep}_\alpha\ Q$ are actually the α-dimensional representations of a certain *universal localization* of $\mathbb{C}Q$ and that under this identification, θ-stables correspond to simple representations. Therefore, there is a version of the Jordan-Hölder theorem for θ-semistable representations, that is, if V is θ-semistable there is a finite filtration of V with all successive quotients θ-stable.

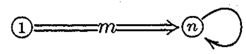
We will see in a moment that the set of all θ-semistable representations in $\mathtt{rep}_\alpha\ Q$ forms a Zariski open subset. Mimicking the characterization of closed orbits in $\mathtt{rep}_n\ A$ we can prove that closed orbits in the open subset of θ-semistables correspond to *direct sums of θ-stables* (the θ-version of semi-simple representations), so if there is a quotient variety its points will classify direct sums of θ-stable representations of total dimension α.

For example, the nonzero representations of

$$① \overset{n+1}{\Longrightarrow} ①$$

are the θ-semistable (actually θ-stable) representations where $\theta = (-1, 1)$ as the only proper subrepresentation of such a representations has dimension vector $(0, 1)$. Several examples we encountered before can also be expressed in this setting.

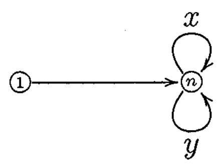
Example 3 (linear control systems) A completely controllable system $(A, B) \in M_{n \times m}(\mathbb{C}) \times M_n(\mathbb{C})$ is a θ-stable representation of the quiver setting



for $\theta = (-n, 1)$ (verify!).

The notions extends to representations of quivers with relations :

Example 4 (Hilbert scheme) Consider the quiver-setting with relations



satisfying the relation $xy - yx = 0$. Then, the orbit space of the θ-(semi)stable representations is the Hilbert scheme $\mathtt{Hilb}_n\ \mathbb{C}^2$ where $\theta = (-n, 1)$.

4. The moduli space

In this section we will construct a quotient variety for the θ-semistable representations of a quiver setting (Q, α). Recall that semi-simple representations were separated by polynomial invariants on $\mathtt{rep}_\alpha\ Q$, direct sums of θ-stable representations are separated by *semi-invariant polynomials*. The stability structure θ determines a *character*

$$GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k} \xrightarrow{\chi_\theta} \mathbb{C}^* \qquad (g_1, \ldots, g_k) \mapsto det(g_1)^{p_1} \ldots det(g_k)^{p_k}$$

and a polynomial $f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q]$ is said to be a χ_θ-semi-invariant iff

$$g.f = \chi_\theta(g)^w f \qquad \forall g \in GL(\alpha)$$

for a fixed $w \in \mathbb{N}$, called the *weight* of the semi-invariant. Remark that polynomial invariants are also semi-invariants (of weight 0) and that the weight-function defines a *gradation* on the subring of all polynomial semi-invariants

$$\mathbb{C}[\mathtt{rep}_\alpha]^{GL(\alpha),\chi_\theta} = \bigoplus_{w \in \mathbb{N}} \{f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q \mid g.f = \chi_\theta(g)^w\ \forall g \in GL(\alpha)\}$$

with part of degree zero the ring of invariants $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)} = \mathbb{C}[\mathtt{iss}_\alpha\ Q]$.

Any positively graded commutative $\mathbb{C}$-algebra $R = \oplus_{i \in \mathbb{N}} R_i$ defines its *projective scheme* $\mathtt{proj}\ R$ on the set of all its graded prime ideals (apart from the positive cone $R_+ = \oplus_{i \geq 1} R_i$). The scheme structure is given by defining standard affine open pieces (for f a homogeneous element)

$$\mathbb{X}(f) = \{P \triangleleft_{gr} R \mid f \notin P\}$$

whose ring of sections is the degree zero part of the graded ring of fractions R_f. For example, consider the $\theta = (-1,1)$-(semi)stable representations of

$$\textcircled{1} \xRightarrow{\;n+1\;} \textcircled{1}$$

and let x_i be the i-th arrow, then x_i is a χ_θ-semi-invariant of weight one and the ring of all χ_θ-semi-invariants is $\mathbb{C}[x_0,\ldots,x_n]$ graded as usual. The corresponding projective scheme clearly is

$$\mathtt{proj}\ \mathbb{C}[x_0,x_1,\ldots,x_n] \simeq \mathbb{P}^n$$

the projective n-space which is covered by the affine open pieces $\mathbb{X}(x_i)$ with ring of sections

$$\mathbb{C}[\frac{x_0}{x_i},\ldots,\frac{x_n}{x_i}] = (\mathbb{C}[x_0,\ldots,x_n]_{x_i})_0$$

This case is the archetypical example of

Theorem 5 *The quotient variety of the Zariski open subset of all θ-semistable representations in $\mathtt{rep}_\alpha\ Q$ is the projective scheme*

$$\mathtt{moduli}_\alpha(Q,\theta) = \mathtt{proj}\ \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha),\chi_\theta}$$

the points of which classify the direct sums of θ-stable representations of total dimension α.

We call $\mathtt{moduli}_\alpha(Q,\theta)$ the *moduli space* of θ-semistable α-dimensional representations of Q, we refer to ng@n §6.3 for the proof of this result. For this reason it is important to know a generating set of the algebra of all semi-invariants. Again this problem essentially reduces to classical invariant theory, see ng@n §7.2.

Let l and r be natural numbers and consider maps

$$\{1,\ldots,l\} \xrightarrow{L} \{1,\ldots,k\} \quad \text{and} \quad \{1,\ldots,r\} \xrightarrow{R} \{1,\ldots,k\}$$

Further, consider natural numbers $x_1,\ldots,x_l$ and $y_1,\ldots,y_r$ such that

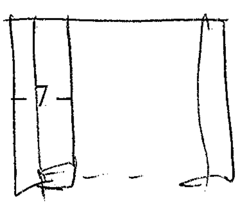
$$\sum_{i=1}^{l} x_i a_{L(i)} = \sum_{j=1}^{r} y_j a_{R(j)}$$

$r \times l$

Now consider an $l \times r$ block matrix where at place (j,i) we place a $x_i \times y_j$ matrix with all its entries linear combinations of paths from $v_{L(i)}$ to $v_{R(j)}$ in the quiver Q. Evaluating this matrix on $\mathtt{rep}_\alpha\ Q$ we obtain a square matrix whose determinant is a semi-invariant with corresponding character

$$\phi = (f_1,\ldots,f_k) \quad \text{such that} \quad f_l = \sum_{j \in R^{-1}(l)} y_j - \sum_{i \in L^{-1}(l)} x_i$$

We call such a semi-invariant a *determinantal* semi-invariant of character ϕ.



Theorem 6 *The ring of semi-invariants*

$$\mathbb{C}[\mathtt{rep}_\alpha\, Q]^{GL(\alpha),\chi}$$

is generated by traces of oriented cycles in the quiver Q and by determinantal semi-invariants of character $n\chi$ for $n \in \mathbb{N}$.

Example 5 For the quiver setting of linear control systems, take $l = n$ and $r = 1$ and $x_1 = \ldots = x_n = 1$ and $y_1 = 1$ and consider the $1 \times n$ matrix

$$\begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix}$$

where p_i is a path from the first vertex to the second, so of the form $B^c A_d$ for some c, d, so we see that the corresponding determinant is a semi-invariant of character $\theta = (-n, 1)$.

The inclusion of the degree zero part $\mathbb{C}[\mathtt{rep}_\alpha\, Q]^{GL(\alpha)}$ into the full ring of semi-invariants defines a *projective space bundle*

$$\mathtt{moduli}_\alpha(Q,\theta) \twoheadrightarrow \mathtt{iss}_\alpha\, Q$$

that is, all fibers are projective varieties. Similar results hold for quivers with relations. If V and W are θ-stable representations and $f : V \longrightarrow W$ a morphism of representations, consider the exact sequence

$$0 \longrightarrow Ker\ f \longrightarrow V \xrightarrow{f} W \longrightarrow Coker\ f \longrightarrow 0$$

It follows that either f is an isomorphism or f is the zero map. Indeed, from the short exact sequences

$$0 \longrightarrow Ker\ f \longrightarrow V \longrightarrow Im\ f \longrightarrow 0 \quad 0 \longrightarrow Im\ f \longrightarrow W \longrightarrow Coker\ f \longrightarrow 0$$

it follows that $\theta.dim\ Ker\ f = \theta.dim\ Im\ f = \theta.dim\ Coker\ f = 0$ and θ-stability of V and W does the rest. In particular, we have a Schur lemma for θ-stable representations

$$Hom_{\mathbb{C}Q}(V,W) = \begin{cases} \mathbb{C} \text{if } V \simeq W \\ 0 \text{otherwise} \end{cases}$$

Exercise 2 Let (Q,α) be a quiver setting such that $gcd(a_1,\ldots,a_k) = 1$. Prove that there is a stability structure θ such that

$$\mathtt{moduli}_\alpha(Q,\theta)$$

is a smooth variety. (Hint : find θ such that all θ-semistables are θ-stable).

relaties!

Exercise 3 Consider the quiver setting of the Hilbert schemes and prove that we obtain a projective space bundle

$$\begin{array}{ccc}
\mathtt{moduli}_\alpha(Q,\theta) & \twoheadrightarrow & \mathtt{iss}_\alpha\, Q \\
\| & & \| \\
\mathtt{Hilb}_n\, \mathbb{C}^2 & \xrightarrow{\pi} & S^n\, \mathbb{C}^2
\end{array}$$

where $S^n\, \mathbb{C}^2$ is the n-th symmetric power of $\mathbb{C}^2$ (that is, n-unordered points in $\mathbb{C}^2$. Prove moreover that this is a *resolution of singularities*, the so called Hilbert-Chow map.

5. Noncommutative desingularization

We now come to a major application of noncommutative geometry to stringtheory : the construction of (partial) commutative desingularizations of quotient singularities. Normally one constructs desingularizations by blowing-up and blowing-down, that is, extending the singular variety. The noncommutative approach is drastically different : if there is a noncommutative algebra A with a smooth component $\mathtt{rep}_\alpha\, A$ such that $\mathtt{iss}_\alpha\, A$ is the singular variety, then one can *restrict* to a Zariski open subset of semistable representations such that the corresponding moduli space is a desingularization.

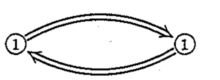
Example 6 (the conifold singularity) We have seen before that for the conifold algebra A_c and $\alpha = (1,1)$ we have that $\mathtt{rep}_\alpha\, A_c$ is a smooth variety and $\mathtt{iss}_\alpha\, A_c$ is the conifold singularity (the three dimensional $\mathbb{Z}/2\mathbb{Z}$-quotient singularity)

$$\frac{\mathbb{C}[x,y,u,v]}{(xy-uv)}$$

Moreover, we have an isomorphism

$$\mathtt{rep}_\alpha\, A_c \simeq GL_2 \times^{GL(\alpha)} \mathtt{rep}_\alpha\, Q$$

for the quiver setting



We have essentially two different stability structures on $\mathtt{rep}_\alpha\, Q$, $\theta = (-1,1)$ and $\theta' = (1,-1)$. All θ-semistable representations are θ-stable and they form the Zariski open subset of $\mathtt{rep}_\alpha\, Q$ for which at least one of the two top arrows is non-zero. The algebra of semi-invariants is generated by the traces (the conifold singularity) together with these two arrows x_1, x_2, that is,

$$\mathbb{C}[\mathtt{rep}_\alpha\, Q]^{GL(\alpha),\chi_\theta} = \frac{\mathbb{C}[x,y,u,v]}{(xy-uv)}[x_1,x_2]$$

graded by $deg(x_1) = deg(x_2) = 1$. Because $\mathtt{rep}_\alpha\ Q$ is smooth, so is the subset of θ-semistable representations and as they are all θ-stable the corresponding quotient map

$$\mathtt{rep}_\alpha(Q,\theta) \twoheadrightarrow \mathtt{moduli}_\alpha(Q,\theta)$$

is a principal $PGL(\alpha)$-fibration whence the moduli space $\mathtt{moduli}_\alpha(Q,\theta)$ is a smooth variety. The inclusion of the degree zero part gives a morphism

$$\mathtt{moduli}_\alpha(Q,\alpha) \xrightarrow{\pi} \mathtt{iss}_\alpha\ A_c$$

which is a desingularization of the conifold singularity. The *exceptional fiber*

$$\pi^{-1}(0,0,0,0) \simeq \mathbb{P}^1$$

as it corresponds to $\mathtt{proj}\mathbb{C}[x_1,x_2]$. Similarly, if y_1,y_2 are the two lower maps, the θ'-semistables are θ'-stable and form the Zariski open subset of representations such that at least one of these two maps is nonzero. Mimicking the above argument we obtain that also

$$\mathtt{moduli}_\alpha(Q,\theta') \xrightarrow{\pi'} \mathtt{iss}_\alpha\ A_c$$

is a desingularization with exceptional fiber $\mathbb{P}^1$. The *rational map*

$$\begin{tikzcd} \mathtt{moduli}_\alpha(Q,\theta) \arrow[rr, dotted] \arrow[dr, "\pi"'] & & \mathtt{moduli}_\alpha(Q,\theta') \arrow[dl, "\pi'"] \\ & \mathtt{iss}_\alpha\ A_c & \end{tikzcd}$$

is called a *flop* in physics literature (actually, the *Atiyah flop*).

This procedure is quite general. In the following (not entirely trivial) exercise we present the main steps to produce the desingularization of two-dimensional quotient singularities (due to Peter Kronheimer) by noncommutative geometry.

Exercise 4 The two-dimensional quotient singularities $\mathbb{C}^2/G$ where G is a finite subgroup of $SL_2(\mathbb{C})$ correspond to the *tame quiver settings* of figure 1. Replace each solid edge e in these diagrams by an arrow-pair (x_e, x_e^*) in both directions. Consider the *preprojective algebra*

$$\Pi_0(Q) = \frac{\mathbb{C}Q}{(\sum_e [x_e, x_e^*])}$$

It is well known that for a tame quiver setting (Q,α) the quotient variety

$$\mathtt{iss}_\alpha\ \Pi_0(Q)$$

is a two-dimensional quotient singularity and that there is a Zariski open subset of simple $\Pi_0(Q)$-representations in $\mathtt{rep}_\alpha\ Q$.

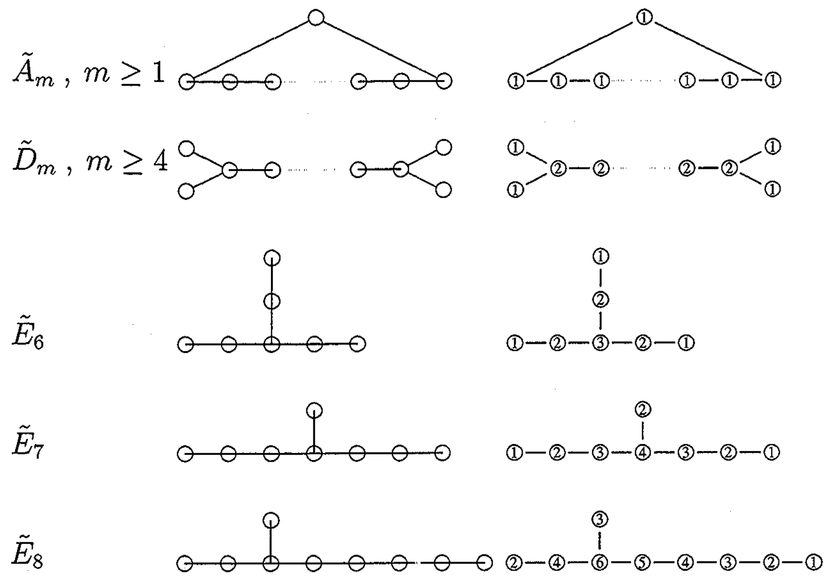


Figure 1: The tame quiver settings.

1. Use this to determine the dimension of $\mathtt{rep}_\alpha\ \Pi_0(Q)$.
2. Bill Crawley-Boevey proved that for V and W representations of $\Pi_0(Q)$ of dimension vectors α and β one has

$$dim_{\mathbb{C}}\ Ext^1_{\Pi_0}(V,W) = dim_{\mathbb{C}}\ Hom_{\Pi_0}(V,W) + dim_{\mathbb{C}}\ Hom_{\Pi_0}(W,V) - \chi(\alpha,\beta) - \chi(\beta,\alpha)$$

where χ is the Euler form of a quiver where we replace every solid edge by one directed arrow (in whatever direction). Use this to prove that for a tame quiver setting (Q,α) the representation variety

$$\mathtt{rep}_\alpha\ \Pi_0(Q)$$

is *singular* but one can choose a stability structure θ such that the Zariski open subset

$$\mathtt{rep}_\alpha(\Pi_0(Q),\theta)$$

is a smooth variety.

3. Conclude to get a desingularization

$$\mathtt{moduli}_\alpha(\Pi_0(Q),\theta) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}^2/G$$

Research problem 2 (open) : In which generality can one use noncommutative geometry to resolve commutative singularities. That is, given a singular variety X, can we find a noncommutative algebra A, a component $\mathtt{rep}_\alpha\ A$ and a stability structure θ such that $\mathtt{iss}_\alpha\ A = X$ and

$$\mathtt{moduli}_\alpha(A,\theta) \longrightarrow\!\!\!\!\rightarrow X$$

a (partial) resolution of singularities. Can one iterate this procedure to obtain a desingularization? Partial results are known for singularities of quiver quotients but the general case remains open. In particular, can one do this for three dimensional quotient singularities (important in string theory) ?

Xtra examples 1

Lieven Le Bruyn

Departement Wiskunde en Informatica, Universiteit Antwerpen (UIA)
B-2610 Antwerp (Belgium)
E-mail : lebruyn@uia.ua.ac.be

ABSTRACT: In this part we give some more examples which may help you to understand the power of the Luna slice result and why the theory of local quivers is useful.

Contents

1. Two local quivers

Let $F = \mathbb{C}\langle x, y\rangle$ the free algebra on two variables and consider the representation scheme of 2-dimensional representations of F

$$\mathtt{rep}_2\, F = M_2(\mathbb{C}) \times M_2(\mathbb{C}) = \{(A, B) \mid A, B \in M_2(\mathbb{C})\}$$

where the correspondence is given by the algebra maps

$$\mathbb{C}\langle x, y\rangle \longrightarrow M_2(\mathbb{C}) \qquad x \mapsto A \quad y \mapsto B$$

The action by GL_2 on $\mathtt{rep}_2\, F$ is given by *simultaneous conjugation*

$$GL_2 \times \mathtt{rep}_2\, F \longrightarrow \mathtt{rep}_2\, F \qquad (g, (A, B)) \mapsto (gAg^{-1}, gBg^{-1})$$

We will consider two semi-simple dimensional representations M and S and investigate the local structure near their orbits. S is the *simple* two-dimensional representation

$$S : \qquad x \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(verify that this is indeed a simple representation, that is, the image is the whole of $M_2(\mathbb{C})$). the stabilizer subgroup of S consists only of the scalar matrices

$$Stab(S) = \{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \mid \lambda \in \mathbb{C}^*\} \simeq \mathbb{C}^*$$

M will be a semi-simple representation having two distinct one-dimensional components

$$M = M_+ \oplus M_i : \qquad x \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It stabilizer subgroup (that is the matrices $g \in GL_2$ commuting with the images of x and y) is the subgroup of diagonal matrices

$$Stab(M) = \{\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix} \mid \lambda, \mu \in \mathbb{C}^*\} \simeq \mathbb{C}^* \times \mathbb{C}^*$$

(verify!). As $\mathtt{rep}_2\ F$ is an affine space, the tangent space in each point is the 8-dimensional affine space, that is

$$T_S(\mathtt{rep}_2\ F) = \{(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} + \epsilon \begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \epsilon \begin{bmatrix}y_1 & y_2\\ y_3 & y_4\end{bmatrix})\}$$

and the stabilizer subgroup $Stab(S) \simeq \mathbb{C}^*$ acts trivially on the tangents pace, that is, as a $\mathbb{C}^*$-representation we have

$$T_S(\mathtt{rep}_2\ F) \simeq M_0^{\oplus 8}$$

with M_0 the one-dimensional trivial $\mathbb{C}^*$-representation, $\lambda.m = m, \forall \lambda \in \mathbb{C}^*, m \in M_0$. As for the semi-simple M we have

$$T_M(\mathtt{rep}_2\ F) = \{(\epsilon \begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \epsilon \begin{bmatrix}y_1 & y_2\\ y_3 & y_4\end{bmatrix})\}$$

This time the stabilizer subgroup acts non-trivially on certain entries, for example

$$\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}\begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix}, \begin{bmatrix}\lambda^{-1} & 0\\ 0 & \mu^{-1}\end{bmatrix} = \begin{bmatrix}x_1 & \lambda\mu^{-1}x_2\\ \lambda^{-1}\mu x_3 & x_4\end{bmatrix}$$

whence as a $Stab(M) \simeq \mathbb{C}^* \times \mathbb{C}^*$-representation we have a decomposition

$$\begin{aligned} T_M(\mathtt{rep}_2\ F) \simeq & \quad M_{(0,0)}^{\oplus 4} \oplus M_{(1,-1)}^{\oplus 2} \oplus M_{(-1,1)}^{\oplus 2} \\ &= (\mathbb{C}x_1 + \mathbb{C}x_4 + \mathbb{C}y_1 + \mathbb{C}y_4) \oplus (\mathbb{C}x_2 + \mathbb{C}y_2) \oplus (\mathbb{C}x_3 + \mathbb{C}y_3) \end{aligned}$$

where $M_{(k,l)}$ is the one dimensional representation such that $(\lambda, \mu).m = \lambda^k \mu^l m$.

Next, we bring in the tangent space to the orbit. To do this we have to compute the image of the differential of the action map. For a general representation $(A, B) \in \mathtt{rep}_2\ F$ this amounts to computing

$$(\mathbb{1}_2 + \epsilon \begin{bmatrix}a & b\\ c & d\end{bmatrix})(A, B)(\mathbb{1}_2 - \epsilon \begin{bmatrix}a & b\\ c & d\end{bmatrix})$$

as $T_{\mathbb{1}_2}(GL_2) = M_2(\mathbb{C})$. Apply this to the simple representation S to get

$$(\mathbb{1}_2 + \epsilon \begin{bmatrix}a & b\\ c & d\end{bmatrix})(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix})(\mathbb{1}_2 - \epsilon \begin{bmatrix}a & b\\ c & d\end{bmatrix}) =$$

$$(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} + \epsilon \begin{bmatrix}-b-c & a-d\\ -d+a & c+b\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \epsilon \begin{bmatrix}0 & -2b\\ 2c & 0\end{bmatrix})$$

which is a three-dimensional subspace of $T_S(\mathtt{rep}_2\ F)$. Therefore,as a $Stab(S) = \mathbb{C}^*\mathbb{1}_2$-representation, the normal space to the orbit in S can be represented as

$$N_S \simeq M_0^{\oplus 5} = \{(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} x_1 & x_2 \\ 0 & x_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \epsilon \begin{bmatrix} y_1 & 0 \\ 0 & y_4 \end{bmatrix})\}$$

Applied to the semi-simple representation M we obtain

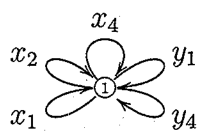
$$(\mathbb{1}_2 + \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix})(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})(\mathbb{1}_2 - \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix}) =$$

$$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & -2b \\ 2c & 0 \end{bmatrix})$$

which is a two-dimensional subspace of $T_M(\mathtt{rep}_2\ F)$ which as a $Stab(M) = \mathbb{C}^* \times \mathbb{C}^*$-representation is isomorphic to $M_{(1,-1)} \oplus M_{(-1,1)}$. Therefore, the normal space to the orbit can be represented as a $Stab(M)$-representation by

$$N_M = M_{(0,0)}^{\oplus 4} \oplus M_{(1,-1)} \oplus M_{(-1,1)} = \{\ (\epsilon \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \epsilon \begin{bmatrix} y_1 & 0 \\ 0 & y_4 \end{bmatrix})\ \}$$

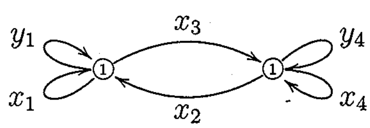
These calculations are compatible with the *local quiver settings*. For S the local quiver setting (Q, β) is



Clearly, $GL(\beta) = \mathbb{C}^*$ and it acts trivially on $\mathtt{rep}_\beta\ Q = \mathbb{C}^5$ whence as $Stab(S) = \mathbb{C}^*$-representation we have an isomorphism

$$N_S \simeq \mathtt{rep}_\beta\ Q$$

For M the local quiver setting (Q, β) is (verify!)



This time the basechange group $GL(\beta) = \mathbb{C}^* \times \mathbb{C}^* = \{(\lambda, \mu)\}$ and its action on the quiver-representation space $\mathtt{rep}_\beta\, Q$ is given by the rules

$$(\lambda,\mu).\begin{cases} x_1 \mapsto x_1 \\ y_1 \mapsto y_1 \\ x_4 \mapsto x_4 \\ y_4 \mapsto y_4 \\ x_2 \mapsto \lambda\mu^{-1}x_2 \\ x_3 \mapsto \lambda^{-1}\mu x_3 \end{cases}$$

and hence as a $Stab(M)$-representation we have an isomorphism

$$N_M \simeq \mathtt{rep}_\beta\, Q$$

2. Two étale maps

Let $N = (A, B)$ be a semi-simple representation in $\mathtt{rep}_2\, F$ with local quiver setting (Q, β). Then, the *slice* in N is given by the normal space to the orbit $N_N = \mathtt{rep}_\beta\, Q$ and the Luna slice theorem asserts that the horizontal maps of the commuting diagram

$$\begin{tikzcd} GL_2 \times^{GL(\beta)} \mathtt{rep}_\beta\, Q \arrow[r, "action"] \arrow[d, "\pi'"'] & \mathtt{rep}_2\, F \arrow[d, "\pi"] \\ \mathtt{iss}_\beta\, Q \arrow[r] & \mathtt{iss}_2\, F \end{tikzcd}$$

are étale maps in a Zariski neighborhood of $\mathcal{O}(N)$ resp. of $\pi(N)$. We know that $\mathbb{C}[\mathtt{iss}_2\, F]$ is generated by traces of monomials in generic 2×2 matrices X and Y, that is,

$$\mathbb{C}[\mathtt{iss}_2\, F] = \mathbb{C}[tr(X), tr(Y), tr(X^2), tr(Y^2), tr(XY)]$$

a polynomial ring (verify!) whence $\mathtt{iss}_2\, F \simeq \mathbb{C}^5$ and the map π is given by

$$(A, B) \mapsto (tr(A), tr(B), tr(A^2), tr(B^2), tr(AB))$$

On the other hand we will see next time that the ring of polynomial *quiver invariants*

$$\mathbb{C}[\mathtt{iss}_\beta\, Q] = \mathbb{C}[\mathtt{rep}_\beta\, Q]^{GL(\beta)}$$

is generated by taking traces *along oriented cycles* in Q (actually this is a more or less direct consequence of the Procesi-Razmyslov theorem), explaining the map π'.We will explain the upper horizontal maps and verify the étale property for the lower horizontal maps for the representations S and M.

With the above conventions for the normalspace in S and the quiver setting (Q, β) we have that the *action map* in S is induced by the map

$$GL_2 \times \mathtt{rep}_\beta\, Q \longrightarrow \mathtt{rep}_2\, F$$

$$(g, (x_1, x_2, x_4, y_1, y_4)) \mapsto g.(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \\ 0 & x_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} y_1 & 0 \\ 0 & y_4 \end{bmatrix}).g^{-1}$$

As the action of $GL(\beta) = \mathbb{C}^*$ is trivial on $\mathtt{rep}_\beta\, Q$ we have that the quotient map

$$GL_2 \times^{\mathbb{C}^*} \mathtt{rep}_\beta\, Q \simeq PGL_2 \times \mathtt{rep}_\beta\, Q \xrightarrow{\pi'} \mathtt{iss}_\beta\, Q$$

is just projection on the second factor (all coordinates of $\mathtt{rep}_\beta\, Q$ are (traces of) loops). Hence, to understand the map

$$\mathtt{iss}_\beta\, Q \xrightarrow{f} \mathtt{iss}_2\, F$$

we have to compute $(tr(A), tr(B), tr(A^2), tr(B^2), tr(AB))$ for

$$(A, B) = (\begin{bmatrix} x_1 & 1 + x_2 \\ -1 & x_4 \end{bmatrix}, \begin{bmatrix} 1 + y_1 & 0 \\ 0 & -1 + y_4 \end{bmatrix})$$

Hence, the map f is defined by sending $(x_1, x_2, x_4, y_1, y_4)$ to

$$(x_1 + x_4, y_1 + y_4, x_1^2 + x_4^2 - 2 - 2x_2, y_1^2 + y_4^2 + 2(y_1 - y_4) + 2, x_1 y_1 + x_4 y_4 + x_1 - x_4)$$

On the level of coordinate algebras, we have an embedding

$$\mathbb{C}[\mathtt{iss}_2\, F] = \mathbb{C}[a, b, c, d, e] \longrightarrow \mathbb{C}[\mathtt{iss}_\beta\, Q] = \mathbb{C}[\overline{x_1}, \overline{x_2}, \overline{x_4}, \overline{y_1}, \overline{y_4}] = \frac{\mathbb{C}[\mathtt{iss}_2\, F][x_1, x_2, x_4, y_1, y_4]}{(f_1, f_2, f_3, f_4, f_5)}$$

where

$$\begin{cases} f_1 & = x_1 + x_4 - a \\ f_2 & = y_1 + y_4 - b \\ f_3 & = x_1^2 + x_4^2 - 2 - 2x_2 - c \\ f_4 & = y_1^2 + y_4^2 + 2(y_1 - y_4) + 2 - d \\ f_5 & = x_1 y_1 + x_4 y_4 + x_1 - x_4 - e \end{cases}$$

To verify that this is an étale morphism we have to compute the Jacobian matrix

$$det \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 2x_1 & -2 & 2x_4 & 0 & 0 \\ 0 & 0 & 0 & 2y_1 + 2 & 2y_4 - 2 \\ y_1 + 1 & 0 & y_4 - 1 & x_1 & x_4 \end{bmatrix} = 4(2 + y_1 - y_4)^2$$

That is, if we consider the Zariski open subset U of $\mathtt{iss}_\beta\, Q$ where $2+y_1-y_4$ nonzero we have that $U \xrightarrow{f} \mathtt{iss}_2\, F$ is an étale morphism. Also observe that $\pi(S)$ lies in the image of U as the zero representation $(0,0,0,0,0)$ lies in U. The condition $2+y_1-y_4 \neq 0$ means that the B matrix is not a scalar matrix. Note however that the map f is *not* an isomorphism as there are several points in U having the same image (verify and compute the degree, that is, the number of points in U having the same image).

For the semi-simple representation M we have with the conventions of the previous section that the action map is induced by the map

$$GL_2 \times \mathtt{rep}_\beta\, Q \longrightarrow \mathtt{rep}_2\, F$$

$$(g,(x_1,x_2,x_3,x_4,y_1,y_2)) \mapsto g.((\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} y_1 & 0 \\ 0 & y_4 \end{bmatrix}).g^{-1}$$

This time, the polynomial quiver invariants are

$$\mathbb{C}[\mathtt{iss}_\beta\, Q] = \mathbb{C}[x_1,x_4,y_1,y_4,x_2x_3]$$

and under the quotient map π' a point $\overline{(g,(x_1,x_2,x_3,x_4,y_1,y_4)}$ is mapped to $(x_1,x_4,y_1,y_4,x_2x_3) \in \mathbb{C}^5 = \mathtt{iss}_\beta\, Q$. The map

$$\mathtt{iss}_\beta\, Q \xrightarrow{f} \mathtt{iss}_2\, F$$

is determined by calculating $(tr(A), tr(B), tr(A^2), tr(B^2), tr(AB))$ for

$$(A,B) = (\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \begin{bmatrix} 1+y_1 & 0 \\ 0 & -1+y_4 \end{bmatrix})$$

That is, the image of a point $(x_1,x_4,y_1,y_4,z) \in \mathtt{iss}_\beta\, Q$ is the point

$$(x_1+x_4, y_1+y_4, x_1^2+x_4^2+2z, y_1y_4+y_4-y_1, x_1y_1+x_4y_4+x_1-x_4) \in \mathtt{iss}_2\, F$$

the induced morphism between the coordinate rings is

$$\mathbb{C}[\mathtt{iss}_2\, F] = \mathbb{C}[a,b,c,d,e] \longrightarrow \mathbb{C}[\mathtt{iss}_\beta\, Q] = \frac{\mathbb{C}[\mathtt{iss}_2\, F][x_1,x_4,y_1,y_4,z]}{(f_1,f_2,f_3,f_4,f_5)}$$

where

$$\begin{cases} f_1 &= x_1+x_4-a \\ f_2 &= y_1+y_4-b \\ f_3 &= x_1^2+x_4^2+2z-c \\ f_4 &= y_1y_4+y_4-y_1-d \\ f_5 &= x_1y_1+x_4y_4+x_1-x_4-e \end{cases}$$

The corresponding Jacobian matrix is

$$det \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 2x_1 & 2x_4 & 0 & 0 & 2 \\ 0 & 0 & y_4 - 1 & y_1 + 1 & 0 \\ y_1 + 1 & y_4 - 1 & x_1 & x_4 & 0 \end{bmatrix} = -2(2 + y_1 - y_4)^2$$

proving again that the map is an étale map provided we restrict to the Zariski open subset U where $2 + y_1 - y_4 \neq 0$.

3. A Clifford algebra

Consider the Clifford algebra A associated to the quadratic form

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$$

over the polynomial algebra $\mathbb{C}[u, v]$. That is, A is generated by x and y satisfying the relations

$$x^2 = u \qquad y^2 = v \qquad xy + yx = 0$$

Let us compute the dimension of $\mathtt{rep}_2\, A$. First we determine the coordinate ring of $\mathtt{iss}_2\, A$. As $tr(x) = 0 = tr(y)$ and from the equation $xy + yx = 0$ it follows that $tr(xy) = 0$ whence $\mathbb{C}[\mathtt{iss}_2\, A] = \mathbb{C}[tr(x^2), tr(y^2)] = \mathbb{C}[u, v]$ and $\mathtt{iss}_2\, A$ is two dimensional. Repeating the Clifford algebra techniques we used to classify the simple representations of the conifold algebra we see that if $uv \neq 0$, the corresponding semi-simple 2-dimensional representation is actually simple hence there is a Zariski open subset in $\mathtt{iss}_2\, A$ consisting of simple representations. Therefore,

$$dim\ \mathtt{rep}_2\, A = dim\ \mathtt{iss}_2\, A + 2^2 - 1 = 5$$

Also remark that both M and S (the representations of the previous sections) belong to $\mathtt{rep}_2\, A$, S corresponds to the point $(1, 1)$ and is simple whereas M corresponds to the point $(0, 1)$ and is semi-simple.

We claim that M is a smooth point in $\mathtt{rep}_2\, A$. Embed $\mathtt{rep}_2\, A \hookrightarrow \mathtt{rep}_2\, F$, then we must determine which tangent vectors in $T_M(\mathtt{rep}_2\, F)$ actually are tangent to $\mathtt{rep}_2\, A$. Applying the equation $xy + yx = 0$ to

$$(\epsilon \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \epsilon \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix})$$

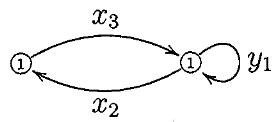
we find that $x_1 = x_4 = 0$. Moreover, $tr(y) = 0$ whence $y_4 = -y_1$ so we get as the tangent space

$$T_M(\mathtt{rep}_2\, A) = \{(\begin{bmatrix} 0 & x_2 \\ x_3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \epsilon \begin{bmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{bmatrix})\}$$

so it is 5-dimensional whence M is smooth. As a representation over the stabilizer subgroup $Stab(M) = \mathbb{C}^* \times \mathbb{C}^*$ it decomposes into

$$T_M(\mathtt{rep}\ A) = M_{(0,0)} \oplus M_{(1,-1)}^{\oplus 2} \oplus M_{(-1,1)}^{\oplus 2}$$

The tangent space to the orbit of M we already computed to be isomorphic to $M_{(1,-1)} \oplus M_{(-1,1)}$ and hence the local quiver setting (Q,β) for M in $\mathtt{rep}_2\ A$ is



There is no need for a genuine *slice variety* as the image of the usual action map for $\mathtt{rep}_2\ F$

$$GL_2 \times \mathtt{rep}_2\ Q \longrightarrow \mathtt{rep}_2\ F$$

$$(g,(x_2,x_3,y_1)) \mapsto g.(\begin{bmatrix} 0 & x_2 \\ x_3 & 0 \end{bmatrix}, \begin{bmatrix} 1+y_1 & 0 \\ 0 & -1-y_1 \end{bmatrix}).g^{-1}$$

has its image in $\mathtt{rep}_2\ A$.

The induced morphism on the quotient varieties, (use the fact that $\mathbb{C}[\mathtt{rep}_\beta\ Q] = \mathbb{C}[x_2x_3, y_1]$ (the oriented cycles in Q) and that $\mathbb{C}[\mathtt{iss}_2\ A] = \mathbb{C}[u,v]$)

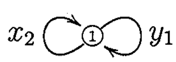
$$\mathtt{iss}_\beta\ Q \longrightarrow \mathtt{iss}_2\ A \qquad (x_2x_3, y_1) \mapsto (x_2x_3, (1+y_1)^2)$$

Clearly the Jacobian of this map is

$$det \begin{bmatrix} 1 & 0 \\ 0 & 2(1+y_1) \end{bmatrix} = 2(1+y_1)$$

so the maps is étale on the Zariski open subset where $y_1 \neq -1$. In this case it is clear that it is a two-to-one map (and *not* an isomorphism in the Zariski topology!).

Calculate for yourself that the local quiver setting for the simple representation S in $\mathtt{rep}_z\ A$ is



Again, there is no need for a real slice as the action map for $\mathtt{rep}_2\ F$ sends

$$(g,(x_2,y_1)) \mapsto g.(\begin{bmatrix} 0 & 1+x_2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1+y_1 & 0 \\ 0 & -1-y_1 \end{bmatrix}) \in \mathtt{rep}_2\ A$$

most So in mot easy cases there is no need for the other square in the formulation of the Luna slice theorem. Here is a short explanation of what you have to do if the natural action map maps outside $\mathtt{rep}_n\ A$:

If $F_m = \mathbb{C}\langle x_1, \ldots, x_m\rangle \twoheadrightarrow A$ then $\mathtt{rep}_n\ A \xhookrightarrow{i} \mathtt{rep}_n\ F_n$ and consider the normal space N_M to the orbit $\mathcal{O}(M)$ in $wisrep_n\ F_n$ and define the *slice variety* $S = N_M \cap i(\mathtt{rep}_n\ A)$. We know that M is a smooth point of S so we can always find a $Stab(M)$-invariant polynomial in $\mathbb{C}[S]$ such that the Zariski open subset of S it determines consists only of smooth points and contains M. Then the trick is to find a $Stab(M)$-equivariant lift for the natural $Stab(M)$-exact sequence

$$0 \longrightarrow \mathfrak{m}^2 \longrightarrow \mathfrak{m} \longrightarrow N'^*_M \longrightarrow 0$$

where N'_M is the normal space to the orbit in $\mathtt{rep}_n\ A$. This lift

$$N'^*_M \xrightarrow{\phi^*} \mathfrak{m} \hookrightarrow \mathbb{C}[S]$$

will then define a morphism of varieties $S \xrightarrow{\phi} N'_M$ which is étale in M and hence defines an étale map in a Zariski neighborhood.