noncommutative geometry@n

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Abstract. This is yet another version of the book **noncommutative geometry@n** by Lieven Le Bruyn. Constructive criticism is, as always, wellcome at lieven.lebruyn@ua.ac.be.

Introduction

"... La suite est trop confuse dans les notes pour être exploitable telle quelle."

Bellaiche, Dat, Marin, Racinet, Randriambololona in [**33**].

Rather than adding to the plethora of pet-proposals for a noncommutative geometry, we will focus in this book on some methods that are likely to prove useful in the 'final theory'. Whereas the details of this theory are unclear at the time of writing, the rough outline is slowly emerging.

The starting point is that a lot of interesting (families of) moduli spaces in algebraic geometry are special cases of the isomorphism problem in suitable Abelian categories $\mathtt{ab}$

$$\mathtt{moduli} \hookrightarrow \mathtt{iso(ab)}$$

In recent years one has come to realize that many of these naturally occurring Abelian categories are locally controlled by noncommutative algebras

$$\mathtt{ab} = \cup_i \ \mathtt{rep}\ A_i$$

where $\mathtt{rep}\ A_i$ is the Abelian category of all finite dimensional representations of the affine noncommutative algebra A_i and where the covering is compatible with the natural notions of isomorphism on both sides. However, one should *not* view $\mathtt{rep}\ A$ as an *affine* noncommutative scheme. This is only justified under extra conditions on A.

Among these noncommutative schemes $\mathtt{rep}\ A$ one singles out the *smooth* varieties by imposing a noncommutative regularity condition on the algebra A. There are several characterizations of commutative regular algebras. Generalizing these to the world of noncommutative algebras leads to quite different notions of noncommutative smoothness. We choose Grothendieck's characterization in terms of algebra lifts through nilpotent ideals. This approach has the advantage that the resulting $\mathtt{alg}$-smooth algebras behave well with respect to noncommutative differential forms and connections. An obvious disadvantage is that examples quickly lead us away from the cosy setting of Noetherian algebras and into the exotic wilderness of universal algebra constructions.

Basic examples of $\mathtt{alg}$-smooth algebras include coordinate rings of smooth affine curves as well as path algebras of finite quivers. More intricate examples are constructed from these by applying universal algebra constructions such as free products, universal localization, passing to a Morita equivalent algebra, taking the n-th root, and so on.

In this book we will present methods to tackle the isomorphism problem for smooth noncommutative varieties, that is, we want to describe

$$\texttt{iso}(\texttt{rep}\ A)$$

for A an $\texttt{alg}$-smooth algebra. Clearly, this is a wild problem so sooner or later we will hit the wall. All we can do is to try to push the wall a bit further. The methods we will use are drawn from two classical sources : geometric invariant theory and the theory of orders in central simple algebras.

We can partition $\texttt{rep}\ A$ with respect to the dimension of the representation

$$\texttt{rep}\ A = \bigsqcup_n \texttt{rep}_n\ A$$

where $\texttt{rep}_n A$ is the affine scheme of n-dimensional representations of A. If A is $\texttt{alg}$-smooth, each $\texttt{rep}_n A$ is a smooth affine scheme (in particular, it is reduced). The direct sum $\oplus$ on A-representation induces *sum*-morphisms

$$\texttt{rep}_n A \times \texttt{rep}_m B \longrightarrow \texttt{rep}_{m+n} A$$

Whereas $\texttt{rep}_n A$ is reduced, it usually decomposes into several irreducible components

$$\texttt{rep}_n A = \bigsqcup_{|\alpha|=n} \texttt{rep}_\alpha A$$

The *component semigroup* $\texttt{comp}A$ is the set of all occurring α, the addition is induced by the sum morphisms and the dimension $|\alpha|$ defines an augmentation $\texttt{comp}A \longrightarrow \mathbb{N}$.

Consider the subset $\texttt{simp}A$ (resp. $\texttt{schur}A$) of all components containing a simple (resp. a Schur) representation. The *empire* of the algebra A is the (infinite) quiver $\texttt{Emp}A$ with a vertex v_α for every $\alpha \in \texttt{schur}A$. The number of directed arrows from the vertex v_α to v_β is equal to $ext(\alpha, \beta)$ which is the minimal dimension of $Ext^1_A(V, W)$ for $V \in \texttt{rep}_\alpha A$ and $W \in \texttt{rep}_\beta A$. With $\texttt{emp}A$ we denote the full subquiver on the vertices v_α for $\alpha \in \texttt{simp}A$. These quivers contains the information about the noncommutative étale structure of $\texttt{rep}A$.

The *wall* of the $\texttt{alg}$-smooth algebra A is the (usually finite) full subquiver $\texttt{wall}A$ of $\texttt{Emp}A$ on the vertices corresponding to semigroup generators of $\texttt{comp}A$. Without being too dogmatic about it, let us define $\texttt{rep}A$ to be *affine* if and only if $\texttt{wall}A$ is strongly connected, that is, every pair of vertices v_α, v_β belongs to an oriented cycle in $\texttt{wall}A$. This means that there are 'enough' simple representations to allow a meaningful reduction. In general, one can reduce to an affine setting by taking suitable universal localizations of A.

We assume that $\texttt{rep}A$ is affine and that $\texttt{wall}A$ is a finite quiver on the semigroup generators $\{\alpha_1, \ldots, \alpha_k\}$. Then, A is said to be isomorphic in the *noncommutative étale topology* to

$$B \between \langle \texttt{wall}A \rangle$$

where the algebra B is Morita equivalent to the path algebra $\langle \texttt{wall}A \rangle$ of the finite quiver $\texttt{wall}A$. The Morita equivalence is determined by the dimensions $|\alpha|$ of the semigroup generators α. We claim that path algebras of quivers (or Morita equivalent algebras) play the role of affine spaces as being the only analytic local structure for manifolds.

The structure of $\mathtt{emp}A$ is determined by the Euler form χ_A of $\mathtt{wall}A$. If $\beta \in \mathtt{comp}A$ it can be written as

$$\beta = e_1\alpha_1 + \ldots + e_k\alpha_k \qquad \text{with } e_i \in \mathbb{N}$$

and $\beta \in \mathtt{simp}A$ if and only if

$$\chi_A(\epsilon, \delta_i) \leq 0 \quad \text{and} \quad chi_A(\delta_i, \epsilon) \leq 0$$

for all $1 \leq i \leq k$ with $\epsilon = (e_1, \ldots, e_k)$ (unless, $\mathtt{wall}A$ is just one oriented cycle in which case ϵ must be $(1, \ldots, 1)$). Further, the arrows from v_β to v_γ in $\mathtt{emp}A$ are determined by the wall as

$$ext(\beta, \gamma) = \delta_{\beta\gamma} - \chi_A(\epsilon, \eta)$$

if $\eta = (f_1, \ldots, f_k)$ and $\gamma = f_1\alpha_1 + \ldots + f_k\alpha_k$. In particular, if A and A' are in the same étale isomorphismclass, then $\mathtt{simp}A = \mathtt{simp}A'$. We next define when such A and A' are *birational* in the *noncommutative Zariski topology.*

Let $\alpha \in \mathtt{comp}A \cap \mathtt{comp}A'$ with $|\alpha| = n$, and consider the natural basechange action of GL_n on $\mathtt{rep}_\alpha A$ and on $\mathtt{rep}_\alpha A'$ of which the orbits are precisely the isomorphism classes of representations. From invariant theory we recall that the *closed orbits* can be classified by the affine scheme corresponding to the ring of polynomial GL_n-invariants. Michael Artin proved that the closed orbits determine the isoclasses of *semisimple* n-dimensional representations and Claudio Procesi proved that the ring of polynomial invariants is generated by traces of monomials in the algebra generators. The corresponding quotient maps

$$\mathtt{rep}_\alpha \xrightarrow{\pi} \mathtt{iss}_\alpha A \qquad \mathtt{rep}_\alpha A' \xrightarrow{\pi'} \mathtt{iss}_\alpha A'$$

send the n-dimensional representation to the isomorphism class of the direct sum of its Jordan-Hölder components. If $\alpha \in \mathtt{simp}A = \mathtt{simp}A'$ then the induced PGL_n-action is generically free, whence there is an open subset in the quotient varieties over which the representation variety is a principal PGL_n-fibration. Hence they define two central simple algebras Σ_α resp. Σ'_α of dimension n^2 over the functionfield $\mathbb{C}(\mathtt{iss}_\alpha A)$ resp. $\mathbb{C}(\mathtt{iss}_\alpha A')$. We call A and A' birational in the noncommutative Zariski topology if and only if for all $\alpha \in \mathtt{simp}A = \mathtt{simp}A'$ we have that $\mathtt{iss}_\alpha A$ is birational to $\mathtt{iss}_\alpha A'$ (hence they have the isomorphic functionfields and respecting this isomorphism we have that

$$\Sigma_\alpha \simeq \Sigma'_\alpha$$

We can express this condition without reference to geometry. Define $\int_\alpha A$ to be the algebra obtained from A by first adjoining formally all traces of monomials in the algebra generators, then modding out all relations coming from the Cayley-Hamilton identity for $n \times n$ matrices ($|\alpha| = n$) and finally taking the direct summand corresponding to the irreducible component $\mathtt{rep}_\alpha A$. These algebras natural come equipped with a trace map and we define its image $tr \int_\alpha A$ to be the commutative algebra $\oint_\alpha A$. Reformulating the above, we have that A and A' are birational if and only if

$$\int_\alpha A \qquad \text{and} \qquad \int_\alpha A'$$

are orders in the same central simple algebra for all $\alpha \in \mathtt{simp}A = \mathtt{simp}A'$.

The solution to the isomorphism problem for finite dimensional representation of the $\mathtt{alg}$-smooth algebra A combines the étale and the Zariski invariants of A.

The main result asserts that

$$\mathtt{iso}(\mathtt{rep}A) = \bigsqcup_{(Q,\alpha)} \underbrace{\mathtt{iso}(\mathtt{null}_\alpha\, Q)}_{\text{étale}} \times \underbrace{\mathtt{azu} \int_{\beta_1} A \times \ldots \times \mathtt{azu} \int_{\beta_l} A}_{\text{Zariski}}$$

where the disjoint union is taken over all quiver settings (Q, α) with Q a *finite* subquiver of $\mathtt{emp}A$ on the vertices $\{v_{\beta_1}, \ldots, v_{\beta_l}\} \subset \mathtt{simp}A$ and where $\mathtt{azu} \int_{\beta_i} A$ is the *Azumaya locus* of $\int_{\beta_i} A$. The correspondence is given as follows. Let $M \in \mathtt{rep}_\beta A$, then its image $\xi = \pi_\beta(M)$ in $\mathtt{iss}_\beta A$ is given by the semi-simplification

$$M^{ss} = S_1^{\oplus e_1} \oplus \ldots \oplus S_l^{\oplus e_l}$$

where the S_i are non-isomorphic simples lying in $\mathtt{rep}_{\beta_i} A$ and occurring with multiplicity e_i in M^{ss}. This already accounts for the Zariski part. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}[\mathtt{iss}_\beta A]$ corresponding to $\pi_\beta(M)$, then the $\mathfrak{m}$-adic completion of the order $\int_\beta A$ is fully determined by a quiver-setting

$$\int_\beta^{\hat{\mathfrak{m}}} A \between \int_\alpha^{\hat{0}} \langle Q \rangle$$

where Q is a quiver on l vertices (corresponding to the distinct simple components of M^{ss}) and $\alpha = (e_1, \ldots, e_l)$ (the multiplicities of the simple components). In Q, the number of arrows from the vertex corresponding to S_i to that of S_j is given by

$$\#\ \{ \text{ⓙ} \longleftarrow \text{ⓘ} \} = dim\ Ext^1_A(S_i, S_j)$$

and one verifies that this number is $ext(\beta_i, \beta_j)$ whence Q is the full subquiver of $\mathtt{emp}A$ on the vertices $\{v_{\beta_1}, \ldots, v_{b_l}\}$. In geometric terms, the local description implies that the fiber of the quotient map in ξ is isomorphic as GL_n-variety to

$$\pi_\beta^{-1}(\xi) \simeq GL_n \times^{GL(\alpha)} \mathtt{null}_\alpha\, Q$$

where $\mathtt{null}_\alpha\, Q$ is the *nullcone* for the basechange action group $GL(\alpha)$ on the space of α-dimensional representations of Q. In particular, GL_n-orbits in the fiber $\pi^{-1}(\xi)$ correspond one-to-one to $GL(\alpha)$-orbits in the nullcone, which accounts for the étale part in the above description.

If there are not enough simple representations, that is if $\mathtt{rep}A$ is *not* affine, it is better to consider the bigger empire $\mathtt{Emp}A$ on the Schur roots of A. One replaces the process of semisimplification by that of taking the Jordan-Hölder components with respect to a suitable *stability structure* on $\mathtt{rep}A$. Using Schofield's theory of universal localization at Sylvester rank function one can usually reduce to the case treated before, that is,

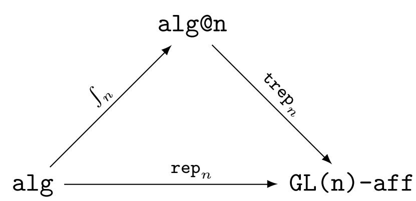
$$\mathtt{ress}A = \cup_i \mathtt{rep}A_{\Sigma_i}$$

where the universal localizations are taken such that $\mathtt{rep}A_\Sigma$ is affine and where $\mathtt{ress}A$ are the finite dimensional representations of A which are semistable for some stability structure on $\mathtt{rep}A$. If one fixes a stability structure θ, then the substitute for $\mathtt{iss}_\alpha A$ is the moduli space $\mathtt{moss}_\alpha(A, \theta)$ whose points parametrize direct sums of θ-stable representations of A of total dimension α. The basis idea of the above local description of $\mathtt{ress}A$ is that a θ-stable representation becomes simple in a suitable universal localization A_Σ.

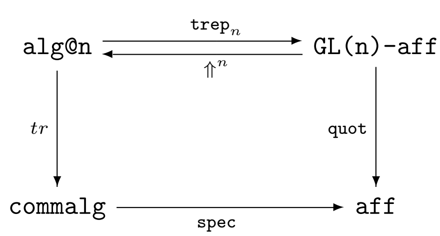
In the special, but important, case of path algebras of quivers, these moduli spaces also play a crucial role in the remaining combinatorial problem of describing the orbits in $\mathtt{null}_\alpha Q$. We give a representation theoretic interpretation of the Hesselink stratification of these nullcones in terms of associated *string* quiver settings where the underlying quiver is directed. Non-emptiness of a potential stratum is decided by the non-emptiness of the corresponding moduli space where the stability structure is determined by the coweight. Unfortunately, there is a twist in the tail. Whereas the moduli spaces account for most of the orbits in a given stratum, to describe all orbits one has to enlarge the quiver and study the orbits under a *parabolic* group. Here, we hit the wall with the methods presented in this book. After all, describing $\mathtt{rep}A$, even in the special case of the free algebra, is a *hopeless problem.*

This book is organized as follows. The first two chapters set the main stage, we define `alg`-smooth algebras, give examples of them and show that they are the natural class of noncommutative smooth algebras to consider from a noncommutative differential geometric perspective. Then, we introduce representation schemes of affine algebras as the main tool to study these `alg`-smooth algebras. Whereas for `alg`-smooth algebras one often gets by using only the reduced structure, for arbitrary algebras the scheme structure is needed. This scheme structure contains all information about algebra morphisms $A \longrightarrow M_n(C)$ where C is a commutative algebra. In fact, one can even equip these schemes with a thickening structure, inspired by the work of Kapranov, to include all algebra morphisms $A \longrightarrow M_n(B)$ where B is a noncommutative infinitesimal extension of a commutative ring.

In the third and fourth chapter we show that one can develop a geometry *at level n* having all the sophistication of ordinary commutative geometry (which is level 1). More precisely, if `GL(n)-aff` is the category of all commutative affine schemes equipped with a linear GL_n-action, then there is a triangle



The fundamental anti-equivalence `spec` : `commalg` $\longrightarrow$ `aff` of commutative algebraic geometry extends to a *left* inverse $\Uparrow^n$ assigning to an affine GL_n-scheme `afX` its *witness algebra* which is the algebra of GL_n-equivariant polynomial maps `afX` $\longrightarrow M_n(\mathbb{C})$. There is the commuting diagram of functors



where `quot` is the quotient functor which assigns to an affine scheme with GL_n-action `afX` the affine scheme determined by the ring of polynomial invariants $\mathbb{C}[\mathtt{afX}]^{GL_n}$.

The fifth chapter is pivotal in our approach to $\mathtt{iso}(\mathtt{rep}A)$. We recall enough of étale cohomology to describe the Brauer group of functionfields by the coniveau spectral sequence and to describe orders by cohomology pointed sets of automorphism schemes provided we know their étale local description. The latter is given by applying the Luna-Knop theory of étale slices to the setting of representation schemes. As an illustration of the force of these two methods we characterize all central simple algebras over a projective smooth surface having a noncommutative smooth model.

The last two chapters apply the machinery developed so far to the isomorphism problem of finite dimensional representations of `alg`-smooth algebras and their contents was described above.

A major conceptual problem in writing this book was that it assumes some familiarity with quite different topics : commutative algebraic geometry, invariant theory, representation theory, étale cohomology, Brauer groups, universal algebra, Azumaya algebras and p.i.-theory to name of few. Whereas I tried to include as many details as feasible, the reader may want to consult some standard texts for more details. I recommend, respectively, the books by Robin Hartshorne [**22**], Hanspeter Kraft [**36**], Peter Gabriel and Andrei Roiter [**19**], J.S. Milne [**47**], Ina Kersten [**29**], Aidan Schofield [**60**], Maxim Knus and Manuel Ojanguren [**32**] and Claudio Procesi [**52**].

Notation

Machines.

- `commalg` the category of commutative $\mathbb{C}$-algebras.
- `alg` the category of all $\mathbb{C}$-algebras.
- $\langle m \rangle = \mathbb{C}\langle x_1, \ldots, x_m \rangle$ the free algebra in m variables.
- $\langle \infty \rangle = \mathbb{C}\langle x_1, x_2, \ldots \rangle$ the free algebra in infinitely many variables.
- $\langle Q \rangle = \mathbb{C}Q$ the path algebra of a finite quiver Q.
- $A \between A'$: A is Morita-equivalent to A'.
- $A * A'$ the algebra free product of A and A'.
- `mod` A the category of left A-modules.
- `projnod` A the finitely generated projective left A-modules.
- A_Σ the universal localization of A at a set Σ of maps in `projmod` A.
- $u(\Sigma)$: the upper envelope of a set Σ of maps in `projmod` A.
- `Brat` A : the Bratelli diagram of an inductive limit A of semi-simple algebras.
- `dgalg` the category of differential graded $\mathbb{C}$-algebras.
- $\Omega\ A$ the ring of noncommutative differential forms of A.
- $\Omega^{ev}\ A$ the ring of even noncommutative differential forms of A.
- $T(A)$ the tensor algebra of A.
- $\perp_A$ the universal algebra for based linear maps from A.
- ∇_r (resp. ∇_l) a right (resp. left) connection.
- $\mathtt{Der}_{\mathbb{C}}\ A$ the Lie algebra of $\mathbb{C}$-derivations of A.
- $\Omega_B\ A$ the ring of B-relative noncommutative differential forms of A.

Thickenings.

- $\sqrt[n]{A}$ the n-th root algebra of A.
- $\mathbb{C}[F]$ the coordinate ring of the affine scheme F.
- $\mathtt{rep}_n\ A$ the n-dimensional representation functor of A.
- i_A the universal map $A \longrightarrow M_n(\sqrt[n]{A})$.
- `poisson` the category of commutative Poisson algebras.
- A_{ab} the Abelianization $\frac{A}{[A,A]}$ of A.
- A^{Lie} the Lie algebra structure on A given by commutators.
- $F^k\ A$ the k-th part of the commutator filtration on A.
- `gr` A the associated graded algebra for the commutator filtration on A.
- $Q^\mu_S(A)$ the micro-localization of A at S wrt. the commutator filtration on A.
- $\mathcal{O}^\mu_A$ the formal structure defined by A on `spec` A_{ab}.
- $\langle d \rangle_{[[\mathsf{ab}]]}$, the formal structure on $\mathbb{A}^d$ determined by $\langle d \rangle$.

- $\mathfrak{f}_d$ the free Lie algebra on d variables.
- `thick` the category of thickenings of commutative algebras.
- `thick.d` the category of d-thickenings of commutative algebras.
- $\int_1^d$: `alg` $\longrightarrow$ `thick.d`, the d-th thickening functor.
- $\int_1^\infty$: `alg` $\longrightarrow$ `thick`, the thickening functor.

Necklaces.

- `vect` the category of $\mathbb{C}$-vectorspaces.
- $[A, A]_v$ the subspace spanned by all commutators of A.
- $n_{\widehat{w}}$ the necklace associated to the word w.
- s_i the i-th Newton symmetric function.
- $\mathtt{neck}_d$ the space spanned by all necklaces in $X_d = \{x_1, \ldots, x_d\}$.
- $\{-, -\}_K$ the Kontsevich bracket on necklaces.
- $\oint$: `alg` $\longrightarrow$ `commalg` the necklace functor.
- `alg@` the category of $\mathbb{C}$-algebras with trace.
- $\int$: `alg` $\longrightarrow$ `alg@` the trace functor.
- σ_i the i-th elementary symmetric function.
- $\chi_a^{(n)}(t)$ the formal Cayley-Hamilton polynomial of degree n.
- `alg@n` the category of Cayley-Hamilton algebras of degree n.
- $\int_n$: `alg` $\longrightarrow$ `alg@n` the Cayley-Hamilton functor of degree n.
- $\oint_n$: `alg` $\longrightarrow$ `commalg` the necklace functor of degree n.
- $\downarrow_n$: `alg` $\longrightarrow$ `commalg` the n-th invariant functor.
- $\mathtt{DR}^*$ the Karoubi complex.
- H^*_{dR} noncommutative de Rham cohomology.
- $\mathtt{DR}^*_B$ the B-relative Karoubi complex.
- $H^*_{B,dR}$ noncommutative B-relative de Rham cohomology.
- $\mathtt{neck}_Q$ the necklace Lie algebra of a symmetric quiver.

Witnesses.

- S_d the symmetric group on d letters.
- λ a partition (or conjugacy class of S_d).
- λ^* the dual partition.
- c_λ the Young symmetrizer.
- $\mathtt{fund}_n$ fundamental n-th necklace relation.
- $\uparrow_n$: `alg` $\longrightarrow$ `alg@n` the n-th equivariant functor.
- $\mathtt{cha}_n$ fundamental n-th trace relation.
- $\mathtt{trep}_n A$ scheme of trace preserving n-dimensional representations.
- `GL(n)-aff` the category of affine schemes with GL_n-action.
- $\Uparrow^n$: `GL(n)-aff` $\longrightarrow$ `alg@n` the witness algebra functor.
- $\mathtt{simp}GL_n$ the isomorphism classes of irreducible GL_n-representations.
- $V_{(s)}$ the isotypical component of V of type $s \in \mathtt{simp}GL_n$.
- $\mathtt{iss}_n A$ the quotient scheme of $\mathtt{rep}_n A$ under the action of GL_n.
- $\mathtt{rrep}_n A$ the reduced variety of $\mathtt{rep}_n A$.
- $\mathtt{riss}_n A$ the reduced variety of $\mathtt{iss}_n A$.
- $\mathcal{O}(M)$ the GL_n-orbit of $M \in \mathtt{rep}_n A$.
- $\overline{\mathcal{O}(M)}$ the Zariski closure of the orbit $\mathcal{O}(M)$.

- $\mathtt{rep}_n A \xrightarrow{\pi} \mathtt{iss}_n A$ the quotient map.
- U_n unitary $n \times n$ matrices.
- $\mu_{\mathbb{R}}$ the real moment map.

Coverings.

- $\mathtt{spec} C$ the prime spectrum of $C \in \mathtt{commalg}$.
- G_K the absolute Galois group of the field K.
- $\mathtt{C}_{\mathtt{et}}$ the étale site of $C \in \mathtt{commalg}$.
- $\mathbb{G}_m$ the multiplicative group scheme.
- $\boldsymbol{\mu}_n$ the group scheme of n-th roots of unity.
- $R^i f$ the right derived functors of f.
- $\mathtt{S}(\mathtt{C}_{\mathtt{et}})$ the sheaves on the étale site.
- $\mathtt{S}^{ab}(\mathtt{C}_{\mathtt{et}})$ the sheaves of Abelian groups on the étale site.
- $H^i_{et}(C, \mathbb{G})$ the étale cohomology groups for $\mathbb{G} \in \mathtt{S}^{ab}(\mathtt{C}_{\mathtt{et}})$.
- $H^1_{et}(C, \mathbb{G})$ the cohomology pointed set for $\mathbb{G} \in \mathtt{S}(\mathtt{C}_{\mathtt{et}})$.
- $Tw_C(A)$ twisted forms of the C-algebra A.
- $\mathtt{Tsen.d}$ the d-th Tsen property for fields.
- $\mathtt{Tate.d}$ the d-th Tate property for fields.
- $E_2^{p,q} \Rightarrow E^n$ spectral sequence data.
- $\mathtt{afX}, \mathtt{afY}, ...$ the affine scheme $X, Y, ...$
- $Stab$ the stabilizer subgroup.
- $T_x X$ the tangent space in x to a variety (scheme) X.
- $N_x X$ the normal space to the orbit in $x \in X$.
- $\mathtt{smooth}_n A$ the n-th smooth locus of A.
- $\int_n^{\hat{\mathfrak{m}}} A$ the $\mathfrak{m}$-adic completion of $\int_n A$ for $\mathfrak{m}$ a maximal ideal of $\oint_n A$.
- $Q^\bullet$ a marked quiver.
- $\mathtt{ram}\mathcal{A}$ the ramification locus of an order $\mathcal{A}$.

Empires.

- $\mathtt{rep} A$ the Abelian category of finite dimensional representations of A.
- $\mathtt{comp} A$ the semigroup of connected components of $\mathtt{rep} A$.
- $X^{(n)}$ the n-th symmetric product of a variety (scheme) X.
- $\mathtt{simp} A$ the simple roots of A.
- $\mathtt{supp}\alpha$ the support of a dimension vector α.
- $\mathtt{azu}_n A$ the n-th Azumaya locus of A.
- $C^{sh}_{\mathfrak{p}}$ the strict Henselization of C at $\mathfrak{p} \in \mathtt{spec} C$.
- $\mathbb{C}\{x_1, \ldots, x_d\}$ the ring of algebraic functions in d variables.
- $Br(C)$ the Brauer group of $C \in \mathtt{commalg}$.
- $ext(\alpha, \beta)$ the minimal dimension of extension groups.
- $\mathtt{empire} A$ the empire of A.
- $\mathtt{nullempire} A$ the nullcone of the empire of A.
- $\mathtt{types}_\alpha A$ all representation types of $\alpha \in \mathtt{comp} A$.
- $<<$ the ordering on $\mathtt{types}_\alpha\ Q$.
- $\mathtt{wall} A$ the wall of A.
- $\mathtt{azu}_\alpha A$ the Azumaya locus of A wrt. $\alpha \in \mathtt{simp} A$.
- $\mathtt{ram}_\alpha A$ the ramification locus of A wrt. $\alpha \in \mathtt{simp} A$.

- $\int_n^{**} A$ the reflexive closure of the order $\int_n A$.
- $\beta(C)$ the reflexive Brauer group of a normal commutative domain C.
- $\mathtt{neck}_\alpha$ the Poisson Lie algebra on $\mathbb{C}[\mathtt{iss}_\alpha Q]$.

Nullcones.

- $\mathtt{schur}A$ the Schur roots of A.
- $\mathtt{Emp}A$ the bigger empire of A.
- $\mathtt{ress}A$ finitre dimensional semistable representations of A.
- $\asymp$ confused in the stability structure.
- $\Delta(V)$ special semistable subrepresentation of V.
- $\nabla(V)$ special semistable factorrepresentation of V.
- $\mu(V)$ slope stability structure.
- $K_0(A)$ Grothendieck group of f.g. projective A-modules.
- $G_0(A)$ Grothendieck group of f.p. A-modules.
- $\mathtt{schof}A$ Schofield fractal of A.
- T_Q Tits form of quiver Q.
- q_Q quadratic form of quiver Q.
- $\mathtt{ind}Q$ indecomposable roots of Q.
- $\mathtt{itypes}_\alpha Q$ decomposition types into indecomposable roots for α.
- F_Q fundamental set of roots of Q.
- $\mathtt{Grass}_k(l)$ Grassmann manifold of k-dimensional subspaces of $\mathbb{C}^l$.
- Δ_{re} real roots.
- Δ_{im} imaginary roots.
- $hom(\alpha.\beta)$ minimal dimension of homomorphisms.
- $\mathtt{Grass}_\alpha(\beta)$ quiver Grassmannian.
- $\alpha \perp \beta$ left orthogonal relation.
- $\mathtt{moss}_\alpha(Q,\theta)$ moduli space of θ-semistable α-dimensional representations of Q.
- Q^b bipartite double of Q.
- $c(\Sigma), o(\Sigma), K(\Sigma)$ control matrix, observation matrix and Kalman code of system Σ.
- $\mathtt{brauer}A$ Brauer stable representations of A.
- $\mathtt{bs}A$ Brauer-Severi scheme of A.
- $\mathtt{null}_n^m$ nullcone of GL_n-action on M_n^m.
- $\mathtt{null}_\alpha Q$ nullcone of $GL(\alpha)$-action on $\mathtt{rep}_\alpha Q$.

CHAPTER 1

Machines

"I propose to consider smooth algebras (that is, formally smooth finitely generated algebras) as machines for producing an infinite system of usual smooth schemes $(M_n)_{n=1,2,\dots}$."
Maxim Kontsevich in [**34**].

There are several characterizations of commutative regular algebras. Generalizing these to the world of noncommutative algebras leads to quite different notions of noncommutative smoothness. We choose Grothendieck's characterization in terms of algebra lifts through nilpotent ideals. This approach has the advantage that the resulting `alg`-smooth algebras behave well with respect to noncommutative differential forms and connections. An obvious disadvantage is that examples quickly lead us away from the setting of Noetherian algebras and into the exotic wilderness of universal algebra constructions.

In later chapters we will study `alg`-smooth algebras via associated Noetherian algebras determined by their schemes of finite dimensional representations. These representation schemes are (commutative) smooth varieties. In this way we view `alg`-smooth algebras as machines producing a family of manifolds and connecting morphisms.

Commutative manifolds are locally diffeomorphic to affine spaces. We will see that path algebras of quivers are to `alg`-smooth algebras what affine spaces are to manifolds. For this reason we give explicit descriptions of all constructions for this class of `alg`-smooth algebras.

1.1. Smooth algebras.

In this section we will define `alg`-smoooth algebras and give some elementary examples : coordinate rings of smooth affine curves and path algebras of quivers. From these building blocks one can construct more complicated examples by two methods : algebra free products and universal localizations. We will restrict attention to affine algebras. However, in the theory of C^*-algebras there are many (non-affine) `alg`-smooth algebras for which our methods fail as they have very few, if any, finite dimensional representations. We present one example coming from the aperiodic Penrose tilings of the plane.

Throughout, we fix an algebraically closed field of characteristic zero and denote it with $\mathbb{C}$. All algebras will be associative $\mathbb{C}$-algebras with a unit element.

With `cat` we will denote a category of $\mathbb{C}$-algebras. For example. `commalg` is the category of all *commutative* $\mathbb{C}$-algebras and `alg` is the category of *all* $\mathbb{C}$-algebras and $\mathbb{C}$-algebra morphisms as morphisms.

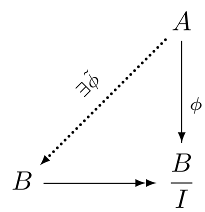
Definition 1. A *test-object* in a category of $\mathbb{C}$-algebras `cat` is a pair (B, I) such that B is an object in `cat`, $I \triangleleft B$ is a *nilpotent* ideal of B such that the quotient map

$$B \twoheadrightarrow \frac{B}{I}$$

is a morphism in `cat`. In particular, the quotient algebra $\frac{B}{I}$ is an object in `cat`.

For a fixed category `cat` of $\mathbb{C}$-algebras we define `cat`-smooth algebras by a lifting property with respect to test-objects.

Definition 2. An object A in `cat` is said to be `cat`-smooth if and only if for all test-objects (B, I) in `cat` and all morphisms $A \xrightarrow{\phi} \frac{B}{I}$ in `cat` the diagram



can be completed with a morphism $A \xrightarrow{\tilde{\phi}} B$ in `cat`.

The terminology is motivated by the characterization of commutative regular algebras, due to Alexander Grothendieck.

Theorem 1 (Grothendieck). *Let C be a commutative affine $\mathbb{C}$-algebra, then C is regular if and only if C is* `commalg`*-smooth.*

In this case, the affine scheme `spec`C *with coordinate ring C is a smooth scheme. In particular,* `spec`C *is a reduced variety, that is, C has no non-zero nilpotent elements.*

Proof. See for example [**25**] or [**22**, Exercise 8.6]. □

`alg`-smooth algebras were first studied by Bill Schelter in [**59**] and subsequently in the framework of noncommutative differential geometry by Joachim Cuntz and Daniel Quillen in [**10**].

Example 1. The archetypical example of an `alg`-smooth algebra is the *free algebra* in m-variables $\langle m \rangle = \mathbb{C}\langle x_1, \ldots, x_m \rangle$. Let (B, I) be a test-object in `alg` and consider an algebra morphism

$$\mathbb{C}\langle x_1, \ldots, x_m \rangle \xrightarrow{\phi} \frac{B}{I}$$

If $\phi(x_i) = \overline{b_i}$ then taking any representant $b_i \in B$ of the class $\overline{b_i} \in \frac{B}{I}$, $\tilde{\phi}(x_i) = b_i$ defines an algebra lift as there are no relations among the x_i.

The free algebra on infinitely many variables $\langle \infty \rangle = \mathbb{C}\langle x_1, x_2, \ldots \rangle$ is also a `alg`-smooth algebra though not affine.

A commutative `alg`-smooth algebra is clearly `commalg`-smooth. However, the converse is not true.

Example 2. Consider the polynomial algebra $\mathbb{C}[x_1, \ldots, x_m]$ and the 4-dimensional noncommutative local algebra

$$B = \frac{\mathbb{C}\langle x, y \rangle}{(x^2, y^2, xy + yx)} = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy$$

B has a one-dimensional nilpotent ideal $I = \mathbb{C}(xy - yx)$ such that the 3-dimensional quotient $\frac{B}{I}$ is commutative. Take the algebra morphism $\mathbb{C}[x_1, \ldots, x_d] \xrightarrow{\phi} \frac{B}{I}$ defined by $x_1 \mapsto x, x_2 \mapsto y$ and $x_i \mapsto 0$ for $i \geq 2$. This morphism admits no lift to B as for any potential lift $[\tilde{\phi}(x), \tilde{\phi}(y)] \neq 0$ in B. Therefore, $\mathbb{C}[x_1, \ldots, x_d]$ can only be smooth if $d = 1$.

EXAMPLE 3. The $k+1$-dimensional semi-simple algebra

$$C_k = \frac{\mathbb{C}[e_1, \ldots, e_k]}{(e_i^2 - e_i, e_i e_j, \sum_{i=1}^k e_i - 1)} = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

is `alg`-smooth because one can lift a decomposition of the unit element in mutual orthogonal idempotents through a nilpotent ideal. Indeed, let (B, I) be a test-object with $I^l = 0$ and let $1 = \overline{e_1} + \ldots + \overline{e_k}$ be a decomposition of 1 into orthogonal idempotents of $\frac{B}{I}$. Any element $1 - i$ with $i \in I$ is invertible in B as

$$(1 - i)(1 + i + i^2 + \ldots + i^{l-1}) = 1 - i^l = 1.$$

If $\overline{e}$ is an idempotent of B/I and $x \in B$ such that $\pi(x) = \overline{e}$. Then, $x - x^2 \in I$ whence

$$0 = (x - x^2)^l = x^l - l x^{l+1} + \binom{l}{2} x^{l+2} - \ldots + (-1)^l x^{2l}$$

and therefore $x^l = a x^{l+1}$ with $a = l - \binom{l}{2} x + \ldots + (-1)^{l-1} x^{l-1}$. Observe that $ax = xa$. If we take $e = (ax)^l$, then e is an idempotent in B as

$$e^2 = (ax)^{2l} = a^l (a^l x^{2l}) = a^l x^l = e$$

the next to last equality follows from $x^l = a x^{l+1} = a^2 x^{l+2} = \ldots = a^l x^{2l}$. Moreover,

$$\pi(e) = \pi(a)^l \pi(x)^l = \pi(a)^l \pi(x)^{2l} = \pi(a^l x^{2l}) = \pi(x)^l = \overline{e}.$$

If $\overline{f}$ is another idempotent in B/I such that $\overline{e}\,\overline{f} = 0 = \overline{f}\,\overline{e}$ then we can lift $\overline{f}$ to an idempotent f' of B. Because $f'e \in I$ we have

$$f = (1 - e)(1 - f'e)^{-1} f'(1 - f'e).$$

Because $f'(1 - f'e) = f'(1 - e)$ one verifies that f is idempotent, $\pi(f) = \overline{f}$ and $e.f = 0 = f.e$. Assume by induction that we have already lifted the pairwise orthogonal idempotents $\overline{e}_1, \ldots, \overline{e}_{k-1}$ to pairwise orthogonal idempotents $e_1, \ldots, e_{k-1}$ of B, then $e = e_1 + \ldots + e_{k-1}$ is an idempotent of B such that $\overline{e}\overline{e}_k = 0 = \overline{e}_k \overline{e}$. Hence, we can lift $\overline{e_k}$ to an idempotent $e_k \in B$ such that $e e_k = 0 = e_k e$. But then also

$$e_i e_k = (e_i e) e_k = 0 = e_k (e e_i) = e_k e_i.$$

Finally, as $e_1 + \ldots + e_k - 1 = i \in I$ we have that

$$e_1 + \ldots + e_k - 1 = (e_1 + \ldots + e_k - 1)^l = i^l = 0$$

This decomposition defines the required lift $C_k \longrightarrow B$.

DEFINITION 3. A finite *quiver* Q is a directed graph determined by

- a finite set $Q_v = \{v_1, \ldots, v_k\}$ of *vertices*, and
- a finite set $Q_a = \{a_1, \ldots, a_l\}$ of *arrows* where we allow multiple arrows between vertices and loops in vertices.

Every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ has a *starting vertex* $s(a) = i$ and a *terminating vertex* $t(a) = j$. The description of the quiver Q is encoded in the integral $k \times k$ matrix

$$\chi_Q = \begin{bmatrix} \chi_{11} & \dots & \chi_{1k} \\ \vdots & & \vdots \\ \chi_{k1} & \dots & \chi_{kk} \end{bmatrix} \qquad \text{where} \qquad \chi_{ij} = \delta_{ij} - \# \{ \textcircled{j} \longleftarrow \textcircled{i} \}$$

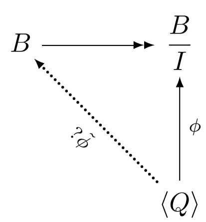
The corresponding bilinear form on $\mathbb{Z}^k$ is called the *Euler form* of the quiver Q.

The underlying vectorspace of the *path algebra* $\langle Q \rangle = \mathbb{C}Q$ of the quiver Q has as basis the directed paths in Q. Multiplication is induced by (left) concatenation of paths. More precisely, $1 = v_1 + \ldots + v_k$ is a decomposition of 1 into mutually orthogonal vertex-idempotents and we define

- $v_j.a$ is always zero unless $\textcircled{j} \xleftarrow{a} \bigcirc$ in which case it is the path a,
- $a.v_i$ is always zero unless $\bigcirc \xleftarrow{a} \textcircled{i}$ in which case it is the path a,
- $a_i.a_j$ is always zero unless $\bigcirc \xleftarrow{a_i} \bigcirc \xleftarrow{a_j} \bigcirc$ in which case it is the path $a_i a_j$.

Path algebras of quivers are of crucial importance in the study of `alg`-smooth algebras. We will show that they are to noncommutative manifolds what affine spaces are to commutative manifolds.

Example 4. For any finite quiver Q, the path algebra $\langle Q \rangle$ is `alg`-smooth. Let (B, I) be a test-object in `alg` and consider



The decomposition $1 = \phi(v_1) + \ldots + \phi(v_k)$ into mutually orthogonal idempotents in $\frac{B}{I}$ can be lifted though the nilpotent ideal I to a decomposition $1 = \tilde{\phi}(v_1) + \ldots + \tilde{\phi}(v_k)$ into mutually orthogonal idempotents in B by example 3. But then, taking for every arrow a

$$\textcircled{j} \xleftarrow{a} \textcircled{i} \quad \text{an arbitrary element} \quad \tilde{\phi}(a) \in \tilde{\phi}(v_j)(\phi(a) + I)\tilde{\phi}(v_i)$$

gives a required lift $\langle Q \rangle \xrightarrow{\tilde{\phi}} B$.

Observe that all examples of `alg`-smooth algebras constructed so far are path algebras of quivers. The free algebra $\langle m \rangle$ of example 1 is the path algebra of the quiver with one vertex and m loops. The semi-simple algebra C_k of example 3 is the path algebra of the quiver on k vertices having no arrows.

Before we can construct more examples of `alg`-smooth algebras we need a ring-theoretic characterization of them.

Definition 4. An *A-bimodule* M is a left and right A-module such that $a(ma') = (am)a'$ for all $a, a' \in A$ and all $m \in M$. A-bimodules are the same as left $A \otimes A^{op}$-modules where A^{op} is A with the opposite multiplication. The correspondence is given by $a \otimes a'.m = ama'$.

Let A a $\mathbb{C}$-algebra and I an A-bimodule. The Hochschild chain and cochain complexes are defined by

$$C_\bullet(A,I) = \{I \otimes A^{\otimes n} \ , \ n \in \mathbb{N}\} \quad \text{and} \quad C^\bullet(A,I) = \{\mathtt{Hom}_{\mathbb{C}}(A^{\otimes n}, I) \ , \ n \in \mathbb{N}\}$$

and the associated Hochschild homology and cohomology groups are denoted by $H_n(A,I)$ and $H^n(A,I)$.

Hence, the i-th *Hochschild cohomology* group $H^i(A,M)$ of an A-bimodule M is the i-th extension $\mathtt{Ext}^i_{A\otimes A^{op}}(A,M)$ in the category of left $A \otimes A^{op}$-modules. In particular, an A-bimodule M is projective as bimodule if and only if $H^1(M,M') = 0$ for all A-bimodules M'.

By iteration on the degree of nilpotency, an algebra A is $\mathtt{alg}$-smooth if it satisfies the lifting property for test-objects (B,I) with $I^2 = 0$. Given such a test-object, consider the pull-back diagram

$$\begin{array}{ccc} A \times_{\frac{B}{I}} B & \xrightarrow{pr_1} & A \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow{\pi} & \frac{B}{I} \end{array}$$

where $A \times_{\frac{B}{I}} B = \{(a,b) \ : \ \phi(a) = \pi(b)\}$ is an *infinitesimal extension* of A, that is, the kernel of pr_1, say M has square zero.

For M a fixed bimodule over A consider all infinitesimal extensions of A by M. A basic result about Hochschild cohomology (see for example [**51**, Chap. 11]) identifies isomorphism classes of these extensions with the second Hochschild cohomology group $H^2(A,M)$.

Recall that $\Omega^1 A$ is the kernel of the multiplication A-bimodule map.

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \xrightarrow{m} A \longrightarrow 0$$

THEOREM 2 (Schelter). *The following statements are equivalent.*

(1) *A is $\mathtt{alg}$-smooth.*
(2) *A has cohomological dimension ≤ 1 for Hochschild cohomology.*
(3) *$\Omega^1 A$ is a projective A-bimodule.*
(4) *Every infinitesimal extension $R \twoheadrightarrow A$ has a splitting $A \longrightarrow R$.*

PROOF. If an infinitesimal extension $R \twoheadrightarrow A$ has a splitting then it determines an isomorphism of R with the semidirect product $A \oplus M$ and the splitting becomes the inclusion of A. Therefore (4) implies that $H^2(A,M) = 0$ for all A-bimodules M. The defining sequence of $\Omega^1 A$ asserts that

$$H^2(A,M) = \mathtt{Ext}^2_{A\otimes A^{opp}}(A,M) = \mathtt{Ext}^1_{A\otimes A^{opp}}(\Omega^1 A, M)$$

from which it follows that $\Omega^1 A$ is a projective A-bimodule. □

DEFINITION 5. Two $\mathbb{C}$-algebras A and A' are called *Morita-equivalent* if and only if there is an equivalence of categories

$$\mathtt{mod}\ A \sim \mathtt{mod}\ A'$$

where $\mathtt{mod}\ A$ is the category of left A-modules. This is equivalent to

$$A' \simeq \mathtt{End}_A\ P$$

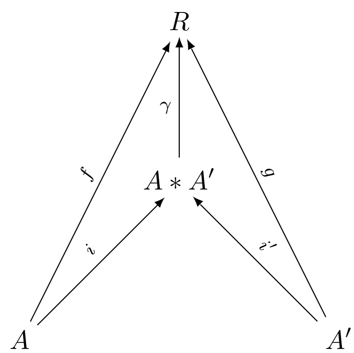
with P a finitely generated *progenerator* for the category $A - \mathtt{mod}$. That is, P is a finitely generated projective left A-module and for any $M \in A - \mathtt{mod}$ there is an epimorphism $P^I \longrightarrow\!\!\!\!\rightarrow M$. If A and A' are Morita-equivalent, we denote this property by $A \between A'$.

Definition 6. For $\mathbb{C}$-algebras A and A', let $\mathcal{B}$ be a vectorspace basis for $A - \mathbb{C}1$ and $\mathcal{B}'$ a vectorspace basis for $A' - \mathbb{C}1$. The *free algebra product* $A * A'$ is the $\mathbb{C}$-vectorspace with basis all words of the form

$$w = a_1 b_1 a_2 b_2 \ldots a_k b_k \qquad \text{or} \qquad w = a_1 b_1 a_2 b_2 \ldots a_k$$

for some k, all $a_i \in \mathcal{B}$ and all $b_j \in \mathcal{B}'$. Multiplication is defined by concatenation of words and if the end term of the first word belongs to the same set of the starting term of the second word one uses the multiplication table in the relevant algebra to reduce to a linear combination of allowed words.

The free algebra product is universal with respect to pairs of $\mathbb{C}$-algebra morphisms $A \xrightarrow{f} R \xleftarrow{g} A'$. That is, with the natural inclusion maps i and i' any $\mathbb{C}$-algebra morphism $\gamma : A * A' \longrightarrow R$ is of the form $f * g$



making the diagram commute.

Theorem 3. *Let A and A' be two $\mathbb{C}$-algebras.*

1. *If A is* $\mathtt{alg}$*-smooth and $A \between A'$, then A' is* $\mathtt{alg}$*-smooth.*
2. *If A and A' are* $\mathtt{alg}$*-smooth, then so are $A * A'$ and $A \oplus A'$.*
3. *If A is a commutative affine domain which is* $\mathtt{alg}$*-smooth, then $A \simeq \mathbb{C}$ or A is the coordinate ring of a smooth affine curve.*

Proof. (1) : If $A \between A'$ then their categories of bimodules are equivalent and the conclusion follows from theorem 2.

(2) : By the universal property of free products any algebra map $A * A' \longrightarrow \frac{B}{I}$ is of the form $\phi * \psi$ for $A \xrightarrow{\phi} \frac{B}{I}$ and $A' \xrightarrow{\psi} \frac{B}{I}$. By assumption there exist lifts $\tilde{\phi}$ and $\tilde{\psi}$ but then the original map has a lifting $\tilde{\phi} * \tilde{\psi}$. The second case is obvious.

(3) : For a commutative affine $\mathbb{C}$-algebra, Hochschild dimension coincides with homological dimension, whence the result follows. □

Example 5. A finite dimensional semi-simple $\mathbb{C}$-algebra

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

is $\mathtt{alg}$-smooth. Indeed, $A \between C_k$ and C_k is $\mathtt{alg}$-smooth by example 3.

Taking `alg`-smooth algebras to be the affine building blocks, we want to construct noncommutative manifolds by gluing these blocks along 'open subsets'. In commutative algebraic geometry, the algebra of functions on a *Zariski open* subset is given by a localization of the coordinate ring. For this reason we have to consider localizations of noncommutative algebras.

In noncommutative ringtheory one usually considers the localization of an algebra A at a multiplicative subset S satisfying the (left) *Ore conditions*

- If $as = 0$ for $a \in A$ and $s \in S$, then there is an $s' \in S$ such that $s'a = 0$.
- For all $s_1 \in S$ and $a_1 \in A$, there are $s_2 \in S$ and $a_2 \in A$ such that $s_2 a_1 = a_2 s_1$.

If these conditions are satisfied, one can form a ring of fractions A_S by taking equivalence classes on $S \times A$ (leading to left quotients $s^{-1}a$) with respect to the relation

$$(s_1, a_1) \sim (s_2, a_2) \Leftrightarrow \exists a, a' \in A \ : \ aa_1 = a'a_2 \text{ and } as_1 = a's_2 \in S$$

However, for general `alg`-smooth algebras (such as free algebras or path algebras of quivers) there are very few multiplicatively closed sets satisfying the Ore conditions.

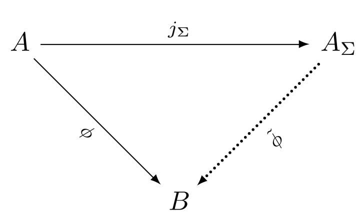
EXAMPLE 6. Consider in the free algebra $\langle m \rangle = \mathbb{C}\langle x_1, \ldots, x_m \rangle$ the multiplicatively closed subset $\{1, x_1, x_1^2, \ldots\}$. As there are no relations in $\langle m \rangle$ we can never satisfy the second Ore condition for $s_1 = x_1$ and $a_1 = x_j$ when $j \neq 1$. Therefore, there is no Ore set in $\langle m \rangle$ containing the powers of x_1.

For this reason we have to consider another localization theory : *universal localization* .

DEFINITION 7. If A is a $\mathbb{C}$-algebra we denote by `projmod` A the category of finitely generated projective left A-modules. The *universal localization* A_Σ with respect to a set Σ of maps in `projmod` A is the algebra having an algebra morphism $j_\Sigma \ : \ A \longrightarrow A_\Sigma$ such that the extended maps

$$A_\Sigma \otimes_A \sigma \quad \text{in} \quad \texttt{projmod}\ A_\Sigma$$

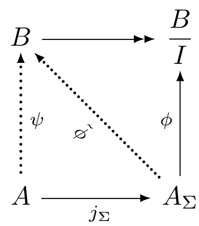
are isomorphisms for all $\sigma \in \Sigma$ and is universal as such. That is, if $A \longrightarrow B$ is an algebra map such that all extended maps $B \otimes_A \sigma$ are isomorphisms in `projmod` B, then there is an algebra map



making the diagram commute.

THEOREM 4. *Let A be* `alg`*-smooth and Σ a set of maps in* `projmod` A*. Then, the universal localization A_Σ is* `alg`*-smooth.*

Proof. Consider a test-object (B, I) in `alg` and an algebra map $A \xrightarrow{\phi} \frac{B}{I}$, then we have the following diagram



Here, ψ exists because A is `alg`-smooth. By *Nakayama's lemma* (see for example [**51**, §4.2]) all maps $\sigma \in \Sigma$ become isomorphisms under tensoring with ψ. But then, $\tilde{\phi}$ exists by the universal property of A_Σ. □

Unlike Ore localizations, it is often quite hard to give a precise generator/relation description of universal localizations. Observe that if we invert the maps Σ in `projmod` A, we also invert all maps lying in the *upper envelope* $u(\Sigma)$, that is all maps in `projmod` A which can be written as

$$\begin{bmatrix} \sigma_1 & u_{12} & \dots & u_{1l} \\ 0 & \sigma_2 & & u_{2l} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_l \end{bmatrix}$$

for some l with $\sigma_i \in \Sigma$ and the u_{ij} arbitrary maps.

A description of an equivalence relation giving the elements of A_Σ, even of maps between induced projective modules of A_Σ, was given by Peter Malcolmson [**45**] (see also [**60**, Chp. 4] for more details).

Theorem 5 (Malcolmson). *Let Σ be a set of maps in* `projmod` A. *Then,*

(1) *Every map between induced projective A_Σ-modules has the form*

$$f\gamma^{-1}g \qquad \text{with } \gamma \in u(\Sigma)$$

(2) *Two such maps $f_1\gamma_1^{-1}g_1$ and $f_2\gamma_2^{-1}g_2$ are equal if and only if there is a solution to the matrix equation*

$$\begin{bmatrix} \gamma_1 & 0 & 0 & 0 & g_1 \\ 0 & \gamma_2 & 0 & 0 & -g_2 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & g_4 \\ f_1 & f_2 & f_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \gamma_5 \\ f_5 \end{bmatrix} \cdot \begin{bmatrix} \gamma_6 & g_6 \end{bmatrix}$$

where all maps are defined over A and $\gamma_i \in u(\Sigma)$.

Example 7. Let Σ be a set of square matrices over A such that $1 \in \Sigma$ and $\Sigma = u(\Sigma)$. An element of A_Σ is determined by a triple (f, γ, g) where $\gamma \in \Sigma$ is a square matrix (say $n \times n$), f a $1 \times n$ row vector and g an $n \times 1$ column vector and we denote the corresponding element of A_Σ by $f\gamma^{-1}g$. To understand the above equivalence relation, assume that $(f_1, \gamma_1, g_1) \sim (f_2, \gamma, g_2)$ with the matrix-equation given in theorem 5. If all matrices in Σ are invertible, then

$$\begin{aligned} 0 = f_5 g_6 = & \quad f_5\gamma_6(\gamma_5\gamma_6)^{-1}\gamma_5 g_6 \\ = & \quad f_1\gamma_1^{-1}g_1 - f_2\gamma_2^{-1}g_2 + f_3\gamma_3^{-1}0 + 0\gamma_4^{-1}g_4 \\ = & \quad f_1\gamma_1^{-1}g_1 - f_2\gamma_2^{-1}g_2 \end{aligned}$$

whence $f_1\gamma_1^{-1}g_1 = f_2\gamma_2^{-1}g_2$.

An algebra structure on A_Σ is induced by the following operations :

- $(f_1,\gamma_1,g_1) + (f_2,\gamma_2,g_2) = (\begin{bmatrix} f_1 & f_2 \end{bmatrix}, \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix})$
- $(f_1,\gamma_1,g_1).(f_2,\gamma_2,g_2) = (\begin{bmatrix} f_1 & 0 \end{bmatrix}, \begin{bmatrix} \gamma_1 & -g_1 f_2 \\ 0 & \gamma_2 \end{bmatrix}, \begin{bmatrix} 0 \\ g_2 \end{bmatrix})$
- $-(f,\gamma,g) = (f,\gamma,-g)$

and the canonical map $A \longrightarrow A_\Sigma$ is defined by $a \mapsto (1,1,a)$. For proofs and more details, see [**46**].

Fortunately, one can give an explicit description of universal localizations of path algebras of quivers.

EXAMPLE 8. Let Q be a finite quiver on k vertices and consider the path algebra $\langle Q \rangle$. Then, we can identify the isomorphism classes in `projmod` $\langle Q \rangle$ with $\mathbb{N}^k$. To each vertex v_i corresponds an *indecomposable projective* left $\langle Q \rangle$-ideal $P_i = \mathbb{C}Qv_i$ having as $\mathbb{C}$-vectorspace basis all paths in Q *starting at* v_i.

The homomorphisms between these projectives are given by

$$Hom_{\mathbb{C}Q}(P_i,P_j) = \bigoplus_{(i) \overset{p}{\leftsquigarrow} (j)} \mathbb{C}p$$

where p is an oriented path in Q starting at v_j and ending at v_i.

Therefore, any $\langle Q \rangle$-module morphism σ between two projective left modules

$$P_{i_1} \oplus \ldots \oplus P_{i_u} \xrightarrow{\sigma} P_{j_1} \oplus \ldots \oplus P_{j_v}$$

can be represented by an $u \times v$ matrix M_σ whose (p,q)-entry m_{pq} is a linear combination of oriented paths in Q starting at v_{j_q} and ending at v_{i_p}.

Form a $v \times u$ matrix N_σ with entries free variables y_{pq}. The universal localization at $\{\sigma\}$ is then the *affine* algebra

$$\langle Q \rangle_\sigma = \frac{\langle Q \rangle * \mathbb{C}\langle y_{11},\ldots,y_{uv}\rangle}{I_\sigma}$$

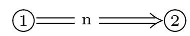
where I_σ is the ideal determined by the matrix equations

$$M_\sigma.N_\sigma = \begin{bmatrix} v_{i_1} & & 0 \\ & \ddots & \\ 0 & & v_{i_u} \end{bmatrix} \qquad N_\sigma.M_\sigma = \begin{bmatrix} v_{j_1} & & 0 \\ & \ddots & \\ 0 & & v_{j_v} \end{bmatrix}$$

Equivalently, $\langle Q \rangle_\sigma$ is the path algebra of a quiver *with relations* where the quiver is Q extended with arrows y_{pq} from v_{i_p} to v_{j_q} for all $1 \leq p \leq u$ and $1 \leq q \leq v$ and the relations are the above matrix entry relations.

Repeating this procedure for every $\sigma \in \Sigma$ we obtain the universal localization $\langle Q \rangle_\Sigma$. In particular, if Σ is a finite set of maps, then the universal localization $\langle Q \rangle_\Sigma$ is an *affine* $\mathbb{C}$-algebra, that is finitely generated.

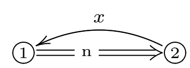
EXAMPLE 9. When we take the universal localization $\langle Q\rangle_a$ of the path algebra with respect to one arrow $(i) \xrightarrow{a} (j)$ we obtain an algebra, Morita equivalent to the path algebra of the contracted quiver Q' obtained by identifying the vertices v_i and v_j. For example, consider the quiver Q



having n arrows, say $a_1, \ldots, a_n$ from v_1 to v_2. The path algebra $\mathbb{C}Q$ is the $n+2$-dimensional algebra

$$\mathbb{C}Q = \begin{bmatrix} \mathbb{C} & \mathbb{C}a_1 + \ldots + \mathbb{C}a_n \\ 0 & \mathbb{C} \end{bmatrix}$$

The universal localization with respect to a_1 is the path algebra of the quiver



with relations $xa_1 = v_1$ and $a_1x = v_2$. The elements v_1, v_2, a_1 and x generate the the matrixalgebra $M_2(\mathbb{C})$ and the centralizer of this subring is isomorphic to $v_1\langle Q\rangle_{a_1}v_1$ which is freely generated by the paths xa_i for $i \neq 1$. Therefore,

$$\langle Q\rangle_{a_1} \simeq M_2(\langle a-1\rangle)$$

where $\langle a-1\rangle$ is the path algebra of the contracted quiver, obtained from Q by removing the arrow a_1 and identifying the vertices. It is clear that this argument extends to more general quivers.

EXAMPLE 10. With the few facts we know so far we can build a huge class of `alg`-smooth algebras. Take as the elementary building blocks the `alg`-smooth algebras

- The coordinate ring $\mathbb{C}[C]$ of an affine smooth curve, see theorem 3(3).
- The path algebra $\langle Q\rangle$ of a finite quiver Q, see example 4.

The basic operations to create new `alg`-smooth algebras from known ones are

- Taking the algebra free product $A * A'$.
- Passing to a Morita equivalent algebra.
- Taking the universal localization A_Σ for a set of maps in `projmod` A.

To describe universal localizations we have to keep track of `projmod`, the finitely generated projective modules. For the building blocks we have a complete description.

- The isomorphism classes of `projmod`$\mathbb{C}[C]$ are

  $$\mathbb{Z} \oplus \texttt{Pic}\ C$$

  where `Pic` C is the *Picard group* , that is, the ideal class group of the Dedekind domain $\mathbb{C}[C]$.
- Every finitely generated projective modules of $\langle Q\rangle$ is isomorphic to

  $$P_1^{\oplus n_1} \oplus \ldots \oplus P_k^{n_k}$$

  where P_i is the indecomposable projective corresponding to vertex v_i and all $n_i \in \mathbb{N}$.

We can also follow `projmod` through the constructions :

- A finitely generated projective module of $A * A'$ is isomorphic to

$$A * A' \otimes_A P \oplus A * A' \otimes_{A'} P'$$

where P (resp. P') is a finitely generated projective of A (resp. A'), see [**60**, Thm 2.13].
- A finitely generated projective module of A_Σ is isomorphic to

$$A_\Sigma \otimes_A P$$

where P is a finitely generated projective of A, see [**60**, Cor. 4.5].

We will study affine `alg`-smooth algebras by investigating their schemes of finite dimensional representations. In the theory of C^*-algebras, there is a class of (non-affine) `alg`-smooth algebras which often have no finite dimensional representations at all. We present one such example, connected to Penrose aperiodic tilings of the plane, in detail.

THEOREM 6 (Cuntz-Quillen). *The inductive limit of a countable system*

$$\ldots \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \ldots$$

of finite dimensional semi-simple algebras is `alg`*-smooth.*

PROOF. (Sketch) We know from example 5 that every A_n is `alg`-smooth. Hence it suffices to show that one can choose the liftings in a compatible way. This can be deduced from the fact that for a finite-dimensional semi-simple algebra A there is a uniqueness for the lifting morphism. Suppose we have a square-zero extension $A = B/I$ and two lifting morphisms $l, l' : A \longrightarrow B$. Using l we can identify $B = A \oplus I$ with $l(a) = a$. But then,

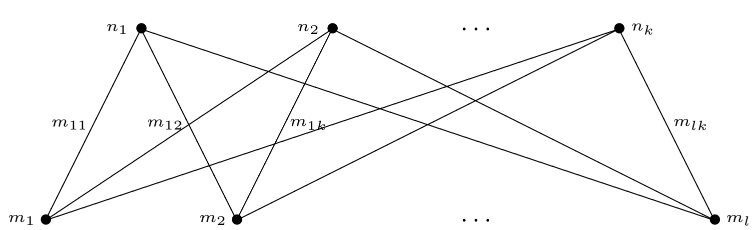
$$l'(a) = a + D(a)$$

where $D : A \longrightarrow I$ is a derivation which must be *inner* by semi-simplicity (see for example [**51**, §11.5]), that is, $D(a) = [a, i]$ for some $i \in I$. But then, because

$$l'(a) = a + [a, i] = (1 + i)^{-1} l(a)(1 + i)$$

the two lifts in an infinitesimal extension are conjugate by an element congruent to one modulo I. □

Let $A_n = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ and $A_{n+1} = M_{m_1}(\mathbb{C}) \oplus \ldots \oplus M_{m_l}(\mathbb{C})$. A $\mathbb{C}$-algebra morphism $A_n \longrightarrow A_{n+1}$ determines non-negative integers m_{ij} such that $m_i = \sum_j m_{ij} n_j$ and hence be a labeled graph



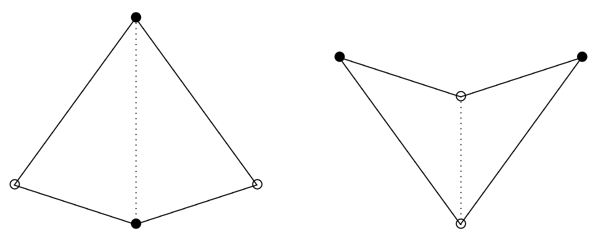
We delete an edge whenever $m_{ij} = 0$ and delete the label if $m_{ij} = 1$. If we put these labeled graphs on top of each other for all $n \in \mathbb{N}$ we obtain the *Bratelli diagram* `Brat` A of the `alg`-smooth algebra $A = \varinjlim A_n$.

Much structural information of A can be read off from the Bratelli diagram. For example, closed twosided ideals of A are in one-to-one correspondence with subsets D of the vertices of `Brat` A satisfying the following two properties

(1) If $v \in D$ and w is a vertex in a lower layer which can be connected by a path to v, then $w \in D$, and
(2) If all vertices $w_1, \ldots, w_z$ in the next layer which are connected to v belong to D, then so does v.

We give one concrete example of such an `alg`-smooth algebra without finite dimensional representations.

EXAMPLE 11. The *Penrose algebra* A_{Pen} is the `alg`-smooth algebra connected to aperiodic Penrose tilings of the plane. The tiles, which are usually called *Penrose kites and darts*, are quadrangles with two sides of length 1 and two sides of length $\tau = \frac{1+\sqrt{5}}{2}$, the golden ratio. The corners are colored with two colors and the matching condition to produce Penrose tilings is that we must put equal edges together and also match the colors at the vertices.



Kite Dart

Using these tiles and the matching condition one obtains uncountable many aperiodic tilings of the plane, properties of which are proved using the operations of *composition, decomposition* and *inflation* of tilings. These operations naturally lead to an inductive limit of semi-simple algebra, see [**9**, §II.3] for more details. Consider

$$K_n = \{(z_0, z_1, \ldots, z_n) \in \{0,1\}^{n+1} \text{ satisfying } \; z_i = 1 \Rightarrow z_{i+1} = 0\}$$

The projection morphism $K_{n+1} \twoheadrightarrow K_n$ is the obvious one forgetting the final z_{n+1}. On the finite set K_n we have the equivalence relation $\mathcal{R}_n$ defined by $z \sim z'$ if and only if $z_n = z'_n$. A function $a = a_{(z_0,\ldots,z_n),(z'_0,\ldots,z'_n)}$ on the finite set $\mathcal{R}_n$ defines an element $\tilde{a}$ of the Penrose algebra A_{Pen} by the rule

$$\begin{cases} \tilde{a}_{z,z'} = a_{(z_0,\ldots,z_n),(z'_0,\ldots,z'_n)} & \text{if } ((z_0,\ldots,z_n),(z'_0,\ldots,z'_n)) \in \mathcal{R}_n, \\ \tilde{a}_{z,z'} = 0 & \text{if } ((z_0,\ldots,z_n),(z'_0,\ldots,z'_n)) \notin \mathcal{R}_n. \end{cases}$$

The structure of the subalgebra A_n of A_{Pen} generated by the complex-valued functions on $\mathcal{R}_n$ is

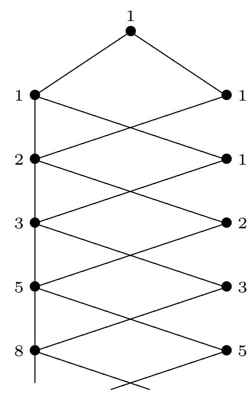
$$A_n \simeq M_{0_n}(\mathbb{C}) \oplus M_{1_n}(\mathbb{C})$$

where 0_n is the number of elements of K_n that end with 0 and 1_n the number of elements ending with 1. The projection $K_{n+1} \twoheadrightarrow K_n$ induces an inclusion $A_n \hookrightarrow A_{n+1}$ which is

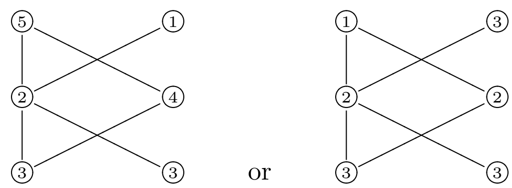
$$A_n = M_{0_n}(\mathbb{C}) \oplus M_{1_n}(\mathbb{C}) \hookrightarrow A_{n+1} = M_{0_n+1_n}(\mathbb{C}) \oplus M_{0_n}(\mathbb{C})$$

$$m_0 \oplus m_1 \longrightarrow \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix} \oplus m_0$$

as the coherence condition implies that $0_{n+1} = 0_n + 1_n$ and $1_{n+1} = 0_n$. Observe that if we add $0_0 = 1$, then the sequence of numbers $\{0_0, 0_1, 0_2, \ldots\}$ is the *Fibonacci series* $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$. Therefore, the Bratelli diagram `Brat` A_{Pen} is of the form



Consequently, the Penrose algebra A_{Pen} is a simple `alg`-smooth algebra. To prove this we consider the set D describing a closed ideal. We indicate in the pictures below the order by which the properties dictate the inclusion of a vertex starting from a given vertex ① in D.



In particular, A_{Pen} does not have finite dimensional representations.

1.2. Differential forms.

In this section we run through the formal theory of noncommutative differential forms. We have two specific aims in mind. First, we will prove that the existence of a connection on the 1-forms $\Omega^1 A$ forces the algebra A to be `alg`-smooth.Secondly, we prove that free algebras and path algebras of quivers have the homology of contractible spaces, consistent with their role of noncommutative affine spaces.

At this point you may wonder why on earth we take the exotic class of `alg`-smooth algebras as the building blocks for noncommutative algebraic manifolds. There are two compelling reasons.

First, we want the noncommutative algebra A to control a family of (commutative) manifolds. If A is `alg`-smooth we will see in the next chapter that such a family is given by $\texttt{rep}_n A$, $n = 1, 2, \ldots$, the schemes of finite dimensional representations of A.

Secondly, we will prove in this section that in order to have a decent theory of noncommutative differential forms on A allowing for connections on the cotangent bundle (the 1-forms $\Omega^1 A$), the algebra A must be `alg`-smooth. Later we will prove that these noncommutative differential forms induce ordinary GL_n-invariant differential forms on the smooth varieties $\texttt{rep}_n A$ when A is `alg`-smooth.

DEFINITION 8. $\texttt{dgalg}$ is the category of *differential graded* $\mathbb{C}$*-algebras* , that is, an object $R \in \texttt{dgalg}$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra

$$R = \oplus_{i \in \mathbb{Z}} R^i$$

endowed with a *differential* d of degree one

$$\dots \xrightarrow{d} R^{i-1} \xrightarrow{d} R^i \xrightarrow{d} R^{i+1} \xrightarrow{d} \dots$$

satisfying $d \circ d = 0$ and for all $r \in R^i$ and $s \in R$ we have

$$d(rs) = (dr)s + (-1)^i r(ds).$$

Morphisms in $\texttt{dgalg}$ are $\mathbb{C}$-algebra morphisms $R \xrightarrow{\phi} S$ which are graded and commute with the differentials.

DEFINITION 9. For $A \in \texttt{alg}$, the differential graded algebra ΩA of *noncommutative differential forms* is constructed as follows. Let $\overline{A}$ be the quotient vector space $A/\mathbb{C}.1$ and

$$\Omega^n A = A \otimes \underbrace{\overline{A} \otimes \dots \otimes \overline{A}}_{n}$$

for $n \geq 0$ and $\Omega^n A = 0$ for $n < 0$. For all $a_i \in A$ we denote the image of $a_0 \otimes a_1 \otimes \dots \otimes a_n$ in $\Omega^n A$ by $a_0 da_1 \dots da_n$.

A multiplication is defined on $\Omega A = \oplus_{n \in Z} \Omega^n A$ by

$$\begin{aligned} (a_0 da_1 \dots da_n)(a_{n+1} da_{n+2} \dots da_m) = \\ (-1)^n a_0 a_1 da_2 \dots da_m + (-1)^{n-1} a_0 d(a_1 a_2) da_3 \dots da_m + \\ \sum_{i=2}^{n-1} (-1)^{n-i} a_0 \dots da_{i-1} d(a_i a_{i+1}) da_{i+1} \dots da_m + \\ a_0 da_1 \dots da_{n-1} d(a_n a_{n+1}) da_{n+2} \dots da_m \end{aligned}$$

The differential d of degree one

$$\dots \xrightarrow{d} \Omega^{n-1} A \xrightarrow{d} \Omega^n A \xrightarrow{d} \Omega^{n+1} A \xrightarrow{d} \dots$$

is defined by

$$d(a_0 da_1 \dots da_n) = 1 da_0 da_1 \dots da_n.$$

EXAMPLE 12 (Cuntz-Quillen). These formulas define the unique $\texttt{dgalg}$ structure on $\Omega\ A$ such that

$$a_0 da_1 \dots da_n = (a_0, a_1, \dots, a_n).$$

In *any* $R = \oplus_i R_i \in \texttt{dgalg}$ containing A as an even degree subalgebra we have the following identities

$$\begin{aligned} d(a_0 da_1 \dots da_n) &= && da_0 da_1 \dots da_n \\ (a_0 da_1 \dots da_n)(a_{n+1} da_{n+2} \dots da_m) &= && (-1)^n a_0 a_1 da_2 \dots da_m \\ & && + \textstyle\sum_{i=1}^n (-1)^{n-i} a_0 da_1 \dots d(a_i a_{i+1}) \dots da_m \end{aligned}$$

which proves uniqueness.

To prove existence, we define d on $\Omega\ A$ as above making the $\mathbb{Z}$-graded $\mathbb{C}$-vectorspace $\Omega\ A$ into a complex as $d \circ d = 0$. Consider the *graded endomorphism ring* of the complex

$$\texttt{End} = \oplus_{n \in \mathbb{Z}} \texttt{End}_n = \oplus_{n \in \mathbb{Z}} \texttt{Hom}_{complex}(\Omega^\bullet\ A, \Omega^{\bullet + n}\ A).$$

With the composition as multiplication, $\texttt{End}$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra and we make it into an object in $\texttt{dgalg}$ by defining a differential

$$\ldots \xrightarrow{D} \texttt{End}_{n-1} \xrightarrow{D} \texttt{End}_n \xrightarrow{D} \texttt{End}_{n+1} \xrightarrow{D} \ldots$$

by the formula on any homogeneous ϕ

$$D\phi = d \circ \phi - (-1)^{deg\ \phi} \phi \circ d.$$

Now define the morphism $A \xrightarrow{l} \texttt{End}_0$ which assigns to $a \in A$ the left multiplication operator

$$la(a_0, \ldots, a_n) = (aa_0, \ldots, a_n)$$

and extend it to a map

$$\Omega\ A \xrightarrow{l_*} \texttt{End} \qquad \text{by} \qquad l_*(a_0, \ldots, a_n) = la_0 \circ D\ la_1 \circ \ldots \circ D\ la_n.$$

Applying the general formulae given at the beginning of the proof to the subalgebra $l(A) \hookrightarrow \texttt{End}$ we see that the image of l_* is a differential graded subalgebra of $\texttt{End}$ and is the differential graded subalgebra generated by $l(A)$.

Define an evaluation map $\texttt{End} \xrightarrow{ev} \Omega\ A$ by $ev(\phi) = \phi(1)$. Because

$$\begin{aligned} D\ la_i(1, a_{i+1}, \ldots, a_n) &= d(a_i, a_{i-1}, \ldots, a_n) - la_i d(1, a_{i+1}, \ldots, a_n) \\ &= (1, a_i, \ldots, a_n) \end{aligned}$$

we have that

$$ev(la_0 \circ D\ la_1 \circ \ldots \circ D\ la_n) = (a_0, \ldots, a_n)$$

showing that ev is a left inverse for l_* whence l_* in injective.

Hence we can use the isomorphism $\Omega\ A \simeq Im(l_*)$ to transport the $\texttt{dgalg}$ structure to $\Omega\ A$.

Definition 10. For A, R in $\texttt{alg}$ a $\mathbb{C}$-linear map $A \xrightarrow{\rho} R$ satisfying $\rho(1_A) = 1_R$ is called a *based linear map* .

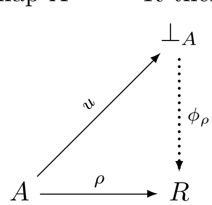
The universal algebra for based linear maps from A is the quotient algebra of the *tensor algebra* $T(A) = \oplus_{n \geq 0} A^{\otimes n}$

$$\perp_A = \frac{T(A)}{T(A)(1 - 1_A)T(A)}$$

where 1_A is the degree one element of $T(A)$ determined by the unit element of A. There is a universal based linear map

$$A \xrightarrow{u} \perp_A \qquad a \mapsto \overline{a}$$

such that for any based linear map $A \xrightarrow{\rho} R$ there is a unique *algebra map* ϕ_ρ



making the diagram commute. If we apply this to the identity map $A \xrightarrow{id} A$ we obtain an ideal $I_A = \texttt{Ker}\ \phi_{id}$ of $\perp_A$. Clearly, $A \simeq \frac{\perp_A}{I_A}$.

Example 13. Let $C_k = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$ (k factors) with idempotents $e_1, \ldots, e_k$, then $T(C_k) = \langle k \rangle$ and as $1_{C_k} = e_1 + \ldots + e_k$ we deduce that $\perp_{C_k} = \langle k - 1 \rangle$.

Definition 11. There are two canonical embeddings $A \overset{i_1}{\underset{i_2}{\rightrightarrows}} A * A$. For $a \in A$ define the elements in $A * A$:

$$\begin{cases} p(a) & = \frac{1}{2}(i_1(a) + i_2(a)) \\ q(a) & = \frac{1}{2}(i_1(a) - i_2(a)) \end{cases}$$

and let $Q_A \triangleleft A * A$ be the ideal generated by the elements $q(a)$ for $a \in A$. Clearly $A \simeq \dfrac{A * A}{Q_A}$.

We will relate the algebra of all noncommutative differential forms ΩA, respectively $\Omega^{ev} A = \oplus_{n \geq 0} \Omega^{2n} A$ the algebra of all even noncommutative differential forms, to the algebra $A * A$, respectively $\perp_A$, by defining a new multiplication on ΩA.

Definition 12. For $R \in \texttt{dgalg}$ define the *Fedosov product* on R to be the one induced by defining on homogeneous $r, s \in R$ the product

$$r \circ s = rs - (-1)^{deg\ r} dr ds$$

R equipped with the Fedosov product will be denoted by $(R, \circ)$ and is again an object in $\texttt{alg}$.

Theorem 7 (Cuntz-Quillen). *With notations as above we have :*

1. $(\Omega^{ev} A, \circ) \simeq \perp_A$ *and under this isomorphism* $I_A^n \simeq \oplus_{k \geq n} \Omega^{2k} A$.
2. $(\Omega A, \circ) \simeq A * A$ *and under this isomorphism* $Q_A^n \simeq \oplus_{k \geq n} \Omega^k A$.

Proof. (1) : The inclusion $A \subset (\Omega^{ev} A, \circ)$ is a based linear map and by the universal property of $\perp_A$ there is an algebra morphism

$$\perp_A \xrightarrow{\phi} (\Omega^{ev} A, \circ) \quad \text{with } \phi(u(a)) = a$$

Define for all $a, a' \in A$ the element $\omega(a, a') = u(aa') - u(a)u(a') \in \perp_A$ and observe that

$$\phi(\omega(a, a')) = aa' - a \circ a' = da da'$$

From the fact that the Fedosov product coincides with the usual product on ΩA if one of the terms t is a *closed form* (that is, if $dt = 0$) it follows that ϕ is surjective as

$$\begin{aligned} \phi(u(a_0)\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n})) = & \quad a_0 \circ da_1 da_2 \circ \ldots \circ da_{2n-1} da_{2n} \\ = & \quad a_0 da_1 da_2 \ldots da_{2n} \end{aligned}$$

There is a section to ϕ, the linear map $(\Omega^{ev} A, \circ) \xrightarrow{\psi} \perp_A$ sending $a_0 da_1 \ldots da_{2n}$ to $u(a_0)\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n})$. The image is closed under left multiplication by $u(a)$ for $a \in A$ as $\psi(a \circ a_0 da_1 \ldots da_{2n})$

$$\begin{aligned} = & \quad \psi(aa_0 da_1 \ldots da_{2n}) - \psi(da da_0 da_1 \ldots da_{2n}) \\ = & \quad u(aa_0)\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n}) - \omega(a, a_0)\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n}) \\ = & \quad u(a)u(a_0)\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n}) = u(a)\psi(a_0 da_1 \ldots da_{2n}) \end{aligned}$$

Because the image contains the unit element and the $u(a)$ generate $\perp_A$ it follows that ψ is surjective whence ϕ is an isomorphism. The last statement follows from this identification.

(2) : We have two algebra map $A \xrightarrow{u} (\Omega A, \circ)$ given by $a \mapsto a \pm da$ because

$$\begin{aligned}(a \pm da) \circ (a' \pm da') = &\quad aa' - dada' \pm a(da') \pm (da)a' + dada' \\ = &\quad aa' \pm d(aa')\end{aligned}$$

By the universal property of $A * A$ there is an algebra morphism

$$A * A \xrightarrow{\psi} (\Omega A, \circ) \qquad \psi(p(a)) = a \quad \text{and} \quad \psi(q(a)) = da$$

Because the Fedosov product coincides with the usual product when one of the forms is closed we have

$$\psi(p(a_0)q(a_1)\dots q(a_n)) = a_0 da_1 \dots da_n$$

Conversely, we have a section to ψ defined by

$$\Omega A \xrightarrow{\phi} A * A \qquad a_0 da_1 \dots da_n \mapsto p(a_0)q(a_1)\dots q(a_n)$$

and we only have to prove that ϕ is surjective. The image `Im` ϕ is closed under left multiplication by $p(a)$ and $q(a)$ as $p(1) = 1$ and

$$\begin{cases} p(a)p(a_0)q(a_1)\dots q(a_n) = & p(aa_0)q(a_1)\dots q(a_n) - q(a)q(a_0)q(a_1)\dots q(a_n) \\ q(a)p(a_0)q(a_1)\dots q(a_n) = & q(aa_0)q(a_1)\dots q(a_n) - p(a)q(a_0)q(a_1)\dots q(a_n) \end{cases}$$

Because the elements $p(a)$ and $q(a)$ generate $A * A$, the image `Im` ϕ is a left ideal containing 1, whence ψ is surjective. Again, the last statement follows. □

Having defined noncommutative differential forms, we can consider connections on bimodules and the relation to `alg`-smoothness.

DEFINITION 13. For E an A-bimodule, connections on E are given by linear maps.

- A *right connection* : $E \xrightarrow{\nabla_r} E \otimes_A \Omega^1 A$ satisfying
$$\nabla_r(aea') = a(\nabla_r e)a' + aeda',$$
- A *left connection* : $E \xrightarrow{\nabla_l} \Omega^1 A \otimes_A E$ satisfying
$$\nabla_l(aea') = a(\nabla_l e)a' + daea'$$

We say that E has a *connection* if it has both a left and a right connection.

THEOREM 8 (Cuntz-Quillen). *The following are equivalent :*

1. *A is `alg`-smooth.*
2. *There is an algebra morphism $A \longrightarrow \frac{\perp_A}{I_A^2}$.*
3. *There is a linear map $\overline{A} \xrightarrow{\phi} \Omega^2 A$ satisfying*
$$\phi(a_1 a_2) = a_1 \phi(a_2) + \phi(a_1) a_2 + da_1 da_2$$
4. *There is a right connection on the A-bimodule $\Omega^1 A$.*
5. *There is a connection on the A-bimodule $\Omega^1 A$.*

PROOF. (1) ⇒ (2) : Consider the test-object $(B, I) = (\frac{\perp_A}{I_A^2}, \frac{I_A}{I_A^2})$. As $A \simeq \frac{\perp_A}{I_A} = B/I$ we can lift the identity morphism to an algebra morphism.

(2) ⇒ (3) : From theorem 7 we recall that

$$\frac{\perp_A}{I_A^2} \simeq A \oplus \Omega^2 A$$

where multiplication on the right-hand side is given by the Fedosov product modulo forms of degree > 2. Because we lift the identity morphism, the algebra morphism $A \longrightarrow \frac{\perp A}{I_A^2}$ must have the form $a \mapsto a - \phi(a)$ for some $\phi : \overline{A} \longrightarrow \Omega^2 A$. We have (because the Fedosov product coincides with the ordinary product on ΩA if one of the terms is a closed form)

$$(a_1 - \phi(a_1)) \circ (a_2 - \phi(a_2)) = a_1 a_2 - da_1 da_2 - a_1 \phi(a_2) - \phi(a_1) a_2$$

which is an algebra morphism if and only if it satisfies the required condition.

$(3) \Rightarrow (4)$: Observe that $\Omega^1 A \otimes_A \Omega^1 \simeq \Omega^2 A$. Define, using the map ϕ a linear map

$$\nabla_r : \Omega^1 A \longrightarrow \Omega^2 A \qquad \nabla_r(a_0 da_1) = a_0 \phi(a_1)$$

This satisfies the required condition as

$$\begin{aligned}
\nabla_r(a_0(da_1)a) = & \quad \nabla_r(a_0 d(a_1 a) - a_0 a_1 da) \\
= & \quad a_0 \phi(a_1 a) - a_0 a_1 \phi(a) \\
= & \quad a_0 a_1 \phi(a) + a_0 \phi(a_1) a + a_0 da_1 da - a_0 a_1 \phi(a) \\
= & \quad a_0 (\nabla_r da_1) a + a_0 da_1 da
\end{aligned}$$

$(4) \Rightarrow (5)$: A connection on $\Omega^1 A$ is the datum of three maps

$$\Omega^1 A \underset{\nabla_r}{\overset{\nabla_l}{\underset{d}{\rightrightarrows}}} \Omega^2 A$$

satisfying the following properties

$$\begin{array}{rclll}
\nabla_l(aea') & = & a\nabla_l(e)a' & +(da)ea' & \\
d(aea') & = & a(de)a' & +(da)ea' & -ae(da') \\
\nabla_r(aea') & = & a\nabla_r(e)a' & & +ae(da')
\end{array}$$

Hence, if ∇_r is a right connection then $d + \nabla_r$ is a left connection and if ∇_l is a left connection then $\nabla_l - d$ is a right connection.

$(5) \Rightarrow (1)$: For any A-bimodule E, a right connection ∇_r on E defines a bimodule splitting s_r of the right multiplication map m_r

$$E \otimes A \underset{s_r}{\overset{m_r}{\rightleftarrows}} E$$

by the formula

$$s_r(e) = e \otimes 1 - j(\nabla_r e) \quad \text{where} \quad j(e \otimes da) = ea \otimes 1 - e \otimes a$$

Similarly, a left connection gives a bimodule splitting s_l to the left multiplication map. Consequently, if a connection exists on E, then E must be a *projective* bimodule. If we apply this to the A-bimodule $\Omega^1 A$ we obtain the result from theorem 2. $\square$

EXAMPLE 14. A connection on $\langle m \rangle = \mathbb{C}\langle x_1, \ldots, x_m \rangle$. Let $\phi(x_i) = 0$ for all $1 \leq i \leq m$, then we can define by induction of the length n of a word in the generators, the image of

$$\begin{aligned}
\phi(x_{i_1} \ldots x_{i_n}) = & \quad x_{i_1} \phi(x_{i_2} \ldots x_{i_n}) + \phi(x_{i_1}) x_{i_2} \ldots x_{i_n} + dx_{i_1} d(x_{i_2} \ldots x_{i_n}) \\
= & \quad dx_{i_1} d(x_{i_2} \ldots x_{i_n}) + x_{i_1} \phi(x_{i_2} \ldots x_{i_n})
\end{aligned}$$

Whence we obtain the description of the map $\phi : \overline{\langle m \rangle} \longrightarrow \Omega^2\langle m \rangle$

$$\phi(x_{i_1}\dots x_{i_n}) = \sum_{k=1}^{n-1} x_{i_1}\dots x_{i_{k-1}} dx_{i_k} d(x_{i_{k+1}}\dots x_{i_n})$$

From this map we define the connection $\nabla_r : \Omega^1\langle m \rangle \longrightarrow \Omega^2\langle m \rangle$ by

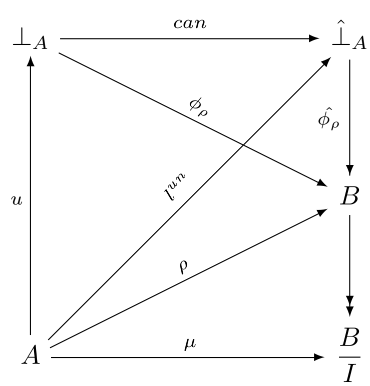
$$\nabla_r\ d(x_{i_1}\dots x_{i_n}) = \sum_{k=1}^{n-1} x_{i_1}\dots x_{i_{k-1}} dx_{i_k} d(x_{i_{k+1}}\dots x_{i_n})$$

EXAMPLE 15 (Cuntz-Quillen). The *Yang-Mills derivation* on a `alg`-smooth algebra A. The *I_A-adic completion* of $\perp_A$ is by definition the inverse limit

$$\hat{\perp}_A = \underset{\overleftarrow{n}}{\texttt{lim}} \ \frac{\perp_A}{I_A^n}$$

If A is `alg`-smooth then there is a collection of compatible lifted algebra morphisms $A \longrightarrow \frac{\perp_A}{I_A^n}$. These compatible lifts define a universal algebra lift $A \xrightarrow{l^{un}} \hat{\perp}_A$. This map can be used to construct algebra lifts modulo nilpotent ideals in a systematic way.

Let (B, I) be a test-object in `alg` and $A \xrightarrow{\mu} \frac{B}{I}$ an algebra map. We can lift μ to B as a based linear map, say ρ and have the following situation



Here, ϕ_ρ is the algebra map coming from the universal lifting property of $\perp_A$ and $\hat{\phi_\rho}$ is its extension to the completion. But then, $\tilde{\mu} = \hat{\phi_\rho} \circ l^{un}$ is an *algebra* lift of μ.

One can construct the universal lift l^{un} from the linear map $\overline{A} \xrightarrow{\phi} \Omega^2 A$ of the previous theorem. Because $\perp_A$ is freely generated by the $a \in A - \mathbb{C}1$, we define the *Yang-Mills derivation* on $\perp_A$ by

$$\perp_A \xrightarrow{D} \perp_A \qquad D(a) = \phi(a) \quad \forall a \in A.$$

Let L be the degree two operator on $\Omega^{ev} A$ defined by

$$L(a_0 da_1 \dots da_{2n}) = \phi(a_0) da_1 \dots da_{2n} + \sum_{j=1}^{2n} a_0 da_1 \dots da_{j-1} d\phi(a_j) da_{j+1} \dots da_{2n}$$

and let H denote the degree zero operator on even forms which is multiplication by n on $\Omega^{2n} A$. Then, we have the relations

$$[H, L] = L \qquad \text{and} \qquad D = H + L$$

and as a consequence we have on $\hat{\perp}_A \simeq \hat{\Omega}^{ev}\ A = \prod_n \Omega^{2n} A$ that

$$e^{-L} H e^{L} = H + e^{-L}[H, e^{L}] = H + \int_0^1 e^{-tL}[H, L] e^{tL} dt = D$$

Therefore, the universal lift for all $a \in A$ is given by

$$l^{un}(a) = e^{-L} a = a - \phi(a) + \frac{1}{2} L\phi(a) - \dots$$

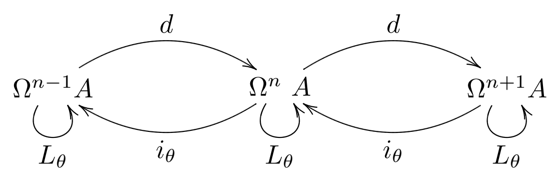
For more details we refer to [**10**, p.280].

As path algebras of quivers are similar to affine spaces we want to compute the homology groups and prove that they are the same as the de Rham cohomology of affine space.

Definition 14. For A in `alg`, a *derivation* θ is a $\mathbb{C}$-linear map $A \longrightarrow A$ satisfying for $\theta(aa') = \theta(a)a' + a\theta(a')$ for all $a, a' \in A$. The set of all $\mathbb{C}$-linear derivations $\mathtt{Der}_{\mathbb{C}}\ A$ is a Lie algebra with bracket $[\theta, \theta'] = \theta \circ \theta' - \theta' \circ \theta$ where $\circ$ is composition of maps.

For B in `dgalg` a *super-derivation* is a linear map $s : B \longrightarrow B$ such that for all homogeneous $b, b' \in B$ we have $s(bb') = s(b)b' + (-1)^i b s(b')$ where i is the degree of b.

Given $\theta \in \mathtt{Der}_{\mathbb{C}}\ A$ we define a degree preserving derivation L_θ and a degree -1 super-derivation i_θ on ΩA



by the rules

$$\begin{cases} L_\theta(a) = \theta(a) & L_\theta(da) = d\ \theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

for all $a \in A$.

Theorem 9 (Cartan homotopy formulas). *For $\theta, \gamma \in \mathtt{Der}_{\mathbb{C}}\ A$ we have*

$$L_\theta = i_\theta \circ d + d \circ i_\theta$$

and we have the following equalities of operators

$$\begin{cases} L_\theta \circ i_\gamma - i_\gamma \circ L_\theta = [L_\theta, i_\gamma] & = i_{[\theta,\gamma]} = i_{\theta\circ\gamma - \gamma\circ\theta} \\ L_\theta \circ L_\gamma - L_\gamma \circ L_\theta = [L_\theta, L_\gamma] & = L_{[\theta,\gamma]} = L_{\theta\circ\gamma - \gamma\circ\theta} \end{cases}$$

Proof. For the first equality, observe that both sides are derivations on ΩA which agree on all the generators a, da ($a \in A$) for ΩA.

By definition,both sides of the second identity are degree -1 super-derivations on ΩA so it suffices to check that they agree on generators. Clearly, both sides give zero when evaluated on $a \in A$ and for da we have

$$(L_\theta \circ i_\gamma - i_\gamma \circ L_\theta) da = L_\theta\ \gamma(a) - i_\gamma\ d\ \theta(a) = \theta\ \gamma(a) - \gamma\ \theta(a) = i_{[\theta,\gamma]}(da)$$

A similar argument proves the last identity. $\square$

DEFINITION 15. An algebra $A \in \mathtt{alg}$ is said to be *contractible* if the homology of the differential forms complex

$$H^n\ A = \frac{\mathtt{Ker}(\Omega^n A \longrightarrow \Omega^{n+1} A)}{\mathtt{Im}(\Omega^{n-1} A \longrightarrow \Omega^n A)}$$

is concentrated in degree zero and $H^0\ A = \mathbb{C}$.

Recall from commutative differential geometry that affine spaces are $\mathbb{C}$-contractible. We want to generalize this to free algebras and to path algebras of quivers.

EXAMPLE 16 (Kontsevich). The free algebra $\langle m \rangle = \mathbb{C}\langle x_1, \ldots, x_m \rangle$ is contractible. Define the *Euler derivation* E on $\langle m \rangle$ by defining it one the generators to be

$$E(x_i) = x_i \qquad \text{for all } 1 \leq i \leq m.$$

By induction on the length k of a word w in the variables x_i one proves that

$$E(w) = kw$$

We claim that L_E is a total degree preserving linear automorphism on

$$\Omega^n \langle m \rangle \qquad \text{for } n \geq 1.$$

For if w_i for $0 \leq i \leq n$ are words in the x_i of degree k_i with $k_i \geq 1$, then one verifies that

$$L_E(w_0 dw_1 \ldots dw_n) = (k_0 + \ldots + k_n) w_0 dw_1 \ldots dw_n.$$

Using the words of length ≥ 1 in the x_i as a basis for $\overline{\langle m \rangle}$, we see that the kernel and image of the differential d must be homogeneous. But then, if ω is a multi-homogeneous element in $\Omega^n \langle m \rangle$ and in $\mathtt{Ker}\ d$ we have for some integer $k \neq 0$ that

$$k\omega = L_E(\omega) = (i_E \circ d + d \circ i_E)\omega = d(i_E\ \omega)$$

and hence ω lies in $\mathtt{Im}\ d$. That is,

$$\begin{cases} H^0\ \langle m \rangle = & \mathbb{C} \\ H^n\ \langle m \rangle = & 0 \end{cases}$$

for all $n \geq 1$.

In order to generalize this argument to the case of path algebras of quivers we have to get rid of the forms dv_i for the vertex-idempotents v_i. As $\langle Q \rangle$ is even a smooth algebra in $\mathtt{alg}_{C_k}$, the category of all C_k-algebras, it makes sense to consider the *relative* differential forms, defined as follows.

DEFINITION 16. For a $\mathbb{C}$-subalgebra $B \subset A$ define the *relative differential forms* of degree n with respect to B to be

$$\Omega^n_B\ A = A \otimes_B \underbrace{\overline{A}_B \otimes_B \ldots \otimes_B \overline{A}_B}_{n}$$

where $\overline{A}_B$ is the cokernel of the B-bimodule inclusion $B \subset A$. $\Omega^n_B\ A$ is the quotient space of $\Omega^n A$ by the relations

$$\begin{aligned} a_0 da_1 \ldots d(a_{i-1} b) da_i \ldots da_n =& a_0 da_1 \ldots da_{i-1} d(b a_i) \ldots da_n \\ a_0 da_1 \ldots da_{i-1} db da_{i+1} \ldots da_n =& 0 \end{aligned}$$

for all $b \in B$ and $1 \leq i \leq n$. One verifies that the multiplication and differential defined on ΩA are compatible with these relations, making $\Omega_B\ A$ an object in `dgalg`.

$\Omega_B\ A$ has the following universal property. Given $\Gamma = \oplus \Gamma^n$ in `dgalg` and an algebra map $A \xrightarrow{f} \Gamma^0$ such that $d(f\ B) = 0$, then there is a unique morphism in `dgalg` making the diagram commute

$$\begin{array}{ccc} \Omega_B\ A & \overset{\exists f_*}{\dashrightarrow} & \Gamma \\ \uparrow & & \uparrow \\ A & \xrightarrow{\quad f \quad} & \Gamma^0 \end{array}$$

Because of this, we have an isomorphism in `dgalg`

$$\Omega_B\ A = \frac{\Omega A}{\Omega A (dB) \Omega A}$$

The homology of the relative differential complex will be denoted by $H^n_B\ A$ and an algebra A is said to be *B-relative contractible* if $H^0_B\ A = B$ and $H^n_B\ A = 0$ for all $n \geq 1$.

EXAMPLE 17. The path algebra of a finite quiver on k-vertices is C_k-relative contractible. We claim that a basis for $\Omega^n_{C_k}\ \langle Q \rangle$ is given by the elements

$$p_0 dp_1 \ldots dp_n$$

where p_i is an oriented path in the quiver such that $l(p_0) \geq 0$ and $l(p_i) \geq 1$ (where $l(p)$ is the length of the path p) for $1 \leq i \leq n$ and such that the starting point of p_i is the endpoint of p_{i+1} for all $1 \leq i \leq n-1$. Clearly $l(p_i) \geq 1$ when $i \geq 1$ or p_i would be a vertex-idempotent whence in C_k. Let v be the starting point of p_i and w the end point of p_{i+1} and assume that $v \neq w$, then

$$p_i \otimes_B p_{i+1} = p_i v \otimes_B w p_{i+1} = p_i v w \otimes_B p_{i+1} = 0$$

from which the claim follows.

But then we can define a C_k-derivation E on $\langle Q \rangle$ by $E(v_i) = 0$ for all vertex-idempotents and $E(a) = a$ for all arrows in Q. By induction on the length $l(p)$ of a path p in the quiver Q we verify that $E(p) = l(p)p$. Using the description of a basis of $\Omega^n_{C_k}\ \langle Q \rangle$, we can repeat the argument of example 16. If follows that

$$H^0_{C_k}\ \langle Q \rangle = C_k \qquad \text{and} \qquad H^n_{C_k}\ \langle Q \rangle = 0$$

for all $n \geq 1$.

CHAPTER 2

Thickenings

"The naive aim of noncommutative algebraic geometry would be to associate to the surjection $R \twoheadrightarrow R_{ab}$ an embedding of $\texttt{spec}\ R_{ab}$ *into some noncommutative space* $\texttt{spec}\ R$*. The essence of our perturbative approach is not to worry about the whole* $\texttt{spec}\ R$ *but concentrate on the formal neighborhood of* $\texttt{spec}\ R_{ab}$ *in* $\texttt{spec}\ R$*."*

Mikhail Kapranov in [**27**]

For $A \in \texttt{alg}$ we denote with $\texttt{rep}A$ the Abelian category of all finite dimensional representations of A . We use the dimension function to decompose

$$\texttt{rep}A = \bigsqcup_n \texttt{rep}_n A$$

where $\texttt{rep}_n A$ is the affine scheme of n-dimensional representations of A. We will see that if A is $\texttt{alg}$-smooth, then each $\texttt{rep}_n A$ is a smooth reduced variety. In this way we view $\texttt{alg}$-smooth algebras as machines producing a family of smooth (affine) varieties. For general A however, the scheme structure of $\texttt{rep}_n A$ will be important in chapter 3 to reconstruct certain Cayley-Hamilton quotients from it.

We will introduce the coordinate ring of the representation scheme $\texttt{rep}_n A$ as the Abelianization of the *n-th root algebra* $\sqrt[n]{A}$ represents the functor

$$\texttt{alg} \longrightarrow \texttt{sets} \qquad B \mapsto Hom_{\texttt{alg}}(A, M_n(B))$$

If A is $\texttt{alg}$-smooth, then so is $\sqrt[n]{A}$ giving yet another method to construct $\texttt{alg}$-smooth algebras. Moreover, these algebras form a semigroup. By this we mean that there are connecting algebra morphisms

$$\sqrt[k]{A} \longrightarrow \sqrt[k_1]{A} * \ldots * \sqrt[k_r]{A}$$

whenever $k = \sum k_i$. Abelianizing these morphisms we will obtain the sum maps on the representation schemes. There is also a natural GL_n action on $\sqrt[n]{A}$ which after Abelianization gives a GL_n-action on $\texttt{rep}_n A$ the orbits of which are precisely the isomorphism classes of n-dimensional representations.

We will endow $\texttt{rep}_n A$ with a sheaf $\mathcal{O}^{\mu}_{\sqrt[n]{A}}$ of noncommutative algebras, which encodes all algebra morphisms $A \longrightarrow M_n(B)$ when B is a noncommutative infinitesimal extension of a commutative algebra. For A an $\texttt{alg}$-smooth algebra, this construction coincides with the formal structure of Mikhail Kapranov [**27**]. This associates to an affine $\texttt{commalg}$-smooth algebra C (which we know is not $\texttt{alg}$-smooth, unless the Krull dimension is one) a $\texttt{thick}$-smooth algebra A with Abelianization C.

2.1. Representing algebras.

In this section we will introduce and study the n-th root algebra $\sqrt[n]{A}$ representing the functor

$$\texttt{alg} \longrightarrow \texttt{sets} \qquad B \mapsto Hom_{\texttt{alg}}(A, M_n(B))$$

If A is $\texttt{alg}$-smooth, then so is $\sqrt[n]{A}$ giving yet another method to construct $\texttt{alg}$-smooth algebras. Moreover, there are connecting algebra morphisms

$$\sqrt[k]{A} \longrightarrow \sqrt[k_1]{A} * \ldots * \sqrt[k_r]{A}$$

whenever $k = \sum k_i$. These morphisms will induce the sum maps on the representation schemes.

Definition 17. A functor $F : \texttt{alg} \longrightarrow \texttt{sets}$ is said to be *representable* by the algebra D if and only if F is isomorphic to the functor

$$Hom_{\texttt{alg}}(D, -) \; : \; \texttt{alg} \longrightarrow \texttt{sets}$$

which assigns to a $\mathbb{C}$-algebra B the set of $\mathbb{C}$-algebra morphisms $Hom_{\texttt{alg}}(D, B)$ and to an algebra morphism $B \xrightarrow{f} B'$ the mapping

$$Hom_{\texttt{alg}}(D, B) \xrightarrow{f \circ -} Hom_{\texttt{alg}}(D, B')$$

obtained by composition.

Theorem 10 (Bergman). *The functor*

$$Hom_{\texttt{alg}}(A, M_n(-)) \; : \; \texttt{alg} \longrightarrow \texttt{sets} \qquad B \mapsto Hom_{\texttt{alg}}(A, M_n(B))$$

*is represented by an algebra $\sqrt[n]{A}$, the n-*th root *of A.*

That is, for any $\mathbb{C}$-algebra B, there is a functorial one-to-one correspondence between the sets

$$\begin{cases} \mathbb{C}\text{-}algebra\ maps & A \longrightarrow M_n(B) \\ \mathbb{C}\text{-}algebra\ maps & \sqrt[n]{A} \longrightarrow B \end{cases}$$

Proof. Consider the canonical embedding $M_n(\mathbb{C}) \overset{i}{\hookrightarrow} A * M_n(\mathbb{C})$ and define

$$\sqrt[n]{A} = C_{M_n(\mathbb{C})}(A * M_n(\mathbb{C})) = \{ r \in A * M_n(\mathbb{C}) \;\mid\; ri(m) = i(m)r \; \forall m \in M_n(\mathbb{C}) \}$$

One can use the *separability idempotent* $\sum_{i=1}^n e_{ij} \otimes e_{ji} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^{op}$ to prove that for any $\mathbb{C}$-algebra map $M_n(\mathbb{C}) \longrightarrow R$ we have that $M_n(C_{M_n(\mathbb{C})}(R)) \simeq R$. In the special of interest to us we have

$$M_n(\sqrt[n]{A}) \simeq A * M_n(\mathbb{C})$$

We claim that $\sqrt[n]{A}$ represents the functor $Hom_{\texttt{alg}}(A, M_n(-))$. If $\sqrt[n]{A} \xrightarrow{f} B$ is an algebra map, we obtain an algebra map $A \longrightarrow M_n(B)$ by composition

$$A \xrightarrow{id_A * 1} A * M_n(\mathbb{C}) \simeq M_n(\sqrt[n]{A}) \xrightarrow{M_n(f)} M_n(B)$$

Conversely, given an algebra map $A \xrightarrow{g} M_n(B)$ and the canonical map $M_n(\mathbb{C}) \xrightarrow{i} M_n(B)$ which centralizes B in $M_n(B)$. Then, by the universal property of free algebra products we have an algebra map $A * M_n(\mathbb{C}) \xrightarrow{g * i} M_n(B)$ and restricting to $\sqrt[n]{A}$ this maps factors

$$\begin{array}{ccc} A * M_n(\mathbb{C}) & \xrightarrow{g*i} & M_n(B) \\ \uparrow & & \uparrow \\ \sqrt[n]{A} & \dashrightarrow & B \end{array}$$

One verifies that these two operations are each others inverses. $\square$

EXAMPLE 18. The n-th root of $\langle m \rangle$, $\sqrt[n]{\langle m \rangle} \simeq \langle mn^2 \rangle$.

Assign to an algebra map $\langle m \rangle = \mathbb{C}\langle x_1, \ldots, x_m \rangle \xrightarrow{f} M_n(B)$ the algebra map

$$\langle mn^2 \rangle = \mathbb{C}\langle x_{ij,k} \mid 1 \leq i,j \leq n, 1 \leq k \leq m \rangle \longrightarrow B$$

by sending the variable $x_{ij,k}$ to the (i,j)-entry of $f(x_k) \in M_n(B)$.

Conversely, assign to an algebra map $\langle mn^2 \rangle \xrightarrow{g} B$ the algebra map $\langle m \rangle \xrightarrow{f} M_n(B)$ defined by

$$f(x_k) = \begin{bmatrix} g(x_{11,k}) & \ldots & g(x_{1n,k}) \\ \vdots & & \vdots \\ g(x_{n1,k}) & \ldots & g(x_{nn,k}) \end{bmatrix}$$

and verify that both operations are each others inverses.

Taking n-th roots is yet another method to construct new `alg`-smooth algebras.

THEOREM 11. *(1) If A is an affine $\mathbb{C}$-algebra, then so is $\sqrt[n]{A}$.*
(2) If A is `alg`-smooth, then so is $\sqrt[n]{A}$.

PROOF. (1) : Consider the canonical map

$$A \xrightarrow{id_A * \mathbb{1}_n} A * M_n(\mathbb{C}) \simeq M_n(\sqrt[n]{A})$$

By the universal property of the construction it is clear that the matrix entries of $id_A * \mathbb{1}_n(a)$ for all $a \in A$ generate the algebra $\sqrt[n]{A}$ as a $\mathbb{C}$-algebra. Hence, if A is generated by at most m elements, then $\sqrt[n]{A}$ is generated by at most mn^2 elements.

(2) : $M_n(\mathbb{C})$ is `alg`-smooth whence so is $A * M_n(\mathbb{C})$ by theorem 3. As $\sqrt[n]{A}$ is Morita equivalent to $A * M_n(\mathbb{C})$, it follows again from theorem 3 that $\sqrt[n]{A}$ is `alg`-smooth. $\square$

EXAMPLE 19. (The n-th root of a path algebra of a finite quiver Q on k vertices) Consider the extended quiver $Q^{(n)}$ on the left of figure 1 That is, add to the vertices and arrows of Q one extra vertex v_0 and for every vertex v_i in Q add n directed arrows from v_0 to v_i. We will denote the j-th arrow $1 \leq j \leq n$ from v_0 to v_i by x_{ij}. Consider the morphism between projective left $\langle Q^{(n)} \rangle$-modules

$$P_1 \oplus P_2 \oplus \ldots \oplus P_k \xrightarrow{\sigma} \underbrace{P_0 \oplus \ldots \oplus P_0}_{n}$$

determined by the matrix

$$M_\sigma = \begin{bmatrix} x_{11} & \ldots & \ldots & x_{1n} \\ \vdots & & & \vdots \\ x_{k1} & \ldots & \ldots & x_{kn} \end{bmatrix}.$$

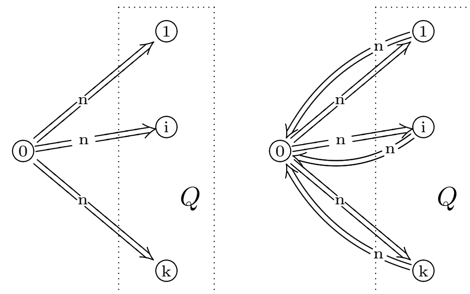


FIGURE 1. The extended quiver $Q^{(n)}$ and the universal localization.

Consider the universal localization $\langle Q^{(n)}\rangle_\sigma$, that is, add for each vertex v_i in Q another n arrows y_{ij} with $1 \leq j \leq n$ from v_i to v_0 as on the right of figure 1. With these arrows y_{ij} one forms the $n \times k$ matrix

$$N_\sigma = \begin{bmatrix} y_{11} & \dots & y_{k1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ y_{1n} & \dots & y_{kn} \end{bmatrix}$$

and the universal localization $\langle Q^{(n)}\rangle_\sigma$ is described by the relations

$$M_\sigma.N_\sigma = \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_k \end{bmatrix} \quad \text{and} \quad N_\sigma.M_\sigma = \begin{bmatrix} v_0 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & v_1 \end{bmatrix}.$$

We can now determine the n-th root of the path algebra

$$\sqrt[n]{\langle Q\rangle} = v_0 \ \langle Q^{(n)}\rangle_\sigma \ v_0.$$

The right hand side is generated by all the oriented cycles in $Q^{(n)}_\sigma$ starting and ending in v_0 and is therefore generated by the $y_{ip}x_{iq}$ and the $y_{ip}ax_{jq}$ where a is an arrow in Q starting in v_j and ending in v_i. For an algebra morphism

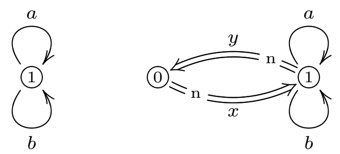
$$\langle Q\rangle \xrightarrow{\phi} M_n(B)$$

there is an algebra morphism

$$v_0 \ \langle Q^{(n)}\rangle_\sigma \ v_0 \xrightarrow{\psi} B$$

by sending $y_{ip}ax_{jq}$ to the (p,q)-entry of the $n \times n$ matrix $\phi(a)$ and $y_{ip}x_{iq}$ to the (p,q)-entry of $\phi(v_i)$. The defining relations among the x_{ip} and y_{iq} imply that ψ is indeed an algebra morphism. For example, consider the special case $\langle 2\rangle = \mathbb{C}\langle a,b\rangle$, that is the path algebra of the quiver on the left of figure 2 In order to describe $\sqrt[n]{\langle 2\rangle}$ we consider the quiver with relations as on the right of figure 2. We see that the algebra of oriented cycles in v_0 in this quiver with relations is isomorphic to the free algebra in $2n^2$ free variables

$$\mathbb{C}\langle y_1ax_1, \dots, y_nax_n, y_1bx_1, \dots, y_nbx_n\rangle$$



$$: \quad y_i x_j = \delta_{ij} v_0, \qquad \sum_i x_i y_i = v_1.$$

FIGURE 2. The quiver and extended quiver for $\langle 2 \rangle = \mathbb{C}\langle x, y \rangle$.

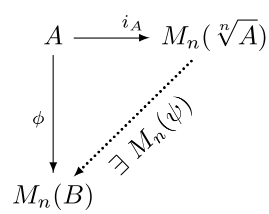
which agrees with the description of $\sqrt[n]{\langle 2 \rangle}$ given in example 18.

THEOREM 12. *There is a natural action of GL_n by algebra automorphisms on the n-th root algebra $\sqrt[n]{A}$.*

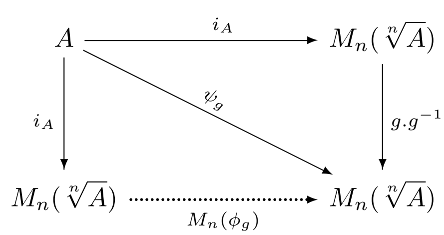
PROOF. The natural map $A \xrightarrow{id_A * 1} A * M_n(\mathbb{C})$ gives a canonical $\mathbb{C}$-algebra map

$$A \xrightarrow{i_A} M_n(\sqrt[n]{A}) \simeq A * M_n(\mathbb{C})$$

with the following universal property. For any $\mathbb{C}$-algebra morphism $A \xrightarrow{\phi} M_n(B)$, there is a $\mathbb{C}$-algebra map $\sqrt[n]{A} \xrightarrow{\psi} B$ completing the commuting diagram



For $g \in GL_n$ we consider conjugation on the first component of $M_n(\sqrt[n]{A}) = M_n(\mathbb{C}) \otimes \sqrt[n]{A}$. Then, g acts on $\sqrt[n]{A}$ via the automorphism $\sqrt[n]{A} \xrightarrow{\phi_g} \sqrt[n]{A}$ corresponding the the composition ψ_g



□

EXAMPLE 20. The GL_n-action on $\sqrt[n]{\langle m \rangle}$. We have seen that $\sqrt[n]{\langle m \rangle} = \langle mn^2 \rangle = \mathbb{C}\langle x_{11,1}, \ldots, x_{nn,m} \rangle$. The universal map $\langle m \rangle \xrightarrow{i} M_n(\sqrt[n]{\langle m \rangle})$ is given by

$$x_k \mapsto \begin{bmatrix} x_{11,k} & \ldots & x_{1n,k} \\ \vdots & & \vdots \\ x_{n1,k} & \ldots & x_{nn,k} \end{bmatrix}$$

It follows that the action of $g \in GL_n$ on $\sqrt[n]{\langle m \rangle}$ is given by the automorphism sending the variable $x_{ij,k}$ to the (i,j)-th entry of the matrix

$$g.\begin{bmatrix} x_{11,k} & \dots & x_{1n,k} \\ \vdots & & \vdots \\ x_{n1,k} & \dots & x_{nn,k} \end{bmatrix}.g^{-1}$$

The connecting morphisms compatible with the GL_n-actions are induced by a canonical map between free products of roots. This map, in turn, follows from the universal property of $\sqrt[n]{A}$.

There is some arithmetic associated with the root construction.

THEOREM 13. *For all strictly positive natural numbers a_i, k_i we have*

(1) *For $k = \sum_i a_i k_i$ there is a* connecting morphism

$$\sqrt[k]{A} \xrightarrow{c} \sqrt[k_1]{A} * \sqrt[k_2]{A} * \dots * \sqrt[k_z]{A}$$

(2) *For $k = \prod_i k_i$ there is a natural isomorphism*

$$\sqrt[k]{A} \longrightarrow \sqrt[k_1]{\sqrt[k_2]{\dots \sqrt[k_z]{A}}}$$

PROOF. (1) : Assume $k = a_1k_1 + \dots + a_zk_z$ and consider the algebra

$$U = \sqrt[k_1]{A} * \sqrt[k_2]{A} * \dots * \sqrt[k_z]{A}$$

The canonical maps $\sqrt[k_i]{A} \xrightarrow{c_i} U$ correspond to algebra maps $A \xrightarrow{f_i} M_{k_i}(U)$. This gives algebra maps

$$A \xrightarrow{f_i \mathbb{1}_{a_i}} M_{a_i k_i}(U) \qquad a \mapsto \begin{bmatrix} f_i(a) & & 0 \\ & \ddots & \\ 0 & & f_i(a) \end{bmatrix}$$

Consider the algebra map

$$A \xrightarrow{f} M_k(U) \qquad a \mapsto \begin{bmatrix} f_1(a)\mathbb{1}_{a_1} & & 0 \\ & \ddots & \\ 0 & & f_z(a)\mathbb{1}_{a_z} \end{bmatrix}$$

which gives the required morphism $\sqrt[k]{A} \longrightarrow U$.

(2) : This follows from the defining property of $\sqrt[n]{A}$ using the canonical isomorphism $M_{ab}(B) \simeq M_a(M_b(B))$. □

The n-th root algebra $\sqrt[n]{A}$ is a fairly mysterious ring, the precise structure of which is obscure. An intriguing property was proved by A. Schofield [**60**, Thm. 2.19].

THEOREM 14 (Schofield). *For any algebra $A \in$ `alg`, the n-th root algebra $\sqrt[n]{A}$ is a domain.*

PROOF. Assume $a, b \in \sqrt[n]{A}$ with $ab = 0$. By Morita equivalence, we may regard a and b as endomorphisms of the induced projective module

$$P = (M_n(\mathbb{C}) * A) \otimes_{M_n(\mathbb{C})} S$$

where S is the simple module of $M_n(\mathbb{C})$. By the *coproduct theorems* of George Bergman (see [**2**] or [**60**, Thms. 2.1,2.2,2.3]) `Im` b is an induced module (being a submodule of an induced module). Moreover, there is an isomorphism of induced modules $P' \longrightarrow P$ such that the composition with the surjection $P \twoheadrightarrow \texttt{Im}\, b$ is an induced map $P' \longrightarrow \texttt{Im}\, b$. Because $(M_n(\mathbb{C}) * A) \otimes_{M_n(\mathbb{C})} S$ is the only representation of P as an induced module, we have a commuting diagram

$$\begin{array}{ccccc} P & \xrightarrow{\quad a \quad} & P & \xrightarrow{\quad b \quad} & P \\ \downarrow & & \| & & \uparrow \\ M_n(\mathbb{C}) * A \otimes_{M_n} Q_1 & \xrightarrow{id \otimes \alpha} & M_n(\mathbb{C}) * A \otimes_{M_n} S & \xrightarrow{id \otimes \alpha} & M_n(\mathbb{C}) * A) \otimes_{M_n} Q_2 \end{array}$$

where Q_1, Q_2 are (projective) $M_n(\mathbb{C})$-modules. The only possibility for the $M_n(\mathbb{C})$-morphism

$$Q_1 \xrightarrow{\alpha} S \xrightarrow{\beta} Q_2$$

to be zero is that either α or β is the zero map. But this implies that a or b must be zero. $\square$

2.2. Representation schemes.

In this section we restrict attention to algebra morphisms from A to $M_n(C)$ when C is a commutative $\mathbb{C}$-algebra. This functor is representable by an affine scheme $\texttt{rep}_n A$, the scheme of n-dimensional representations of A. The coordinate ring $\mathbb{C}[\texttt{rep}_n A]$ is the Abelianization of $\sqrt[n]{A}$. Moreover, the natural GL_n-action as well as the connecting sum maps are induced by those on $\sqrt[n]{A}$.

Definition 18. A functor $\texttt{af} : \texttt{commalg} \longrightarrow \texttt{sets}$ is said to be an *affine scheme* if there is an affine commutative $\mathbb{C}$-algebra D which represents $\texttt{af}$, that is, $\texttt{af}$ is isomorphic to the functor

$$Hom_{\texttt{commalg}}(D, -) \; : \; \texttt{commalg} \longrightarrow \texttt{sets}$$

The algebra D is then said to be the *coordinate ring* of the scheme $\texttt{af}$ and will be denoted by $\mathbb{C}[\texttt{af}]$.

Definition 19. Let A be an affine $\mathbb{C}$-algebra. The *n-th representation functor* of A is the functor

$$\texttt{rep}_n A \; : \; \texttt{commalg} \longrightarrow \texttt{sets}$$

which assigns to a commutative $\mathbb{C}$-algebra C the set of all $\mathbb{C}$-algebra morphisms $A \longrightarrow M_n(C)$. Equivalently, $\texttt{rep}_n\, A$ is the set of all left $A \otimes C$-module structures on the free C-module $C^{\oplus n}$ of rank n. The correspondence is given by defining a module structure on $C^{\oplus n}$ by left multiplication

$$(a \otimes c).\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \phi_C(a) \begin{bmatrix} cc_1 \\ \vdots \\ cc_n \end{bmatrix}$$

Theorem 15. *If A is an affine $\mathbb{C}$-algebra, then the Abelianization of the n-th root $\sqrt[n]{A}$ represents the functor $\texttt{rep}_n A$, that is,*

$$\mathbb{C}[\texttt{rep}_n A] \simeq \frac{\sqrt[n]{A}}{[\sqrt[n]{A}, \sqrt[n]{A}]}$$

Proof. By theorem 11 $\sqrt[n]{A}$ is an affine $\mathbb{C}$-algebra representing the functor

$$Hom_{\mathtt{alg}}(A,-)\ :\ \mathtt{alg} \longrightarrow \mathtt{sets}$$

Therefore, its Abelianization represents the functor

$$Hom_{\mathtt{alg}}(A,-)\ :\ \mathtt{commalg} \longrightarrow \mathtt{sets}$$

which is the functor $\mathtt{rep}_n\ A$. □

Example 21. We know that $\sqrt[n]{\langle m\rangle} = \langle mn^2\rangle = \mathbb{C}\langle x_{11,1},\ldots,x_{nn,m}\rangle$. Therefore, the Abelianization is the polynomial ring $\mathbb{C}[x_{11}(1),\ldots,x_{nn}(m)]$ in the mn^2 commuting variables $x_{ij}(k)$ for $1 \leq i,j \leq n$ and $1 \leq k \leq m$. The representation scheme $\mathtt{rep}_n\langle m\rangle$ is the affine space of dimension mn^2

$$M_n^m = \underbrace{M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})}_{m}$$

The coordinate functions of the k-th component are given by the entries of the *generic $n \times n$ matrix*

$$X_k = \begin{bmatrix} x_{11}(k) & \ldots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \ldots & x_{nn}(k) \end{bmatrix}$$

The functorial construction and the foregoing example give us a method to compute the coordinate ring of $\mathtt{rep}_n A$ for any affine $\mathbb{C}$-algebra A.

Example 22. If A has a generating set $\{a_1,\ldots,a_m\}$, then we have a presentation

$$A \simeq \frac{\mathbb{C}\langle x_1,\ldots,x_m\rangle}{I_A} \simeq \frac{\langle m\rangle}{I_A}$$

where I_A is the twosided ideal of relations holding among the a_i. Consider the ideal $I_A(n)$ of the polynomial ring $\mathbb{C}[M_n^d] = \mathbb{C}[\mathtt{rep}_n\langle m\rangle]$ in the variables $x_{ij}(k)$ generated by all the entries of the matrices

$$r(X_1,\ldots,X_m) \in M_n(\mathbb{C}[\mathtt{rep}_n\langle m\rangle])$$

for all $r(x_1,\ldots,x_d) \in I_A$. It follows from the functorial description of the n-th root that

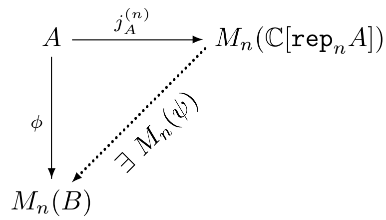
$$\mathbb{C}[\mathtt{rep}_n A] = \frac{\sqrt[n]{A}}{[\sqrt[n]{A},\sqrt[n]{A}]} = \frac{\mathbb{C}[x_{11}(1),\ldots,x_{nn}(m)]}{I_A(n)}$$

Even when A is not finitely presented, the ideals $I_A(n)$ are always finitely generated. In general however, the ideal $I_A(n)$ need not be radical, so the functor $\mathtt{rep}_n A$ is not always determined by the set of zeroes of $I_A(n)$ in the affine space M_n^m.

Theorem 16. *Composing the universal map of the n-th root with Abelianization we have a universal algebra map*

$$j_A^{(n)}\ :\ A \xrightarrow{i_A} M_n(\sqrt[n]{A}) \twoheadrightarrow M_n(\mathbb{C}[\mathtt{rep}_n A])$$

For every $\mathbb{C}$-algebra morphism $A \xrightarrow{\phi} M_n(B)$ with B a commutative $\mathbb{C}$-algebra, there is a morphism of commutative $\mathbb{C}$-algebras $\mathbb{C}[\mathtt{rep}_n A] \xrightarrow{\psi} B$ making the diagram

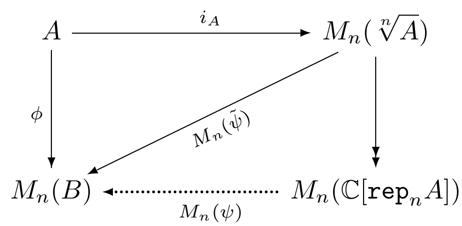


commute. Moreover, $\mathtt{rep}_n A$ has a natural action of the algebraic group scheme $\mathtt{GL}_n$. That is, for all commutative algebras C there is a group action

$$GL_n(C) \times \mathtt{rep}_n A(C) \longrightarrow \mathtt{rep}_n A(C)$$

and the orbits under this action are precisely the isomorphism classes of left $A \otimes C$-module structures on $C^{\oplus n}$.

Proof. By the universal property of the map i_A there is a $\mathbb{C}$-algebra morphism $\tilde{\psi} : \sqrt[n]{A} \longrightarrow B$ making the upper left triangle of the diagram below commute



Because B is commutative, the map $\tilde{\psi}$ factors through the Abelianization $\frac{\sqrt[n]{A}}{[\sqrt[n]{A}, \sqrt[n]{A}]} = \mathbb{C}[\mathtt{rep}_n A]$ giving the required map ψ.

By theorem 12 there is an action of GL_n by algebra automorphisms on $\sqrt[n]{A}$. As any algebra automorphism preserves the commutator ideal, it induces an action on the Abelianization which is $\mathbb{C}[\mathtt{rep}_n\ A]$. The action of $GL_n(C)$ is given by basechange in the free C-module $C^{\oplus n}$ whence orbits correspond to isomorphism classes of $A \otimes C$-modules. □

Example 23. It follows from example 20 that the action of GL_n on $\mathtt{rep}_n\langle m\rangle = M_n^m$ is *simultaneous conjugation*

$$GL_n \times M_n^m \longrightarrow M_n^m \qquad (g, (Y_1, \dots, Y_m)) \mapsto (gY_1g^{-1}, \dots, gY_mg^{-1})$$

Theorem 17. *Let $k = \sum a_i k_i$ be a solution in strict positive integers, then there is a connecting morphism of representation schemes*

$$\mathtt{rep}_{k_1} A \times \mathtt{rep}_{k_2} A \times \dots \times \mathtt{rep}_{k_z} A \longrightarrow \mathtt{rep}_k A$$

compatible with the actions.

Proof. Abelianizing the connecting morphisms of theorem 13

$$\sqrt[k]{A} \longrightarrow \sqrt[k_1]{A} * \sqrt[k_2]{A} * \dots * \sqrt[k_z]{A}$$

we obtain a morphism of commutative algebras

$$\frac{\sqrt[k]{A}}{[\sqrt[k]{A},\sqrt[k]{A}]} \longrightarrow \frac{\sqrt[k_1]{A}}{[\sqrt[k_1]{A},\sqrt[k_1]{A}]} \otimes \frac{\sqrt[k_2]{A}}{[\sqrt[k_2]{A},\sqrt[k_2]{A}]} \otimes \ldots \otimes \frac{\sqrt[k_z]{A}}{[\sqrt[k_z]{A},\sqrt[k_z]{A}]}$$

and this morphism gives us a morphism of the affine schemes

$$\mathtt{rep}_{k_1} A \times \mathtt{rep}_{k_2} A \times \ldots \times \mathtt{rep}_{k_z} A \longrightarrow \mathtt{rep}_k A$$

For a commutative algebra C this map sends a z-tuple

$$(V_1, \ldots, V_z) \in \mathtt{rep}_{k_1} A(C) \times \ldots \times \mathtt{rep}_{k_z} A(C)$$

to the $A \otimes C$-module structure

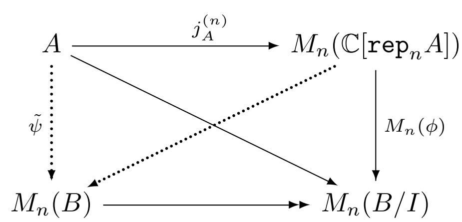
$$V_1^{\oplus a_1} \oplus \ldots \oplus V_z^{\oplus a_z}$$

Hence the image consists of *decomposable* modules. Conversely, if we bring in the $GL_n(C)$-action we see that a module structure on $C^{\oplus n}$ is decomposable if and only if its orbit contains an image of one of these connecting morphisms. $\square$

Affine $\mathtt{alg}$-smooth algebras are machines producing an infinite system of affine smooth varieties $\mathtt{rep}_n\ A$, $n = 1, 2, \ldots$.

THEOREM 18. *If A is an affine* $\mathtt{alg}$*-smooth $\mathbb{C}$-algebra, then the n-th representation scheme $\mathtt{rep}_n A$ is a smooth affine variety (in particular, it is reduced) for all n.*

PROOF. By Grothendieck's criterium we have to prove that the coordinate ring $\mathbb{C}[\mathtt{rep}_n A] = \frac{\sqrt[n]{A}}{[\sqrt[n]{A},\sqrt[n]{A}]}$ is a $\mathtt{commalg}$-smooth $\mathbb{C}$-algebra. That is, we have to find an algebra lift $\tilde{\phi}$ for every algebra map $\mathbb{C}[\mathtt{rep}_n A] \xrightarrow{\phi} B/I$ with (B, I) a test-object in $\mathtt{commalg}$. Consider the diagram



where $j_A^{(n)}$ is the n-th universal map and where ψ is the composition $M_n(\phi) \circ j_A^{(n)}$. Because A is $\mathtt{alg}$-smooth, we have an algebra lift

$$A \xrightarrow{\tilde{\psi}} M_n(B)$$

making the lower left triangle commute. By the universal property of the map $j_A^{(n)}$ we then deduce the existence of an algebra map of commutative $\mathbb{C}$-algebras

$$\mathbb{C}[\mathtt{rep}_n A] \xrightarrow{\tilde{\phi}} B$$

making the upper left triangle commute. But then it follows that the lower right triangle commutes and hence that $\tilde{\phi}$ is an algebra lift for ϕ. $\square$

EXAMPLE 24. Let A be the finite dimensional semi-simple algebra

$$A = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C})$$

Because A is $\mathtt{alg}$-smooth, $\mathtt{rep}_n A$ is reduced so we only have to study the $\mathbb{C}$-points.

A has precisely k distinct simple modules $\{S_1, \ldots, S_k\}$ of dimensions $\{d_1, \ldots, d_k\}$. Here, S_i can be viewed as column vectors of size d_i on which the component $M_{d_i}(\mathbb{C})$ acts by left multiplication and the other factors act trivially. Because A is semi-simple every n-dimensional A-representation M is isomorphic to

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

for certain multiplicities e_i satisfying the numerical condition

$$n = e_1 d_1 + \ldots + e_k d_k$$

Therefore, $\mathtt{rep}_n\ A$ is the disjoint union of a finite number of (necessarily closed) GL_n-orbits, each determined by an integral vector $(e_1, \ldots, e_k)$ satisfying the condition.

To understand the structure of the orbits we need to determine the stabilizer subgroup of M, that is, the group of A-module isomorphisms of M. By *Schur's lemma* we know that

$$Hom_A(S_i, S_i) \simeq \mathbb{C} id_{S_i} \quad \text{and} \quad Hom_A(S_i, S_j) = 0 \text{ when } i \neq j$$

Choose a basis of M by first fixing a basis for S_1 and taking e_1 copies of it, one for each of the S_1 components of M, then fixing a basis for S_2 and taking e_2 copies of it, one for each S_2 component, and so on. In this basis, any A-module isomorphism of M is an element of the stabilizer subgroup $Stab_{GL_n}(M)$

$$\begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{d_1}) & 0 & \ldots & 0 \\ 0 & GL_{e_2}(\mathbb{C} \otimes \mathbb{1}_{d_2}) & & 0 \\ \vdots & & \ddots & \ldots \\ 0 & 0 & \ldots & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{d_k}) \end{bmatrix} \hookrightarrow GL_n(\mathbb{C})$$

Therefore, the n-dimensional representation scheme of A decomposes into connected components, one for each solution $(e_1, \ldots, e_k)$ to the numerical condition $n = e_1 d_1 + \ldots + e_k d_k$

$$\mathtt{rep}_n\ A \simeq \bigsqcup_{(e_1, \ldots, e_k)} GL_n / (GL_{e_1} \times \ldots \times GL_{e_k})$$

Definition 20. Let Q be a finite quiver. A *representation* V of the quiver Q is given by

- a finite dimensional $\mathbb{C}$-vector space V_i for each vertex $v_i \in Q_v$, and
- a linear map $V_j \xleftarrow{V_a} V_i$ for every arrow $(j) \xleftarrow{a} (i)$ in Q_a.

If $dim\ V_i = d_i$ we call the integral vector $\alpha = (d_1, \ldots, d_k) \in \mathbb{N}^k$ the *dimension vector* of V and denote it with $dim\ V$.

The set $\mathtt{rep}_\alpha\ Q$ of all representations V of Q such that $dim(V) = \alpha$ is an affine space

$$\mathtt{rep}_\alpha\ Q = \bigoplus_{(j) \xleftarrow{a} (i)} M_{d_j \times d_i}(\mathbb{C}) \simeq \mathbb{C}^r$$

where $r = \sum_{a \in Q_a} d_{s(a)} d_{t(a)}$.

A *morphism* $V \xrightarrow{\phi} W$ between two representations V and W of Q is determined by a set of linear maps

$$V_i \xrightarrow{\phi_i} W_i \quad \text{for all vertices } v_i \in Q_v$$

satisfying the following compatibility conditions. For every arrow there is a commuting diagram $ⓙ \xleftarrow{a} ⓘ$ in Q_a

$$\begin{array}{ccc} V_i & \xrightarrow{V_a} & V_j \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_j} \\ W_i & \xrightarrow{W_a} & W_j \end{array}$$

Basechange in all the vertex spaces induces an action of the algebraic group $GL(\alpha) = GL_{d_1} \times \ldots \times GL_{d_k}$ on the affine space $\mathtt{rep}_\alpha Q$. That is, if $g = (g_1, \ldots, g_k) \in GL(\alpha)$ and if $V = (V_a)_{a \in Q_a}$ then $g.V$ is determined by the matrices

$$(g.V)_a = g_{t(a)} V_a g_{s(a)}^{-1}.$$

If V and W in $\mathtt{rep}_\alpha\, Q$ are isomorphic as representations of Q, such an isomorphism is determined by invertible matrices $g_i : V_i \longrightarrow W_i \in GL_{d_i}$ and therefore they belong to the same orbit under $GL(\alpha)$.

Example 25. Let $\alpha = (d_1, \ldots, d_k)$ be a dimension vector such that $n = |\alpha| = \sum_i d_i$. Fixing on ordering of the vertices and fixing a basis in every vertexspace we obtain an embedding of algebraic groups

$$GL(\alpha) = GL_{d_1} \times \ldots \times GL_{d_k} \hookrightarrow GL_n$$

Using this embedding we have an action of $GL(\alpha)$ on the product $GL_n \times \mathtt{rep}_\alpha Q$

$$h.(g, V) = (gh^{-1}, h.V)$$

and the *associated fiber bundle*

$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha Q$$

is the set of orbits under this action. It is a smooth affine variety.

We claim that the n-th representation scheme of $\langle Q \rangle$ decomposes

$$\mathtt{rep}_n \langle Q \rangle = \bigsqcup_{|\alpha| = n} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha Q$$

into smooth connected components.

Recall that $C_k \simeq \mathbb{C} \oplus \ldots \oplus \mathbb{C}$ is the subalgebra of $\mathbb{C}Q$ generated by the vertex idempotents. The inclusion $C_k \hookrightarrow \mathbb{C}Q$ induces a morphism

$$\mathtt{rep}_n \mathbb{C}Q \overset{\psi}{\twoheadrightarrow} \mathtt{rep}_n C_k = \bigsqcup_{|\alpha| = n} GL_n / GL(\alpha)$$

where the decomposition is given by the previous example. The α-component corresponds to the semisimple C_k-module $S_1^{\oplus d_1} \oplus \ldots \oplus S_k^{\oplus d_k}$ with S_i the simple one-dimensional module concentrated in vertex v_i. Take the point $p \in \mathtt{rep}_n C_k$

sending the vertex idempotent v_i to the matrix

$$v_i \mapsto \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

with 1's from position $\sum_{l=1}^{i-1} d_l + 1$ to position $\sum_{l=1}^{i} d_l$. The fiber $\psi^{-1}(p)$ consists of all representations of Q of dimension vector α. As the basespace is an orbit with stabilizer $GL(\alpha)$ and ψ is GL_n-equivariant it follows that the inverse image of the component $GL_n/GL(\alpha)$ is the homogeneous space

$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha Q$$

as claimed.

EXAMPLE 26. It is not always true that $\mathtt{rep}_n\ A$ is a reduced variety. Take

$$A = \frac{\mathbb{C}[x]}{(x^2)}$$

and consider the representation scheme $\mathtt{rep}_2\ A$. The coordinate ring $\mathbb{C}[\mathtt{rep}_2\ A]$ is the quotient of the polynomial ring $\mathbb{C}[x_1, x_2, x_3, x_4]$ by the ideal generated by the entries of the matrix

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^2 = \begin{bmatrix} x_1^2 + x_2x_3 & x_2(x_1 + x_4) \\ x_3(x_1 + x_4) & x_4^2 + x_2x_3 \end{bmatrix}$$

That is, the ideal with generators

$$I_A = (x_1^2 + x_2x_3, x_2(x_1 + x_4), x_3(x_1 + x_4), (x_1 - x_4)(x_1 + x_4))$$

The reduced variety of $\mathtt{rep}_2\ A$ consists of all matrices X such that $X^2 = 0$. Conjugating X to an upper triangular form we see that the variety is the union of two GL_2-orbits

$$\mathcal{O}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \cup \mathcal{O}(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$$

This variety is a cone in $\mathbb{C}^3$ with top the zero matrix and defining radical ideal

$$(x_1 + x_4, x_1^2 + x_2x_3)$$

I_A is properly contained in this ideal. Still, we have that

$$rad(I_A) = (x_1 + x_4, x_1^2 + x_3x_4)$$

because $\overline{x_1 + x_4}^3 = 0 \in \mathbb{C}[x_1, x_2, x_3, x_4]/I_A$.

EXAMPLE 27. We generalize the previous example and study the representation scheme $\mathtt{rep}_n\ \frac{\mathbb{C}[x]}{(x^r)}$. Any algebra morphism $\mathbb{C}[x] \longrightarrow M_n(\mathbb{C})$ is determined by the image of x, whence $\mathtt{rep}_n(\mathbb{C}[x]) = M_n(\mathbb{C})$. We know that conjugacy classes in

$M_n(\mathbb{C})$ are classified by the *Jordan normalform* . Let A is conjugated to a matrix in normalform

$$\begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_s} \end{bmatrix}$$

where J_i is a Jordan block of size d_i, hence $n = d_1 + d_2 + \ldots + d_s$. Then, the n-dimensional $\mathbb{C}[x]$-module M determined by A can be decomposed uniquely as

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_s$$

where M_i is a $\mathbb{C}[x]$-module of dimension d_i which is *indecomposable* , that is, cannot be decomposed as a direct sum of proper submodules.

Consider the quotient algebra $A = \mathbb{C}[x]/(x^r)$, then the ideal I_A of $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$ is generated by the n^2 entries of the matrix

$$\begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix}^r.$$

Observe that when J is a Jordan block of size d with eigenvalue zero we have

$$J^{d-1} = \begin{bmatrix} 0 & \ldots & 0 & d-1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix} \quad \text{and} \quad J^d = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}$$

Therefore, the representation scheme $\mathtt{rep}_n\ \mathbb{C}[x]/(x^r)$ is the union of all conjugacy classes of matrices having 0 as only eigenvalue and all of which Jordan blocks have size $\leq r$. Expressed in module theoretic terms, any n-dimensional $\mathbb{C}[x]/(x^r)$-module M is isomorphic to a direct sum of indecomposables

$$M = I_1^{\oplus e_1} \oplus I_2^{\oplus e_2} \oplus \ldots \oplus I_r^{\oplus e_r}$$

where I_j is the unique indecomposable j-dimensional $\mathbb{C}[x]/(x^r)$-module (corresponding to the Jordan block of size j). Of course, the multiplicities e_i of the factors must satisfy the equation

$$e_1 + 2e_2 + 3e_3 + \ldots + re_r = n$$

In M we can consider the subspaces for all $1 \leq i \leq r-1$

$$M_i = \{m \in M \mid x^i.m = 0\}$$

the dimension of which can be computed knowing the powers of Jordan blocks

$$t_i = dim_{\mathbb{C}}\ M_i = e_1 + 2e_2 + \ldots (i-1)e_i + i(e_i + e_{i+1} + \ldots + e_r)$$

Giving n and the $r-1$-tuple $(t_1, t_2, \ldots, t_{n-1})$ is the same as giving the multiplicities e_i because

$$\begin{cases} 2t_1 & = t_2 + e_1 \\ 2t_2 & = t_3 + t_1 + e_2 \\ 2t_3 & = t_4 + t_2 + e_3 \\ & \vdots \\ 2t_{n-2} & = t_{n-1} + t_{n-3} + e_{n-2} \\ 2t_{n-1} & = n + t_{n-2} + e_{n-1} \\ n & = t_{n-1} + e_n \end{cases}$$

Let n-dimensional $\mathbb{C}[x]/(x^r)$-modules M and M' be determined by the $r-1$-tuples $(t_1, \ldots, t_{r-1})$ respectively $(t'_1, \ldots, t'_{r-1})$ then we have that

$$\mathcal{O}(M') \hookrightarrow \overline{\mathcal{O}(M)} \quad \text{if and only if} \quad t_1 \leq t'_1, t_2 \leq t'_2, \ldots, t_{r-1} \leq t'_{r-1}$$

Therefore, we have an inverse order isomorphism between the orbits in $\mathtt{rep}_n(\mathbb{C}[x]/(x^r))$ and the $r-1$-tuples of natural numbers $(t_1, \ldots, t_{r-1})$ satisfying the following linear inequalities (which follow from the above system)

$$2t_1 \geq t_2, 2t_2 \geq t_3 + t_1, 2t_3 \geq t_4 + t_2, \ldots, 2t_{n-1} \geq n + t_{n-2}, n \geq t_{n-2}.$$

First, consider $r = 2$, then the orbits in $\mathtt{rep}_n\ \mathbb{C}[x]/(x^2)$ are parameterized by a natural number t_1 satisfying the inequalities $n \geq t_1$ and $2t_1 \geq n$, the multiplicities are given by $e_1 = 2t_1 - n$ and $e_2 = n - t_1$. Moreover, the orbit of the module $M(t'_1)$ lies in the closure of the orbit of $M(t_1)$ whenever $t_1 \leq t'_1$. That is, if $n = 2k + \delta$ with $\delta = 0$ or 1, then $\mathtt{rep}_n\ \mathbb{C}[x]/(x^2)$ is the union of $k+1$ orbits and the orbitclosures form a linear order as follows (from big to small)

$$I_1^{\delta} \oplus I_2^{\oplus k} \;\text{——}\; I_1^{\oplus \delta+2} \oplus I_2^{\oplus k-1} \;\text{——}\; \ldots \;\text{——}\; I_1^{\oplus n}$$

If $r = 3$, orbits in $\mathtt{rep}_n\ \mathbb{C}[x]/(x^3)$ are determined by couples of natural numbers (t_1, t_2) satisfying the following three linear inequalities

$$\begin{cases} 2t_1 & \geq t_2 \\ 2t_2 & \geq n + t_1 \\ n & \geq t_2 \end{cases}$$

For example, for $n = 8$ we obtain the situation of figure 3 Therefore, $\mathtt{rep}_8\ \mathbb{C}[x]/(x^3)$ consists of 10 orbits with orbitclosure diagram as in figure 3 (the nodes represent the multiplicities $[e_1 e_2 e_3]$). Here we used the equalities $e_1 = 2t_1 - t_2$, $e_2 = 2t_2 - n - t_1$ and $e_3 = n - t_2$. For general n and r this result shows that $\mathtt{rep}_n\ \mathbb{C}[x]/(x^r)$ is the closure of the orbit of the module with decomposition

$$M_{gen} = I_r^{\oplus e} \oplus I_s \quad \text{if} \quad n = er + s$$

EXAMPLE 28. For $A, A' \in \mathtt{alg}$ we have that

$$\mathtt{rep}_n A * A' \simeq \mathtt{rep}_n A \times \mathtt{rep}_n A'$$

Indeed, by the universal property of algebra free products, any algebra map $A * A' \longrightarrow M_n(C)$ with $C \in \mathtt{commalg}$ is determined by the restrictions $A \longrightarrow M_n(C)$ and $A' \longrightarrow M_n(C)$.

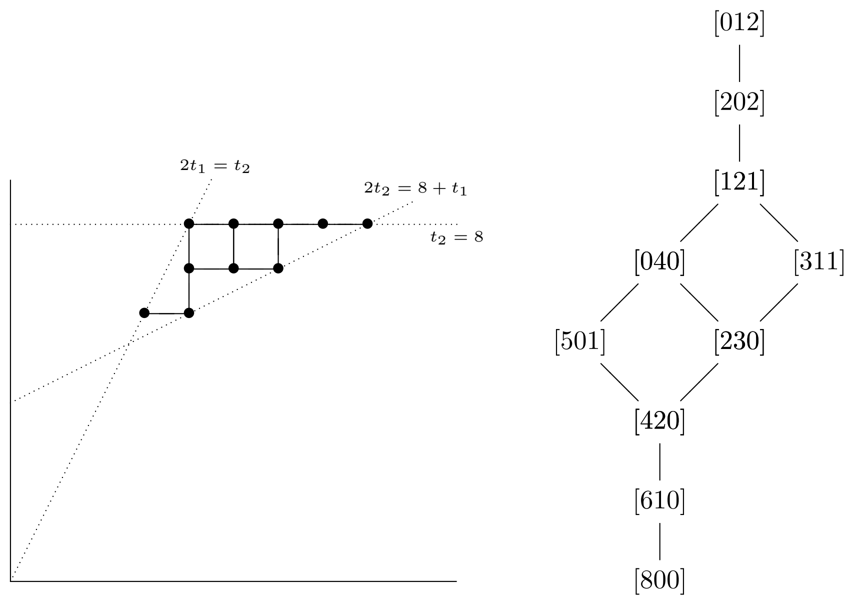


FIGURE 3. Inequalities and orbit closures in $\mathtt{rep}_8 \frac{\mathbb{C}[x]}{(x^3)}$.

Further, if Σ is a finite set of maps in $\mathtt{projmod}A$, then $\mathtt{rep}_n A_\Sigma$, the representation scheme of the universal localization of A at Σ is a Zariski open subscheme of $\mathtt{rep}_n A$. Note however that this open subscheme may be empty.

2.3. Formal structure.

In this section we will define the formal structure on $\mathtt{rep}_n A$. The motivation is that we want to endow $\mathtt{rep}_n A$ with a sheaf (for gluing purposes) of noncommutative algebras encoding as much information as possible about algebra morphisms from A to $M_n(B)$ with B a noncommutative algebra.

By microlocalization we will be able to recover all information for B a noncommutative infinitesimal extension of a commutative algebra. If A is $\mathtt{alg}$-smooth we will connect this sheaf to the canonical formal structure Mikhail Kapranov defined on affine smooth varieties.

DEFINITION 21. A commutative $\mathbb{C}$-algebra C is said to be a *Poisson algebra* provided there there is an alternating bilinear bracket $\{f,g\}$ on C, called the *Poisson bracket*, which satisfies the *Jacobi identity*

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \quad \forall f,g,h \in C$$

and is a *derivation* with respect to each argument, that is

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \quad \text{and} \quad \{fg,h\} = f\{g,h\} + \{f,h\}g$$

A Poisson algebra C is negatively graded if $C = \oplus_{k=0}^{\infty} C_{-k}$ is a graded commutative algebra and the decomposition into homogeneous components is compatible with the Poisson bracket

$$\{C_{-k}, C_{-l}\} \subset C_{-(k+l)}$$

Let `poisson` be the category of all commutative Poisson algebras with morphisms $\mathbb{C}$-algebra morphisms preserving the Poisson brackets.

We will assign to any $\mathbb{C}$-algebra A a Poisson algebra by considering the *commutator filtration.*

DEFINITION 22. For $A \in$ `alg` , A^{Lie} its natural Lie algebra structure defined by

$$[a, a'] = aa' - a'a$$

Let A^{Lie}_m be the subspace spanned by the expressions $[a_1, [a_2, \ldots, [a_m, a_{m+1}] \ldots]$ containing m instances of Lie brackets.

The *commutator filtration* of A is the (increasing) filtration by ideals $(F^k\ A)_{k \in \mathbb{Z}}$ with $F^k\ A = A$ for $d \in \mathbb{N}$ and

$$F^{-k}\ A = \sum_m \sum_{i_1+\ldots+i_m=k} AA^{Lie}_{i_1} A \ldots AA^{Lie}_{i_m} A$$

Observe that all $\mathbb{C}$-algebra morphisms preserve the commutator filtration.

The main properties of the commutator filtration are that for all $k, l \in \mathbb{N}$ we have

$$F^{-k}A.F^{-l}A \subset F^{-(k+l)}A \quad \text{and} \quad [F^{-k}A, F^{-l}A] \subset F^{-(k+l+1)}A$$

The first inclusion asserts that the commutator filtration is an algebra filtration, the second implies that the *associated graded* of the commutator filtration, that is,

$$\mathtt{gr}\ A = \oplus_{k=0}^{\infty} \frac{F^{-k}A}{F^{-(k+1)}A}$$

is a negatively graded commutative Poisson algebra with part of degree zero the Abelianization $A_{ab} = \frac{A}{[A,A]}$.

Indeed, define the *degree* of an element $a \in A$, $deg(a)$, to be the maximal k such that $a \in F^{-k}A$, then we define the *principal symbol* of an element of $a \in A$ to be the homogeneous element of degree $-deg(a)$ of $\mathtt{gr}\ A$

$$\sigma a = \overline{a} \in \frac{F^{-deg(a)}A}{F^{-(deg(a)+1)}A}$$

With these definition, define a Poisson bracket on the associated graded, let $f, g \in \mathtt{gr}\ A$ and take f', g' preimages of f and g and define

$$\{f, g\} = \sigma[f', g']$$

This definition does not depend on the choice of the preimages and is indeed a Poisson bracket.

We will assume throughout that the commutator filtration on A is *separated* , that is,

$$\bigcap_i F^{-i}\ A = 0$$

However, this is not always the case. In the exceptional cases one has to replace A by the quotient $\frac{A}{\cap_i F^{-i}\ A}$ in what follows.

EXAMPLE 29. Taking the total degree of an element in $\langle m \rangle$ or the length of a path in $\langle Q \rangle$ and observing that the minimal degree (length) of an element in F^{-i} is at least $i+1$, the commutator filtration on free algebras and on path algebras of quivers is separated.

EXAMPLE 30. Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra. Then, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. If $U(\mathfrak{g})$ denotes the *enveloping algebra* of $\mathfrak{g}$, then

$$F^{-i}\ U(\mathfrak{g}) = [U(\mathfrak{g}), U(\mathfrak{g})]$$

for all $i < 0$. Therefore, the commutator filtration on $U(\mathfrak{g})$ is *not* separated.

If the commutator filtration on A is separated, we construct a sheaf of non-commutative algebras $\mathcal{O}^{\mu}_A$ on the commutative affine scheme $\mathtt{spec}\ A_{ab} = \mathtt{rep}_1 A$ by *micro-localization*.

DEFINITION 23. Define the *Rees ring* of the commutator filtration to be the algebra

$$\tilde{A} = \bigoplus_{i \in \mathbb{Z}} (F^{-i}A)t^i \hookrightarrow A[t, t^{-1}]$$

where t is an extra central variable. The two basic properties of the Rees ring construction are

$$\frac{\tilde{A}}{(t)} \simeq \mathtt{gr}\ A \quad \text{and} \quad \frac{\tilde{A}}{(t-1)} \simeq A$$

Let π_d denote the gradation preserving quotient map $\tilde{A} \twoheadrightarrow \frac{\tilde{A}}{(t^d)}$.

Let S_c be a multiplicatively closed subset of A_{ab}, then

$$S = S_c + [A, A]$$

is a multiplicatively closed subset of A with $\sigma S = S_c$. Note that all elements of S have degree zero. Then, $\pi_1(S)$ is a left and right Ore set consisting of homogeneous elements in $\pi_1(\tilde{A}) = \mathtt{gr}\ A$.

Because one can lift Ore sets through nilpotent ideals, it follows that $\pi_d(S)$ is a left and right Ore set of homogeneous degree zero elements in $\pi_d(\tilde{A})$ for every d. Therefore, we have for every d a graded localization

$$\pi_d(S)^{-1}\frac{\tilde{A}}{(t^d)}$$

These algebras form an inverse system of graded algebras and we consider its inverse limit

$$Q^{\mu}_S(\tilde{A}) = \underset{\leftarrow}{lim}\ \pi_d(S)^{-1}\frac{\tilde{A}}{(t^d)}$$

The central element t acts torsion free on this graded algebra and the filtered algebra

$$Q^{\mu}_S(A) = \frac{Q^{\mu}_S(\tilde{A})}{(t-1)Q^{\mu}_S(\tilde{A})}$$

is called the *micro-localization* of A at the multiplicatively closed set S, see for example [**65**] for more details.

It follows from general theory, see for example [**65**], that the associated graded algebra of the micro-localization is isomorphic to the graded localization

$$\mathtt{gr}\ Q^{\mu}_S(A) \simeq \sigma(A)^{-1}\mathtt{gr}\ A = S_c^{-1}\mathtt{gr}\ A$$

Let $\mathtt{spec}\ A_{ab} = \mathtt{rep}_1 A$ be the affine scheme with coordinate ring $A_{ab} = \frac{A}{[A,A]} = \int_1 A$, the latter definition will be explained in the next chapter. Recall that the *Zariski topology* on $\mathtt{spec}\ A_{ab}$ has a basis of open sets

$$X(f) = \{P \text{ a prime ideal of } A_{ab} \text{ such that } f \notin P\ \}$$

Consider the multiplicatively closed set $\{1, f, f^2, \ldots\}$ in A_{ab} and the corresponding set $S_f = \{1, f, f^2, \ldots\} + [R, R]$.

DEFINITION 24 (Kapranov). The *formal structure* of the affine algebra A on the affine scheme of its Abelianization is the sheaf of noncommutative algebras $\mathcal{O}_A^\mu$ on $\mathtt{rep}_1 A$ defined by its sections on the basis open set $X(f)$

$$\Gamma(X(f), \mathcal{O}_A^\mu) = Q_{S_f}^\mu(A)$$

Let $\mathfrak{f}_d$ be the free Lie algebra in d variables $x_1, \ldots, x_d$. We will give an explicit Poincaré-Birkhoff-Witt basis for the enveloping algebra $U(\mathfrak{f}_d)$.

DEFINITION 25. Let $X_d = \{x_1, \ldots, x_d\}$. Order the variables by $x_1 < x_2 < \ldots < x_d$ and induce the alphabetic ordering on all the words in X_d, that is

$$u < v \qquad \begin{cases} v = ux \quad \mathit{or} \\ u = xau' \text{ and } v = xbv' \quad \text{with } a < b \end{cases}$$

for nonempty words x, words u' and v' and letters a en b. This is the total ordering on the words such as they would appear in a dictionary.

A *Lyndon word* is a nonempty word w such that w is smaller in the ordering than all its nontrivial right factors, that is if $w = uv$ for nonempty words u and v then $w < v$.

EXAMPLE 31. The Lyndon words in two variables $a < b$ of length ≤ 4 are in order

$$a < a^3b < a^2b < a^2b^2 < ab < ab^2 < ab^3 < b$$

DEFINITION 26. The *standard factorization* of a Lyndon word is a decomposition $w = uv$ where v is the smallest proper rightfactor of w. Inductively, associate a Lie element L_w of $\mathfrak{f}_d$ to w : if $w = a$ is a letter from $X_d = \{x_1, \ldots, x_d\}$ then $L_w = a$. Otherwise, for the standard factorization $w = uv$ of w we define $L_w = [L_u, L_v]$.

Observe that if w is a word of length l, then L_w involves $l - 1$ Lie brackets. With B_l we denote the set of Lie elements L_w where w is a Lyndon word of length l.

EXAMPLE 32. The Lyndon words of length 5 together with their standard factorization and the corresponding Lie algebra elements in $\mathfrak{f}_2$ are

$$\begin{aligned} a^4b = a(a^3b) &\quad = [a, [a, [a, [a, b]]]] \\ a^3b^2 = a(a^2b^2) &\quad = [a, [a, [[a, b], b]]] \\ a^2b^3 = a(ab^3) &\quad = [a, [[[a, b], b], b]] \\ ab^2ab = (ab^2)(ab) &\quad = [[[a, b], b], [a, b]] \\ abab^2 = (ab)(ab^2) &\quad = [[a, b], [a, [a, b]]] \\ ab^4 = (ab^3)b &\quad = [[[[a, b], b], b], b] \end{aligned}$$

It is well known, see for example [**57**, §4] that $B = \cup_{k \geq 1} B_k$ is an ordered $\mathbb{C}$-basis of the Lie algebra $\mathfrak{f}_d$ and that its enveloping algebra

$$U(\mathfrak{f}_d) = \mathbb{C}\langle x_1, \ldots, x_d \rangle = \langle d \rangle$$

is the free associative algebra in the variables x_i.

EXAMPLE 33. The commutator filtration on $\langle d\rangle$. Number the elements of $\cup_{k\geq 2}B_k$ according to the order $\{b_1, b_2, \dots\}$ and for $b_i \in B_k$ we define $ord(b_i) = k-1$ (the number of brackets needed to define b_i). Let Λ be the set of all functions with finite support $\lambda : \cup_{k\geq 2}B_k \longrightarrow \mathbb{N}$ and define $ord(\lambda) = \sum \lambda(b_i) ord(b_i)$.

Rephrasing the *Poincaré-Birkhoff-Witt* result for $U(\mathfrak{f}_d)$ we have that any non-commutative polynomial $p \in \mathbb{C}\langle x_1, \dots, x_d\rangle$ can be written uniquely as a finite sum

$$p = \sum_{\lambda\in\Lambda} [\![f_\lambda]\!]\, M_\lambda$$

where $[\![f_\lambda]\!] \in \mathbb{C}[x_1, \dots, x_d] = S(B_1)$ and $M_\lambda = \prod_i b_i^{\lambda(b_i)}$. In fact, by [**57**, Thm. 4.9] a $\mathbb{C}$-basis for the enveloping algebra $U(\mathfrak{f}_d) = \langle d\rangle$ is given by the decreasing products

$$L_{w_1} L_{w_2} \dots L_{w_z} \quad w_i \text{ a Lyndon words and } w_1 \geq w_2 \geq \dots \geq w_z$$

With this notation we have that the commutator filtration on $\langle d\rangle$ has components

$$F^{-k}\ \langle d\rangle = \{\sum_{\lambda} [\![f_\lambda]\!]\, M_\lambda, \forall \lambda : ord(\lambda) \geq k\}$$

EXAMPLE 34 (Kapranov). The formal structure on $\mathbb{A}^d$ induced by $\langle d\rangle$. For every $\lambda, \mu, \nu \in \Lambda$, there is a unique bilinear differential operator with polynomial coefficients

$$C^{\nu}_{\lambda\mu} : \mathbb{C}[x_1, \dots, x_d] \otimes_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_d] \longrightarrow \mathbb{C}[x_1, \dots, x_d]$$

defined by expressing the product $[\![f]\!]\ M_\lambda.\ [\![g]\!]\ M_\mu$ in $\langle d\rangle$ uniquely as $\sum_{\nu\in\Lambda} [\![C^{\nu}_{\lambda\mu}(f, g)]\!]\ M_\nu$. There is an algorithm to compute these coefficients, see example 35.

By associativity of $\langle d\rangle$, the $C^{\nu}_{\lambda\mu}$ satisfy the *associativity constraint* . That is, we have equality of the trilinear differential operators

$$\sum_{\mu_1} C^{\nu}_{\mu_1\lambda_3} \circ (C^{\mu_1}_{\lambda_1\lambda_2} \otimes id) = \sum_{\mu_2} C^{\nu}_{\lambda_1\mu_2} \circ (id \otimes C^{\mu_2}_{\lambda_2\lambda_3})$$

for all $\lambda_1, \lambda_2, \lambda_3, \nu \in \Lambda$. One defines the algebra $\langle d\rangle_{[[\mathtt{ab}]]}$ to be the $\mathbb{C}$-vectorspace of possibly *infinite formal sums* $\sum_{\lambda\in\Lambda} [\![f_\lambda]\!]\ M_\lambda$ with multiplication defined by the operators $C^{\nu}_{\lambda\mu}$. We have

$$\Gamma(\mathbb{A}^d, \mathcal{O}^{\mu}_{\langle d\rangle}) = \langle d\rangle_{[[\mathtt{ab}]]}$$

We compute now the sections on an arbitrary open subset. Let $A_d(\mathbb{C})$ be the d-th *Weyl algebra* ,

$$A_d(\mathbb{C}) = \frac{\mathbb{C}\langle x_1, \dots, x_d, y_1, \dots, y_d\rangle}{([x_i, x_j], [y_i, y_j], [x_i, y_j] - \delta_{ij})}$$

Let $\mathcal{O}_{\mathbb{A}^d}$ be the structure sheaf on $\mathbb{A}^d$. It is well-known that the ring of sections $\mathcal{O}_{\mathbb{A}^d}(U)$ on any Zariski open subset $U \hookrightarrow \mathbb{A}^d$ is a left $A_d(\mathbb{C})$-module. Define a sheaf $\mathcal{O}^f_{\mathbb{A}^d}$ of noncommutative algebras on $\mathbb{A}^d$ by taking as its sections over U the algebra

$$\mathcal{O}^f_{\mathbb{A}^d}(U) = \mathbb{C}\langle x_1, \dots, x_d\rangle_{[[\mathtt{ab}]]} \underset{\mathbb{C}[x_1,\dots,x_d]}{\otimes} \mathcal{O}_{\mathbb{A}^d}(U)$$

That is the $\mathbb{C}$-vectorspace of possibly infinite formal sums $\sum_{\lambda\in\Lambda} [\![f_\lambda]\!]\ M_\lambda$ with $f_\lambda \in \mathcal{O}_{\mathbb{A}^d}(U)$ and the multiplication is given as before by the action of the bilinear differential operators $C^{\nu}_{\lambda\mu}$ on the left $A_d(\mathbb{C})$-module $\mathcal{O}_{\mathbb{A}^d}(U)$. That is, for all $f, g \in \mathcal{O}_{\mathbb{A}^d}(U)$ we have

$$[\![f]\!]\ M_\lambda.[\![g]\!]\ M_\mu = \sum_\nu [\![C^{\nu}_{\lambda\mu}(f,g)]\!]\ M_\nu$$

This sheaf of noncommutative algebras $\mathcal{O}^f_{\mathbb{A}^d}$ is the formal structure on $\mathbb{A}^d$ defined by $\langle d \rangle$.

EXAMPLE 35. One can give an algorithm to compute the coefficients $[\![C^{\nu}_{\lambda\mu}(f,g)]\!]$. A standard sequence of Lyndon words is a sequence

$$s = (w_1, \ldots, w_n)$$

where the $w_i = u_i v_i$ are the standard factorizations of the Lyndon words and we have for each i that either w_i is a letter from X_d or

$$v_i \geq w_{i+1}, \ldots, w_n$$

A rise of s is an index i such that $w_i < w_{i+1}$, an inversion is a couple (i,j) with $i < j$ such that $w_i < w_j$. A legal rise is a rise i such that

$$w_{i+1} \geq w_{i+2}, \ldots, w_n$$

Define a rewriting system on the set of all standard sequences. If i is a legal rise then $s \rightarrow s'$ where

$$s' = (w_1, \ldots, w_{i-1}, (w_i.w_{i+1}), w_{i+2}, \ldots, w_n)$$

where $w_i.w_{i+1}$ is again a Lyndon word. Call this operation $\lambda_i(s)$ and let $\xrightarrow{*}$ be the reflexive and transitive closure of the binary operation $\longrightarrow$. By [**57**, Thm. 4.3] for every standard sequence s there exists a decreasing standard sequence t such that $s \xrightarrow{*} t$.

For a legal rise i of s

$$\rho_i(s) = (w_1, \ldots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \ldots, w_n)$$

For any standard sequence s define the derivation tree $T(s)$ of s to be the labelled rooted tree with the following properties : if s is decreasing then $T(s)$ is its own root, labelled s. If not, $T(s)$ is the tree with root labelled s and with left subtree $T(\lambda_i(s))$ and right subtree $T(\rho_i(s))$ for the rightmost legal rise i. For any decreasing sequence $t = (t_1, \ldots, t_k)$ define $P(t) = L_{t_1} L_{t_2} \ldots L_{t_k}$. Let m be a monomial in the variables $x_1, \ldots, x_d$, then m defines a standard sequence by writing the letter components from left to right and consider the derivation tree $T(m)$, then in $U(\mathfrak{f}_d) = \langle d \rangle$ it follows from [**57**, lemma 4.11] that

$$m = \sum_t P(t) \qquad t \text{ a leaf of } T(m)$$

To compute the coefficients $[\![C^{\nu}_{\lambda\mu}(f,g)]\!]$, use the fact that any word in the variables determines a standard sequence by writing the letter factors from left to right. So assume f and g are monomials in the commuting variables, write their product by concatenating these monomials (forgetting the commutativity). Then apply the first part and for each of the terms collect together all terms L_a and L_b. This gives the coefficients for the remaining product.

For a fixed commutative affine algebra C there are many algebras A with $A_{ab} \simeq C$, so there are several formal structures on $\texttt{spec}\ C$. If C is $\texttt{commalg}$-smooth, Mikhail Kapranov proved in [**27**] that there is a canonical choice determined by smooth algebras in appropriate categories of $\mathbb{C}$-algebras.

If I is a *central* A-bimodule, that is $a.i = i.a$ for all $a \in A$ and $i \in I$, we have that $C_\bullet(A,I)$ and $C^\bullet(A,I)$ are complexes of A_{ab}-modules and, in particular, $H_n(A,I)$ and $H^n(A,I)$ are A_{ab}-modules, see [**27**, Prop. 1.3.2]. Moreover, from the identification of the complexes

$$C^\bullet(A,I) \simeq \texttt{Hom}_{A_{ab}}(C_\bullet(A,A_{ab}),I)$$

and the fact that $C_\bullet(A,A_{ab})$ consists of free A_{ab}-modules, there is a *spectral sequence*

$$E_2^{ij} = \texttt{Ext}^j_{A_{ab}}(H_i(A,A_{ab}),I) \Rightarrow H^{i+j}(A,I)$$

THEOREM 19 (Kapranov). *Let A be a $\mathbb{C}$-algebra such that A_{ab} is $\texttt{commalg}$-smooth.*

1. *For any central A-bimodule I : $H^2(A,I) = \texttt{Hom}_{A_{ab}}(H_2(A,A_{ab}),I)$.*
2. *There is a* universal infinitesimal extension *of A*

$$0 \longrightarrow H_2(A,A_{ab}) \longrightarrow A^\tau \longrightarrow A \longrightarrow 0$$

such that for any infinitesimal extension $0 \longrightarrow I \longrightarrow B \longrightarrow A$ there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(A,A_{ab}) & \longrightarrow & A^\tau & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

identical on A.

PROOF. (1) : Because A_{ab} is $\texttt{commalg}$-smooth, we know that $H_1(A,A_{ab}) = \Omega^1_{A_{ab}}$ is a projective A_{ab}-module whence $\texttt{Ext}^j_{A_{ab}}(H_1(A,A_{ab}),I) = 0$ for all $j > 0$. Moreover,

$$\texttt{Ext}^j_{A_{ab}}(H_0(A,A_{ab}),I) = \texttt{Ext}^j_{A_{ab}}(A_{ab},I) = 0$$

Therefore, the only nontrivial term E_2^{ij} with $i+j=2$ is $\texttt{Hom}_{A_{ab}}(H_2(A,A_{ab}),I)$ and as there are no differentials coming into this term we only have to consider outgoing differentials. But d_2 with values in $\texttt{Ext}^2_{A_{ab}}(H_0(A,A_{ab}),I) = 0$ and d_3 with values in $\texttt{Ext}^3_{A_{ab}}(H_0(A,A_{ab}),I) = 0$, proving the claim.

(2) : Apply part (1) to $I = H_2(A,A_{ab})$ then the identity map gives a specific element in $H^2(A,H_2(A,A_{ab}))$ which classifies infinitesimal extensions of A with kernel $H_2(A,A_{ab})$ giving us the extension A^τ. The infinitesimal extension $0 \longrightarrow I \longrightarrow B \longrightarrow A$ of A is determined by an element of $H^2(A,I)$ which by (1) gives a morphism $H_2(A,A_{ab}) \longrightarrow I$ which determines a morphism of extensions. □

DEFINITION 27. A *thickening* of a commutative algebra C is a $\mathbb{C}$-algebra A such that $F^{-i}A = 0$ for i large enough and such that $A_{ab} = C$. The full subcategory of $\texttt{alg}$ consisting of all thickenings of commutative algebras will be denoted by $\texttt{thick}$.

For any $d \in \mathbb{N}$ we denote by `thick.d` the full subcategory of `alg` consisting of all *thickenings of degree* d, that is, all $\mathbb{C}$-algebras A such that $F^{-(d+1)}A = 0$.

The *d-th thickening functor*, respectively the *thickening functor*

$$\int_1^d \; : \; \mathtt{alg} \longrightarrow \mathtt{thick.d} \quad \text{resp.} \quad \int_1^\infty \; : \; \mathtt{alg} \longrightarrow \mathtt{thick}$$

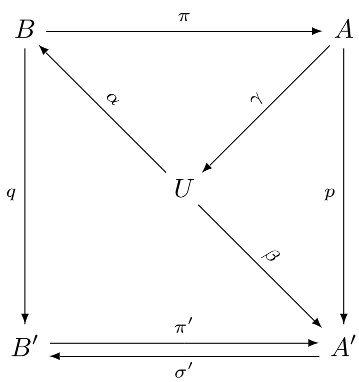
assigns to a $\mathbb{C}$-algebra A the (completed) thickening of A_{ab}

$$\int_1^d A = \frac{A}{F^{-(d+1)}A} \quad \text{resp.} \quad \int_1^\infty A = \underset{\leftarrow}{lim} \; \frac{A}{F^{-d}A}$$

THEOREM 20 (Kapranov). *Let C be an affine `commalg`-smooth algebra. Then,*

(1) *For every $d \in \mathbb{N}$ there is a unique (upto isomorphism identical on C) `thick.d`-smooth algebra $A.d$ with $A.d_{ab} \simeq C$.*
(2) *There is a unique (upto isomorphism identical on C) `thick`-smooth algebra A with $A_{ab} \simeq C$.*

PROOF. (1) : Observe that `thick.1` = `commalg` and as C is `commalg`-smooth we will prove existence by induction on d. Assume A' is `thick.d-1`-smooth with $A'_{ab} \simeq C$ and consider the universal infinitesimal extension $A = {A'}^{\tau}$. We claim that A is `thick.d`-smooth. It suffices to prove a splitting for all $B \overset{\pi}{\twoheadrightarrow} A$ in `thick.d` with nilpotent kernel. Let $B' = \frac{B}{F_{-d}B}$, then $B' \in$ `thick.d-1` and consider the natural projection $B \overset{q}{\twoheadrightarrow} B'$. Consider the diagram



Here, π' is the surjection induced by π and as $B' \in$ `thick.d-1` and A' is `thick.d-1`-smooth there is a splitting σ'. The algebra U is taken to be the fiber product

$$\begin{array}{ccc} U = B \times_{B'} A' & \xrightarrow{\beta} & A' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \sigma'} \\ B & \xrightarrow{q} & B' \end{array}$$

with α and β the natural projections. Then, $U \overset{\beta}{\twoheadrightarrow} A'$ is an infinitesimal extension with kernel $I = \mathtt{Ker}\; q$. Moreover, $\pi \circ \alpha \; : \; U \longrightarrow A$ is a morphism of infinitesimal extensions because

$$p \circ \pi \circ \alpha = \pi' \circ q \circ \alpha = \pi \circ \sigma' \circ \beta = \beta$$

But then, by the universal property of $A = A^{'\tau}$ there is a morphism of infinitesimal extensions $A \xrightarrow{\gamma} U$ and we can define a morphism $\sigma = \alpha \circ \gamma$ which one verifies to be a splitting of π.

To prove uniqueness, assume there are two `thick.d`-smooth thickenings A_1, A_2 of C. By induction on d we may assume that

$$\frac{A_1}{F_{-d}A_1} \simeq A^{d-1} \simeq \frac{A_2}{F_{-d}A_2}$$

where A^{d-1} is the unique `thick.d-1`-smooth thickening of C. But then the lifting property of `thick.d`-smooth algebras provides us with a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_1 & \longrightarrow & A_1 & \longrightarrow & A^{d-1} & \longrightarrow & 0 \\
 & & \downarrow\uparrow & & f\downarrow\uparrow g & & \| & & \\
0 & \longrightarrow & I_2 & \longrightarrow & A_2 & \longrightarrow & A^{d-1} & \longrightarrow & 0
\end{array}$$

Remains to prove that $f \circ g$ and $gcircf$ are automorphisms. Let $h = g \circ f$ then as $A_{1,ab} \simeq C \simeq A_{2,ab}$ we have that $\pi_1 \circ h = \pi_1$ where $A_1 \xrightarrow{\pi_1} C$ is the Abelianization map with kernel I_1. There is an isomorphism of algebras

$$A_1 \times_C A_1 \xrightarrow{\simeq} A_1 \times_C (C \oplus I_1) \qquad (a, a') \mapsto (a, \pi_1(a) + a' - a)$$

Consider the commuting diagram

$$\begin{array}{ccccc}
A_1 & \xrightarrow{(id,h)} & A_1 \times_C A_1 & \xrightarrow{\simeq} & A_1 \times_C (C \oplus I_1) \\
\pi_1 \downarrow & & & & \downarrow \\
C & & \overset{\overline{h}}{\cdots\cdots\cdots\cdots\rightarrow} & & C \oplus I_1
\end{array}$$

where $\overline{h}$ exists because $C \oplus I_1$ is commutative. Moreover, the projection of $\overline{h}$ to C is the identity map and such algebra morphisms are classified by the module $\mathtt{Der}(C, I)$ of I-valued derivations of C. But then,

$$h(a) = a + D(\pi_1(a)) \qquad \forall a \in A_1$$

for some $D \in \mathtt{Der}(C, I)$ and is therefore an algebra isomorphism.

(2) follows immediately from (1). □

Definition 28. Let C be an affine `commalg`-smooth algebra. The unique formal structure $\mathcal{O}_A^{\mu}$ on the affine smooth variety `spec` C determined by the unique `thick`-smooth algebra A with $A_{ab} \simeq C$ is called the *thickening structure* on `spec` C.

We can extend the n-dimensional representation functor `commalg` $\longrightarrow$ `sets` of A (see definition 19) to the categories of thickenings of commutative algebras and show that they are representable. Moreover, we can relate Kapranov's thickening structure on the smooth representation scheme $\mathtt{rep}_n$ A when A is `alg`-smooth to the root construction.

Theorem 21. *With notations as above, we have :*

(1) *The functors*

$$\texttt{thick.d} \longrightarrow \texttt{sets} \quad \textit{resp.} \quad \texttt{thick} \longrightarrow \texttt{sets}$$

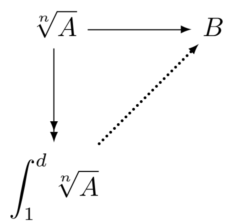
which assigns to a thickening B the set $Hom_{\texttt{alg}}(A, M_n(B))$ of all $\mathbb{C}$ algebra morphisms are representable in thick.d *respectively in* thick *by the algebras*

$$\int_1^d \sqrt[n]{A} \quad \textit{respectively} \quad \int_1^\infty \sqrt[n]{A}$$

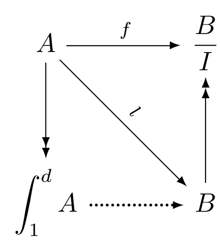
(2) *If A is* alg*-smooth, then $\int_1^d A$ is* thick.d*-smooth and $\int_1^\infty A$ is* thick*-smooth.*

(3) *If A is* alg*-smooth, then the thickening structure on the smooth affine variety $\texttt{rep}_n\ A$ coincides with $\mathcal{O}^\mu_{\sqrt[n]{A}}$.*

Proof. (1) : If $B \in \texttt{thick.d}$, then any algebra morphism $A \longrightarrow M_n(B)$ is determined by an algebra map $\sqrt[n]{A} \longrightarrow B$ which factors



(2) : Let (B, I) be a test-object in thick.d, then for every algebra morphism $A \longrightarrow \frac{B}{I}$ we have a commuting diagram



Here, the lift l exists because A is alg-smooth and this map factors through $\int_1^d A$ as $B \in \texttt{thick.d}$. The thick-case is similar.

(3) : Because A is alg-smooth, so is $\sqrt[n]{A}$ and hence $\int_1^\infty \sqrt[n]{A}$ is thick-smooth. Moreover, the Abelianization of $\sqrt[n]{A}$ and of $\int_1^\infty \sqrt[n]{A}$ is the coordinate ring of $\texttt{rep}_n\ A$. Therefore, the result follows from the uniqueness of thickening structures. □

For a general algebra $A \in \texttt{alg}$, the formal structure $\mathcal{O}^\mu_{\sqrt[n]{A}}$ determined by the n-th root algebra $\sqrt[n]{A}$ on $\texttt{rep}_n\ A$ contains all information about algebra maps $A \longrightarrow M_n(B)$ where B is a thickening of a commutative algebra.

CHAPTER 3

Necklaces

"I will take the Ring", he said
"though I do not know the way."
J.R.R. Tolkien in 'Lord of the Rings".

In this chapter we will finally get some algebraic grip on the `alg`-smooth algebra A by associating to it a family of affine Noetherian algebras $\int_n A$, $n \in \mathbb{N}$. These algebras are all quotients of a fixed algebra with trace $\int A$ obtained by dividing out the formal Cayley-Hamilton identities of degree n. The trace algebra $\int A$ is obtained from A by adjoining the polynomial algebra on all necklaces in the generators of A, that is, equivalence classes of monomials in the generators under cyclic permutation.

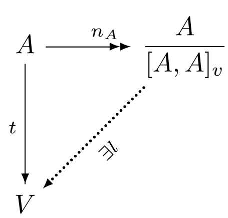
Further, we will equip the space spanned by all necklaces with a Lie algebra structure coming from noncommutative symplectic geometry.

3.1. Algebras with trace.

In this section we associate to the algebra $A \in$ `alg` an algebra with a trace map $\int A \in$ `alg@`. If $A = \langle d \rangle$ this algebra is obtained by tensoring with the polynomial algebra on all necklaces in $X_d = \{x_1, \ldots, x_d\}$. If d is even, we define a Lie bracket on the space spanned by all necklaces.

Definition 29. Let $A \in$ `alg` and $V \in$ `vect`, the category of $\mathbb{C}$-vectorspaces. A *trace map* $A \xrightarrow{t} V$ is a linear map satisfying $t(ab) = t(ba)$ for all $a, b \in A$.

The *universal trace map* $A \xrightarrow{n_A} \frac{A}{[A,A]_v}$, where $[A,A]_v$ is the subspace of A spanned by all the commutators $[a,b]$ with $a, b \in A$, has the property that any trace map factors



Definition 30. Let $A \in$ `alg` with algebra generators $a_i, i \in I$. Let $w = a_{i_1} a_{i_2} \ldots a_{i_l}$ be a word of length l in the a_i. The corresponding *necklace* word $n_{\overset{\frown}{w}}$ is the equivalence class of w in all monomials of length l under cyclic permutation. That is, $w \sim w_k = a_{i_k} a_{i_{k+1}} \ldots a_{i_l} a_1 a_2 \ldots a_{i_{k-1}}$ for all $k \leq l$.

The class $n_{\overset{\frown}{w}}$ can be depicted by viewing the consecutive terms a_i of w as i-colored beads of a necklace as in figure 1 Two words w and w' are equivalent if their necklaces differ only upto a rotation.

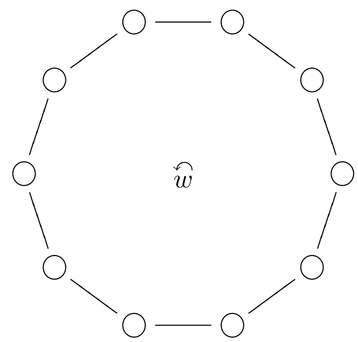


FIGURE 1. The necklace of a word w.

EXAMPLE 36. The necklaces for $\langle d \rangle = \mathbb{C}\langle x_1, \ldots, x_d \rangle$. Order the variables in $X_d = \{x_1, \ldots, x_d\}$ by $x_1 < \ldots < x_d$. From [**57**, Thm. 5.1 & Cor. 7.5] we recall that a complete set of representatives of the necklace words in X_d is given by the words

$$\{ l^n \mid l \text{ a Lyndon word in } X_d \text{ and } n \geq 1 \}$$

A necklace is said to be *primitive* if no nontrivial rotation leaves it invariant. More generally, every necklace has a minimal *period* p dividing its length l. The number of primitive necklaces of length l for $\langle d \rangle$ is given by

$$\frac{1}{l} \sum_{p|l} \mu(p) d^{\frac{l}{p}}$$

where μ is the Möbius function. Indeed, let X_d^l be the set of words in X_d of length l, then the generating function of X_d^l is

$$(x_1 + \ldots + x_d)^l = s_1(a_1, \ldots, a_d)^l$$

where $a_1, \ldots, a_d$ are commuting variables and where we define the evaluation of a necklace word w to be $a_1^{n_1} \ldots a_d^{n_d}$ provided w contains n_i occurrences of the letter x_i. Also recall the definition of the *Newton functions*

$$s_i(a_1, \ldots, a_d) = a_1^i + \ldots + a_d^i$$

Further, $X_d^l = \sqcup_{p|l} C_e(l)$ where $C_p(l)$ is the set of words of length l and period e (that is, of the form $w^{l/e}$ with w a primitive necklace). Let $P(e)$ be the set of primitive words of length e, then each word in $P(e)$ has e equivalents having the same necklace and the map $u \mapsto u^{l/e}$ from $P(e)$ to $C_e(l)$ is a bijection. Let $l_e(a_1, \ldots, a_d)$ denote the generating function of primitive necklaces of length e, then

$$s_1(a_1, \ldots, a_d)^l = \sum_{e|n} e l_e(a_1^{l/e}, \ldots, a_d^{l/e})$$

and because $s_i(a_1, \ldots, a_d) = s_1(a_1^i, \ldots, a_d^i)$ we deduce that

$$\begin{aligned} \frac{1}{l} \sum_{i|l} \mu(i) s_i(a_1, \ldots, a_d)^{l/i} = & \quad \tfrac{1}{l} \textstyle\sum_{i|l} \mu(i) s_1(a_1^i, \ldots, a_d^i)^{l/i} \\ = & \quad \tfrac{1}{l} \textstyle\sum_{i|l} \mu(i) \sum_{e|\frac{l}{i}} e l_e(a_1^{l/e}, \ldots, a_d^{l/e}) \end{aligned}$$

As $i|l$ and $e|\frac{l}{i}$ is the same as $e|l$ and $i|\frac{l}{e}$ the last expression is equal to

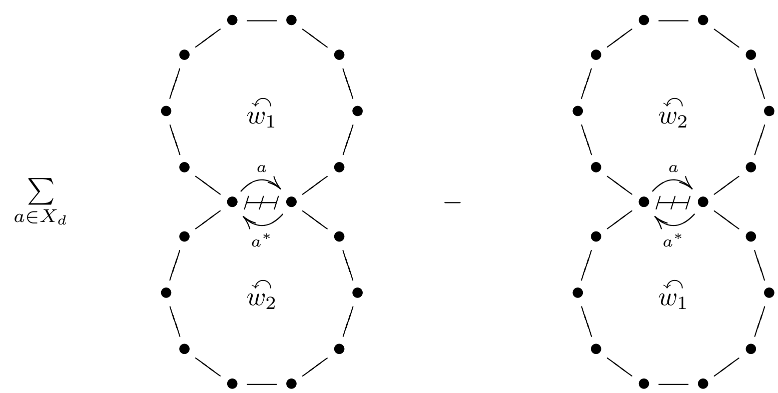
$$\frac{1}{l}\sum_{e|l} el_e(a_1^{l/e},\ldots,a_d^{l/e})\sum_{i|\frac{l}{e}}\mu(i)$$

and the second term vanishes unless $l = e$ (where it is 1) so this is just $l_l(a_1,\ldots,a_d)$. Finally, substitute in the generating function $a_1 = \ldots = a_d = 1$ to obtain the result. For more details we refer to [**57**, §7.1]. As a consequence, the number of all necklaces of length l for $\langle d\rangle$ is equal to

$$\frac{1}{l}\sum_{p|l}\phi(p)d^{\frac{l}{d}}$$

with ϕ the Euler function.

Definition 31. Let $\texttt{neck}_d$ be the $\mathbb{C}$-vectorspace spanned by the necklaces in $X_d = \{x_1,\ldots,x_d\}$ and let $* : X_d \longrightarrow X_d$ be an involution, that is, $(x_i^*)^* = x_i$ for all $1 \leq i \leq d$. The $*$-*Kontsevich bracket* on $\texttt{neck}_d$ is induced by the bracket on necklaces defined by



To compute the bracket $\{w_1, w_2\}_K$ for two necklaces w_1 and w_2 we consider for all letters a from X_d all occurrences of a in w_1 and all occurrences of a^* in w_2. Open up the necklaces by removing these factors and glue the open ends together to form a new necklace. Next, replace the roles of a^* and $a$and redo this operation with a minus sign and all all these terms.

The $*$-Kontsevich bracket defines a Lie-algebra structure on $\texttt{neck}_d$, see figure 2 for a graphical proof. We call $\texttt{neck}_d$ with this Lie bracket the $*$-*necklace Lie algebra* of $\langle d\rangle$.

For $V \in \texttt{vect}$ we denote by $S(V)$ the *symmetric algebra* of V, that is the Abelianization of the *tensor algebra* $T(V) = \oplus_{i=0}^{\infty} V^{\otimes i}$.

Definition 32. The *necklace functor*

$$\oint : \texttt{alg} \longrightarrow \texttt{commalg}$$

assigns to a $\mathbb{C}$-algebra A its *necklace algebra*

$$\oint A = S(\frac{A}{[A,A]_v})$$

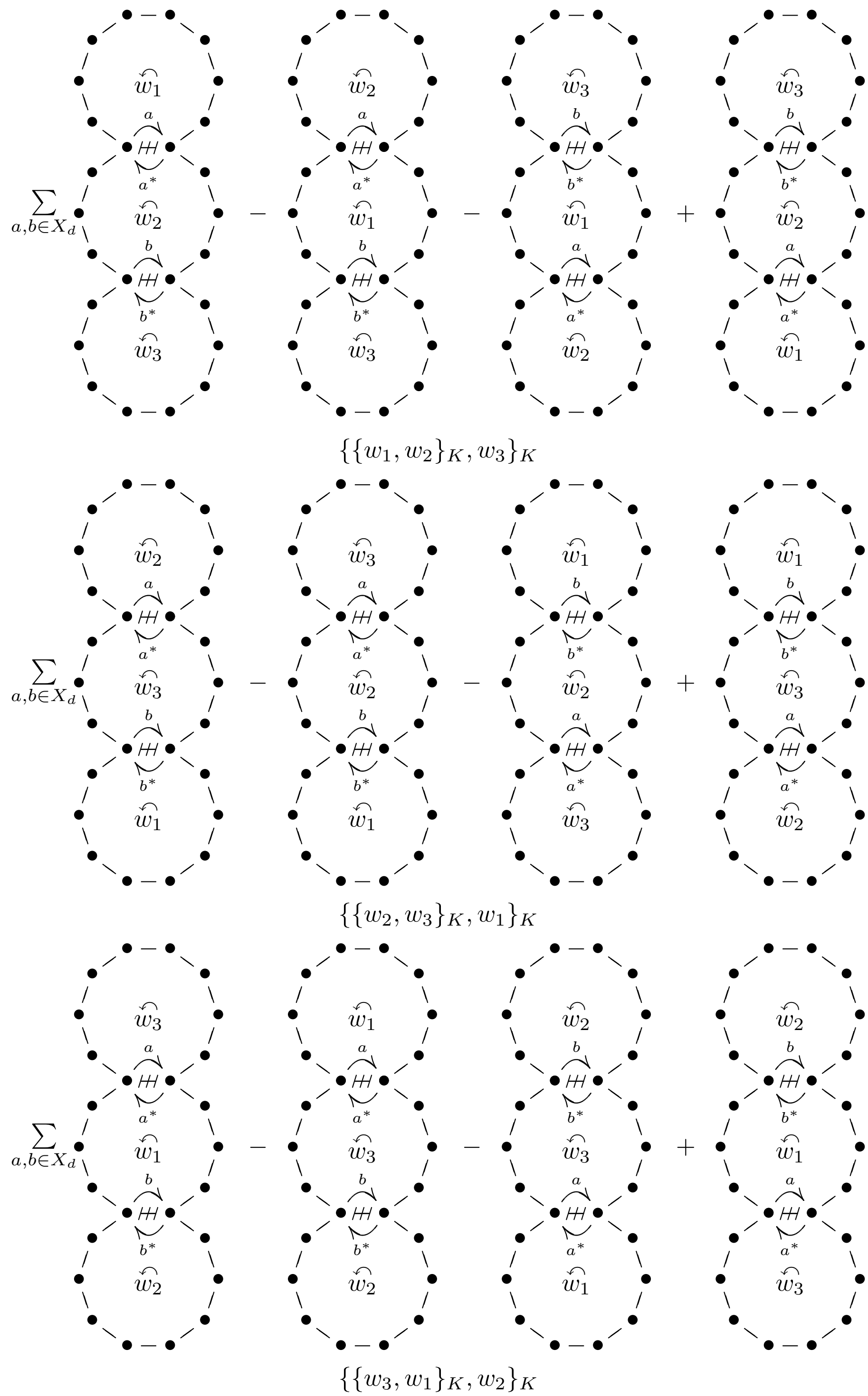


FIGURE 2. Jacobi identity for the Kontsevich bracket. Term $1a$ vanishes against $2c$, term $1b$ against $3d$, $1c$ against $3a$, $1d$ against $2b$, $2a$ against $3c$ and $2d$ against $3b$.

EXAMPLE 37. For $A \in \texttt{commalg}$, the necklace algebra $\oint A$ is isomorphic to the symmetric algebra $S(A)$ on A because $[A, A]_v = 0$.

Let $A = M_n(\mathbb{C})$ and recall that all $n \times n$ matrices of trace zero form the simple Lie algebra sl_n for the commutator bracket. In particular, we have that $[M_n(\mathbb{C}), M_n(\mathbb{C})]_v = [sl_n, sl_n] = sl_n$. The universal trace map on $M_n(\mathbb{C})$ is

$$M_n(\mathbb{C}) \xrightarrow{n} \frac{M_n(\mathbb{C})}{[M_n(\mathbb{C}), M_n(\mathbb{C})]_v} = \mathbb{C}.\mathbb{1}_n \qquad \text{with} \qquad M \mapsto Tr(M)\mathbb{1}_n$$

with Tr the usual trace on matrices. But then,

$$\oint M_n(\mathbb{C}) = S(\mathbb{C}.\mathbb{1}_n) \simeq \mathbb{C}[x]$$

with x corresponding to the class of the identity matrix $\mathbb{1}_n$.

DEFINITION 33. $\texttt{alg@}$ is the category of $\mathbb{C}$-*algebras with trace* . Its objects are pairs (A, tr_A) with $A \in \texttt{alg}$ and a linear trace map

$$tr_A \; : \; A \longrightarrow A$$

satisfying the following properties for all $a, b \in A$:

(1) $tr_A(a)b = btr_A(a)$,
(2) $tr_A(ab) = tr_A(ba)$ and
(3) $tr_A(tr_A(a)b) = tr_A(a)tr_A(b)$.

Note that the first property asserts that the image $tr_A(A)$ of the trace map is contained in the *center* of A.

Morphisms in $\texttt{alg@}$ are *trace preserving* algebra maps. That is, if (A, tr_A) and (B, tr_B) are two objects in $\texttt{alg@}$ we only consider algebra maps making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr_B} \\ A & \xrightarrow{\phi} & B \end{array}$$

commute.

DEFINITION 34. The *trace functor*

$$\int \; : \; \texttt{alg} \longrightarrow \texttt{alg@}$$

assigns to an algebra $A \in \texttt{alg}$ its *trace algebra* $\int A = \oint A \otimes_{\mathbb{C}} A$. $\int A \in \texttt{alg@}$ with trace

$$\int A \xrightarrow{tr} \int A \quad \text{defined by} \quad c \otimes a \mapsto cn_A(a) \otimes 1$$

where n_A is the universal trace map. $\oint A \otimes 1$ is a central subalgebra of $\int A$ and we have

$$tr \int A = \oint A$$

EXAMPLE 38. If $A \in \texttt{commalg}$ we have seen in example 37 that $\oint A = S(A)$. Therefore,

$$\int A \simeq S(A) \otimes_{\mathbb{C}} A$$

and the trace map is given by the multiplication map $tr(a \otimes a') = aa' \otimes 1$.

EXAMPLE 39. For $A = M_n(\mathbb{C})$ we know from example 37 that $\oint M_n(\mathbb{C}) \simeq \mathbb{C}[x]$. As a consequence,

$$\int M_n(\mathbb{C}) = \oint M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) = \mathbb{C}[x] \otimes M_n(\mathbb{C})$$

and the trace is given by $tr(x^i \otimes M) = x^i n(M) \otimes 1 = x^i Tr(M) x = Tr(M) x^{i+1}$.

EXAMPLE 40. By example 37 we have $\oint \mathbb{C}[x] = S(\mathbb{C}[x]) \simeq \mathbb{C}[x_0, x_1, \ldots, x_i, \ldots]$ with x_i corresponding to x^i for all $i \in \mathbb{N}$. But then,

$$\int \mathbb{C}[x] = \oint \mathbb{C}[x] \otimes \mathbb{C}[x]$$

and the trace is induced by $tr(x_i \otimes x^j) = x_i n(x^j) \otimes 1 = x_i x_j$.

THEOREM 22. *The forgetful functor*

$$\texttt{alg@} \xrightarrow{i} \texttt{alg}$$

has the trace functor as a left adjoint .

PROOF. We have to show that for any $A \in \texttt{alg}$ and any $(B, tr_B) \in \texttt{alg@}$ there is a functorial bijection

$$Hom_{\texttt{alg}}(A, iB) \xrightarrow{t(A,B)} Hom_{\texttt{alg@}}(\int A, B)$$

To a trace preserving algebra map $\int A \xrightarrow{\Phi} B$ we assign the restriction ϕ to the subalgebra $1 \otimes A$ of $\int A$. Conversely, if $A \xrightarrow{\phi} B$ is an algebra map, then

$$A \xrightarrow{tr_B \circ \phi} tr_B B$$

is a trace map to the commutative $\mathbb{C}$-algebra $tr_B B$. By the universal property of the symmetric algebra, it factors through an algebra map

$$\oint A \xrightarrow{\tilde{\phi}} tr_B B \hookrightarrow B$$

But then, we have a trace preserving algebra map

$$\Phi = \tilde{\phi} \otimes \phi \ : \ \int A = \oint A \otimes A \longrightarrow B$$

One verifies that these two constructions are each others inverses.

Functoriality of the bijections means that for any algebra morphism $A \xrightarrow{f} A'$ and any trace preserving algebra morphism $(B, tr_B) \xrightarrow{g} (B', tr_{B'})$ the following diagrams of sets are commutative

$$\begin{array}{ccc} Hom_{\texttt{alg}}(A', iB) & \xrightarrow{t(A',B)} & Hom_{\texttt{alg@}}(\int A', B) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle -\circ \int f} \\ Hom_{\texttt{alg}}(A, iB) & \xrightarrow{t(A,B)} & Hom_{\texttt{alg@}}(\int A, B) \end{array}$$

respectively,

$$\begin{array}{ccc} Hom_{\texttt{alg}}(A, iB) & \xrightarrow{t(A,B)} & Hom_{\texttt{alg@}}(\int A, B) \\ \downarrow{\scriptstyle ig\circ -} & & \downarrow{\scriptstyle g\circ -} \\ Hom_{\texttt{alg}}(A, iB') & \xrightarrow{t(A,B')} & Hom_{\texttt{alg@}}(\int A, B') \end{array}$$

□

EXAMPLE 41. The *one Ring to rule them all* : $\int \langle \infty \rangle$. It is often convenient to have an infinite supply of variables $X = \{x_1, x_2, \ldots, x_i, \ldots\}$ and to consider the corresponding free algebra $\langle \infty \rangle = \mathbb{C}\langle x_1, x_2, \ldots \rangle$. Totally order the set X, induce a total order on all the words in X and define Lyndon words in X as before. Let $w = x_{i_1} x_{i_2} \ldots x_{i_l} \in \texttt{Lyndon}^*$, the set of powers of Lyndon words in X, and denote the represented necklace by $\overset{\frown}{w} = [i_1, i_2, \ldots, i_l]$. Define a new variable $t_{\overset{\frown}{w}}$. With these notations we have that the *free necklace algebra* is the commutative polynomial ring

$$\oint \langle \infty \rangle = \mathbb{C}[t_{\overset{\frown}{w}} \mid w \in \texttt{Lyndon}^*]$$

in infinitely many commuting variables.

The *free trace algebra* is the algebra

$$\int \langle \infty \rangle = \mathbb{C}[t_{\overset{\frown}{w}} \mid w \in \texttt{Lyndon}^*] \otimes \mathbb{C}\langle x_1, x_2, \ldots \rangle$$

with coefficients in $\oint \langle \infty \rangle$. The trace map on $\int \langle \infty \rangle$ is defined to be

$$tr(\sum_i a_i \otimes w_i) = \sum_i a_i t_{\overset{\frown}{w_i}} \otimes 1$$

where the a_i are polynomials in the variables $t_{\overset{\frown}{w}}$.

EXAMPLE 42. The *free necklace algebra on m variables* $\oint \langle m \rangle$ is the quotient of $\oint \langle \infty \rangle$ where we divide out the ideal generated by all necklaces involving a term x_i with $i > m$. Similarly, we have a description of the *free trace algebra on m variables* $\int \langle m \rangle$ as a (trace preserving) quotient of $\int \langle \infty \rangle$.

Though $\int \langle m \rangle$ is *not* an affine $\mathbb{C}$-algebra, it is a *trace affine algebra* , that is, there are finitely many of its elements (in this case X_m) which together with all the traces of words in these elements generate the algebra.

THEOREM 23. *Every trace affine algebra (A, tr_A) is an epimorphic image*

$$\int \langle \infty \rangle \twoheadrightarrow A \qquad \text{and} \qquad \int \langle m \rangle \twoheadrightarrow A$$

if A is generated by m elements in the category `alg@` *of algebras with trace.*

PROOF. Assume that A is trace generated by the elements $\{a_1, \ldots, a_m\}$ and forget the trace, then there is a morphism $\langle m \rangle \longrightarrow A$ defined by sending x_i to a_i. Applying the trace functor we obtain a trace preserving algebra map

$$\int \langle m \rangle \longrightarrow \int A \xrightarrow{v_A} A$$

where v_A is the *universal map*. By assumption on the trace generation of A the composition is an epimorphism. □

3.2. Cayley-Hamilton algebras.

In this section we define for every $n \in \mathbb{N}$ the quotient $\int_n A$ of $\int A$ by dividing out the ideal of all Cayley-Hamilton identities of degree n. If A is affine, then $\int_n A$ is an affine Noetherian algebra. In the next chapter we will relate $\int_n A$ to the GL_n-geometry of the scheme of n-dimensional representations $\mathtt{rep}_n A$.

Take n commuting variables $\lambda_1, \ldots, \lambda_n$ and consider the polynomial

$$f_n(t) = \prod_{i=1}^{n}(t - \lambda_i) = t^n + \sum_{i=1}^{n}(-1)^i \sigma_i t^{n-i}$$

where σ_i is the i-th *elementary symmetric polynomial* in the λ_j. These polynomials are algebraically independent and generate the *ring of symmetric polynomials* in the λ_j,

$$\mathbb{C}[\sigma_1, \ldots, \sigma_n] = \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{S_n}$$

Here, S_n is the *symmetric group* on n letters acting by automorphisms on the polynomial ring $\mathbb{C}[\lambda_1, \ldots, \lambda_n]$ by permuting the variables.

THEOREM 24. *The symmetric* Newton functions $s_i = \lambda_1^i + \ldots + \lambda_n^i$. *form another generating set for the symmetric polynomials. That is,*

$$\mathbb{C}[\sigma_1, \ldots, \sigma_n] = \mathbb{C}[s_1, \ldots, s_n].$$

PROOF. It suffices to express each σ_i as a polynomial in the s_j. We claim that the following identities hold for all $1 \leq j \leq n$

$$s_j - \sigma_1 s_{j-1} + \sigma_2 s_{j-2} - \ldots + (-1)^{j-1}\sigma_{j-1}s_1 + (-1)^j \sigma_j.j = 0 \tag{3.1}$$

For $j = n$ this identity holds because we have

$$0 = \sum_{i=1}^{n} f_n(\lambda_i) = s_n + \sum_{i=1}^{n}(-1)^i \sigma_i s_{n-i}$$

if we take $s_0 = n$. Assume now $j < n$ then the left hand side of equation 3.1 is a symmetric function in the λ_i of degree $\leq j$ and is therefore a polynomial $p(\sigma_1, \ldots, \sigma_j)$ in the first j elementary symmetric polynomials. Let ϕ be the algebra epimorphism

$$\mathbb{C}[\lambda_1, \ldots, \lambda_n] \xrightarrow{\phi} \mathbb{C}[\lambda_1, \ldots, \lambda_j]$$

defined by mapping $\lambda_{j+1}, \ldots, \lambda_j$ to zero. Clearly, $\phi(\sigma_i)$ is the i-th elementary symmetric polynomial in $\{\lambda_1, \ldots, \lambda_j\}$ and $\phi(s_i) = \lambda_1^i + \ldots + \lambda_j^i$. Repeating the above $j = n$ argument (replacing n by j) we have

$$0 = \sum_{i=1}^{j} f_j(\lambda_i) = \phi(s_j) + \sum_{i=1}^{j}(-1)^i \phi(\sigma_i)\phi(s_{n-i})$$

(this time with $s_0 = j$). But then, $p(\phi(\sigma_1), \ldots, \phi(\sigma_j)) = 0$ and as the $\phi(\sigma_k)$ for $1 \leq k \leq j$ are algebraically independent we must have that p is the zero polynomial finishing the proof of the identity. □

Let M be an $n \times n$ matrix with *eigenvalues* $\{\lambda_1, \ldots, \lambda_n\}$, then the *characteristic polynomial* of M is

$$det\ (t\mathbb{1}_n - M) = \prod_{i=1}^{n}(t - \lambda_i) = f_n(t)$$

If we conjugate M to an upper triangular matrix, we see that the Newton functions are

$$s_i = \lambda_1^i + \ldots + \lambda_n^i = tr(M^i)$$

for all $1 \leq i \leq n$. By the foregoing theorem, there exist polynomials in the traces of powers of M

$$\sigma_i = g_i(tr(M), tr(M^2), \ldots, tr(M^n))$$

such that the characteristic polynomial of M can be expressed as

$$det\ (t\mathbb{1}_n - M) = t^n + \sum_{i=1}^{n} (-1)^n g_i(tr(M), \ldots, tr(M^n)) t^{n-i}$$

Definition 35 (Procesi). For $(A, tr_A) \in$ `alg@` we define for $a \in A$ the (formal) *Cayley-Hamilton polynomial of degree n*

$$\chi_a^{(n)}(t) = t^n + \sum_{i=1}^{n} g_i(tr_A(a), tr_A(a^2), \ldots, tr_A(a^n)) t^{n-i} \in A[t]$$

where the g_i are the polynomials introduced above.

An algebra with trace $(A, tr_A) \in$ `alg@` is said to be a *Cayley-Hamilton algebra of degree n* if the following two properties are satisfied :

(1) $tr_A(1) - n = 0$, and
(2) For all $a \in A$ we have $\chi_a^{(n)}(a) = 0$ in A.

`alg@n` is the category with objects the Cayley-Hamilton algebras of degree n and with morphisms trace preserving $\mathbb{C}$-algebra maps.

Example 43. The archetypical example of a Cayley-Hamilton algebra of degree n is the ring of $n \times n$ matrices $M_n(C)$ over a commutative algebra C equipped with the usual trace map.

Definition 36. The *Cayley-Hamilton functor of degree n*

$$\int_n : \texttt{alg} \longrightarrow \texttt{alg@n}$$

assigns to an algebra A its *n-th trace algebra* . This is the quotient in `alg` of $\int A$ by dividing out the trace closure of the ideal generated by all the left-hand terms of the formal Cayley-Hamilton polynomials of degree n of elements of A

$$\int_n A = \frac{\int A}{(\ tr(1) - n, \chi_a^{(n)}(a)\ \forall a \in \int A\ )}$$

Thus, $\int_n A \in$ `alg@n` and we define the *necklace functor of degree n*

$$\oint_n : \texttt{alg} \longrightarrow \texttt{commalg}$$

This functor assigns to an algebra A its *n-th necklace algebra* $\oint_n A = tr\ \int_n A$.

Example 44. The Cayley-Hamilton functor of degree one is just Abelianization. The first Cayley-Hamilton equation is $\chi_a^{(1)}(x) = x - tr(a)$. Hence, in the quotient $\int_1 A$ we have that $a = tr(a)$ for all $a \in \int_1 A$. By the trace property $tr(a)b = btr(a)$ we deduce that $\int_1 A$ is commutative. So the universal algebra map $A \longrightarrow \int_1 A$ factors through the Abelianization. This is an isomorphism as $\oint A$ is generated by $tr(a)$ for $a \in A$ (which are equal to a in the quotient $\int_1 A$). This also explains the notation $\int_1^d A$ used in the previous chapter.

Example 45. With the notation of example 40 we have that

$$\int_n \langle 1 \rangle = \int_n \mathbb{C}[x] \simeq \mathbb{C}[x, x_1, \ldots, x_{n-1}] \quad \text{and} \quad \oint_n \langle 1 \rangle = \oint_n \mathbb{C}[x] \simeq \mathbb{C}[x_1, \ldots, x_n]$$

Indeed, in the quotient $\int_n \mathbb{C}[x]$ we have to satisfy the equations

$$tr(1) = x_0 = n \quad \text{and} \quad x^n - x_1 x^{n-1} + p_2(x_1, x_2) x^{n-2} - \ldots \pm p_n(x_1, \ldots, x_n) = 0$$

where x_n appears linearly in $p_n(x_1, \ldots, x_n)$. Also, the higher x_m for $m > n$ can be written as polynomials in $x, x_1, \ldots, x_{n-1}$ by induction and multiplying the second equality by powers of x and taking traces. But then we see that

$$\oint_n \mathbb{C}[x] = tr \int_n \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$$

Example 46. Recall that $\int M_n(\mathbb{C}) = M_n(\mathbb{C}[x])$ and $tr(1 \otimes \mathbb{1}_n) = nx$. In the quotient $\int_m M_n(\mathbb{C})$ we must have $nx = m$ so $\int_m M_n(\mathbb{C})$ is an epimorphic image of $M_n(\mathbb{C})$. Assume that the quotient is $M_n(\mathbb{C})$, then the composition

$$M_n(\mathbb{C}) \longrightarrow \int_m M_n(\mathbb{C}) \xrightarrow{tr_m} \mathbb{C}$$

gives a trace map on $M_n(\mathbb{C})$ satisfying the formal m-th Cayley-Hamilton equation. The last coefficient of $\chi_a^{(m)}(x)$ gives a multiplicative map on $M_n(\mathbb{C})$ so it gives a character on GL_n which must therefore be of the form det^k for some integer k. But then by polarization (to be discussed in the next chapter) we must have that $tr(M) = kTr(M)$ for all $M \in M_n(\mathbb{C})$. But then, $x = k$ and m is a multiple of n. As a consequence we have

$$\int_m M_n(\mathbb{C}) = \begin{cases} M_n(\mathbb{C}) & \text{if } n|m \\ 0 & \text{otherwise} \end{cases}$$

and that $\oint_m M_n(\mathbb{C})$ is $\mathbb{C}$ resp. 0.

We aim to prove that $\int_n A$ is an affine $\mathbb{C}$-algebra whenever A is. First we make a small detour into one of the more exotic realms of noncommutative algebra : the *Nagata-Higman problem* .

Theorem 25 (Nagata-Higman). *Let R be an associative algebra* without a unit element. *Assume there is a fixed natural number n such that $x^n = 0$ for all $x \in R$. Then, $R^{2^n-1} = 0$, that is*

$$x_1.x_2.\ldots.x_{2^n-1} = 0$$

for all $x_j \in R$.

Proof. We use induction on n, the case $n = 1$ being obvious. Consider for all $x, y \in R$

$$f(x, y) = yx^{n-1} + xyx^{n-2} + x^2yx^{n-3} + \ldots + x^{n-2}yx + x^{n-1}y.$$

Because for all $c \in \mathbb{C}$ we must have that

$$0 = (y + cx)^n = x^n c^n + f(x, y)c^{n-1} + \ldots + y^n$$

it follows that all the coefficients of the c^i with $1 \leq i < n$ must be zero, in particular $f(x, y) = 0$. But then we have for all $x, y, z \in R$ that

$$\begin{aligned} 0 &= f(x, z)y^{n-1} + f(x, zy)y^{n-2} + f(x, zy^2)y^{n-3} + \ldots + f(x, zy^{n-1}) \\ &= nx^{n-1}zy^{n-1} + zf(y, x^{n-1}) + xzf(y, x^{n-2}) + x^2zf(y, x^{n-3}) + \ldots + x^{n-2}zf(y, x) \end{aligned}$$

and therefore $x^{n-1}zy^{n-1} = 0$. Let $I \triangleleft R$ be the twosided ideal of R generated by all elements x^{n-1}, then we have that $I.R.I = 0$. In the quotient algebra $\overline{R} = R/I$ every element $\overline{x}$ satisfies $\overline{x}^{n-1} = 0$.

By induction we may assume that $\overline{R}^{2^{n-1}-1} = 0$, or equivalently that $R^{2^{n-1}-1}$ is contained in I. But then,

$$R^{2^n-1} = R^{2(2^{n-1}-1)+1} = R^{2^{n-1}-1}.R.R^{2^{n-1}-1} \subset I.R.I = 0$$

finishing the proof. □

The *Nagata-Higman problem* asks for the optimal function $l(n)$ such that $R^{l(n)} = 0$ but $R^{l(n)-1} \neq 0$. It is conjectured that $l(n) = \frac{n(n+1)}{2}$. In the next chapter, we will prove Razmyslov's bound $l(n) \leq n^2$. For more details on this problem we refer to the lecture notes by E. Formanek [**16**].

Definition 37. Giving the variables x_i all degree one defines a *positively graded* $\mathbb{C}$-algebra structure on $\langle \infty \rangle$. This gradation induces a positive gradation on the necklace algebra $\oint \langle \infty \rangle$ by taking as the degree of a generator $t_{\widehat{w}}$ to be the length of the word w in the variables x_i. This induces a gradation on the trace algebra $\int \langle \infty \rangle$ such that the trace map is degree preserving.

Because all the Cayley-Hamilton relations are homogeneous, it follows that the generic n-th trace algebra $\int_n \langle \infty \rangle$ and the generic n-th necklace algebra $\oint_n \langle \infty \rangle$ are positively graded algebras. We will call the gradation on each of these algebras the *generator gradation* .

Similarly, for a fixed number d of generators, $\int \langle d \rangle$, $\oint \langle d \rangle$, $\int_n \langle d \rangle$ and $\oint_n \langle d \rangle$ are positively graded $\mathbb{C}$-algebras with respect to the generator gradation.

Theorem 26 (Procesi). *The generic n-trace algebra on $\int_n \langle \infty \rangle$ is spanned as a module over the generic n-th necklace algebra $\oint_n \langle \infty \rangle$ by all monomials*

$$x_{i_1} x_{i_2} \dots x_{i_l}$$

of length $l \leq 2^n - 1$. In particular, for a fixed number d of variables $\int_n \langle d \rangle$ is a finitely generated module over $\oint_n \langle d \rangle$.

Proof. Let $\int_+$ be the strict positive part of $\int_n \langle \infty \rangle$ in the generator gradation and $\oint_+$ the strict positive part of $\oint_n \langle \infty \rangle$. Form the graded associative $\mathbb{C}$-algebra (*without unit element*)

$$R = \frac{\int_+}{\oint_+ \cdot \int_+}$$

Every element $t \in \int_+$ satisfies a Cayley-Hamilton relation of degree n of the form

$$t^n + c_1 t^{n-1} + c_2 t^{n-2} + \dots + c_n = 0$$

with the $c_i \in \oint_+$. Hence, $x^n = 0$ for all $x \in R$. By the Nagata-Higman theorem we know that $R^{2^n-1} = (R_1)^{2^n-1} = 0$.

Let $\int'$ be the graded $\oint_n \langle \infty \rangle$-submodule of $\int_n \langle \infty \rangle$ spanned by all monomials in the (images of the) variables x_i of degree at most $2^n - 1$. Then,

$$\int_n \langle \infty \rangle = \int' + \oint_+ . \int_n \langle \infty \rangle.$$

We claim that $\int_n \langle\infty\rangle = \int'$. If not, there is a homogeneous $t \in \int_n \langle\infty\rangle$ *of minimal degree* k not contained in $\int'$. Still, we have a description

$$t = t' + c_1.t_1 + \ldots + c_s.t_s$$

with t' and all c_i, t_i homogeneous elements of positive degree. As $deg(t_i) < k$, $t_i \in \int'$ for all i. Whence $t \in \int'$, a contradiction. The second part follows. □

We have reduced the problem of finite algebra generation of $\int_n \langle d\rangle$ to that of the generic n-th necklace algebra $\oint_n \langle d\rangle$.

Theorem 27 (Procesi). *The generic n-th necklace algebra* $\oint_n \langle\infty\rangle$ *is generated by the necklaces* $t_{\widehat{w}}$ *where* w *is a necklace word of length* $l \leq 2^n$. *In particular, for a fixed number* d *of variables* $\oint_n \langle d\rangle$ *is an affine* $\mathbb{C}$*-algebra and so is* $\int_n \langle d\rangle$.

Proof. Let $\int'$ be the $\mathbb{C}$-subalgebra of $\int_n \langle\infty\rangle$ generated by the (images of the) variables x_i. Then, $tr(\int'_+)$ generates the ideal $\oint_+$. Let $\mathbb{S}$ be the set of all monomials in the x_i of degree at most $2^n - 1$. By the foregoing theorem we know that $\int' \subset \oint_n \langle\infty\rangle.\mathbb{S}$. The trace map

$$tr : \int_n \langle\infty\rangle \longrightarrow \oint_n \langle\infty\rangle$$

is $\oint_n \langle\infty\rangle$-linear. Therefore, as $\int'_+ \subset \int'.(\mathbb{C}x_1 + \mathbb{C}x_2 + \ldots)$ we have

$$tr(\int'_+) \subset tr(\oint_n \langle\infty\rangle.\mathbb{S}.(\mathbb{C}x_1 + \mathbb{C}x_2 + \ldots)) \subset \oint_n \langle\infty\rangle.tr(\mathbb{S}')$$

where $\mathbb{S}'$ is the set of monomials in the x_i of degree at most 2^n. If $\oint'$ is the $\mathbb{C}$-subalgebra of $\oint_n \langle\infty\rangle$ generated by all $tr(\mathbb{S}')$, then we have $tr(\int'_+) \subset \oint_n \langle\infty\rangle.\oint'_+$. Finally, we deduce

$$\oint_+ = \oint_n \langle\infty\rangle.tr(\int_+) \subset \oint_n \langle\infty\rangle.\oint'_+$$

and thus $\oint_n \langle\infty\rangle = \oint' + \oint_n \langle\infty\rangle \oint'_+$. It follows that $\oint_n \langle\infty\rangle = \oint'$ by an argument similar to that of the foregoing proof. The other statements follow from this and the previous theorem. □

Example 47. In a Cayley-Hamilton algebra of degree 2 the following identities are valid for all a, b

$$\begin{aligned} 0 &= a^2 - tr(a)a + \tfrac{1}{2}(tr(a)^2 - tr(a^2)) \\ a.b + b.a &= tr(ab) - tr(a)tr(b) + tr(a)b + tr(b)a \end{aligned}$$

The second identity follows from the first by replacing $a + b$ for a. Consider the free algebra on two generators $\langle 2\rangle = \mathbb{C}\langle x, y\rangle$ and consider in $\oint_2 \langle 2\rangle$ the subalgebra $\oint'$ generated the necklaces

$$\{tr(x), tr(y), tr(x^2), tr(y^2), tr(xy)\}$$

Using the two identities and $\oint_2 \langle 2\rangle$-linearity of the trace on $\int_2 \langle 2\rangle$ we see that the trace of any monomial in x and y of degree $k \geq 3$ can be expressed in elements of $\oint'$ and traces of monomials of degree $\leq k - 1$. We deduce that

$$\oint_2 \langle 2\rangle = \mathbb{C}[tr(x), tr(y), tr(x^2), tr(y^2), tr(xy)].$$

Note that there can be no algebraic relations between these generators because we can specialize to the 2×2 matrices in $\mathbb{C}[a,b,c,d,e,f]$

$$x \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad y \mapsto \begin{bmatrix} c & d \\ e & f \end{bmatrix}$$

and obtain algebraic independent polynomials. From the identities it follows that over $\oint_2 \langle 2 \rangle$ any monomial in x and y of degree $k \geq 3$ can be expressed as a linear combination of $1, x, y$ and xy and so these elements generate $\int_2 \langle 2 \rangle$ as a $\oint_2 \langle 2 \rangle$-module. In fact, they form a free basis. If not, there would be a relation say

$$xy = \alpha 1 + \beta x + \gamma y$$

with $\alpha, \beta, \gamma \in \oint_2 \langle 2 \rangle$. However, specializing

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{whence} \quad xy \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

we obtain a contradiction. Therefore,

$$\int_2 \langle 2 \rangle = \oint_2 \langle 2 \rangle.1 \oplus \oint_2 \langle 2 \rangle.x \oplus \oint_2 \langle 2 \rangle.y \oplus \oint_2 \langle 2 \rangle.xy$$

EXAMPLE 48. Consider the subalgebra R of $\oint_2 \langle 3 \rangle$ generated by the elements $tr(x), tr(y), tr(z), tr(x^2), tr(y^2), tr(z^2)$ and $tr(xy), tr(xz), tr(yz)$. Let Λ be the subalgebra of $\int_2 \langle 3 \rangle$ generated by R and x, y and z. It follows from the identities given in example 47 that Λ is a finitely generated module over R generated by the elements

$$\{1, x, y, z, xy, yz, xz, xyz\}$$

We will see in chapter 6 that the Krull dimension of $\oint_2 \langle 3 \rangle$ is 9 whence the generators of R are algebraically independent, that is

$$R = \mathbb{C}[tr(x), tr(y), tr(z), tr(xy), tr(xz), tr(yz), tr(x^2), tr(y^2), tr(z^2)]$$

Further, $\oint_2 \langle 3 \rangle$ is a quadratic extension of R as $tr(xyz) \notin R$ for otherwise there would be an homogeneous multilinear identity

$$tr(xyz) = \alpha tr(x)tr(y)tr(z) + \beta(tr(x)tr(yz) + tr(y)tr(xz) + tr(z)tr(xy))$$

which cannot exist by specializing the generators to the 2×2 matrices

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Moreover, taking traces of the identity

$$(xyz)^2 - tr(xyz)xyz + det(x)det(y)det(z) = 0$$

and simplifying the first term we get that $tr(xyz)$ satisfies a quadratic equation over R,

$$\oint_2 \langle 3 \rangle \simeq R.1 \oplus R.tr(xyz)$$

Let K be the field of fractions of R, then $K \otimes_R \int_2 \langle 3 \rangle$ has dimension 8 (again, this will follow from results from chapter 6) over K so the 8 module generators given before are a basis. Finally, as $tr(\Lambda) \subset \Lambda$ we obtain that $\Lambda = \int_2 \langle 3 \rangle$,

$$\int_2 \langle 3 \rangle = R.1 \oplus R.x \oplus R.y \oplus R.z \oplus R.xy \oplus R.xz \oplus R.yz \oplus R.xyz$$

For more details we refer to the paper [**40**]. For a description of $\oint_2 \langle d \rangle$ and $\int_2 \langle d \rangle$ we refer to the monograph [**38**] where $\int_2 \langle d \rangle$ (resp. $\oint_2 \langle d \rangle$) are called the trace ring (resp.the center of the trace ring) of d generic 2×2 matrices.

Example 49. Consider generic 3×3 trace zero matrices

$$X = x - \frac{1}{3}tr(x) \quad Y = y - \frac{1}{3}tr(y)$$

then it follows from the Cayley-Hamilton identity of $X + Y$ that the following relations hold

$$\begin{array}{lc} g_1 & X^3 + CX + F = 0 \\ g_2 & X^2Y + XYX + YX^2 + CY + DX + H = 0 \\ g_3 & Y^2X + YXY + XY^2 + DY + EX + G = 0 \\ g_4 & Y^3 + EY + I = 0 \end{array}$$

where we denote

$$C = -\frac{1}{2}tr(X^2) \quad D = -tr(XY) \quad E = -\frac{1}{2}tr(Y^2) \quad G = -tr(XY^2)$$

$$H = -tr(YX^2) \quad F = -\frac{1}{3}tr(X^3) \quad I = -\frac{1}{3}tr(Y^3)$$

Then one can prove that

$$\int_3 \langle 2 \rangle \simeq \frac{\mathbb{C}[tr(x), tr(y), C, D, E, F, G, H, I]\langle X, Y \rangle}{(g_1, g_2, g_3, g_4)}$$

which is a free module of rank 18 over the polynomial subalgebra of $\oint_3 \langle 2 \rangle$

$$\mathbb{C}[tr(x), tr(y), C, D, E, F, G, H, I, J]$$

where J is the central element

$$J = 2XYXY + X^2Y^2 + YX^2Y + YXYX + XY^2X + 2DXY + DYX + GX + HY$$

We refer to the paper [**41**] for full details.

3.3. Invariants of representations.

Recall from theorem 16 that there is an action of GL_n on $\mathtt{rep}_n A$, the orbits of which correspond to isomorphism classes of n-dimensional A-representations. Hence, GL_n acts by algebra isomorphisms on the coordinate ring $\mathbb{C}[\mathtt{rep}_n A]$. In this section we will prove that the algebra of invariant polynomials is generated by (traces of) necklaces.

Definition 38. The n-th invariant functor

$$\downarrow_n : \mathtt{alg} \longrightarrow \mathtt{commalg}$$

assigns to a $\mathbb{C}$-algebra A the ring of invariants of the GL_n-action on the n-th representation scheme $\mathtt{rep}_n\ A$

$$\downarrow_n A = \mathbb{C}[\mathtt{rep}_n A]^{GL_n}$$

The strategy we will use is to prove the generator result first for $A = \langle m \rangle$ and deduce the general result by applying the Reynold's operator. In the following chapter we will identify $\downarrow_n A$ with $\oint_n A$.

Recall that

$$\mathtt{rep}_n \langle m \rangle = \underbrace{M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})}_{m} = M_n^m$$

the affine space of m-tuples of $n \times n$ matrices on which GL_n acts by simultaneous conjugation. We have to determine the ring of all polynomial maps f

$$M_n^m = M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C}) \xrightarrow{f} \mathbb{C}$$

which are constant along orbits for this action. The strategy we follow is standard in invariant theory.

- First, we will determine the *multilinear* maps which are constant along orbits, equivalently, the *linear* maps

$$M_n^{\otimes m} = \underbrace{M_n(\mathbb{C}) \otimes \ldots \otimes M_n(\mathbb{C})}_{m} \longrightarrow \mathbb{C}$$

  which are constant along GL_n-orbits where GL_n acts by the diagonal action, that is,

$$g.(A_1 \otimes \ldots \otimes A_m) = gA_1g^{-1} \otimes \ldots \otimes gA_mg^{-1}.$$

- Next, we will be able to obtain from them all polynomial invariant maps by using *polarization* and *restitution* operations.

First, we translate the problem into classical invariant theory of GL_n. Let $V_n \simeq \mathbb{C}^n$ be the n-dimensional vectorspace of column vectors on which GL_n acts naturally by left multiplication

$$V_n = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \\ \vdots \\ \mathbb{C} \end{bmatrix} \qquad \text{with action} \qquad g. \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix}$$

In order to define an action on the dual space $V_n^* = Hom(V_n, \mathbb{C}) \simeq \mathbb{C}^n$ of *covectors* (or, row vectors) we have to use the *contragradient* action

$$V_n^* = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \ldots & \mathbb{C} \end{bmatrix} \qquad \text{with action} \qquad \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \end{bmatrix} .g^{-1}$$

Observe, that we have an *evaluation* map $V_n^* \times V_n \longrightarrow \mathbb{C}$ which is given by the scalar product $f(v)$ for all $f \in V_n^*$ and $v \in V_n$

$$\begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \end{bmatrix} . \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix} = \phi_1\nu_1 + \phi_2\nu_2 + \ldots + \phi_n\nu_n$$

which is invariant under the diagonal action of GL_n on $V_n^* \times V_n$. Further, we have the natural identification

$$M_n(\mathbb{C}) = V_n \otimes V_n^* = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \\ \vdots \\ \mathbb{C} \end{bmatrix} \otimes \begin{bmatrix} \mathbb{C} & \mathbb{C} & \ldots & \mathbb{C} \end{bmatrix}.$$

Under this identification, a *pure tensor* $v \otimes f$ corresponds to the rank one matrix or rank one endomorphism of V_n defined by

$$v \otimes f : V_n \longrightarrow V_n \quad \text{with} \;\; w \mapsto f(w)v$$

and observe that the rank one matrices span $M_n(\mathbb{C})$. The diagonal action of GL_n on $V_n \otimes V_n^*$ is then determined by its action on the pure tensors where it coincides with the action of conjugation on M_n.

Consider the identification

$$(V_n^{*\otimes m} \otimes V_n^{\otimes m})^* \simeq End(V_n^{\otimes m})$$

obtained from the nondegenerate pairing

$$End(V_n^{\otimes m}) \times (V_n^{*\otimes m} \otimes V_n^{\otimes m}) \longrightarrow \mathbb{C}$$

given by

$$\langle \lambda, f_1 \otimes \ldots \otimes f_m \otimes v_1 \otimes \ldots \otimes v_m \rangle = f_1 \otimes \ldots \otimes f_m(\lambda(v_1 \otimes \ldots \otimes v_m))$$

GL_n acts diagonally on $V_n^{\otimes m}$ and hence again by conjugation on $End(V_n^{\otimes m})$ after embedding $GL_n \hookrightarrow GL(V_n^{\otimes m}) = GL_{mn}$. Therefore, the identifications are isomorphism as vectorspaces with GL_n-action. Hence, the space of GL_n-invariant linear maps

$$V_n^{*\otimes m} \otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

is the space $End_{GL_n}(V_n^{\otimes m})$ of GL_n-linear endomorphisms of $V_n^{\otimes m}$.

There is a different presentation of this vectorspace relating it to the symmetric group. Recall that the diagonal action of GL_n on $V_n^{\otimes m}$ is given by

$$g.(v_1 \otimes \ldots \otimes v_m) = g.v_1 \otimes \ldots \otimes g.v_m$$

The symmetric group S_m on m letters on $V_n^{\otimes m}$ given by

$$\sigma.(v_1 \otimes \ldots \otimes v_m) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)}$$

These two actions commute with each other and give embeddings of GL_n and S_m in $End(V_n^{\otimes m})$. The subspace of $V_n^{\otimes m}$ spanned by the image of GL_n will be denoted by $\langle GL_n \rangle$. Similarly, with $\langle S_m \rangle$ we denote the subspace spanned by the image of S_m.

THEOREM 28 (Schur). *With notations as above we have :*

1. $\langle GL_n \rangle = End_{S_m}(V_n^{\otimes m})$
2. $\langle S_m \rangle = End_{GL_n}(V_n^{\otimes m})$

PROOF. (1) : Under the identification $End(V_n^{\otimes m}) = End(V_n)^{\otimes m}$ an element $g \in GL_n$ is mapped to the symmetric tensor $g \otimes \ldots \otimes g$. On the other hand, the image of $End_{S_m}(V_n^{\otimes m})$ in $End(V_n)^{\otimes m}$ is the subspace of all symmetric tensors in $End(V)^{\otimes m}$. We can give a basis of this subspace as follows. Let $\{e_1, \ldots, e_{n^2}\}$ be a basis of $End(V_n)$, then the vectors $e_{i_1} \otimes \ldots \otimes e_{i_m}$ form a basis of $End(V_n)^{\otimes m}$ which is stable under the S_m-action. Further, any S_m-orbit contains a unique representative of the form

$$e_1^{\otimes h_1} \otimes \ldots \otimes e_{n^2}^{\otimes h_{n^2}}$$

with $h_1 + \ldots + h_{n^2} = m$. If we denote by $r(h_1, \ldots, h_{n^2})$ the sum of all elements in the corresponding S_m-orbit then these vectors are a basis of the symmetric tensors in $End(V_n)^{\otimes m}$.

The claim follows if we can show that every linear map λ on the symmetric tensors which is zero on all $g \otimes \ldots \otimes g$ with $g \in GL_n$ is the zero map. Write $e = \sum x_i e_i$, then

$$\lambda(e \otimes \ldots \otimes e) = \sum x_1^{h_1} \ldots x_{n^2}^{h_{n^2}} \lambda(r(h_1, \ldots, h_{n^2}))$$

is a polynomial function on $End(V_n)$. As GL_n is a Zariski open subset of $End(V)$ on which by assumption this polynomial vanishes, it must be the zero polynomial. Therefore, $\lambda(r(h_1, \ldots, h_{n^2})) = 0$ for all $(h_1, \ldots, h_{n^2})$ finishing the proof.

(2) : Recall that the groupalgebra $\mathbb{C}S_m$ of S_m is a *semisimple algebra* . Any epimorphic image of a semisimple algebra is semisimple. Therefore, $\langle S_m \rangle$ is a semisimple subalgebra of the matrixalgebra $End(V_n^{\otimes m}) \simeq M_{nm}$. By the *double centralizer theorem* (see for example [**51**, §12.7]), it is therefore equal to the centralizer of $End_{S_m}(V_m^{\otimes m})$. By the first part, it is the centralizer of $\langle GL_n \rangle$ in $End(V_n^{\otimes m})$ and therefore equal to $End_{GL_n}(V_n^{\otimes m})$. □

By (2), every GL_n-endomorphism of $V_n^{\otimes m}$ can be written as a linear combination of the morphisms λ_σ describing the action of $\sigma \in S_m$ on $V_n^{\otimes m}$. We will trace back these morphisms λ_σ through the canonical identifications until we can express them in terms of matrices.

THEOREM 29 (Procesi-Razmyslov). .

Let $\sigma = (i_1 i_2 \ldots i_\alpha)(j_1 j_2 \ldots j_\beta) \ldots (z_1 z_2 \ldots z_\zeta)$ be a decomposition of $\sigma \in S_m$ into cycles (including those of length one). Then, under the above identification we have

$$\mu_\sigma(A_1 \otimes \ldots \otimes A_m) = tr(A_{i_1} A_{i_2} \ldots A_{i_\alpha}) tr(A_{j_1} A_{j_2} \ldots A_{j_\beta}) \ldots tr(A_{z_1} A_{z_2} \ldots A_{z_\zeta})$$

where μ_σ is the linear invariant $V_n^{*\otimes m} \otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$ corresponding to λ_σ under the identification $(V_n^{*\otimes m} \otimes V_n^{\otimes m})^* \simeq End(V_n^{\otimes m})$. That is, $\mu_\sigma(f_1 \otimes \ldots f_m \otimes v_1 \otimes \ldots \otimes v_m)$ is equal to

$$\begin{aligned} \langle \lambda_\sigma, f_1 \otimes \ldots f_m \otimes v_1 \otimes \ldots \otimes v_m \rangle &= f_1 \otimes \ldots \otimes f_m(v_{\sigma(1)} \otimes \ldots v_{\sigma(m)}) \\ &= \prod_i f_i(v_{\sigma(i)}) \end{aligned}$$

PROOF. We know that every multilinear GL_n-invariant map

$$\gamma : V_n^{*\otimes m} \otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

is a linear combination of the invariants μ_σ, $\sigma \in S_m$. Under $M_n(\mathbb{C}) = V_n \otimes V_n^*$ a multilinear GL_n-invariant map

$$(V_n^* \otimes V)_n^{\otimes m} = V_n^{*\otimes m} \otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

corresponds to a multilinear GL_n-invariant map

$$M_n(\mathbb{C}) \otimes \ldots \otimes M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

Under the identification, matrix multiplication is induced by composition on rank one endomorphisms and here the rule is given by

$$v \otimes f . v' \otimes f' = f(v') v \otimes f'$$

$$\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \ldots & \phi_n \end{bmatrix} . \begin{bmatrix} \nu'_1 \\ \vdots \\ \nu'_n \end{bmatrix} \otimes \begin{bmatrix} \phi'_1 & \ldots & \phi'_n \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} f(v') \otimes \begin{bmatrix} \phi'_1 & \ldots & \phi'_n \end{bmatrix} .$$

Moreover, the trace map on M_n is induced by that on rank one endomorphisms where it is given by the rule

$$tr(v \otimes f) = f(v)$$

$$tr(\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \dots & \phi_n \end{bmatrix}) = tr(\begin{bmatrix} \nu_1\phi_1 & \dots & \nu_1\phi_n \\ \vdots & \ddots & \vdots \\ \nu_n\phi_1 & \dots & \nu_n\phi_n \end{bmatrix}) = \sum_i \nu_i\phi_i = f(v)$$

Both sides of identity in the statement are multilinear hence it suffices to verify the equality for rank one matrices. Write $A_i = v_i \otimes f_i$, then we have that

$$\begin{aligned} \mu_\sigma(A_1 \otimes \dots \otimes A_m) = & \quad \mu_\sigma(v_1 \otimes \dots v_m \otimes f_1 \otimes \dots \otimes f_m) \\ = & \quad \prod_i f_i(v_{\sigma(i)}) \end{aligned}$$

Consider the subproduct

$$f_{i_1}(v_{i_2}) f_{i_2}(v_{i_3}) \dots f_{i_{\alpha-1}}(v_{i_\alpha}) = S$$

Now, look at the matrixproduct

$$v_{i_1} \otimes f_{i_1}.v_{i_2} \otimes f_{i_2}. \ \dots \ .v_{i_\alpha} \otimes f_{i_\alpha}$$

which is by the product rule equal to

$$f_{i_1}(v_{i_2}) f_{i_2}(v_{i_3}) \dots f_{i_{\alpha-1}}(v_{i_\alpha}) v_{i_1} \otimes f_{i_\alpha}$$

Hence, by the trace rule we have that

$$tr(A_{i_1} A_{i_2} \dots A_{i_\alpha}) = \prod_{j=1}^{\alpha} f_{i_j}(v_{\sigma(i_j)}) = S$$

□

Having found a description of the multilinear invariant polynomial maps

$$M_n^m = \underbrace{M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})}_{m} \longrightarrow \mathbb{C}$$

we will now describe all polynomial maps which are constant along orbits by polarization.

Theorem 30 (First fundamental theorem of matrix invariants). *Any invariant function from $\mathbb{C}[M_n^m]^{GL_n} = \mathbb{C}[\mathtt{rep}_m \langle m \rangle]^{GL_n}$ is a polynomial in the invariants*

$$tr(X_{i_1} \dots X_{i_l})$$

where $X_{i_1} \dots X_{i_l}$ run over all possible noncommutative polynomials in the generic matrices $\{X_1, \dots, X_m\}$. In particular, there is an algebra epimorphism

$$\oint \langle m \rangle \longrightarrow\!\!\!\!\!\to \ \downarrow_n \langle m \rangle$$

Proof. The coordinate algebra $\mathbb{C}[\mathtt{rep}_n \langle m \rangle]$ is the polynomial ring in mn^2 variables $x_{ij}(k)$ where $1 \leq k \leq m$ and $1 \leq i, j \leq n$. Consider the m generic $n \times n$ matrices

$$X_k = \begin{bmatrix} x_{11}(k) & \dots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \dots & x_{nn}(k) \end{bmatrix} \in M_n(\mathbb{C}[M_n^m]) = M_n(\mathbb{C}[\mathtt{rep}_n \ \langle m \rangle]).$$

The action of GL_n on polynomial maps $f \in \mathbb{C}[M_n^m]$ is fully determined by the action on the coordinate functions $x_{ij}(k)$ given by

$$g.x_{ij}(k) = (g^{-1}.X_k.g)_{ij}.$$

This action preserves the subspaces spanned by the entries of any of the generic matrices.

Hence, we can define a $\mathbb{Z}^m$-gradation on $\mathbb{C}[M_n^m]$ by $deg(x_{ij}(k)) = (0,\dots,0,1,0,\dots,0)$ (with 1 at place k) and decompose

$$\mathbb{C}[M_n^m] = \bigoplus_{(d_1,\dots,d_m)\in\mathbb{N}^m} \mathbb{C}[M_n^m]_{(d_1,\dots,d_m)}$$

where $\mathbb{C}[M_n^m]_{(d_1,\dots,d_m)}$ is the subspace of all multihomogeneous forms f in the $x_{ij}(k)$ of degree $(d_1,\dots,d_m)$, that is, in each monomial term of f there are exactly d_k factors coming from the entries of the generic matrix X_k for all $1 \leq k \leq m$. The action of GL_n stabilizes each of these subspaces, that is,

$$\text{if} \quad f \in \mathbb{C}[M_n^m]_{(d_1,\dots,d_m)} \quad \text{then} \quad g.f \in \mathbb{C}[M_n^m]_{(d_1,\dots,d_m)} \quad \text{for all } g \in GL_n.$$

In particular, if f determines a polynomial map on M_n^m which is constant along orbits, that is, if f belongs to the ring of invariants $\mathbb{C}[M_n^m]^{GL_n}$ then each of its multihomogeneous components is also an invariant and therefore it suffices to determine all multihomogeneous invariants.

Let $f \in \mathbb{C}[M_n^m]_{(d_1,\dots,d_m)}$ and take for each $1 \leq k \leq m$ d_k new variables $t_1(k),\dots,t_{d_k}(k)$. Expand

$$f(t_1(1)A_1(1) + \dots + t_{d_1}A_{d_1}(1), \dots, t_1(m)A_1(m) + \dots + t_{d_m}(m)A_{d_m}(m))$$

as a polynomial in the variables $t_i(k)$, then we get an expression

$$\sum t_1(1)^{s_1(1)} \dots t_{d_1}^{s_{d_1}(1)} \dots t_1(m)^{s_1(m)} \dots t_{d_m}(m)^{s_{d_m}(m)}. \\ f_{(s_1(1),\dots,s_{d_1}(1),\dots,s_1(m),\dots,s_{d_m}(m))}(A_1(1),\dots,A_{d_1}(1),\dots,A_1(m),\dots,A_{d_m}(m))$$

such that for all $1 \leq k \leq m$ we have $\sum_{i=1}^{d_k} s_i(k) = d_k$. Moreover, each of the $f_{(s_1(1),\dots,s_{d_1}(1),\dots,s_1(m),\dots,s_{d_m}(m))}$ is a multi-homogeneous polynomial function on

$$\underbrace{M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})}_{d_1} \oplus \underbrace{M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})}_{d_2} \oplus \dots \oplus \underbrace{M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})}_{d_m}$$

of multi-degree $(s_1(1),\dots,s_{d_1}(1),\dots,s_1(m),\dots,s_{d_m}(m))$. Observe that if f is an invariant polynomial function on M_n^m, then each of these multi homogeneous functions is an invariant polynomial function on M_n^D where $D = d_1 + \dots + d_m$.

In particular, we consider the multi-*linear* function

$$f_{1,\dots,1} : M_n^D = M_n^{d_1} \oplus \dots \oplus M_n^{d_m} \longrightarrow \mathbb{C}$$

which we call the *polarization* of the polynomial f and denote with $Pol(f)$. Observe that $Pol(f)$ in symmetric in each of the entries belonging to a block $M_n^{d_k}$ for every $1 \leq k \leq m$. If f is invariant under GL_n, then so is the multilinear function $Pol(f)$ and we know the form of all such functions by the results given before (replacing M_n^m by M_n^D).

We want to recover f back from its polarization. We claim to have the equality

$$Pol(f)(\underbrace{A_1,\dots,A_1}_{d_1},\dots,\underbrace{A_m,\dots,A_m}_{d_m}) = d_1!\dots d_m! f(A_1,\dots,A_m)$$

and hence we recover f. This process is called *restitution* . The claim follows from the observation that

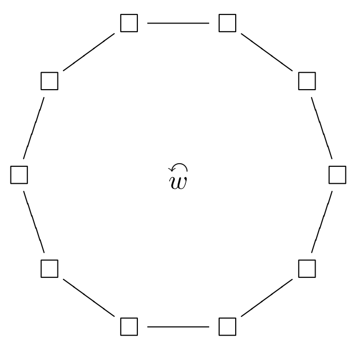
$$f(t_1(1)A_1 + \ldots + t_{d_1}(1)A_1, \ldots, t_1(m)A_m + \ldots + t_{d_m}(m)A_m) =$$
$$f((t_1(1) + \ldots + t_{d_1}(1))A_1, \ldots, (t_1(m) + \ldots + t_{d_m}(m))A_m) =$$
$$(t_1(1) + \ldots + t_{d_1}(1))^{d_1} \ldots (t_1(m) + \ldots + t_{d_m}(m))^{d_m} f(A_1, \ldots, A_m)$$

and the definition of $Pol(f)$. Hence we have proved that any multi-homogeneous invariant polynomial function f on M_n^m of multidegree $(d_1, \ldots, d_m)$ can be obtained by restitution of a multilinear invariant function

$$Pol(f) : M_n^D = M_n^{d_1} \oplus \ldots \oplus M_n^{d_m} \longrightarrow \mathbb{C}$$

If we combine this with the description of all multilinear invariant functions we obtain the first part of the theorem.

The last statement follows from the observation that the generators $tr(X_{i_1}X_{i_2}\ldots X_{i_l})$ are only determined up to cyclic permutation of the factors X_j. That is, they correspond to a necklace word w



where each i-colored bead corresponds to a generic matrix X_i. These bead-matrices are cyclically multiplied to obtain an $n \times n$ matrix with coefficients in $M_n(\mathbb{C}[M_n^m])$. The trace of this matrix is called $tr(w)$ and they generate the ring of polynomial invariants. $\square$

Example 50. The Jordan normalform can be used to give a direct proof of the fact that the polynomial functions on $\mathtt{rep}_1\langle 1\rangle = M_n^1 = M_n(\mathbb{C})$ which are constant along orbits are polynomials in the traces of the generic $n \times n$ matrix

$$X = \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix}$$

Construct the continuous map

$$M_n \xrightarrow{\pi} \mathbb{C}^n$$

sending a matrix $A \in M_n$ to the point $(\sigma_1(A), \ldots, \sigma_n(A))$ in $\mathbb{C}^n$, where $\sigma_i(A)$ is the i-th elementary symmetric function in the eigenvalues of A (which is a polynomial in the traces of powers of A). Clearly, this map is constant along orbits. We claim that π is surjective. Take any point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ and consider the matrix $A \in M_n$

$$A = \begin{bmatrix} 0 & & & & a_n \\ -1 & 0 & & & a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & a_2 \\ & & & -1 & a_1 \end{bmatrix} \tag{3.2}$$

then $\pi(A) = (a_1, \ldots, a_n)$, that is,

$$det(t\mathbb{1}_n - A) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \ldots + (-1)^n a_n.$$

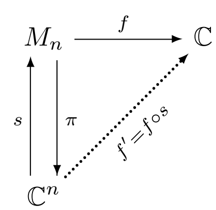
We call a matrix $B \in M_n$ *cyclic* if there is a (column) vector $v \in \mathbb{C}^n$ such that $\mathbb{C}^n$ is spanned by the vectors $\{v, B.v, B^2.v, \ldots, B^{n-1}.v\}$. Let $g \in GL_n$ be the basechange transforming the standard basis to the ordered basis

$$(v, -B.v, B^2.v, -B^3.v, \ldots, (-1)^{n-1}B^{n-1}.v).$$

In this new basis, the linear map determined by B (or equivalently, $g.B.g^{-1}$) is equal to the matrix in canonical form

$$\begin{bmatrix} 0 & & & & b_n \\ -1 & 0 & & & b_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & b_2 \\ & & & -1 & b_1 \end{bmatrix}$$

where $B^n.v$ has coordinates $(b_n, \ldots, b_2, b_1)$ in the new basis. Conversely, any matrix in this form is a cyclic matrix. By taking the determinant of the $n \times n$ matrix with columns $v, B.v, \ldots, B^{n-1}.v$ for a generic vector v we see that the set of all cyclic matrices B forms a Zariski open subset of $M_n(\mathbb{C})$. Let f be a polynomial function on $M_n(\mathbb{C})$ which is constant along orbits and consider the diagram



where s is the *section* of π (that is, $\pi \circ s = id_{\mathbb{C}^n}$) determined by sending a point $(a_1, \ldots, a_n)$ to the cyclic matrix in canonical form A as in equation (3.2). We claim that $f = f' \circ \pi$ for $f' = f \circ s$ a polynomial in the σ_i (or equivalently in the traces of powers of the generic matrix X). By continuity, it suffices to check equality on the dense open set of cyclic matrices in M_n. There it is a consequence of the following three facts we have proved before : (1) : any cyclic matrix lies in the same orbit as one in standard form, (2) : s is a section of π and (3) : f is constant along orbits.

THEOREM 31. *For any affine $\mathbb{C}$-algebra A, there is an algebra epimorphism*

$$\oint A \xrightarrow{\pi_n} \downarrow_n A$$

That is, the ring of invariants $\mathbb{C}[\mathtt{rep}_n A]^{GL_n}$, is generated by traces of necklaces words.

Proof. We only need to recall the construction of the Reynolds operator. Let V and W be two $\mathbb{C}$-vectorspaces with a locally finite GL_n-action and let $V \xrightarrow{f} W$ be a GL_n-equivariant linear map. The *Reynolds operator* R is the canonical projection to the isotypical component of the trivial representation (for unexplained terminology refer to section 4.2). There is a commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow R & & \downarrow R \\ V^{GL_n} & \xrightarrow{f_0} & W^{GL_n} \end{array}$$

and it follows from complete reducibility of GL_n-representations that f_0 is surjective (resp. injective) if f is surjective (resp. injective). The statement then follows from the surjection $\mathbb{C}[\mathtt{rep}_n \langle m \rangle] \twoheadrightarrow \mathbb{C}[\mathtt{rep}_n A]$ and the previous theorem. □

Example 51. (Invariants of quiver-representations) Recall from example 25 that the n-th representation scheme of $\langle Q \rangle$ decomposes into smooth connected components

$$\mathtt{rep}_n \langle Q \rangle = \bigsqcup_{|\alpha|=n} GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha Q$$

Therefore, $\downarrow_n \langle Q \rangle = \mathbb{C}[\mathtt{rep}_n \langle Q \rangle]^{GL_n}$ decomposes into

$$\bigoplus_{|\alpha|=n} \mathbb{C}[GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha Q]^{GL_n}$$

If $H \subset G$ are reductive groups and V an H-representation, then we have for the invariants of the associated fiber product

$$\mathbb{C}[G \times^H V]^G \simeq \mathbb{C}[V]^H$$

Applying this to the action of the basechange group $GL(\alpha)$ on $\mathtt{rep}_\alpha Q$ we get

$$\downarrow_n \langle Q \rangle = \bigoplus_{|\alpha|=n} \mathbb{C}[\mathtt{rep}_\alpha Q]^{GL(\alpha)}$$

where the components are called the *invariants of α-dimensional quiver representations* .

A generating set for the path algebra $\langle Q \rangle$ is given by the vertex-idempotents $v_1, \ldots, v_k$ and the arrows $a_1, \ldots, a_l$ giving an epimorphism $\langle m \rangle \twoheadrightarrow \langle Q \rangle$ with $m = k + l$. This epimorphism induces the epimorphism

$$\downarrow_n \langle m \rangle \overset{\pi}{\twoheadrightarrow} \downarrow_n \langle Q \rangle \twoheadrightarrow \mathbb{C}[\mathtt{rep}_\alpha Q]^{GL(\alpha)}$$

and to determine the generators of the quiver invariants we have to follow the image of the generic matrices under these maps and take traces of necklace words. For the first k variables, the generic matrix X_i is mapped to the $n \times n$ matrix

$$v_i \mapsto \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

with 1's from position $\sum_{l=1}^{i-1} d_l + 1$ to position $\sum_{l=1}^{i} d_l$ for $\alpha = (d_1, \ldots, d_k)$. For the last l variables, the generic matrix X_a corresponding to the arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ is mapped to the block matrix in $M_n(\mathbb{C}[\mathtt{rep}_\alpha\ Q])$

$$\begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & & & \vdots \\ \vdots & \boxed{M_a} & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}$$

where M_a is the $d_j \times d_i$ matrix of variables from $\mathbb{C}[\mathtt{rep}_\alpha\ Q]$

$$\boxed{M_a} = \begin{bmatrix} x_{11}(a) & \ldots & \ldots & x_{1d_i}(a) \\ \vdots & & & \vdots \\ x_{d_j1}(a) & \ldots & \ldots & x_{d_jd_i}(a) \end{bmatrix}$$

We know that the ring of quiver invariants $\mathbb{C}[\mathtt{rep}_\alpha Q]^{GL(\alpha)}$ is generated by the images of the traces of necklacewords $tr(X_{i_1} \ldots X_{i_r})$. Using the explicit block-form of the matrices, we see that such a trace is zero unless the induced path $\pi(x_{i_1} \ldots x_{i_r})$ is an *oriented cycle* in the quiver Q. We recover the result, proved in [**42**] that

$$\mathbb{C}[\mathtt{rep}_\alpha Q]^{GL(\alpha)}$$

is generated by traces along oriented cycles in the quiver Q. By this we mean that we multiply the $n \times n$ matrices corresponding to the vertices and arrows in order and take the trace of the obtained $n \times n$ matrix with coefficients in $\mathbb{C}[\mathtt{rep}_\alpha Q]$.

EXAMPLE 52. If $A \in \mathtt{alg}$ is affine it has a presentation $A = \langle m \rangle / R_A$ where R_A is the ideal of relations holding in A. It follows from the Reynold operator that

$$\downarrow_n A = \mathbb{C}[\mathtt{rep}_n A]^{GL_n}$$

is generated by (traces of) necklaces of words in the generators $a_1, \ldots, a_m$ of A. If A is the path algebra of a quiver with relations, then we can restrict attention to necklaces in the quiver. In particular this applies to the universal localization $\langle Q \rangle_\Sigma$ and the n-th root algebra $\sqrt[n]{\langle Q \rangle}$ of the path algebra of a finite quiver Q.

3.4. Necklace Lie algebras.

In this section we will size down the complex of noncommutative differential forms by dividing out the super-commutators. The zero term in this Karoubi complex can be viewed as the noncommutative functions and is the space spanned by all necklaces. More generally, terms in the Karoubi complex induce GL_n-invariant differential forms on the representation schemes. Moreover, for a symmetric quiver we will define a Poisson structure on this space.

We have seen that the $*$-Kontsevich bracket induces a Lie algebra structure on $\mathtt{neck}_d$ the space spanned by all necklace words of $\langle d \rangle$. In this section we will extend this to necklace Lie algebras of quivers and relate them to noncommutative differential forms and to noncommutative symplectic geometry.

For $A \in \mathtt{alg}$, we define for $\omega \in \Omega^i A$ and $\omega' \in \Omega^j A$ the *super-commutator* to be

$$[\omega, \omega'] = \omega\omega' - (-1)^{ij}\omega'\omega$$

That is, it is the usual commutator unless both i and j are odd in which case it is the sum $\omega\omega' + \omega'\omega$.

Definition 39. The differential d is a super-derivation on ΩA whence

$$d([\omega, \omega']) = [d\omega, \omega'] + (-1)^i [\omega, d\omega']$$

Therefore, if we define

$$\mathtt{DR}^n\ A = \frac{\Omega^n A}{\sum_{i=0}^n [\Omega^i A, \Omega^{n-i} A]}$$

Then the $\mathtt{dgalg}$-structure on $\Omega\ A$ induces one on the complex

$$\mathtt{DR}^0\ A \xrightarrow{d} \mathtt{DR}^1\ A \xrightarrow{d} \mathtt{DR}^2\ A \xrightarrow{d} \dots$$

which is called the *Karoubi complex* of A.

Example 53. Terms of the Karoubi complex induce ordinary differential forms on the smooth manifolds $\mathtt{rep}_n A$ whenever A is $\mathtt{alg}$-smooth. $\mathtt{DR}^0 A = \frac{A}{[A,A]_v}$ can be viewed as the space of *noncommutative functions* on A. Elements of A induce matrix valued functions on $\mathtt{rep}_n A$ hence taking traces gives a linear map

$$\mathtt{DR}^0 A = \frac{A}{[A,A]_v} \xrightarrow{tr} \mathbb{C}[\mathtt{rep}_n A]$$

which are even GL_n-invariant. More generally, any element of ΩA induces a matrix valued differential form on $\mathtt{rep}_n A$ and taking traces gives a differential form on $\mathtt{rep}_n A$. Using the vanishing of the trace of commutators we see that this map factors through the Karoubi complex

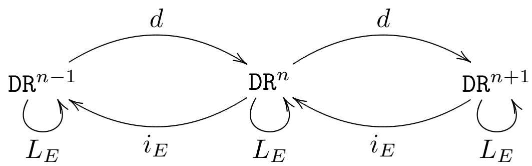
$$\Omega A \twoheadrightarrow \mathtt{DR}^* A \longrightarrow \Omega^* \mathbb{C}[\mathtt{rep}_n A]$$

That is, taking traces of noncommutative differential forms gives a uniform way to define GL_n-invariant differential forms on all the representation spaces $\mathtt{rep}_n A$ for $n \in \mathbb{N}$.

Definition 40. We define the (noncommutative) *de Rham cohomology groups* of A to be the homology of the Karoubi complex, that is

$$H^n_{dR}\ A = \frac{\mathtt{Ker}\ \mathtt{DR}^n\ A \xrightarrow{d} \mathtt{DR}^{n+1}\ A}{\mathtt{Im}\ \mathtt{DR}^{n-1}\ A \xrightarrow{d} \mathtt{DR}^n\ A}$$

Example 54. The de Rham cohomology of $\langle m \rangle$. Let E be the Eulerian derivation on $\langle m \rangle$ and verify that d is compatible with the subspaces of super-commutators for i_E and L_E. The induced operations



are such that L_E is an isomorphism on $\mathtt{DR}^n\ \langle m \rangle$ whenever $n \geq 1$ and still satisfy $L_E = i_E \circ d + d \circ i_E$. Therefore,

$$\mathtt{H}^n_{dR}\ \langle m \rangle = \begin{cases} \mathbb{C} & \text{when } n = 0\ , \\ 0 & \text{when } n \geq 1. \end{cases}$$

Definition 41. For a $\mathbb{C}$-subalgebra $B \subset A$, define a *relative Karoubi complex*

$$\mathtt{DR}^0_B\ A \xrightarrow{d} \mathtt{DR}^1_B\ A \xrightarrow{d} \mathtt{DR}^2_B\ A \xrightarrow{d} \ \dots$$

where

$$\mathtt{DR}^n_B\ A = \frac{\Omega^n_B\ A}{\sum_{i=0}^n [\ \Omega^i_B\ A, \Omega^{n-i}_B\ A\ ]}$$

The (noncommutative) *relative de Rham cohomology groups* of A with respect to B is the homology of this complex

$$\mathtt{H}^n_{B,dR}\ A = \frac{\mathtt{Ker}\ \mathtt{DR}^n_B\ A \xrightarrow{d} \mathtt{DR}^{n+1}_B\ A}{\mathtt{Im}\ \mathtt{DR}^{n-1}_B\ A \xrightarrow{d} \mathtt{DR}^n_B\ A}$$

Example 55. The C_k-relative de Rham cohomology of $\langle Q \rangle$. Again one can use the Eulerian C_k-derivation on $\langle Q \rangle$ to prove that

$$\begin{cases} H^0_{C_k,dR}\ \mathbb{C}Q & \simeq \mathbb{C} \times \dots \times \mathbb{C}\ (k \text{ factors}) \\ H^n_{C_k,dR}\ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

Definition 42. A *quiver necklace word* w in the quiver Q is an equivalence class of an oriented cycle $c = a_1 \dots a_l$ of length $l \geq 0$ in Q. Here, $c \sim c'$ if c' is obtained from c by cyclicly permuting the composing arrows a_i.

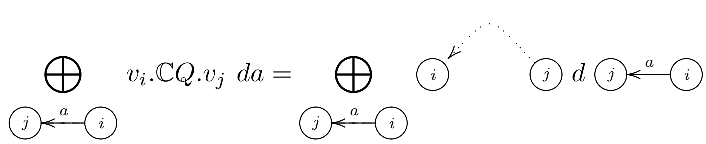
Theorem 32. *With notations as before, we have*

(1) *A $\mathbb{C}$-basis for the* noncommutative quiver functions

$$\mathtt{DR}^0_{C_k}\ \langle Q \rangle \simeq \frac{\langle Q \rangle}{[\ \langle Q \rangle, \langle Q \rangle\ ]_v}$$

is given by the quiver necklace words in the quiver Q.

(2) *The space of* noncommutative quiver 1-forms $\mathtt{DR}^1_{C_k}\ \langle Q \rangle$ *is*



Proof. (1) : Let $\mathtt{neck}_Q$ be the $\mathbb{C}$-space spanned by all quiver necklace words w in Q and define a linear map

$$\langle Q \rangle \overset{n}{\twoheadrightarrow} \mathbb{W} \qquad \begin{cases} p \mapsto w_p & \text{if } p \text{ is a cycle} \\ p \mapsto 0 & \text{if } p \text{ is not} \end{cases}$$

for all oriented paths p in the quiver Q, where w_p is the necklace word in Q determined by the oriented cycle p. Because $w_{p_1p_2} = w_{p_2p_1}$ it follows that the commutator subspace $[\langle Q \rangle, \langle Q \rangle]$ belongs to the kernel of this map. Conversely, let

$$x = x_0 + x_1 + \ldots + x_m$$

be in the kernel where x_0 is a linear combination of non-cyclic paths and x_i for $1 \leq i \leq m$ is a linear combination of cyclic paths mapping to the same necklace word w_i, then $n(x_i) = 0$ for all $i \geq 0$. Clearly, $x_0 \in [\langle Q \rangle, \langle Q \rangle]$ as we can write every noncyclic path $p = a.p' = a.p' - p'.a$ as a commutator. If $x_i = a_1p_1 + a_2p_2 + \ldots + a_lp_l$ with $n(p_i) = w_i$, then $p_1 = q.q'$ and $p_2 = q'.q$ for some paths q, q' whence $p_1 - p_2$ is a commutator. But then, $x_i = a_1(p_1 - p_2) + (a_2 - a_1)p_2 + \ldots + a_lp_l$ is a sum of a commutator and a linear combination of strictly fewer elements. By induction, this shows that $x_i \in [\langle Q \rangle, \langle Q \rangle]$.

(2) : If $p.q$ is not a cycle, then $pdq = [p, dq]$ and so vanishes in $\mathtt{DR}^1_{C_k}\ \langle Q \rangle$ so we only have to consider terms pdq with $p.q$ an oriented cycle in Q. For any three paths p, q and r in Q we have the equality

$$[p.qdr] = pqdr - qd(rp) + qrdp$$

whence in $\mathtt{DR}^1_{C_k}\ \langle Q \rangle$ we have relations allowing to reduce the length of the differential part

$$qd(rp) = pqdr + qrdp$$

so $\mathtt{DR}^1_{C_k}\ \langle Q \rangle$ is spanned by terms of the form pda with $a \in Q_a$ and $p.a$ an oriented cycle in Q. Therefore, we have a surjection

$$\Omega^1_{C_k}\ \langle Q \rangle \twoheadrightarrow \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} v_i.\mathbb{C}Q.v_j\ da$$

By construction, it is clear that $[\Omega^0_{rel}\ \mathbb{C}Q, \Omega^1_{rel}\ \mathbb{C}Q]$ lies in the kernel of this map and using an argument as in the lemma above one shows also the converse inclusion. □

Definition 43. The description of $\mathtt{DR}^i_{C_k}\ \langle Q \rangle$ for $i = 0, 1$ and the differential $\mathtt{DR}^0_{C_k}\ \langle Q \rangle \xrightarrow{d} \mathtt{DR}^1_{C_k}\ \langle Q \rangle$ allow us to define *quiver partial derivatives* associated to an arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in Q.

$$\frac{\partial}{\partial a}\ :\ \mathtt{DR}^0_{C_k}\ \langle Q \rangle \longrightarrow v_i \langle Q \rangle v_j \qquad \text{by} \qquad df = \sum_{a \in Q_a} \frac{\partial f}{\partial a} da$$

To compute the partial derivative of a quiver necklace word w with respect to an arrow a, we run through w and each time we encounter a we open the necklace by removing that occurrence of a and then take the sum of all the paths obtained.

To define a Kontsevich bracket on $\mathtt{neck}_d$ we needed an involution $*$ on the generators. In particular, d must be even. A similar restriction will be needed in order to define a Lie algebra structure on the space $\mathtt{neck}_Q$.

Definition 44. A quiver Q is said to be *symmetric* if for all vertices v_i and v_j we have

$$\#\ \{\ \textcircled{j} \longleftarrow \textcircled{i}\ \} = \#\ \{\ \textcircled{i} \longleftarrow \textcircled{j}\ \}$$

or, equivalently, if the Euler form of Q is symmetric.

If Q is symmetric, a *quiver involution* $*$ is an involution on the set $\{a_1, \ldots, a_l\}$ of arrows of Q such that if

$$\textcircled{\scriptsize j} \xleftarrow{a} \textcircled{\scriptsize i} \qquad \text{then} \qquad \textcircled{\scriptsize i} \xleftarrow{a^*} \textcircled{\scriptsize j}$$

Given a quiver involution $*$ we can partition the arrows of Q

$$Q_a = L \sqcup R \qquad \text{such that} \qquad L^* = R$$

We call such a partition a *symplectic quiver structure* on Q.

Let us recall the relevant notions in the commutative case. A *symplectic structure* on a (commutative) manifold M is given by a closed differential 2-form. The non-degenerate 2-form ω gives a canonical isomorphism

$$T\ M \simeq T^*\ M$$

that is, between vector fields on M and differential 1-forms. Further, there is a unique $\mathbb{C}$-linear map from functions f on M to vectorfields ξ_f by the requirement that $-df = i_{\xi_f}\omega$ where i_ξ is the contraction of n-forms to $n-1$-forms using the vectorfield ξ. We can make the functions on M into a *Poisson algebra* by defining

$$\{f, g\} = \omega(\xi_f, \xi_g)$$

and one verifies that this bracket satisfies the Jacobi and Leibnitz identities.

The *Lie derivative* L_ξ with respect to ξ is defined by the Cartan homotopy formula

$$L_\xi\ \varphi = i_\xi d\varphi + d i_\xi \varphi$$

for any differential form φ. A vectorfield ξ is said to be *symplectic* if it preserves the symplectic form, that is, $L_\xi \omega = 0$. In particular, for any function f on M we have that ξ_f is symplectic. The assignment

$$f \longrightarrow \xi_f$$

defines a Lie algebra morphism from the functions $\mathcal{O}(M)$ on M equipped with the Poisson bracket to the Lie algebra of symplectic vectorfields, $Vect_\omega\ M$. This map fits into the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(M) \longrightarrow Vect_\omega\ M \longrightarrow H^1_{dR}\ M \longrightarrow 0$$

DEFINITION 45. A *noncommutative quiver vectorfield* is a C_k-derivation θ of $\langle Q \rangle$. the set of all quiver vectorfields will be denoted by $Der_{C_k}\langle Q \rangle$.

If $*$ is a quiver involution and $Q_a = L \sqcup R$ a quiver symplectic structure we define the *symplectic 2-form*

$$\omega = \sum_{a \in L} da da^* \in \mathtt{DR}^2_{C_k}\langle Q \rangle$$

A vectorfield $\theta \in Der_{C_k}\langle Q \rangle$ is said to be *symplectic* if $L_\theta \omega = 0$ in $\mathtt{DR}^2_{C_k}\langle Q \rangle$. The set of all symplectic vectorfields is denoted by $Der_\omega\langle Q \rangle$.

THEOREM 33. *Given a symplectic structure $Q_a = L \sqcup R$ on a symmetric quiver, there is a one-to-one correspondence between*

(1) *noncommutative quiver 1-forms, and*
(2) *noncommutative quiver vectorfields.*

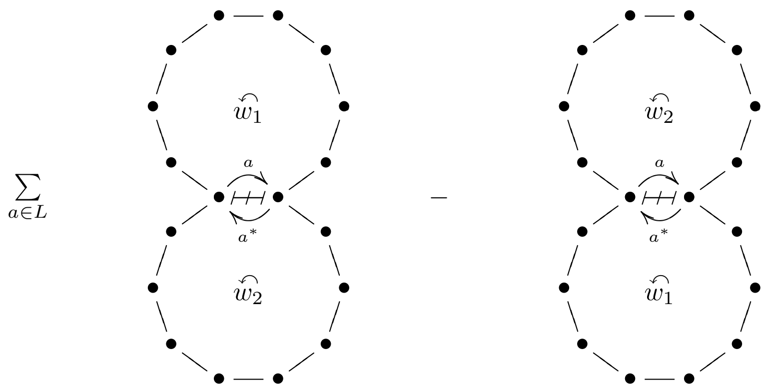


FIGURE 3. Kontsevich bracket $\{w_1, w_2\}_K$.

PROOF. For $\theta \in Der_{C_k}\langle Q\rangle$ define operators L_θ and i_θ on $\Omega_{C_k}\langle Q\rangle$ and on $\mathtt{DR}_{C_k}\langle Q\rangle$ by

$$\begin{cases} L_\theta(a) = \theta(a) & L_\theta(da) = d\theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

These operators allow us to define a linear map

$$Der_{C_k}\langle Q\rangle \xrightarrow{\tau} \mathtt{DR}^1_{C_k}\langle Q\rangle \qquad \text{by} \qquad \tau(\theta) = i_\theta(\omega)$$

Every C_k-derivation θ on $\langle Q\rangle$ is fully determined by its image on the arrows in Q and if $a = \textcircled{j} \xleftarrow{a} \textcircled{i}$

$$\theta(a) = \theta(v_j a v_i) = v_j \theta(a) v_i \in v_j\langle Q\rangle v_i$$

so determines an element $\theta(a)da^* \in \mathtt{DR}^1_{C_k}\langle Q\rangle$.

$$\begin{aligned} i_\theta(\omega) &= \sum_{a\in L} i_\theta(da)da^* - i_\theta(da^*)da \\ &= \sum_{a\in L} \theta(a)da^* - \theta(a^*)da \end{aligned}$$

lies in $\mathtt{DR}^1_{C_k}\langle Q\rangle$. As both C_k-derivations and 1-forms are determined by their coefficients, τ is indeed bijective. □

DEFINITION 46. Let $*$ be a quiver involution with a symplectic structure $Q_a = L \sqcup R$. The $*$-*Kontsevich bracket* on the noncommutative quiver functions $\mathtt{DR}^0_{C_k}\langle Q\rangle$ is defined by

$$\{w_1, w_2\}_K = \sum_{a\in L} \left(\frac{\partial w_1}{\partial a}\frac{\partial w_2}{\partial a^*} - \frac{\partial w_1}{\partial a^*}\frac{\partial w_2}{\partial a}\right) \mathtt{mod}\ [\langle Q\rangle, \langle Q\rangle]$$

That is, to compute $\{w_1, w_2\}_K$ we consider for every arrow $a \in L$ all occurrences of a in w_1 and a^* in w_2. We then open up the necklaces removing these factors and gluing the open ends together to form a new necklace word. We then replace the roles of a^* and a and redo this operation (with a minus sign), see figure 3. Finally, we add all the obtained necklace words.

THEOREM 34. *Let Q be a symmetric function, $*$ a quiver involution and $Q_a = L \sqcup R$ a symplectic structure.*

The noncommutative quiver functions $\mathtt{DR}^0_{C_k}\langle Q \rangle$ equipped with the Kontsevich bracket is a Lie algebra and the sequence

$$0 \longrightarrow C_k \longrightarrow \mathtt{DR}^0_{C_k}\langle Q \rangle \xrightarrow{\tau^{-1}d} Der_\omega\langle Q \rangle \longrightarrow 0$$

to be defined below is an exact sequence (hence a central extension) of Lie algebras.

PROOF. One proves the first statement with the graphical argument given before for the free algebra. The Cartan homotopy formula

$$L_\theta = i_\theta \circ d + d \circ i_\theta$$

and the fact that ω is a closed form imply when $\theta \in Der_\omega\langle Q \rangle$ that

$$L_\theta \omega = d i_\theta \omega = \tau(\theta) = 0$$

That is, $\tau(\theta)$ is a closed form which by vanishing of the cohomology of the Karoubi complex shows that it must be an exact form. That is we have an isomorphism of exact sequences of $\mathbb{C}$-vectorspaces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_k & \longrightarrow & \mathtt{DR}^0_{C_k}\langle Q \rangle & \xrightarrow{d} & (\mathtt{DR}^1_{C_k}\langle Q \rangle)_{exact} & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{\simeq} & & \downarrow{\tau^{-1}} & & \\
0 & \longrightarrow & C_k & \longrightarrow & \dfrac{\langle Q \rangle}{[\langle Q \rangle, \langle Q \rangle]} & \longrightarrow & Der_\omega\langle Q \rangle & \longrightarrow & 0
\end{array}$$

The symplectic derivations $Der_\omega\langle Q \rangle$ is a Lie algebra with bracket $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - \theta_2 \circ \theta_1$.

For every necklace word w we have a derivation $\theta_w = \tau^{-1} dw$ which is defined by

$$\begin{cases} \theta_w(a) & = \frac{\partial w}{\partial a^*} \\ \theta_w(a^*) & = -\frac{\partial w}{\partial a} \end{cases}$$

With this notation we get the following interpretations of the Kontsevich bracket

$$\{w_1, w_2\}_K = i_{\theta_{w_1}}(i_{\theta_{w_2}}\omega) = L_{\theta_{w_1}}(w_2) = -L_{\theta_{w_2}}(w_1)$$

where the next to last equality follows because $i_{\theta_{w_2}}\omega = dw_2$ and the fact that $i_{\theta_{w_1}}(dw) = L_{\theta_{w_1}}(w)$ for any w. More generally, for any C_k-derivation θ and any necklace word w we have the equation

$$i_\theta(i_{\theta_w}\omega) = L_\theta(w)$$

By the commutation relations for the operators L_θ and i_θ we have for all C_k-derivations θ_i the equalities

$$\begin{aligned} L_{\theta_1} i_{\theta_2} i_{\theta_3}\omega - i_{\theta_2} i_{\theta_3} L_{\theta_1}\omega &= [L_{\theta_1}, i_{\theta_2}] i_{\theta_3}\omega + i_{\theta_2} L_{\theta_1} i_{\theta_3}\omega \\ &\quad - i_{\theta_2} L_{\theta_1} i_{\theta_3}\omega + i_{\theta_2}[L_{\theta_1}, i_{\theta_3}]\omega \\ &= i_{[\theta_1, \theta_2]} i_{\theta_3}\omega + i_{\theta_2} i_{[\theta_1, \theta_3]}\omega \end{aligned}$$

By the homotopy formula we have $L_{\theta_w}\omega = 0$ for every necklace word w, whence we get

$$L_{\theta_{w_1}} i_{\theta_2} i_{\theta_3}\omega = i_{[\theta_{w_1}, \theta_2]} i_{\theta_3}\omega + i_{\theta_2} i_{[\theta_{w_1}, \theta_3]}\omega$$

Take $\theta_2 = \theta_{w_2}$, then the left hand side is equal to

$$\begin{aligned} L_{\theta_{w_1}} i_{\theta_{w_2}} i_{\theta_3} \omega &= -L_{\theta_{w_1}} i_{\theta_3} i_{\theta_{w_2}} \omega \\ &= -L_{\theta_{w_1}} L_{\theta_3} w_2 \end{aligned}$$

whereas the last term on the right equals

$$\begin{aligned} i_{\theta_{w_2}} i_{[\theta_{w_1},\theta_3]} \omega &= -i_{[\theta_{w_1},\theta_3]} i_{\theta_{w_2}} \omega \\ &= -L_{[\theta_{w_1},\theta_3]} w_2 = -L_{\theta_{w_1}} L_{\theta_3} w_2 + L_{\theta_3} L_{\theta_{w_1}} w_2 \end{aligned}$$

and substituting this we obtain that

$$\begin{aligned} i_{[\theta_{w_1},\theta_{w_2}]} i_{\theta_3} \omega &= -L_{\theta_{w_1}} L_{\theta_3} w_2 + L_{\theta_{w_1}} L_{\theta_3} w_2 - L_{\theta_3} L_{\theta_{w_1}} w_2 \\ &= -L_{\theta_3} L_{\theta_{w_1}} w_2 = -L_{\theta_3} \{w_1, w_2\}_K \\ &= -i_{\theta_3} i_{\theta_{\{w_1,w_2\}_K}} \omega = i_{\theta_{\{w_1,w_2\}_K}} i_{\theta_3} \omega \end{aligned}$$

Finally, if we take $\theta = [\theta_{w_1}, \theta_{w_2}] - \theta_{\{w_1,w_2\}_K}$ we have that $i_\theta \omega$ is a closed 1-form and that $i_\theta i_{\theta_3} \omega = -i_{\theta_3} i_\theta \omega = 0$ for all θ_3. But then by the homotopy formula $L_{\theta_3} i_\theta \omega = 0$ whence $i_\theta \omega = 0$, which finally implies that $\theta = 0$. □

CHAPTER 4

Witnesses

> *"In the spirit of Weyl's book, we then take the problem of describing the relations among such invariants and concomitants. The result is quite striking in that it basically says that any relation among invariants and matrix concomitants is a consequence of the theorem of Hamilton-Cayley."*
>
> C. Procesi in [**53**].

In this chapter we prove the fundamental reconstruction results due to Claudio Procesi [**54**]. Let `alg@n` be the category of all algebras with trace satisfying the formal n-th Cayley-Hamilton identities and assign to $B \in \texttt{alg@n}$ the commutative affine scheme $\texttt{trep}_n B$ of trace preserving representations. Let `GL(n)-aff` be the category of all commutative affine schemes equipped with a linear GL_n-action, then there is a triangle

$$\begin{array}{ccc} & \texttt{alg@n} & \\ {\scriptstyle \int_n}\nearrow & & \searrow{\scriptstyle \texttt{trep}_n} \\ \texttt{alg} & \xrightarrow{\ \texttt{rep}_n\ } & \texttt{GL(n)-aff} \end{array}$$

The fundamental anti-equivalence $\texttt{spec} : \texttt{commalg} \longrightarrow \texttt{aff}$ of commutative algebraic geometry extends to a *left* inverse $\Uparrow^n$ assigning to an affine GL_n-scheme `fun` its *witness algebra* which is the algebra of GL_n-equivariant polynomial maps $\texttt{fun} \longrightarrow M_n(\mathbb{C})$. There is the commuting diagram of functors

$$\begin{array}{ccc} \texttt{alg@n} & \underset{\Uparrow^n}{\overset{\texttt{trep}_n}{\rightleftarrows}} & \texttt{GL(n)-aff} \\ \Big\downarrow{\scriptstyle tr} & & \Big\downarrow{\scriptstyle \texttt{quot}} \\ \texttt{commalg} & \xrightarrow[\texttt{spec}]{} & \texttt{aff} \end{array}$$

where `quot` is the quotient functor which assigns to an affine scheme with GL_n-action `fun` the affine scheme determined by the ring of polynomial invariants $\mathbb{C}[\texttt{fun}]^{GL_n}$. In particular, there is a geometric reconstruction result for the n-th trace algebra of an algebra A

$$\int_n A = M_n(\mathbb{C}[\texttt{rep}_n A])^{GL_n} \quad \text{and} \quad tr \int_n A = \mathbb{C}[\texttt{rep}_n A]^{GL_n}$$

That is, the n-th trace algebra can be recovered from the representation scheme $\texttt{rep}_n A$ as the ring of GL_n-equivariants and the n-th *necklace* algebra $\oint_n A = tr \int_n A$

is the coordinate ring of the quotient scheme which by the result of M. Artin [**1**] parametrizes the n-dimensional semisimple representations of A. Note however that because $\Uparrow^n$ is only a left inverse (and not an equivalence of categories) `noncommutative geometry@n` is not merely GL_n-equivariant geometry. In fact, equivariant constructions (such as equivariant desingularization) quickly lead us away from representation schemes.

4.1. Necklace relations.

In this section we will determine the kernel of the epimorphism

$$\oint \langle \infty \rangle \xrightarrow{\mu} \downarrow_n \langle \infty \rangle$$

which will be crucial to relate the invariant ring $\downarrow_n A$ to the n-th necklace algebra $\oint_n A$. The result is proved using the representation theory of the symmetric group. We recall some of the basics of this theory and refer the reader to [**17**, Ch.4] for more details.

Definition 47. S_d is the *symmetric group* of all permutations on d letters. Conjugacy classes in S_d correspond to *partitions* $\lambda = (\lambda_1, \ldots, \lambda_k)$ of d, that is, decompositions in natural numbers

$$d = \lambda_1 + \ldots + \lambda_k \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$$

The correspondence assigns to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ the conjugacy class of a permutation consisting of disjoint cycles of lengths $\lambda_1, \ldots, \lambda_k$.

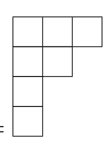
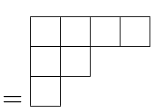
One assigns to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ a *Young diagram* with λ_i boxes in the i-th row, the rows of boxes lined up to the left. The *dual partition* $\lambda^* = (\lambda_1^*, \ldots, \lambda_r^*)$ to λ is defined by interchanging rows and columns in the Young diagram of λ.

A *Young tableau* is a numbering of the boxes of a Young diagram by the integers $\{1, 2, \ldots, d\}$. For a fixed Young tableau T of type λ one defines subgroups of S_d by

$$P_\lambda = \{\sigma \in S_d \mid \sigma \text{ preserves each row } \}$$

$$Q_\lambda = \{\sigma \in S_d \mid \sigma \text{ preserves each column } \}$$

Example 56. To the partition $\lambda = (3, 2, 1, 1)$ of 7 we assign the Young diagram

$\lambda =$  $\lambda^* =$ 

with dual partition $\lambda^* = (4, 2, 1)$. Two distinct Young tableaux of type λ are

1	2	3
4	5	
6		
7		

1	3	5
2	4	
6		
7		

For the second Young tableau we obtain the subgroups

$$\begin{cases} P_\lambda & = S_{\{1,3,5\}} \times S_{\{2,4\}} \times \{(6)\} \times \{(7)\} \\ Q_\lambda & = S_{\{1,2,6,7\}} \times S_{\{3,4\}} \times \{(5)\} \end{cases}$$

The group algebra $\mathbb{C}S_d$ is a semisimple algebra. In particular, any simple S_d-representation is isomorphic to a minimal left ideal of $\mathbb{C}S_d$ which is generated by an idempotent.

Definition 48. Given a tableau T of type λ, define the elements of $\mathbb{C}S_d$

$$a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma \quad , \quad b_\lambda = \sum_{\sigma \in Q_\lambda} sgn(\sigma) e_\sigma \quad \text{and} \quad c_\lambda = a_\lambda . b_\sigma$$

c_λ is called the *Young symmetrizer* corresponding to T.

There is a one-to-one correspondence between the simple representations of $\mathbb{C}S_d$ and the conjugacy classes in S_d (or, equivalently, Young diagrams).

Theorem 35 (Young). *For every partition λ of d the left ideal $\mathbb{C}S_d.c_\lambda = V_\lambda$ is a simple S_d-representations and, conversely, any simple S_d-representation is isomorphic to V_λ for a unique partition λ.*

Proof. Observe that $P_\lambda \cap Q_\lambda = \{e\}$ (any permutation preserving rows as well as columns preserves all boxes) and so any element of S_d can be written in at most one way as a product $p.q$ with $p \in P_\lambda$ and $q \in Q_\lambda$. In particular, the Young symmetrizer can be written as $c_\lambda = \sum \pm e_\sigma$ with $\sigma = p.q$ for unique p and q and the coefficient $\pm 1 = sgn(q)$. From this it follows that for all $p \in P_\lambda$ and $q \in Q_\lambda$ we have

$$p.a_\lambda = a_\lambda.p = a_\lambda \quad , \quad sgn(q)q.b_\lambda = b_\lambda.sgn(q)q = b_\lambda \quad , \quad p.c_\lambda.sgn(q)q = c_\lambda$$

Moreover, we claim that c_λ is the unique element in $\mathbb{C}S_d$ (up to a scalar factor) satisfying the last property. This requires a few preparations.

Assume $\sigma \notin P_\lambda.Q_\lambda$ and consider the tableaux $T' = \sigma T$, that is, replacing the label i of each box in T by $\sigma(i)$. We claim that there are two distinct numbers which belong to the same row in T and to the same column in T'. If this were not the case, then all the distinct numbers in the first row of T appear in different columns of T'. But then we can find an element q_1' in the subgroup $\sigma.Q_\lambda.\sigma^{-1}$ preserving the columns of T' to take all these elements to the first row of T'. But then, there is an element $p_1 \in T_\lambda$ such that $p_1 T$ and $q_1' T'$ have the same first row. We can proceed to the second row and so on and obtain elements $p \in P_\lambda$ and $q' \in \sigma.Q_\lambda, \sigma^{-1}$ such that the tableaux pT and $q'T'$ are equal. Hence, $pT = q'\sigma T$ entailing that $p = q'\sigma$. Further, $q' = \sigma.q.\sigma^{-1}$ but then $p = q'\sigma = \sigma q$ whence $\sigma = p.q^{-1} \in P_\lambda.Q_\lambda$, a contradiction. Therefore, to $\sigma \notin P_\lambda.Q_\lambda$ we can assign a *transposition* $\tau = (ij)$ (replacing the two distinct numbers belonging to the same row in T and to the same column in T') for which $p = \tau \in P_\lambda$ and $q = \sigma^{-1}.\tau.\sigma \in Q_\lambda$.

After these preliminaries, assume that $c' = \sum a_\sigma e_\sigma$ is an element such that

$$p.c'.sgn(q)q = c' \quad \text{for all} \quad p \in P_\lambda, q \in Q_\lambda$$

We claim that $a_\sigma = 0$ whenever $\sigma \notin P_\lambda.Q_\lambda$. For take the transposition τ found above and $p = \tau$, $q = \sigma^{-1}.\tau.\sigma$, then $p.\sigma.q = \tau.\sigma.\sigma^{-1}.\tau.\sigma = \sigma$. However, the coefficient of σ in c' is a_σ and that of $p.c'.q$ is $-a_\sigma$ proving the claim. That is,

$$c' = \sum_{p,q} a_{pq} e_{p.q}$$

but then by the property of c' we must have that $a_{pq} = sgn(q) a_e$ whence $c' = a_e c_\lambda$ finishing the proof of the claimed uniqueness of the element c_λ.

As a consequence we have for all elements $x \in \mathbb{C}S_d$ that $c_\lambda.x, c_\lambda = \alpha_x c_\lambda$ for some scalar $\alpha_x \in \mathbb{C}$ and in particular that $c_\lambda^2 = n_\lambda c_\lambda$, for,

$$\begin{aligned} p.(c_\lambda.x.c_\lambda).sgn(q)q &= p.a_\lambda.b_\lambda.x.a_\lambda.b_\lambda.sgn(q)q \\ &= a_\lambda.b_\lambda.x.a_\lambda.b_\lambda = c_\lambda.x.c_\lambda \end{aligned}$$

and the statement follows from the uniqueness result for c_λ.

Define $V_\lambda = \mathbb{C}S_d.c_\lambda$ then we have $c_\lambda.V_\lambda \subset \mathbb{C}c_\lambda$. We claim that V_λ is a simple S_d-representation. Let $W \subset V_\lambda$ be a simple subrepresentation, then being a left ideal of $\mathbb{C}S_d$ we can write $W = \mathbb{C}S_d.x$ with $x^2 = x$ (note that W is a direct summand). Assume that $c_\lambda.W = 0$, then $W.W \subset \mathbb{C}S_d.c_\lambda.W = 0$ implying that $x = 0$ whence $W = 0$, a contradiction. Hence, $c_\lambda.W = \mathbb{C}c_\lambda \subset W$, but then

$$V_\lambda = \mathbb{C}S_d.c_\lambda \subset W \quad \text{whence} V_\lambda = W$$

is simple. Remains to show that for different partitions, the corresponding simple representations cannot be isomorphic.

We put a *lexicographic* ordering on the partitions by the rule that

$$\lambda > \mu \quad \text{if the first nonvanishing } \lambda_i - \mu_i \text{ is positive}$$

We claim that if $\lambda > \mu$ then $a_\lambda.\mathbb{C}S_d.b_\mu = 0$. It suffices to check that $a_\lambda.\sigma.b_\mu = 0$ for $\sigma \in S_d$. As $\sigma.b_\mu.\sigma^{-1}$ is the "b-element" constructed from the tableau $b.T'$ where T' is the tableaux fixed for μ, it is sufficient to check that $a_\lambda.b_\mu = 0$. As $\lambda > \mu$ there are distinct numbers i and j belonging to the same row in T and to the same column in T'. If not, the distinct numbers in any fixed row of T must belong to different columns of T', but this can only happen for all rows if $\mu \geq \lambda$. So consider $\tau = (ij)$ which belongs to P_λ and to Q_μ, whence $a_\lambda.\tau = a_\lambda$ and $\tau.b_\mu = -b_\mu$. But then,

$$a_\lambda.b_\mu = a_\lambda.\tau, \tau, b_\mu = -a_\lambda.b_\mu$$

proving the claim.

If $\lambda \neq \mu$ we claim that V_λ is not isomorphic to V_μ. Assume that $\lambda > \mu$ and ϕ a $\mathbb{C}S_d$-isomorphism with $\phi(V_\lambda) = V_\mu$, then

$$\phi(c_\lambda V_\lambda) = c_\lambda \phi(V_\lambda) = c_\lambda V_\mu = c_\lambda \mathbb{C}S_d c_\mu = 0$$

Hence, $c_\lambda V_\lambda = \mathbb{C}c_\lambda \neq 0$ lies in the kernel of an isomorphism which is clearly absurd.

Summarizing, we have constructed to distinct partitions of d, λ and μ nonisomorphic simple $\mathbb{C}S_d$-representations V_λ and V_μ. As we know that there are as many isomorphism classes of simples as there are conjugacy classes in S_d (or partitions), the V_λ form a complete set of isomorphism classes of simple S_d-representations. □

Recall that the free necklace algebra $\oint \langle \infty \rangle$ is the commutative polynomial ring on variables $t_{\widehat{w}}$ where w varies over all necklace words in the noncommuting variables $X = \{x_1, x_2, \ldots, x_i, \ldots\}$. If $w = x_{i_1} \ldots x_{il}$ we will write $t_{\widehat{w}} = t(x_{i_1} \ldots x_{i_l})$.

Definition 49. For $\sigma \in S_d$ let

$$\sigma = (i_1 i_1 \ldots i_\alpha)(j_1 j_2 \ldots j_\beta) \ldots (z_1 z_2 \ldots z_\zeta)$$

be a decomposition into cycles including those of length one. Define a linear map

$$T : \mathbb{C}S_d \longrightarrow \oint \langle \infty \rangle$$

which assigns to σ the *formal necklace* $T_\sigma(x_1, \ldots, x_d)$ defined by

$$T_\sigma(x_1, \ldots, x_d) = t(x_{i_1} x_{i_2} \ldots x_{i_\alpha}) t(x_{j_1} x_{j_2} \ldots x_{j_\beta}) \ldots t(x_{z_1} x_{z_2} \ldots x_{z_\zeta})$$

A linear combination $\sum a_\sigma T_\sigma(x_1, \ldots, x_d)$ is said to be *n-th necklace relation* if it belongs to the kernel of

$$\oint \langle \infty \rangle \overset{\mu}{\twoheadrightarrow} \downarrow_n \langle \infty \rangle$$

given in theorem 30.

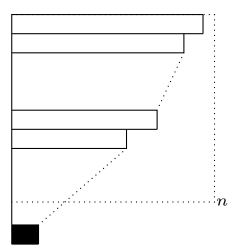
Theorem 36 (Second fundamental theorem of matrix invariants). *A formal necklace*

$$\sum_{\sigma \in S_d} a_\sigma T_\sigma(x_1, \ldots, x_d)$$

is a necklace relation (for $n \times n$ matrices) if and only if the element

$$\sum a_\sigma e_\sigma \in \mathbb{C}S_d$$

belongs to the ideal of $\mathbb{C}S_d$ spanned by the Young symmetrizers c_λ relative to partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$



with a least $n+1$ rows, that is, $k \geq n+1$.

Proof. Let $V = V_n$ be again the n-dimensional vectorspace of column vectors, then S_d acts naturally on $V^{\otimes d}$ via

$$\sigma.(v_1 \otimes \ldots \otimes v_d) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$$

hence determines a linear map $\lambda_\sigma \in End(V^{\otimes d})$. In the previous chapter we have seen that under the natural identifications

$$(M_n^{\otimes d})^* \simeq (V^{*\otimes d} \otimes V^{\otimes d})^* \simeq End(V^{\otimes d})$$

the map λ_σ defines the multilinear map

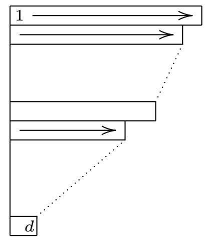
$$\mu_\sigma : \underbrace{M_n(\mathbb{C}) \otimes \ldots \otimes M_n(\mathbb{C})}_{d} \longrightarrow \mathbb{C}$$

defined by (using the cycle decomposition of σ as before)

$$\mu_\sigma(A_1 \otimes \ldots \otimes A_d) = tr(A_{i_1} A_{i_2} \ldots A_{i_\alpha}) tr(A_{j_1} A_{j_2} \ldots A_{j_\beta}) \ldots tr(A_{z_1} A_{z_2} \ldots A_{z_\zeta}) \quad .$$

Therefore, a linear combination $\sum a_\sigma T_\sigma(x_1, \ldots, x_d)$ is an n-th necklace relation if and only if the multilinear map $\sum a_\sigma \mu_\sigma : M_n^{\otimes d} \longrightarrow \mathbb{C}$ is zero. This, in turn, is equivalent to the endomorphism $\sum a_\sigma \lambda_\sigma \in End(V^{\otimes m})$, induced by the action of the element $\sum a_\sigma e_\sigma \in \mathbb{C}S_d$ on $V^{\otimes d}$, being zero. In order to answer the latter problem we have to understand the action of a Young symmetrizer $c_\lambda \in \mathbb{C}S_d$ on $V^{\otimes d}$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of d and equip the corresponding Young diagram with the standard tableau (that is, order first the boxes in the first row from left to right, then the second row from left to right and so on).



The subgroup P_λ of S_d which preserves each row then becomes

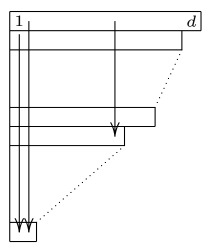
$$P_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k} \hookrightarrow S_d.$$

As $a_\lambda = \sum_{p \in P_\lambda} e_p$ we see that the image of the action of a_λ on $V^{\otimes d}$ is the subspace

$$Im(a_\lambda) = Sym^{\lambda_1}\ V \otimes Sym^{\lambda_2}\ V \otimes \dots \otimes Sym^{\lambda_k}\ V \hookrightarrow V^{\otimes d} \quad .$$

Here, $Sym^i\ V$ denotes the subspace of symmetric tensors in $V^{\otimes i}$.

Similarly, equip the Young diagram of λ with the tableau by ordering first the boxes in the first column from top to bottom, then those of the second column from top to bottom and so on.



Equivalently, give the Young diagram corresponding to the dual partition of λ

$$\lambda^* = (\mu_1, \mu_2, \dots, \mu_l)$$

the standard tableau. Then, the subgroup Q_λ of S_d which preserves each row of λ (or equivalently, each column of λ^*) is

$$Q_\lambda = S_{\mu_1} \times S_{\mu_2} \times \dots \times S_{\mu_l} \hookrightarrow S_d$$

As $b_\lambda = \sum_{q \in Q_\lambda} sgn(q) e_q$ we see that the image of b_λ on $V^{\otimes d}$ is the subspace

$$Im(b_\lambda) = \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \dots \otimes \bigwedge^{\mu_l} V \hookrightarrow V^{\otimes d} \quad .$$

Here, $\bigwedge^i\ V$ is the subspace of all anti-symmetric tensors in $V^{\otimes i}$. Note that $\bigwedge^i\ V = 0$ whenever i is greater than the dimension $dim\ V = n$. That is, the image of the action of b_λ on $V^{\otimes d}$ is zero whenever the dual partition λ^* contains a row of length $\geq n+1$, or equivalently, whenever λ has $\geq n+1$ rows. Because the Young symmetrizer $c_\lambda = a_\lambda . b_\lambda \in \mathbb{C}\ S_d$ this finishes the proof. □

Example 57. (Fundamental necklace relation)

Consider the partition $\lambda = (1, 1, \ldots, 1)$ of $n+1$, with corresponding Young tableau

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n+1 \\ \hline \end{array}$$

Then, $P_\lambda = \{e\}$, $Q_\lambda = S_{n+1}$ and we have the Young symmetrizer

$$a_\lambda = 1 \qquad b_\lambda = c_\lambda = \sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma.$$

The corresponding element is called the *fundamental necklace relation*

$$\mathtt{fund}_n(x_1, \ldots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} sgn(\sigma) T_\sigma(x_1, \ldots, x_{n+1}).$$

Clearly, $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$ is multilinear of degree $n+1$ in the variables $\{x_1, \ldots, x_{n+1}\}$. Conversely, any multilinear necklace relation of degree $n+1$ must be a scalar multiple of $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$. This follows from the theorem as the ideal described there is for $d = n+1$ just the scalar multiples of $\sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma$.

Theorem 37 (Procesi-Razmyslov). *The n-th necklace relations form the ideal of $\oint \langle \infty \rangle$ generated by all the elements*

$$\mathtt{fund}_n(m_1, \ldots, m_{n+1})$$

where the m_i run over all monomials in the variables $\{x_1, x_2, \ldots, x_i, \ldots\}$.

Proof. Take a homogeneous necklace relation $f \in Ker\ \mu$ of degree d and polarize it to get a multilinear element $f' \in \oint \langle \infty \rangle$. Clearly, f' is also an n-th necklace relation and if we can show that f' belongs to the described ideal, then so does f as the process of restitution maps this ideal into itself.

We may thus assume that f is multilinear of degree d. A priori f may depend on more than d variables x_k, but we can separate f as a sum of multilinear polynomials f_i each depending on precisely d variables such that for $i \neq j$ f_i and f_j do not depend on the same variables. Setting some of the variables equal to zero, we see that each of the f_i is again a necklace relation.

Thus, we may assume that f is a multilinear n-th necklace relation of degree d depending on the variables $\{x_1, \ldots, x_d\}$. But then we know from theorem 36 that we can write

$$f = \sum_{\tau \in S_d} a_\tau T_\tau(x_1, \ldots, x_d)$$

where $\sum a_\tau e_\tau \in \mathbb{C} S_d$ belongs to the ideal spanned by the Young symmetrizers of Young diagrams λ having at least $n+1$ rows.

We claim that this ideal is generated by the Young symmetrizer of the partition $(1, \ldots, 1)$ of $n+1$ under the natural embedding of S_{n+1} into S_d. Let λ be a Young diagram having $k \geq n+1$ boxes and let c_λ be a Young symmetrizer with respect to a tableau where the boxes in the first column are labeled by the numbers $I = \{i_1, \ldots, i_k\}$ and let S_I be the obvious subgroup of S_d. As $Q_\lambda = S_I \times Q'$ we see that $b_\lambda = (\sum_{\sigma \in S_I} sgn(\sigma) e_\sigma).b'$ with $b' \in \mathbb{C} Q'$. Hence, c_λ belongs to the twosided ideal generated by $c_I = \sum_{\sigma \in S_I} sgn(\sigma) e_\sigma$ but this is also the twosided ideal generated by $c_k = \sum_{\sigma \in S_k} sgn(\sigma)e_\sigma$ as one verifies by conjugation with a partition sending I to $\{1,\ldots,k\}$. Moreover, by induction one shows that the twosided ideal generated by c_k belongs to the twosided ideal generated by $c_d = \sum_{\sigma \in S_d} sgn(\sigma)e_\sigma$, finishing the proof of the claim.

Hence, we can write

$$\sum_{\tau \in S_d} a_\tau e_\tau = \sum_{\tau_i,\tau_j \in S_d} a_{ij} e_{\tau_i}.(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_{\tau_j}$$

so it suffices to analyze the form of the necklace identity associated to an element of the form

$$e_\tau.(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_{\tau'} \quad \text{with} \quad \tau,\tau' \in S_d$$

Now, if a groupelement $\sum_{\mu \in S_d} b_\mu e_\mu$ corresponds to the formal necklace polynomial $\mathtt{neck}(x_1,\ldots,x_d)$, then the element $e_\tau.(\sum_{\mu \in S_d} b_\mu e_\mu).e_{\tau^{-1}}$ corresponds to the formal necklace polynomial $\mathtt{neck}(x_{\tau(1)},\ldots,x_{\tau(d)})$.

Therefore, we may replace the element $e_\tau.(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_{\tau'}$ by the element

$$(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\eta \quad \text{with} \quad \eta = \tau'.\tau \in S_d$$

We claim that we can write $\eta = \sigma'.\theta$ with $\sigma' \in S_{n+1}$ and $\theta \in S_d$ such that each cycle of θ contains at most one of the elements from $\{1,2,\ldots,n+1\}$. Indeed assume that η contains a cycle containing more than one element from $\{1,\ldots,n+1\}$, say 1 and 2, that is

$$\eta = (1 i_1 i_2 \ldots i_r 2 j_1 j_2 \ldots j_s)(k_1 \ldots k_\alpha)\ldots(z_1 \ldots z_\zeta)$$

then we can express the product $(12).\eta$ in cycles as

$$(1 i_1 i_2 \ldots i_r)(2 j_1 j_2 \ldots j_s)(k_1 \ldots k_\alpha)\ldots(z_1 \ldots z_\zeta)$$

Continuing in this manner we reduce the number of elements from $\{1.\ldots,n+1\}$ in every cycle to at most one.

But then as $\sigma' \in S_{n+1}$ we have seen that $(\sum sgn(\sigma)e_\sigma).e_{\sigma'} = sgn(\sigma')(\sum sgn(\sigma)e_\sigma)$ and consequently

$$(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\eta = \pm(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\theta$$

where each cycle of θ contains at most one of $\{1,\ldots,n+1\}$. Let us write

$$\theta = (1 i_1 \ldots i_\alpha)(2 j_1 \ldots j_\beta)\ldots(n+1 s_1 \ldots s_\kappa)(t_1 \ldots t_\lambda)\ldots(z_1 \ldots z_\zeta)$$

Now, let $\sigma \in S_{n+1}$ then the cycle decomposition of $\sigma.\theta$ is obtained as follows : substitute in each cycle of σ the element 1 formally by the string $1 i_1 \ldots i_\alpha$, the element 2 by the string $2 j_1 \ldots j_\beta$, and so on until the element $n+1$ by the string $n+1 s_1 \ldots s_\kappa$ and finally adjoin the cycles of θ in which no elements from $\{1,\ldots,n+1\}$ appear.

Finally, we can write out the formal necklace element corresponding to the element $(\sum_{\sigma \in S_{n+1}} sgn(\sigma)e_\sigma).e_\theta$ as

$$\mathtt{fund}_n(x_1 x_{i_1} \ldots x_{i_\alpha}, x_2 x_{j_1} \ldots x_{j_\beta}, \ldots, x_{n+1} x_{s_1} \ldots x_{s_\kappa}) t(x_{t_1} \ldots x_{t_\lambda}) \ldots t(x_{z_1} \ldots x_{z_\zeta})$$

finishing the proof of the theorem. □

4.2. Trace relations.

In this section we will introduce the n-th ring of equivariant maps $\uparrow_n A$ and study the kernel of the trace preserving map $\int \langle \infty \rangle \longrightarrow\!\!\!\!\rightarrow \uparrow_n \langle \infty \rangle$ which is called the ideal of *trace relations*. This description will be crucial in the next section to relate the algebras $\oint_n A$ resp. $\int_n A$ to $\downarrow_n A$ resp. $\uparrow_n A$.

Recall that GL_n acts by algebra automorphisms on the coordinate ring $\mathbb{C}[\mathtt{rep}_n A]$ and by conjugation on the matrixring $M_n(\mathbb{C})$. The diagonal action on

$$M_n(\mathbb{C}[\mathtt{rep}_n A]) = M_n(\mathbb{C}) \otimes \mathbb{C}[\mathtt{rep}_n A]$$

is given by the formula

$$g. \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix} = g^{-1} \begin{bmatrix} g.c_{11} & \dots & g.c_{1n} \\ \vdots & & \vdots \\ g.c_{n1} & \dots & g.c_{nn} \end{bmatrix} g$$

DEFINITION 50. The n-th equivariant functor

$$\uparrow_n : \mathtt{alg} \longrightarrow \mathtt{alg@n}$$

assigns to a $\mathbb{C}$-algebra A the ring of GL_n-equivariant maps

$$\uparrow_n A = M_n(\mathbb{C}[\mathtt{rep}_n A])^{GL_n}$$

That is, $\uparrow_n A$ is the algebra of all polynomial maps $\mathtt{rep}_n A \longrightarrow M_n(\mathbb{C})$ which are *equivariant* , that is, commute with the GL_n action on both spaces

$$\begin{array}{ccc} \mathtt{rep}_n A & \xrightarrow{f} & M_n(\mathbb{C}) \\ {\scriptstyle g.}\downarrow & & \downarrow{\scriptstyle g \; g^{-1}} \\ \mathtt{rep}_n A & \xrightarrow{f} & M_n(\mathbb{C}) \end{array}$$

The matrixalgebra $M_n(\mathbb{C}[\mathtt{rep}_n A])$ with the natural trace map is a Cayley-Hamilton algebra of degree n. The restriction of this trace to the subalgebra $\uparrow_n A$ makes $\uparrow_n A$ an object of $\mathtt{alg@n}$.

We have already used the *Reynolds operator* so it is about time to introduce it formally.

DEFINITION 51. GL_n is a *reductive group* , that is, every finite dimensional GL_n-representation is completely reducible, that is, a direct sum of irreducible GL_n-representations.

Let $\mathtt{simp}GL_n$ be the set of isomorphism classes of irreducible GL_n-representations. An irreducible GL_n-representation W belonging to the class $s \in \mathtt{simp}GL_n$ is said to be of *type* s.

Let X be a vectorspace (not necessarily finite dimensional) with a linear GL_n-action. The GL_n-action on X is said to be *locally finite* if every finite dimensional subspace $Y \subset X$ is contained in a finite dimensional GL_n-subrepresentation $Y' \subset X$. In this case we can use reductivity of GL_n to decompose

$$X = \bigoplus_{s \in \mathtt{simp}GL_n} X_{(s)}$$

into its *isotypical components* , that is,

$$X_{(s)} = \sum \{W | W \subset X, W \in s\}$$

If $X \xrightarrow{\phi} X'$ is a GL_n-linear map, then for all $s \in \mathtt{simp}GL_n$ we have linear maps $X_{(s)} \xrightarrow{\phi_s} X'_{(s)}$. If ϕ is injective (resp. surjective) then each ϕ_s is injective (resp. surjective).

If $0 \in \mathtt{simp}GL_n$ is the class of the trivial GL_n-representation, then

$$X_{(0)} = X^{GL_n} = \{x \in X \mid g.x = x \ \forall g \in GL_n\}$$

The *Reynolds operator* is the projection $X \xrightarrow{R} X_{(0)}$ the isotypical component of the trivial representation, or equivalently, the GL_n-invariant elements of X.

EXAMPLE 58. $M_n(\mathbb{C}) = V \otimes V^*$ is a GL_n-representation, hence so is M_n^m and all symmetric powers $S^i \ M_n^m$. Therefore $\mathbb{C}[\mathtt{rep}_n\langle m\rangle] = \mathbb{C}[M_n^m] = \oplus S^i \ M_n^m$ has a locally finite GL_n-action.

If A is an affine algebra generated by m elements, then the kernel of the epimorphism $\mathbb{C}[\mathtt{rep}_n\langle m\rangle] \twoheadrightarrow \mathbb{C}[\mathtt{rep}_n A]$ is GL_n-stable. Therefore, the GL_n-action on the coordinate ring $\mathbb{C}[\mathtt{rep}_n A]$ of the scheme of n-dimensional representations of the affine algebra A is locally finite.

Using the Reynolds operator, it suffices in order to determine the algebra generators of $\uparrow_n A$ to find those $\uparrow_n \langle \infty \rangle$ (or $\uparrow_n \langle m \rangle$).

THEOREM 38 (Procesi). *As an algebra over the n-th invariant algebra $\downarrow_n \langle m \rangle$, the n-th equivariant algebra $\uparrow_n \langle m \rangle$ is generated by the monomials in the generic matrices $\{X_1, \ldots, X_m\}$ of degree $\leq 2^n - 1$.*

PROOF. Recall that $\mathtt{rep}_n\langle m\rangle = M_n^m = M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})$. Consider a GL_n-equivariant map $M_n^m \xrightarrow{f} M_n(\mathbb{C})$ and associate to it the polynomial map

$$M_n^{m+1} = M_n^m \oplus M_n(\mathbb{C}) \xrightarrow{tr(fX_{m+1})} \mathbb{C}$$

defined by sending $(A_1, \ldots, A_m, A_{m+1})$ to $tr(f(A_1, \ldots, A_m).A_{m+1})$.

For all $g \in GL_n$ we have that $f(g.A_1.g^{-1}, \ldots, g.A_m.g^{-1})$ is equal to $g.f(A_1, \ldots, A_m).g^{-1}$ and hence

$$\begin{aligned} tr(f(g.A_1.g^{-1}, \ldots, g.A_m.g^{-1}).g.A_{m+1}.g^{-1}) &= tr(g.f(A_1, \ldots, A_m).g^{-1}.g.A_{m+1}.g^{-1}) \\ &= tr(g.f(A_1, \ldots, A_m).A_{m+1}.g^{-1}) \\ &= tr(f(A_1, \ldots, A_m).A_{m+1}) \end{aligned}$$

so $tr(fX_{m+1})$ is an invariant polynomial function on M_n^{m+1} which is *linear* in X_{m+1}. By the first fundamental theorem of matrix invariants, we can write

$$tr(fX_{m+1}) = \sum \underbrace{g_{i_1 \ldots i_l}}_{\in \downarrow_n \langle m \rangle} tr(X_{i_1} \ldots X_{i_l} X_{m+1})$$

Here, we used the necklace property allowing to permute cyclically the trace terms in which X_{m+1} occurs such that X_{m+1} occurs as the last factor. But then, $tr(fX_{m+1}) = tr(gX_{m+1})$ where

$$g = \sum g_{i_1 \ldots i_l} X_{i_1} \ldots X_{i_l}.$$

Finally, from the *nondegeneracy* of the trace map on $M_n(\mathbb{C})$ (that is, if $A, B \in M_n$ such that $tr(AC) = tr(BC)$ for all $C \in M_n(\mathbb{C})$, then $A = B$) it follows that $f = g$. $\square$

DEFINITION 52. By the foregoing theorem, there is a trace preserving epimorphism

$$\int \langle \infty \rangle \xrightarrow{\tau} \uparrow_n \langle \infty \rangle$$

The elements of $\mathtt{Ker}\tau$ are called *trace relations* .

EXAMPLE 59. (Fundamental trace relation)
As the fundamental necklace relation

$$\mathtt{fund}_n(x_1, \ldots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} sgn(\sigma) T_\sigma(x_1, \ldots, x_{n+1}).$$

is multilinear in the variables x_i we can use the necklace property of the formal trace t to write it in the form

$$\mathtt{fund}_n(x_1, \ldots, x_{n+1}) = t(\mathtt{cha}_n(x_1, \ldots, x_n)x_{n+1}) \quad \text{with} \quad \mathtt{cha}_n(x_1, \ldots, x_n) \in \int \langle \infty \rangle$$

Observe that $\mathtt{cha}_n(x_1, \ldots, x_n)$ is multilinear in the variables x_i. Moreover, by the nondegeneracy of the trace map tr and the fact that $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$ is a necklace relation, it follows that $\mathtt{cha}_n(x_1, \ldots, x_n)$ is a trace relation.

Any multilinear trace relation of degree n in the variables $\{x_1, \ldots, x_n\}$ is a scalar multiple of $\mathtt{cha}_n(x_1, \ldots, x_n)$. This follows from the corresponding uniqueness result for $\mathtt{fund}_n(x_1, \ldots, x_{n+1})$.

An explicit expression of this *fundamental trace relation* is

$$\mathtt{cha}_n(x_1, \ldots, x_n) = \sum_{k=0}^{n} (-1)^k \sum_{i_1 \neq i_2 \neq \ldots \neq i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \sum_{\sigma \in S_J} sgn(\sigma) T_\sigma(x_{j_1}, \ldots, x_{j_{n-k}})$$

where $J = \{1, \ldots, n\} - \{i_1, \ldots, i_k\}$.

For x one of the variables x_i, the formal n-th Cayley-Hamilton polynomial $\chi_x^{(n)}(x)$ is a homogeneous element of degree n of $\int \langle \infty \rangle$. It follows from Cayley-Hamilton theorem for $M_n(\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle])$ that $\chi_x^{(n)}(x)$ is a trace relation. Fully polarizing $\chi_x^{(n)}(x)$ (say, using the variables $\{x_1, \ldots, x_n\}$) one obtains a multilinear trace relation of degree n which must be a scalar multiple of $\mathtt{fund}_n(x_1, \ldots, x_n)$.

EXAMPLE 60. For $n = 2$, the formal Cayley-Hamilton polynomial of an element $x \in \int \langle \infty \rangle$ is

$$\chi_x^{(2)}(x) = x^2 - t(x)x + \frac{1}{2}(t(x)^2 - t(x^2))$$

Polarization with respect to the variables x_1 and x_2 gives the expression

$$x_1x_2 + x_2x_1 - t(x_1)x_2 - t(x_2)x_1 + t(x_1)t(x_2) - t(x_1x_2)$$

which is $\texttt{cha}_2(x_1,x_2)$. Indeed, multiplying by x_3 on the right and taking the formal trace t we obtain

$$\begin{aligned} & t(x_1x_2x_3)+t(x_2x_1x_3)-t(x_1)t(x_2x_3)-t(x_2)t(x_1x_3) \\ & \quad +t(x_1)t(x_2)t(x_3)-t(x_1x_2)t(x_3) \\ = \; & T_{(123)}(x_1,x_2,x_3)+T_{(213)}(x_1,x_2,x_3)-T_{(1)(23)}(x_1,x_2,x_3)-T_{(2)(13)}(x_1,x_2,x_3) \\ & \quad +T_{(1)(2)(3)}(x_1,x_2,x_3)-T_{(12)(3)}(x_1,x_2,x_3) \\ & \quad = \textstyle\sum_{\sigma\in S_3} T_\sigma(x_1,x_2,x_3)=\texttt{fund}_2(x_1,x_2,x_3) \end{aligned}$$

Theorem 39 (Procesi). *The trace relations* $\texttt{Ker}\tau$ *is the twosided ideal of the trace algebra* $\int\langle\infty\rangle$ *generated by all elements*

$$\texttt{fund}_n(m_1,\ldots,m_{n+1}) \quad \textit{and} \quad \texttt{cha}_n(m_1,\ldots,m_n)$$

where the m_i *run over all monomials in the variables* $\{x_1,x_2,\ldots,x_i,\ldots\}$.

Proof. Consider an n-th trace relation $\texttt{trace}(x_1,\ldots,x_d)\in Ker\ \tau$. Then, we have a necklace relation

$$t(\texttt{trace}(x_1,\ldots,x_d)x_{d+1})\in Ker\ \nu$$

By theorem 37 we know that this element must be of the form

$$\sum n_{i_1\ldots i_{n+1}}\texttt{fund}_n(m_{i_1},\ldots,m_{i_{n+1}})$$

with the m_i monomials, the $n_{i_1\ldots i_{n+1}}\in\oint\langle\infty\rangle$ and the expression linear in the variable x_{d+1}. That is, x_{d+1} appears linearly in each of the terms

$$n_{1\ldots n+1}\texttt{fund}_n(m_1,\ldots,m_{n+1})$$

so appears linearly in $n_{1\ldots n+1}$ or in precisely one of the monomials m_i. If x_{d+1} appears linearly in $n_{1\ldots n+1}$ we can write

$$n_{1\ldots n+1}=t(n'_{1\ldots n}.x_{d+1}) \quad \text{with} \quad n'_{1\ldots n}\in\int\langle\infty\rangle$$

If x_{d+1} appears linearly in one of the monomials m_i we may assume that it does so in m_{n+1}, permuting the monomials if necessary. That is, we may assume $m_{n+1}=m'_{n+1}.x_{d+1}.m"_{n+1}$ with m,m' monomials. But then, we can write

$$\begin{aligned} n_{1\ldots n+1}\texttt{fund}_n(m_1,\ldots,m_{n+1}) = \; & n_{1\ldots n+1}t(\texttt{cha}_n(m_1,\ldots,m_n).m'_{n+1}.x_{d+1}.m"_{n+1}) \\ = \; & t(n_{1\ldots n+1}.m"_{n+1}.\texttt{cha}_n(m_1,\ldots,m_n).m'_{n+1}.x_{d+1}) \end{aligned}$$

using $\oint\langle\infty\rangle$-linearity and the necklace property of the formal trace t. Separating the two cases, one can write the total expression

$$\begin{aligned} t(\texttt{trace}(x_1,\ldots,x_d)x_{d+1}) = t([& \sum_{\underline{i}} n'_{i_1\ldots i_{n+1}}\texttt{fund}_n(m_{i_1},\ldots,m_{i_{n+1}}) \\ & + \sum_{\underline{j}} n_{j_1\ldots j_{n+1}}.m"_{j_{n+1}}.\texttt{cha}_n(m_{j_1},\ldots,m_{j_n}).m'_{j_{n+1}}]x_{d+1}) \end{aligned}$$

Two formal trace elements $\texttt{trace}(x_1,\ldots,x_d)$ and $\texttt{trace}'(x_1,\ldots,x_d)$ are equal iff

$$t(\texttt{trace}(x_1,\ldots,x_d)x_{d+1})=t(\texttt{trace}'(x_1,\ldots,x_d)x_{d+1})$$

finishing the proof. □

DEFINITION 53. For $m_1, m_2, \ldots, m_i, \ldots \in \int \langle \infty \rangle$, the *substitution*

$$f \mapsto f(m_1, m_2, \ldots, m_i, \ldots)$$

is the uniquely determined algebra endomorphism of $\int \langle \infty \rangle$ which maps the variable x_i to m_i and is compatible with the trace t.

That is, the substitution sends a monomial $x_{i_1} x_{i_1} \ldots x_{i_k}$ to the element $m_{i_1} m_{i_2} \ldots m_{i_k}$ and the trace of a necklace word $t(x_{i_1} x_{i_2} \ldots x_{i_k})$ to the element $t(m_{i_1} m_{i_2} \ldots m_{i_k})$.

A *substitution invariant ideal* of $\int \langle \infty \rangle$ is a twosided ideal of $\int \langle \infty \rangle$ closed under all possible substitutions as well as under the formal trace t.

For a subset of elements $E \subset \int \langle \infty \rangle$ there is a minimal substitution invariant ideal containing E. We will refer to this ideal as the *substitution invariant ideal generated by* E.

The algebra $\int_n \langle \infty \rangle$ is the free algebra in the generators $\{x_1, x_2, \ldots, x_i, \ldots\}$ in the category `alg@n`. That is, if $(B, tr) \in$ `alg@n` is trace generated by $\{b_1, b_2, \ldots, b_i, \ldots\}$, then there is a trace preserving algebra epimorphism in `alg@n`

$$\int_n \langle \infty \rangle \twoheadrightarrow B$$

by mapping $x_i \mapsto b_i$ and $t(x_{i_1} \ldots x_{i_l})$ to $tr(b_{i_1} \ldots b_{i_l})$.

The kernel of the natural quotient morphism

$$\int \langle \infty \rangle \overset{\pi_n}{\twoheadrightarrow} \int_n \langle \infty \rangle$$

is a substitution invariant ideal. For, consider a substitution endomorphism ϕ of $\int \langle \infty \rangle$

$$\begin{array}{ccc} \int \langle \infty \rangle & \xrightarrow{\phi} & \int \langle \infty \rangle \\ \downarrow & \searrow^{\psi} & \downarrow \pi_n \\ \frac{\int \langle \infty \rangle}{Ker\ \psi} & \hookrightarrow & \int_n \langle \infty \rangle \end{array}$$

Because $\psi = \pi_n \circ \phi$ preserves traces, $Ker\ \psi$ is an ideal closed under traces and the quotient $\frac{\int \langle \infty \rangle}{Ker\ \psi} \in$ `alg@n` (being a subalgebra of $\int_n \langle \infty \rangle$). The claim that $\int_n \langle \infty \rangle$ is free means that $Ker\ \pi_n$ is the minimal ideal of $\int \langle \infty \rangle$ such that the quotient is an object in `alg@n`. Therefore, $Ker\ \psi \subset Ker\ \pi_n$ and ψ factors through $\int_n \langle \infty \rangle$. Therefore, the substitution ϕ induces an endomorphism of $\int_n \langle \infty \rangle$ proving the claim.

THEOREM 40 (Procesi). *There are natural isomorphisms in* `alg@n`

$$\int_n \langle \infty \rangle \simeq \uparrow_n \langle \infty \rangle \quad \textit{and} \quad \int_n \langle m \rangle \simeq \uparrow_n \langle m \rangle$$

As a consequence we have isomorphisms in `commalg`

$$\oint_n \langle \infty \rangle \simeq \downarrow_n \langle \infty \rangle \quad \textit{and} \quad \oint_n \langle m \rangle \simeq \downarrow_n \langle m \rangle$$

PROOF. We claim that the ideal of n-th trace relations $Ker\ \tau$ is the substitution invariant ideal of $\int \langle\infty\rangle$ generated by the formal Cayley-Hamilton polynomials

$$\chi_x^{(n)}(x) \quad \text{for all} \quad x \in \int \langle\infty\rangle$$

This follows from theorem 39 and the definition of a substitution invariant ideal once we can show that the full polarization of $\chi_x^{(n)}(x)$, which we have seen is $\mathtt{cha}_n(x_1,\ldots,x_n)$, lies in the substitution invariant ideal generated by the $\chi_x^{(n)}(x)$.

This follows as we can replace the process of polarization by the process of multilinearization, the first step is to replace, for instance

$$\chi_x^{(n)}(x) \quad \text{by} \quad \chi_{x+y}^{(n)}(x+y) - \chi_x^{(n)}(x) - \chi_y^{(n)}(y)$$

The final result of multilinearization is the same as of full polarization and multilinearizing a polynomial in a substitution invariant ideal remains in the ideal.

Because $\chi_x^{(n)}(x)$ for $x \in \int \langle\infty\rangle$ maps to zero under π_n, it follows from substitution invariance of $Ker\ \pi_n$ that $Ker\ \tau \subset Ker\ \pi_n$. Because the quotient by $Ker\ \tau$ is $\uparrow_n \langle\infty\rangle \in \mathtt{alg@n}$, we obtain by minimality of $Ker\ \pi_n$ that $Ker\ \tau = Ker\ \pi_n$ proving the first statement. The second follows. $\square$

The foregoing can be used to improve the bound of $2^n - 1$ in the Nagata-Higman problem to n^2.

THEOREM 41 (Razmyslov). *Let R be an associative $\mathbb{C}$-algebra without unit element. Assume that $r^n = 0$ for all $r \in R$. Then, for all $r_i \in R$ we have*

$$r_1 r_2 \ldots r_{n^2} = 0$$

PROOF. Consider the positive part $\langle\infty\rangle_+$ of the free $\mathbb{C}$-algebra $\langle\infty\rangle$ which is an associative $\mathbb{C}$-algebra without unit. Let $T(n)$ be the twosided ideal of $\langle\infty\rangle_+$ generated by all n-powers f^n for $f \in \langle\infty\rangle_+$. Observe that the ideal $T(n)$ is invariant under all substitutions of $\langle\infty\rangle_+$. The Nagata-Higman problem then asks for a number $N(n)$ such that the product

$$x_1 x_2 \ldots x_{N(n)} \in T(n).$$

An alternative description of the quotient algebra $\langle\infty\rangle_+ / T(n)$ is the following. Let $\oint_+$ be the positive part of the n-th necklace algebra $\oint_n \langle\infty\rangle$ and $\int_+$ the positive part of the n-th trace algebra $\int_n \langle\infty\rangle$. Consider the associative $\mathbb{C}$-algebra without unit

$$\overline{\int_+} = \frac{\int_+}{\oint_+ \int_n \langle\infty\rangle}$$

Observe the following facts about $\overline{\int_+}$: as $\mathbb{C}$-algebra it is generated by the variables $X_1, X_2, \ldots$ because all the other algebra generators of the form $t(x_{i_1} \ldots x_{i_r})$ of $\int \langle\infty\rangle$ are mapped to zero in $\overline{\int_+}$. Further, from the Cayley-Hamilton polynomial it follows that every $t \in \overline{\mathbb{T}_+}$ satisfies $t^n = 0$. Hence, we have an algebra epimorphism

$$\frac{\langle\infty\rangle_+}{T(n)} \twoheadrightarrow \overline{\int_+}$$

Observe that the quotient

$$\frac{\int \langle\infty\rangle}{\oint_+ \langle\infty\rangle \int \langle\infty\rangle}$$

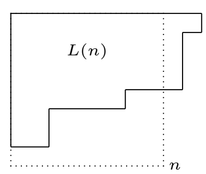
(where $\oint_+ \langle\infty\rangle$ is the positive part of the graded algebra $\oint\langle\infty\rangle$) is the free $\mathbb{C}$-algebra on the variables $\{x_1, x_2, \ldots\}$. To obtain $\overline{\int_+}$ we have to factor out the ideal of trace relations. A formal n-th Cayley-Hamilton polynomial $\chi_x^{(n)}(x)$ is mapped to x^n in $\int\langle\infty\rangle / \oint_+ \langle\infty\rangle \int\langle\infty\rangle$. That is, to obtain $\overline{\int_+}$ we factor out the substitution invariant ideal (observe that t is zero here) generated by the elements x^n, but this is just the definition of $\langle\infty\rangle_+/T(n)$, hence the above epimorphism is actually an isomorphism.

Therefore, a reformulation of the Nagata-Higman problem is to find a number $N = N(n)$ such that the product of the first N generic matrices

$$X_1X_2\ldots X_N \in \oint_+ \langle\infty\rangle \int_n \langle\infty\rangle \quad \text{or, equivalently that} \quad tr(X_1X_2\ldots X_N X_{N+1})$$

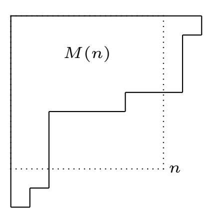
can be expressed as a linear combination of products of traces of lower degree. Using the description of the necklace relations given in theorem 36 we can reformulate this conditions in terms of the group algebra $\mathbb{C}S_{N+1}$. Let us introduce the following subspaces of the groupalgebra :

- A will be the subspace spanned by all $N+1$ cycles in S_{N+1},
- B will be the subspace spanned by all elements except $N+1$ cycles,
- $L(n)$ will be the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions



with $\leq n$ rows, and

- $M(n)$ will be the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions



having more than n rows.

With these notations, we can reformulate the above condition as

$$(12\ldots NN+1) \in B + M(n) \quad \text{and consequently} \quad \mathbb{C}S_{N+1} = B + M(n)$$

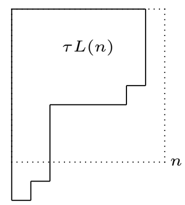
Define an inner product on the groupalgebra $\mathbb{C}S_{N+1}$ such that the groupelements form an orthonormal basis, then A and B are orthogonal complements and also $L(n)$ and $M(n)$ are orthogonal complements. But then, taking orthogonal complements the condition can be rephrased as

$$(B + M(n))^{\perp} = A \cap L(n) = 0.$$

Finally, let us define an automorphism τ on $\mathbb{C}S_{N+1}$ induced by sending e_σ to $sgn(\sigma)e_\sigma$. Clearly, τ is just multiplication by $(-1)^N$ on A and therefore the above condition is equivalent to

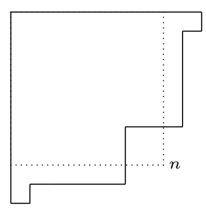
$$A \cap L(n) \cap \tau L(n) = 0.$$

Moreover, for any Young tableau λ we have that $\tau(a_\lambda) = b_{\lambda^*}$ and $\tau(b_\lambda) = a_{\lambda^*}$. Hence, the automorphism τ sends the Young symmetrizer associated to a partition to the Young symmetrizer of the dual partition. This gives the following characterization : $\tau L(n)$ is the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions



with $\leq n$ columns.

Now, specialize to the case $N = n^2$. Clearly, any Young diagram having $n^2 + 1$ boxes must have either more than n columns or more than n rows



and consequently we indeed have for $N = n^2$ that

$$A \cap L(n) \cap \tau L(n) = 0$$

finishing the proof. □

4.3. Witness algebras.

In this section we will show that the n-th necklace algebra $\oint_n A$ and the n-th trace algebra $\int_n A$ can be reconstructed from the GL_n-action on the n-th representation scheme $\mathtt{rep}_n A$ by proving that the functors

$$\begin{cases} \oint_n \\ \downarrow_n \end{cases} : \mathtt{alg} \longrightarrow \mathtt{commalg} \quad \text{and} \quad \begin{cases} \int_n \\ \uparrow_n \end{cases} : \mathtt{alg} \longrightarrow \mathtt{alg@n}$$

are paired equivalent. Moreover we will give a geometric reconstruction result for Cayley-Hamilton algebras of degree n.

Definition 54. The *n-th trace preserving representation functor* of a $\mathbb{C}$-algebra with trace (A, tr) in $\mathtt{alg@}$ is the functor

$$\mathtt{trep}_n A \;:\; \mathtt{commalg} \longrightarrow \mathtt{sets}$$

which assigns to a commutative $\mathbb{C}$-algebra B the set $Hom_{\mathtt{alg@}}(A, M_n(B))$.

Theorem 42. *For A an affine algebra in $\mathtt{alg@}$, the functor $\mathtt{trep}_n A$ is represented by the affine commutative algebra*

$$\mathbb{C}[\mathtt{trep}_n A] = \frac{\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle]}{I_A}$$

with I_A a stable ideal under the GL_n-action on $\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle]$.

PROOF. The functor $\mathtt{trep}_n A$ is representable by the quotient $\mathbb{C}[\mathtt{trep}_n A] = \frac{\mathbb{C}[\mathtt{rep}_n A]}{I_A}$ where I_A is the ideal of $\mathbb{C}[\mathtt{rep}_n A]$ minimal with respect to the condition that the composition

$$A \xrightarrow{j_A^{(n)}} M_n(\mathbb{C}[\mathtt{rep}_n A]) \twoheadrightarrow M_n(\frac{\mathbb{C}[\mathtt{rep}_n A]}{I_A})$$

is trace preserving. The ideal I_A can be described as follows. Consider the quotient in $\mathtt{alg@}$

$$A' = \frac{A}{(tr(1) - n, \chi_a^{(n)}(a) \ \forall a \in A)}$$

then A' is a Cayley-Hamilton algebra of degree n and we have that $\mathtt{trep}_n A = \mathtt{trep}_n A'$. As A is an affine algebra in $\mathtt{alg@}$ we have trace preserving epimorphisms

$$\int \langle \infty \rangle \twoheadrightarrow A \quad \text{and} \quad \int_n \langle \infty \rangle \xrightarrow{p_A} A'$$

where the kernel T_A of p_A is the ideal of trace relations of degree n of A'.

By the universal embedding $\int_n \langle \infty \rangle \hookrightarrow M_n(\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle])$ we can extend the ideal T_A to the matrixalgebra and obtain

$$M_n(\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle]) T_A M_n(\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle]) = M_n(I_A)$$

for some ideal I_A of $\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle]$. □

EXAMPLE 61. Let A be the *quantum plane* of order two,

$$A = \frac{\mathbb{C}\langle x, y \rangle}{(xy + yx)}$$

One verifies that $u = x^2$ and $v = y^2$ are central elements of A and that A is a free module of rank 4 over $\mathbb{C}[u, v]$. In fact, A is a $\mathbb{C}[u, v]$-order in the quaternion division algebra

$$\Delta = \begin{pmatrix} u & & v \\ & \mathbb{C}(u, v) & \end{pmatrix}$$

and the reduced trace map on Δ makes A into a Cayley-Hamilton algebra of degree 2. More precisely, tr is the linear map on A such that

$$\begin{cases} tr(x^i y^j) = 0 & \text{if either } i \text{ or } j \text{ are odd, and} \\ tr(x^i y^j) = 2x^i y^j & \text{if } i \text{ and } j \text{ are even.} \end{cases}$$

In particular, a trace preserving 2-dimensional representation is determined by a couple of 2×2 matrices

$$(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} , \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix}) \quad \text{with } tr(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} . \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix}) = 0$$

That is,$\mathtt{trep}_2 A$ is the hypersurface in $\mathbb{C}^6$ determined by the equation

$$\mathtt{trep}_2 A = \mathbb{V}(2x_1 x_4 + x_2 x_6 + x_3 x_5) \hookrightarrow \mathbb{C}^6$$

and is therefore irreducible of dimension 5 with an isolated singularity at $p = (0, \ldots, 0)$.

THEOREM 43 (Procesi). *Let* $(A, tr) \in$ `alg@` *be an algebra with trace and let* $(A', tr) \in$ `alg@n` *be the quotient which is a Cayley-Hamilton algebra of degree* n*. Then, we can reconstruct* A' *and its central subalgebra* $tr(A')$ *as algebras of equivariant resp. invariant polynomial maps*

$$A' \simeq M_n(\mathbb{C}[\mathtt{trep}_n A])^{GL_n} \quad \textit{and} \quad tr(A') = \mathbb{C}[\mathtt{trep}_n A]^{GL_n}$$

PROOF. In the previous chapter we have proved that $\int_n \langle \infty \rangle \simeq M_n(\mathbb{C}[\mathtt{rep}_n \langle \infty \rangle])^{GL_n}$ and we can apply the Reynolds operator R to the situation

$$\begin{array}{ccc} M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle]) & \xrightarrow{\pi} & M_n(\mathbb{C}[\mathtt{trep}_n A]) \\ \downarrow R & & \downarrow R \\ \int_n \langle\infty\rangle & \xrightarrow{\pi_0} & M_n(\mathbb{C}[\mathtt{trep}_n A])^{GL_n} \end{array}$$

The epimorphism π_0 factors through $\frac{\int_n \langle\infty\rangle}{T_A}$ inducing an epimorphism

$$A' \longrightarrow M_n(\mathbb{C}[\mathtt{trep}_n A])^{GL_n}$$

We claim that this map is also injective, or equivalently, that

$$M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle])T_A M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle]) \cap \int_n \langle\infty\rangle = T_A$$

Using functoriality of the Reynolds operator with respect to multiplication in $M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle])$ by an element $x \in \int_n \langle\infty\rangle$ or with respect to the trace map (both commuting with the GL_n-action) we deduce the following identities :

- For all $x \in \int_n \langle\infty\rangle$ and all $z \in M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle])$ we have
$$R(xz) = xR(z) \quad \text{and} \quad R(zx) = R(z)x$$
- For all $z \in M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle])$ we have
$$R(tr(z)) = tr(R(z))$$

Assume that $z = \sum_i t_i x_i n_i \in M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle])T_A M_n(\mathbb{C}[\mathtt{rep}_n\langle\infty\rangle]) \cap \int_n \langle\infty\rangle$ with $m_i, n_i \in M_n(\mathbb{C}[\mathtt{rep}_n\langle m\rangle])$ and $t_i \in T_A$. Now, consider the generic matrix $X_{m+1} \in \int_n \langle\infty\rangle$ which does not occur in any of these elements. By the necklace property of traces we have

$$tr(zX_{m+1}) = \sum_i tr(m_i t_i n_i X_{m+1}) = \sum_i tr(n_i X_{m+1} m_i t_i)$$

and if we apply the Reynolds operator to it we obtain the equality

$$tr(zX_{m+1}) = tr(\sum_i R(n_i X_{m+1} m_i) t_i)$$

For any i, the term $R(n_i X_{m+1} m_i)$ is invariant so belongs to $\int_n \langle m+1\rangle$ and is linear in X_{m+1}. Knowing the generating elements of $\int_n \langle m+1\rangle$ we can write

$$R(n_i X_{m+1} m_i) = \sum_j s_{ij} X_{m+1} t_{ij} + \sum_k tr(u_{ik} X_{m+1}) v_{ik}$$

with all of the elements s_{ij}, t_{ij}, u_{ik} and v_{ik} in $\int_n \langle m \rangle$. Substituting this information and again using the necklace property we obtain

$$tr(zX_{m+1}) = tr((\sum_{i,j,k} s_{ij}t_{ij}t_i + tr(v_{ik}t_i))X_{m+1})$$

From nondegeneracy of the trace map we deduce

$$z = \sum_{i,j,k} s_{ij}t_{ij}t_i + tr(v_{ik}t_i)$$

Because $t_i \in T_A$ and T_A is stable under taking traces we obtain $z \in T_A$ finishing the proof of the first statement.

Apply functoriality of the Reynolds operator to the setting

$$\begin{array}{ccc} M_n(\mathbb{C}[\mathtt{trep}_n A]) & \overset{tr}{\rightleftarrows} & \mathbb{C}[\mathtt{trep}_n A] \\ \downarrow R & & \downarrow R \\ A' & \overset{tr_A}{\rightleftarrows} & \mathbb{C}[\mathtt{trep}_n A]^{GL_n} \end{array}$$

from which the second statement follows. $\square$

THEOREM 44. *When applied to affine $\mathbb{C}$-algebras, the functors*

$$\begin{cases} \oint_n \\ \downarrow_n \end{cases} : \mathtt{alg} \longrightarrow \mathtt{commalg} \quad \textit{and} \quad \begin{cases} \int_n \\ \uparrow_n \end{cases} : \mathtt{alg} \longrightarrow \mathtt{alg@n}$$

are paired equivalent.

PROOF. Because the trace functor $\int : \mathtt{alg} \longrightarrow \mathtt{alg@}$ is a left adjoint functor of the forgetful functor $i : \mathtt{alg@} \longrightarrow \mathtt{alg}$ and because $n \times n$ matrices over commutative algebras are Cayley-Hamilton algebras of degree n we have functorial bijections for any algebra A and any commutative algebra B

$$Hom_{\mathtt{alg}}(A, M_n(B)) = Hom_{\mathtt{alg@}}(\int A, M_n(B)) = Hom_{\mathtt{alg@n}}(\int_n A, M_n(B))$$

Therefore, we have equivalence between the functors

$$\mathtt{rep}_n A \simeq \mathtt{trep}_n \int A \simeq \mathtt{trep}_n \int_n A$$

and the result follows from theorem 43 applied to the Cayley-Hamilton algebra $\int_n A$ of degree n. $\square$

EXAMPLE 62. If $A \in \mathtt{alg}$ is an affine $\mathbb{C}$-algebra, then for all n

$$\int_n A \in \mathtt{alg@n} \qquad \text{and} \qquad \oint_n A \in \mathtt{commalg}$$

are affine $\mathbb{C}$-algebras. Moreover, $\int_n A$ is a finite module over the Noetherian commutative algebra $\oint_n A$ hence is itself Noetherian. Indeed, this follows from the generic case and the Reynolds operator.

EXAMPLE 63. Let A be a Cayley-Hamilton algebra of degree n with trace map tr, then we can define a *norm map* on A by defining

$$N(a) = \sigma_n(a) \quad \text{for all } a \in A.$$

Recall that the elementary symmetric function σ_n is a polynomial function $f(t_1, t_2, \ldots, t_n)$ in the Newton functions $t_i = \sum_{j=1}^n x_j^i$. Then, $\sigma(a) = f(tr(a), tr(a^2), \ldots, tr(a^n))$. Because, we have a trace preserving embedding $A \hookrightarrow M_n(\mathbb{C}[\mathtt{trep}_n A])$ and the norm map N coincides with the determinant in this matrix-algebra, we have that

$$N(1) = 1 \quad \text{and} \quad \forall a, b \in A \quad N(ab) = N(a)N(b).$$

Furthermore, the norm map extends to a polynomial map on $A[t]$ and we have that $\chi_a^{(n)}(t) = N(t - a)$, in particular we can obtain the trace by polarization of the norm map. For the finite dimensional semi-simple $\mathbb{C}$-algebra

$$A = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C}),$$

let tr be a trace map on A making it into a Cayley-Hamilton algebra of degree n with $tr(A) = \mathbb{C}$. Then, we claim that there exist a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}_+^k$ such that $n = \sum_{i=1}^k m_i d_i$ and for any $a = (A_1, \ldots, A_k) \in A$ with $A_i \in M_{d_i}(\mathbb{C})$ we have that

$$tr(a) = m_1 Tr(A_1) + \ldots + m_k Tr(A_k)$$

where Tr are the usual trace maps on matrices.

The norm-map N on A defined by the trace map tr induces a group morphism on the invertible elements of A

$$N : A^* = GL_{d_1}(\mathbb{C}) \times \ldots \times GL_{d_k}(\mathbb{C}) \longrightarrow \mathbb{C}^*$$

that is, a *character*. Now, any character is of the following form, let $A_i \in GL_{d_i}(\mathbb{C})$, then for $a = (A_1, \ldots, A_k)$ we must have

$$N(a) = det(A_1)^{m_1} det(A_2)^{m_2} \ldots det(A_k)^{m_k}$$

for certain integers $m_i \in \mathbb{Z}$. Since N extends to a polynomial map on the whole of A we must have that all $m_i \geq 0$. By polarization it then follows that

$$tr(a) = m_1 Tr(A_1) + \ldots m_k Tr(A_k)$$

and it remains to show that no $m_i = 0$. Indeed, if $m_i = 0$ then tr would be the zero map on $M_{d_i}(\mathbb{C})$, but then we would have for any $a = (0, \ldots, 0, A, 0, \ldots, 0)$ with $A \in M_{d_i}(\mathbb{C})$ that

$$\chi_a^{(n)}(t) = t^n$$

whence $\chi_a^{(n)}(a) \neq 0$ whenever A is not nilpotent. This contradiction finishes the proof of the claim. Recall from §4.3 that

$$\mathtt{rep}_n A = \bigsqcup_{(m_1, \ldots, m_k)} GL_n / (GL_{m_1} \times \ldots \times GL_{m_k})$$

That is, the representation scheme is the disjoint union of the different trace preserving representation schemes

$$\mathtt{trep}_n A = GL_n / (GL_{m_1} \times \ldots \times GL_{m_k})$$

for the trace map $tr = m_1 Tr_1 + \ldots + m_k Tr_k$.

EXAMPLE 64. Let A be a finite dimensional algebra with radical J and assume there is a trace map tr on A making A into a Cayley-Hamilton algebra of degree n and such that $tr(A) = \mathbb{C}$. We claim that the norm map $N : A \longrightarrow \mathbb{C}$ is zero on J. Indeed, any $j \in J$ satisfies $j^l = 0$ for some l whence $N(j)^l = 0$. But then, polarization gives that $tr(J) = 0$ and we have that the semisimple algebra

$$A^{ss} = A/J = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C})$$

is a semisimple Cayley-hamilton algebra of degree n on which we can apply the foregoing exercise. Because $A \simeq A^{ss} \oplus J$ as $\mathbb{C}$-vectorspaces we deduce that if $tr : A \longrightarrow \mathbb{C}$ is a trace map such that A is a Cayley-Hamilton algebra of degree n, there exists a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}^k_+$ such that for all $a = (A_1, \ldots, A_k, j)$ with $A_i \in M_{d_i}(\mathbb{C})$ and $j \in J$ we have

$$tr(a) = m_1 Tr(A_1) + \ldots m_k Tr(A_k)$$

with Tr the usual traces on $M_{d_i}(\mathbb{C})$ and $\sum_i m_i d_i = n$.

Fix a trace map tr on A determined by a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}^k$. Then, the trace preserving variety $\mathtt{trep}_n\ A$ is the scheme of A-modules of *dimension vector* α, that is, those A-modules M such that

$$M^{ss} = S_1^{\oplus m_1} \oplus \ldots \oplus S_k^{\oplus m_k}$$

where S_i is the simple A-module of dimension d_i determined by the i-th factor in A^{ss}. By theorem 43 A can be recovered from the GL_n-structure of the affine scheme $\mathtt{trep}_n\ A$ of all A-modules of dimension vector α.

Still, there can be other trace maps on A making A into a Cayley-Hamilton algebra of degree n. For example let C be a finite dimensional commutative $\mathbb{C}$-algebra with radical N, then $A = M_n(C)$ is finite dimensional with radical $J = M_n(N)$ and the usual trace map $tr : M_n(C) \longrightarrow C$ makes A into a Cayley-Hamilton algebra of degree n such that $tr(J) = N \neq 0$. Still, if A is semi-simple, the center $Z(A) = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$ (as many terms as there are matrix components in A) and any subring of $Z(A)$ is of the form $\mathbb{C} \oplus \ldots \oplus \mathbb{C}$. In particular, $tr(A)$ has this form and composing the trace map with projection on the j-th component we have a trace map tr_j on which we can apply the foregoing.

DEFINITION 55. `GL(n)-aff` will be the category of all affine schemes with a GL_n-action. A reformulation of theorem 43 is that the contravariant functor

$$\mathtt{trep}_n : \mathtt{alg@n} \longrightarrow \mathtt{GL(n)\text{-}aff}$$

which assigns to a Cayley-Hamilton algebra of degree n its trace preserving representation scheme has a *left* inverse

$$\Uparrow^n : \mathtt{GL(n)\text{-}aff} \longrightarrow \mathtt{alg@n}$$

assigning to an affine GL_n-scheme `fun` the equivariants $\Uparrow^n \mathtt{fun} = M_n(\mathbb{C}[\mathtt{fun}])^{GL_n}$. The Cayley-Hamilton algebra $\Uparrow^n \mathtt{fun}$ is called the *witness algebra* of `fun`.

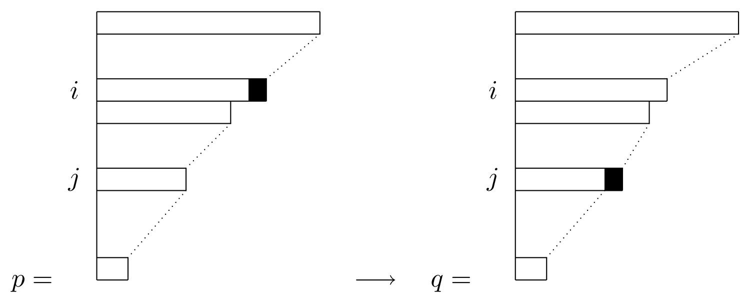
Note however that this is *not* an equivalence of categories. There are many GL_n-varieties having the same witness algebra.

EXAMPLE 65. To give some easy examples we need to recall some facts about orbitclosures of nilpotent $n \times n$ matrices.

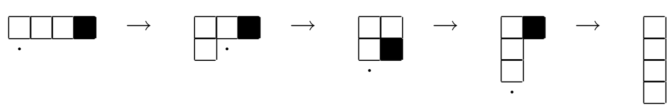
Denote a partition p of n by an integral n-tuple $(a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ with $\sum_{i=1}^n a_i = n$. As before, we represent a partition by a Young diagram by omitting rows corresponding to zeroes.If $q = (b_1, \ldots, b_n)$ is another partition of n we say that p *dominates* q and write

$$p > q \quad \text{if and only if} \quad \sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i \quad \text{for all } 1 \leq r \leq n.$$

The *dominance order* is induced by the *Young move* of throwing a row-ending box down the diagram. Indeed, let p and q be partitions of n such that $p > q$ and assume there is no partition r such that $p > r$ and $r > q$. Let i be the minimal number such that $a_i > b_i$. Then by the assumption $a_i = b_i + 1$. Let $j > i$ be minimal such that $a_j \neq b_j$, then we have $b_j = a_j + 1$ because p dominates q. But then, the remaining rows of p and q must be equal. That is, a Young move can be depicted as



For example, the Young moves between the partitions of 4 given above are as indicated



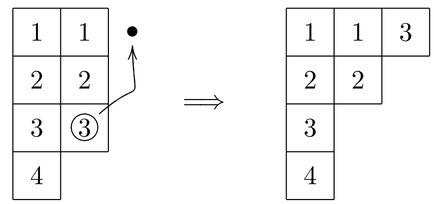
A *Young p-tableau* is the Young diagram of p with the boxes labeled by integers from $\{1, 2, \ldots, s\}$ for some s such that each label appears at least ones. A Young p-tableau is said to be *of type q* for some partition $q = (b_1, \ldots, b_n)$ of n if the following conditions are met :

- the labels are non-decreasing along rows,
- the labels are strictly increasing along columns, and
- the label i appears exactly b_i times.

For example, if $p = (3, 2, 1, 1)$ and $q = (2, 2, 2, 1)$ then the p-tableau below

1	1	3
2	2	
3		
4		

is of type q (observe that $p > q$ and even $p \to q$). In general, let $p = (a_1, \ldots, a_n)$ and $q = (b_1, \ldots, b_n)$ be partitions of n and assume that $p \to q$. Then, there is a Young p-tableau of type q. For, fill the Young diagram of q by putting 1's in the first row, 2's in the second and so on. Then, upgrade the fallen box together with its label to get a Young p-tableau of type q. In the example above



Conversely, assume there is a Young p-tableau of type q. The number of boxes labeled with a number $\leq i$ is equal to $b_1 + \ldots + b_i$. Further, any box with label $\leq i$ must lie in the first i rows (because the labels strictly increase along a column). There are $a_1 + \ldots + a_i$ boxes available in the first i rows whence

$$\sum_{j=1}^{i} b_i \leq \sum_{j=1}^{i} a_i \quad \text{for all} \quad 1 \leq i \leq n$$

and therefore $p > q$. After these preliminaries on partitions, let us return to nilpotent matrices.

Let A be a nilpotent matrix of type $p = (a_1, \ldots, a_n)$, that is, conjugated to a matrix with Jordan blocks (all with eigenvalue zero) of sizes a_i. We have seen before that the subspace V_l of column vectors $v \in \mathbb{C}^n$ such that $A^l.v = 0$ has dimension

$$\sum_{h=1}^{l} \#\{j \mid a_j \geq h\} = \sum_{h=1}^{l} a_h^*$$

where $p^* = (a_1^*, \ldots, a_n^*)$ is the dual partition of p. Choose a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$ such that for all l the first $a_1^* + \ldots + a_l^*$ base vectors span the subspace V_l. For example, if A is in Jordan normal form of type $p = (3, 2, 1, 1)$

$$\begin{bmatrix} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & & & & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

then $p^* = (4, 2, 1)$ and we can choose the standard base vectors ordered as follows

$$\{\underbrace{e_1, e_4, e_6, e_7}_{4}, \underbrace{e_2, e_5}_{2}, \underbrace{e_3}_{1}\}.$$

Take a partition $q = (b_1, \ldots, b_n)$ with $p \to q$ (in particular, $p > q$), then for the dual partitions we have $q^* \to p^*$ (and thus $q^* > p^*$). But then there is a Young q^*-tableau of type p^*. In the example with $q = (2, 2, 2, 1)$ we have $q^* = (4, 3)$ and $p^* = (4, 2, 1)$ and we can take the Young q^*-tableau of type p^*

1	1	1	1
2	2	3	

Now label the boxes of this tableau by the base vectors $\{v_1, \ldots, v_n\}$ such that the boxes labeled i in the Young q^*-tableau of type p^* are filled with the base vectors from $V_i - V_{i-1}$. Call this tableau T. In the example, we can take

$$T = \begin{array}{|c|c|c|c|} \hline e_1 & e_4 & e_6 & e_7 \\ \hline e_2 & e_5 & e_3 & \\ \cline{1-3} \end{array}$$

Define a linear operator F on $\mathbb{C}^n$ by the rule that $F(v_i) = v_j$ if v_j is the label of the box in T just above the box labeled v_i. In case v_i is a label of a box in the first row of T we take $F(v_i) = 0$. Obviously, F is a nilpotent $n \times n$ matrix and by construction we have that

$$rk\ F^l = n - (b_1^* + \ldots + b_l^*)$$

That is, F is nilpotent of type $q = (b_1, \ldots, b_n)$. Moreover, F satisfies $F(V_i) \subset V_{i-1}$ for all i by the way we have labeled the tableau T and defined F.

In the example above, we have $F(e_2) = e_1$, $F(e_5) = e_4$, $F(e_3) = e_6$ and all other $F(e_i) = 0$. That is, F is the matrix

$$\begin{bmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & & & 0 & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & 1 & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

which is seen to be of type $(2, 2, 2, 1)$ after performing a few Jordan moves.

Returning to the general case, consider for all $\epsilon \in \mathbb{C}$ the $n \times n$ matrix :

$$F_\epsilon = (1 - \epsilon)F + \epsilon A.$$

We claim that for all but finitely many values of ϵ we have $F_\epsilon \in \mathcal{O}(A)$. Indeed, we have seen that $F(V_i) \subset V_{i-1}$ where V_i is defined as the subspace such that $A^i(V_i) = 0$. Hence, $F(V_1) = 0$ and therefore

$$F_\epsilon(V_1) = (1 - \epsilon)F + \epsilon A(V_1) = 0.$$

Assume by induction that $F_\epsilon^i(V_i) = 0$ holds for all $i < l$, then we have that

$$\begin{array}{rcl} F_\epsilon^l(V_l) & = & F_\epsilon^{l-1}((1 - \epsilon)F + \epsilon A)(V_l) \\ & \subset & F_\epsilon^{l-1}(V_{l-1}) = 0 \end{array}$$

because $A(V_l) \subset V_{l-1}$ and $F(V_l) \subset V_{l-1}$. But then we have for all l that

$$rk\ F_\epsilon^l \leq dim\ V_l = n - (a_1^* + \ldots + a_l^*) = rk\ A^l \overset{def}{=} r_l.$$

Then for at least one $r_l \times r_l$ submatrix of F_ϵ^l its determinant considered it as a polynomial of degree r_l in ϵ is not identically zero (as it is nonzero for $\epsilon = 1$). But then this determinant is non-zero for all but finitely many ϵ. Hence, $rk\ F_\epsilon^l = rk\ A^l$ for all l for all but finitely many ϵ. As these numbers determine the dual partition p^* of the type of A, F_ϵ is a nilpotent $n \times n$ matrix of type p for all but finitely many values of ϵ, proving the claim. But then, $F_0 = F$ which we have proved to be a nilpotent matrix of type q belongs to the closure of the orbit $\mathcal{O}(A)$. That is, we have proved the difficult part of the *Gerstenhaber-Hesselink theorem* which asserts that if A is a nilpotent $n \times n$ matrix of type $p = (a_1, \ldots, a_n)$ and B nilpotent of type $q = (b_1, \ldots, b_n)$ then, B belongs to the closure of the orbit $\mathcal{O}(A)$, that is,

$$B \in \overline{\mathcal{O}(A)} \quad \text{if and only if} \quad p > q$$

in the domination order on partitions of n.

To prove this theorem we only have to observe that if B is contained in the closure of A, then B^l is contained in the closure of A^l and hence $rk\ A^l \geq rk\ B^l$ (because $rk\ A^l < k$ is equivalent to vanishing of all determinants of $k \times k$ minors which is a closed condition). But then,

$$n - \sum_{i=1}^{l} a_i^* \geq n - \sum_{i=1}^{l} b_i^*$$

for all l, that is, $q^* > p^*$ and hence $p > q$. The other implication was proved above if we remember that the domination order was induced by the Young moves and clearly we have that if $B \in \overline{\mathcal{O}(C)}$ and $C \in \overline{\mathcal{O}(A)}$ then also $B \in \overline{\mathcal{O}(A)}$.

We are now in a position to give the promised examples of affine GL_n-schemes having the same witness algebra. Consider the action by conjugation of GL_n on $M_n(\mathbb{C}) = \mathtt{rep}_n\langle 1 \rangle$ and take a nilpotent matrix A. All eigenvalues of A are zero, so the conjugacy class of A is fully determined by the sizes of its Jordan blocks. These sizes determine a partition $\lambda(A) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$.

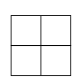
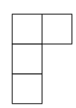
$$\mathcal{O}(B) \subset \overline{\mathcal{O}(A)} \iff \lambda(B)^* \geq \lambda(A)^*.$$

where λ^* denotes the dual partition. The witness algebra of $\overline{\mathcal{O}(A)}$ is equal to

$$M_n(\mathbb{C}[\overline{\mathcal{O}(A)}])^{GL_n} = \mathbb{C}[X]/(X^k)$$

where k is the number of columns of the Young diagram $\lambda(A)$.

Hence, the orbit closures of nilpotent matrices such that their associated Young diagrams have equal number of columns have the same witness algebras. For example, if $n = 4$ then the closures of the orbits corresponding to


and


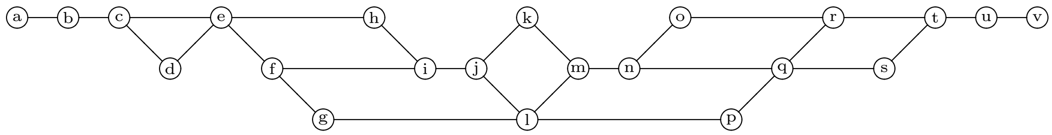
have the same witness algebra, although the closure of the second is a proper closed subscheme of the closure of the first.

Example 66. The following table lists all partitions (and their dual in the other column)

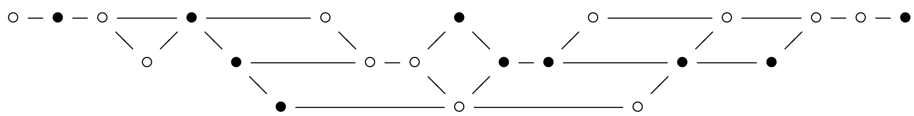
The partitions of 8.

a	(8)	v	(1,1,1,1,1,1,1,1)
b	(7,1)	u	(2,1,1,1,1,1,1)
c	(6,2)	t	(2,2,1,1,1,1)
d	(6,1,1)	s	(3,1,1,1,1,1)
e	(5,3)	r	(2,2,2,1,1)
f	(5,2,1)	q	(3,2,1,1,1)
g	(5,1,1,1)	p	(4,1,1,1,1)
h	(4,4)	o	(2,2,2,2)
i	(4,3,1)	n	(3,2,2,1)
j	(4,2,2)	m	(3,3,1,1)
k	(3,3,2)	k	(3,3,2)
l	(4,2,1,1)	l	(4,2,1,1)

The domination order between these partitions can be depicted as follows where all the Young moves are from left to right



Of course, from this graph we can read off the dominance order graphs for partitions of $n \leq 8$. In the picture below, the closures of orbits corresponding to connected nodes of the same color have the same witness algebra.



We can use the reconstruction result to characterize the smooth algebras in `alg@n` as those Cayley-Hamilton algebras A of degree n for which $\mathtt{trep}_n A$ is a smooth variety.

THEOREM 45. *If A is* `alg`*-smooth, then the n-th trace algebra $\int_n A$ is* `alg@n`*-smooth.*

PROOF. If (B, I) is a testobject in `alg@n`, then it is also a testobject in `alg`. Hence, there is a lifting $\tilde{\phi} : A \longrightarrow B$ to the map $A \longrightarrow \int_n A \xrightarrow{\phi} \frac{B}{I}$. Because the trace functor is the left adjoint to the inclusion `alg@` $\xrightarrow{i}$ `alg` there is a corresponding *trace preserving* algebra morphism

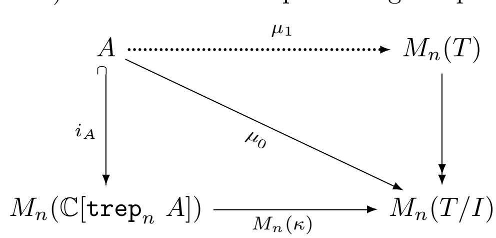
$$\psi = t(A,B)(\tilde{\phi}) \ : \ \int A \longrightarrow B$$

As B is a Cayley-Hamilton algebra of degree n, this map factors through the quotient $\int_n A$. $\square$

THEOREM 46 (Procesi). *For $(A, tr_A) \in$* `alg@n` *the following are equivalent :*

(1) *A is* `alg@n`*-smooth.*
(2) $\mathtt{trep}_n A$ *is a smooth scheme.*

PROOF. (1) $\Rightarrow$ (2) : We have to show that $\mathbb{C}[\mathtt{trep}_n\ A]$ is `commalg`-smooth. Take a commutative test-object (T, I) with I nilpotent and an algebra map $\kappa : \mathbb{C}[\mathtt{trep}_n\ A] \longrightarrow T/I$. Composing with the universal embedding i_A (coming from the reconstruction result) we obtain a trace preserving morphism μ_0



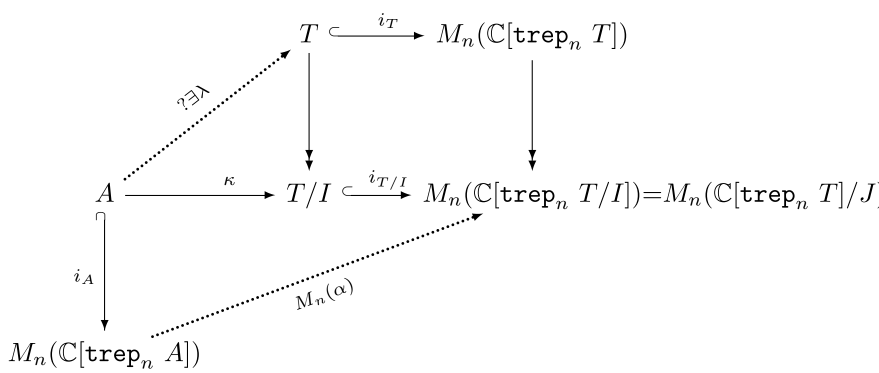
Because $M_n(T)$ with the usual trace is a Cayley-Hamilton algebra of degree n and $M_n(I)$ a trace stable ideal there is a trace preserving algebra map μ_1 because A

is alg@n-smooth. By the universal property of the embedding i_A there exists a $\mathbb{C}$-algebra morphism

$$\lambda : \mathbb{C}[\mathtt{trep}_n\ A] \longrightarrow T$$

such that $M_n(\lambda)$ completes the diagram. The morphism λ is the required lift.

(2) $\Rightarrow$ (1) : Let (T, I) be a test-object in **alg@n** and take a trace preserving $\mathbb{C}$-algebra map $\kappa : A \longrightarrow T/I$. We obtain the diagram



Here, $J = M_n(\mathbb{C}[\mathtt{trep}_n T])IM_n(\mathbb{C}[\mathtt{trep}_n T])$ and we know already that $J \cap T = I$. By the universal property of the embedding i_A we obtain a $\mathbb{C}$-algebra map

$$\mathbb{C}[\mathtt{trep}_n\ A] \xrightarrow{\alpha} \mathbb{C}[\mathtt{trep}_n\ T]/J$$

which we would like to lift to $\mathbb{C}[\mathtt{trep}_n\ T]$. This does *not* follow from the fact that $\mathbb{C}[\mathtt{trep}_n\ A]$ is commalg-smooth as J is usually not nilpotent.

We need the technical result that if I is an ideal of B closed under taking traces and if $E(I)$ denotes the extended ideal

$$E(I) = M_n(\mathbb{C}[\mathtt{trep}_n B])IM_n(\mathbb{C}[\mathtt{trep}_n B])$$

then for all powers k we have the inclusion $E(I)^{kn^2} \cap B \subset I^k$.

Write $\frac{B}{I} = \overline{B} = \frac{\int_n \langle m \rangle}{T}$ and consider the extended ideal $E_{\overline{B}} = M_n(\mathbb{C}[\mathtt{rep}_n \langle m \rangle])TM_n(\mathbb{C}[\mathtt{rep}_n \langle m \rangle]) = M_n(N)$ then we know already that $\mathbb{C}[\mathtt{trep}_n \overline{B}] = \frac{\mathbb{C}[\mathtt{rep}_n \langle m \rangle]}{N}$. We claim that for all k we have $E^k_{\overline{B}} \cap \int_n \langle m \rangle \subset T^k$.

Indeed, let $\int_n \langle m \rangle$ be the trace algebra on the generic $n \times n$ matrices $\{X_1, \ldots, X_m\}$ and $\int_n \langle l + m \rangle$ the trace algebra on the generic matrices $\{Y_1, \ldots, Y_l, X_1, \ldots, X_m\}$. Let $\{U_1, \ldots, U_l\}$ be elements of $\int_n \langle m \rangle$ and consider the trace preserving map $\int_n \langle l + m \rangle \xrightarrow{u} \int_n \langle m \rangle$ induced by the map defined by sending Y_i to U_i. Then, by the universal property we have a commutative diagram of Reynold operators

$$\begin{array}{ccc}
M_n(\mathbb{C}[M_n^{l+m}]) & \xrightarrow{\tilde{u}} & M_n(\mathbb{C}[M_n^m]) \\
\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} \\
\int_n \langle l + m \rangle & \xrightarrow{u} & \int_n \langle m \rangle
\end{array} .$$

Now, let $A_1, \ldots, A_{l+1}$ be elements from $M_n(\mathbb{C}[M_n^m])$, then we can calculate $R(A_1U_1A_2U_2A_3 \ldots A_lU_lA_{l+1})$ by first computing

$$r = R(A_1Y_1A_2Y_2A_3 \ldots A_lY_lA_{l+1})$$

and then substituting the Y_i with U_i. The Reynolds operator preserves the degree in each of the generic matrices, therefore r will be linear in each of the Y_i and is a sum of trace algebra elements. By our knowledge of the generators of necklaces and the trace algebra we can write each term of the sum as an expression

$$tr(M_1)tr(M_2) \ldots tr(M_z)M_{z+1}$$

where each of the M_i is a monomial of degree $\leq n^2$ in the generic matrices $\{Y_1, \ldots, Y_l, X_1, \ldots, X_m\}$. Now, look at how the generic matrices Y_i are distributed among the monomials M_j. Each M_j contains *at most* n^2 of the Y_i's, hence the monomial M_{z+1} contains *at least* $l - vn^2$ of the Y_i where $v \leq z$ is the number of M_i with $i \leq z$ containing at least one Y_j.

Now, assume all the U_i are taken from the ideal $T \triangleleft \int_n \langle m \rangle$ which is closed under taking traces, then it follows that

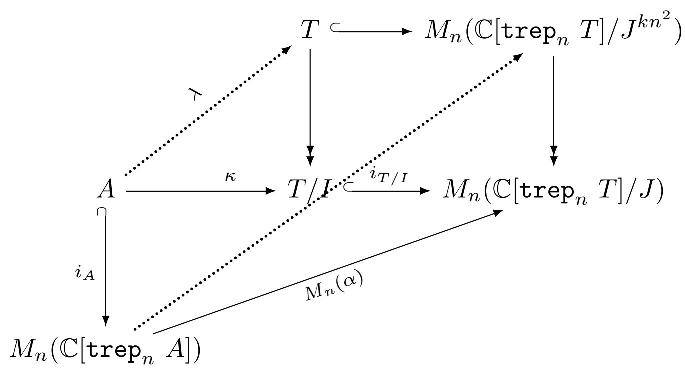
$$R(A_1U_1A_2U_2A_3 \ldots A_lU_lA_{l+1}) \in T^{v+(l-vn^2)} \subset T^k$$

if we take $l = kn^2$ as $v + (k-v)n^2 \geq k$. But this finishes the proof of the claim.

Returning to the main line of argument, as I is a nilpotent ideal of T there is some h such that $I^h = 0$. As I is closed under taking traces we know by the claim that

$$E(I)^{hn^2} \cap T \subset I^h = 0.$$

Now, by definition $E(I) = M_n(\mathbb{C}[\mathtt{trep}_n\ T])IM_n(\mathbb{C}[\mathtt{trep}_n\ T])$ which is equal to $M_n(J)$. That is, the inclusion can be rephrased as $M_n(J)^{hn^2} \cap T = 0$, whence there is a trace preserving embedding $T \hookrightarrow M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2})$. Now, we have the following situation



This time we *can* lift α to a $\mathbb{C}$-algebra morphism

$$\mathbb{C}[\mathtt{trep}_n\ A] \longrightarrow \mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2}.$$

which gives us a trace preserving morphism

$$A \xrightarrow{\lambda} M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2})$$

with image contained in the algebra of GL_n-invariants. Because $T \hookrightarrow M_n(\mathbb{C}[\mathtt{trep}_n\ T]/J^{hn^2})$ and by surjectivity of invariants under surjective maps, the GL_n-equivariants are equal to T, giving the required lift λ. □

Whereas GL_n-equivariant geometry provides us with powerful tools to study n-dimensional (trace preserving) representation schemes, the methods sometimes lead us away from Cayley-Hamilton algebras.

The foregoing theorem may suggest a method to construct `alg@n`-smooth algebras. Start with an arbitrary $A \in$ `alg@n`. If A is not `alg@n`-smooth the scheme $\mathtt{trep}_n A$ contains singularities. There is a GL_n-equivariant desingularization. Cover this desingularization by affine GL_n-invariant opens and take their witness algebra which morally should give us `alg@n`-smooth algebras. However, this is not the case.

Example 67. Let A be the quantum plane of order two. In example 61 we have seen that

$$\mathtt{trep}_2 A = \mathbb{V}(2x_1x_4 + x_2x_6 + x_3x_5) \hookrightarrow \mathbb{C}^6$$

is a hypersurface with an isolated singularity at the origin $p = (0,0,0,0,0,0)$.

Consider the blow-up of $\mathbb{C}^6$ at p which is the closed subvariety of $\mathbb{C}^6 \times \mathbb{P}^5$ defined by

$$\tilde{\mathbb{C}}^6 = \mathbb{V}(x_iX_j - x_jX_i)$$

with the X_i the projective coordinates of $\mathbb{P}^5$. The strict transform of $\mathtt{trep}_2 A$ is the subvariety

$$\tilde{\mathtt{trep}} = \mathbb{V}(x_iX_j - x_jX_i, 2X_1X_4 + X_2X_6 + X_3X_5) \hookrightarrow \mathbb{C}^6 \times \mathbb{P}^5$$

which is a smooth variety. Moreover, there is a natural GL_2-action on it induced by simultaneous conjugation on the fourtuple of 2×2 matrices

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \quad \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix} \quad \begin{bmatrix} X_1 & X_2 \\ X_3 & -X_1 \end{bmatrix} \quad \begin{bmatrix} X_4 & X_5 \\ X_6 & -X_4 \end{bmatrix}$$

As the projection $\tilde{\mathtt{trep}} \twoheadrightarrow \mathtt{trep}_2 A$ is a GL_2-isomorphism outside the exceptional fiber, we only need to investigate the semi-stable points over p. Take the particular point x

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

which is semi-stable and has as stabilizer

$$Stab(x) = \boldsymbol{\mu}_2 = \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rangle \hookrightarrow PGL_2$$

Hence, there is no affine GL_2-stable open of $\tilde{\mathtt{trep}}$ containing x such that it is of the form $\mathtt{trep}_2 B$ for some $B \in$ `alg@2` as this would contradict the fact that the stabilizer subgroup of a module is connected. Connectivity follows from the fact that the stabilizer subgroup is the group of module automorphisms, which in turn is the group of units of the endomorphism ring of the module.

4.4. Semisimple modules.

If A is an affine algebra, its n-th necklace algebra $\oint_n A = \downarrow_n A$ is an affine commutative algebra whence it is the coordinate ring of an affine scheme which we denote with $\mathtt{iss}_n A$. In this section we will justify this notation by showing that the $\mathbb{C}$-points of $\mathtt{iss}_n A$ classify the isomorphism classes of semi-simple n-dimensional A-representations. Information on the $\mathbb{C}$-points is contained in the *reduced* variety structure of the scheme so we will restrict to varieties in this section. If we want to stress this fact we denote by $\mathtt{rrep}_n A$ the reduced variety of the scheme $\mathtt{rep}_n A$ of n-dimensional representations, and by $\mathtt{riss}_n A$ the reduced variety of the affine scheme $\mathtt{iss}_n A$. Note that in case A is $\mathtt{alg}$-smooth, then $\mathtt{rep}_n A$ is smooth whence a reduced variety. But then, $\mathtt{iss}_n A$ is also reduced in this case. For arbitrary algebras however the two structures can be different,

Example 68. Let $A = \frac{\mathbb{C}[x]}{(x^2)}$, then the coordinate ring of $\mathtt{rep}_1 A = \mathtt{iss}_1 A$ (note that $GL_1 = \mathbb{C}^*$ acts trivially on $\mathtt{rep}_1 A$) is the ring of dual numbers $\mathbb{C}[\epsilon] = \mathbb{C}[x]/(x^2)$. However, the coordinate ring of the *reduced* varieties $\mathtt{rrep}_1 A = \mathtt{riss}_n A$ is $\mathbb{C}$.

Because of their relevance to the *reduced* structure of representation schemes, we quickly run through the proofs of the dimension formula, Chevalley's theorem and the relation between analytic and Zariski closures. More details can be found in the excellent textbook by Hanspeter Kraft [**36**].

Definition 56. A morphism $X \xrightarrow{\phi} Y$ between two affine irreducible varieties is said to be *dominant* if the image $\phi(X)$ is Zariski dense in Y. On the level of the coordinate algebras, dominance is equivalent to $\phi^* : \mathbb{C}[Y] \longrightarrow \mathbb{C}[X]$ being injective and hence inducing a fieldextension $\phi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ between the functionfields.

A morphism $X \xrightarrow{\phi} Y$ between two affine varieties is said to be *finite* if under the algebra morphism ϕ^* the coordinate algebra $\mathbb{C}[X]$ is a finite $\mathbb{C}[Y]$-module.

A finite and surjective morphism with X irreducible and

$$X \xrightarrow{\phi} Y$$

Y is normal satisfies the *going-down property* . That is, let $Y' \hookrightarrow Y$ be an irreducible Zariski closed subvariety and $x \in X$ with $\phi(x) = y' \in Y'$. Then, there is an irreducible Zariski closed subvariety $X' \hookrightarrow X$ such that $x \in X'$ and $\phi(X') = Y'$.

Example 69. Let X be an irreducible affine variety of dimension d. By the *Noether normalization* result $\mathbb{C}[X]$ is a finite module over a polynomial subalgebra $\mathbb{C}[f_1, \ldots, f_d]$. But then, the finite inclusion $\mathbb{C}[f_1, \ldots, f_d] \hookrightarrow \mathbb{C}[X]$ determines a finite surjective morphism

$$X \xrightarrow{\phi} \mathbb{C}^d$$

Example 70. An important source of finite morphisms is given by integral extensions. Recall that, if $R \hookrightarrow S$ is an inclusion of domains we call S *integral* over R if every $s \in S$ satisfies an equation

$$s^n = \sum_{i=0}^{n-1} r_i s^i \quad \text{with} \quad r_i \in R.$$

A *normal* domain R has the property that any element of its field of fractions which is integral over R belongs already to R. If $X \xrightarrow{\phi} Y$ is a dominant morphism between two irreducible affine varieties, then ϕ is finite if and only if $\mathbb{C}[X]$ in integral over $\mathbb{C}[Y]$ for the embedding coming from ϕ^*.

THEOREM 47 (Dimension formula). *Let $X \xrightarrow{\phi} Y$ be a dominant morphism between irreducible affine varieties. Then, for any $x \in X$ and any irreducible component C of the fiber $\phi^{-1}(\phi(z))$ we have*

$$dim\ C \geq dim\ X - dim\ Y.$$

Moreover, there is a nonempty open subset U of Y contained in the image $\phi(X)$ such that for all $u \in U$ we have

$$dim\ \phi^{-1}(u) = dim\ X - dim\ Y.$$

PROOF. Let $d = dim\ X - dim\ Y$ and apply the Noether normalization result to the affine $\mathbb{C}(Y)$-algebra $\mathbb{C}(Y)\mathbb{C}[X]$. Then, we can find a function $g \in \mathbb{C}[Y]$ and algebraic independent functions $f_1, \ldots, f_d \in \mathbb{C}[X]_g$ (g clears away any denominators that occur after applying the normalization result) such that $\mathbb{C}[X]_g$ is *integral* over $\mathbb{C}[Y]_g[f_1, \ldots, f_d]$. That is, we have the commutative diagram

$$\begin{array}{ccccc}
\mathbb{X}_X(g) & & \xrightarrow{\quad\rho\quad} & & \mathbb{X}_Y(g) \times \mathbb{C}^d \\
\downarrow & & & & \downarrow{\scriptstyle pr_1} \\
X & \xrightarrow{\phi} & Y & \hookleftarrow & \mathbb{X}_Y(g)
\end{array}$$

where we know that ρ is finite and surjective. But then we have that the open subset $\mathbb{X}_Y(g)$ lies in the image of ϕ and in $\mathbb{X}_Y(g)$ all fibers of ϕ have dimension d. For the first part of the statement we have to recall the statement of *Krull's Hauptideal result* : if X is an irreducible affine variety and $g_1, \ldots, g_r \in \mathbb{C}[X]$ with $(g_1, \ldots, g_r) \neq \mathbb{C}[X]$, then any component C of $\mathbb{V}_X(g_1, \ldots, g_r)$ satisfies the inequality

$$dim\ C \geq dim\ X - r.$$

If $dim\ Y = r$ apply this result to the g_i determining the morphism

$$X \xrightarrow{\phi} Y \twoheadrightarrow \mathbb{C}^r$$

where the latter morphism is the one from example 69. □

THEOREM 48 (Chevalley's theorem). *Let $X \xrightarrow{\phi} Y$ be a morphism between affine varieties, the function*

$$X \longrightarrow \mathbb{N} \quad \textit{defined by} \quad x \mapsto dim_x\ \phi^{-1}(\phi(x))$$

is upper-semicontinuous. That is, for all $n \in \mathbb{N}$, the set

$$\{x \in X \mid dim_x\ \phi^{-1}(\phi(x)) \leq n\}$$

is Zariski open in X.

PROOF. Let $Z(\phi, n)$ be the set $\{x \in X \mid dim_x\ \phi^{-1}(\phi(x)) \geq n\}$. We will prove that $Z(\phi, n)$ is closed by induction on the dimension of X. We first make some reductions. We may assume that X is irreducible. For, let $X = \cup_i X_i$ be the decomposition of X into irreducible components, then $Z(\phi, n) = \cup Z(\phi \mid X_i, n)$.

Next, we may assume that $Y = \overline{\phi(X)}$ whence Y is also irreducible and ϕ is a dominant map. Now, we are in the setting of theorem 47. Therefore, if $n \leq dim\ X - dim\ Y$ we have $Z(\phi, n) = X$, so it is closed. If $n > dim\ X - dim\ Y$ consider the open set U in Y of theorem 47. Then, $Z(\phi, n) = Z(\phi \mid (X - \phi^{-1}(U)), n)$. the dimension of the closed subvariety $X - \phi^{-1}(U)$ is strictly smaller that $dim\ X$ hence by induction we may assume that $Z(\phi \mid (X - \phi^{-1}(U)), n)$ is closed in $X - \phi^{-1}(U)$ whence closed in X. □

EXAMPLE 71. For A an affine $\mathbb{C}$-algebra, denote the reduced structure of the n-th representation scheme by $\mathtt{rrep}_n\ A$. We claim that for any $\mathbb{C}$-point $V \in \mathtt{rrep}_n\ A$ its orbit $\mathcal{O}(V)$ is open in the closure $\overline{\mathcal{O}(V)}$ and the closure contains a closed orbit.

Consider the *orbit-map* $GL_n \xrightarrow{\phi} \mathtt{rrep}_n\ A$ defined by $g \mapsto g.V$. Because the image contains an open dense subset of the closure of the image for any morphism between affine varieties, $\overline{\mathcal{O}(V)} = \overline{\phi(GL_n)}$ contains a Zariski open subset U of $\overline{\mathcal{O}(V)}$ contained in the image of ϕ which is the orbit $\mathcal{O}(V)$. But then,

$$\mathcal{O}(V) = GL_n.V = \cup_{g \in GL_n} g.U$$

is also open in $\overline{\mathcal{O}(V)}$. $\overline{\mathcal{O}(V)}$ contains a closed orbit. Indeed, assume $\mathcal{O}(V)$ is not closed, then the complement $C_M = \overline{\mathcal{O}(V)} - \mathcal{O}(V)$ is a proper Zariski closed subset whence $\mathtt{dim}\ C < \mathtt{dim}\ \overline{\mathcal{O}(V)}$. But, C is the union of GL_n-orbits $\mathcal{O}(V_i)$ with $\mathtt{dim}\ \overline{\mathcal{O}(V_i)} < \mathtt{dim}\ \overline{\mathcal{O}(V)}$. Repeating the argument with the V_i and induction on the dimension we will obtain a closed orbit in $\overline{\mathcal{O}(V)}$.

DEFINITION 57. A subset Z of an affine variety X is said to be *locally closed* if Z is open in its Zariski closure $\overline{Z}$. A subset Z is said to be *constructible* if Z is the union of finitely many locally closed subsets.

Finite unions, finite intersections and complements of constructible subsets are again constructible. Further, if $X \xrightarrow{\phi} Y$ be a morphism between affine varieties and if Z is a constructible subset of X, then $\phi(Z)$ is a constructible subset of Y.

EXAMPLE 72. The subset $\mathtt{rind}_n\ A$ of the reduced representation variety $\mathtt{rrep}_n\ A$ consisting of the *indecomposable* n-dimensional representations of A is constructible.

Indeed, consider for any pair k, l such that $k + l = n$ the morphism

$$GL_n \times \mathtt{rrep}_k\ A \times \mathtt{rrep}_l\ A \longrightarrow \mathtt{rrep}_n\ A$$

by sending a triple (g, M, N) to $g.(M \oplus N)$. The image of this map is constructible. The decomposable n-dimensional A-modules belong to one of these finitely many sets whence are constructible, but then so is its complement which in $\mathtt{rep}_n^{ind}\ A$.

DEFINITION 58. Let the basefield be the field of complex numbers and X a closed subvariety of $\mathbb{C}^k$. The induced $\mathbb{C}$-topology on X is called the *analytic topology*. It is much finer than the Zariski topology. For Z a subset in X we denote the *analytic closure* by $\overline{Z}^{\mathbb{C}}$.

THEOREM 49. *If Z is a constructible subset of an affine variety X, then*

$$\overline{Z}^{\mathbb{C}} = \overline{Z}$$

PROOF. Consider an embedding $X \hookrightarrow \mathbb{C}^k$ then Z is a constructible subset of $\mathbb{C}^k$. As a constructible subset, Z contains a subset U which is open and dense (in the Zariski topology) in $\overline{Z}$.

By reducing to irreducible components, we may assume that $\overline{Z}$ is irreducible. If $dim\ \overline{Z} = 1$, consider Z_s, the subset of points where $\overline{Z}$ is a complex manifold. Then, $\overline{Z} - Z_s$ is finite and by the *implicit function theorem* every $u \in Z_s$ has a $\mathbb{C}$-open neighborhood which is $\mathbb{C}$-homeomorphic to the complex line $\mathbb{C}^1$, whence the result holds in this case.

In general, let $z \in \overline{Z}$ and consider an irreducible curve $C \hookrightarrow \overline{Z}$ containing z and such that $C \cap U \neq \emptyset$. Such a curve always exists, for if $d = dim\ Z$ consider the finite surjective morphism $Z \xrightarrow{\phi} \mathbb{C}^d$ of example 69. Let $y \in \mathbb{C}^d - \phi(Z - U)$ and consider the line L through y and $\phi(z)$. By the going-down property there is an irreducible curve $C \hookrightarrow Z$ containing z such that $\phi(C) = L$ and by construction $C \cap V \neq \emptyset$.

Then, $C \cap V$ is Zariski open and dense in C and by the one dimensional argument, $z \in \overline{(C \cap V)}^{\mathbb{C}} \subset \overline{V}^{\mathbb{C}}$. We can do this for any $z \in \overline{V}$ finishing the proof. □

EXAMPLE 73. Let V be an n-dimensional representation of an affine $\mathbb{C}$-algebra A. The Zariski closure $\overline{\mathcal{O}(V)}$ of its orbit in the reduced representation variety $\mathtt{rrep}_n\ A$ coincides with its closure $\overline{\mathcal{O}(V)}^{\mathbb{C}}$ in the analytic topology.

DEFINITION 59. A *one parameter subgroup* of a linear group G is a morphism $\lambda\ :\ \mathbb{C}^* \longrightarrow G$ of algebraic groups.

EXAMPLE 74. Let $\lambda\ :\ \mathbb{C}^* \longrightarrow GL_n$ be a one-parameter subgroup of GL_n. Let H be the image under λ of the subgroup $\boldsymbol{\mu}_\infty$ of roots of unity in $\mathbb{C}^*$. We claim that there is a $g \in GL_n$ such that

$$g.H.g^{-1} \hookrightarrow \begin{bmatrix} \mathbb{C}^* & & 0 \\ & \ddots & \\ 0 & & \mathbb{C}^* \end{bmatrix}$$

Assume $h \in H$ not a scalar matrix, then h has a proper eigenspace decomposition $V \oplus W = \mathbb{C}^n$. We use that $h^l = \mathbb{1}_n$ and hence all its Jordan blocks must have size one. Because H is commutative, both V and W are stable under H. By induction on n we may assume that the images of H in $GL(V)$ and $GL(W)$ are diagonalizable, but then the same holds in GL_n.

As $\boldsymbol{\mu}_\infty$ is infinite, it is Zariski dense in $\mathbb{C}^*$ and because the diagonal matrices are Zariski closed in GL_n we have

$$g.\lambda(\mathbb{C}^*).g^{-1} = g.\overline{H}.g^{-1} \hookrightarrow T_n$$

Further, any one-parameter subgroup of T_n is determined by an n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ and maps t to $(t^{r_1}, \ldots, t^{r_n}$.

Summarizing, if $\lambda\ :\ \mathbb{C}^* \longrightarrow GL_n$ is a one-parameter subgroup, then there is a $g \in GL_n$ and an n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ such that

$$\lambda(t) = g^{-1}. \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} .g$$

THEOREM 50. *Let V be a GL_n-representation, $v \in V$ and a point $w \in V$ lying in the orbitclosure $\overline{\mathcal{O}(v)}$.*

Then, there exists a matrix g with coefficients in the field $\mathbb{C}((t))$ such that $det(g) \neq 0$ and

$$(g.v)_{t=0} \quad \textit{is well defined and is equal to} \quad w$$

PROOF. Note that $g.v$ is a vector with coordinates in the field $\mathbb{C}((t))$. If all coordinates belong to $\mathbb{C}[[t]]$ we can set $t = 0$ in this vector and obtain a vector in V. It is this vector that we denote with $(g.v)_{t=0}$.

Consider the orbit map $\mu : GL_n \longrightarrow V$ defined by $g' \mapsto g'.v$. As $w \in \overline{\mathcal{O}(v)}$ we have seen that there is an irreducible curve $C \hookrightarrow GL_n$ such that $w \in \overline{\mu(C)}$. We obtain a diagram of $\mathbb{C}$-algebras

$$\begin{array}{ccccc}
\mathbb{C}[GL_n] & \longrightarrow & \mathbb{C}[C] & \hookrightarrow & \mathbb{C}(C) \\
\uparrow \mu^* & & \mu^* \uparrow & & \uparrow \\
\mathbb{C}[V] & \longrightarrow & \mathbb{C}[\overline{\mu(C)}] & \hookrightarrow & \mathbb{C}[C']
\end{array}$$

Here, $\mathbb{C}[C]$ is defined to be the integral closure of $\mathbb{C}[\overline{\mu(C)}]$ in the functionfield $\mathbb{C}(C)$ of C. Two things are important to note here : $C' \longrightarrow \overline{\mu(C)}$ is finite, so surjective and take $c \in C'$ be a point lying over $w \in \overline{\mu(C)}$. Further, C' having an integrally closed coordinate ring is a complex manifold. Hence, by the implicit function theorem polynomial functions on C can be expressed in a neighborhood of c as power series in one variable, giving an embedding $\mathbb{C}[C'] \hookrightarrow \mathbb{C}[[t]]$ with $(t) \cap \mathbb{C}[C'] = M_c$. This inclusion extends to one on the level of their fields of fractions. That is, we have a diagram of $\mathbb{C}$-algebra morphisms

$$\begin{array}{ccccccc}
\mathbb{C}[GL_n] & \longrightarrow & \mathbb{C}(C) & = & \mathbb{C}(C') & \hookrightarrow & \mathbb{C}((t)) \\
\uparrow \mu^* & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{C}[V] & \longrightarrow & \mathbb{C}[\overline{\mu(C)}] & \hookrightarrow & \mathbb{C}[C'] & \hookrightarrow & \mathbb{C}[[t]]
\end{array}$$

The upper composition defines an invertible matrix $g(t)$ with coefficients in $\mathbb{C}((t))$, its (i, j)-entry being the image of the coordinate function $x_{ij} \in \mathbb{C}[GL_n]$. Moreover, the inverse image of the maximal ideal $(t) \triangleleft \mathbb{C}[[t]]$ under the lower composition gives the maximal ideal $M_w \triangleleft \mathbb{C}[V]$. This proves the claim. $\square$

EXAMPLE 75. Let g be an $n \times n$ matrix with coefficients in $\mathbb{C}((t))$ and $det\ g \neq 0$. Then there exist $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ such that

$$g = u_1. \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} .u_2$$

with $r_i \in \mathbb{Z}$ and $r_1 \leq r_2 \leq \ldots \leq r_n$. Indeed, by multiplying g with a suitable power of t we may assume that $g = (g_{ij}(t))_{i,j} \in M_n(\mathbb{C}[[t]])$. If $f(t) = \sum_{i=0}^{\infty} f_i t^i \in \mathbb{C}[[t]]$ define $v(f(t))$ to be the minimal i such that $a_i \neq 0$. Let (i_0, j_0) be an entry where $v(g_{ij}(t))$ attains a minimum, say r_1. That is, for all (i, j) we have $g_{ij}(t) = t^{r_1} t^r f(t)$ with $r \geq 0$ and $f(t)$ an invertible element of $\mathbb{C}[[t]]$.

By suitable row and column interchanges we can take the entry (i_0, j_0) to the $(1,1)$-position. Then, multiplying with a unit we can replace it by t^{r_1} and by elementary row and column operations all the remaining entries in the first row and column can be made zero. That is, we have invertible matrices $a_1, a_2 \in GL_n(\mathbb{C}[[t]])$ such that

$$g = a_1. \begin{bmatrix} t^{r_1} & \underline{0}^\tau \\ \underline{0} & \boxed{g_1} \end{bmatrix} .a_2$$

Repeat the same idea on the submatrix g_1 and continue.

Theorem 51 (Hilbert criterium). *Let V be a GL_n-representation, $X \subset V$ a closed GL_n-stable subvariety and $\mathcal{O}(x) = GL_n.x$ the orbit of a point $x \in X$.*

If $Y \subset \overline{\mathcal{O}(x)}$ is a closed GL_n-stable subset, then there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ such that

$$\underset{t \to 0}{lim} \lambda(t).x \in Y$$

Proof. It suffices to prove the result for $X = V$. By the foregoing theorem, there is an invertible matrix $g \in M_n(\mathbb{C}((t)))$ such that

$$(g.x)_{t=0} = y \in Y$$

By the foregoing example, we can find $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ such that

$$g = u_1.\lambda'(t).u_2 \quad \text{with} \quad \lambda'(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}$$

a one-parameter subgroup. There exist $x_i \in V$ such that $u_2.x = \sum_{i=0}^{\infty} z_i t^i$ in particular $u_2(0).x = x_0$. But then,

$$\begin{aligned} (\lambda'(t).u_2.x)_{t=0} &= \sum_{i=0}^{\infty} (\lambda'(t).x_i t^i)_{t=0} \\ &= (\lambda'(t).x_0)_{t=0} + (\lambda'(t).x_1 t)_{t=0} + \dots \end{aligned}$$

But one easily verifies (using a basis of eigenvectors of $\lambda'(t)$) that

$$\underset{s \to 0}{lim} \lambda'^{-1}(s).(\lambda'(t) x_i t^i)_{t=0} = \begin{cases} (\lambda'(t).x_0)_{t=0} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases}$$

As $(\lambda'(t).u_2.x)_{t=0} \in Y$ and Y is a closed GL_n-stable subset, we also have that

$$\underset{s \to 0}{lim} \lambda'^{-1}(s).(\lambda'(t).u_2.x)_{t=0} \in Y \quad \text{that is,} \quad (\lambda'(t).x_0)_{t=0} \in Y$$

But then, we have for the one-parameter subgroup $\lambda(t) = u_2(0)^{-1}.\lambda'(t).u_2(0)$ that

$$\underset{t \to 0}{lim} \lambda(t).x \in Y$$

finishing the proof. □

Definition 60. The *nullcone* of a GL_n-representation V is the set of points x such that the fixed point $0 \in V$ lies in the orbit closure of x.

Theorem 52. *Let V be a finite dimensional GL_n-representation and $v \in V$ a point in the nullcone. Then, there is a one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ such that*

$$\underset{t \to 0}{lim}\ \lambda(t).v = 0$$

Definition 61. A *finite filtration* F on an n-dimensional representation M is a sequence of A-submodules

$$F \quad : \quad 0 = M_{t+1} \subset M_t \subset \ldots \subset M_1 \subset M_0 = M.$$

The *associated graded* A-module is the n-dimensional module

$$gr_F \; M = \oplus_{i=0}^{t} M_i / M_{i+1}.$$

Theorem 53. *The following two statements are equivalent for n-dimensional A-modules M and N.*

(1) *There is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that*

$$\underset{t \to 0}{lim} \; \lambda(t).M = N$$

(2) *There is a finite filtration F on the A-module M such that*

$$gr_F \; M \simeq N$$

as A-modules.

Proof. (1) $\Rightarrow$ (2) : If V is any GL_n-representation and $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ a one-parameter subgroup, we have an induced *weight space decomposition* of V

$$V = \oplus_i V_{\lambda,i} \quad \text{where} \quad V_{\lambda,i} = \{ v \in V \mid \lambda(t).v = t^i v, \forall t \in \mathbb{C}^* \}.$$

In particular, we apply this to the underlying vectorspace of M. We define

$$M_j = \oplus_{i > j} V_{\lambda,i}$$

and claim that this defines a finite filtration on M with associated graded A-module N. For any $a \in A$ (it suffices to vary a over the generators of A) we can consider the linear maps

$$\phi_{ij}(a) : V_{\lambda,i} \hookrightarrow V = M \xrightarrow{a.} M = V \twoheadrightarrow V_{\lambda,j}$$

(that is, we express the action of a in a blockmatrix Φ_a with respect to the decomposition of V). Then, the action of a on the module corresponding to $\lambda(t).\psi$ is given by the matrix $\Phi'_a = \lambda(t).\Phi_a.\lambda(t)^{-1}$ with corresponding blocks

$$\begin{array}{ccc} V_{\lambda,i} & \xrightarrow{\phi_{ij}(a)} & V_{\lambda,j} \\ \uparrow {\scriptstyle \lambda(t)^{-1}} & & \downarrow {\scriptstyle \lambda(t)} \\ V_{\lambda,i} & \xrightarrow[\phi'_{ij}(a)]{} & V_{\lambda,j} \end{array}$$

that is $\phi'_{ij}(a) = t^{j-i}\phi_{ij}(a)$. Therefore, if $lim_{t \to 0}\lambda(t).\psi$ exists we must have that

$$\phi_{ij}(a) = 0 \quad \text{for all} \quad j < i.$$

But then, the action by a sends any $M_k = \oplus_{i>k} V_{\lambda,i}$ to itself, that is, the M_k are A-submodules of M. Moreover, for $j > i$ we have

$$\underset{t \to 0}{lim} \; \phi'_{ij}(a) = \underset{t \to 0}{lim} \; t^{j-i}\phi_{ij}(a) = 0$$

Consequently, the action of a on the limit-module is given by the diagonal blockmatrix with blocks $\phi_{ii}(a)$, but this is precisely the action of a on $V_i = M_{i-1}/M_i$, that is, the limit corresponds to the associated graded module.

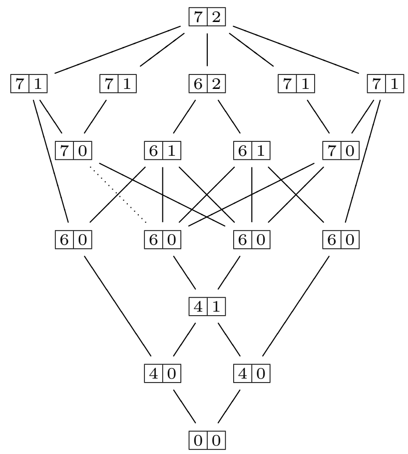


FIGURE 1. Kraft's diamond describing the nullcone of M_3^2.

(2) $\Rightarrow$ (1) : Given a finite filtration on M

$$F \quad : \quad 0 = M_{t+1} \subset \ldots \subset M_1 \subset M_0 = M$$

we have to find a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ which induces the filtration F as in the first part of the proof. Clearly, there exist subspaces V_i for $0 \leq i \leq t$ such that

$$V = \oplus_{i=0}^t V_i \quad \text{and} \quad M_j = \oplus_{j=i}^t V_i.$$

Then we take λ to be defined by $\lambda(t) = t^i Id_{V_i}$ for all i and it verifies the claims. $\square$

EXAMPLE 76. In the statement of the Hilbert criterium it is important that Y is a closed subset. In general, it does *not* follow that any orbit $\mathcal{O}(y) \hookrightarrow \overline{\mathcal{O}(x)}$ can be reached via a one-parameter subgroup. Consider two modules $M, N \in \texttt{rrep}_n\ A$. Assume that $\mathcal{O}(N) \hookrightarrow \overline{\mathcal{O}(M)}$ and that we can reach the orbit of N via a one-parameter subgroup. Then, by the equivalence of the foregoing theorem we know that N must be *decomposable* as it is the associated graded of a nontrivial filtration on M. This gives us a criterium to construct examples showing that the closedness assumption in the formulation of Hilbert's criterium is essential.

The nullcone of $\texttt{rrep}_3\langle 2\rangle$ has been worked out by Hanspeter Kraft in [**35**, p.202]. The orbits are depicted in figure 1 In this picture, each node corresponds to a torus. The right hand number is the dimension of the torus and the left hand number is the dimension of the orbit represented by a point in the torus. The solid or dashed lines indicate orbitclosures. For example, the dashed line corresponds to the following two points in $M_3^2 = M_3 \oplus M_3$

$$M = (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) \qquad N = (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

We claim that N is an indecomposable 3-dimensional module of $\diamondsuit_2$. Indeed, the only subspace of the column vectors $\mathbb{C}^3$ left invariant under both x and y is equal to

$$\begin{bmatrix} \mathbb{C} \\ 0 \\ 0 \end{bmatrix}$$

hence M_ρ cannot have a direct sum decomposition of two or more modules. Next, we claim that $\mathcal{O}(N) \hookrightarrow \overline{\mathcal{O}(M)}$. Indeed, simultaneous conjugating ψ with the invertible matrix

$$\begin{bmatrix} 1 & \epsilon^{-1}-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon^{-1} \end{bmatrix} \quad \text{we obtain the couple} \quad (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \epsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

and letting $\epsilon \longrightarrow 0$ we see that the limiting point is N.

THEOREM 54 (M. Artin). *The orbit $\mathcal{O}(M)$ of a $\mathbb{C}$-point M of $\mathtt{rrep}_n A$ is closed in $\mathtt{rrep}_n A$ if and only if M is an n-dimensional semisimple A-module.*

PROOF. Assume that the orbit $\mathcal{O}(M)$ is Zariski closed. Let $gr\ M$ be the associated graded module for a composition series of M. By the above equivalence we know that $\mathcal{O}(gr\ M)$ is contained in $\overline{\mathcal{O}(M)} = \mathcal{O}(M)$. But then $gr\ M \simeq M$ whence M is semisimple.

Conversely, assume M is semisimple. We know that the orbitclosure $\overline{\mathcal{O}(M)}$ contains a closed orbit, say $\mathcal{O}(N)$. By the Hilbert criterium we have a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that

$$\lim_{t \rightarrow 0} \lambda(t).M = N' \simeq N.$$

By the equivalence this means that there is a finite filtration F on M with associated graded module $gr_F\ M \simeq N$. For the semisimple module M the only possible finite filtrations are such that each of the submodules is a direct sum of simple components, so $gr_F\ M \simeq M$, whence $M \simeq N$ and hence the orbit $\mathcal{O}(M)$ is closed. □

DEFINITION 62. The inclusion $\oint_n A = \mathbb{C}[\mathtt{rep}_n A]^{GL_n} \subset \mathbb{C}[\mathtt{rep}_n A]$ induces the *quotient maps*

$$\mathtt{rep}_n A \xrightarrow{\pi} \mathtt{iss}_n A \qquad \mathtt{rrep}_n A \xrightarrow{\pi} \mathtt{riss}_n A$$

THEOREM 55. *For A an affine algebra, the quotient map*

$$\mathtt{rrep}_n A \overset{\pi}{\twoheadrightarrow} \mathtt{riss}_n A$$

is surjective and the $\mathbb{C}$-points of $\mathtt{iss}_n A$ classify the isomorphism classes of semi-simple n-dimensional representations of A.

PROOF. First, we prove these statements for $A = \langle m \rangle$. As $\mathtt{rep}_n \langle m \rangle = M_n^m$ is a GL_n-representation we prove a few general facts valid for *any* finite dimensional GL_n-representation V.

(1) : Let $I \triangleleft \mathbb{C}[V]$ be a GL_n-stable ideal, that is, $g.I \subset I$ for all $g \in GL_n$, then

$$(\mathbb{C}[V]/I)^{GL_n} \simeq \mathbb{C}[V]^{GL_n}/(I \cap \mathbb{C}[V]^{GL_n}).$$

Indeed, as I has the induced GL_n-action which is locally finite we have the isotypical decomposition $I = \oplus I_{(s)}$ and clearly $I_{(s)} = \mathbb{C}[V]_{(s)} \cap I$. But then also, taking quotients we have

$$\oplus_s (\mathbb{C}[V]/I)_{(s)} = \mathbb{C}[V]/I = \oplus_s \mathbb{C}[V]_{(s)}/I_{(s)}.$$

Therefore, $(\mathbb{C}[V]/I)_{(s)} = \mathbb{C}[V]_{(s)}/I_{(s)}$ and taking the special case $s = 0$ is the statement.

(2) : Let $J \lhd \mathbb{C}[V]^{GL_n}$ be an ideal, then we have a lying-over property

$$J = J\mathbb{C}[V] \cap \mathbb{C}[V]^{GL_n}.$$

Hence, $\mathbb{C}[V]^{GL_n}$ is Noetherian and is finitely generated.

For any $f \in \mathbb{C}[V]^{GL_n}$ left-multiplication by f in $\mathbb{C}[V]$ commutes with the GL_n-action, whence $f.\mathbb{C}[V]_{(s)} \subset \mathbb{C}[V]_{(s)}$. That is, $\mathbb{C}[V]_{(s)}$ is a $\mathbb{C}[V]^{GL_n}$-module. But then, as $J \subset \mathbb{C}[V]^{GL_n}$ we have

$$\oplus_s (J\mathbb{C}[V])_{(s)} = J\mathbb{C}[V] = \oplus_s J\mathbb{C}[V]_{(s)}.$$

Therefore, $(J\mathbb{C}[V])_{(s)} = J\mathbb{C}[V]_{(s)}$ and again taking the special value $s = 0$ we obtain $J\mathbb{C}[V] \cap \mathbb{C}[V]^{GL_n} = (J\mathbb{C}[V])_{(0)} = J$. Noetherianity follows from the fact that $\mathbb{C}[V]$ is Noetherian. Because the action of GL_n on $\mathbb{C}[V]$ preserves the gradation, the ring of invariants is also graded

$$\mathbb{C}[V]^{GL_n} = R = \mathbb{C} \oplus R_1 \oplus R_2 \oplus \ldots \quad .$$

Because $\mathbb{C}[V]^{GL_n}$ is Noetherian, the ideal $R_+ = R_1 \oplus R_2 \oplus \ldots$ is finitely generated $R_+ = Rf_1 + \ldots + Rf_l$ by homogeneous elements $f_1, \ldots, f_l$. We claim that as a $\mathbb{C}$-algebra $\mathbb{C}[V]^{GL_n}$ is generated by the f_i. Indeed, we have $R_+ = \sum_{i=1}^l \mathbb{C}f_i + R_+^2$ and then also

$$R_+^2 = \sum_{i,j=1}^l \mathbb{C}f_i f_j + R_+^3$$

and iterating this procedure we obtain

$$R_+^m = \sum_{\sum m_i = m} \mathbb{C}f_1^{m_1} \ldots f_l^{m_l} + R_+^{m+1}.$$

Consider the subalgebra $\mathbb{C}[f_1, \ldots, f_l]$ of $R = \mathbb{C}[V]^{GL_n}$, then for any power $d > 0$

$$\mathbb{C}[V]^{GL_n} = \mathbb{C}[f_1, \ldots, f_l] + R_+^d.$$

For any i we then have for the homogeneous components of degree i

$$R_i = \mathbb{C}[f_1, \ldots, f_l]_i + (R_+^d)_i.$$

Now, if $d > i$ we have that $(R_+^d)_i = 0$ and hence that $R_i = \mathbb{C}[f_1, \ldots, f_l]_i$. As this holds for all i we proved the claim.

(3) : Let I_j be a family of GL_n-stable ideals of $\mathbb{C}[V]$, then

$$(\sum_j I_j) \cap \mathbb{C}[V]^{GL_n} = \sum_j (I_j \cap \mathbb{C}[V]^{GL_n}).$$

Indeed, for any j we have the decomposition $I_j = \oplus_s (I_j)_{(s)}$. But then, we have

$$\oplus_s (\sum_j I_j)_{(s)} = \sum_j I_j = \sum_j \oplus_s (I_j)_{(s)} = \oplus_s \sum_j (I_j)_{(s)}.$$

Therefore, $(\sum_j I_j)_{(s)} = \sum_j (I_j)_{(s)}$ and taking $s = 0$ gives the required statement.

Returning to the case of interest to us : we claim that the algebraic quotient

$$\mathtt{rep}_n \langle m \rangle \xrightarrow{\pi} \mathtt{iss}_n \langle m \rangle$$

is surjective on $\mathbb{C}$-points and if $Z \hookrightarrow \texttt{rep}_n\langle m\rangle$ is a closed GL_n-stable subset (such as $\texttt{rep}_n A$), then $\pi(Z)$ is closed in $\texttt{iss}_n\langle m\rangle$ and the morphism

$$\pi \mid Z : Z \longrightarrow \pi(Z)$$

is an algebraic quotient, that is, $\mathbb{C}[\pi(Z)] \simeq \mathbb{C}[Z]^{GL_n} = \mathbb{C}[\texttt{iss}_n A]$.

For, let $y \in \texttt{iss}_n\langle m\rangle$ with maximal ideal $M_y \triangleleft \mathbb{C}[\texttt{iss}_n\langle m\rangle]$. By (2) we have $M_y\mathbb{C}[\texttt{rep}_n\langle m\rangle] \neq \mathbb{C}[\texttt{rep}_n\langle m\rangle]$ and hence there is a maximal ideal M_x of $\mathbb{C}[\texttt{rep}_n\langle m\rangle]$ containing $M_y\mathbb{C}[\texttt{rep}_n\langle m\rangle]$, but then $\pi(x) = y$.

Let $Z = \mathbb{V}(I)$ for a G-stable ideal I of $\mathbb{C}[\texttt{rep}_n\langle m\rangle]$, then $\overline{\pi(Z)} = \mathbb{V}_{\texttt{iss}_n\langle m\rangle}(I \cap \mathbb{C}[\texttt{iss}_n\langle m\rangle])$. That is, $\mathbb{C}[\overline{\pi(Z)}] = \mathbb{C}[\texttt{iss}_n\langle m\rangle]/(I \cap \mathbb{C}[\texttt{iss}_n\langle m\rangle])$. However, by (1) we have that

$$\mathbb{C}[\texttt{iss}_n\langle m\rangle]/(\mathbb{C}[\texttt{iss}_n\langle m\rangle] \cap I) \simeq (\mathbb{C}[\texttt{rep}_n\langle m\rangle]/I)^{GL_n} = \mathbb{C}[Z]^{GL_n}$$

and hence $\mathbb{C}[\overline{\pi(Z)}] = \mathbb{C}[Z]^{GL_n}$. Finally, surjectivity of $\pi \mid Z$ is proved as before.

An immediate consequence is that the Zariski topology on $\texttt{iss}_n\langle m\rangle$ is the quotient topology of that on $\texttt{rep}_n\langle m\rangle$. For, take $U \subset \texttt{iss}_n\langle m\rangle$ with $\pi^{-1}(U)$ Zariski open in $\texttt{rep}_n\langle m\rangle$. Then, $\texttt{rep}_n\langle m\rangle - \pi^{-1}(U)$ is a GL_n-stable closed subset of $\texttt{rep}_n\langle m\rangle$ and $\pi(\texttt{rep}_n\langle m\rangle - \pi^{-1}(U)) = \texttt{iss}_n\langle m\rangle - U$ is Zariski closed in $\texttt{iss}_n\langle m\rangle$.

We claim that the quotient $\texttt{rep}_n\langle m\rangle \xrightarrow{\pi} \texttt{iss}_n\langle m\rangle$ separates disjoint closed GL_n-stable subvarieties of $\texttt{rep}_n\langle m\rangle$. Let Z_j be closed GL_n-stable subvarieties of $\texttt{rep}_n\langle m\rangle$ with defining ideals $Z_j = \mathbb{V}(I_j)$. Then, $\cap_j Z_j = \mathbb{V}(\sum_j I_j)$. Applying (3) we obtain

$$\begin{aligned}\overline{\pi(\cap_j Z_j)} &= \mathbb{V}_{\texttt{iss}_n\langle m\rangle}((\sum_j I_j) \cap \mathbb{C}[\texttt{iss}_n\langle m\rangle]) = \mathbb{V}_{\texttt{iss}_n\langle m\rangle}(\sum_j (I_j \cap \mathbb{C}[\texttt{iss}_n\langle m\rangle])) \\ &= \cap_j \mathbb{V}_{\texttt{iss}_n\langle m\rangle}(I_j \cap \mathbb{C}[\texttt{iss}_n\langle m\rangle]) = \cap_j \overline{\pi(Z_j)}\end{aligned}$$

The onto property implies that $\overline{\pi(Z_j)} = \pi(Z_j)$ from which the claim follows.

$\mathbb{C}$-points of $\texttt{iss}_n\langle m\rangle$ classify the closed GL_n-orbits in $\texttt{rep}_n\langle m\rangle$ (whence the isomorphism classes of semi-simple n-dimensional representations). In fact, every fiber $\pi^{-1}(y)$ contains exactly one closed orbit C and we have

$$\pi^{-1}(y) = \{x \in \texttt{rep}_n\langle m\rangle \mid C \subset \overline{\mathcal{O}(x)}\}$$

Indeed, the fiber $F = \pi^{-1}(y)$ is a GL_n-stable closed subvariety of $\texttt{rep}_n\langle m\rangle$. Take any orbit $\mathcal{O}(x) \subset F$ then either it is closed or contains in its closure an orbit of strictly smaller dimension. Induction on the dimension then shows that $\overline{\mathcal{O}(x)}$ contains a closed orbit C. On the other hand, assume that F contains two closed orbits, then they have to be disjoint contradicting the separation property. □

Example 77. For $M \in M_n(\mathbb{C})$ it is usually very difficult to describe the ideal of relations of the orbit $\mathcal{O}(M)$ of M. If M is semisimple (that is, M diagonalizable) we can invoke the reconstruction theorem 43 to describe this ideal. Consider the semisimple commutative algebra generated by X, that is,

$$A = \frac{\mathbb{C}[t]}{(f(t))}$$

where $f(t)$ is the minimal polynomial of M and the trace map on A is given once we give the elements $a_k = Tr(M^k)$, or equivalently the coefficients of the characteristic polynomial. The equations defining the closed orbit of M are then $Tr(M^k) = a_k$ for $1 \leq k < deg f(t)$ and the entries of $f(X)$ for a generic $n \times n$ matrix X.

EXAMPLE 78. Let $A \in \texttt{alg}$ be finite dimensional, then there are only a finite number of simple A-representations. Therefore, $\texttt{riss}_n A$ is a finite number of points. As a consequence $\texttt{rep}_n A$ is the disjoint union of a finite number of connected components, each consisting of those n-dimensional representations of A having the same Jordan-Hölder decomposition. Connectivity follows from the fact that the semi-simple representation of the sum of the Jordan-Hölder components lies in the closure of each orbit.

EXAMPLE 79. Let $C \in \texttt{commalg}$ be an affine algebra with corresponding reduced variety $X = \texttt{rspec}C$. Every simple C-representation is one-dimensional, that is, determines a point of X. Applying the Jordan-Hölder theorem we obtain that

$$\texttt{riss}_n C \simeq X^{(n)} = (\underbrace{X \times \ldots \times X}_{n})/S_n$$

the n-th symmetric product of X.

In particular, if X is an affine smooth curve, then its coordinate ring $C = \mathbb{C}[X]$ is $\texttt{alg}$-smooth and therefore $\texttt{rep}_n C$ is smooth and therefore reduced. Hence, $\texttt{iss}_n C$ is also reduced and we have that $\texttt{iss}_n C = X^{(n)}$ the n-symmetric product of the curve X.

CHAPTER 5

Coverings

"When Michael Artin got interested in the topic he was able to use the powerful ideas of faithfully flat descent which were unknown to the specialists in non-commutative algebra, also that was the time of revival of geometric invariant theory and the invariant interpretation has changed completely the point of view."
Claudio Procesi in [**55**]

This chapter describes two applications of the étale machinery to noncommutative algebras : description of Brauer groups of functionfields and the local structure of orders. First, we introduce cohomology on the étale site of a commutative ring and relate it to classical Galois cohomology. We aim for a handle on the size of the Brauer group of a functionfield $\mathbb{C}(X)$ of a d-dimensional variety X which is provided by the coniveau spectral sequence. In this sequence étale cohomology is related to Galois cohomology for the functionfields of all irreducible subvarieties of X. We include classical work of Tsen and Tate on the cohomological size of the resulting Galois groups as they give an indication of the huge variety of noncommutative orders over a fixed variety X. In the special case when X is a smooth projective surface, the Artin-Mumford exact sequence determines the Brauer group of $\mathbb{C}(X)$ in terms of all curves on X and their (ramified) covers.

In the second section we give an important application of étale extensions to invariant theory. If a reductive group G is acting on a smooth variety X and if $\mathcal{O}(x)$ is a closed G-orbit one would like the local G-structure of X around x to be the product of the orbit $\mathcal{O}(x)$ with the normal space N_x to the orbit. Surprisingly, this is true if we view isomorphism in the étale topology and replace product by fiber bundle, as was proved by Domingo Luna [**44**]. We give the proof due to Friedrich Knop as it is valid even when the variety X is not smooth, nor even reduced (as is often the case in representation schemes).

We then apply this result to the local description of representation varieties of `alg`-smooth and `alg@n`-smooth algebras. It turns out that the normal space is isomorphic (as a representation over the stabilizer subgroup) to the representation space of a particular (marked) quiver setting : the *local quiver*. The étale local structure of $\oint_n A$ and $\int_n A$ is fully encoded in the local quiver.

In the final section we combine these two different applications to the problem of characterizing those central simple algebras Σ over a projective (normal) variety X having a noncommutative smooth model, that is, a sheaf of `alg@n`-smooth algebras. The coniveau spectral sequence describing $Br_n(\mathbb{C}(X))$ gives us information on the ramification of maximal orders in Σ whereas the étale local description of `alg@n`-smooth orders given by the local quiver allows us to compute the ramification possible for `alg@n`-smooth orders. Combining these two algebra-geometric data we are able to prove that such a central simple algebra has a model with only a finite number of noncommutative singularities, each of which is of quantum-plane type and we characterize the ones without singularities.

5.1. Etale cohomology.

A closed subvariety $X \hookrightarrow \mathbb{C}^m$ can be equipped with the Zariski topology or with the much finer analytic or complex topology. A major disadvantage of the coarseness of the Zariski topology is the failure to have an implicit function theorem in algebraic geometry. Etale morphisms are introduced to bypass this problem. These morphisms determine the *étale topology* which is no longer a topology determined by subsets but rather a *Grothendieck topology* determined by *covers* . In this section, algebras $C \in$ `commalg` will not necessarily be affine and with `spec`C we denote the *prime spectrum* of C, that is the set of prime ideals of C, equipped with the Zariski topology.

Definition 63. A finite morphism $C \xrightarrow{f} B$ of commutative $\mathbb{C}$-algebras is said to be *étale* if and only if $B = C[t_1, \ldots, t_k]/(f_1, \ldots, f_k)$ such that the *Jacobian matrix*

$$\begin{bmatrix} \frac{\partial f_1}{\partial t_1} & \ldots & \frac{\partial f_1}{\partial t_k} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial t_1} & \ldots & \frac{\partial f_k}{\partial t_k} \end{bmatrix}$$

has a determinant which is a unit in B. The corresponding map on the prime spectra

$$\texttt{spec}B \xrightarrow{f_*} \texttt{spec}C$$

should be viewed as a finite cover.

Example 80. Consider the inclusion $\mathbb{C}[x, x^{-1}] \subset \mathbb{C}[x, x^{-1}][\sqrt[n]{x}]$ and the induced map on the affine schemes

$$\texttt{spec}\ \mathbb{C}[x, x^{-1}][\sqrt[n]{x}] \xrightarrow{\psi} \texttt{spec}\ \mathbb{C}[x, x^{-1}] = \mathbb{C} - \{0\}.$$

Every point $\lambda \in \mathbb{C} - \{0\}$ has exactly n preimages $\lambda_i = \zeta^i \sqrt[n]{\lambda}$. Moreover, in a neighborhood of λ_i, the map ψ is a diffeomorphism. Still, we do not have an inverse map in algebraic geometry as $\sqrt[n]{x}$ is not a polynomial map. However, $\mathbb{C}[x, x^{-1}][\sqrt[n]{x}]$ is an étale extension of $\mathbb{C}[x, x^{-1}]$. In this way étale morphisms can be seen as an algebraic substitute for the failure of an inverse function theorem in algebraic geometry.

Example 81. Let K be a field of characteristic zero, choose an algebraic closure $\mathbb{K}$ with *absolute Galois group* $G_K = Gal(\mathbb{K}/K)$. Then, the following are equivalent

(1) $K \longrightarrow A$ is étale
(2) $A \otimes_K \mathbb{K} \simeq \mathbb{K} \times \ldots \times \mathbb{K}$
(3) $A = \prod L_i$ where L_i/K is a finite field extension

Indeed, assume (1), then $A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ where f_i have invertible Jacobian matrix. Then $A \otimes \mathbb{K}$ is a `commalg`-smooth algebra (hence reduced) of dimension 0 so (2) holds. Assume (2), then

$$Hom_{K-alg}(A, \mathbb{K}) \simeq Hom_{\mathbb{K}-alg}(A \otimes \mathbb{K}, \mathbb{K})$$

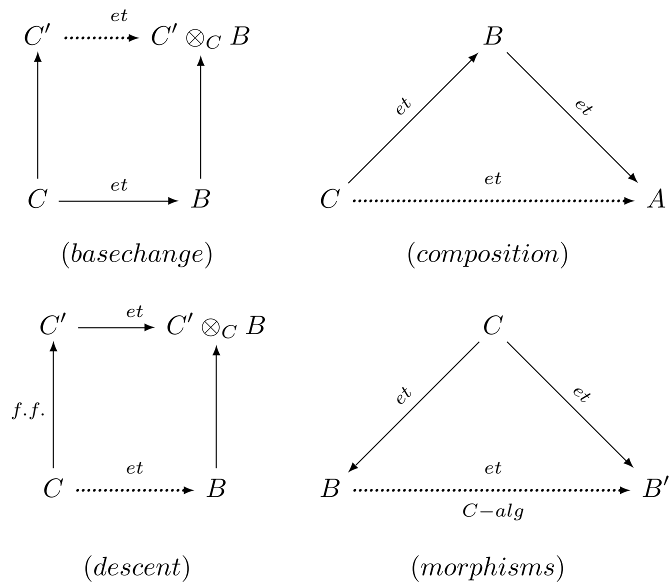


FIGURE 1. Sorite for étale morphisms

has $dim_{\mathbb{K}}(A \otimes \mathbb{K})$ elements. On the other hand we have by the *Chinese remainder theorem* that

$$A/Jac\ A = \prod_i L_i$$

with L_i a finite field extension of K. However,

$$dim_{\mathbb{K}}(A \otimes \mathbb{K}) = \sum_i dim_K(L_i) = dim_K(A/Jac\ A) \leq dim_K(A)$$

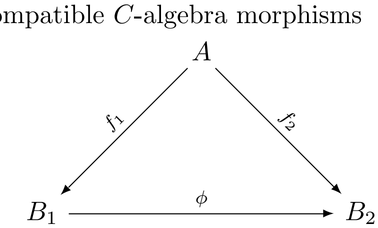
and as both ends are equal A is reduced and hence $A = \prod_i L_i$ whence (3). Assume (3), then each $L_i = K[x_i]/(f_i)$ with $\partial f_i/\partial x_i$ invertible in L_i. But then $A = \prod L_i$ is étale over K whence (1) holds.

THEOREM 56. *Etale morphisms satisfy 'sorite', that is, they satisfy the commutative diagrams of figure 1. In these diagrams, et denotes an étale morphism, f.f. denotes a faithfully flat morphism and the dashed arrow is the étale morphism implied by 'sorite'.*

PROOF. See for example [**47**, I]. □

DEFINITION 64. The *étale site* of C, which we will denote by $\mathtt{C_{et}}$ is the category with

- objects : the étale extensions $C \xrightarrow{f} B$ of C
- morphisms : compatible C-algebra morphisms



In view of theorem 56 all morphisms in $\mathtt{C_{et}}$ are étale morphisms.

$\mathtt{C_{et}}$ is equipped with a *Grothendieck topology* by defining a *cover* to be a collection $\mathcal{C} = \{B \xrightarrow{f_i} B_i\}$ in $\mathtt{C_{et}}$ such that

$$\mathtt{spec}\ B = \cup_i\ Im\ (\mathtt{spec}\ B_i \xrightarrow{f_{i*}} \mathtt{spec}\ B\ )$$

Observe that all the properties of a Grothendieck category as in [**47**, II.§1] follow from this definition and theorem 56.

EXAMPLE 82. With the notation of example 81, we associate to every finite étale extension $A = \prod L_i$ the finite set $rts(A) = Hom_{K-alg}(A, \mathbb{K})$ on which the Galois group G_K acts via a finite quotient group. If we write $A = K[t]/(f)$, then $rts(A)$ is the *set of roots* in $\mathbb{K}$ of the polynomial f with obvious action by G_K. Galois theory, in the interpretation of Grothendieck, can now be stated by observing that the functor

$$\mathtt{K_{et}} \xrightarrow{rts(-)} \mathtt{finite}\ G_K - \mathtt{sets}$$

is an anti-equivalence of categories.

DEFINITION 65. An *étale presheaf* of groups on $\mathtt{C_{et}}$ is a functor

$$\mathbb{G} : \mathtt{C_{et}} \longrightarrow \mathtt{groups}$$

In analogy with usual (pre)sheaf notation we denote for every object $B \in \mathtt{C_{et}}$ the *global sections* $\Gamma(B, \mathbb{G}) = \mathbb{G}(B)$ and for every morphism $B \xrightarrow{\phi} B'$ in $\mathtt{C_{et}}$ the *restriction map* $Res^B_{B'} = \mathbb{G}(\phi) : \mathbb{G}(B) \longrightarrow \mathbb{G}(B')$ with $g \mid B' = \mathbb{G}(\phi)(g)$.

An étale presheaf $\mathbb{G}$ is an *étale sheaf* provided for every $B \in \mathtt{C_{et}}$ and every cover $\{B \longrightarrow B_i\}$ we have exactness of the equalizer diagram

$$0 \longrightarrow \mathbb{G}(B) \longrightarrow \prod_i \mathbb{G}(B_i) \rightrightarrows \prod_{i,j} \mathbb{G}(B_i \otimes_B B_j)$$

A sequence of sheaves of Abelian groups on $\mathtt{C_{et}}$ is said to be *exact*

$$\mathbb{G}' \xrightarrow{f} \mathbb{G} \xrightarrow{g} \mathbb{G}"$$

if for every $B \in \mathtt{C_{et}}$ and $s \in \mathbb{G}(B)$ such that $g(s) = 0 \in \mathbb{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in $\mathtt{C_{et}}$ and sections $t_i \in \mathbb{G}'(B_i)$ such that $f(t_i) = s \mid B_i$.

EXAMPLE 83. For a group G we define the constant sheaf

$$\mathbb{G} : \mathtt{C_{et}} \longrightarrow \mathtt{groups} \quad B \mapsto G^{\oplus \pi_0(B)}$$

where $\pi_0(B)$ is the number of connected components of $\mathtt{spec}\ B$.

EXAMPLE 84. The multiplicative group $\mathbb{G}_m$. The functor

$$\mathbb{G}_m : \mathtt{C_{et}} \longrightarrow \mathtt{groups} \quad B \mapsto B^*$$

is a sheaf on $\mathtt{C_{et}}$.

EXAMPLE 85. the roots of unity μ_n. There is a sheaf morphism

$$\mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m$$

and we denote its kernel by μ_n. As $C \in \mathtt{commalg}$ is a $\mathbb{C}$-algebra we can identify μ_n with the constant sheaf $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ via the isomorphism $\zeta^i \mapsto i$ after choosing a primitive n-th root of unity $\zeta \in \mathbb{C}$. The *Kummer sequence* of sheaves of Abelian groups

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 0$$

is exact on $\mathtt{C_{et}}$ (but not necessarily on $\mathtt{spec}\ C$ with the Zariski topology). We only need to verify surjectivity. Let $B \in \mathtt{C_{et}}$ and $b \in \mathbb{G}_m(B) = B^*$. Consider the étale extension $B' = B[t]/(t^n - b)$ of B, then b has an n-th root over in $\mathbb{G}_m(B')$. Observe that this n-th root does not have to belong to $\mathbb{G}_m(B)$.

Example 86. Using the notation of example 81 we have the following interpretation of Abelian sheaves on $\mathtt{K_{et}}$. Let $\mathbb{G}$ be a presheaf on $\mathtt{K_{et}}$. Define

$$M_{\mathbb{G}} = \underrightarrow{lim}\ \mathbb{G}(L)$$

where the limit is taken over all subfields $L \subset \mathbb{K}$ which are finite over K. The Galois group G_K acts on $\mathbb{G}(L)$ on the left through its action on L whenever L/K is Galois. Hence, G_K acts an $M_{\mathbb{G}}$ and $M_{\mathbb{G}} = \cup M_{\mathbb{G}}^H$ where H runs through the *open subgroups* (that is, containing a normal subgroup having a finite quotient) of G_K. That is, $M_{\mathbb{G}}$ is a *continuous G_K-module* . Conversely, given a continuous G_K-module M we define a presheaf $\mathbb{G}_M$ on $\mathtt{K_{et}}$ by taking $\mathbb{G}_M(L) = M^H$ where $H = G_L = Gal(\mathbb{K}/L)$ and $\mathbb{G}_M(\prod L_i) = \prod \mathbb{G}_M(L_i)$ for an arbitrary étale extension. One verifies that $\mathbb{G}_M$ is a sheaf of Abelian groups on $\mathtt{K_{et}}$.

There is an equivalence of categories between sheaves an $\mathtt{K_{et}}$ and continuous G_K-modules

$$\mathbf{S}(\mathtt{K_{et}}) \rightleftarrows G_K - \mathtt{mod}$$

induced by the correspondences $\mathbb{G} \mapsto M_{\mathbb{G}}$ and $M \mapsto \mathbb{G}_M$. Indeed, a G_K-morphism $M \longrightarrow M'$ induces a morphism of sheaves $\mathbb{G}_M \longrightarrow \mathbb{G}_{M'}$. Conversely, if H is an open subgroup of G_K with $L = \mathbb{K}^H$, then if $\mathbb{G} \xrightarrow{\phi} \mathbb{G}'$ is a sheafmorphism, $\phi(L) : \mathbb{G}(L) \longrightarrow \mathbb{G}'(L)$ commutes with the action of G_K by functoriality of ϕ. Therefore, $\underrightarrow{lim}\ \phi(L)$ is a G_K-morphism $M_{\mathbb{G}} \longrightarrow M_{\mathbb{G}'}$. One verifies easily that $Hom_{G_K}(M, M') \longrightarrow Hom(\mathbb{G}_M, \mathbb{G}_{M'})$ is an isomorphism and that the canonical map $\mathbb{G} \longrightarrow \mathbb{G}_{M_{\mathbb{G}}}$ is an isomorphism.

Definition 66. Let $\mathcal{A}$ be an Abelian category. An object I of $\mathcal{A}$ is said to be *injective* if the functor

$$\mathcal{A} \longrightarrow \mathtt{abelian} \quad M \mapsto Hom_{\mathcal{A}}(M, I)$$

is exact. We say that $\mathcal{A}$ has enough injectives if, for every object M in $\mathcal{A}$, there is a monomorphism $M \hookrightarrow I$ into an injective object. If $\mathcal{A}$ has enough injectives and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor from $\mathcal{A}$ into a second Abelian category $\mathcal{B}$, then there is an essentially unique sequence of functors

$$R^i\ f : \mathcal{A} \longrightarrow \mathcal{B} \quad i \geq 0$$

called the *right derived functors* of f satisfying the following properties

(1) $R^0\ f = f$
(2) $R^i\ I = 0$ for I injective and $i > 0$
(3) For every short exact sequence in $\mathcal{A}$

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

there are connecting morphisms $\delta^i : R^i\ f(M") \longrightarrow R^{i+1}\ f(M')$ for $i \geq 0$ such that we have a long exact sequence

$$\ldots \longrightarrow R^i\ f(M) \longrightarrow R^i\ f(M") \xrightarrow{\delta^i} R^{i+1}\ f(M') \longrightarrow R^{i+1}\ f(M) \longrightarrow \ldots$$

(4) For any morphism $M \longrightarrow N$ there are morphisms $R^i\ f(M) \longrightarrow R^i\ f(N)$ for $i \geq 0$

To compute the objects $R^i\ f(M)$ define an object N in $\mathcal{A}$ to be *f-acyclic* if $R^i\ f(M) = 0$ for all $i > 0$. If we have an *acyclic resolution* of M

$$0 \longrightarrow M \longrightarrow N_0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \dots$$

by f-acyclic object N_i, then the objects $R^i\ f(M)$ are canonically isomorphic to the cohomology objects of the complex

$$0 \longrightarrow f(N_0) \longrightarrow f(N_1) \longrightarrow f(N_2) \longrightarrow \dots$$

One can show that all injectives are f-acyclic and hence that derived objects of M can be computed from an injective resolution of M.

DEFINITION 67. Let $\mathbf{S}^{ab}(\mathtt{C_{et}})$ be the category of all sheaves of Abelian groups on $\mathtt{C_{et}}$. This is an Abelian category having enough injectives whence we can form right derived functors of left exact functors. In particular, consider the global section functor

$$\Gamma : \mathbf{S}^{ab}(\mathtt{C_{et}}) \longrightarrow \text{Ab} \quad \mathbb{G} \mapsto \mathbb{G}(C)$$

which is left exact. The right derived functors of Γ will be called the *étale cohomology functors* and we denote

$$R^i\ \Gamma(\mathbb{G}) = H^i_{et}(C, \mathbb{G})$$

In particular, if we have an exact sequence of sheaves of Abelian groups $0 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 0$, then we have a long exact cohomology sequence

$$\dots \longrightarrow H^i_{et}(C, \mathbb{G}) \longrightarrow H^i_{et}(C, \mathbb{G}") \longrightarrow H^{i+1}_{et}(C, \mathbb{G}') \longrightarrow \dots$$

EXAMPLE 87. The category $G_K - \mathtt{mod}$ of continuous G_K-modules is Abelian having enough injectives. Therefore, the left exact functor

$$(-)^G : G_K - \mathtt{mod} \longrightarrow \mathtt{abelian}$$

admits right derived functors. They are called the *Galois cohomology groups* and denoted

$$R^i\ M^G = H^i(G_K, M)$$

For any sheaf of Abelian groups $\mathbb{G}$ on $\mathtt{K_{et}}$ we have a group isomorphism

$$H^i_{et}(K, \mathbb{G}) \simeq H^i(G_K, \mathbb{G}(\mathbb{K}))$$

Therefore, étale cohomology is a natural extension of Galois cohomology to arbitrary algebras.

For applications in noncommutative algebra and geometry $\mathbb{G}$ will often be a sheaf of automorphism groups which are usually not Abelian. In this case we cannot define cohomology groups. Still, we can define a *first cohomology pointed set* $H^1_{et}(C, \mathbb{G})$.

DEFINITION 68. If $\mathbb{G}$ is a sheaf of not necessarily Abelian groups on $\mathtt{C_{et}}$, then for an étale cover $\mathcal{C} = \{C \longrightarrow C_i\}$ of C define a 1-cocycle for $\mathcal{C}$ with values in $\mathbb{G}$ to be a family

$$g_{ij} \in \mathbb{G}(C_{ij}) \text{ with } C_{ij} = C_i \otimes_C C_j$$

satisfying the cocycle condition

$$(g_{ij} \mid C_{ijk})(g_{jk} \mid C_{ijk}) = (g_{ik} \mid C_{ijk})$$

where $C_{ijk} = C_i \otimes_C C_j \otimes_C C_k$.

Two cocycles g and g' for $\mathcal{C}$ are said to be cohomologous if there is a family $h_i \in \mathbb{G}(C_i)$ such that for all $i, j \in I$ we have

$$g'_{ij} = (h_i \mid C_{ij}) g_{ij} (h_j \mid C_{ij})^{-1}$$

This is an equivalence relation and the set of cohomology classes is written as $H^1_{et}(\mathcal{C}, \mathbb{G})$. It is a pointed set having as its distinguished element the cohomology class of $g_{ij} = 1 \in \mathbb{G}(C_{ij})$ for all $i, j \in I$.

We then define the non-Abelian first *cohomology pointed set* to be

$$H^1_{et}(C, \mathbb{G}) = \underrightarrow{lim} \; H^1_{et}(\mathcal{C}, \mathbb{G})$$

where the limit is taken over all étale coverings of C. It coincides with the previous definition in case $\mathbb{G}$ is Abelian.

A sequence $1 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 1$ of sheaves of non necessarily Abelian groups on $\mathtt{C_{et}}$ is said to be exact if for every $B \in \mathtt{C_{et}}$ we have that $\mathbb{G}'(B) = Ker\ \mathbb{G}(B) \longrightarrow \mathbb{G}"(B)$ and for every $g" \in \mathbb{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in $\mathtt{C_{et}}$ and sections $g_i \in \mathbb{G}(B_i)$ such that g_i maps to $g" \mid B$.

Theorem 57. *For an exact sequence of groups on* $\mathtt{C_{et}}$

$$1 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 1$$

there is associated an exact sequence of pointed sets

$$1 \longrightarrow \mathbb{G}'(C) \longrightarrow \mathbb{G}(C) \longrightarrow \mathbb{G}"(C) \xrightarrow{\delta} H^1_{et}(C, \mathbb{G}') \longrightarrow$$
$$\longrightarrow H^1_{et}(C, \mathbb{G}) \longrightarrow H^1_{et}(C, \mathbb{G}") \dashrightarrow H^2_{et}(C, \mathbb{G}')$$

where the last map exists when $\mathbb{G}'$ *is contained in the center of* $\mathbb{G}$ *(and therefore is Abelian whence* H^2 *is defined).*

Proof. The connecting map δ is defined as follows. Let $g" \in \mathbb{G}"(C)$ and let $\mathcal{C} = \{C \longrightarrow C_i\}$ be an étale covering of C such that there are $g_i \in \mathbb{G}(C_i)$ that map to $g \mid C_i$ under the map $\mathbb{G}(C_i) \longrightarrow \mathbb{G}"(C_i)$. Then, $\delta(g)$ is the class determined by the one cocycle

$$g_{ij} = (g_i \mid C_{ij})^{-1} (g_j \mid C_{ij})$$

with values in $\mathbb{G}'$. The last map can be defined in a similar manner, the other maps are natural and one verifies exactness. □

Let A be a not necessarily commutative C-algebra and M an C-module. Consider the sheaves of groups $\mathtt{Aut}(A)$ resp. $\mathtt{Aut}(M)$ on $\mathtt{C_{et}}$ associated to the presheaves

$$B \mapsto Aut_{B-alg}(A \otimes_C B) \text{ resp. } B \mapsto Aut_{B-mod}(M \otimes_C B)$$

for all $B \in \mathtt{C_{et}}$. A *twisted form* of A (resp. M) is an C-algebra A' (resp. an C-module M') such that there is an étale cover $\mathcal{C} = \{C \longrightarrow C_i\}$ of C such that there are isomorphisms

$$\begin{cases} A \otimes_C C_i \xrightarrow{\phi_i} A' \otimes_C C_i \\ M \otimes_C C_i \xrightarrow{\psi_i} M' \otimes_C C_i \end{cases}$$

of C_i-algebras (resp. C_i-modules). The set of C-algebra isomorphism classes (resp. C-module isomorphism classes) of twisted forms of A (resp. M) is denoted by $Tw_C(A)$ (resp. $Tw_C(M)$). To a twisted form A' one associates a cocycle on $\mathcal{C}$

$$\alpha_{A'} = \alpha_{ij} = \phi_i^{-1} \circ \phi_j$$

with values in Aut(A). Moreover, one verifies that two twisted forms are isomorphic as C-algebras if their cocycles are cohomologous. That is, there are embeddings

$$\begin{cases} Tw_C(A) \hookrightarrow H^1_{et}(C, \mathtt{Aut}(A)) \\ Tw_C(M) \hookrightarrow H^1_{et}(C, \mathtt{Aut}(M)) \end{cases}$$

In favorable situations one can even show bijectivity. In particular, this is the case if the automorphisms group is a smooth affine algebraic group-scheme.

EXAMPLE 88. GL_r is an affine smooth algebraic group defined over K and is the automorphism group of a vectorspace of dimension r. It defines a sheaf of groups on $\mathtt{K_{et}}$ that we will denote by $\mathtt{GL}_r$. Because the first cohomology classifies twisted forms of vectorspaces of dimension r and there is just one such class, we have

$$H^1_{et}(K, \mathtt{GL}_r) = H^1(G_K, GL_r(\mathbb{K})) = 0$$

In particular, we have *'Hilbert's theorem 90'*

$$H^1_{et}(K, \mathbb{G}_m) = H^1(G_K, \mathbb{K}^*) = 0$$

EXAMPLE 89. Let A be a finite dimensional K-algebra. It is classical, see for example [**51**], that the following are equivalent :

(1) A has no proper twosided ideals and the center of A is K.
(2) $A_{\mathbb{K}} = A \otimes_K \mathbb{K} \simeq M_n(\mathbb{K})$ for some n.
(3) $A_L = A \otimes_K L \simeq M_n(L)$ for some n and some finite Galois extension L/K.
(4) $A \simeq M_k(D)$ for some k where D is a division algebra of dimension l^2 with center K.

An algebra satisfying these properties is said to be a *central simple algebra* over K.

PGL_n is an affine smooth algebraic group defined over K and is the automorphism group of the K-algebra $M_n(K)$. It defines a sheaf of groups on $\mathtt{K_{et}}$ denoted by $\mathtt{PGL}_n$. By the above equivalences any central simple K-algebra Δ of dimension n^2 is a twisted form of $M_n(K)$. Therefore, the pointed set

$$H^1_{et}(K, \mathtt{PGL}_n) = H^1(G_K, PGL_n(\mathbb{K}))$$

classifies the central simple K-algebras of dimension n^2.

DEFINITION 69. If A and B are central simple K-algebras, then so is $A \otimes_K B$ by example 89 (2). We say that two central simple K-algebras A and B are equivalent iff

$$A \simeq M_a(D) \qquad \text{and} \qquad B \simeq M_b(D)$$

for a finite dimensional division algebra D with center K. The tensorproduct induces a groupstructure on the equivalence classes of central simple K-algebras, $Br(K)$, called the *Brauer group* of the field K.

The unit element of $Br(K)$ is $[K]$ and the inverse of $[A]$ is the equivalence class of the opposite algebra $[A^{op}]$ as $A \otimes_K A^{op} \simeq M_{l^2}(K)$ if A is of dimension l^2.

THEOREM 58. *There is a natural inclusion*

$$H^1_{et}(K, \mathtt{PGL}_n) \hookrightarrow H^2_{et}(K, \mu_n) = Br_n(K)$$

where $Br_n(K)$ is the n-torsion part of the Brauer group of K. Moreover,

$$Br(K) = H^2_{et}(K, \mathbb{G}_m)$$

is a torsion group.

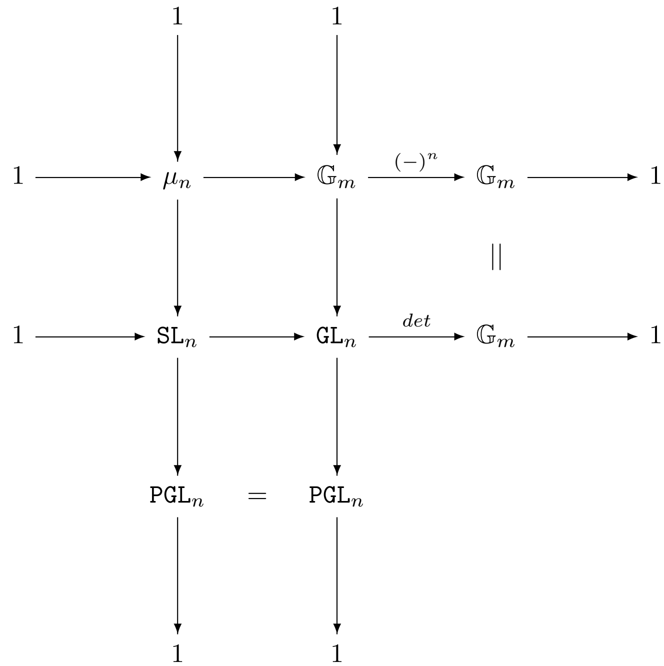


FIGURE 2. Brauer group diagram.

PROOF. Consider the exact commutative diagram of sheaves of groups on $\mathtt{K_{et}}$ of figure 2. Taking cohomology of the second exact sequence we obtain

$$GL_n(K) \xrightarrow{det} K^* \longrightarrow H^1_{et}(K, \mathtt{SL}_n) \longrightarrow H^1_{et}(K, \mathtt{GL}_n)$$

where the first map is surjective and the last term is zero, whence

$$H^1_{et}(K, \mathtt{SL}_n) = 0$$

Taking cohomology of the first vertical exact sequence we get

$$H^1_{et}(K, \mathtt{SL}_n) \longrightarrow H^1_{et}(K, \mathtt{PGL}_n) \longrightarrow H^2_{et}(K, \mu_n)$$

from which the first claim follows.

As for the second assertion, taking cohomology of the first exact sequence we get

$$H^1_{et}(K, \mathbb{G}_m) \longrightarrow H^2_{et}(K, \mu_n) \longrightarrow H^2_{et}(K, \mathbb{G}_m) \xrightarrow{n.} H^2_{et}(K, \mathbb{G}_m)$$

By Hilbert 90, the first term vanishes and hence $H^2_{et}(K, \mu_n)$ is equal to the n-torsion of the group

$$H^2_{et}(K, \mathbb{G}_m) = H^2(G_K, \mathbb{K}^*) = Br(K)$$

where the last equality is the *crossed product theorem* , see for example [**51**]. □

In noncommutative geometry, the field K will be the functionfield of an algebraic variety and $Br(K)$ gives a measure for the noncommutative function skew-fields having center K. These Brauer groups are usually huge objects and their description contains a lot of geometric/combinatorial data about a smooth model of K. The dimension of the variety puts a bound on the size of the Galois group G_K and hence limits the non-zero Galois cohomology groups.

DEFINITION 70. A field K is said to be a `Tsen.d`-field if every homogeneous form of degree deg with coefficients in K and $n > deg^d$ variables has a non-trivial zero in K.

EXAMPLE 90. An algebraically closed field $\mathbb{K}$ is a `Tsen.0`-field as any form in n-variables defines a hypersurface in $\mathbb{P}^{n-1}_{\mathbb{K}}$. In fact, algebraic geometry tells us a stronger story : if $f_1, \ldots, f_r$ are forms in n variables over $\mathbb{K}$ and $n > r$, then these forms have a common non-trivial zero in $\mathbb{K}$. Indeed, every f_i defines a hypersurface $V(f_i) \hookrightarrow \mathbb{P}^{n-1}_{\mathbb{K}}$. The intersection of r hypersurfaces has dimension $\geq n-1-r$ from which the claim follows.

In fact, one can extend this to higher Tsen-fields. Let K be a `Tsen.d`-field and $f_1, \ldots, f_r$ forms in n variables of degree deg. If $n > r deg^d$, then they have a non-trivial common zero in K. See [**64**] for a proof.

THEOREM 59. *Let K be of transcendence degree d over an algebraically closed field $\mathbb{C}$, then K is* `Tsen.d`.

PROOF. First we claim that the purely transcendental field $\mathbb{C}(t_1, \ldots, t_d)$ is a `Tsen.d`. By induction we have to show that if L is `Tsen.k`, then $L(t)$ is `Tsen.k+1`.

By homogeneity we may assume that $f(x_1, \ldots, x_n)$ is a form of degree deg with coefficients in $L[t]$ and $n > deg^{k+1}$. For fixed s we introduce new variables $y^{(s)}_{ij}$ with $i \leq n$ and $0 \leq j \leq s$ such that

$$x_i = y^{(s)}_{i0} + y^{(s)}_{i1} t + \ldots + y^{(s)}_{is} t^s$$

If r is the maximal degree of the coefficients occurring in f, then we can write

$$f(x_i) = f_0(y^{(s)}_{ij}) + f_1(y^{(s)}_{ij})t + \ldots + f_{deg.s+r}(y^{(s)}_{ij}) t^{deg.s+r}$$

where each f_j is a form of degree deg in $n(s+1)$-variables. By the previous example, these forms have a common zero in L provided

$$n(s+1) > deg^k(ds + r + 1) \Longleftrightarrow (n - deg^{i+1})s > deg^i(r+1) - n$$

which can be satisfied by taking s large enough. the common non-trivial zero in L of the f_j, gives a non-trivial zero of f in $L[t]$.

By assumption, K is an algebraic extension of $\mathbb{C}(t_1, \ldots, t_d)$ which by the above argument is `Tsen.d`. As the coefficients of any form over K lie in a finite extension E of $\mathbb{C}(t_1, \ldots, t_d)$ it suffices to prove that E is `Tsen.d`.

Let $f(x_1, \ldots, x_n)$ be a form of degree deg in E with $n > deg^d$. Introduce new variables y_{ij} with

$$x_i = y_{i1}e_1 + \ldots y_{ik}e_k$$

where e_i is a basis of E over $\mathbb{C}(t_1, \ldots, t_d)$. Then,

$$f(x_i) = f_1(y_{ij})e_1 + \ldots + f_k(y_{ij})e_k$$

where the f_i are forms of degree deg in $k.n$ variables over $\mathbb{C}(t_1, \ldots, t_d)$. Because $\mathbb{C}(t_1, \ldots, t_d)$ is `Tsen.d`, these forms have a common zero as $k.n > k.deg^d$. Finding a non-trivial zero of f in E is equivalent to finding a common non-trivial zero to the $f_1, \ldots, f_k$ in $\mathbb{C}(t_1, \ldots, t_d)$, done. □

THEOREM 60 (Tsen's theorem). *Let K be the functionfield of a curve C defined over an algebraically closed field. Then, the only central simple K-algebras are $M_n(K)$. That is, $Br(K) = 1$.*

PROOF. Assume there exists a central division algebra Δ of dimension n^2 over K. There is a finite Galois extension L/K such that $\Delta \otimes L = M_n(L)$. If $x_1, \ldots, x_{n^2}$ is a K-basis for Δ, then the reduced norm of any $x \in \Delta$,

$$N(x) = det(x \otimes 1)$$

is a form in n^2 variables of degree n. Moreover, as $x \otimes 1$ is invariant under the action of $Gal(L/K)$ the coefficients of this form actually lie in K.

By the previous theorem, K is `Tsen.1` and $N(x)$ has a non-trivial zero whenever $n^2 > n$. As the reduced norm is multiplicative, this contradicts $N(x)N(x^{-1}) = 1$. Hence, $n = 1$ and the only central division algebra is K itself. □

EXAMPLE 91. If K is the functionfield of a surface, and if Δ is a central simple K-algebra of dimension n^2, then the reduced norm map

$$N \ : \ \Delta \longrightarrow K$$

is surjective. For, let $e_1, \ldots, e_{n^2}$ be a K-basis of Δ and $k \in K$, then

$$N(\sum x_i e_i) - k x_{n^2+1}^n$$

is a form of degree n in $n^2 + 1$ variables. Since K is `Tsen.2`, it has a non-trivial solution (x_i^0), but then, $\delta = (\sum x_i^0 e_i) x_{n^2+1}^{-1}$ has reduced norm equal to k.

DEFINITION 71. The *cohomological dimension* of a group G, $cd(G) \leq d$ if and only if $H^r(G, A) = 0$ for all $r > d$ and all torsion modules $A \in G$-mod.

A field K is said to be a `Tate.d`-field if the absolute Galois group $G_K = Gal(\mathbb{K}/K)$ has cohomological dimension d.

EXAMPLE 92. We claim that $cd(G) \leq d$ if $H^{d+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for the simple G-modules with trivial action $\mathbb{Z}/p\mathbb{Z}$.

To start, one can show that a profinite group G (that is, a projective limit of finite groups, see [**64**] for more details) has $cd(G) \leq d$ if and only if

$$H^{d+1}(G, A) = 0 \text{ for all torsion } G\text{-modules } A$$

Further, as all Galois cohomology groups of profinite groups are torsion, we can decompose the cohomology in its p-primary parts and relate their vanishing to the cohomological dimension of the p-Sylow subgroups G_p of G. This problem can then be verified by computing cohomology of finite simple G_p-modules of p-power order, but for a profinite p-group there is just one such module namely $\mathbb{Z}/p\mathbb{Z}$ with the trivial action proving the claim.

We will encounter many *spectral sequences* so it may be useful to recall their definition in some detail.

DEFINITION 72. A *spectral sequence* $E_2^{p.q} \Longrightarrow E^n$ (or $E_1^{p,q} \Longrightarrow E^n$) consists of the following data :

A family of objects $E_r^{p,q}$ in an Abelian category for $p, q, r \in \mathbb{Z}$ such that $p, q \geq 0$ and $r \geq 2$ (or $r \geq 1$).

A family of morphisms in the Abelian category

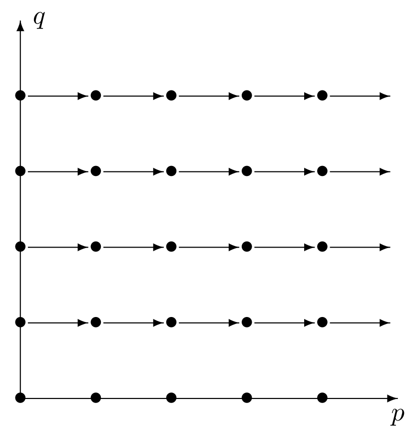
$$d_r^{p.q} : E_r^{p.q} \longrightarrow E_r^{p+r,q-r+1}$$

satisfying the complex condition

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$$

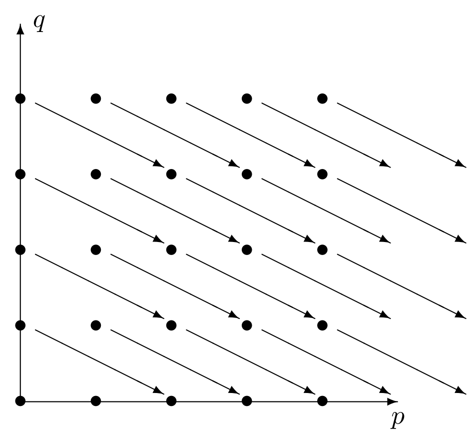
and where we assume that $d_r^{p.q} = 0$ if any of the numbers $p, q, p+r$ or $q-r+1$ is < 1. At level one we have the following

$E_1^{p,q} \quad =$



At level two we have the following

$E_2^{p,q} \quad =$



The objects $E_{r+1}^{p,q}$ on level $r+1$ are derived from those on level r by taking the cohomology objects of the complexes, that is,

$$E_{r+1}^{p} = \frac{Ker\ d_r^{p,q}}{Im\ d_r^{p-r,q+r-1}}$$

At each place (p, q) this process converges as there is an integer r_0 depending on (p, q) such that for all $r \geq r_0$ we have $d_r^{p.q} = 0 = d_r^{p-r,q+r-1}$. We then define

$$E_\infty^{p,q} = E_{r_0}^{p,q} (= E_{r_0+1}^{p,q} = \ldots)$$

Observe that there are injective maps $E_\infty^{0,q} \hookrightarrow E_2^{0,q}$.

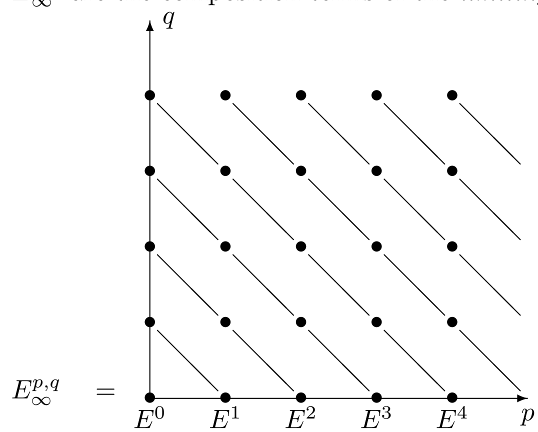
A family of objects E^n for integers $n \geq 0$ and for each we have a filtration

$$0 \subset E_n^n \subset E_{n-1}^n \subset \ldots \subset E_1^n \subset E_0^n = E^n$$

such that the successive quotients are given by

$$E_p^n \ / \ E_{p+1}^n = E_\infty^{p,n-p}$$

That is, the terms $E_\infty^{p,q}$ are the composition terms of the *limiting terms* E^{p+q}.



THEOREM 61. *(Hochschild-Serre spectral sequence) If N is a closed normal subgroup of a profinite group G, then*

$$E_2^{p,q} = H^p(G/N, H^q(N,A)) \Longrightarrow H^n(G,A)$$

holds for every continuous G-module A.

THEOREM 62 (Tate). *Let K be a field of transcendence degree d over an algebraically closed field, then K is* `Tate.d`*. In particular, if $\mathbf{A}$ is a constant sheaf of an Abelian torsion group A on* $\mathtt{K_{et}}$*, then*

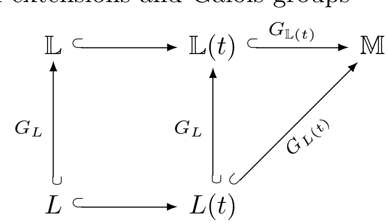
$$H^i_{et}(K, \mathbf{A}) = 0$$

whenever $i > trdeg_{\mathbb{C}}(K)$.

PROOF. Let $\mathbb{C}$ denote the algebraically closed basefield, then K is algebraic over $\mathbb{C}(t_1, \ldots, t_d)$ and therefore

$$G_K \hookrightarrow G_{\mathbb{C}(t_1,\ldots,t_d)}$$

Thus, K is `Tate.d` if $\mathbb{C}(t_1, \ldots, t_d)$ is `Tate.d`. By induction it suffices to prove that if $cd(G_L) \leq k$ then $cd(G_{L(t)}) \leq k+1$. Let $\mathbb{L}$ be the algebraic closure of L and $\mathbb{M}$ the algebraic closure of $L(t)$. As $L(t)$ and $\mathbb{L}$ are linearly disjoint over L we have the following diagram of extensions and Galois groups



where $G_{L(t)}/G_{\mathbb{L}(t)} \simeq G_L$.

We claim that $cd(G_{\mathbb{L}(t)}) \leq 1$. Consider the exact sequence of $G_{L(t)}$-modules

$$0 \longrightarrow \boldsymbol{\mu}_p \longrightarrow \mathbb{M}^* \xrightarrow{(-)^p} \mathbb{M}^* \longrightarrow 0$$

where $\boldsymbol{\mu}_p$ is the subgroup (of $\mathbb{C}^*$) of p-roots of unity. As $G_{L(t)}$ acts trivially on $\boldsymbol{\mu}_p$ it is after a choice of primitive p-th root of one isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Taking cohomology with respect to the subgroup $G_{\mathbb{L}(t)}$ we obtain

$$0 = H^1(G_{\mathbb{L}(t)}, \mathbb{M}^*) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{M}^*) = Br(\mathbb{L}(t))$$

But the last term vanishes by Tsen's theorem as $\mathbb{L}(t)$ is the functionfield of a curve defined over the algebraically closed field $\mathbb{L}$. Therefore, $H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all simple modules $\mathbb{Z}/p\mathbb{Z}$, whence $cd(G_{\mathbb{L}(t)}) \leq 1$.

By the inductive assumption we have $cd(G_L) \leq k$ and now we are going to use exactness of the sequence

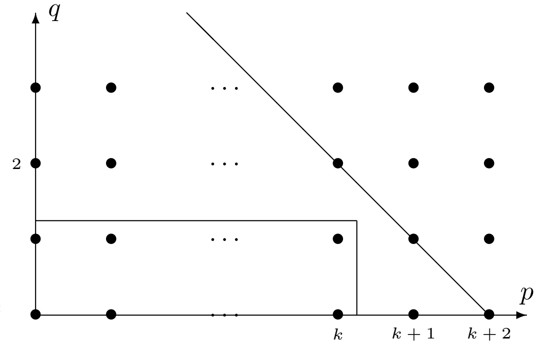
$$0 \longrightarrow G_L \longrightarrow G_{L(t)} \longrightarrow G_{\mathbb{L}(t)} \longrightarrow 0$$

to prove that $cd(G_{L(t)}) \leq k+1$. For, let A be a torsion $G_{L(t)}$-module and consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_L, H^q(G_{\mathbb{L}(t)}, A)) \Longrightarrow H^n(G_{L(t)}, A)$$

By the restrictions on the cohomological dimensions of G_L and $G_{\mathbb{L}(t)}$ the level two term has following shape

$E_2^{p,q} \quad =$



where the only non-zero groups are lying in the lower rectangular region. Therefore, all $E_\infty^{p,q} = 0$ for $p+q > k+1$. Now, all the composition factors of $H^{k+2}(G_{L(t)}, A)$ are lying on the indicated diagonal line and hence are zero. Thus, $H^{k+2}(G_{L(t)}, A) = 0$ for all torsion $G_{L(t)}$-modules A and hence $cd(G_{L(t)}) \leq k+1$. □

We need to classify all central simple algebras Σ of dimension n^2 over the function field K of transcendence degree d. For large dimensions d this is a hopeless task. Still, étale cohomology can be used to go a long way towards this goal and for small d one does get a nice description.

The first tool we need is the *Leray spectral sequence* . Assume we have an algebra morphism $C \xrightarrow{f} C'$ and a sheaf of groups $\mathbb{G}$ on $\mathtt{C'}_{\mathtt{et}}$. We define the *direct image* of $\mathbb{G}$ under f to be the sheaf of groups $f_*\ \mathbb{G}$ on $\mathtt{C}_{\mathtt{et}}$ defined by

$$f_*\ \mathbb{G}(B) = \mathbb{G}(B \otimes_C C')$$

for all $B \in \mathtt{A}_{\mathtt{et}}$ (recall that $B \otimes_C C' \in \mathtt{C'}_{\mathtt{et}}$ so the right hand side is well defined).

This gives us a left exact functor

$$f_* : \mathbf{S}^{ab}(\mathtt{C'}_{\mathtt{et}}) \longrightarrow \mathbf{S}^{ab}(\mathtt{C}_{\mathtt{et}})$$

and therefore there exist right derived functors $R^i\ f_*$. If $\mathbb{G}$ is an Abelian sheaf on $\mathtt{C'}_{\mathtt{et}}$, then $R^i\ f_*\mathbb{G}$ is a sheaf on $\mathtt{C}_{\mathtt{et}}$. One verifies that its stalk in a prime ideal $\mathfrak{p}$ is equal to

$$(R^i\ f_*\mathbb{G})_{\mathfrak{p}} = H^i_{et}(C^{sh}_{\mathfrak{p}} \otimes_C C', \mathbb{G})$$

where the right hand side is the direct limit of cohomology groups taken over all étale neighborhoods of $\mathfrak{p}$. We can relate cohomology of $\mathbb{G}$ and $f_*\mathbb{G}$ by the following

THEOREM 63. *(Leray spectral sequence) If $\mathbb{G}$ is a sheaf of Abelian groups on* $\texttt{C'}_{\texttt{et}}$ *and* $C \xrightarrow{f} C'$ *an algebra morphism. Then, there is a spectral sequence*

$$E_2^{p,q} = H^p_{et}(C, R^q\ f_*\mathbb{G}) \Longrightarrow H^n_{et}(C', \mathbb{G})$$

In particular, if $R^j\ f_*\mathbb{G} = 0$ for all $j > 0$, then for all $i \geq 0$ we have isomorphisms

$$H^i_{et}(C, f_*\mathbb{G}) \simeq H^i_{et}(C', \mathbb{G})$$

PROOF. See for example [**47**, III.Thm.1.18]. □

We will use it to relate étale cohomology over K to that over a discrete valuation ring C in K with residue field $k = \frac{C}{m}$, that is we have algebra morphisms

$$\begin{array}{ccc} C & \xrightarrow{i} & K \\ \downarrow{\scriptstyle \pi} & & \\ k & & \end{array}$$

From section 5.1 we recall that the n-torsion part of the Brauer groups of K and k are given by the étale (or Galois) cohomology groups

$$H^2_{et}(K, \mu_n) \qquad \text{resp.} \qquad H^2_{et}(k, \mu_n)$$

and we like to deduce information on $Br_n(C)$.

THEOREM 64. *There is a long exact sequence of groups*

$$0 \longrightarrow H^1_{et}(C, \mu_n^{\otimes l}) \longrightarrow H^1_{et}(K, \mu_n^{\otimes l}) \longrightarrow H^0_{et}(k, \mu_n^{\otimes l-1}) \longrightarrow$$
$$H^2_{et}(A, \mu_n^{\otimes l}) \longrightarrow H^2_{et}(K, \mu_n^{\otimes l}) \longrightarrow H^1_{et}(k, \mu_n^{\otimes l-1}) \longrightarrow \ \dots$$

PROOF. By considering the Leray spectral sequence for the inclusion i we claim that the following equalities hold :

(1) $R^0\ i_*\mu_n^{\otimes l} \simeq \mu_n^{\otimes l}$ on $\texttt{C}_{\texttt{et}}$.
(2) $R^1\ i_*\mu_n^{\otimes l} \simeq \mu_n^{\otimes l-1}$ concentrated in m.
(3) $R^j\ i_*\mu_n^{\otimes l} \simeq 0$ whenever $j \geq 2$.

Indeed, the strict Henselizations of C at the two primes $\{0, m\}$ are resp.

$$C_0^{sh} \simeq \mathbb{K} \text{ and } C_m^{sh} \simeq \mathbf{k}\{t\}$$

where $\mathbb{K}$ (resp. $\mathbf{k}$) is the algebraic closure of K (resp. k). Therefore,

$$(R^j\ i_*\mu_n^{\otimes l})_0 = H^j_{et}(\mathbb{K}, \mu_n^{\otimes l})$$

which is zero for $i \geq 1$ and $\mu_n^{\otimes l}$ for $j = 0$. Further, $C_m^{sh} \otimes_C K$ is the field of fractions of $\mathbf{k}\{t\}$ and hence is of transcendence degree one over the algebraically closed field $\mathbf{k}$, whence

$$(R^j\ i_*\mu_n^{\otimes l})_m = H^j_{et}(L, \mu_n^{\otimes l})$$

which is zero for $j \geq 2$ because L is a `Tate.1`-field.

For the field-tower $K \subset L \subset \mathbb{K}$ we have that $G_L = \hat{\mathbb{Z}} = \underset{\leftarrow}{lim} \mu_m$ because the only Galois extensions of L are the Kummer extensions obtained by adjoining $\sqrt[m]{t}$. But then,

$$H^1_{et}(L, \mu_n^{\otimes l}) = H^1(\hat{Z}, \mu_n^{\otimes l}(\mathbb{K})) = Hom(\hat{Z}, \mu_n^{\otimes l}(\mathbb{K})) = \mu_n^{\otimes l-1}$$

from which the claims follow.

0	0	0	...
$H^0_{et}(k, \mu_n^{\otimes l-1})$	$H^1_{et}(k, \mu_n^{\otimes l-1})$	$H^2_{et}(k, \mu_n^{\otimes l-1})$	...
$H^0_{et}(C, \mu_n^{\otimes l})$	$H^1_{et}(C, \mu_n^{\otimes l})$	$H^2_{et}(C, \mu_n^{\otimes l})$	...

FIGURE 3. Second term of Leray spectral sequence.

0	0	0	...
$Ker\ \alpha_1$	$Ker\ \alpha_2$	$Ker\ \alpha_3$	...
$H^0_{et}(C, \mu_n^{\otimes l})$	$H^1_{et}(C, \mu_n^{\otimes l})$	$Coker\ \alpha_1$	...

FIGURE 4. Limiting term of the Leray spectral sequence.

Therefore, the second term of the Leray spectral sequence for $i_*\mu_n^{\otimes l}$ has the shape given in figure 3 with connecting morphisms

$$H^{i-1}_{et}(k, \mu_n^{\otimes l-1}) \xrightarrow{\alpha_i} H^{i+1}_{et}(C, \mu_n^{\otimes l})$$

The spectral sequences converges to its limiting term given in figure 4 The previous theorem gives us the short exact sequences

$$0 \longrightarrow H^1_{et}(C, \mu_n^{\otimes l}) \longrightarrow H^1_{et}(K, \mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_1 \longrightarrow 0$$

$$0 \longrightarrow Coker\ \alpha_1 \longrightarrow H^2_{et}(K, \mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_2 \longrightarrow 0$$

$$0 \longrightarrow Coker\ \alpha_{i-1} \longrightarrow H^i_{et}(K, \mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_i \longrightarrow 0$$

Gluing these sequences gives us the required result. □

We will extend the definition of étale cohomology to the setting of arbitrary (non-affine) schemes.

DEFINITION 73. A morphism of schemes

$$Y \xrightarrow{f} X$$

is said to be an *étale extension (resp. cover)* if locally f is of the form

$$f^a \mid U_i : C_i = \Gamma(U_i, \mathcal{O}_X) \longrightarrow C_i' = \Gamma(f^{-1}(U_i), \mathcal{O}_Y)$$

with $C_i \longrightarrow C_i'$ an étale extension (resp. cover) of algebras.

The *étale site of* X is also defined locally and will be denoted by X_{et}. Presheaves and sheaves of groups on X_{et} are defined similarly and the right derived functors of the left exact global sections functor

$$\Gamma : \mathbf{S}^{ab}(X_{et}) \longrightarrow \texttt{abelian}$$

will be called the cohomology functors and denoted

$$R^i\ \Gamma(\mathbb{G}) = H^i_{et}(X, \mathbb{G})$$

If X is a smooth, irreducible projective variety of dimension d over $\mathbb{C}$, we can initiate the computation of the cohomology groups $H^i_{et}(X, \mu_n^{\otimes l})$ via Galois cohomology of functionfields of subvarieties using the *coniveau spectral sequence* :

THEOREM 65 (Coniveau spectral sequence). *Let X be a smooth irreducible variety over $\mathbb{C}$. Let $X^{(p)}$ denote the set of irreducible subvarieties x of X of codimension p with functionfield $\mathbb{C}(x)$. Then, there exists a* coniveau spectral sequence

$$E_1^{p.q} = \bigoplus_{x \in X^{(p)}} H^{q-p}_{et}(\mathbb{C}(x), \mu_n^{\otimes l-p}) \Longrightarrow H^{p+q}_{et}(X, \mu_n^{\otimes l})$$

relating Galois cohomology of the functionfields to the étale cohomology of X.

PROOF. Unlike the spectral sequences used before, the existence of the coniveau spectral sequence by no means follows from general principles. A lot of heavy machinery on étale cohomology of schemes is used in the proof. In particular, the cohomology groups with support of a closed subscheme, see for example [**47**, p. 91-94], and cohomological purity and duality, see [**47**, p. 241-252]. For a detailed exposition we refer the reader to [**8**]. □

EXAMPLE 93. By the results of section 5.1 on cohomological dimension and vanishing of Galois cohomology of $\mu_n^{\otimes k}$ when the index is larger than the transcendence degree, we deduce that the non-zero terms of the coniveau spectral are restricted to the triangular shaped region of figure 5

EXAMPLE 94. Consider the connecting morphisms of the coniveau spectral sequence, a typical instance of which is

$$\bigoplus_{x \in X^{(p)}} H^i(\mathbb{C}(x), \mu_n^{\oplus l-p}) \longrightarrow \bigoplus_{y \in X^{(p+1)}} H^{i-1}(\mathbb{C}(y), \mu_n^{\oplus l-p-1})$$

Take one of the closed irreducible subvarieties x of X of codimension p and one y of codimension $p+1$. Then, either y is not contained in x in which case the component map

$$H^i_{et}(\mathbb{C}(x), \mu_n^{\oplus l-p}) \longrightarrow H^{i-1}_{et}(\mathbb{C}(y), \mu_n^{\oplus l-p-1})$$

is the zero map, or, y is contained in x and hence defines a codimension one subvariety of x. That is, y defines a discrete valuation on $\mathbb{C}(x)$ with residue field $\mathbb{C}(y)$. In this case, the component map is the connecting morphism of theorem 64.

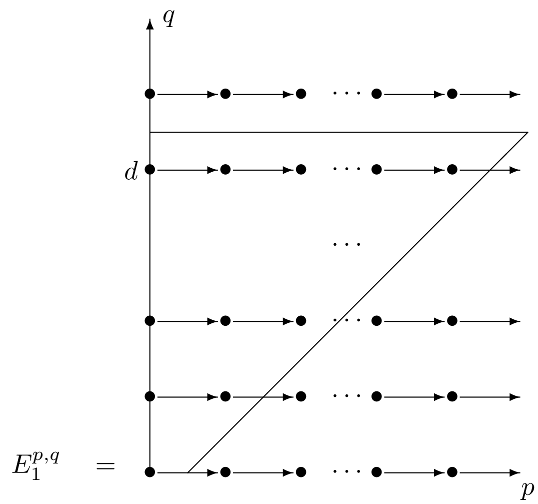


FIGURE 5. Coniveau spectral sequence

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	0	0	0	$\cdots$
$H^2(\mathbb{C}(S), \mu_n)$	$\oplus_C H^1(\mathbb{C}(S), \mathbb{Z}_n)$	$\oplus_P\ \mu_n^{-1}$	0	$\cdots$
$H^1(\mathbb{C}(S), \mu_n)$	$\oplus_C\ \mathbb{Z}_n$	0	0	$\cdots$
μ_n	0	0	0	$\cdots$

FIGURE 6. The coniveau spectral sequence for a surface S.

EXAMPLE 95. In particular, let K be the functionfield of X. Then we can define the *unramified cohomology groups*

$$F_n^{i,l}(K/\mathbb{C}) = Ker\ H^i(K, \mu_n^{\otimes l}) \xrightarrow{\oplus \partial_{i,A}} \oplus H^{i-1}(k_A, \mu_n^{\otimes l-1})$$

where the sum is taken over all discrete valuation rings A of K (or equivalently, the irreducible codimension one subvarieties of X) with residue field k_A. By definition, this is a (stable) birational invariant of X. In particular, if X is (stably) rational over $\mathbb{C}$, then

$$F_n^{i,l}(K/\mathbb{C}) = 0 \text{ for all } i, l \geq 0$$

EXAMPLE 96. The Brauer group of the function field of a smooth irreducible projective surface S. The first term of the coniveau spectral sequence for S has the shape of figure 6 where C runs over all irreducible curves on S and P over all points of S. We claim that the connecting morphism

$$H^1(\mathbb{C}(S), \mu_n) \xrightarrow{\gamma} \oplus_C \mathbb{Z}_n$$

is surjective. Indeed, from the Kummer sequence describing μ_n as the kernel of $\mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m$ and Hilbert 90 we have that

$$H^1_{et}(\mathbb{C}(S), \mu_n) = \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$$

The claim follows from the exact diagram below describing divisors of rational functions

$$\begin{array}{ccccccccc}
 & & \boldsymbol{\mu}_n & \simeq & \boldsymbol{\mu}_n & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathbb{C}(S)^* & \xrightarrow{div} & \oplus_C \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (-)^n} & & \downarrow{\scriptstyle n.} & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathbb{C}(S)^* & \xrightarrow{div} & \oplus_C \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \oplus_C \mathbb{Z}_n & \simeq & \oplus_C \mathbb{Z}_n & &
\end{array}$$

By the coniveau spectral sequence $H^1_{et}(S, \mu_n)$ is the kernel of γ and in particular, $H^1_{et}(S, \mu_n) \hookrightarrow H^1(\mathbb{C}(S), \mu_n)$.

Assume in addition that S is *simply connected* , that is, every étale cover $Y \twoheadrightarrow S$ is trivial (Y is the finite disjoint union of copies of S). As an element in $H^1_{et}(S, \mu_n)$ determines a cyclic extension $L = \mathbb{C}(S)\sqrt[n]{f}$ with $f \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$ such that in each fieldcomponent L_i of L there is an étale cover $T_i \longrightarrow S$ with $\mathbb{C}(T_i) = L_i$. If S is simply connected, nontrivial étale covers do not exist whence $f = 1 \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$ or $H^1_{et}(S, \mu_n) = 0$.

We now invoke another major tool in étale cohomology of schemes, *Poincaré duality* , see for example [**47**, VI,§11]. If S is simply connected, then

(1) $H^0_{et}(S, \mu_n) = \mu_n$
(2) $H^1_{et}(S, \mu_n) = 0$
(3) $H^3_{et}(S, \mu_n) = 0$
(4) $H^4_{et}(S, \mu_n) = \mu_n^{-1}$

The third claim follows from the second as both groups are dual to each other. The last claim follows from the fact that for any smooth irreducible projective variety X of dimension d one has that

$$H^{2d}_{et}(X, \mu_n) \simeq \mu_n^{\otimes 1-d}$$

We are now in a position to state and prove the important

THEOREM 66. *(Artin-Mumford exact sequence) Let S be a simply connected smooth projective surface. Then, there is an exact sequence of groups*

$$0 \longrightarrow Br_n(S) \longrightarrow Br_n(\mathbb{C}(S)) \longrightarrow \oplus_C \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	0	0	0	$\cdots$
$Ker\ \alpha$	$Ker\ \beta/Im\ \alpha$	$Coker\ \beta$	0	$\cdots$
$Ker\ \gamma$	$Coker\ \gamma$	0	0	$\cdots$
μ_n	0	0	0	$\cdots$

FIGURE 7. Limiting term for a surface S.

$$\longrightarrow \oplus_P\ \mu_n^{-1} \longrightarrow \mu_n^{-1} \longrightarrow 0$$

PROOF. The top complex in the first term of the coniveau spectral sequence for S is

$$H^2(\mathbb{C}(S), \mu_n) \xrightarrow{\alpha} \oplus_C\ H^1(\mathbb{C}(C), \mathbb{Z}_n) \xrightarrow{\beta} \oplus_P\ \mu_n$$

The second term of the spectral sequence (which is also the limiting term) is given in figure 7 We know already that $Coker\ \gamma = 0$. By Poincare duality we have that $Ker\ \beta = Im\ \alpha$ and $Coker\ \beta = \mu_n^{-1}$. Hence, the top complex is exact in its middle term and can be extended to an exact sequence

$$0 \longrightarrow H^2_{et}(S, \mu_n) \longrightarrow H^2_{et}(\mathbb{C}(S), \mu_n) \longrightarrow \oplus_C\ H^1_{et}(\mathbb{C}(C), \mathbb{Z}_n) \longrightarrow$$
$$\oplus_P \mu_n^{-1} \longrightarrow \mu_n^{-1} \longrightarrow 0$$

The third term is equal to $\oplus_C \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$ and the second term we remember to be the n-torsion part of the Brauer group $Br_n(\mathbb{C}(S)$. The identification of $Br_n(S)$ with $H^2_{et}(S, \mu_n)$ follows from Gabber's theorem and will be explained below. □

EXAMPLE 97. The Brauer group of $\mathbb{C}(x, y)$. If $S = \mathbb{P}^2$ we have that $Br_n(\mathbb{P}^2) = 0$ as $Br_n(\mathbb{P}^2)$ is the birational invariant $F_n^{2,1}(\mathbb{C}(x, y)/\mathbb{C})$. From the exact sequence

$$0 \longrightarrow Br_n\ \mathbb{C}(x, y) \longrightarrow \oplus_C\ \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow \oplus_P\ \mu_n^{-1} \longrightarrow \mu_n \longrightarrow 0$$

we obtain a description of $Br_n\ \mathbb{C}(x, y)$ by a certain geo-combinatorial package which we call a $\mathbb{Z}_n$-*wrinkle* over $\mathbb{P}^2$. A $\mathbb{Z}_n$-wrinkle is determined by

- A finite collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of *irreducible curves* in $\mathbb{P}^2$, that is, $C_i = V(F_i)$ for an irreducible form in $\mathbb{C}[X, Y, Z]$ of degree d_i.
- A finite collection $\mathcal{P} = \{P_1, \ldots, P_l\}$ of *points* of $\mathbb{P}^2$ where each P_i is either an intersection point of two or more C_i or a singular point of some C_i.
- For each $P \in \mathcal{P}$ the *branch-data* $b_P = (b_1, \ldots, b_{i_P})$ with $b_i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\{1, \ldots, i_P\}$ the different branches of $\mathcal{C}$ in P. These numbers must satisfy the admissibility condition

$$\sum_i b_i = 0 \in \mathbb{Z}_n$$

for every $P \in \mathcal{P}$

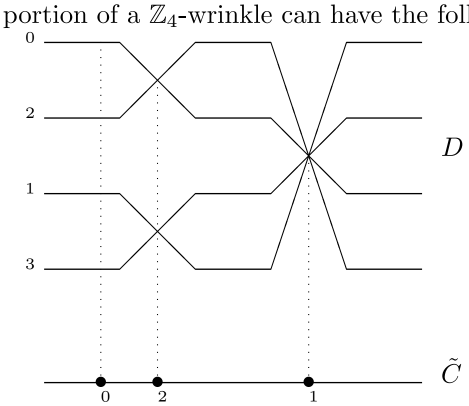
- for each $C \in \mathcal{C}$ we fix a cyclic $\mathbb{Z}_n$-cover of smooth curves

$$D \twoheadrightarrow \tilde{C}$$

of the desingularization $\tilde{C}$ of C which is compatible with the branch-data. That is, if $Q \in \tilde{C}$ corresponds to a $\mathcal{C}$-branch b_i in P, then D is ramified in Q with stabilizer subgroup

$$Stab_Q = \langle b_i \rangle \subset \mathbb{Z}_n$$

For example, a portion of a $\mathbb{Z}_4$-wrinkle can have the following picture



Clearly, the cover-data is the most intractable part of a $\mathbb{Z}_n$-wrinkle, so we need some control on the covers $D \twoheadrightarrow \tilde{C}$. Let $\{Q_1, \ldots, Q_z\}$ be the points of $\tilde{C}$ where the cover ramifies with branch numbers $\{b_1, \ldots, b_z\}$, then D is determined by a continuous module structure (that is, a cofinite subgroup acts trivially) of

$$\pi_1(\tilde{C} - \{Q_1, \ldots, Q_z\}) \text{ on } \mathbb{Z}_n$$

where the fundamental group of the Riemann surface $\tilde{C}$ with z punctures is known (topologically) to be equal to the group

$$\langle u_1, v_1, \ldots, u_g, v_g, x_1, \ldots, x_z \rangle / ([u_1, v_1] \ldots [u_g, v_g] x_1 \ldots x_z)$$

where g is the genus of $\tilde{C}$. The action of x_i on $\mathbb{Z}_n$ is determined by multiplication with b_i. In fact, we need to use the étale fundamental group, see [**47**], but this group has the same finite continuous modules as the topological fundamental group.

For example, if $\tilde{C} = \mathbb{P}^1$ then $g = 0$ and hence $\pi_1(\mathbb{P}^1 - \{Q_1, \ldots, Q_z\}$ is zero if $z \leq 1$ (whence no covers exist) and is $\mathbb{Z}$ if $z = 2$. Hence, there exists a unique cover $D \twoheadrightarrow \mathbb{P}^1$ with branch-data $(1, -1)$ in say $(0, \infty)$ namely with D the normalization of $\mathbb{P}^1$ in $\mathbb{C}(\sqrt[n]{x})$.

If $\tilde{C} = E$ an elliptic curve, then $g = 1$. Hence, $\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$ and there exist unramified $\mathbb{Z}_n$-covers. They are given by the isogenies

$$E' \twoheadrightarrow E$$

where E' is another elliptic curve and $E = E'/\langle \tau \rangle$ where τ is an n-torsion point on E'.

In general, an n-fold cover $D \twoheadrightarrow \tilde{C}$ is determined by a function $f \in \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$. This allows us to put a group-structure on the equivalence classes of $\mathbb{Z}_n$-wrinkles. In particular, we call a wrinkle *trivial* provided all coverings $D_i \twoheadrightarrow \tilde{C}_i$ are trivial (that is, D_i is the disjoint union of n copies of $\tilde{C}_i$). The Artin-Mumford theorem for $\mathbb{P}^2$ can now be stated as :

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	?	?	?	$\cdots$
0	$Ker\ \beta/Im\ \alpha$	$Coker\ \beta$	0	$\cdots$
0	0	0	0	$\cdots$
μ_n	0	0	0	$\cdots$

FIGURE 8. Second term for a smooth rational projective X.

If Δ is a central simple $\mathbb{C}(x,y)$-algebra of dimension n^2, then Δ determines uniquely a $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$. Conversely, any $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$ determines a unique division $\mathbb{C}(x,y)$- algebra whose class in the Brauer group has order n.

EXAMPLE 98. The Brauer group of a smooth irreducible rational projective variety X of dimension d. Using the fact that the birational invariants $F_n^{i,j}(\mathbb{C}(X)/\mathbb{C})$ vanish when $i > 0$ we deduce, with notation as below for the second row of the first term of the coniveau spectral sequence

$$H^2_{et}(\mathbb{C}(X).\mu_n) \xrightarrow{\alpha} \bigoplus_{codim(H)=1} H^1_{et}(\mathbb{C}(H), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\beta} \bigoplus_{codimh=2} \mu_n^{-1}$$

that the second term is given by figure 8 The terms on the third diagonal are also the limiting terms. That is, by the coniveau spectral theorem we have that the obstruction to the exactness of the sequence

$$0 \longrightarrow H^2_{et}(\mathbb{C}(X).\mu_n) \xrightarrow{\alpha} \bigoplus_{codim(H)=1} H^1_{et}(\mathbb{C}(H), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\beta} \bigoplus_{codimh=2} \mu_n^{-1}$$

in the H^1-term is isomorphic to the étale cohomology group $H^3_{et}(X, \mu_n)$. That is, this group describes the obstruction to $\mathbb{Z}_n$-wrinkles on X describing $Br_n(\mathbb{C}(X))$.

5.2. Etale slices.

In this section we will prove that the étale local structure of `alg`-smooth algebras is determined by path algebras of quivers. The proof uses the étale slice theorem due to Domingo Luna [**44**]. We start by recalling the formulation of the slice theorem in differential geometry.

Let M be a compact $\mathcal{C}^\infty$-manifold with a smooth action of a compact Lie group G. By the usual averaging process we can define a G-invariant Riemannian metric on M. For a point $m \in M$ we define

- The G-orbit $\mathcal{O}(m) = G.m$ of m in M,
- the stabilizer subgroup $H = Stab_G(m) = \{g \in G \mid g.m = m\}$ and
- the normal space N_m defined to be the orthogonal complement to the tangent space in m to the orbit in the tangent space to M. That is, we

have a decomposition of H-vectorspaces

$$T_m\ M = T_m\ \mathcal{O}(m) \oplus N_m$$

The normal spaces N_x when x varies over the points of the orbit $\mathcal{O}(m)$ define a vectorbundle $\mathcal{N} \xrightarrow{p} \mathcal{O}(m)$ over the orbit. We identify the bundle with the associated fiber bundle

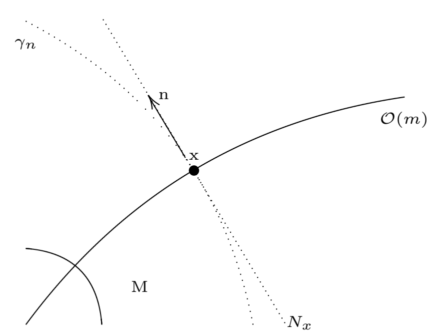
$$\mathcal{N} \simeq G \times^H N_m$$

Any point $n \in \mathcal{N}$ in the normal bundle determines a geodesic

$$\gamma_n : \mathbb{R} \longrightarrow M \quad \text{defined by} \quad \begin{cases} \gamma_n(0) & = p(n) \\ \frac{d\gamma_n}{dt}(0) & = n \end{cases}$$

Using this geodesic we define a G-equivariant exponential map from the normal bundle $\mathcal{N}$ to the manifold M via

$$\mathcal{N} \xrightarrow{exp} M \qquad \text{where} \qquad exp(n) = \gamma_n(1)$$



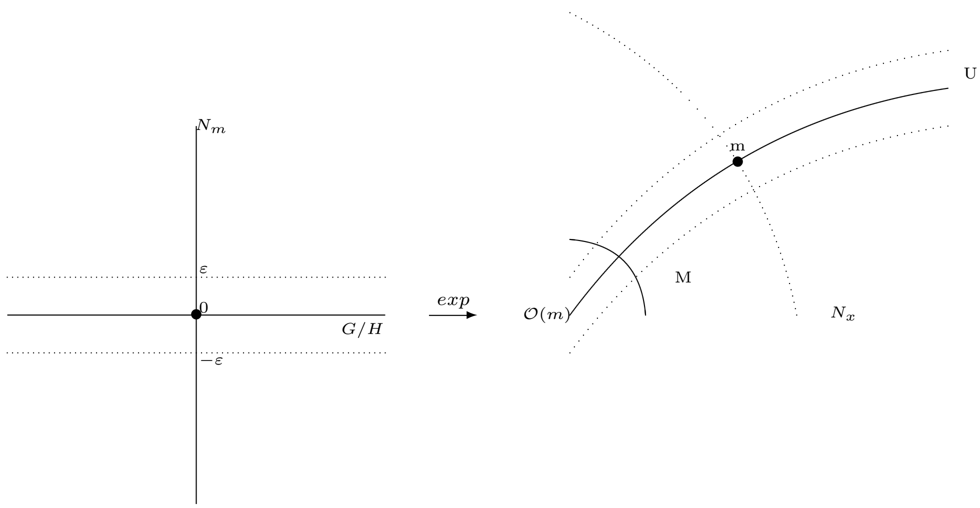
Now, take $\varepsilon > 0$ and define the $\mathcal{C}^\infty$ slice S_ε to be

$$S_\varepsilon = \{n \in N_m \ \mid \parallel n \parallel < \varepsilon \ \}$$

then $G \times^H S_\varepsilon$ is a G-stable neighborhood of the zero section in the normal bundle $\mathcal{N} = G \times^H N_m$. But then we have a G-equivariant exponential

$$G \times^H S_\varepsilon \xrightarrow{exp} M$$

which for small enough ε gives a diffeomorphism with a G-stable tubular neighborhood U of the orbit $\mathcal{O}(m)$ in M.



If we assume moreover that the action of G on M and the action of H on N_m are such that the orbit-spaces are manifolds M/G and N_m/H, then we have the situation

$$\begin{array}{ccccc} G \times^H S_\varepsilon & \xrightarrow[\simeq]{exp} & U & \hookrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ S_\varepsilon/H & \xrightarrow[\simeq]{} & U/G & \hookrightarrow & M/G \end{array}$$

giving a local diffeomorphism between a neighborhood of $\overline{0}$ in N_m/H and a neighborhood of the point $\overline{m}$ in M/G corresponding to the orbit $\mathcal{O}(m)$.

We want to have a similar description for the action of GL_n by basechange on the representation scheme $\mathtt{rep}_n A$ for an affine algebra $A \in \mathtt{alg}$. Because the exponential map is not a morphism in algebraic geometry we would like to replace it by an étale map. Moreover, as the étale slices relate the local structure of two quotient varieties, it is natural to restrict to points in which the stabilizer subgroup is again a reductive group. This is the case for *closed* orbits. Surprisingly, these mild restrictions allow the existence of an algebraic (étale) slice. This was first proved by Domingo Luna [**44**] in the case of reduced varieties and later, in general, by Friedrich Knop [**31**]. Because representation schemes are often not reduced we will follow Knop's proof in the special case of interest to us, that is, when the acting group is GL_n (or a product $GL(\alpha) = GL_{e_1} \times \ldots \times GL_{e_k}$).

We fix the following setting : $\mathtt{afX}$ and $\mathtt{afY}$ will be two affine GL_n-schemes and $\mathtt{afY} \xrightarrow{\psi} \mathtt{afX}$ will be a GL_n-equivariant morphism. Consider points $y \in \mathtt{afY}$ and

$x = \psi(y) \in \mathtt{afX}$. We have the diagram of quotients

$$\begin{array}{ccccccc} x = \psi(y) & \in & \mathtt{afX} & \xleftarrow{\psi} & \mathtt{afY} & \ni & y \\ & & \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} & & \\ & & \mathtt{afX}/GL_n & & \mathtt{afY}/GL_n & & \end{array}$$

and assume the following restrictions :

- ψ is étale in y,
- the GL_n-orbits $\mathcal{O}(y)$ in $\mathtt{afY}$ and $\mathcal{O}(x)$ in $\mathtt{afX}$ are closed.
- the stabilizer subgroups are equal $Stab(x) = Stab(y)$.

In algebraic terms : consider the coordinate rings $R = \mathbb{C}[\mathtt{afX}]$ and $S = \mathbb{C}[\mathtt{afY}]$ and the dual morphism $R \xrightarrow{\psi^*} S$. Let $I \triangleleft R$ be the ideal describing the Zariski closed set $\mathcal{O}(x)$ and $J \triangleleft S$ the ideal describing $\mathcal{O}(y)$. Let $\widehat{R} = \underset{\leftarrow}{lim} \frac{R}{I^n}$ and $\widehat{S} = \underset{\leftarrow}{lim} \frac{S}{J^n}$ be the I-adic resp. J-adic completions.

Theorem 67. *With notations and restrictions as above, we have :*

(1) *The morphism ψ^* induces an isomorphism*

$$\frac{R}{I^n} \xrightarrow{\psi^*} \frac{S}{J^n}$$

for all n. In particular, $\widehat{R} \simeq \widehat{S}$.

(2) *There are natural numbers $m \geq 1$ (independent of the type $s \in \mathtt{simp}GL_n$) and $n \geq 0$ such that*

$$I^{mk+n} \cap R_{(s)} \hookrightarrow (I^{GL_n})^k R_{(s)} \hookrightarrow I^k \cap R_{(s)}$$

for all $k \in \mathbb{N}$.

(3) *The morphism ψ^* induces an isomorphism*

$$R \otimes_{R^{GL_n}} \widehat{R^{GL_n}} \xrightarrow{\simeq} S \otimes_{S^{GL_n}} \widehat{S^{GL_n}}$$

where $\widehat{R^{GL_n}}$ is the I^{GL_n}-adic completion of R^{GL_n} and $\widehat{S^{GL_n}}$ the J^{GL_n}-adic completion of S^{GL_n}.

Proof. (1) : Let $\mathtt{afZ}$ be the closed GL_n-stable subvariety of $\mathtt{afY}$ where ψ is not étale. By the separation property, there is an invariant function $f \in S^{GL_n}$ vanishing on $\mathtt{afZ}$ such that $f(y) = 1$ because the two closed GL_n-subschemes $\mathtt{afZ}$ and $\mathcal{O}(y)$ are disjoint. Replacing S by S_f we may assume that ψ^* is an étale morphism. Because $\mathcal{O}(x)$ is smooth, $\psi^{-1}\,\mathcal{O}(x)$ is the disjoint union of its irreducible components and restricting $\mathtt{afY}$ if necessary we may assume that $\psi^{-1}\,\mathcal{O}(x) = \mathcal{O}(y)$. But then $J = \psi^*(I)S$ and as $\mathcal{O}(y) \xrightarrow{\simeq} \mathcal{O}(x)$ we have $\frac{R}{I} \simeq \frac{S}{J}$ so the result holds for $n = 1$.

Because étale maps are flat, we have $\psi^*(I^n)S = I^n \otimes_R S = J^n$ and an exact sequence

$$0 \longrightarrow I^{n+1} \otimes_R S \longrightarrow I^n \otimes_R S \longrightarrow \frac{I^n}{I^{n+1}} \otimes_R S \longrightarrow 0$$

But then we have

$$\frac{I^n}{I^{n+1}} = \frac{I^n}{I^{n+1}} \otimes_{R/I} \frac{S}{J} = \frac{I^n}{I^{n+1}} \otimes_R S \simeq \frac{J^n}{J^{n+1}}$$

and the result follows from induction on n and the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{I^n}{I^{n+1}} & \longrightarrow & \frac{R}{I^{n+1}} & \longrightarrow & \frac{R}{I^n} & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \vdots & & \downarrow \simeq & & \\
0 & \longrightarrow & \frac{J^n}{J^{n+1}} & \longrightarrow & \frac{S}{J^{n+1}} & \longrightarrow & \frac{S}{J^n} & \longrightarrow & 0
\end{array}$$

(2) : Consider $A = \oplus_{i=0}^{\infty} I^n t^n \hookrightarrow R[t]$, then A^{GL_n} is affine so certainly finitely generated as R^{GL_n}-algebra say by

$$\{r_1 t^{m_1}, \ldots, r_z t^{m_z}\} \quad \text{with } r_i \in R \text{ and } m_i \geq 1.$$

Further, $A_{(s)}$ is a finitely generated A^{GL_n}-module, say generated by

$$\{s_1 t^{n_1}, \ldots, s_y t^{n_y}\} \quad \text{with } s_i \in R_{(s)} \text{ and } n_i \geq 0.$$

Take $m = max\ m_i$ and $n = max\ n_i$ and $r \in I^{mk+n} \cap R_{(s)}$, then $rt^{mk+n} \in A_{(s)}$ and

$$rt^{mk+n} = \sum_j p_j(r_1 t^{m_1}, \ldots, r_z t^{m_z}) s_j t^{n_j}$$

with p_j a homogeneous polynomial of t-degree $mk + n - n_j \geq mk$. But then each monomial in p_j occurs at least with ordinary degree $\frac{mk}{m} = k$ and therefore is contained in $(I^{GL_n})^k R_{(s)} t^{mk+n}$.

(3) : Let s be an irreducible GL_n-module, then the I^{GL_n}-adic completion of $R_{(s)}$ is equal to $\widehat{R_{(s)}} = R_{(s)} \otimes_{R^{GL_n}} \widehat{R^{GL_n}}$. Moreover,

$$\widehat{R}_{(s)} = \underset{\leftarrow}{lim} (\frac{R}{I^k})_{(s)} = \underset{\leftarrow}{lim} \frac{R_{(s)}}{(I^k \cap R_{(s)})}$$

which is the I-adic completion of $R_{(s)}$. By the foregoing lemma both topologies coincide on $R_{(s)}$ and therefore

$$\widehat{R_{(s)}} = \widehat{R}_{(s)} \quad \text{and similarly} \quad \widehat{S_{(s)}} = \widehat{S}_{(s)}$$

Because $\widehat{R} \simeq \widehat{S}$ it follows that $\widehat{R}_{(s)} \simeq \widehat{S}_{(s)}$ from which the result follows as the foregoing holds for all s. □

THEOREM 68. *Take a GL_n-equivariant map* $\mathtt{afY} \xrightarrow{\psi} \mathtt{afX}$*, points $y \in \mathtt{afY}$, $x = \psi(y)$ and assume that ψ is étale in y. Assume that the orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are closed and that ψ is injective on $\mathcal{O}(y)$.*

Then, there is an affine open subset $U \hookrightarrow \mathtt{afY}$ containing y such that

(1) *$U = \pi_Y^{-1}(\pi_Y(U))$ and $\pi_Y(U) = U/GL_n$.*
(2) *ψ is étale on U with affine image.*
(3) *The induced morphism $U/GL_n \xrightarrow{\overline{\psi}} \mathtt{afX}/GL_n$ is étale.*

$$\begin{array}{ccccc} GL_n \times^H N_x & \xleftarrow{GL_n \times^H \phi} & GL_n \times^H \mathtt{afS} & \xrightarrow{\psi} & \mathtt{afX} \\ \downarrow & & \downarrow & & \downarrow \pi \\ N_x/H & \xleftarrow{\phi/H} & \mathtt{afS}/H & \xrightarrow{\psi/GL_n} & \mathtt{afX}/GL_n \end{array}$$

FIGURE 9. Etale slice diagram

(4) *The diagram below is commutative*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & \mathtt{afX} \\ \pi_U \downarrow & & \downarrow \pi_X \\ U/GL_n & \xrightarrow{\overline{\psi}} & \mathtt{afX}/GL_n \end{array}$$

PROOF. By the foregoing lemma we have $\widehat{R^{GL_n}} \simeq \widehat{S^{GL_n}}$ which means that $\overline{\psi}$ is étale in $\pi_Y(y)$. As étaleness is an open condition, there is an open affine neighborhood V of $\pi_Y(y)$ on which $\overline{\psi}$ is étale. If $\overline{R} = R \otimes_{R^{GL_n}} S^{GL_n}$ then the above lemma implies that

$$\overline{R} \otimes_{S^{GL_n}} \widehat{S^{GL_n}} \simeq S \otimes_{S^{GL_n}} \widehat{S^{GL_n}}$$

Let $S^{GL_n}_{loc}$ be the local ring of S^{GL_n} in J^{GL_n}, then as the morphism $S^{GL_n}_{loc} \longrightarrow \widehat{S^{GL_n}}$ is faithfully flat we deduce that

$$\overline{R} \otimes_{S^{GL_n}} S^{GL_n}_{loc} \simeq S \otimes_{S^{GL_n}} S^{GL_n}_{loc}$$

but then there is an $f \in S^{GL_n} - J^{GL_n}$ such that $\overline{R}_f \simeq S_f$. Now, intersect V with the open affine subset where $f \neq 0$ and let U' be the inverse image under π_Y of this set. Remains to prove that the image of ψ is affine. As $U' \xrightarrow{\psi} \mathtt{afX}$ is étale, its image is open and GL_n-stable. By the separation property we can find an invariant $h \in R^{GL_n}$ such that h is zero on the complement of the image and $h(x) = 1$. But then we take U to be the subset of U' of points u such that $h(u) \neq 0$. □

THEOREM 69 (Knop-Luna slice theorem). *Let* $\mathtt{afX}$ *be an affine* GL_n*-scheme with quotient map* $\mathtt{afX} \xrightarrow{\pi} \mathtt{afX}/GL_n$. *Take a point* $x \in \mathtt{afX}$ *such that the orbit* $\mathcal{O}(x)$ *is closed and the stabilizer subgroup* $Stab(x) = H$ *is reductive.*

Then, there is a locally closed affine subscheme $\mathtt{afS} \hookrightarrow \mathtt{afX}$ *(the slice) containing* x *with the following properties*

1. $\mathtt{afS}$ *is an affine* H*-scheme,*
2. *the action map* $GL_n \times \mathtt{afS} \longrightarrow \mathtt{afX}$ *induces an étale* GL_n*-equivariant morphism* $GL_n \times^H \mathtt{afS} \xrightarrow{\psi} \mathtt{afX}$ *with affine image,*
3. *the induced quotient map* ψ/GL_n *is étale*

$$(GL_n \times^H \mathtt{afS})/GL_n \simeq \mathtt{afS}/H \xrightarrow{\psi/GL_n} \mathtt{afX}/GL_n$$

and the right hand side of figure 9 is commutative.

If we assume moreover that $\mathtt{afX}$ *is smooth in* x*, then we can choose the slice* $\mathtt{afS}$ *such that in addition the following properties are satisfied*

(1) $\mathtt{afS}$ *is smooth,*
(2) *there is an* H*-equivariant morphism* $\mathtt{afS} \xrightarrow{\phi} T_x\ \mathtt{afS} = N_x$ *with* $\phi(x) = 0$ *having an affine image,*
(3) *the induced morphism is étale*

$$\mathtt{afS}/H \xrightarrow{\phi/H} N_x/H$$

and the left hand side of figure 9 is commutative.

Proof. Choose a finite dimensional GL_n-subrepresentation V of $\mathbb{C}[\mathtt{afX}]$ that generates the coordinate ring as algebra. This gives a GL_n-equivariant embedding

$$\mathtt{afX} \overset{i}{\hookrightarrow} W = V^*$$

Choose in the vectorspace W an H-stable complement S_0 of $\mathfrak{gl}_n.i(x) = T_{i(x)}\ \mathcal{O}(x)$ and denote $S_1 = i(x) + S_0$ and $\mathtt{afS}_2 = i^{-1}(S_1)$.Then, the diagram below is commutative

$$\begin{array}{ccc} GL_n \times^H \mathtt{afS}_2 & \hookrightarrow & GL_n \times^H S_1 \\ \downarrow \psi & & \downarrow \psi_0 \\ \mathtt{afX} & \overset{i}{\hookrightarrow} & W \end{array}$$

By construction we have that ψ_0 induces an isomorphism between the tangent spaces in $\overline{(1, i(x))} \in GL_n \times^H S_0$ and $i(x) \in W$ which means that ψ_0 is étale in $i(x)$, whence ψ is étale in $\overline{(1, x)} \in GL_n \times^H \mathtt{afS}_2$. By the foregoing theorem we have an affine neighborhood U which must be of the form $U = GL_n \times^H \mathtt{afS}$ giving a slice $\mathtt{afS}$ with the required properties.

Assume that $\mathtt{afX}$ is smooth in x, then S_1 is transversal to $\mathtt{afX}$ in $i(x)$ as

$$T_{i(x)}\ i(\mathtt{afX}) + S_0 = W$$

Therefore, $\mathtt{afS}$ is smooth in x. Again using the separation property we can find an invariant $f \in \mathbb{C}[\mathtt{afS}]^H$ such that f is zero on the singularities of $\mathtt{afS}$ (which is a H-stable closed subscheme) and $f(x) = 1$. Then replace $\mathtt{afS}$ with its affine reduced subvariety of points s such that $f(s) \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}[\mathtt{afS}]$ in x, then we have an exact sequence of H-modules

$$0 \longrightarrow \mathfrak{m}^2 \longrightarrow \mathfrak{m} \xrightarrow{\alpha} N_x^* \longrightarrow 0$$

Choose a H-equivariant section $\phi^* : N_x^* \longrightarrow \mathfrak{m} \hookrightarrow \mathbb{C}[\mathtt{afS}]$ of α then this gives an H-equivariant morphism $\mathtt{afS} \xrightarrow{\phi} N_x$ which is étale in x. Applying the foregoing theorem to this setting finishes the proof. □

In order to apply this slice machinery to the case of interest to us, we give a representation theoretic interpretations in case the affine GL_n-scheme is $\mathtt{rep}_n A$ for $A \in \mathtt{alg}$. We have seen that an orbit $\mathcal{O}(M)$ is closed if and only if M is a semi-simple representation, say with decomposition

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

The stabilizer subgroup in M is isomorphic to $GL(\alpha) = GL_{e_1} \times \ldots \times GL_{e_k}$. The normal space we will identify with $Ext^1_A(M, M)$ and we will see that the action of the stabilizer subgroup on it is the same as the action of $GL(\alpha)$ on the local quiver setting determined by M.

DEFINITION 74. For $A \in \texttt{alg}$, let M and N be two representations of dimensions m and n. A representation P of dimension $m+n$ is said to be an *extension of N by M* if there exists a short exact sequence of left A-modules

$$e: \qquad 0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

Define an equivalence relation on extensions (P,e) of N by M : $(P,e) \cong (P',e')$ if and only if there is an isomorphism $P \xrightarrow{\phi} P'$ of left A-modules such that the diagram below is commutative

$$\begin{array}{lccccccccc} e: & 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle id_M} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id_N} & & \\ e': & 0 & \longrightarrow & M & \longrightarrow & P' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

The set of equivalence classes of extensions of N by M will be denoted by $Ext^1_A(N,M)$.

EXAMPLE 99. An alternative description of $Ext^1_A(N,M)$ is as follows. Let $\rho : A \longrightarrow M_m(\mathbb{C})$ and $\sigma : A \longrightarrow M_n(\mathbb{C})$ be the representations defining M and N. For an extension (P,e) we identify the $\mathbb{C}$-vectorspace with $M \oplus N$ and the A-module structure on P gives a algebra map $\mu : A \longrightarrow M_{m+n}(\mathbb{C})$. We represent the action of a on P by left multiplication of the block-matrix

$$\mu(a) = \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix},$$

where $\lambda(a)$ is an $m \times n$ matrix and hence defines a linear map

$$\lambda : A \longrightarrow Hom_{\mathbb{C}}(N,M).$$

The condition that μ is an algebra morphism is equivalent to the condition

$$\lambda(aa') = \rho(a)\lambda(a') + \lambda(a)\sigma(a')$$

and we denote the set of all liner maps $\lambda : A \longrightarrow Hom_{\mathbb{C}}(N,M)$ by $Z(N,M)$ and call it the space of *cycle* .

The extensions of N by M corresponding to two cycles λ and λ' from $Z(N,M)$ are equivalent if and only if there is an A-module isomorphism in block form

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{with } \beta \in Hom_{\mathbb{C}}(N,M)$$

between them. A-linearity of this map translates to the matrix relation

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \cdot \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix} = \begin{bmatrix} \rho(a) & \lambda'(a) \\ 0 & \sigma(a) \end{bmatrix} \cdot \begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{for all } a \in A$$

or equivalently, that $\lambda(a) - \lambda'(a) = \rho(a)\beta - \beta\sigma(a)$ for all $a \in A$. We will define the subspace of $Z(N,M)$ of *boundaries* $B(N,M)$

$$\{\delta \in Hom_{\mathbb{C}}(N,M) \mid \exists \beta \in Hom_{\mathbb{C}}(N,M) : \forall a \in A : \delta(a) = \rho(a)\beta - \beta\sigma(a)\}.$$

Therefore, $Ext^1_A(N,M) = \frac{Z(N,M)}{B(N,M)}$.

In general, extensions between representations are more difficult to compute than homomorphisms. However, there is one important case where the two are related, path algebras of quivers. Recall that the *Euler form* of a quiver Q on k vertices is the bilinear form on $\mathbb{Z}^k$

$$\chi_Q(.,.) : \mathbb{Z}^k \times \mathbb{Z}^k \longrightarrow \mathbb{Z} \quad \text{defined by} \quad \chi_Q(\alpha,\beta) = \alpha.\chi_Q.\beta^\tau$$

for all row vectors $\alpha, \beta \in \mathbb{Z}^k$.

THEOREM 70. *Let V resp. W be representations of $\langle Q \rangle$ of dimension vector α resp. β, then*

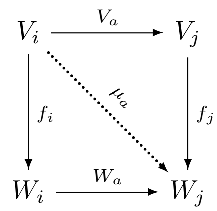
$$dim_{\mathbb{C}}\ Hom_{\langle Q\rangle}(V,W) - dim_{\mathbb{C}}\ Ext^1_{\langle Q\rangle}(V,W) = \chi_Q(\alpha,\beta)$$

PROOF. There is an exact sequence of $\mathbb{C}$-vectorspaces

$$0 \longrightarrow Hom_{\langle Q\rangle}(V,W) \xrightarrow{\gamma} \oplus_{v_i \in Q_v} Hom_{\mathbb{C}}(V_i,W_i) \xrightarrow{d^V_W}$$

$$\xrightarrow{d^V_W} \oplus_{a \in Q_a} Hom_{\mathbb{C}}(V_{s(a)},W_{t(a)}) \xrightarrow{\epsilon} Ext^1_{\langle Q\rangle}(V,W) \longrightarrow 0$$

Here, $\gamma(\phi) = (\phi_1,\ldots,\phi_k)$ and d^V_W maps a family of linear maps $(f_1,\ldots,f_k)$ to the linear maps $\mu_a = f_jV_a - W_af_i$ for any arrow $(j) \xleftarrow{a} (i)$ in Q, that is, to the obstruction of the following diagram to be commutative



By the definition of morphisms between representations of Q it is clear that the kernel of d^V_W coincides with $Hom_{\langle Q\rangle}(V,W)$.

The map ϵ is defined by sending a family of maps $(g_1,\ldots,g_s) = (g_a)_{a\in Q_a}$ to the equivalence class of the exact sequence

$$0 \longrightarrow W \xrightarrow{i} E \xrightarrow{p} V \longrightarrow 0$$

where for all $v_i \in Q_v$ we have $E_i = W_i \oplus V_i$ and the inclusion i and projection map p are the obvious ones and for each arrow $a \in Q_a$ the action of a on E is defined by the matrix

$$E_a = \begin{bmatrix} W_a & g_a \\ 0 & V_a \end{bmatrix} : E_i = W_i \oplus V_i \longrightarrow W_j \oplus V_j = E_j$$

This makes E into a $\langle Q\rangle$-representation and one verifies that the above short exact sequence is one of $\langle Q\rangle$-representations. Remains to prove that the cokernel of d^V_W can be identified with $Ext^1_{\langle Q\rangle}(V,W)$.

A set of algebra generators of $\langle Q\rangle$ is given by $\{v_1,\ldots,v_k,a_1,\ldots,a_l\}$. A cycle is given by a linear map $\lambda : \langle Q\rangle \longrightarrow Hom_{\mathbb{C}}(V,W)$ such that for all $f, f' \in \mathbb{C}Q$ we have the condition

$$\lambda(ff') = \rho(f)\lambda(f') + \lambda(f)\sigma(f')$$

where ρ determines the action on W and σ that on V. For any v_i the condition is $\lambda(v_i^2) = \lambda(v_i) = p^W_i\lambda(v_i) + \lambda(v_i)p^V_i$ whence $\lambda(v_i) : V_i \longrightarrow W_i$ but then applying again the condition we see that $\lambda(v_i) = 2\lambda(v_i)$ so $\lambda(v_i) = 0$. Similarly, for the arrow

$\textcircled{j} \xleftarrow{a} \textcircled{i}$ the condition on $a = v_j a = a v_i$ implies that $\lambda(a) : V_i \longrightarrow W_j$. That is, we can identify $\oplus_{a \in Q_a} Hom_{\mathbb{C}}(V_i, W_j)$ with $Z(V, W)$ under the map ϵ. Moreover, the image of δ gives rise to a family of morphisms $\lambda(a) = f_j V_a - W_a f_i$ for a linear map $f = (f_i) : V \longrightarrow W$ so this image coincides precisely to the subspace of boundaries $B(V, W)$ proving that indeed the cokernel of d^V_W is $Ext^1_{\langle Q \rangle}(V, W)$.

If $dim(V) = \alpha = (r_1, \ldots, r_k)$ and $dim(W) = \beta = (s_1, \ldots, s_k)$, then $dim\ Hom(V, W) - dim\ Ext^1(V, W)$ is equal to

$$\begin{aligned}
&\sum_{v_i \in Q_v} dim\ Hom_{\mathbb{C}}(V_i, W_i) - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} dim\ Hom_{\mathbb{C}}(V_i, W_j) \\
&= \sum_{v_i \in Q_v} r_i s_i - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} r_i s_j \\
&= (r_1, \ldots, r_k) \chi_Q (s_1, \ldots, s_k)^\tau = \chi_Q(\alpha, \beta)
\end{aligned}$$

finishing the proof. □

EXAMPLE 100. Two α-dimensional representations of $\langle Q \rangle$ are isomorphic if and only if they belong to the same orbit under $GL(\alpha)$. Therefore,

$$Stab_{GL(\alpha)}\ V \simeq Aut_{\langle Q \rangle}\ V$$

and the latter is an open subvariety of the affine space $End_{\langle Q \rangle}(V) = Hom_{\langle Q \rangle}(V, V)$ whence they have the same dimension. The dimension of the orbit $\mathcal{O}(V)$ of V in $\mathtt{rep}_\alpha\ Q$ is equal to

$$dim\ \mathcal{O}(V) = dim\ GL(\alpha) - dim\ Stab_{GL(\alpha)}\ V.$$

We have a geometric reformulation of the previous theorem

$$dim\ \mathtt{rep}_\alpha\ Q - dim\ \mathcal{O}(V) = dim\ End_{\langle Q \rangle}(V) - \chi_Q(\alpha, \alpha) = dim\ Ext^1_{\langle Q \rangle}(V, V)$$

Indeed, $dim\ \mathtt{rep}_\alpha\ Q - dim\ \mathcal{O}(V)$ is equal to

$$\sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} d_i d_j - (\sum_i d_i^2 - dim\ End_{\langle Q \rangle}(V)) = dim\ End_{\langle Q \rangle}(V) - \chi_Q(\alpha, \alpha)$$

and by the foregoing theorem the latter term is equal to $dim\ Ext^1_{\langle Q \rangle}(V, V)$. In particular it follows that the orbit $\mathcal{O}(V)$ is *open* in $\mathtt{rep}_\alpha\ Q$ if and only if V has no self-extensions. As $\mathtt{rep}_\alpha\ Q$ is irreducible there can be at most one isomorphism class of a representation without self-extensions.

Because $\mathtt{rep}_\alpha\ Q$ is smooth, the previous example shows that the self-extensions $Ext^1_{\langle Q \rangle}(V, V)$ have the same dimension as the normal space to the orbit in V. We will now show that, in general, the normal space is isomorphic (as representation over the stabilizer subgroup) to the space of self-extensions.

EXAMPLE 101. Let A be an affine $\mathbb{C}$-algebra generated by $\{a_1, \ldots a_m\}$ and $\rho : A \longrightarrow M_n(\mathbb{C})$ an algebra morphism, that is, $\rho \in \mathtt{rep}_n A$. We call a linear map $A \xrightarrow{D} M_n(\mathbb{C})$ a *ρ-derivation* if and only if for all $a, a' \in A$

$$D(aa') = D(a).\rho(a') + \rho(a).D(a').$$

Denote the vectorspace of all ρ-derivations of A by $Der_\rho(A)$. Observe that any ρ-derivation is determined by its image on the generators a_i, hence $Der_\rho(A) \subset M_n^m$. We claim that

$$T_\rho(\mathtt{rep}_n A) = Der_\rho(A).$$

We know that $\mathtt{rep}_n A(\mathbb{C}[\varepsilon])$ is the set of algebra morphisms

$$A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$$

By the foregoing characterization of tangentspaces, $T_\rho(\mathtt{rep}_n A)$ is equal to

$$\{D : A \longrightarrow M_n(\mathbb{C}) \text{ linear} \mid \rho + D\varepsilon : A \longrightarrow M_n(\mathbb{C}[\varepsilon]) \text{ is an algebra map}\}.$$

Because ρ is an algebra morphism, the algebra map condition

$$\rho(aa') + D(aa')\varepsilon = (\rho(a) + D(a)\varepsilon).(\rho(a') + D(a')\varepsilon)$$

is equivalent to D being a ρ-derivation.

Let $\mathtt{afX} \xrightarrow{\phi} \mathtt{afY}$ be a morphism of affine schemes corresponding to the algebra morphism $\mathbb{C}[\mathtt{afY}] \xrightarrow{\phi^*} \mathbb{C}[\mathtt{afX}]$. Let x be a geometric point of $\mathtt{afX}$ and $y = \phi(x)$. Because $\phi^*(m_y) \subset m_x$, ϕ induces a linear map $\frac{m_y}{m_y^2} \longrightarrow \frac{m_x}{m_x^2}$ and taking the dual map gives the *differential of ϕ in x* which is a linear map

$$d\phi_x : T_x(\mathtt{afX}) \longrightarrow T_{\phi(x)}(\mathtt{afY}).$$

Let $D \in T_x(\mathtt{afX}) = Der_x(\mathbb{C}[\mathtt{afX}])$ and x_D the corresponding element of $\mathtt{afX}(\mathbb{C}[\varepsilon])$ defined by $x_D(f) = f(x) + D(f)\varepsilon$, then $x_D \circ \phi^* \in \mathtt{afY}(\mathbb{C}[\varepsilon])$ is

$$x_D \circ \phi^*(g) = g(\phi(x)) + (D \circ \phi^*)\varepsilon = g(\phi(x)) + d\phi_x(D)\varepsilon$$

giving us the ε-interpretation of the differential

$$\phi(x + v\varepsilon) = \phi(x) + d\phi_x(v)\varepsilon$$

for all $v \in T_x(\mathtt{afX})$.

Example 102. Let $X \xrightarrow{\phi} Y$ be a dominant morphism between irreducible affine varieties. There is a Zariski open dense subset $U \hookrightarrow X$ such that $d\phi_x$ is surjective for all $x \in U$.

Indeed, we may assume that ϕ factorizes into

$$\begin{array}{ccc} X & \xrightarrow{\rho} & Y \times \mathbb{C}^d \\ & \searrow^{\phi} & \downarrow{\scriptstyle pr_Y} \\ & & Y \end{array}$$

with ϕ a finite and surjective morphism. Because the tangent space of a product is the sum of the tangent spaces of the components we have that $d(pr_W)_z$ is surjective for all $z \in Y \times \mathbb{C}^d$, hence it suffices to verify the claim for a *finite* morphism ϕ. That is, we may assume that $S = \mathbb{C}[Y]$ is a finite module over $R = \mathbb{C}[X]$ and let L/K be the corresponding extension of the function fields. By the *principal element theorem* we know that $L = K[s]$ for an element $s \in L$ which is integral over R with minimal polynomial

$$F = t^n + g_{n-1}t^{n-1} + \ldots + g_1 t + g_0 \quad \text{with } g_i \in R$$

Consider the ring $S' = R[t]/(F)$ then there is an element $r \in R$ such that the localizations S'_r and S_r are isomorphic. By restricting we may assume that $X = \mathbb{V}(F) \hookrightarrow Y \times \mathbb{C}$ and that

$$\begin{array}{ccc} X = \mathbb{V}(F) & \hookrightarrow & Y \times \mathbb{C} \\ & \searrow^{\simeq} & \downarrow pr_Y \\ & & Y \end{array}$$

Let $x = (y, c) \in X$ then we have (again using the identification of the tangent space of a product with the sum of the tangent spaces of the components) that

$$T_x(X) = \{(v, a) \in T_y(Y) \oplus \mathbb{C} \mid c\frac{\partial F}{\partial t}(x) + vg_{n-1}c^{n-1} + \ldots + vg_1c + vg_0 = 0\}.$$

But then, $d\phi_x$ i surjective whenever $\frac{\partial F}{\partial t}(x) \neq 0$. This condition determines a *non-empty* open subset of X as otherwise $\frac{\partial F}{\partial t}$ would belong to the defining ideal of X in $\mathbb{C}[Y \times \mathbb{C}]$ (which is the principal ideal generated by F) which is impossible by a degree argument

Example 103. Let $\mathtt{afX}$ be a closed GL_n-stable subscheme of a GL_n-representation V and x a geometric point of $\mathtt{afX}$. Consider the orbitclosure $\overline{\mathcal{O}(x)}$ of x in V. As the orbit map

$$\mu : \ GL_n \twoheadrightarrow GL_n.x \hookrightarrow \overline{\mathcal{O}(x)}$$

is dominant we have that $\mathbb{C}[\overline{\mathcal{O}(x)}] \hookrightarrow \mathbb{C}[GL_n]$ and hence a domain, so $\overline{\mathcal{O}(x)}$ is an irreducible affine variety. The *stabilizer subgroup* $Stab(x)$ is the fiber $\mu^{-1}(x)$ and is a closed subgroup of GL_n. We claim that the differential of the orbit map in the identity matrix $e = \mathbb{1}_n$

$$d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\mathtt{afX})$$

satisfies the following properties

$$Ker\ d\mu_e = \mathfrak{stab}(x) \quad \text{and} \quad Im\ d\mu_e = T_x(\overline{\mathcal{O}(x)}).$$

By the previous example we know that there is a dense open subset U of GL_n such that $d\mu_g$ is surjective for all $g \in U$. By GL_n-equivariance of μ it follows that $d\mu_g$ is surjective for all $g \in GL_n$, in particular $d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\overline{\mathcal{O}(x)})$ is surjective. Further, all fibers of μ over $\mathcal{O}(x)$ have the same dimension. It follows from the *dimension formula* that

$$dim\ GL_n = dim\ Stab(x) + dim\ \overline{\mathcal{O}(x)}$$

Combining this with the above surjectivity, a dimension count proves that $Ker\ d\mu_e = \mathfrak{stab}(x)$, the Lie algebra of $Stab(x)$.

Example 104. (The normalspace to orbitclosures in $\mathtt{rep}_n A$) Let A be an affine $\mathbb{C}$-algebra generated by $\{a_1, \ldots, a_m\}$ and $\rho : A \longrightarrow M_n(\mathbb{C})$ an algebra morphism determining the n-dimensional A-representation M. We have the following description of the *normal space* to the orbitclosure $C_\rho = \overline{\mathcal{O}(\rho)}$ of ρ

$$N_\rho(\mathtt{rep}_n A) \overset{def}{=} \frac{T_\rho(\mathtt{rep}_n A)}{T_\rho(C_\rho)} = Ext^1_A(M, M).$$

We have already seen that the space of cycles $Z(M,M)$ is the space of ρ-derivations of A in $M_n(\mathbb{C})$, $Der_\rho(A)$, which we know to be the tangent space $T_\rho(\mathtt{rep}_n A)$.

Moreover, we know that the differential $d\mu_e$ of the orbit map $GL_n \xrightarrow{\mu} C_\rho \hookrightarrow M_n^m$

$$d\mu_e : \quad \mathfrak{gl}_n = M_n \longrightarrow T_\rho(C_\rho)$$

is surjective. $\rho = (\rho(a_1), \ldots, \rho(a_m)) \in M_n^m$ and the action of action of GL_n is given by simultaneous conjugation. But then we have for any $M \in \mathfrak{gl}_n = M_n$ that

$$(I_n + M\varepsilon).\rho(a_i).(I_n - M\varepsilon) = \rho(a_i) + (M\rho(a_i) - \rho(a_i)M)\varepsilon.$$

By definition of the differential we have that

$$d\mu_e(M)(a) = M\rho(a) - \rho(a)M \qquad \text{for all } a \in A.$$

that is, $d\mu_e(M) \in B(M,M)$ and as by surjectivity we conclude $T_\rho(C_\rho) = B(M,M)$.

We have now all information to apply the Knop-Luna slice theorem to the setting of representation schemes.

Definition 75. For $A \in \mathtt{alg}$ be an affine algebra generated by $\{a_1, \ldots, a_m\}$, let $\xi \in \mathtt{iss}_n A$ be a point of the quotient variety and $M_\xi \in \mathtt{rep}_n A$ the n-dimensional semisimple A-module corresponding to it. We can decompose M_ξ into simple components

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

with the S_i distinct simple A-representations, say of dimension d_i. In particular we have

$$n = d_1 e_1 + \ldots + d_k e_k$$

Choosing a basis of M_ξ adapted to this decomposition gives us a point $x = (X_1, \ldots, X_m) \in M_n^m$ in the orbit $\mathcal{O}(M_\xi)$ such that

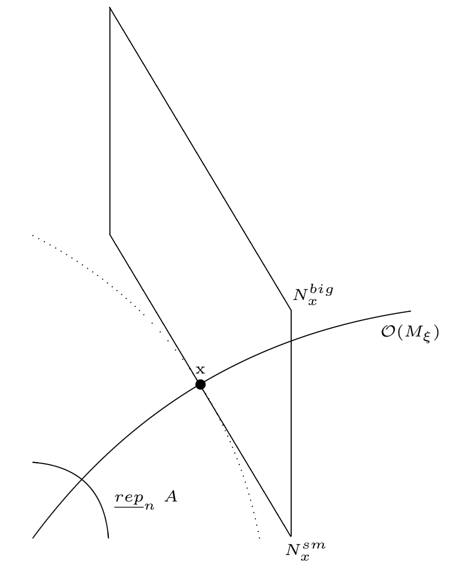
$$X_i = \begin{bmatrix} m_1^{(i)} \otimes \mathbb{1}_{e_1} & 0 & \ldots & 0 \\ 0 & m_2^{(i)} \otimes \mathbb{1}_{e_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_k^{(i)} \otimes \mathbb{1}_{e_k} \end{bmatrix}$$

with each $m_j^{(i)} \in M_{d_j}(\mathbb{C})$. The stabilizer subgroup $Stab(x)$ of GL_n are those invertible matrices $g \in GL_n$ commuting with every X_i. By Schur's lemma we have that the $Stab(x)$ is isomorphic to $GL(\alpha) = GL_{e_1} \times \ldots \times GL_{e_k} = GL(\alpha_\xi)$ for the dimension vector $\alpha_\xi = (e_1, \ldots, e_k)$ determined by the multiplicities of the simple components of M_ξ. The embedding of $Stab(x)$ into GL_n (in the chosen basis) is given by

$$GL(\alpha) = \begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{d_1}) & & \\ & \ddots & \\ & & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{d_k}) \end{bmatrix} \hookrightarrow GL_n$$

We say that $\xi \in \mathtt{iss}_n A$ (or that $M_\xi \in \mathtt{rep}_n A$ is of *representation type*

$$\tau = (e_1, d_1; \ldots; e_k, d_k)$$



$$N_x^{sm} = \frac{T_x\ \mathtt{rep}_n\ A}{T_x\ \mathcal{O}(M_\xi)} \triangleleft N_x^{big} = \frac{T_x\ M_n^m}{T_x\ \mathcal{O}(M_\xi)}$$

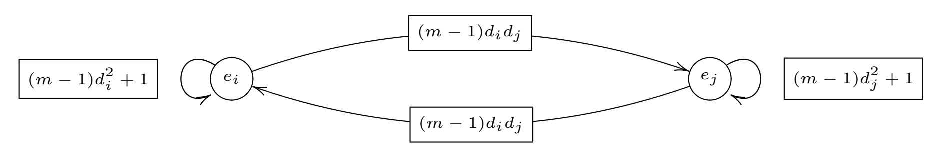
FIGURE 10. Big and small normal spaces to the orbit.

We know that the normal space N_x^{sm} can be identified with the self-extensions $Ext^1_A(M,M)$ and we will give a quiver-description of this space. The idea is to describe first the $GL(\alpha)$-module structure of N_x^{big}, the normal space to the orbit $\mathcal{O}(M_\xi)$ in $\mathtt{rep}_n\langle m\rangle = M_n^m$ (see figure 10) and then to identify the direct summand N_x^{sm}.

THEOREM 71. *Let $\xi \in \mathtt{iss}_n A$ be of representation type $\tau = (e_1, d_1; \ldots ; e_k, d_k)$ and let $\alpha = (e_1, \ldots, e_k)$. The $GL(\alpha)$-module structure of the normal space N_x^{big} in $\mathtt{rep}_n\langle m\rangle = M_n^m$ to the orbit of the semi-simple n-dimensional representation $\mathcal{O}(M_\xi)$ is isomorphic to*

$$\mathtt{rep}_\alpha\ Q_\xi^{big}$$

where the quiver Q_ξ^{big} has k vertices (the number of distinct simple summands of M_ξ) and the subquiver on any two vertices v_i, v_j for $1 \leq i \neq j \leq k$ has the following shape



That is, in each vertex v_i there are $(m-1)d_i^2 + 1$-loops and there are $(m-1)d_i d_j$ arrows from vertex v_i to vertex v_j for all $1 \leq i \neq j \leq k$.

PROOF. The description of N_x^{big} follow from a book-keeping operation involving $GL(\alpha)$-representations. For $x = (X_1, \ldots, X_m)$, the tangent space $T_x\ \mathcal{O}(M_\xi)$ in M_n^m

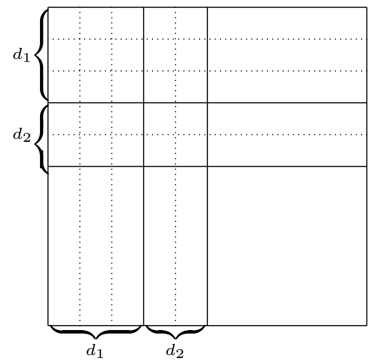


FIGURE 11. Decomposition of the $GL(\alpha)$-action on M_n.

to the orbit is equal to the image of the linear map

$$\begin{array}{ccc} \mathfrak{gl}_n = M_n & \longrightarrow & M_n \oplus \ldots \oplus M_n = T_x\ M_n^m \\ A & \mapsto & ([A, X_1], \ldots, [A, X_m]) \end{array}$$

Observe that the kernel of this map is the centralizer of the subalgebra generated by the X_i, so we have an exact sequence of $Stab(x) = GL(\alpha)$-modules

$$0 \longrightarrow \mathfrak{gl}(\alpha) = Lie\ GL(\alpha) \longrightarrow \mathfrak{gl}_n = M_n \longrightarrow T_x\ \mathcal{O}(x) \longrightarrow 0$$

Because $GL(\alpha)$ is a reductive group every $GL(\alpha)$-module is completely reducible and so the sequence splits. But then, the normal space in $M_n^m = T_x\ M_n^m$ to the orbit is isomorphic as $GL(\alpha)$-module to

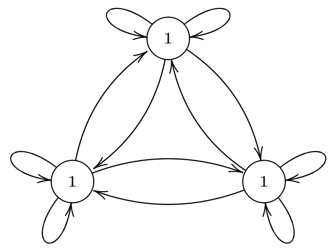
$$N_x^{big} = \underbrace{M_n \oplus \ldots \oplus M_n}_{m-1} \oplus \mathfrak{gl}(\alpha)$$

with the action of $GL(\alpha)$ (embedded as above in GL_n) is given by simultaneous conjugation. If we consider the $GL(\alpha)$-action on M_n we see that it decomposes into a direct sum of subrepresentations (see figure 11)

- for each $1 \leq i \leq k$ we have d_i^2 copies of the $GL(\alpha)$-module M_{e_i} on which GL_{e_i} acts by conjugation and the other factors of $GL(\alpha)$ act trivially,
- for all $1 \leq i, j \leq k$ we have $d_i d_j$ copies of the $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ acts via $g.m = g_i m g_j^{-1}$ and the other factors of $GL(\alpha)$ act trivially.

These $GL(\alpha)$ components are precisely the modules appearing in representation spaces of quivers. $\square$

EXAMPLE 105. If $m = 2$ and $n = 3$ and the representation type is $\tau = (1, 1; 1, 1; 1, 1)$ (that is, M_ξ is the direct sum of three distinct one-dimensional simple representations) then the quiver Q_ξ is



THEOREM 72. *Let $A \in$ `alg` be an affine algebra generated by m elements. Let $\xi \in$ $\mathtt{iss}_n A$ be a point of representation type*

$$\tau = (e_1, d_1; \ldots ; e_k, d_k)$$

and $M_\xi \in \mathtt{rep}_n A$ a corresponding semisimple n-dimensional A-module.

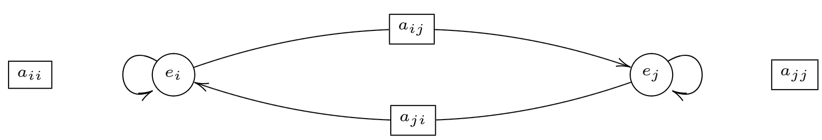
The normal space N_x^{sm} in a point $x \in \mathcal{O}(M_\xi)$ to the orbit in $\mathtt{rep}_n A$ is isomorphic as module over the stabilizer subgroup

$$Stab(x) = GL(\alpha) = GL_{e_1} \times \ldots \times GL_{e_k}$$

(with $\alpha = (e_1, \ldots, e_k)$) to the representation space

$$\mathtt{rep}_\alpha\ Q_\xi$$

where the local quiver *Q_ξ has k vertices (corresponding to the distinct simple components of M_ξ) and is such that for any two vertices $v_i \neq v_j$ the full subquiver is of the form*



where

$$a_{ij} = dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j) \leq (m-1) d_i d_j + \delta_{ij}$$

for all $1 \leq i, j \leq k$.

PROOF. We have GL_n-equivariant embeddings

$$\mathcal{O}(M_\xi) \hookrightarrow \mathtt{rep}_n A \hookrightarrow \mathtt{rep}_n \langle m \rangle = M_n^m$$

and corresponding embeddings of the tangent spaces in x

$$T_x\ \mathcal{O}(M_\xi) \hookrightarrow T_x\ \mathtt{rep}_n A \hookrightarrow T_x\ M_n^m$$

Because $GL(\alpha)$ is reductive, the normal spaces to the orbit is a direct summand of $GL(\alpha)$-modules.

$$N_x^{sm} = \frac{T_x\ \mathtt{rep}_n A}{T_x\ \mathcal{O}(M_\xi)} \triangleleft N_x^{big} = \frac{T_x\ M_n^m}{T_x\ \mathcal{O}(M_\xi)}$$

The isotypical decomposition of N_x^{big} as the $GL(\alpha)$-module $rep_\alpha\ Q_\xi$ allows us to control N_x^{sm}. On the other hand we know that

$$N_x^{sm} = Ext^1_A(M_\xi, M_\xi) = \oplus_{1 \leq i,j \leq k} Ext^1_A(S_i, S_j)^{\oplus e_i e_j}$$

and a comparison finishes the proof. $\square$

$$\begin{array}{ccccc}
GL_n \times^{GL(\alpha)} \mathtt{rep}_{\alpha_\xi}\ Q_\xi & \xleftarrow{GL_n \times^{GL(\alpha)} \phi} & GL_n \times^{GL(\alpha)} S_x & \xrightarrow{\psi} & \mathtt{rep}_n A \\
\downarrow & & \downarrow & & \downarrow \\
\mathtt{iss}_\alpha\ Q_\xi & \xleftarrow{\phi/GL(\alpha)} & S_x/GL(\alpha) & \xrightarrow{\psi/GL(\alpha)} & \mathtt{iss}_n A
\end{array}$$

FIGURE 12. Slice diagram for $\mathtt{rep}_n A$.

We have all the necessary ingredients to complete the prove of the étale local structure of alg-algebras. This result can be seen as analogous to the fact that manifolds are locally affine spaces. In the case of alg-smooth algebras, path algebras of quivers play the role of noncommutative affine spaces. Observe that the étale local structure result can be proved whenever the semi-simple representation is a smooth point of $\mathtt{rep}_n A$.

DEFINITION 76. For $A \in \mathtt{alg}$ let $\xi \in \mathtt{iss}_n A$ be a geometric point with corresponding n-dimensional semisimple module $M_\xi \in \mathtt{rep}_n A$. ξ is said to belong to the *n-th smooth locus* $\mathtt{smooth}_n A$ of A iff $\mathtt{rep}_n A$ is smooth at M_ξ. If A is alg-smooth, then $\mathtt{smooth}_n A = \mathtt{iss}_n A$ for all n.

DEFINITION 77. For $A \in \mathtt{alg}$ and $\mathfrak{m} \triangleleft \oint_n A$ we denote with

$$\oint_n^{\hat{\mathfrak{m}}} A \qquad \text{(resp. with } \int_n^{\hat{\mathfrak{m}}} A)$$

the $\mathfrak{m}$-adic *completion* of $\oint_n A$ (resp. of $\int_n A$).

The following result implies in particular that alg-smooth algebras are locally (in the étale topology) determined by path algebras of quivers.

THEOREM 73. *Let $A \in \mathtt{alg}$ and $\xi \in \mathtt{smooth}_n A$ be a point of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ with corresponding maximal ideal $\mathfrak{m} \triangleleft \oint_n A$. Let Q_ξ be the local quiver and $\alpha = (e_1, \ldots, e_k)$ and let $\mathfrak{m}_0$ be the maximal ideal of $\oint_\alpha \langle Q_\xi \rangle$ corresponding to the trivial representation $0 \in \mathtt{rep}_\alpha\ Q_\xi$. Then,*

$$\oint_n^{\hat{\mathfrak{m}}} A \simeq \oint_\alpha^{\hat{\mathfrak{m}}_0} \langle Q_\xi \rangle \qquad \textit{and} \qquad \int_n^{\hat{\mathfrak{m}}} A \between \int_\alpha^{\hat{\mathfrak{m}}_0} \langle Q_\xi \rangle$$

Moreover, the Morita equivalence is determined by the embedding of the stabilizer subgroup $GL(\alpha)$ in GL_n.

PROOF. Consider the slice diagram of figure 12 for the representation scheme $\mathtt{rep}_n A$. The left hand side exists because $x \in \mathcal{O}(M_\xi)$ is a smooth point of $\mathtt{rep}_n A$, the right hand side exists always. The horizontal maps are étale and the upper ones GL_n-equivariant.

By theorem 72 we know that the normal space to the orbit N_x^{sm} is isomorphic to $\mathtt{rep}_\alpha\ Q_\xi$ from which the first claim follows. To prove the second, observe that the algebra of GL_n-equivariant maps

$$GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ Q_\xi \longrightarrow M_n(\mathbb{C})$$

is Morita equivalent to the algebra of $GL(\alpha)$-equivariant maps

$$\mathtt{rep}_\alpha\ Q_\xi \longrightarrow M_{|\alpha|}(\mathbb{C})$$

where $|\alpha| = e_1 + \ldots + e_k$. □

Example 106. Let X be a smooth affine curve and $A = \mathbb{C}[X]$. The only simple A-representations are one-dimensional and correspond to a point $x \in X$,S_x. We have for all $x, y \in X$

$$Ext^1_A(S_x, S_y) = \delta_{xy}\mathbb{C}$$

We know from example 79 that $\mathtt{iss}_n\ A \simeq X^{(n)}$ so take a point ξ with corresponding semisimple representation

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

with $e_1 + \ldots + e_k = n$. The local quiver Q_ξ has the form

and the dimension vector is $\alpha_\xi = (e_1, \ldots, e_k)$. The quotient variety of this quiver-setting is

$$\mathtt{iss}_{\alpha_\xi}\ Q_\xi \simeq \mathbb{C}^{(e_1)} \times \ldots \times \mathbb{C}^{(e_k)}$$

and we see that the étale map

$$\mathtt{iss}_{\alpha_\xi}\ Q_\xi \longrightarrow \mathtt{iss}_n A = X^{(n)}$$

is in general not an isomorphism in the Zariski topology, but a finite cover.

Even when the left hand sides of the slice diagrams are not defined for $\xi \notin \mathtt{smooth}_n A$ the dimension of the normal spaces to the orbit give a numerical measure of the 'badness' of the noncommutative singularity.

Definition 78. Let $A \in \mathtt{alg}$ be an affine algebra and $\xi \in \mathtt{iss}_n A$ a point of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ with corresponding semisimple representation $x = M_\xi \in \mathtt{rep}_n A$. The *measure of singularity* in ξ is given by the non-negative number

$$ms(\xi) = n^2 + dim_{\mathbb{C}}\ Ext^1_A(M_\xi, M_\xi) - e_1^2 - \ldots - e_k^2 - dim_x\ \mathtt{rep}_n A$$

Clearly, $\xi \in \mathtt{smooth}_n A$ if and only if $ms(\xi) = 0$.

5.3. Smooth models.

In this section we will illustrate how the étale local structure given in the previous section can be combined with the étale cohomological description of Brauer groups to characterize the central simple algebras allowing an `alg@n`-smooth model.

Definition 79. Let Σ be a central simple algebra of dimension n^2 over its center K which is a field of transcendence degree d. We say that Σ has a *smooth model* if there is a projective variety X (not necessarily smooth) with $\mathbb{C}(X) = K$ and a sheaf of $\mathcal{O}_X$-orders $\mathcal{A}$ in Σ such that for an affine open cover $\{U_i\}$ of X we have that

$$A_i = \Gamma(U_i, \mathcal{A})$$

is `alg@n`-smooth for all i.

Our strategy to arrive at a characterization is the following. First we determine the étale local structure of $\mathtt{alg@n}$-smooth algebras using only minor modifications to the arguments used in the previous section. For low dimensions we are then able to give a complete description of all local quiver-settings which do arise. Computing the witness algebras we obtain information on the étale splitting behavior and on the local ramification locus of $\mathtt{alg@n}$-smooth orders. This information can then be combined with the coniveau spectral sequence to give necessary conditions on the classes $[\Sigma]$ allowing an $\mathtt{alg@n}$-smooth model. In the case of surfaces we can even give a complete characterization.

We begin by giving variants of the étale local structure for Cayley-Hamilton algebras. Again, this comes down to describing the normal space to a closed orbit in $\mathtt{trep}_n A$.

EXAMPLE 107. Let $(A, tr_A) \in \mathtt{alg@n}$ and trace generated by $\{a_1, \ldots, a_m\}$. Let $\rho \in \mathtt{trep}_n A$, that is, $\rho : A \longrightarrow M_n(\mathbb{C})$ is a *trace preserving* algebra morphism. As $\mathtt{trep}_n A(\mathbb{C}[\varepsilon])$ is the set of all trace preserving algebra morphisms $A \longrightarrow M_n(\mathbb{C}[\varepsilon])$ (with the usual trace map tr on $M_n(\mathbb{C}[\varepsilon])$) one verifies using the foregoing example that

$$T_\rho(\mathtt{trep}_n A) = Der^{tr}_\rho(A) \subset Der_\rho(A)$$

the subset of *trace preserving ρ-derivations* D, that is, those satisfying

$$D \circ tr_A = tr \circ D \qquad \begin{array}{ccc} A & \xrightarrow{D} & M_n(\mathbb{C}) \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr} \\ A & \xrightarrow{D} & M_n(\mathbb{C}) \end{array}$$

Again, because A is *trace* generated by $\{a_1, \ldots, a_m\}$, a trace preserving ρ-derivation is determined by its image on the a_i and is a subspace of M_n^m.

EXAMPLE 108. (The normalspace to orbitclosures in $\mathtt{trep}_n A$) Let $(A, tr_A) \in \mathtt{alg@n}$ be trace generated by $\{a_1, \ldots, a_m\}$. Let $\rho \in \mathtt{trep}_n A$, that is, $\rho : A \longrightarrow M_n(\mathbb{C})$ is a trace preserving algebra morphism. Any cycle $\lambda : A \longrightarrow M_n(\mathbb{C})$ in $Z(M, M) = Der_\rho(A)$ determines an algebra morphism

$$\rho + \lambda\varepsilon : A \longrightarrow M_n(\mathbb{C}[\varepsilon])$$

We know that the tangent space $T_\rho(\mathtt{trep}_n A)$ is the subspace $Der^{tr}_\rho(A)$ of trace preserving ρ-derivations, that is, those satisfying

$$\lambda(tr_A(a)) = tr(\lambda(a)) \qquad \text{for all } a \in A$$

Observe that all boundaries $\delta \in B(M, M)$, that is, such that there is an $m \in M_n(\mathbb{C})$ with $\delta(a) = \rho(a).m - m.\rho(a)$ are trace preserving as

$$\begin{aligned} \delta(tr_A(a)) = {} & \rho(tr_A(a)).m - m.\rho(tr_A(a)) = tr(\rho(a)).m - m.tr(\rho(a)) \\ = {} & 0 = tr(m.\rho(a) - \rho(a).m) = tr(\delta(a)) \end{aligned}$$

Hence, we can define the space of *trace preserving self-extensions*

$$Ext^{tr}_A(M, M) = \frac{Der^{tr}_\rho(A)}{B(M, M)}$$

Then, as before we have that the normal space to the orbit closure $C_\rho = \overline{\mathcal{O}(\rho)}$ is equal to

$$N_\rho(\mathtt{trep}_n A) \overset{def}{=} \frac{T_\rho(\mathtt{trep}_n A)}{T_\rho(C_\rho)} = Ext^{tr}_A(M, M)$$

Definition 80. A *marked quiver* $Q^\bullet$ is a finite quiver Q such that some of its loops are marked. If α is a dimension vector for Q, the space of *marked quiver representations* of dimension vector α

$$\mathtt{rep}_\alpha\ Q^\bullet$$

is the subspace of $\mathtt{rep}_\alpha\ Q$ consisting of all representations such that the square matrices corresponding to marked loops have trace zero.

Theorem 74. *Let $(A, tr_A) \in$ `alg@n` be trace generated by m elements. Let $\xi \in \mathtt{tiss}_n A$ be a point of representation type*

$$\tau = (e_1, d_1; \ldots ; e_k, d_k)$$

and $M_\xi \in \mathtt{trep}_n A$ a corresponding semisimple n-dimensional A-module.

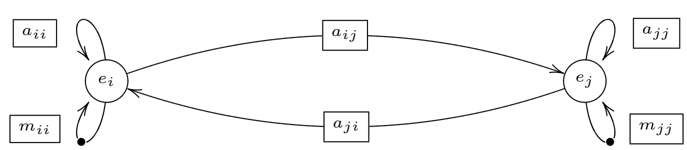
The normal space N^{sm}_x in a point $x \in \mathcal{O}(M_\xi)$ to the orbit in $\mathtt{trep}_n A$ is isomorphic as module over the stabilizer subgroup

$$Stab(x) = GL(\alpha) = GL_{e_1} \times \ldots \times GL_{e_k}$$

(with $\alpha = (e_1, \ldots, e_k)$) to the representation space

$$\mathtt{rep}_\alpha\ Q^\bullet_\xi$$

where the marked local quiver *$Q^\bullet_\xi$ has k vertices (corresponding to the distinct simple components of M_ξ) and is such that for any two vertices $v_i \neq v_j$ the full subquiver is of the form*



where

$$a_{ij} = dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j) \leq (m-1)d_i d_j$$

for all $1 \leq i \neq j \leq k$ and the (marked) vertex loops are determined by the structure of $Ext^{tr}_A(M_\xi, M_\xi)$.

Proof. We only have to observe that arrows in the local quiver Q_ξ of theorem 72 correspond to simple $GL(\alpha)$-modules, whereas a loop at vertex v_i decomposes as $GL(\alpha)$-module into the simples

$$M_{e_i} = M^0_{e_i} \oplus \mathbb{C}_{triv}$$

where $\mathbb{C}_{triv}$ is the one-dimensional simple with trivial $GL(\alpha)$-action and $M^0_{e_i}$ is the space of trace zero matrices in M_{e_i}.

Any $GL(\alpha)$-submodule of N^{big}_x can be thus represented by a marked quiver using the dictionary

- a loop at vertex v_i corresponds to the $GL(\alpha)$-module M_{e_i} on which GL_{e_i} acts by conjugation and the other factors act trivially,

- a marked loop at vertex v_i corresponds to the simple $GL(\alpha)$-module $M^0_{e_i}$ on which GL_{e_i} acts by conjugation and the other factors act trivially,
- an arrow from vertex v_i to vertex v_j corresponds to the simple $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ acts via $g.m = g_i m g_j^{-1}$ and the other factors act trivially,

Combining this with the calculation that the normalspace is the space of trace preserving self-extensions $Ext^{tr}_A(M_\xi, M_\xi)$ we obtain the result. □

With $\texttt{smooth}A$ we denote the set of points $\xi \in \texttt{tiss}_n A$ such that M_ξ is a smooth point of $\texttt{trep}_n A$.

THEOREM 75. *Let $(A, tr_A) \in \texttt{alg@n}$ and $\xi \in \texttt{smooth}A$ be a point of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ with corresponding maximal ideal $\mathfrak{m} \triangleleft tr(A)$. Let $Q^\bullet_\xi$ be the marked local quiver, $\alpha = (e_1, \ldots, e_k)$ and let $\mathfrak{m}_0$ be the maximal ideal of $\oint_\alpha \langle Q^\bullet_\xi \rangle$ corresponding to the trivial representation $0 \in \texttt{rep}_\alpha\ Q^\bullet_\xi$. Then,*

$$tr(\hat{A})_{\mathfrak{m}} \simeq \oint_\alpha^{\hat{\mathfrak{m}}_0} \langle Q^\bullet_\xi \rangle \qquad \textit{and} \qquad \hat{A}_{\mathfrak{m}} \between \int_\alpha^{\hat{\mathfrak{m}}_0} \langle Q^\bullet_\xi \rangle$$

where the Morita equivalence is determined by the embedding of the stabilizer subgroup $GL(\alpha)$ in GL_n. Moreover, if $\{m_1, \ldots, m_l\}$ is the set of marked loops is $Q^\bullet_\xi$ then

$$\oint_\alpha \langle Q^\bullet_\xi \rangle \simeq \frac{\oint_\alpha \langle Q_\xi \rangle}{(tr(m_1), \ldots, tr(m_l))} \qquad \textit{and} \qquad \int_\alpha \langle Q^\bullet_\xi \rangle \simeq \frac{\int_\alpha \langle Q_\xi \rangle}{(tr(m_1), \ldots, tr(m_l))}$$

DEFINITION 81. For $(A, tr_A) \in \texttt{alg@n}$ and $\xi \in \texttt{tiss}_n A$ of type $\tau = (e_1, d_1; \ldots; e_k, d_k)$. The measure of trace singularity in ξ is given by the non-negative number

$$tms(\xi) = n^2 + dim_{\mathbb{C}}\ Ext^{tr}_A(M_\xi, M_\xi) - e_1^2 - \ldots - e_k^2 - dim_x\ \texttt{trep}_n A$$

Clearly, $\xi \in \texttt{smooth}_n A$ (resp. $\xi \in \texttt{smooth}A$) if and only if $ms(\xi) = 0$ (resp. $tms(\xi) = 0$).

Our next job is to determine in low dimensions d the étale local structure of $\texttt{alg@n}$-smooth orders, or more generally, the étale local structure of an order $A \in \texttt{alg@n}$ in a point $\xi \in \texttt{smooth}A$ of its smoooth locus.

THEOREM 76. *Let $A \in \texttt{alg@n}$ over an affine curve $X = \texttt{iss}_n A$. If $\xi \in \texttt{smooth}A$, the étale local structure of A in ξ is determined by the quiver-setting (Q, α) where Q is an oriented cycle on k vertices with $k \leq n$ and $\alpha = \mathbf{1} = (1, \ldots, 1)$. The Morita setting is determined by an unordered partition $p = (d_1, \ldots, d_k)$ having precisely k parts such that $\sum_i d_i = n$ determining the dimensions of the simple components of M_ξ, see figure 13.*

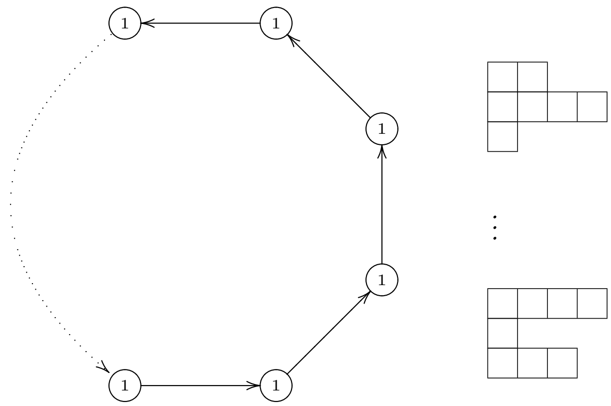


FIGURE 13. Local (marked) quiver-setting of `smooth@n`-algebras over curves.

Further, ξ is a smooth point of $X = \mathtt{iss}_n A$ and the étale local structure of A in ξ is isomorphic to

$$\hat{A}_\xi \simeq \left[\begin{array}{c|c|c|c} M_{d_1}(\mathbb{C}[[x]]) & M_{d_1\times d_2}(\mathbb{C}[[x]]) & \ldots & M_{d_1\times d_k}(\mathbb{C}[[x]]) \\ \hline M_{d_2\times d_1}(x\mathbb{C}[[x]]) & M_{d_2}(\mathbb{C}[[x]]) & \ldots & M_{d_2\times d_k}(\mathbb{C}[[x]]) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_{d_k\times d_1}(x\mathbb{C}[[x]]) & M_{d_k\times d_2}(x\mathbb{C}[[x]]) & \ldots & M_{d_k}(\mathbb{C}[[x]]) \end{array} \right]$$

PROOF. Let $(Q^\bullet, \alpha)$ be the local marked quiver-setting corresponding to $\xi \in \mathtt{smooth}A$. Because $Q^\bullet$ is strongly connected, there exist oriented cycles in $Q^\bullet$. Fix one such cycle of length $s \leq k$ and renumber the vertices of $Q^\bullet$ such that the first s vertices make up the cycle. If $\alpha = (e_1, \ldots, e_k)$, then there exist semi-simple representations in $\mathtt{rep}_\alpha\, Q^\bullet$ with composition

$$\alpha_1 = (\underbrace{1, \ldots, 1}_{s}, \underbrace{0, \ldots, 0}_{k-s}) \oplus \epsilon_1^{\oplus e_1 - 1} \oplus \ldots \oplus \epsilon_s^{\oplus e_s - 1} \oplus \epsilon_{s+1}^{\oplus e_{s+1}} \oplus \ldots \oplus \epsilon_k^{\oplus e_k}$$

where ϵ_i stands for the simple one-dimensional representation concentrated in vertex v_i.

There is a one-dimensional family of simple representations of dimension vector α_1, hence the stratum of semi-simple representations in $\mathtt{iss}_\alpha\, Q^\bullet$ of representation type $\tau = (1, \alpha_1; e_1 - 1, \epsilon_1; \ldots; e_s - 1, \epsilon_s; e_{s+1}, \epsilon_{s+1}; e_k, \epsilon_k)$ is at least one-dimensional. However, as *dim* $\mathtt{iss}_\alpha\, Q^\bullet = 1$ this can only happen if this semi-simple representation is actually simple. That is, $Q = Q^\bullet$, $\alpha = \alpha_1$ and $k = s$ proving the first claim.

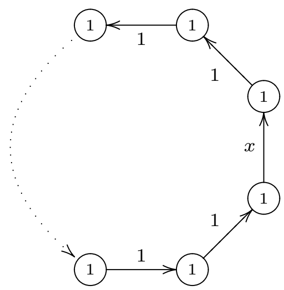
Let M_ξ be the semi-simple n-dimensional representation of A corresponding to ξ, then

$$V_\xi = S_1 \oplus \ldots \oplus S_k \quad \text{with} \quad dim\ S_i = d_i$$

and all S_i distinct. The stabilizer subgroup is $GL(\alpha) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ embedded in GL_n via the diagonal embedding

$$(\lambda_1, \ldots, \lambda_k) \longrightarrow \text{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{d_k})$$

which determines the Morita setting. By basechange in $\mathtt{rep}_\alpha\ Q$ we can bring every simple α-dimensional representation of Q in standard form



where $x \in \mathbb{C}^*$ is the arrow from v_k to v_1.

Therefore, the ring of invariants $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)} \simeq \mathbb{C}[x]$ whence ξ is a smooth point of X by the slice result. Moreover, using the numbering conventions of the vertices the ring of quiver-equivariants has the desired block decomposition. $\square$

Example 109. $\mathtt{alg@n}$-smooth models in dimension one. Let X be a projective curve and $\mathcal{A}$ a sheaf of $\mathcal{O}_X$-orders in a central simple $\mathbb{C}(X)$-algebra Σ of dimension n^2. Then, the following are equivalent

1. $\mathcal{A}$ is a sheaf of $\mathtt{smooth@n}$-algebras, that is, a smooth model of Σ.
2. X is a smooth curve and $\mathcal{A}$ is a sheaf of hereditary $\mathcal{O}_X$-orders.

Smoothness follows from the previous theorem and the above block decomposition combined with the local description of hereditary orders given in [**56**, Thm. 39.14] and étale descent proves the hereditary statement.

Definition 82. Let $(A, tr_A) \in \mathtt{alg@n}$ be a $C = tr_A(A)$-order in a central simple algebra Σ of dimension n^2 over K the field of fractions of C. We say that A is *étale split* in $\xi \in \mathtt{iss}_n A$ if and only if

$$A \otimes_C \hat{K}_\xi \simeq M_n(\hat{K}_\xi)$$

where $\hat{K}_\xi$ is the field of fractions of $\hat{C}_\xi$ the $\mathfrak{m}$-adic completion of C where $\mathfrak{m}$ is the maximal ideal of C corresponding to ξ.

Theorem 77. *Let $A \in \mathtt{alg@n}$ be an order over an affine surface $X = \mathtt{iss}_n A$. If $\xi \in \mathtt{smooth}A$, then the étale local structure of A in ξ is determined by the local quiver-setting (Q, α) where Q is the quiver A_{klm} of figure 14 on $k+l+m \leq n$ vertices and $\alpha = 1 = (1, \ldots, 1)$. The Morita setting is determined by an unordered partition $p = (d_1, \ldots, d_{k+l+m})$ of n with $k+l+m$ non-zero parts determined by the dimensions of the simple components of M_ξ as in figure 14.*

Further, ξ is a smooth point of X, A is étale split in ξ and the étale local structure has the block-decomposition of figure 15 where at spot (i,j) with $1 \leq$

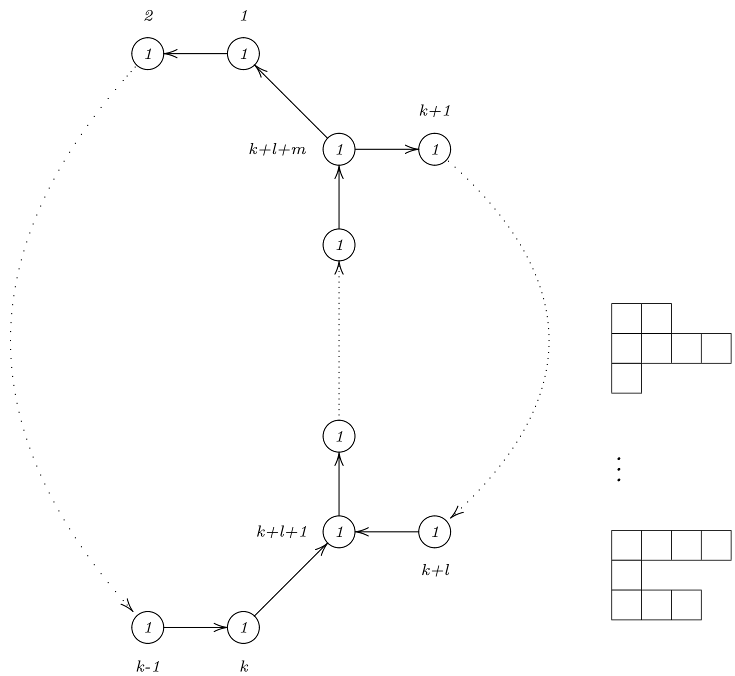


FIGURE 14. Cayley-smooth surface types.

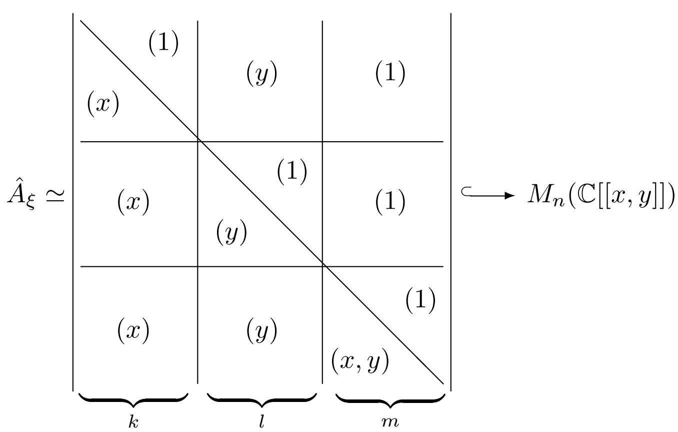


FIGURE 15. Etale local structure of an `alg@n`-smooth order over a surface.

$i, j \leq k+l+m$ there is a block of dimension $d_i \times d_j$ with entries the indicated ideal of $\mathbb{C}[[x,y]]$. In particular, the ramification-type of A in ξ is one of the following :

1. *A is an Azumaya algebra in ξ, or*
2. *ξ is an isolated point (possibly embedded) of the ramification, or*
3. *ξ is a smooth point of the ramification, or*
4. *the ramification has a normal crossing at ξ.*

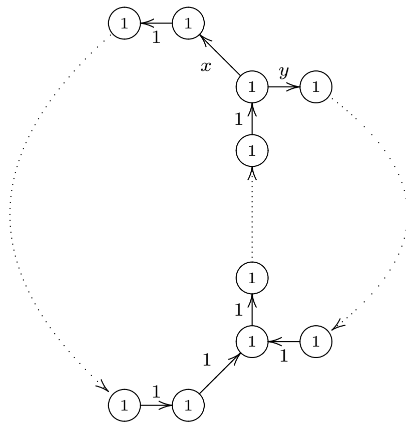


FIGURE 16. Standard form of representations in $\texttt{rep}_1\ A_{klm}$.

PROOF. Let $(Q^\bullet, \alpha)$ be the marked local quiver-setting on r vertices with $\alpha = (e_1, \ldots, e_r)$ corresponding to ξ. As $Q^\bullet$ is strongly connected and the quotient variety is two-dimensional, $Q^\bullet$ must contain more than one oriented cycle, hence it contains a sub-quiver of type A_{klm}, possibly degenerated with k or l equal to zero. Order the first $k+l+m$ vertices of $Q^\bullet$ as indicated then one verifies by theorem 85 that A_{klm} has simple representations of dimension vector $\mathbf{1} = (1, \ldots, 1)$. Assume that A_{klm} is a proper subquiver and $s = k+l+m+1$, then $Q^\bullet$ has semi-simple representations in $\texttt{rep}_\alpha\ Q^\bullet$ of type

$$\alpha_1 = (\underbrace{1, \ldots, 1}_{k+l+m}, 0, \ldots, 0) \oplus \epsilon_1^{\oplus e_1 - 1} \oplus \ldots \oplus \epsilon_{k+l+m}^{\oplus e_{k+l+m}-1} \oplus \epsilon_s^{\oplus e_s} \oplus \ldots \oplus \epsilon_r^{\oplus e_r}$$

The dimension of the quotient variety $\texttt{iss}_1\ A_{klm}$ has dimension 2 so there is a two-dimensional family of such semi-simple representation in the irreducible two-dimensional quotient variety $\texttt{iss}_\alpha\ Q^\bullet$. This is only possible if this semi-simple representation is actually simple, whence $r = k+l+m$, $Q^\bullet = A_{klm}$ and $\alpha = (1, \ldots, 1)$.

If M_ξ is the semi-simple n-dimensional representation of A corresponding to ξ, then

$$M_\xi = S_1 \oplus \ldots \oplus S_r \quad \text{with} \quad dim\ S_i = d_i$$

determining the unordered partition p and the Morita-equivalence because the stabilizer subgroup $GL(\alpha) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ is embedded diagonally in GL_n via

$$(\lambda_1, \ldots, \lambda_r) \mapsto \text{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{d_r})$$

By basechange in $\texttt{rep}_1\ A_{klm}$ every simple α-dimensional representation can be brought in the standard form of figure 16 with $x, y \in \mathbb{C}^*$ and as $\mathbb{C}[\texttt{iss}_1\ A_{klm}] = \mathbb{C}[\texttt{rep}_1\ A_{klm}]^{GL(\alpha)}$ is the ring generated by traces along oriented cycles in A_{klm}, it is isomorphic to $\mathbb{C}[x, y]$ (Alternatively, one can apply theorem 99 to show that the ring of invariants is smooth). It follows from the slice result that ξ is a smooth point of X and that $\hat{A}_\xi$ has the required block-decomposition, in particular A is étale split in ξ.

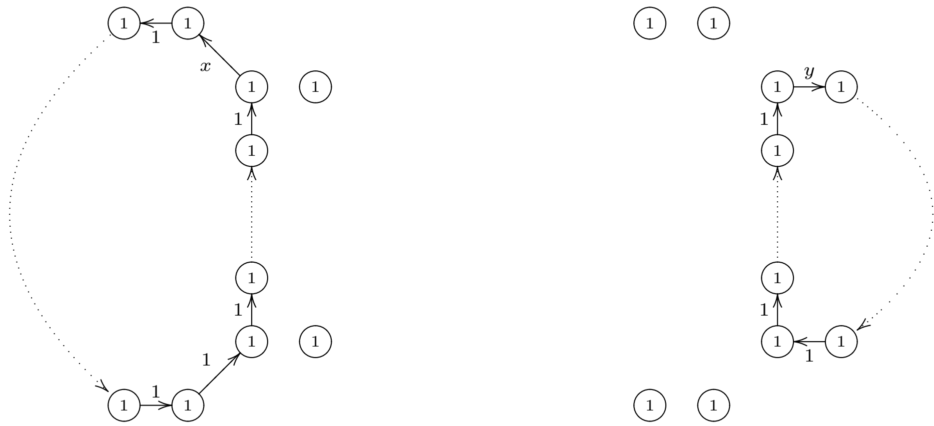


FIGURE 17. Proper semi-simples of A_{klm}.

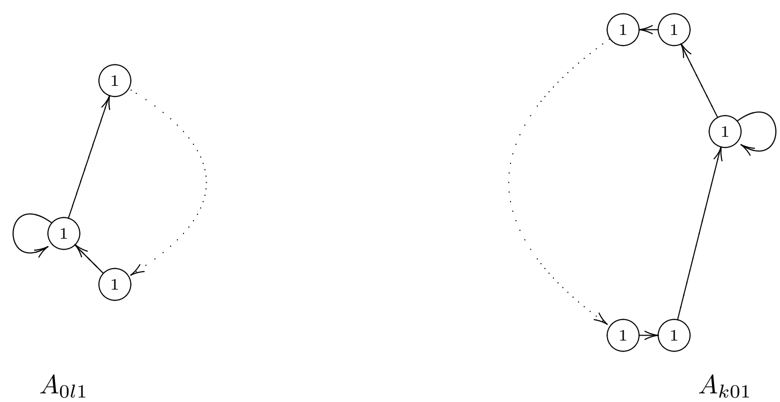


FIGURE 18. Local quivers for A_{klm}.

To prove the ramification statement, we have to compute the local quiver-settings in proper semi-simple representations of $\texttt{rep}_1\ A_{klm}$. Because simples have a strongly connected support, the decomposition types of these proper semi-simples are depicted in figure 17 with $x, y \in \mathbb{C}^*$. The corresponding local quivers local quivers are respectively of the forms in figure 18. Because of the étale local isomorphism between X in a neighborhood of ξ and of $\texttt{iss}_1\ A_{klm}$ in a neighborhood of the trivial representation, the picture of local quiver-settings of A in a neighborhood of ξ is described in figure 19 The Azumaya points are the points in which the quiver-setting is A_{001} (the two-loop quiver). Therefore, the worst case of ramification that can occurs in ξ is that of a normal crossing. The other cases occur for degenerate quiver-settings. □

EXAMPLE 110. $\texttt{alg@n}$-smooth models in dimension two. Let S be a projective surface and $\mathcal{A}$ a sheaf of $\mathcal{O}_S$-algebras in a central simple $\mathbb{C}(S)$-algebra Σ of dimension n^2. If $\mathcal{A}$ is a smooth model of Σ, then the following holds :

(1) S is a smooth surface.

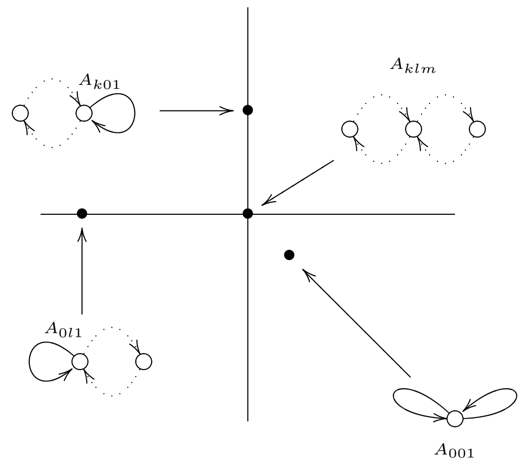


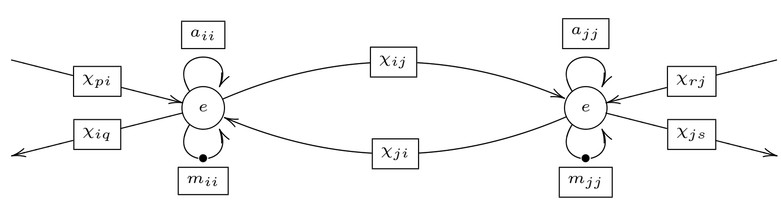
FIGURE 19. Local picture for A_{klm}.

(2) $\mathcal{A}$ is étale split in all points of S.
(3) The ramification locus $\texttt{ram}\mathcal{A} \subset S$ is either empty or consists of a finite number of isolated (possibly embedded) points of S together with a reduced divisor having normal crossings as its worst singularities.

If we want to have similar precise local information on `alg@n`-smooth orders in higher dimensions, we have to compile a list of *admissible* marked quiver settings, that is settings $(Q^\bullet, \alpha)$ satisfying the two properties

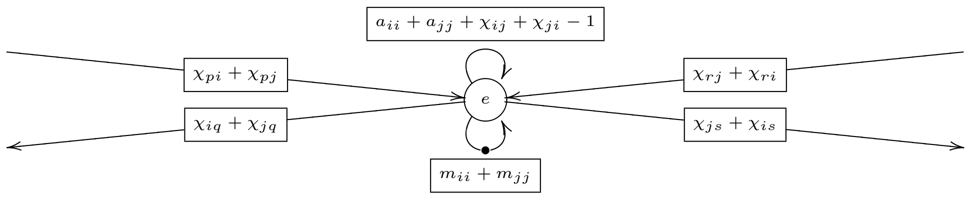
$$\begin{cases} \alpha & \text{is the dimension vector of a simple representation of } Q^\bullet, \text{ and} \\ d & = 1 - \chi_Q(\alpha,\alpha) - \sum_i m_i \end{cases}$$

EXAMPLE 111. The idea is to shrink a marked quiver-setting to its simplest form and classify these simplest forms for given d. By *shrinking* we mean the following process. Let $\alpha = (e_1, \ldots, e_k)$ be the dimension vector of a simple representation of $Q^\bullet$ and let v_i and v_j be two vertices connected with an arrow such that $e_i = e_j = e$. That is, locally we have the following situation



We use one of the arrows connecting v_i with v_j to identify the two vertices. That is, we form the shrinked marked quiver-setting $(Q_s^\bullet, \alpha_s)$ where $Q_s^\bullet$ is the marked quiver on $k-1$ vertices $\{v_1, \ldots, \hat{v}_i, \ldots, v_k\}$ and α_s is the dimension vector with e_i

removed. $Q_s^\bullet$ has the following form in a neighborhood of the contracted vertex



In $Q_s^\bullet$ we have for all $k, l \neq i, j$ that $\chi_{kl}^s = \chi_{kl}$, $a_{kk}^s = a_{kk}$, $m_{kk}^s = m_{kk}$ and the number of arrows and (marked) loops connected to v_j are determined as follows

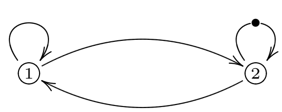
- $\chi_{jk}^s = \chi_{ik} + \chi_{jk}$
- $\chi_{kj}^s = \chi_{ki} + \chi_{kj}$
- $a_{jj}^s = a_{ii} + a_{jj} + \chi_{ij} + \chi_{ji} - 1$
- $m_{jj}^s = m_{ii} + m_{jj}$

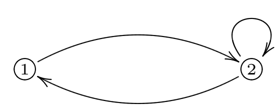
We claim that α is the dimension vector of a simple representation of $Q^\bullet$ if and only if α_s is the dimension vector of a simple representation of $Q_s^\bullet$ and that the dimensions of the corresponding quotient varieties are equal.

Fix an arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$. As $e_i = e_j = e$ there is a Zariski open subset $U \hookrightarrow \mathtt{rep}_\alpha\, Q^\bullet$ of points V such that V_a is invertible. By basechange in either v_i or v_j we can find a point W in its orbit such that $W_a = \mathbb{1}_e$. If we think of W_a as identifying $\mathbb{C}^{e_i}$ with $\mathbb{C}^{e_j}$ we can view the remaining maps of W as a representation in $\mathtt{rep}_{\alpha_s}\, Q_s^\bullet$ and denote it by W^s. The map $U \longrightarrow \mathtt{rep}_{\alpha_s}\, Q_s^\bullet$ is well-defined and maps $GL(\alpha)$-orbits to $GL(\alpha_s)$-orbits. Conversely, given a representation $W' \in \mathtt{rep}_{\alpha_s}\, Q_s^\bullet$ we can uniquely determine a representation $W \in U$ mapping to W'. Both claims follow immediately from this observation.

A marked quiver-setting can uniquely be shrinked to its *simplified form* , which has the characteristic property that no arrow-connected vertices can have the same dimension. The shrinking process has a converse operation which we will call *splitting of a vertex* . However, this splitting operation is usually not uniquely determined.

EXAMPLE 112. Two marked quiver-settings $(Q_1^\bullet, \alpha)$ and $(Q_2^\bullet, \alpha)$ are said to be *equivalent* if and only if their representation spaces $\mathtt{rep}_\alpha\, Q_1^\bullet$ and $\mathtt{rep}_\alpha\, Q_2^\bullet$ are isomorphic $GL(\alpha)$-modules. For example,

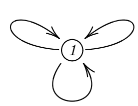


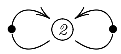
and


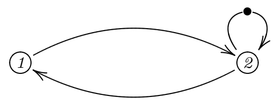
determine the same $\mathbb{C}^* \times GL_2$-module, hence are equivalent.

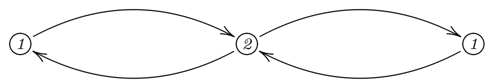
We will merely mention the classification in dimension 3 and 4 and leave the claims as an exercise to the reader.

THEOREM 78. *Let X be a threefold and let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-orders in a central simple $\mathbb{C}(X)$-algebra of dimension n^2. If $\xi \in \mathtt{smooth}\mathcal{A}$, then the local quiver-setting $(Q_\xi^\bullet, \alpha_\xi)$ can be shrinked to one of the following four types*

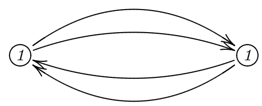
- **type 1 :** 

- **type 2 :** 

- **type 3 :** 

- **type 4 :** 

ξ is a smooth point on X unless $(Q^\bullet_\xi, \alpha_\xi)$ is of type 1 and can be shrinked to a quiver-setting of the form



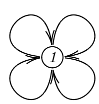
in which case, x is an isolated singularity of X, locally of type $\frac{\mathbb{C}[[u,v,x,y]]}{(uv-xy)}$.
$\mathcal{A}$ is étale split in ξ unless $(Q^\bullet_\xi, \alpha_\xi)$ is of type 2 in which case

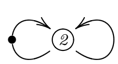
$$\hat{\mathcal{A}}_\xi \between \texttt{Cliff} \begin{bmatrix} x & y \\ y & z \end{bmatrix} \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}[[x,y,z]]$$

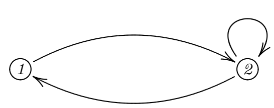
where the Clifford algebra over $\mathbb{C}[x,y,z]$ of the indicated non-degenerate quadratic form is the algebra

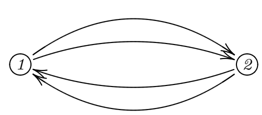
$$\texttt{Cliff} \begin{bmatrix} x & y \\ y & z \end{bmatrix} \simeq \frac{\mathbb{C}\langle a,b\rangle}{(ab^2 - b^2a, a^2b - ba^2)}$$

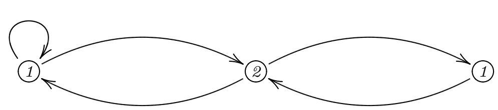
THEOREM 79. *Let X be a fourfold and let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-orders in a central simple $\mathbb{C}(X)$-algebra of dimension n^2. If $\xi \in$ `smooth`$\mathcal{A}$, then the local quiver-setting $(Q^\bullet_\xi, \alpha_\xi)$ can be shrinked to one of the following five equivalence classes of types*

- **type 1 :** 

- **type 2 :** 

- **type 3 :** 

- **type 4 :** 

- **type 5 :** 

Now that we have information on the local ramification locus and the splitting behavior of an `alg@n`-smooth order, the next step is to determine the Brauer classes

0	0	0	0	...
$H^2(K, \mu_n)$	$\oplus_p H^1(k_p, \mathbb{Z}_n)$	μ_n^{-1}	0	...
$H^1(K, \mu_n)$	$\oplus_p \mathbb{Z}_n$	0	0	...
μ_n	0	0	0	...

FIGURE 20. Coniveau spectral sequence for $\mathbb{C}\{x, y\}$.

that have this local behavior. We will perform the calculations in the special case of surfaces, but they can (at least in principle) be generalized to higher dimensions.

THEOREM 80. *Let $\mathbb{C}\{x, y\}$ be the ring of algebraic functions in two variables*

(1) *If $U = \mathtt{spec}\mathbb{C}\{x, y\} - V(x)$, then $Br_n\ U = 0$*
(2) *If $U = \mathtt{spec}\mathbb{C}\{x, y\} - V(xy)$, then $Br_n\ U = \mathbb{Z}_n$ with generator the quantum-plane algebra*

$$\mathbb{C}_\zeta[u, v] = \frac{\mathbb{C}\langle u, v\rangle}{(vu - \zeta uv)}$$

where ζ is a primitive n-th root of one

PROOF. There is only one codimension two subvariety : $m = (x, y)$. Let us compute the coniveau spectral sequence for $\mathtt{spec}\mathbb{C}\{x, y\}$. If K is its field of fractions and if we denote by k_p the field of fractions of $\mathbb{C}\{x, y\}/p$ for p a height one prime, we have the first term as in figure 20 Because $\mathbb{C}\{x, y\}$ is a unique factorization domain, the map

$$H^1_{et}(K, \mu_n) = K^*/(K^*)^n \xrightarrow{\gamma} \oplus_p \mathbb{Z}_n$$

is surjective. Moreover, all fields k_p are isomorphic to the field of fractions of $\mathbb{C}\{z\}$ whose only cyclic extensions are given by adjoining a root of z and hence they are all ramified in m. Therefore, the component maps

$$\mathbb{Z}_n = H^1_{et}(k_p, \mathbb{Z}_n) \xrightarrow{\beta_L} \mu^{-1}$$

are isomorphisms. Hence, we have the form of the second (and limiting) term of the coniveau spectral sequence. Finally, we use the fact that $\mathbb{C}\{x, y\}$ is strict Henselian whence has no proper étale extensions. But then,

$$H^i_{et}(X_{loc}, \mu_n) = 0 \text{ for } i \geq 1$$

and substituting this information in the spectral sequence we obtain that the top sequence of the coniveau spectral sequence

$$0 \longrightarrow Br_n\ K \xrightarrow{\alpha} \oplus_p \mathbb{Z}_n \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

is exact. From this sequence the result follows using the fact (recalled in the next example) that the ramification divisor of the quantum plane is $V(xy)$. □

EXAMPLE 113. (The smooth locus of the quantum plane) Let

$$A = \frac{\mathbb{C}\langle x, y\rangle}{(yx - qxy)}$$

where q is a primitive n-th root of unity. Let $u = x^n$ and $v = y^n$ then A is a free module of rank n^2 over its center $\mathbb{C}[u, v]$. Taking the trace map on the basis

$$tr(x^i y^j) = \begin{cases} 0 & \text{when either } i \text{ or } j \text{ is not a multiple of } n, \\ nx^i y^j & \text{when } i \text{ and } j \text{ are multiples of } n, \end{cases}$$

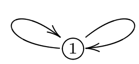
$A \in \mathtt{alg@n}$ with $tr(A) = \mathbb{C}[u, v]$. For $\xi \in \mathtt{iss}_n A = \mathbb{C}^2$ a point (a^n, b) with $a.b \neq 0$, ξ is of representation type $(1, n)$ as the corresponding (semi)simple representation V_ξ is determined by (if m is odd, for even n we replace a by ia and b by $-b$)

$$\rho(x) = \begin{bmatrix} a & & & \\ & qa & & \\ & & \ddots & \\ & & & q^{n-1}a \end{bmatrix} \quad \text{and} \quad \rho(y) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ b & 0 & 0 & \dots & 0 \end{bmatrix}$$

A calculation shows that $Ext^1_A(M_\xi, M_\xi) = \mathbb{C}^2$ where the algebra map $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ corresponding to (α, β) is given by

$$\begin{cases} \phi(x) & = \rho(x) + \varepsilon\ \alpha \mathbb{1}_n \\ \phi(y) & = \rho(y) + \varepsilon\ \beta \mathbb{1}_n \end{cases}$$

and all these algebra maps are trace preserving. That is, $Ext^1_A(M_\xi, M_\xi) = Ext^{tr}_A(M_\xi, M_\xi)$ and because the stabilizer subgroup is $\mathbb{C}^*$ the marked quiver-setting $(Q^\bullet_\xi, \alpha_\xi)$ is



whence $\xi \in \mathtt{smooth}A$, compatible with the fact that over these points the quotient map is a principal PGL_n-fibration.

For $\xi = (a^n, 0)$ with $a \neq 0$ (or, by a similar argument $(0, b^n)$ with $b \neq 0$) the representation type of ξ is $(1, 1; \dots; 1, 1)$ because

$$M_\xi = S_1 \oplus \dots \oplus S_n$$

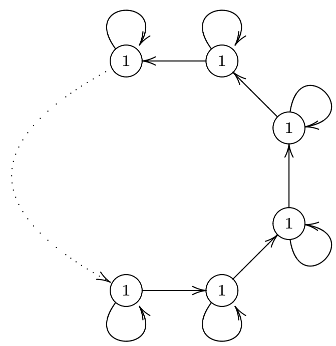
where the simple one-dimensional representation S_i is given by

$$\begin{cases} \rho(x) & = q^i a \\ \rho(y) & = 0 \end{cases}$$

One verifies that

$$Ext^1_A(S_i, S_i) = \mathbb{C} \quad \text{and} \quad Ext^1_A(S_i, S_j) = \delta_{i+1,j}\ \mathbb{C}$$

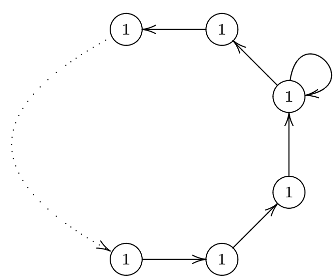
and because the stabilizer subgroup is $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$, the *Ext*-quiver setting is



The algebra map $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ corresponding to the extension $(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) \in Ext^1_A(M_\xi, M_\xi)$ is given by

$$\begin{cases} \phi(x) &= \begin{bmatrix} a + \varepsilon\ \alpha_1 & & & \\ & qa + \varepsilon\ \alpha_2 & & \\ & & \ddots & \\ & & & q^{n-1}a + \varepsilon\ \alpha_n \end{bmatrix} \\ \phi(y) &= \varepsilon \begin{bmatrix} 0 & \beta_1 & 0 & \ldots & 0 \\ 0 & 0 & \beta_2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & \beta_{n-1} \\ \beta_n & 0 & 0 & \ldots & 0 \end{bmatrix} \end{cases}$$

The conditions $tr(x^j) = 0$ for $1 \leq i < n$ impose $n-1$ linear conditions among the α_j, whence the space of trace preserving extensions $Ext^{tr}_A(V_\xi, V_\xi)$ corresponds to the quiver setting



But then, as $\alpha_\xi = (1, \ldots, 1)$

$$1 - \chi_Q(\alpha, \alpha) - \sum_i m_{ii} = 1 - (-1) - 0 = 2 = dim\ \mathtt{iss}_n A$$

whence $\xi \in \mathtt{smooth} A$.

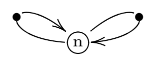
The remaining point is $\xi = (0,0)$ which has representation type $(n, 1)$ as the corresponding semi-simple representation M_ξ is the trivial one. The stabilizer subgroup is GL_n and the (trace preserving) extensions are given by

$$Ext^1_A(M_\xi, M_\xi) = M_n \oplus M_n \quad \text{and} \quad Ext^{tr}_A(M_\xi, M_\xi) = M^0_n \oplus M^0_n$$

determined by the algebra maps $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$

$$\begin{cases} \phi(x) & = \varepsilon\ m_1 \\ \phi(y) & = \varepsilon\ m_2 \end{cases}$$

The local marked quiver-setting $(Q^\bullet_\xi, \alpha_\xi)$ in this point is



$\xi \notin$ $\mathtt{smooth}A$ as the numerical condition fails

$$1 - \chi_Q(\alpha,\alpha) - \sum_i m_{ii} = 1 - (-n^2) - 2 \neq 2 = dim\ \mathtt{iss}_n A$$

That is, $\mathtt{smooth}A = \mathbb{C}^2 - \{(0,0)\}$ and the ramification divisor of A is $V(uv)$.

Let Σ be a central simple K-algebra of dimension n^2 over a field K of transcendence degree 2. If $\mathcal{A}$ is an $\mathtt{alg@n}$-smooth sheaf of $\mathcal{O}_S$-algebras, then we know from example **??** that S is a projective smooth surface, that is, a smooth model for K. By the Artin-Mumford exact sequence, theorem 66, the class of Σ in $Br_n\ \mathbb{C}(S)$ is determined by the following geo-combinatorial data

- A finite collection $\mathcal{C} = \{C_1, \dots, C_k\}$ of *irreducible curves* in S.
- A finite collection $\mathcal{P} = \{P_1, \dots, P_l\}$ of *points* of S where each P_i is either an intersection point of two or more C_i or a singular point of some C_i.
- For each $P \in \mathcal{P}$ the *branch-data* $b_P = (b_1, \dots, b_{i_P})$ with $b_i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\{1, \dots, i_P\}$ the different branches of $\mathcal{C}$ in P. These numbers must satisfy the admissibility condition
$$\sum_i b_i = 0 \in \mathbb{Z}_n$$
for every $P \in \mathcal{P}$
- for each $C \in \mathcal{C}$ we fix a cyclic $\mathbb{Z}_n$-cover of smooth curves
$$D \twoheadrightarrow \tilde{C}$$
of the desingularization $\tilde{C}$ of C which is compatible with the branch-data.

If $\mathcal{B}$ is a maximal $\mathcal{O}_X$-order in Σ, then the ramification locus $\mathtt{ram}\mathcal{B}$ coincides with the collection of curves $\mathcal{C}$.

THEOREM 81. *Let Σ be a central simple K-algebra of dimension n^2 over a field K of transcendence degree 2. Then the following statements hold.*

1. *There is a smooth projective surface S with $\mathbb{C}(S) = K$ such that any maximal $\mathcal{O}_S$-order in Σ has at worst a finite number of noncommutative singularities, all of which are étale locally of quantum-plane type.*
2. *There is a noncommutative smooth model for Σ iff S and $\mathcal{A}$ as in (1) can be chosen such that $\mathtt{ram}\mathcal{A}$ is a disjoint union of smooth curves in S. This holds if and only if for the geo-combinatorial data $(\mathcal{C}, \mathcal{P}, d, \mathcal{D})$ determining $[\Sigma] \in Br_n K$ (in* any *projective smooth model) all branch-data are trivial.*

PROOF. Let X be a projective smooth surface with $\mathbb{C}(X) = K$ and $\mathcal{A}$ a sheaf of maximal $\mathcal{O}_X$-orders in Σ.

claim 1 : For the geo-combinatorial data $(\mathcal{C}, \mathcal{P}, b, \mathcal{D})$ determining the class of Δ in $Br_n\ \mathbb{C}(X)$: if $\xi \in X$ lies in $X - \mathcal{C}$ or if ξ is a non-singular point of $\mathcal{C}$, then $\mathcal{A}$ is $\mathtt{alg@n}$-smooth in ξ.

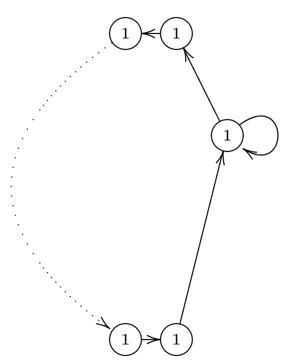
If $\xi \notin \mathcal{C}$, then $\mathcal{A}_\xi$ is an Azumaya algebra over $\mathcal{O}_{X,x}$. As X is smooth in ξ, $\mathcal{A}$ is `alg@n`-smooth in ξ. Alternatively, we know that Azumaya algebras are split by étale extensions, whence $\hat{\mathcal{A}}_\xi \simeq M_n(\mathbb{C}[[x,y]])$ which shows that the behavior of $\mathcal{A}$ near ξ is controlled by the local quiver-setting

and hence $\xi \in$ `smooth`$\mathcal{A}$. If ξ is a nonsingular point of the ramification divisor $\mathcal{C}$, consider the pointed spectrum $X_\xi =$ `spec` $\mathcal{O}_{X,\xi} - \{\mathfrak{m}_\xi\}$. All prime ideals are of height one, corresponding to the curves on X passing through ξ and hence this pointed spectrum is a Dedekind scheme. Further, $\mathcal{A}$ determines a maximal order over X_ξ. But then, tensoring $\mathcal{A}$ with the strict henselization $\mathcal{O}^{sh}_{X,\xi} \simeq \mathbb{C}\{x,y\}$ determines a sheaf of hereditary orders on the pointed spectrum $\hat{X}_\xi = \text{Spec } \mathbb{C}\{x,y\} - \{(x,y)\}$ and we may choose the local variable x such that x is a local parameter of the ramification divisor $\mathcal{C}$ near ξ.

Using the characterization result for hereditary orders over discrete valuation rings, given in [**56**, Thm. 39.14] we know the structure of this extended sheaf of hereditary orders over any height one prime of $\hat{X}_\xi$. Because $\mathcal{A}_\xi$ is a reflexive (even a projective) $\mathcal{O}_{X,\xi}$-module, this height one information determines $\mathcal{A}^{sh}_\xi$ or $\widehat{\mathcal{A}}_\xi$. This proves that $\mathcal{A}^{sh}_\xi$ must be isomorphic to the following blockdecomposition

$$\left[\begin{array}{c|c|c|c}
M_{d_1}(\mathbb{C}\{x,y\}) & M_{d_1\times d_2}(\mathbb{C}\{x,y\}) & \dots & M_{d_1\times d_k}(\mathbb{C}\{x,y\}) \\ \hline
M_{d_2\times d_1}(x\mathbb{C}\{x,y\}) & M_{d_2}(\mathbb{C}\{x,y\}) & \dots & M_{d_2\times d_k}(\mathbb{C}\{x,y\}) \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
M_{d_k\times d_1}(x\mathbb{C}\{x,y\}) & M_{d_k\times d_2}(x\mathbb{C}\{x,y\}) & \dots & M_{d_k}(\mathbb{C}\{x,y\})
\end{array}\right]$$

for a certain unordered partition $p = (d_1, \dots, d_k)$ of n having k parts. (In fact, as we started out with a maximal order $\mathcal{A}$ one can even show that all these integers d_i must be equal.) This corresponds to the local quiver-setting



$$p = (d_1, \dots, d_k)$$

A_{k01}

whence $\xi \in \texttt{smooth}\mathcal{A}$. Hence, a maximal $\mathcal{O}_X$-order in Σ can have at worst non-commutative singularities in the singular points of the ramification divisor $\mathcal{C}$. By changing the smooth model X we can always arrange it that these singularities are at worst normal crossings. To begin, recall the following classical result, see for example [**22**, V.3.8].

(Embedded resolution of curves in surfaces) Let $\mathcal{C}$ be any curve on the surface X. Then, there exists a finite sequence of blow-ups

$$X' = X_s \longrightarrow X_{s-1} \longrightarrow \ldots \longrightarrow X_0 = X$$

and, if $f : X' \twoheadrightarrow X$ is their composition, then the total inverse image $f^{-1}(\mathcal{C})$ is a divisor with normal crossings.

Fix a series of blow-ups $X' \overset{f}{\twoheadrightarrow} X$ such that the inverse image $f^{-1}(\mathcal{C})$ is a divisor on X' having as worst singularities normal crossings. We will replace the $\mathcal{O}_X$-order $\mathcal{A}$ by the $\mathcal{O}_{X'}$-order $\mathcal{A}'$ where $\mathcal{A}'$ is a sheaf of $\mathcal{O}_{X'}$-maximal orders in Σ. In order to determine the ramification divisor of $\mathcal{A}'$ we need to be able to keep track of the ramification divisor $\mathcal{C}$ of Σ through the blow up at a singular point $p \in \mathcal{P}$.

claim 2 : Let $\tilde{X} \twoheadrightarrow X$ be the blow-up of X at a singular point p of $\mathcal{C}$, the ramification divisor of Δ on X. Let $\tilde{\mathcal{C}}$ be the strict transform of $\mathcal{C}$ and E the exceptional line on $\tilde{X}$. Let $\mathcal{C}'$ be the ramification divisor of Δ on the smooth model $\tilde{X}$. Then,

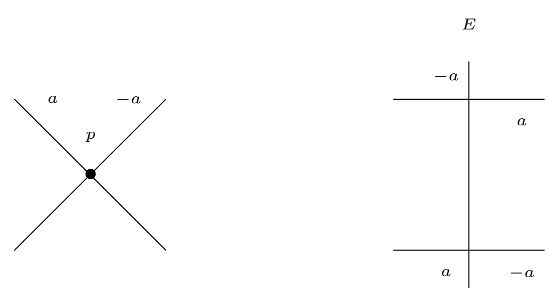
(1) Assume the local branch data at p distribute in an admissible way on $\tilde{\mathcal{C}}$, that is,

$$\sum_{i \text{ at } q} b_{i,p} = 0 \text{ for all } q \in E \cap \tilde{\mathcal{C}}$$

where the sum is taken only over the branches at q. Then, $\mathcal{C}' = \tilde{\mathcal{C}}$.

(2) Assume the local branch data at p do not distribute in an admissible way, then $\mathcal{C}' = \tilde{\mathcal{C}} \cup E$.

Clearly, $\tilde{\mathcal{C}} \hookrightarrow \mathcal{C}' \hookrightarrow \tilde{\mathcal{C}} \cup E$. By the Artin-Mumford sequence applied to X' we know that the branch data of $\mathcal{C}'$ must add up to zero at all points q of $\tilde{\mathcal{C}} \cap E$. We investigate the two cases : (1) : Assume $E \subset \mathcal{C}'$. Then, the E-branch number at q must be zero for all $q \in \tilde{\mathcal{C}} \cap E$. But there are no non-trivial étale covers of $\mathbb{P}^1 = E$ so $ram(\Delta)$ gives the trivial element in $H^1_{et}(\mathbb{C}(E), \mu_n)$, a contradiction. Hence $\mathcal{C}' = \tilde{\mathcal{C}}$.



(2) : If at some $q \in \tilde{\mathcal{C}} \cap E$ the branch numbers do not add up to zero, the only remedy is to include E in the ramification divisor and let the E-branch number be

such that the total sum is zero in $\mathbb{Z}_n$. We are now in a position to prove the first part of the theorem.

Start with any projective smooth surface X with functionfield $\mathbb{C}(X) = L$ and let the class of Σ be determined by the geo-combinatorial data $(\mathcal{C}, \mathcal{P}, b, \mathcal{D})$ in X where $\mathcal{C}$ is the ramification divisor $\mathtt{ram}\Sigma$ and $\mathcal{P}$ is the set of singular points of $\mathcal{C}$. We can split $\mathcal{P}$ in two subsets

- $\mathcal{P}_{unr} = \{P \in \mathcal{P}$ where all the branch-data $b_P = (b_1, \ldots, b_{i_P})$ are trivial, that is, all $b_i = 0$ in $\mathbb{Z}_n\}$
- $\mathcal{P}_{ram} = \{P \in \mathcal{P}$ where some of the branch-data $b_P = (b_1, \ldots, b_{i_P})$ are non-trivial, that is, some $b_i \neq 0$ in $\mathbb{Z}_n\}$

After a finite number of blow-ups we get a birational morphism $S_1 \overset{\pi}{\twoheadrightarrow} X$ such that $\pi^{-1}(\mathcal{C})$ has as its worst singularities normal crossings and all branches in points of $\mathcal{P}$ are separated in S. Let $\mathcal{C}_1$ be the ramification divisor of Δ in S_1. By the foregoing argument we have

- If $P \in \mathcal{P}_{unr}$, then we have that $\mathcal{C}' \cap \pi^{-1}(P)$ consists of smooth points of $\mathcal{C}_1$,
- If $P \in \mathcal{P}_{ram}$, then $\pi^{-1}(P)$ contains at least one singular points Q of $\mathcal{C}_1$ with branch data $b_Q = (a, -a)$ for some $a \neq 0$ in $\mathbb{Z}_n$.

In fact, after blowing-up singular points Q' in $\pi^{-1}(P)$ with trivial branch-data we obtain a smooth surface $S \twoheadrightarrow S_1 \twoheadrightarrow X$ such that the only singular points of the ramification divisor $\mathcal{C}'$ of Δ have non-trivial branch-data $(a, -a)$ for some $a \in \mathbb{Z}_n$. Then, take a maximal $\mathcal{O}_S$-order $\mathcal{A}$ in Σ. By the local calculation of $Br_n\ \mathbb{C}\{x, y\}$ of theorem 80 $\mathcal{A}$ is étale locally of quantum-plane type in these remaining singularities. By example 113 $\mathcal{A}$ is not `alg@n`-smooth in these finite number of points.

In particular, if all branch-data are trivial, this constructs an `alg@n`-smooth model of Σ. Conversely, if $\mathcal{A}$ is an `alg@n`-smooth $\mathcal{O}_S$-order in Σ with S a smooth projective model of $\mathbb{C}(X)$, then $\mathcal{A}$ is locally étale split in every point $s \in S$. But then, so is any maximal $\mathcal{O}_S$-order $\mathcal{A}_{max}$ containing $\mathcal{A}$. By the foregoing arguments this can only happen if all branch-data are trivial. $\square$

CHAPTER 6

Empires

"All information looks like noise until you break the code."
N. Stephenson in "Snow Crash".

This chapter and the next present our approach to the isomorphism problem of finite dimensional representations for an $\mathtt{alg}$-smooth algebra A. We recall the definition, due to Kent Morisson, of the component semigroup $\mathtt{comp}A$ on the set of all connected components of $\mathtt{rep}A$ with addition induced by the direct sum of representations. If A is $\mathtt{alg}$-smooth, the connected components are also the irreducible components and we denote by $\mathtt{rep}_\alpha A$ the component determined by $\alpha \in \mathtt{comp}A$.

With $\mathtt{simp}A$ we denote the subset of $\mathtt{comp}A$ consisting of those irreducible components containing a simple representation. One might view $\#(\mathtt{comp}A - \mathtt{simp}A)$ as a measure for the failure of $\mathtt{rep}A$ to be an affine noncommutative variety. By universal localization one can usually arrive at a situation where this number is finite.

The *empire* $\mathtt{emp}A$ of the $\mathtt{alg}$-smooth algebra A is the (infinite) quiver with vertices v_α for $\alpha \in \mathtt{simp}A$ and where the number of directed arrows from v_α to v_β is $ext(\alpha, \beta)$ the minimal dimension of the extension group $Ext^1_A(M, N)$ where $M \in \mathtt{rep}_\alpha A$ and $N \in \mathtt{rep}_\beta A$. The structure of $\mathtt{emp}A$ is fully determined by a (usually finite) subquiver, the *wall* on the semigroup generators of $\mathtt{comp}A$. The main result asserts that

$$\mathtt{iso}(\mathtt{rep}_\alpha A) = \bigsqcup_{(Q,\alpha)} \mathtt{iso}(\mathtt{null}_\alpha\ Q) \times \mathtt{azu}_{\beta_1} A \times \ldots \times \mathtt{azu}_{\beta_l} A$$

where the disjoint union is taken over all quiver settings (Q, α) with Q a *finite* subquiver of $\mathtt{emp}A$ on the vertices $\{v_{\beta_1}, \ldots, v_{\beta_l}\} \subset \mathtt{simp}A$ and where $\mathtt{azu}_{\beta_i} A$ is the Azumaya locus of $\int_{\beta_i} A$ which is an order in a central simple algebra.

This reduces the study to a combinatorial part, the description of the orbits in nullcones of quiver representations, depending only on the noncommutative étale isomorphism class of A and a geometric part, the description of the Azumaya loci, which contains the noncommutative Zariski information on A. We postpone the description of the nullcones to the last chapter and prove that the orders $\int_\beta A$ usually determine an étale cohomology class on the smooth locus of $\oint_\beta A$.

In the final section we present the results due to Raf Bocklandt characterizing the quiver settings (Q, α) such that $\mathtt{iss}_\alpha\ Q$ is smooth. Combining this with the local étale description, this determines the smooth loci of the irreducible varieties $\mathtt{iss}_\alpha A$ whenever A is $\mathtt{alg}$-smooth.

6.1. Component semigroups.

To start, we recall some results of Kent Morrison [**48**] on the connected component semigroup of an algebra A.

Definition 83 (Morrison). For an affine $\mathbb{C}$-algebra A we denote by $\mathtt{comp}_n A$ the set of connected components of $\mathtt{rep}_n A$ and let

$$\mathtt{comp}\ A = \bigsqcup_n \mathtt{comp}_n A$$

The direct sum maps $\mathtt{rep}_k A \times \mathtt{rep}_l A \longrightarrow \mathtt{rep}_{k+l} A$ make $\mathtt{comp}\ A$ into an Abelian semigroup.

$\mathtt{comp}$ is a contravariant functor $\mathtt{alg} \longrightarrow \mathtt{ab\text{-}semigroups}$. That is, for every $\mathbb{C}$-algebra morphism $A \xrightarrow{f} B$ defines a morphism $\mathtt{rep}_n\ B \xrightarrow{f^*} \mathtt{rep}_n\ A$ by restriction of scalars and hence a semigroup morphism $\mathtt{comp}\ B \xrightarrow{f^*} \mathtt{comp}\ A$.

The dimension function defines a semigroup morphism $\mathtt{comp}\ A \longrightarrow \mathbb{N}$, the *augmentation* map. We call the augmented Abelian semigroup $\mathtt{comp}\ A$ the *component semigroup* of A.

Theorem 82. $\mathtt{comp}\ A$ *also classifies the connected components of the quotient varieties* $\mathtt{iss}_n A$ *for all* $n \in \mathbb{N}$.

Proof. It suffices to show that the fibers of the quotient maps

$$\mathtt{rep}_n A \xrightarrow{\pi} \mathtt{iss}_n A$$

are connected. A point $\xi \in \mathtt{iss}_n A$ corresponds to a semi-simple n-dimensional representation M_ξ of A. The fiber $\pi^{-1}(\xi)$ consists of all n-dimensional representations M having as sum of its Jordan-Hölder components M_ξ. By the Hilbert criterium we can connect M with a point in the orbit of M_ξ by a rational curve $\mathbb{C}$, whence $\pi^{-1}(\xi)$ is connected. □

Example 114. Let A be a finite dimensional algebra. A has finitely many simple representations $S_1, \ldots, S_k$ with $\mathtt{dim} S_i = d_i$. For a fixed natural number n, any semi-simple n-dimensional representation of A is of the form

$$M = S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k}$$

with $\sum a_i d_i = n$. Therefore, $\mathtt{comp}\ A \simeq \mathbb{N}^k$ with k the number of simple representations of A.

Example 115. Let Q be a finite quiver on k-vertices, then $\mathtt{comp}\langle Q \rangle \simeq \mathbb{N}^k$. Indeed, we have seen that $\mathtt{rep}_n\ \langle Q \rangle$ decomposes into connected components corresponding to the dimension vectors α of total dimension n.

Theorem 83 (Morrison). *Let A and B be affine $\mathbb{C}$-algebras, then*

1. $\mathtt{comp}\ A \times B \simeq \mathtt{comp}\ A \times \mathtt{comp}\ B$.
2. $\mathtt{comp}\ A * B \simeq \mathtt{comp}\ A \times_{\mathbb{N}} \mathtt{comp}\ B$.
3. *If* $I \triangleleft A$ *is nilpotent, then* $\mathtt{comp}\ A \simeq \mathtt{comp}\ \frac{A}{I}$.
4. $\mathtt{comp}\ A[x_1, \ldots, x_m] \simeq \mathtt{comp}\ A$.
5. $\mathtt{comp}\ A\langle x_1, \ldots, x_m \rangle \simeq \mathtt{comp}\ A$.

Proof. (1) : The projection maps $A \xleftarrow{p} A \times B \xrightarrow{p'} B$ induce an isomorphism of semigroups $p^* + p'^*\ :\ \mathtt{comp}\ A \times \mathtt{comp}\ B \longrightarrow \mathtt{comp}\ A \times B$ as any $A \times B$

representation is the direct sum of an A-representation and a B-representation. If we take the sum of the two dimension functions, this is an isomorphism as augmented semigroups.

(2) : By the universal property of the free algebra product an n-dimensional representation of $A * B$ consists of an n-dimensional A-representation and an n-dimensional B-representation. Therefore, the inclusions $A \overset{i}{\hookrightarrow} AastB \overset{i'}{\hookleftarrow} B$ induce an isomorphism $i^* \times_{\mathbb{N}} i'^* \ : \ \mathtt{comp}\ A * B \longrightarrow \mathtt{comp}\ A \times_{\mathbb{N}} \mathtt{comp}\ B$.

(3) : A nilpotent ideal acts trivially on a semi-simple representation, whence $\mathtt{iss}_n A = \mathtt{iss}_n \frac{A}{I}$.

(4) : Define a positive gradation on $A[x_1, \ldots, x_k]$ by $deg(a) = 0$ for all $a \in A$ and $deg(x_i) = 1$. The gradation induces a $\mathbb{C}^*$-action on $\mathtt{rep}_n\ A[x_1, \ldots, x_k]$. The limiting point for this action is an n-dimensional representation on which all the x_i act trivially, that is a point in $\mathtt{rep}_n\ A$. Therefore, the inclusion $\mathtt{rep}_n\ A \hookrightarrow \mathtt{rep}_n\ A[x_1, \ldots, x_k]$ gives a one-to-one correspondence between the connected components.

(5) : Again the gradation argument of part (4). □

Example 116. Let A be an affine commutative algebra with corresponding reduced variety $X = \mathtt{spec}A$. As A is commutative, the only epimorphisms $A \twoheadrightarrow M_n(\mathbb{C})$ possible are with $n = 1$. That is, isomorphism classes of simple A-representations are classified by X. The Jordan-Hölder theorem implies that for $n \geq 1$

$$\mathtt{iss}_n A = X^{(n)} = \underbrace{X \times \ldots X}_{n} / S_n$$

the *n-th symmetric product* of X. If X is connected, or equivalently, if A has no non-trivial idempotents, then so is $X^{(n)}$ for every n whence $\mathtt{comp}\ A \simeq \mathbb{N}$. If A decomposes as $A = A_1 \times \ldots \times A_k$ with $\mathtt{spec}A_i$ connected, then

$$\mathtt{comp}\ A \simeq \mathtt{comp}\ A_1 \times \ldots \times \mathtt{comp}\ A_k \simeq \mathbb{N}^k$$

Example 117. The component semigroup for $\langle m \rangle$. Because $\mathtt{comp}\ \mathbb{C}[x] \simeq \mathbb{N}$ by the previous example and

$$\langle m \rangle = \underbrace{\mathbb{C}[x] * \ldots * \mathbb{C}[x]}_{m}$$

it follows from part (2) of theorem 83 that $\mathtt{comp}\ \langle m \rangle \simeq \mathbb{N}$.

Part (5) of theorem 83 are special cases of a more general result.

Theorem 84 (Morrison). *Let A be an affine $\mathtt{alg}$-smooth algebra such that $\mathtt{comp}\ A \simeq \mathbb{N}$ as augmented Abelian semigroups. Then, for any $B \in \mathtt{alg}$ we have*

$$\mathtt{comp}\ A \otimes B \simeq \mathtt{comp}\ B$$

Proof. Let $\rho\ :\ A \otimes B \longrightarrow M_n(\mathbb{C})$ be an n-dimensional representation and let

$$f = \rho(- \otimes 1)\ :\ A \longrightarrow M_n(\mathbb{C})$$

be the induced n-dimensional representation of A. The image $R = f(A)$ is a finite dimensional algebra so is a semidirect sum $R = S \oplus N$ with S semisimple and N the radical of R of nilpotency degree k, that is, $N^k = 0$.

We want to deform f to $g = \pi_S \circ f$ in such a way that all intermediate algebra morphisms h_t have the property that $h_t(A) \subset R$. Let $U = f^{-1}(N)$ and $A_0 = f^{-1}(S)$. Assume by induction we have already constructed an algebra map

$$f_i \ : \ A \longrightarrow R \quad \text{such that} \quad \begin{cases} f_i(N) \subset N^i \\ f_i|A_0 = f|A_0 \end{cases}$$

Consider the projections $R \overset{p_{i+1}}{\twoheadrightarrow} \frac{R}{N^{i+1}} \overset{\pi_i}{\twoheadleftarrow} \frac{R}{N^i}$ and define the family of algebra morphisms

$$\phi_t \ : \ A \longrightarrow \frac{R}{N^{i+1}} \qquad a + u \mapsto f_i(a) + tp_{i+1}(f_i(u))$$

This is an algebra morphism since $p_{i+1}(f_i(u)) \in Ker\ \pi_i$ which is a square zero ideal. Because A is `alg`-smooth we can lift ϕ_t to an algebra morphism

$$\psi_t \ : \ A \longrightarrow R$$

and define $f_{i+1} = \psi_0$. Then, $f_{i+1}(U) \subset N^{i+1}$. Iterating we eventually construct an algebra map $f_k \ : \ A \longrightarrow R$ such that $f_k(U) \subset N^k = 0$ whence $f_k = g$. Thus, f can be deformed to g by a sequence of deformations along the affine line.

The semisimple algebra S is of the form $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_l}(\mathbb{C})$ with $\sum n_i = n$. Therefore, the n-dimensional A-module V_g defined by g is the direct sum

$$V_g = S_1 \oplus \ldots \oplus S_l$$

with S_i a simple of dimension n_i with structure map $A \xrightarrow{g_i} M_{n_i}(\mathbb{C})$. Because `comp` $A \simeq \mathbb{N}$, V lies in the same connected component as the semisimple module $T_{n_i} = S^{\oplus n_i}$ where S is a one-dimensional simple A-module generating `comp` A determined by $A \overset{\epsilon}{\twoheadrightarrow} \mathbb{C}$. That is, we can deform each g_i and hence (by simultaneous deformation) g to the representation

$$A \longrightarrow M_n(\mathbb{C}) \qquad a \mapsto \epsilon(a)\mathbb{1}_n$$

This deformation is taking place inside S and commutes with $\rho(B)$ so we have a deformation of ρ to the n-dimensional representation given by

$$\sigma \ : \ A \otimes B \longrightarrow M_n(\mathbb{C}) \qquad a \otimes b \mapsto \epsilon(a)\sigma(b)$$

proving the result. □

Definition 84. For an affine algebra $A \in$ `alg` let `simp`A be the subset of `comp`A consisting of those connected components containing a *simple* A-module. We call `simp`A the set of *simple roots* of A.

Example 118. The simple roots of $\langle Q \rangle$.

Let Q be a finite quiver with vertices $Q_v = \{v_1, \ldots, v_k\}$. We will give some *necessary* conditions for a dimension vector α to belong to `simp`$\langle Q \rangle$.

For $S \subset Q_v$ we denote with Q_S the full subquiver of Q having S as its set of vertices. A full subquiver Q_S is said to be *strongly connected* if and only if for all $v_i, v_j \in S$ there is an oriented cycle in Q_S passing through v_i and v_j. We can partition

$$Q_v = S_1 \sqcup \ldots \sqcup S_s$$

such that the Q_{S_i} are maximal strongly connected components of Q. Clearly, the direction of arrows in Q between vertices in S_i and S_j is the same by the maximality assumption and can be used to define an orientation between S_i and

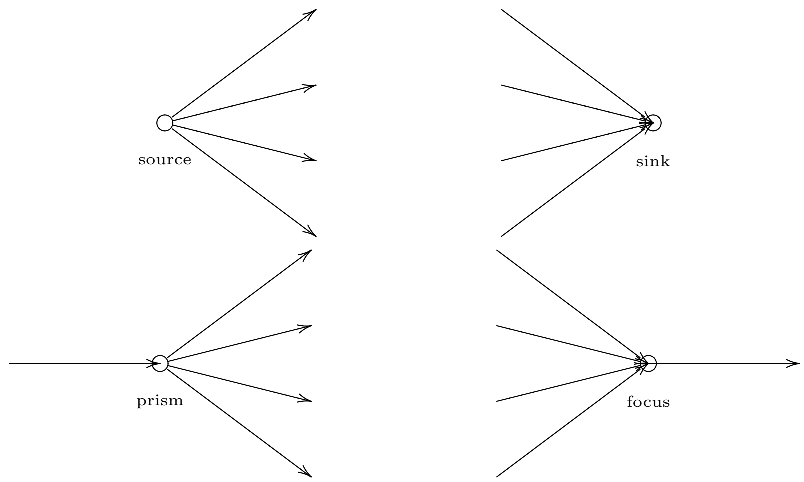


FIGURE 1. Vertex terminology

S_j. The *strongly connected component quiver* $SC(Q)$ is then the quiver on s vertices $\{w_1, \ldots, w_s\}$ with w_i corresponding to S_i and there is one arrow from w_i to w_j if and only if there is an arrow in Q from a vertex in S_i to a vertex in S_j. Observe that when the underlying graph of Q is connected, then so is the underlying graph of $SC(Q)$ and $SC(Q)$ is a quiver without oriented cycles.

condition 1 : If $\alpha = (d_1, \ldots, d_k) \in \mathtt{simp}\langle Q \rangle$, then $Q_{\mathtt{supp}\alpha}$ is a strongly connected subquiver of Q where $\mathtt{supp}\alpha = \{v_i \ : \ d_i \neq 0\}$ is the *support* of the dimension vector. If not, we consider the strongly connected component quiver $SC(Q_{\mathtt{supp}\alpha})$ and by assumption there must be a *sink* (for vertex-terminology see figure 1) in it corresponding to a proper subset $S \overset{\neq}{\hookrightarrow} Q_v$. If $V \in rep_\alpha\, Q$ we can then construct a representation W by

- $W_i = V_i$ for $v_i \in S$ and $W_i = 0$ if $v_i \notin S$,
- $W_a = V_a$ for an arrow a in Q_S and $W_a = 0$ otherwise.

One verifies that W is a proper subrepresentation of V, so V cannot be simple, a contradiction.

condition 2 : If $\alpha \in \mathtt{simp}\langle Q \rangle$, then for all $v_i \in \mathtt{supp}\alpha$

$$\begin{cases} \chi_Q(\alpha, \epsilon_i) & \leq 0 \\ \chi_Q(\epsilon_i, \alpha) & \leq 0 \end{cases}$$

where ϵ_i is the dimension vector of the one-dimensional simple concentrated in v_i. Indeed, let V be a simple representation of Q of dimension vector $\alpha = (d_1, \ldots, d_k)$, then

$$\chi_Q(\epsilon_i, \alpha) = d_i - \sum_{(j) \xleftarrow{a} (i)} d_j$$

If $\chi_Q(\epsilon_i, \alpha) > 0$, then the natural linear map

$$\bigoplus_{(j) \xleftarrow{a} (i)} V_a \ : \ V_i \longrightarrow \bigoplus_{(j) \xleftarrow{a} (i)} V_j$$

has a nontrivial kernel, say K. Consider the representation W of Q determined by

- $W_i = K$ and $W_j = 0$ for all $j \neq i$,
- $W_a = 0$ for all $a \in Q_a$.

then W is a proper subrepresentation of V, a contradiction. Similarly, if $\chi_Q(\alpha, \epsilon_i) = d_i - \sum_{(i) \xleftarrow{a} (j)} d_j > 0$, then the linear map

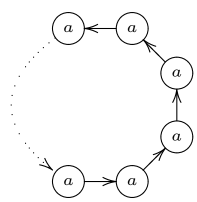
$$\bigoplus_{(i) \xleftarrow{a} (j)} V_a \; : \; \bigoplus_{(i) \xleftarrow{a} (j)} V_j \longrightarrow V_i$$

has an image I which is a proper subspace of V_i. The representation W of Q determined by

- $W_i = I$ and $W_j = V_j$ for $j \neq i$,
- $W_a = V_a$ for all $a \in Q_a$.

is a proper subrepresentation of V, a contradiction. These two conditions are *not* sufficient as we have the following

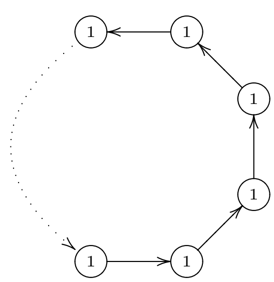
exception : Consider the extended Dynkin quiver of type $\tilde{A}_k$ with cyclic orientation.



and dimension vector $\alpha = (a, \ldots, a)$. All arrow matrices must be invertible if V is simple. In this case, under the action of $GL(\alpha)$, they can be diagonalized. Therefore, $\alpha = (a, \ldots, a) \in \mathtt{simp}\tilde{A}_k$ iff $a = 1$. However, this is the only exceptional case :

THEOREM 85. *$\alpha = (d_1, \ldots, d_k) \in \mathtt{simp}\langle Q \rangle$ if and only if one of the following two cases holds*

(1) $\mathtt{supp}\alpha = \tilde{A}_k$*, the extended Dynkin quiver on k vertices with cyclic orientation and $d_i = 1$ for all $1 \leq i \leq k$*



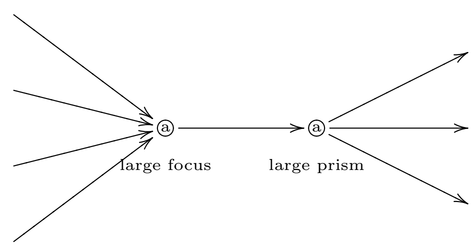
(2) $\mathtt{supp}\alpha \neq \tilde{A}_k$*. Then,* $\mathtt{supp}\alpha$ *is strongly connected and for all $1 \leq i \leq k$ we have*

$$\begin{cases} \chi_Q(\alpha, \epsilon_i) & \leq 0 \\ \chi_Q(\epsilon_i, \alpha) & \leq 0 \end{cases}$$

In either case, $\mathtt{simp}\langle Q \rangle$ *is a cone in* $\mathtt{comp}\langle Q \rangle = \mathbb{N}^k$.

Proof. We will use induction, both on the number of vertices k in $\mathtt{supp}\alpha$ and on the total dimension $n = \sum_i d_i$ of the representation. A vertex v_i is said to be *large* with respect to a dimension vector $\alpha = (d_1, \ldots, d_k)$ whenever d_i is maximal among the d_j. The vertex v_i is said to be *good* if v_i is large and has no direct successor which is a large prism nor a direct predecessor which is a large focus. If $\mathtt{supp}\alpha$ has no good vertex, then either $\mathtt{supp}\alpha$ must have a large prism having no large prism direct successors or it must have a large focus having no large focus direct predecessors. Indeed, if neither of the cases hold, there is an oriented cycle in $\mathtt{supp}\alpha$ consisting of prisms (or consisting of focusses). Assume $(v_{i_1}, \ldots, v_{i_l})$ is a cycle of prisms, then the unique incoming arrow of v_{i_j} belongs to the cycle. As $\mathtt{supp}\alpha \neq \tilde{A}_k$ there is at least one extra vertex v_a not belonging to the cycle. But then, there can be no oriented path from v_a to any of the v_{i_j}, contradicting the assumption that $\mathtt{supp}\alpha$ is strongly connected.

But then take such a large prism (or focus), then because $\chi_Q(\alpha, \epsilon_i) \leq 0$ and $\chi_Q(\epsilon_i, \alpha) \leq 0$ for all $1 \leq i \leq k$, we have the following subquiver in $\mathtt{supp}\alpha$



We can reduce to a quiver situation with strictly less vertices by identifying these two vertices. The resulting quiver is still strongly connected and the dimension vector still satisfies the Euler condition. Therefore, by assumption there is a simple representation and we can extend it to a simple representation on $\mathtt{supp}\alpha$ by putting the identity matrix on the connecting arrow, whence we are done in this case.

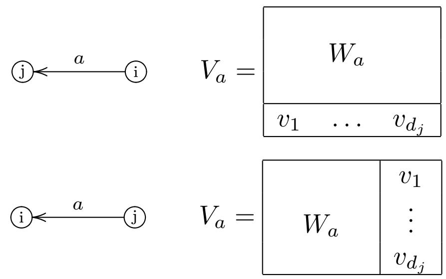
Therefore, we may assume that $\mathtt{supp}\alpha$ has a good vertex v_i. If $d_i = 1$ then all $d_j = 1$ for $v_j \in \mathtt{supp}\alpha$ and we can construct a simple representation by taking $V_b = 1$ for all arrows b in $\mathtt{supp}\alpha$. Simplicity follows from the fact that $\mathtt{supp}\alpha$ is strongly connected.

If $d_i > 1$, consider the dimension vector $\alpha' = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_k)$. Clearly, $\mathtt{supp}\alpha' = \mathtt{supp}\alpha$ is strongly connected and we claim that the Euler-form conditions still hold for α'. The only vertices v_l where things might go wrong are direct predecessors or direct successors of v_i. Assume for one of them $\chi_Q(\epsilon_l, \alpha) > 0$ holds, then

$$d_l = d'_l > \sum_{\textcircled{m} \xleftarrow{a} \textcircled{l}} d'_m \geq d'_i = d_i - 1$$

But then, $d_l = d_i$ whence v_l is a large vertex of α and has to be also a focus with end vertex v_i (if not, $d_l > d_i$), contradicting goodness of v_i.

Hence, by induction on n we may assume that there is a simple representation $W \in \mathtt{rep}_{\alpha'}\ \mathtt{supp}\alpha$. Consider the space $\mathtt{rep}_W$ of representations $V \in \mathtt{rep}_\alpha\ Q$ such that $V \mid \alpha' = W$. That is, for every arrow



rep_W is an affine space consisting of all representations degenerating to $W \oplus S_i$ where S_i is the simple one-dimensional representation concentrated in v_i.

As there are simple representations of Q having a one-dimensional component at each vertex in $\mathtt{supp}\alpha$ and as the subset of simple representations in $\mathtt{rep}_{\alpha'}\ Q$ is open, we can choose W such that $\mathtt{rep}_W$ contains representations V such that a trace of an oriented cycle differs from that of $W \oplus S_i$. As the invariant ring $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)}$ is generated by traces along oriented cycles and classifies the isomorphism classes of semi-simple representations, it follows that the Jordan-Hölder factors of V are different from W and S_i. In view of the definition of $\mathtt{rep}_W$, this can only happen if V is a simple representation, finishing the proof of the theorem. □

Next, we will construct $\mathtt{alg}$-smooth algebras A having as their component semigroup $\mathtt{comp}\ A$ (almost) any sub semigroup of $\mathbb{N}$. We first need to recall some facts on Azumaya algebras and their polynomial identities.

DEFINITION 85. The n-th *Azumaya locus* of an algebra $A \in \mathtt{alg}$ is the Zariski open subscheme (possibly empty) $\mathtt{azu}_n A$ of $\mathtt{iss}_n A$ consisting of the points ξ corresponding to n-dimensional *simple* representations M_ξ.

EXAMPLE 119. If $\mathtt{rep}_n A \xrightarrow{\pi} \mathtt{iss}_n A$ is the quotient map, then we claim that

$$\pi^{-1}(\mathtt{azu}_n A) \twoheadrightarrow \mathtt{azu}_n A$$

is a principal PGL_n-fibration in the étale topology, that is, determines an element in $H^1_{et}(\mathtt{azu}_n A, PGL_n)$.

By assumption, the stabilizer subgroup of $x = M_\xi$ in GL_n is $\mathbb{C}^* \mathbb{1}_n$, that is, PGL_n acts on $\mathtt{rep}_n A$ with trivial stabilizer in x. Let S_x be the slice in x for the PGL_n-action on $\mathtt{rep}_n A$. By taking traces of products of a lifted basis from $M_n(\mathbb{C})$ we find a PGL_n-affine open neighborhood $\mathtt{afU}_\xi$ of ξ contained in $\mathtt{azu}_n A$ and hence by the slice result a commuting diagram

$$\begin{array}{ccc} PGL_n \times S_x & \xrightarrow{\psi} & \pi^{-1}(U_\xi) \\ \downarrow & & \downarrow \pi \\ S_x & \xrightarrow[\psi/PGL_n]{} & U_\xi \end{array}$$

where ψ and ψ/PGL_n are étale maps. That is, ψ/PGL_n is an étale neighborhood of ξ over which π is trivialized. As this holds for all points $\xi \in \mathtt{azu}_n A$ the claim follows.

In particular, if $\texttt{azu}_n A = \texttt{iss}_n A$ and if $C = \mathbb{C}[\texttt{iss}_n A$ this shows that there is an étale cover $\{B_i\}$ of C such that

$$\int_n A \otimes_C B_i \simeq M_n(B_i)$$

We will now explain the terminology by proving the connection with the classical notion of *Azumaya algebras*.

DEFINITION 86. For $C \in \texttt{commalg}$ an algebra $A \in \texttt{alg}_C$ is said to be an *Azumaya algebra* if and only if

(1) A is a finitely generated projective C-module, and,
(2) the natural multiplication map

$$A^e = A \otimes_C A^{op} \xrightarrow{j} End_C(A) \qquad j(a' \otimes a")a = a'aa"$$

is an isomorphism in $\texttt{alg}_C$.

EXAMPLE 120. If A is a central simple algebra of dimension n^2 over K we have seen that $A \otimes_K A^{op} \simeq M_{n^2}(K)$. Hence, Azumaya algebras over a field K are precisely the central simple K-algebras.

EXAMPLE 121. If $P \in \texttt{projmod}C$, then the endomorphismring $A = End_C(P)$ is an Azumaya algebra over C. In particular, if $P = C^{\oplus n}$, then $End_C(P) = M_n(C)$ is an Azumaya algebra. These Azumaya algebras will be called *trivial*.

If A and A' are two Azumaya algebras over C one verifies easily that $A \otimes_C A'$ is also an Azumaya algebra over C. We call two C-Azumaya algebras *equivalent* if there are $P, P' \in \texttt{projmod}C$ such that

$$A \otimes_C End_C(P) \simeq A' \otimes_C End_C(P')$$

Observe that this generalizes the equivalence notion on central simple algebras. Again, the equivalence classes of C-Azumaya algebras form a commutative group under the tensorproduct, in which the class of $End_C(P)$ is the identity element and the inverse of the class of A is the class of A^{op}. This group is called the *Brauer group* $Br(C)$ of the commutative algebra C.

EXAMPLE 122. If $C \longrightarrow C'$ is a morphism in $\texttt{commalg}$ and if A is an Azumaya algebra over C, then $A_{C'} = A \otimes_C C'$ is an Azumaya algebra over C'. Indeed, as $A \in \texttt{projmod}C$, $A \otimes_C C'$ is a finitely projective C'-module and the maps

$$A_{C'} \otimes_{C'} A^{op}_{C'} \simeq (A \otimes_C A^{op})_{C'} \simeq (End_C(P))_{C'} \simeq End_{C'}(P \otimes_C C')$$

give the required isomorphism. Also the notion of trivial Azumaya algebra and of equivalence is preserved, giving a groupmorphism

$$Br(C) \longrightarrow Br(C')$$

on the level of Brauer groups. If $C \longrightarrow C'$ is a *faithfully flat* extension, then we can descend C'-isomorphisms to C-isomorphisms and C'-projective modules to C-projective modules. Hence, in that case, if $A \otimes_C C'$ is a C'-Azumaya algebra, then A is a C-Azumaya algebra.

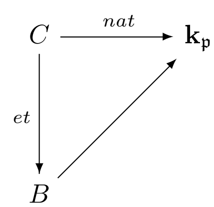
In particular, let $c_i \in C$ be a set of elements generating the unit ideal in C, or equivalently, the open sets $\mathbb{X}(c_i)$ in the Zariski topology cover $\texttt{spec}C$. Then, the direct sum of the corresponding sections

$$\oplus_i C_{c_i}$$

is faithfully flat over C. Therefore, A is an Azumaya algebra over C if and only if all A_{c_i} are Azumaya algebras over C_{c_i}. This means that the Azumaya property is a local property for the Zariski topology (as well as for the étale topology).

We will now investigate the étale local structure of Azumaya algebras. For this we need to know what the local rings are in the étale topology.

Definition 87. Let $\mathfrak{p}$ be a prime ideal of C and denote with $\mathbf{k}_{\mathfrak{p}}$ the algebraic closure of the field of fractions of $A/\mathfrak{p}$. An *étale neighborhood* of $\mathfrak{p}$ is an étale extension $B \in \mathtt{C}_{\mathtt{et}}$ such that the diagram below is commutative



The localization at $\mathfrak{p}$ for the étale topology is the *strict Henselization*

$$C_{\mathfrak{p}}^{sh} = \underrightarrow{lim}\ B$$

where the limit is taken over all étale neighborhoods of $\mathfrak{p}$.

A local algebra L with maximal ideal $\mathfrak{m}$ and residue map $\pi : L \twoheadrightarrow L/\mathfrak{m} = k$ is said to be *Henselian* if for every monic polynomial $f \in L[t]$ allowing a decomposition

$$\pi(f) = g_0.h_0$$

in $k[t]$, then $f = g.h$ with $\pi(g) = g_0$ and $\pi(h) = h_0$. If L is Henselian, tensoring with k induces an equivalence of categories between the étale A-algebras and the étale k-algebras.

An Henselian local algebra is said to be *strict Henselian* if and only if its residue field is algebraically closed. Thus, a strict Henselian ring has no proper finite étale extensions and can be viewed as a local algebra for the étale topology.

Example 123. Consider the local algebra of $\mathbb{C}[x_1,\ldots,x_d]$ in the maximal ideal $(x_1,\ldots,x_d)$, then the Henselization and strict Henselization are both equal to

$$\mathbb{C}\{x_1,\ldots,x_d\}$$

the ring of *algebraic functions* . This is the subalgebra of $\mathbb{C}[[x_1,\ldots,x_d]]$ of formal power-series consisting of those series $\phi(x_1,\ldots,x_d)$ which are algebraically dependent on the coordinate functions x_i over $\mathbb{C}$. In other words, those ϕ for which there exists a non-zero polynomial $f(x_i,y) \in \mathbb{C}[x_1,\ldots,x_d,y]$ with $f(x_1,\ldots,x_d,\phi(x_1,\ldots,x_d)) = 0$.

These algebraic functions may be defined implicitly by polynomial equations. Consider a system of equations

$$f_i(x_1,\ldots,x_d;y_1,\ldots,y_m) = 0 \text{ for } f_i \in \mathbb{C}[x_i,y_j] \text{ and } 1 \leq i \leq m$$

Suppose there is a solution in $\mathbb{C}$ with

$$x_i = 0 \text{ and } y_j = y_j^o$$

such that the Jacobian matrix is non-zero

$$det\ (\frac{\partial f_i}{\partial y_j}(0,\ldots,0;y_1^o,\ldots,y_m^0)) \neq 0$$

Then, the system can be solved uniquely for power series $y_j(x_1,\ldots,x_d)$ with $y_j(0,\ldots,0) = y_j^o$ by solving inductively for the coefficients of the series. One can show that such implicitly defined series $y_j(x_1,\ldots,x_d)$ are algebraic functions and that, conversely, any algebraic function can be obtained in this way.

THEOREM 86. *Azumaya algebras are locally matrixrings in the étale topology. In particular, there is a one-to-one correspondence between the pointed set*

$$H^1_{et}(C, \mathtt{PGL}_n)$$

and isomorphism classes of Azumaya algebras over C of rank n^2.

PROOF. (Sketch) Let $\mathfrak{m}$ be a maximal ideal of C and let $\hat{C}_{\mathfrak{m}}$ be the completion of the local ring $C_{\mathfrak{m}}$ both having residue field k. The strict Hensilization $C^{sh}_{\mathfrak{m}}$ is then a complete local ring with maximal ideal M residue field the algebraic closure $\overline{k}$ of k. If A is an Azumaya algebra over C, then

$$A \otimes_C \frac{C^{sh}_{\mathfrak{m}}}{M} \simeq M_n(\overline{k})$$

for some n as there are no Azumaya algebras (central simple algebras) over an algebraically closed field. Then, the idea is to lift a set of matrix units e_{ij} modulo the various powers of M and by Nakayama's lemma we still get a set of matrix units over $C^{sh}_{\mathfrak{m}}/M^k$ for all k and can pass to the limit whence

$$A \otimes_C C^{sh}_{\mathfrak{m}} \simeq M_n(C^{sh}_{\mathfrak{m}})$$

But then, as $C^{sh}_{\mathfrak{m}}$ is the limit of étale neighborhoods of $\mathfrak{m}$ we can take an étale extension B of C such that $A \otimes B$ is locally a matrixring of locally constant rank.

If A has constant rank n^2 the second statement follows as the automorphism groupscheme of $n \times n$ matrices is $\mathtt{PGL}_n$. □

Having a cohomological description of Azumaya algebras of constant rank we expect a cohomological description of the Brauer group as in the case of fields. This difficult result was proved by Ofer Gabber [**18**].

THEOREM 87 (Gabber). *For $C \in$ `commalg`, there exists an isomorphism*

$$Br(C) \simeq H^2_{et}(C, \mathbb{G}_m)_{tors}$$

between the Brauer group of C and the torsion part *of the cohomology group $H^2_{et}(C, \mathbb{G}_m)$.*

We collect a number of ringtheoretical facts on Azumaya algebras for later use. In particular, an Azumaya algebra of constant rank n^2 is an object in `alg@n`.

THEOREM 88. *Let A be an Azumaya algebra over C. Then*

1. *The center of A is C.*
2. *For any ideal $I \triangleleft A$ we have $I = AJ$ where $J = I \cap C$ and $\frac{A}{I} = A \otimes_C \frac{C}{J}$ is an Azumaya algebra.*
3. *There is a C-linear* reduced trace map

$$A \xrightarrow{tr} C$$

which coincides with the usual trace in any splitting $A \otimes_C B \simeq M_n(B)$.

(4) *There is a canonical element* $s = \sum_i a_i \otimes a_i' \in A \otimes_C A^{op}$ *called the* separability idempotent *such that*

$$tr(a) = \sum_i a_i a a_i' \quad \text{for all } a \in A.$$

(5) *A is a projective A-bimodule.*
(6) *There is an equivalence of categories*

$$C - \texttt{mod} \simeq A - \texttt{bimod} \qquad N \mapsto A^e \otimes_C N$$

with inverse for any $M \in A - \texttt{bimod}$

$$M^A = \{m \in M \mid (1 \otimes a - a \otimes 1)m = 0 \ \forall a \in A\}$$

which is $Hom_{A-\texttt{bimod}}(A, M)$.
(7) *If* $A \subset B$ *for any C-algebra B, then* $B = A \otimes_C cent_B(A)$ *where* $cent_B(A)$ *is the centralizer of A in B.*

Proof. (1) : Let B be a faithfully flat splitting of A, that is, $A \otimes_C B \simeq M_n(B)$. If Z is the center of A, then $C \subset Z$ and $Z \otimes_C B$ is contained in the center of $A \otimes_C B$ which is B.

(2) : To prove $I = AJ$ one extends to B as before and uses the fact that there is a one-to-one correspondence between ideals of B and of $M_n(B)$.

(3) and (4) : One shows that the usual trace $M_n(B) = A \otimes_C B \longrightarrow B$ maps A to C by verifying the faithfully flat descent criterion using that the two isomorphisms

$$A \otimes_C (B \otimes_C B) \rightrightarrows M_n(B \otimes_C B)$$

are conjugate by an automorphism that leaves the trace invariant.

(5) : Because $A \otimes_C A^{op} \simeq End_C(A)$ it suffices to show that $P \in \texttt{projmod}C$ is also projective over $End_C(P)$. This is a Zariski local condition so we may assume that C is local and $P = C^{\oplus k}$ is free. But then, $End_A(P) = M_k(C)$ of which the projectives are the columns which are $P = C^{\oplus k}$.

(6) and (7) : Let left ideal J of $A \otimes_C A^{op}$ annihilating $1 \in A$ is generated by the elements $1 \otimes a - a \otimes 1$ where $a \in A$. Indeed, if $\sum_i a_i b_i = (\sum_i a_i \otimes b_i)1 = 0$ then

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1)$$

Because $A = A \otimes_C A^{op}/J$ the identification $Hom_{A-\texttt{bimod}}(A, M) = M^A$ is given by $\phi \mapsto \phi(1)$. To prove that the natural map

$$(A \otimes_C A^{op}) \otimes_C M^A = (A \otimes_C A^{op}) \otimes_C Hom_{A-\texttt{bimod}}(A, M) \longrightarrow M$$

is an isomorphism it suffices by faithfully flat descent to prove it for $A = M_n(B)$. □

Theorem 89 (Razmyslov). *There is a multilinear noncommutative polynomial* $h(x, y)$ *which is alternating in the x variables and when evaluated in* $n \times n$ *matrices over a field takes all its values in the center and does not vanish identically.*

Proof. If a noncommutative polynomial f is linear in a variable x_i then it is of the form

$$f = \sum_k a_k x_i b_k$$

With this notation and for an extra variable z define

$$A_i(f) = \sum_k b_k x_i a_k \qquad f_{x_i=z} = \sum_k a_k z b_k$$

Evaluating in $n \times n$ matrices we obtain from the necklace property of the trace that

$$tr(zf) = tr(z \sum_k a_k x_i b_k) = tr(x_i \sum_k b_k z a_k) = tr(x_i A_i(f)_{x_i=z})$$

whence, in particular, $tr(f) = tr(x_i A_i(f)_{x_i=1})$. Moreover, we have for any other variable x_j that

$$tr(x_j A_i(f)_{x_i=1}) = tr(x_j \sum_k b_k a_k) = tr(\sum_k a_k x_j b_k) = tr(f_{x_i=x_j}).$$

Consider the multilinear and alternating (at least in the x_i) noncommutative polynomial $F(y_1, \ldots, y_{n^2+1}, x_1, \ldots, x_{n^2})$ to be

$$\sum_{\sigma \in S_{n^2}} sgn(\sigma) y_1 x_{\sigma(1)} y_2 x_{\sigma(2)} \ldots y_{n^2} x_{\sigma(n^2)} y_{n^2+1}$$

Then we deduce from the above and the alternating property that

$$tr(F) = tr(x_i A_i(F)_{x_i=1}) \qquad \forall j \neq i \ : \ tr(x_j A_i(F)_{x_i=1}) = 0.$$

Define for $j \neq i$, $h_i(x_j, y) = A_i(F)_{x_i=1}$ which for a dual basis (up to the scalar factor $tr(F)$) for the non-degenerate trace form on $n \times n$ matrices over a field whenever $x_1, \ldots, x_{n^2}$ are evaluated to be linearly independent $n \times n$ matrices. But then, for

$$h(x, y) = \sum_{i=1}^{n^2} x_i y_0 h_i(x_j, y)$$

we have the identity

$$h(x, y) = tr(y_0) tr(F)$$

from which the properties follow except for the non-vanishing. To prove this use the substitutions

$$x_{i+n(j-1)} \mapsto e_{ij} \qquad y_1 \mapsto e_{11} \quad y_{n^2+1} \mapsto e_{n1}$$

$$i \neq 1, n^2+1 \quad y_i \mapsto e_{kl} \quad \text{if} \quad x_{i-1} \mapsto e_{rk} \quad x_i \mapsto e_{ls}$$

then all monomials appearing in F vanish under this substitution except for the monomial corresponding to the identity permutation where it evaluates to e_{11} whence $tr(F) = 1$ in this case. □

THEOREM 90 (Artin). *The following are equivalent for an affine algebra A :*

1. *A is an Azumaya algebra of constant rank n^2 over its center.*
2. *A satisfies all polynomial identities of $n \times n$ matrices and has no simple representation of dimension $< n$.*

PROOF. (Schelter) (1) $\Rightarrow$ (2) : Because of the splitting $A \otimes_C B \simeq M_n(B)$, A satisfies all polynomial identities of $n \times n$ matrices. Let I be the kernel of a simple representation, then $I \cap C = \mathfrak{m}$ is a maximal ideal of C and $A/A\mathfrak{m}$ is an Azumaya algebra of rank n^2 over $C/\mathfrak{m} \simeq \mathbb{C}$ whence must be $M_n(\mathbb{C})$.

(2) $\Rightarrow$ (1) : Take the polynomial $h(x_1,\ldots,x_{n^2},y)$ of the previous theorem and consider

$$\sum_{j=1}^{n^2+1}(-1)^j h(x_1,\ldots,\hat{x_j},\ldots,x_{n^2+1},y)x_j$$

This is an alternating and multilinear function of $x_1,\ldots,x_{n^2+1}$ so it is an identity of $M_n(\mathbb{C})$ whence of A. For all maximal ideals $M \triangleleft A$ we have that h does not vanish on A/M whence the evaluations of h in A generate the unit ideal. Choose $a_{ij}, b_{ik}, t_i \in A$ such that

$$1 = \sum_{i=1}^{l} h(a_{i1},\ldots,a_{in^2},b_{i1},\ldots,b_{im})t_i$$

For $1 \leq i \leq l$ and $1 \leq j \leq n^2$ define $f_{ij} \in Hom_C(A,C)$ by

$$f_{ij}(a) = (-1)^{j+n^2} h(a_{i1},\ldots,\hat{a_{ij}},\ldots,a_{in^2},a,b_{i1},\ldots,b_{im})$$

Then because h evaluates to central elements and the above equalities we have for $a \in A$

$$\begin{aligned} a = & \sum_i h(a_{i1},\ldots,a_{in^2},b_{i1},\ldots,b_{im})at_i \\ = & \sum_{i,j}(-1)^{j+n^2} h(a_{i1},\ldots,\hat{a_{ij}},\ldots,a_{in^2},a,b_{i1},\ldots,b_{im})a_{ij}t_i \\ = & \sum_{i,j} f_{ij}(a)a_{ij}t_i \end{aligned}$$

which shows that $\{f_{ij}, a_{ij}t_i\}$ are a dual basis for A as a C-module whence $A \in$ `projmod`C. The dual basis implies the existence of C-endomorphisms of A

$$\phi_{ijpq} \; : \; A \longrightarrow A \qquad a \mapsto f_{ij}(a)a_{pq}t_p$$

which generate $End_C(A)$ as a C-module. As the ϕ_{ijpq} are in the image of the natural map

$$A \otimes_C A^{op} \xrightarrow{j} End_C(A)$$

this shows that j is surjective. Remains to prove injectivity. Assume $j(\sum_s a_s \otimes a'_s) = 0$ then $\sum_s a_s a a'_s = 0$ for all $a \in A$. Then,

$$\begin{aligned} \sum_s a_s \otimes a'_s = & \sum_s(\sum_{i,j} f_{ij}(a_s)a_{ij}t_i \otimes a's) \\ = & (\sum_{i,j} a_{ij}t_i) \otimes (\sum_s f_{ij}(a_s)a'_s) \\ = & (\sum_{i,j} a_{ij}t_i) \otimes (\sum_{k,s} d_{ijk}a_s d'_{ijk}a'_s) = 0 \end{aligned}$$

where we denoted $f_{ij}(a) = \sum_k d_{ijk}ad'_{ijk}$. Whence j is an isomorphism so A is an Azumaya algebra over C. $\square$

Example 124. Let A be an affine Azumaya algebra of constant rank over its center C. We have seen that $A \in$ `alg@n` and that $tr(A) = C$. Therefore,

$$C = \mathbb{C}[\mathtt{trep}_n A]^{GL_n}$$

and the quotient map

$$\mathtt{trep}_n A \xrightarrow{\pi} \mathtt{spec} C$$

is by the previous theorem a principal PGL_n fiber bundle. Indeed, let $\xi \in$ `spec`C be a point, then it determines a trace preserving semi-simple n-dimensional representation. However, there are only n-dimensional simples whence $\pi^{-1}(\xi)$ consists of a unique (closed) orbit isomorphic to PGL_n. Moreover, locally we can split the quotient map in the étale topology. That is, there is an étale cover $\{B_i\}$ of C such that

$$A \otimes_C B_i \simeq M_n(B_i)$$

whence the fiber product

$$\begin{array}{ccc} \mathtt{trep}_n A \times_{\mathtt{spec} C} \mathtt{spec} B_i \simeq PGL_n \times \mathtt{spec} B_i & \longrightarrow & \mathtt{spec} B_i \\ \downarrow & & \downarrow{\scriptstyle et} \\ \mathtt{trep}_n A & \xrightarrow{\pi} & \mathtt{spec} C \end{array}$$

Assume that C is a regular commutative algebra, then so is B_i whence $PGL_n \times \mathtt{spec} B_i$ is a smooth variety for all i, but then so is $\mathtt{trep}_n A$ by étale descent. Combining this with theorem 46 we have that for an affine Azumaya algebra A the following are equivalent

1. A is `alg@n`-smooth.
2. the center C is `commalg`-smooth.

This gives us a large supply of `alg@n`-smooth algebras.

Azumaya algebras arise further as the trace algebras of the generators of the semigroup of representation schemes.

Example 125. Let $\alpha \in \mathtt{comp} A$ be a semigroup generator and augmentation n. Let $\mathtt{rrep}_\alpha$ be the component of $\mathtt{rrep}_n\ A$ determined by α then the restriction of the quotient map

$$\begin{array}{ccc} \mathtt{rrep}_\alpha & \hookrightarrow & \mathtt{rrep}_n\ A \\ \vdots & & \downarrow{\scriptstyle \pi} \\ \mathtt{riss}_\alpha & \hookrightarrow & \mathtt{riss}_n\ A \end{array}$$

is a *principal PGL_n-fibration* . Indeed, let M be an n-dimensional A-module in $\mathtt{rep}_\alpha$, then we can deform M to its semisimplification M^{ss} (the sum of the Jordan-Hölder components). Assume $M \neq M^{ss}$ and suppose

$$M^{ss} = S_1 \oplus \ldots \oplus S_l$$

with the S_i simples. Then, $\alpha = \beta_1 + \ldots + \beta_l$ in $\mathtt{comp} A$ where β_i is the element of $\mathtt{comp} A$ corresponding to the connected component containing the simple factor S_i, contradicting the assumption that α is a semigroup generator. But then, the corresponding trace algebra

$$\int_\alpha A = M_n(\mathbb{C}[\mathtt{rep}_\alpha])^{GL_n}$$

is an Azumaya algebra of constant rank n^2 over its center which is $\mathbb{C}[\mathtt{iss}_\alpha]$.

The component semigroup of an `alg`-smooth algebra can be highly complicated. We will give some examples of universal localizations A of $\langle m \rangle$ such that $\mathtt{comp} A$ is any additive sub semigroup of $\mathbb{N}$.

Definition 88. A *special affine algebra* A has a presentation

$$A = \frac{\mathbb{C}\langle x_1, \ldots, x_a, y_1, \ldots, y_b\rangle}{(y_i p_i(x_1, \ldots, x_a, y_1, \ldots, y_{i-1}) - 1 \ , \ 1 \leq i \leq b)}$$

where p_i is a noncommutative polynomial in the variables $x_1, \ldots, x_a, y_1, \ldots, y_{i-1}$.

The *inversion depth* $\mathtt{idp}A$ of a special affine algebra A is the minimal number b required in a special presentation of A.

Example 126. If A is a special affine algebra, then A is $\mathtt{alg}$-smooth as it is a universal localization of $\langle a\rangle$. Further, $\mathtt{rep}_n A$ is a Zariski open (possibly empty) subset of $\mathtt{rep}_n \langle a\rangle = M_n^a$ and is thus connected (even irreducible). Therefore, $\mathtt{comp}A$ is an additive sub semigroup of $\mathbb{N}$. If n is a semigroup generator of $\mathtt{comp}A$, then $\int_n A$ is an Azumaya algebra of rank n^2.

Example 127. Let $c_n(x_1, \ldots, x_a)$ be a central polynomial for $n \times n$ matrices (such as Razmyslov's polynomials of theorem 89 and consider the special affine algebra

$$A = \frac{\mathbb{C}\langle x_1, \ldots, x_a, y\rangle}{(y c_n(x_1, \ldots, x_a) - 1)}$$

of inversion depth 1. Then,

$$\mathtt{comp}A = \{m \in \mathbb{N} \mid n \leq m\}$$

and this semigroup has generators $n, n+1, \ldots, 2n-1$ whence

$$\int_n A, \quad \int_{n+1} A, \quad \ldots, \quad \int_{2n-1} A$$

are Azumaya algebras. Indeed, $\mathtt{comp}A$ is the set of natural numbers m such that $\mathtt{rep}_m A \neq \emptyset$. From the defining relation of A it follows that $\mathtt{rep}_m A \neq \emptyset$ whenever there are $m \times m$ matrices $X_1, \ldots, X_a \in M_m(\mathbb{C})$ such that $c_n(X_1, \ldots, X_a) \in GL_n$. By Artin's theorem we know that $m \geq n$ and as there are $n \times n$ matrices $A_1, \ldots, A_a \in M_n(\mathbb{C})$ such that $c_n(A_1, \ldots, A_a) \neq 0$ and in the center (whence in GL_n) we can find the required matrices for all $m > n$ by taking

$$X_i = \begin{bmatrix} A_i & 0 \\ 0 & \mathbb{1}_{m-n} \end{bmatrix}$$

from which the claims follow.

Example 128. Let A be a special affine algebra and n a semigroup generator of $\mathtt{comp}A$. We claim that for any set

$$m_1 < m_2 < \ldots < m_s \in \mathtt{comp}A - \{n\}$$

we can find an element $a \in A$ such that the image of a is 0 in $\int_n A$ but is non-zero in $\int_{m_j} A$ for all $1 \leq j \leq s$.

Because $\int_n A$ is an Azumaya algebra, there exist elements $R_{ij}, S_i \in \int_n A$ and central polynomials g_i for $n \times n$ matrices such that

$$1 = \sum_i g_i(R_{ij}) S_i \quad \text{in} \quad \int_n A$$

Lift the elements R_{ij} and S_i to elements r_{ij} and s_i in A and consider the element

$$a_0 = 1 - \sum_i g_i(r_{ij}) s_i \in A$$

By construction, the image of a_0 is zero in $\int_n A$ and is equal to 1 for all $m_j < n$ by Artin's theorem. Let m_t be minimal among the m_j such that the image of a is zero in $\int_{m_t} A$, then we take

$$a_1 = a_0 + c_{m_t}(x_1, \ldots, x_a)$$

where $c_{m_t}(x_1, \ldots, x_a)$ is a central polynomial for $m_t \times m_t$ matrices (which evaluates to zero in all $\int_k A$ with $k < m_t$. Repeat this procedure until we reach m_s and take a to be the final element a_j.

THEOREM 91. *Let S be an additive sub semigroup of $\mathbb{N}$ with generators $n_1 < n_2 < \ldots < n_s$. For every integer $a \geq 1$ there is a special affine algebra A_a with $\mathtt{idp}A \leq a$ such that*

$$S \subset \mathtt{comp}A_a \qquad \textit{and} \qquad S \cap [0, an_1] = \mathtt{comp}A \cap [0, an_1]$$

In particular, if $gcd(S) = 1$ there is a special affine algebra A such that $\mathtt{comp}A = S$.

PROOF. The proof proceeds by induction on a. If $a = 1$, example 127 with $n = n_1$ gives the required algebra. Assume the result holds for $a - 1$. That is, we have a special affine algebra A_{a-1} satisfying

$$S \subset \mathtt{comp}A_{a-1} \qquad \text{and} \qquad S \cap [0, (a-1)n_1] = \mathtt{comp}A_{a-1} \cap [0, (a-1)n_1]$$

Define the set of integers

$$\{m_1, \ldots, m_b\} = (\ [(a-1)n_1, an_1] \cap \mathtt{comp}A_{a-1}\ ) - (\ [(a-1)n_1, an_1] \cap S\ )$$

Because S and $\mathtt{comp}A_{a-1}$ are the same set when restricted to $[0, (a-1)n_1]$ all of the m_i are generators of $\mathtt{comp}A_{a-1}$.

By the argument of example 128 there is for each m_i an element $r_i \in A_{a-1}$ such that the

$$\text{image of } r_i \begin{cases} = 0 & \text{in} \quad \int_{m_i} A_{a-1} \\ \neq 0 & \text{in} \quad \int_{m_j} A_{a-1}\ \forall j \neq i \end{cases}$$

Construct the special affine algebra

$$A_a = \frac{A_{a-1} * \mathbb{C}[z]}{(zr_1 r_2 \ldots r_b - 1)}$$

and check that this algebra satisfies the requirements. □

6.2. The wall.

In this section we introduce the *empire* of an $\mathtt{alg}$-smooth algebra as a combinatorial tool to initiate the study of $\mathtt{iso}(\mathtt{rep}A)$. It is a quiver on the set of simple roots $\mathtt{simp}A$ of A. We will prove that this quiver is fully determined by a (usually finite) subquiver, the *wall*, which is the full subquiver on the semigroup generators of $\mathtt{comp}A$. We have seen that $\int_\alpha A$ is an Azumaya algebra if α is a semigroup generator.

In this section we will extend this result by showing that $\int_\alpha A$ determines a *reflexive Azumaya algebra* for most $\alpha \in \mathtt{simp}A$. These reflexive Azumaya algebras determine an étale cohomology class on the smooth locus of the corresponding irreducible component $\mathtt{iss}_\alpha A$. In the next section we will give a characterization of the singular locus of these components.

Although some results extend, we will restrict attention to A an $\mathtt{alg}$-smooth algebra in this section. Recall that in this case $\mathtt{comp}A$ is the set of irreducible components of $\mathtt{rep}A$ and that $\mathtt{rep}_\alpha A$ is a smooth affine variety for every $\alpha \in \mathtt{comp}A$. As a consequence,

$$\mathtt{azu}_\alpha A \subset \mathtt{iss}_\alpha A$$

the set of all simple representations in $\mathtt{iss}_\alpha A$ is a nonempty Zariski open smooth subscheme for all $\alpha \in \mathtt{simp}A$. Further observe that as $\mathtt{rep}_\alpha A$ is smooth, the quotient variety $\mathtt{iss}_\alpha A$ is *normal*, that is, $\oint_\alpha A = \mathbb{C}[\mathtt{iss}_\alpha A]$ is an integrally closed Noetherian domain.

Example 129. ($ext(\alpha, \beta)$) Let $\alpha \neq \beta \in \mathtt{simp}A$ with $dim(\alpha) = n$ and $dim(\beta) = m$. If A is generated by k elements, there is an affine subvariety

$$\begin{array}{ccc} \mathtt{Ext}_A(\alpha,\beta) & \hookrightarrow & \mathtt{rep}_\alpha A \times \mathtt{rep}_\beta A \times M_{m\times}(\mathbb{C})^{\oplus k} \\ \downarrow e & & \\ \mathtt{rep}_\alpha A \times \mathtt{rep}_\beta A & & \end{array}$$

such that the fiber $e^{-1}(V, W)$ over a point $(V, W) \in \mathtt{rep}_\alpha A \times \mathtt{rep}_\beta A$ is the vectorspace $Ext^1_A(V, W)$. Because the fiber dimension is upper-semicontinuous and as the target space is irreducible, there is a non-empty Zariski open subset $\mathtt{ext}_{min}$ of $\mathtt{rep}_\alpha A \times \mathtt{rep}_\beta A$ where $dim_{\mathbb{C}}\ Ext^1_A(V, W)$ attains its minimal value. We denote this minimal dimension with $ext(\alpha, \beta)$.

Observe that as $\alpha, \beta \in \mathtt{simp}A$ there is an open set of couples (V, W) with $V \in \mathtt{azu}_\alpha A$ and $W \in \mathtt{azu}_\beta A$ such that $dim_{\mathbb{C}}\ Ext^1_A(V, W) = ext(\alpha, \beta)$. Interchanging the roles of α and β we have that there is also an open subset such that $dim_{\mathbb{C}}\ Ext^1_A(W, V) = ext(\beta, \alpha)$. In fact, we claim

If A is alg*-smooth, then for all $V \in \mathtt{azu}_\alpha A$ and all $W \in \mathtt{azu}_\beta A$ we have*

$$dim_{\mathbb{C}}\ Ext^1_A(V, W) = ext(\alpha, \beta) \qquad \text{and} \qquad dim_{\mathbb{C}}\ Ext^1_A(W, V) = ext(\beta, \alpha)$$

Indeed, $V \oplus W$ is a smooth point of $\mathtt{rep}_{\alpha+\beta} A$ with stabilizer subgroup $\mathbb{C}^* \times \mathbb{C}^*$. Computing tangent spaces (or normal spaces to orbits) we have the following equalities

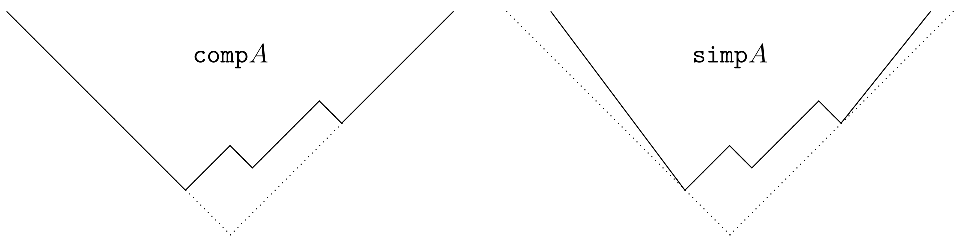
$$\begin{cases} dim\ \mathtt{rep}_\alpha A & = (n^2 - 1) + dim\ Ext^1_A(V, V) \\ dim\ \mathtt{rep}_\beta A & = (m^2 - 1) + dim\ Ext^1_A(W, W) \end{cases}$$

and the dimension of $\mathtt{rep}_{\alpha+\beta} A$ is equal to

$$(n+m)^2 - 2 + dim\ Ext^1_A(V, V) + dim\ Ext^1_A(W, W) + dim\ Ext^1_A(V, W) + dim\ Ext^1_A(W, V)$$

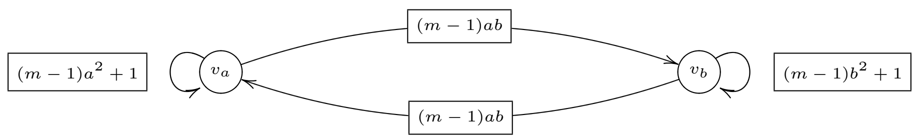
Because there is an open subset where $Ext^1_A(V, W)$ and $Ext^1_A(W, V)$ both attain the minimal value we see that these numbers cannot increase whenever V and W are simple representations, proving the claim.

Definition 89. The *empire* of the alg-smooth algebra A, $\mathtt{emp}\ A$, is the quiver having vertices v_α for every simple root $\alpha \in \mathtt{simp}\ A$



such that for $\alpha \neq \beta$ there are $ext(\alpha, \beta)$ directed arrows from v_α to v_β and there are $dim\ \mathtt{iss}_\alpha A$ loops in every vertex v_α.

EXAMPLE 130. $\mathtt{emp}\langle m \rangle$ is the complete quiver K_∞ on infinitely many vertices v_n, $n \in \mathbb{N}_+$ such that the full subquiver on any two vertices v_a and v_b is of the form



Before we show that $\mathtt{emp}A$ is controlled by a tiny subquiver, we indicate its importance in the study of isomorphism classes of finite dimensional representations of A.

DEFINITION 90. Let $\beta \in \mathbb{N}^{\mathtt{simp}A}$ be a dimension vector with finite support $\mathtt{supp}\beta = \{\alpha_1, \ldots, \alpha_k\}$. With $E_{\mathtt{supp}\beta}$ we denote the full subquiver of $\mathtt{emp}A$ on the vertices of $\mathtt{supp}\beta$. We denote

$$\mathtt{null}_\beta \mathtt{emp}A = \mathtt{null}_\beta\ E_{\mathtt{supp}\beta} \times \mathtt{azu}_{\alpha_1} A \times \ldots \mathtt{azu}_{\alpha_k} A$$

where $\mathtt{null}_\beta E_{\mathtt{supp}\beta}$ is the *nullcone* for the basechange action of $GL(\beta)$ on the representation space $\mathtt{rep}_\beta\ E_{\mathtt{supp}\beta}$. We have the induced action of $GL(\beta)$ on the component $\mathtt{null}_\beta\ E_{\mathtt{supp}\beta}$ and denote the orbits by

$$\mathtt{iso}(\mathtt{null}_\beta \mathtt{emp}A)$$

If β varies over all dimension vectors with finite support, we denote

$$\mathtt{nullemp}A = \bigsqcup_\beta \mathtt{null}_\beta \mathtt{emp}A$$

and denote the orbits for the natural $GL(\beta)$-actions by $\mathtt{iso}(\mathtt{nullemp}A)$.

THEOREM 92. *If A is $\mathtt{alg}$-smooth, there is a natural one-to-one correspondence*

$$\mathtt{iso}(\mathtt{rep}A) \leftrightarrow \mathtt{iso}(\mathtt{nullemp}A)$$

PROOF. Let $M \in \mathtt{rep}_n A$ with Jordan-Hölder decomposition

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

If S_i belongs to the irreducible component $\mathtt{iss}_{\alpha_i} A$ where $\alpha_i \in \mathtt{simp}A$, then $(S_1, \ldots, S_k)$ is a point of $\mathtt{azu}_{\alpha_1} A \times \ldots \times \mathtt{azu}_{\alpha_k} A$. If $\pi : \mathtt{rep}_n A \twoheadrightarrow \mathtt{iss}_n A$ is the quotient map, then $M \in \pi^{-1}(\xi)$. By the étale slice theorem we have a GL_n-equivariant isomorphism

$$\pi^{-1}(\xi) \simeq GL_n \times^{GL(\alpha_\xi)} \mathtt{null}_{\alpha_\xi} Q_\xi$$

in particular, there is a natural one-to-one correspondence between isoclasses of representations in $\pi^{-1}(\xi)$ and $GL(\alpha_\xi)$-orbits in the nullcone of Q_ξ.

Remains to prove that if β is the dimension vector with support $\{\alpha_1, \ldots, \alpha_k\}$ such that $\beta \mid \mathtt{supp}\beta = (e_1, \ldots, e_k)$ then $E_{\mathtt{supp}\beta} \simeq Q_\xi$. But this follows from the definition of Q_ξ and example 129 above. □

This result reduces the study of $\mathtt{iso}(\mathtt{rep}A)$ to (a) the description of the Azumaya loci of the Cayley-Hamilton orders $\int_\alpha A$ for $\alpha \in \mathtt{simp}A$ and (b) a purely quiver-theoretic problem (independent of A) to describe the nullcone of quiverrepresentations. We will investigate these nullcones in the last chapter.

We will prove that the structure of $\mathtt{emp}A$ is determined by a (usually finite) subquiver, the *wall* of A. First we need to derive some consequences of the étale slice theorems in the case of quiver representations.

Definition 91. For a quiver setting (Q, α), $\mathtt{types}_\alpha Q$ will be the set of all semi-simple representation types of points in $\mathtt{iss}_\alpha Q$. That is, $\tau \in \mathtt{types}_\alpha Q$ if and only if

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$$

where $e_i \in \mathbb{N}_0$ are the *multiplicities* and $\alpha_i \in \mathtt{simp}\langle Q\rangle$ such that

$$\alpha = e_1\alpha_1 + \ldots + e_z\alpha_z$$

Theorem 85 gives an algorithm to determine the finite set $\mathtt{types}_\alpha Q$.

We define two representation types

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z) \quad \text{and} \quad \tau' = (e'_1, \alpha'_1; \ldots; e'_{z'}, \alpha'_{z'})$$

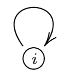
to be direct *successors* $\tau' < \tau$ if and only if one of the following two cases occurs

- (splitting of one simple) : $z' = z + 1$ and for all but one $1 \le i \le z$ we have that $(e_i, \alpha_i) = (e'_j, \alpha'_j)$ for a uniquely determined j and for the remaining i_0 we have that the remaining couples of τ' are
$$(e_i, \alpha'_u; e_i, \alpha'_v) \quad \text{with} \quad \alpha_i = \alpha'_u + \alpha'_v$$
- (combining two simple types) : $z' = z - 1$ and for all but one $1 \le i \le z'$ we have that $(e'_i, \alpha'_i) = (e_j, \alpha_j)$ for a uniquely determined j and for the remaining i we have that the remaining couples of τ are
$$(e_u, \alpha'_i; e_v, \alpha'_i) \quad \text{with} \quad e_u + e_v = e'_i$$

The direct successor relation $<$ induces a partial ordering $\ll$ on $\mathtt{types}_\alpha Q$.

Theorem 93. *For any quiver setting (Q, α) we have :*

(1) *Let $\xi \in \mathtt{iss}_\alpha\ Q$ be a point of representation type*
$$\tau = t(\xi) = (e_1, \alpha_1; \ldots, e_z, \alpha_z) \in \mathtt{types}_\alpha Q$$
The normal space N_x to the orbit in $x \in \mathcal{O}(M_\xi)$ in $\mathtt{rep}_\alpha\ Q$ (where M_ξ is the corresponding semi-simple representation) is determined by the local quiver setting (Q_τ, α_τ), *that is,*
$$N_x \simeq \mathtt{rep}_{\alpha_\tau}\ Q_\tau$$
where (Q_τ, α_τ) depends only on τ. More precisely, Q_τ is the quiver on z vertices (the number of distinct simple components of V_ξ) say $\{w_1, \ldots, w_z\}$ with
$$\#\ \textcircled{\scriptsize j} \xleftarrow{\ a\ } \textcircled{\scriptsize i} \quad = \quad -\chi_Q(\alpha_i, \alpha_j) \quad \textit{for } i \neq j, \textit{ and}$$

$\#$  $= \quad 1 - \chi_Q(\alpha_i, \alpha_i)$

and $\alpha_\tau = (e_1, \ldots, e_z)$.

(2) *The quotient variety* $\mathtt{iss}_\alpha\ Q$ *has a finite stratification into locally closed smooth subvarieties*

$$\mathtt{iss}_\alpha\ Q = \bigsqcup_{\tau \in \mathtt{types}_\alpha Q} \mathtt{iss}_\alpha(\tau)$$

where $\mathtt{iss}_\alpha(\tau)$ *is the set of points* $\xi \in \mathtt{iss}_\alpha\ Q$ *such that* $t(\xi) = \tau$. *Moreover, if* $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$ *then*

$$dim\ \mathtt{iss}_\alpha(\tau) = \sum_{i=1}^{z} (1 - \chi_Q(\alpha_i, \alpha_i))$$

(3) *The closure inclusion ordering of these locally smooth strata is given by*

$$\mathtt{iss}_\alpha(\tau') \subset \overline{\mathtt{iss}_\alpha(\tau)} \qquad \textit{iff} \qquad \tau' \ll \tau$$

Proof. (1) : Take the point $\overline{(1, x)}$ in the irreducible component $GL_n \times^{GL(\alpha)} \mathtt{rep}_\alpha\ Q$ in $\mathtt{rep}_n \langle Q \rangle$ where $n = \sum_{i=1}^z |\alpha_i|$ and apply theorem 72. The Euler-form description follows from theorem 70 and Schur's lemma stating that $Hom_{\langle Q \rangle}(S_i, S_j) = \delta_{ij}\mathbb{C}$ whenever S_i and S_j are simple representations. Alternatively, one can apply the Knop-Luna slice theorem directly to the $GL(\alpha)$-action on $\mathtt{rep}_\alpha\ Q$ and do a book-keeping calculation similar to the proof of theorem 72, see [**42**] for more details.

(2) : Let $\xi \in \mathtt{iss}_\alpha(\tau)$ and consider a nearby point ξ'. By the étale local description of theorem 72 and part (1) we may assume (by étale descent) that ξ' corresponds to a semi-simple α_τ-dimensional representation of Q_τ. If some trace of an oriented cycle in Q_τ of length > 1 is non-zero, then ξ' cannot be of representation type τ. Therefore, if $\xi' \in \mathtt{iss}_\alpha(\tau)$ it is determined by the traces of the loops in Q_τ. Therefore, locally in the étale topology $\mathtt{iss}_\alpha(\tau)$ is an affine space near ξ of dimension the number of loops in Q_τ.

(3) : Observe that $\tau' \ll \tau$ if and only if the stabilizer subgroup $GL(\alpha_\tau)$ is conjugated (in $GL(\alpha)$) to a subgroup of the stabilizer subgroup $GL(\alpha_{\tau'})$. The statement now follows, either from general theory as in [**63**, lemma 5.5] or from a comparison of the local quivers. □

Definition 92. For any quiver setting (Q, α) we have that $\mathtt{rep}_\alpha\ Q$ is an irreducible variety, whence is is the quotient $\mathtt{iss}_\alpha\ Q$. Hence, there is a unique *Zariski open* stratum

$$\mathtt{iss}_\alpha(\tau_{gen})$$

We call $\tau_{gen} \in \mathtt{types}_\alpha Q$ the *generic semi-simple representation type* for (Q, α).

Example 131. There is an algorithm to compute the generic semi-simple representation type τ_{gen} :

`input :` A quiver setting (Q, α) and a semi-simple representation type

$$\tau = (e_1, \alpha_1; \ldots; e_l, \alpha_l) \in \mathtt{types}_\alpha Q$$

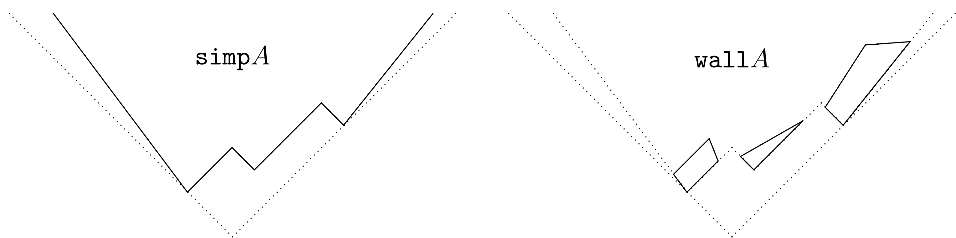
For $\alpha = (a_1, \ldots, a_k)$ one can always start with the type $(a_1, \vec{v_1}; \ldots; a_k, \vec{v_k})$.

`step 1` : Compute the local quiver Q_τ on l vertices and the dimension vector α_τ. If the only oriented cycles in Q_τ are vertex-loops, `stop and output this type`. If not, proceed.
`step 2` : Take a proper oriented cycle $C = (j_1, \ldots, j_r)$ with $r \geq 2$ in Q_τ where j_s is the vertex in Q_τ determined by the dimension vector α_{j_s}. Set $\beta = \alpha_{j_1} + \ldots + \alpha_{j_r}$, $e'_i = e_i - \delta_{iC}$ where $\delta_{iC} = 1$ if $i \in C$ and is 0 otherwise. `replace` τ by the new semi-simple representation type

$$\tau' = (e'_1, \alpha_1; \ldots; e'_l, \alpha_l; 1, \beta)$$

`delete` the terms (e'_i, α_i) with $e'_i = 0$ and set τ to be the resulting type. `goto step 1`.

Definition 93. The *wall* of the algebra A, $\texttt{wall}A$ is the full subquiver of $\texttt{emp}A$ on the vertices v_α where α runs over the semigroup generators of $\texttt{comp}A$



Recall that for a semigroup generator α, the Cayley-Hamilton algebra $\int_\alpha A$ is an Azumaya algebra, whence $\texttt{azu}_\alpha A = \texttt{iss}_\alpha A$.

Example 132. For the path algebra of a quiver Q on k vertices we have that $\texttt{comp}\langle Q \rangle \simeq \mathbb{N}^k$ and hence the semigroup generators are given by the vertex-dimension vectors δ_i. But then,

$$ext(\delta_i, \delta_j) = -\chi_Q(\delta_i, \delta_j) + \delta_{ij}$$

from which it follows that $\texttt{wall}\langle Q \rangle \simeq Q$.

Theorem 94. *If A is an* `alg`*-smooth algebra, then the* $\texttt{wall}A$ *contains enough information to determine the quiver structure of the whole* $\texttt{emp}A$.

Proof. Let $\{\beta_i \ , \ i \in I\}$ be the semigroup generators for $\texttt{comp}A$. First we have find the vertices of $\texttt{emp}A$, that is, to characterize the set $\texttt{simp}A$. Assume $\alpha \in \texttt{comp}A$, then

$$\alpha = e_1 \beta_{i_1} + \ldots + e_l \beta_{i_l}$$

for a finite number of semigroup generators β_{i_j} and $e_j \in \mathbb{N}$. Take a simple representation S_i in $\texttt{rep}_{\beta_i} A$, then

$$M_\xi = S_{i_1}^{\oplus e_1} \oplus \ldots \oplus S_{i_l}^{\oplus e_l} \in \texttt{rep}_\alpha A$$

is a closed orbit and by the étale slice results we know that there is étale isomorphism between a neighborhood of $\xi \in \texttt{iss}_\alpha A$ and a neighborhood of the trivial representation in $\texttt{iss}_{\alpha_\xi} Q_\xi$. If $\alpha \in \texttt{simp}A$, then this neighborhood must contain simple representations, whence $\alpha_\xi = (e_1, \ldots, e_l)$ is a simple root for the quiver Q_ξ which by definition is the full subquiver of $\texttt{wall}A$ on the vertices corresponding to $\beta_{i_1}, \ldots, \beta_{i_l}$.

Therefore, $\mathtt{simp}A$ is the subset of $\mathtt{comp}A$ obtained from those positive integer combinations of the generators $\alpha = \sum_i e_i \beta_i$ such that the dimension vector $e = (e_i)$ is a simple root of the finite quiver $\mathtt{wall}A \mid \mathtt{supp}e$. Observe that by theorem 85 we have an algorithm to determine these simple roots.

Next, we have to determine the number of arrows between v_α and v_β for $\alpha, \beta \in \mathtt{simp}A$. By the foregoing argument we have a finite full subquiver Q of $\mathtt{wall}A$ on the vertices $\beta_{i_1}, \ldots, \beta_{i_l}$ such that $\alpha = \sum e_j \beta_{i_j}$ and $\beta = \sum f_j \beta_{i_j}$ with e and f simple roots of Q. The $\mathtt{wall}A$ determines the local structure of $\mathtt{rep}_{\alpha+\beta}A$ in a point of representation type $(e_1, \beta_{i_1}; \ldots; e_l, \beta_{i_l}; f_1, \beta_{i_1}; \ldots; f_l, \beta_{i_l})$ and as this stratum lies in the closure of the stratum of points of type $(1, \alpha; 1, \beta)$ we know by the foregoing discussion on local quivers for path algebras of quivers the local quiver in those points. Therefore, $ext(\alpha, \beta) = \delta_{\alpha\beta} - \chi_Q(\alpha, \beta)$ and we obtain that the number of arrows from v_α to v_β is also determined by $\mathtt{wall}A$. □

The wall also determines the structure of $\mathtt{iss}_\alpha A$ for $\alpha \in \mathtt{comp}A$. We will denote by χ_W the Euler-form of the possibly infinite quiver $\mathtt{wall}A$ but as we will only apply it to dimension vectors having finite support this causes no problems.

THEOREM 95. *Let A be an $\mathtt{alg}$-smooth algebra. For $\alpha \in \mathtt{comp}$ denote with $\mathtt{types}\alpha$ the collection of all representation types*

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z) \quad \textit{with} \quad \alpha_i \in \mathtt{simp}A$$

then there is a finite stratification into locally closed smooth subvarieties

$$\mathtt{iss}_\alpha A = \bigsqcup_{\tau \in \mathtt{types}\alpha} \mathtt{iss}_\alpha(\tau)$$

where $\mathtt{iss}_\alpha(\tau)$ is the set of all points $\xi \in \mathtt{iss}_\alpha A$ of representation type τ. If $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$ then this strata is isomorphic to

$$\mathtt{iss}_\alpha(\tau) \simeq \mathtt{azu}_{\alpha_1}A \times \ldots \times \mathtt{azu}_{\alpha_z}A$$

and hence has dimension $\sum_{i=1}^z dim\ \mathtt{iss}_{\alpha_i}A = \sum_{i=1}^z (1 - \chi_W(\gamma_i, \gamma_i))$ where $\mathtt{supp}\gamma_i = \{\beta_{i1}, \ldots, \beta_{ik_i}\}$ a subset of vertices of $\mathtt{wall}A$ such that $\sum \gamma_i(j)\beta_{ij} = \alpha_i$.

The local quiver-setting in a point $\xi \in \mathtt{iss}_\alpha A$ of type $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$ is

$$(Q_\xi, \alpha_\xi) = (\mathtt{emp}A \mid \{\alpha_1, \ldots, \alpha_z\}, (e_1, \ldots, e_z))$$

and as the right-hand side is fully determined by the $\mathtt{wall}A$, the wall contains enough information to describe the étale local structure of

$$\oint_\alpha A \qquad \textit{and} \qquad \int_\alpha A$$

for all $\alpha \in \mathtt{comp}A$.

PROOF. Follows from the proof of the previous theorem, the results on quotient varieties of quiver representations and the étale local structure of $\mathtt{alg}$-smooth algebras. □

In particular, this result shows that for most $\alpha \in \mathtt{simp}A$, the *ramification locus*

$$\mathtt{ram}_\alpha A = \mathtt{iss}_\alpha A - \mathtt{azu}_\alpha A$$

has codimension ≥ 2. In this case, the *reflexive closure* $\int_n^{**} A$ of the Cayley-Hamilton order $\int_\alpha A$ is a *reflexive Azumaya algebra* and hence determines an étale cohomology class.

DEFINITION 94. Let $C \in$ `commalg` be a normal affine domain with field of fractions K. An C-subalgebra A of a central simple K-algebra Σ of dimension n^2 is said to be an *order* if A is a finitely generated C-module and contains a K-basis of Σ, that is, $A.K = \Sigma$. Because C is integrally closed, the reduced trace of Σ on elements of A has its values in C. That is, $(A, tr_A) \in$ `alg@n` and $tr(A) = C$. An order is said to be *maximal* if there is no C-order A' in Σ properly containing A.

A is said to be a *reflexive Azumaya algebra* over C iff the center of A is C, A is a reflexive V-module, that is

$$\cap_{\mathfrak{p}} A_{\mathfrak{p}} = A$$

where the intersection is taken over all height one prime ideals $\mathfrak{p}$ of C, and if every $A_{\mathfrak{p}}$ is an Azumaya algebra over the discrete valuation ring $C_{\mathfrak{p}}$.

Two reflexive Azumaya algebras (possibly in different central simple K-algebras) are said to be *equivalent* if there exist *reflexive* C-modules M and N such that

$$A \otimes'_C End_C(M) \simeq A' \otimes'_C End_C(N)$$

where the modified tensor product is the *reflexive closure*, that is, the intersection of all localizations at height one prime ideals of C.

The set of all equivalence classes of reflexive Azumaya algebras, equipped with the modified tensor product, is an Abelian group called the *reflexive Brauer group* of C and denoted $\beta(C)$. One can prove, see for example [**50**], that

$$\beta(C) = \cap_{ht(\mathfrak{p})=1} Br(C_{\mathfrak{p}}) \hookrightarrow Br(K)$$

That is, one can view the reflexive Brauer group as being the subgroup of $Br(K)$ consisting of those central simple algebras containing a maximal order with ramification locus having codimension at least two.

EXAMPLE 133. If $C = \mathbb{C}[X]$ with X an affine smooth curve, then any height one prime is maximal. Therefore, a reflexive Azumaya algebra is Azumaya and $\beta(C) = Br(C)$.

If $C = \mathbb{C}[X]$ with X an affine surface, then there are reflexive Azumaya algebras which are not Azumaya. For example,

$$C = \frac{\mathbb{C}[x,y,z]}{(x^2 - yz)} \qquad A = End_C(C \oplus P)$$

where C is the affine cone and $P = (x, y)$ is a ruling. Then, A is an Azumaya algebra in every point except the top $\mathfrak{m} = (x, y, z)$. Still, every reflexive Azumaya algebra is Azumaya over all smooth points. This follows from the fact that reflexive modules over regular local rings of dimension ≤ 2 are free and because reflexive Azumaya algebras which are projective are Azumaya. Therefore, if C is the coordinate ring of a surface X, then $\beta(C) = Br(X_{sm})$ where X_{sm} is the smooth locus of X.

If $C = \mathbb{C}[X]$ and $dim\ X \geq 3$, then a reflexive Azumaya algebra does not have to be Azumaya on the whole of X_{sm}. For example, take $C = \mathbb{C}[x, y, z]$ and

$$M = \texttt{ker}\ C.a \oplus C.b \oplus C.c \xrightarrow{a \mapsto x, b \mapsto y, c \mapsto z} C$$

then M is a reflexive module which is not projective in the origin, but then $End_C(M)$ is a reflexive Azumaya algebra which is not Azumaya in the origin.

In general, one has the following important result due to R. Hoobler, see [**24**]

$$\beta(C) \simeq Br(C)$$

whenever C is `commalg`-smooth. For arbitrary affine normal domains one has the following cohomological description of the reflexive Brauer group.

THEOREM 96. *If $C \in$ `commalg` is the coordinate ring of a normal affine variety X, then*

$$\beta(C) \simeq H^2_{et}(X_{sm}, \mathbb{G}_m)$$

PROOF. The singular locus of X determines an ideal I which is of height at least 2 because C is a normal domain. Therefore, there are $c_1, c_2 \in I$ such that $ht(Cc_1 + Cc_2) = 2$. Let U be the open set determined by the ideal $Cc_1 + Cc_2$, then U can be covered by two affine open sets $U_1 = \mathbb{X}(c_1)$ and $U_2 = \mathbb{X}(c_2)$. Because the reflexive Brauer group is determined by the Brauer groups in height one primes we obtain $\beta(C)$ as the pullback

$$\begin{array}{ccc} \beta(C) & \longrightarrow & \beta(C_{c_1}) \\ \downarrow & & \downarrow \\ \beta(C_{c_2}) & \longrightarrow & \beta(C_{c_1c_2}) \end{array}$$

Because of Hoobler's result and the fact that $U \subset X_{sm}$ we can replace three corners by Brauer groups

$$\begin{array}{ccc} \beta(C) & \longrightarrow & Br(U_1) \\ \downarrow & & \downarrow \\ Br(U_2) & \longrightarrow & Br(U_1 \cap U_2) \end{array}$$

By Gabber's result, theorem 87, we know that the Brauer group of any affine scheme X is equal to $H^2_{et}(X, \mathbb{G}_m)_{tors}$, the group of torsion elements of $H^2_{et}(X, \mathbb{G}_m)$. If X is in addition smooth, then because of the inclusion

$$H^2_{et}(X, \mathbb{G}_m) \longrightarrow H^2_{et}(\mathbb{C}(X), \mathbb{G}_m) = Br(\mathbb{C}(X))$$

we know that the cohomology group is torsion so we can dispose of the subscript. That is, we obtain $\beta(C)$ as the pullback of the diagram

$$\begin{array}{ccc} \beta(C) & \longrightarrow & H^2_{et}(U_1, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^2_{et}(U_2, \mathbb{G}_m) & \longrightarrow & H^2_{et}(U_1 \cap U_2, \mathbb{G}_m) \end{array}$$

Equivalently, this asserts that $\beta(C) = H^2_{et}(U, \mathbb{G}_m) \simeq Br(U)$ where the last isomorphism follows because Gabber's result is actually valid for the union of two affine schemes. Finally, we invoke Grothendieck's result on cohomological purity of the Brauer group [**21**, III.Thm.6.1] to the situation $U \subset X_{sm}$. This asserts that

$$H^2_{et}(U, \mathbb{G}_m) \simeq H^2_{et}(X_{sm}, \mathbb{G}_m)$$

and the claim follows. □

For this reason it is important to determine the smooth locus of $\mathtt{iss}_\alpha A$ which we will do in the next section. We can also read off from $\mathtt{emp}A$ in which points $\xi \in \mathtt{iss}_\alpha A$ the order $\int_\alpha A$ is étale split.

THEOREM 97. *Let A be* $\mathtt{alg}$*-smooth, $\alpha \in \mathtt{simp}A$ and $\xi \in \mathtt{iss}_\alpha A$ a point of representation type*

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$$

Then, $\int_\alpha A$ is étale split in ξ if and only if $gcd(e_1, \ldots, e_z) = 1$.

PROOF. Firs observe that the local quiver Q_ξ is the full subquiver of $\mathtt{emp}A$ on the vertices corresponding to $\{\alpha_1, \ldots, \alpha_z\}$. In the étale topology $\int_\alpha A$ is locally in ξ Morita equivalent to the algebra

$$B = \int_{\alpha_\xi} \langle Q_\xi \rangle = M_e(\mathbb{C}[\mathtt{rep}_{\alpha_\xi}\ Q_\xi])^{GL(\alpha_\xi)}$$

(where $e = e_1 + \ldots + e_z$) locally at the trivial representation. Therefore, it suffices to investigate the splitting behavior of the latter. The quotient map

$$\mathtt{rep}_{\alpha_\xi} \langle Q_\xi \rangle \twoheadrightarrow \mathtt{iss}_{\alpha_\xi} \langle Q_\xi \rangle$$

is over the Azumaya locus a principal $PGL(\alpha_\xi)$-fibration, that is, it determines an element of

$$H^1_{et}(\mathtt{azu}, PGL(\alpha))$$

This pointed set classifies Azumaya algebras over $\mathtt{azu}$ with a distinguished embedding of $C_z = \mathbb{C} \times \ldots \times \mathbb{C}$ (the vertex-idempotents in B) which are split by an étale cover on which this embedding is conjugated to the standard embedding $C_z \subset M_e(\mathbb{C})$.

If $gcd(e_1, \ldots, e_z) = 1$ then B determines the trivial class in the Brauer group. For, let B' be an Azumaya localization of B. By assumption, the natural map between the K-groups

$$K_0(C_z) \longrightarrow K_0(M_e(\mathbb{C}))$$

is surjective, whence the same is true for B' proving that the class of B' is split by a Zariski cover (and not merely an étale one). In other words,

$$\mathtt{rep}_{\alpha_\xi} B' \simeq \mathtt{iss}_{\alpha_\xi} B' \times PGL(\alpha)$$

If $gcd(e_1, \ldots, e_z) = n > 1$, then we form a new quiver Q' by extending Q_ξ with an extra vertex v_0 and having e_i/n directed arrows from v_0 to v_i. Further, consider the extended dimension vector $\alpha' = (n, e_1, \ldots, e_z)$. There is an open subset of $\mathtt{rep}_{\alpha'}\ Q'$ where the e_i/n maps from v_0 to v_i define an isomorphism from $V_0^{\oplus e_i/n} \longrightarrow V_i$ for all i. This reduces the classification problem for the quiver setting (Q', α') on this set to that of (Q_ξ, α_ξ) where each vertex space is in addition given a fixed representation as the vectorspace $V^{\oplus e_i/n}$ where V is a vectorspace of dimension n. But this is the same problem as studying a large number of $n \times n$ matrices under simultaneous conjugation. This latter problem is not étale split and $\mathtt{iss}_{\alpha'}\ Q'$ is rational over $\mathtt{iss}_{\alpha_\xi}\ Q_\xi$, see [**43**] also the former cannot be split. □

EXAMPLE 134. (Simple representations of torus knot groups) Consider a solid cylinder C with q line segments on its curved face, equally spaced and parallel to the axis. If the ends of C are identified with a twist of $2\pi\frac{p}{q}$ where p is an integer relatively prime to q, we obtain a single curve $K_{p.q}$ on the surface of a solid torus

T. If we assume that T lies in $\mathbb{R}^3$ in the standard way, the curve $K_{p.q}$ is called the (p,q) torus knot.

The fundamental group of the complement $\mathbb{R}^3 - K_{p,q}$ is called the $(p.q)$-torus knot group $G_{p,q}$ which has a presentation

$$G_{p,q} = \pi_1(\mathbb{R}^3 - K_{p,q}) \simeq \langle a, b \mid a^p = b^q \rangle$$

An important special case is $(p,q) = (2,3)$ in which case we obtain the three string braid group, $G_{2,3} \simeq B_3$.

Recall that the center of $G_{p,q}$ is generated by a^p and that the quotient group is the free product of cyclic groups of order p and q

$$\overline{G_{p,q}} = \frac{G_{p,q}}{\langle\ a^p\ \rangle} \simeq \mathbb{Z}_p * \mathbb{Z}_q$$

As the center acts by scalar multiplication on any irreducible representation, the representation theory of $G_{p,q}$ essentially reduces to that of $\mathbb{Z}_p * \mathbb{Z}_q$. Observe that in the special case $(p,q) = (2,3)$ considered above, the quotient group is the modular group $PSL_2(\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$.

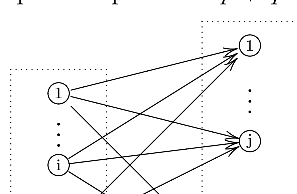
Let V be an n-dimensional representation of $\mathbb{Z}_p * \mathbb{Z}_q$, then the restriction of V to the cyclic subgroups $\mathbb{Z}_p$ and $\mathbb{Z}_q$ decomposes into eigenspaces

$$\begin{cases} V \ \downarrow_{\mathbb{Z}_p} & \simeq S_1^{\oplus a_1} \oplus S_\zeta^{\oplus a_2} \oplus \ldots \oplus S_{\zeta^{p-1}}^{\oplus a_p} \\ V \ \downarrow_{\mathbb{Z}_q} & \simeq T_1^{\oplus b_1} \oplus T_\xi^{\oplus b_2} \oplus \ldots \oplus T_{\xi^{q-1}}^{\oplus b_q} \end{cases}$$

where ζ (resp. ξ) is a primitive q-th (resp. p-th) root of unity and where S_{ζ^i} (resp. T_{ξ^i}) is the one-dimensional space $\mathbb{C}v$ with action $a.v = \zeta^i v$ (resp. $b.v = \xi^i v$). Using these decompositions we define linear maps ϕ_{ij} as follows

$$\begin{array}{ccccc} S_{\zeta^{i-1}}^{\oplus a_i} & & \xrightarrow{\phi_{ij}} & & T_{\xi^{j-1}}^{\oplus b_j} \\ \big\downarrow & & & & \big\uparrow \\ S_1^{\oplus a_1} \oplus S_\zeta^{\oplus a_2} \oplus \ldots \oplus S_{\zeta^{p-1}}^{\oplus a_p} & = & V & = & T_1^{\oplus b_1} \oplus T_\xi^{\oplus b_2} \oplus \ldots \oplus T_{\xi^{q-1}}^{\oplus b_q} \end{array}$$

This means that we can associate to an n-dimensional representation V of $\mathbb{Z}_p * \mathbb{Z}_q$ a representation of the full bipartite quiver on $p+q$ vertices



where we put at the left i-th vertex the space $S_{\zeta^{i-1}}^{\oplus a_i}$, on the right j-th vertex the space $T_{\xi^{j-1}}^{\oplus b_j}$ and the morphism connecting the i-th left vertex to the right j-vertex is the map ϕ_{ij}. That is, to V we associate a representation V_Q of dimension vector $\alpha = (a_1, \ldots, a_p; b_1, \ldots, b_q)$ and of course we have that $a_1 + \ldots + a_p = n = b_1 + \ldots + b_q$.

If V and W are isomorphic as $\mathbb{Z}_p * \mathbb{Z}_q$ representation, they have isomorphic weight space decompositions for the restrictions to $\mathbb{Z}_p$ and $\mathbb{Z}_q$ and fixing bases in these weight spaces gives isomorphic quiver representations $V_Q \simeq W_Q$. Further, observe that if V is a representation of $\mathbb{Z}_p * \mathbb{Z}_q$ then the matrix

$$m(V_Q) = \begin{bmatrix} \phi_{11}(V_Q) & \dots & \phi_{p1}(V_Q) \\ \vdots & & \vdots \\ \phi_{1q}(V_Q) & \dots & \phi_{pq}(V_Q) \end{bmatrix}$$

is invertible. Consider the universal localization $\langle Q \rangle_\sigma$ where σ corresponds to the above map ϕ. Then the variety of semi-simple n-dimensional representations of $\mathbb{Z}_p * \mathbb{Z}_q$ decomposes into components

$$\mathtt{iss}_n\ \mathbb{Z}_p * \mathbb{Z}_q = \bigsqcup_{\sum a_i = \sum b_j = n} \mathtt{iss}_\alpha\ \langle Q \rangle_\sigma$$

We see that $\mathtt{comp}\langle Q \rangle_\sigma$ is the subsemigroup of $\mathbb{Z}^{p+q}$

$$\mathtt{comp}\langle Q \rangle_\sigma = \{(a_1, \dots, a_p; b_1, \dots, b_q)\ \sum a_i = \sum b_j\ \}$$

There are some obvious 1-dimensional irreducible representations of $\mathbb{Z}_p * \mathbb{Z}_q$

$$V_{ij} = \mathbb{C}v \quad \text{with } a.v = \zeta^{i-1}v \text{ and } b.v = \xi^{j-1}v.$$

which have dimension vector $\alpha_{ij} = (\delta_{1i}, \dots, \delta_{pi}; \delta_{1j}, \dots, \delta_{qj})$. This shows that the generators of $\mathtt{comp}\langle Q \rangle_\sigma$ are given by these $p.q$ dimensionvectors.

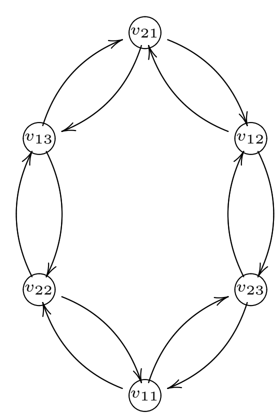
Hence, $\mathtt{wall}\langle Q \rangle_\sigma$ is the quiver on $i.j$ vertices v_{ij} (corresponding to the θ-stable representations V_{ij}) such that the number of arrows from v_{ij} to v_{kl} is equal to

$$\delta_{ij,kl} - \chi_Q(\alpha_{ij}, \alpha_{kl})$$

Given the special form of the full bipartite quiver Q it is easy to verify that

$$\#\ \{\ a \in \mathtt{wall}\langle Q \rangle_\sigma\ \mid\ \textcircled{v_{kl}} \xleftarrow{a} \textcircled{v_{ij}}\ \} = \begin{cases} 1 & \text{if } i \neq k \text{ and } j \neq l \\ 0 & \text{otherwise.} \end{cases}$$

For example, in the modular case $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ the wall has the form



We want to characterize $\mathtt{simp}\langle Q \rangle_\sigma$ and the $\mathtt{emp}\langle Q \rangle_\sigma$. We consider the direct sum of simple representations

$$V = \oplus_{i,j}\ V_{ij}^{\oplus m_{ij}}$$

that is, the dimension vector of V is $\alpha = \sum_{i,j} m_{ij}\alpha_{ij}$. $\alpha \in \texttt{simp}\langle Q\rangle_\sigma$ is equivalent to $\gamma = (m_{11},\ldots,m_{pq})$ being the dimension vector of a simple representation of $\texttt{wall}\langle Q\rangle_\sigma$. We claim

$\alpha = (a_1,\ldots,a_p;b_1,\ldots,b_q)$ with $\sum_i a_i = n = \sum_j b_j \in \texttt{simp}\langle Q\rangle_\sigma$ iff

$$n = 1 \quad \textit{or} \quad a_i + b_j \leq n$$

for all $1 \leq i \leq p$ and $1 \leq j \leq q$.

A moments thought shows that the conditions are necessary. Conversely, assume the numerical condition is satisfied and consider the semi-simple representation of $\langle Q\rangle_\sigma$

$$V = \oplus_{i,j} V_{ij}^{\oplus m_{ij}}$$

We note that

$$a_i = \sum_{j=1}^{q} m_{ij} \qquad \text{and} \qquad b_l = \sum_{i=1}^{p} m_{il}.$$

Let $I_p \in M_p(\mathbb{C})$ be the $p\times p$ identity matrix and let $A_p \in M_p(\mathbb{C})$ be the $p \times p$ matrix of the form

$$A_p = \begin{pmatrix} 0 & -1 & \ldots & -1 & -1 \\ -1 & 0 & \ldots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & 0 & -1 \\ -1 & -1 & \ldots & -1 & 0 \end{pmatrix}.$$

Then the Euler form of the $\texttt{wall}\langle Q\rangle_\sigma$ is the symmetric matrix

$$\chi_W = \begin{pmatrix} I_p & A_p & \ldots & A_p & A_p \\ A_p & I_p & \ldots & A_p & A_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_p & A_p & \ldots & I_p & A_p \\ A_p & A_p & \ldots & A_p & I_p \end{pmatrix} \qquad \in \qquad M_q(M_p(\mathbb{C})).$$

When $n = 1$, we have that $V = V_{11}$ is obviously a simple representation. When $n = 2$, we notice that γ is the dimension vector of a simple representation if and only if

$$\gamma | supp(\gamma) = (1,1;1,1) \qquad \text{because} \qquad supp(\gamma) = \widetilde{A}_2$$

Now, consider $n \geq 3$ and consider the dimension vector $\gamma = (m_{11},\ldots,m_{pq})$. We have to verify that γ is the dimension vector of a simple representation of $\texttt{wall}\langle Q\rangle_\sigma$ which, by symmetry of χ_W, amounts to checking that

$$\chi_W(\gamma,\epsilon_{kl}) = \chi_W(\epsilon_{kl},\gamma) \leq 0$$

for all $1 \leq k \leq p$ and $1 \leq l \leq q$ where $\epsilon_{kl} = (\delta_{ij,kl})$ are the standard base vectors. Computing the left hand term this is equivalent to

$$m_{kl} + \sum_{k\neq i=1}^{p} \sum_{l\neq j=1}^{q} -m_{ij} \leq 0.$$

or

$$m_{kl} + \sum_{k\neq i=1}^{p} m_{il} \leq \sum_{k\neq i=1}^{p} \sum_{l\neq j=1}^{q} m_{ij} + \sum_{k\neq i=1}^{p} m_{il}$$

Resubstituting the values of a_i and b_l in this expression we see that this is equivalent to

$$b_l \leq \sum_{k \neq i=1}^{p} a_i = n - a_k$$

Therefore, the condition is satisfied if for all $1 \leq k \leq p$ and $1 \leq l \leq q$ we have

$$a_k + b_l \leq n.$$

finishing the proof of the claim.

In the special case of $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$, our condition on the dimension vector $\alpha = (a_1, a_2; b_1, b_2, b_3)$ is equivalent to

$$a_i + b_j \leq n = a_1 + a_2 \quad \text{whence} \quad b_j \leq a_i$$

for all $1 \leq i \leq 2$ and $1 \leq j \leq 3$ which was the criterium found by Bruce Westbury in [**66**].

6.3. Smooth loci.

In this section we will prove the theorem due to Raf Bocklandt characterizing the quiver settings (Q, α) such that the ring of polynomial invariants

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] = \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)}$$

is `commalg`-smooth, see [**4**],[**6**] and [**5**]. By the étale local description in terms of local quiver settings this characterization can be used to determine the singular locus of any $\mathtt{iss}_\alpha A$ for $\alpha \in \mathtt{comp} A$.

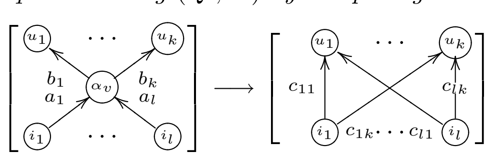
Because $\mathbb{C}[\mathtt{rep}_\alpha\ Q]$ has a natural gradation by defining the degree of all variable matrix coordinates to be one. Therefore, the ring of invariants is a positively graded algebra whence the regularity condition is equivalent to $\mathbb{C}[\mathtt{iss}_\alpha\ Q]$ being a polynomial algebra. We begin by relating rings of invariants of different quiver settings.

Theorem 98 (Bocklandt). *We have the following reductions :*

(1) `b1` *: Let (Q, α) be a quiver setting and v a vertex without loops such that*

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \ \text{or}\ \chi_Q(\epsilon_v, \alpha) \geq 0.$$

Define the quiver setting (Q', α') by composing arrows through v :



(some of the vertices may be the same). Then,

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] \simeq \mathbb{C}[\mathtt{iss}_{\alpha'}\ Q']$$

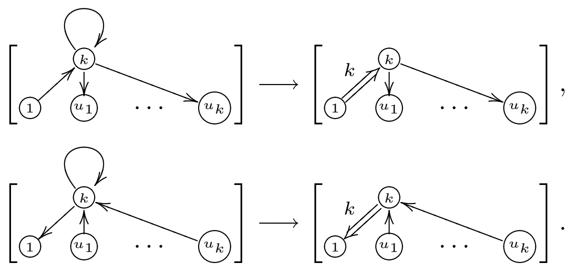
(2) `b2` *: Let (Q, α) be a quiver setting and v a vertex with k loops such that $\alpha_v = 1$. Let (Q', α) be the quiver setting where Q' is the quiver obtained by removing the loops in v, then*

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] \simeq \mathbb{C}[\mathtt{iss}_\alpha\ Q'] \otimes \mathbb{C}[X_1, \cdots, X_k]$$

(3) `b3` *: Let (Q,α) be a quiver setting and v a vertex with one loop such that $\alpha_v = k \geq 2$ and*

$$\chi_Q(\alpha, \epsilon_v) = -1 \text{ or } \chi_Q(\epsilon_v, \alpha) = -1.$$

Define the quiver setting (Q', α) by changing the quiver as below :



Then,

$$\mathbb{C}[\mathtt{iss}_\alpha\ Q] \simeq \mathbb{C}[\mathtt{iss}_\alpha Q'] \otimes \mathbb{C}[X_1, \ldots, X_k]$$

PROOF. (1) : $\mathtt{rep}_\alpha\ Q$ can be decomposed as

$$\begin{aligned}\mathtt{rep}_\alpha\ Q &= \underbrace{\bigoplus_{a,\ s(a)=v} M_{\alpha_{t(a)}\times\alpha_{s(a)}}(\mathbb{C})}_{\text{arrows starting in } v} \oplus \underbrace{\bigoplus_{a,\ t(a)=v} M_{\alpha_{t(a)}\times\alpha_{s(a)}}(\mathbb{C})}_{\text{arrows terminating in } v} \oplus\ \mathtt{rest} \\ &= M_{\sum_{s(a)=v}\alpha_{t(a)}\times\alpha_v}(\mathbb{C}) \oplus M_{\alpha_v\times\sum_{t(a)=v}\alpha_{s(a)}}(\mathbb{C}) \oplus \mathtt{rest} \\ &= M_{\alpha_v-\chi(\alpha,\epsilon_v)\times\alpha_v}(\mathbb{C}) \oplus M_{\alpha_v\times\alpha_v-\chi(\epsilon_v,\alpha)}(\mathbb{C}) \oplus \mathtt{rest}\end{aligned}$$

$GL_{\alpha_v}(\mathbb{C})$ only acts on the first two terms and not on `rest`. Taking the quotient corresponding to $GL_{\alpha_v}(\mathbb{C})$ involves only the first two terms.

We recall the *first fundamental theorem* for GL_n-invariants , see for example [**36**, II.4.1]. The quotient variety

$$(M_{l\times n}(\mathbb{C}) \oplus M_{n\times m})/GL_n$$

where GL_n acts in the natural way, is for all $l, n, m \in \mathbb{N}$ isomorphic to the space of all $l \times m$ matrices of rank $\leq n$. The projection map is induced by multiplication

$$M_{l\times n}(\mathbb{C}) \oplus M_{n\times m}(\mathbb{C}) \xrightarrow{\pi} M_{l\times m}(\mathbb{C}) \qquad (A, B) \mapsto A.B$$

In particular, if $n \geq l$ and $n \geq m$ then π is surjective and the quotient variety is isomorphic to $M_{l\times m}(\mathbb{C})$.

By this fundamental theorem and the fact that either $\chi_Q(\alpha, \epsilon_v) \geq 0$ or $\chi_Q(\epsilon_v, \alpha) \geq 0$, the above quotient variety is isomorphic to

$$M_{\alpha_v-\chi(\alpha,\epsilon_v)\times\alpha_v-\chi(\epsilon_v,\alpha)}(\mathbb{C}) \oplus \mathtt{rest}$$

This space can be decomposed as

$$\bigoplus_{a,\ t(a)=vb,\ s(b)=v} M_{\alpha_{t(b)}\times\alpha_{s(a)}}(\mathbb{C}) \oplus \mathtt{rest} = \mathtt{rep}_{\alpha'}\ Q'$$

Taking quotients for $GL(\alpha')$ then proves the claim.

(2) : Trivial as $GL(\alpha)$ acts trivially on the loop-representations in v.

(3) : We only prove this for the first case. Call the loop in the first quiver ℓ and the incoming arrow a. Call the incoming arrows in the second quiver $c_i, i = 0, \ldots, k-1$.

There is a map

$$\pi : \texttt{rep}_\alpha\ Q \to \texttt{rep}_{\alpha'}\ Q' \times \mathbb{C}^k : V \mapsto (V', Tr(V_\ell), \dots, Tr(V_{\ell^k})) \text{ with } V'_{c_i} := V_\ell^i V_a$$

Suppose $(V', x_1, \dots, x_k) \in \texttt{rep}_{\alpha'} Q' \times \mathbb{C}^k \in$ such that $(x_1, \dots, x_k)$ correspond to the traces of powers of an invertible diagonal matrix D with k different eigenvalues $(\lambda_i, i = 1, \dots, k)$ and the matrix A made of the columns $(V_{c_i}, i = 0, \dots, k-1)$ is invertible. The image of the representation

$$V \in \texttt{rep}_\alpha\ Q : V_a = V'_{c_0}, V_\ell = A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1}$$

under π is $(V', x_1, \dots, x_k)$ because

$$\begin{aligned} V_\ell^i V_a &= A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D^i \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1} V'_{c_0} \\ &= A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1^i \\ \vdots \\ \lambda_k^i \end{pmatrix} \\ &= V_{c_i} \end{aligned}$$

and the traces of V_ℓ are the same as those of D. The conditions on $(V', x_1, \dots, x_k)$, imply that the image of π, U, is dense, and hence π is a dominant map.

There is a bijection between the generators of $\mathbb{C}[\texttt{iss}_\alpha Q]$ and $\mathbb{C}[\texttt{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \dots, X_k]$ by identifying

$$f_{\ell^i} \mapsto X_i, i = 1, \dots, k\ , f_{\cdots a \ell^i \cdots} \mapsto f_{\cdots c_i \cdots}, i = 0, \dots, k-1$$

Notice that higher orders of ℓ don't occur by the Caley Hamilton identity on V_ℓ. If n is the number of generators of $\mathbb{C}[\texttt{iss}_\alpha Q]$, we have two maps

$$\phi : \mathbb{C}[Y_1, \cdots Y_n] \to \mathbb{C}[\texttt{iss}_\alpha Q] \subset \mathbb{C}[\texttt{rep}_\alpha Q],$$

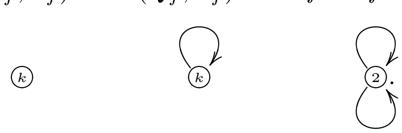
$$\phi' : \mathbb{C}[Y_1, \cdots Y_n] \to \mathbb{C}[\texttt{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \dots, X_k] \subset \mathbb{C}[\texttt{rep}_{\alpha'} Q' \times \mathbb{C}^k].$$

Note that $\phi'(f) \circ \pi \equiv \phi(f)$ and $\phi(f) \circ \pi^{-1}|_U \equiv \phi'(f)|_U$. So if $\phi(f) = 0$ then also $\phi'(f)|_U = 0$. Because U is zariski-open and dense in $\texttt{rep}_{\alpha'} Q' \times \mathbb{C}^2$, $\phi'(f) \equiv 0$. A similar argument holds for the inverse implication whence $Ker(\phi) = Ker(\phi')$. □

Definition 95. A quiver setting (Q, α) is said to be *final* iff none of the reduction steps `b1`, `b2` or `b3` of theorem 98 can be applied. Every quiver setting can be reduced to a final quiver setting which we denote $(Q, \alpha) \rightsquigarrow (Q_f, \alpha_f)$.

Theorem 99 (Bocklandt). *For a quiver setting (Q, α) with $Q = \texttt{supp}\alpha$ strongly connected, the following are equivalent :*

1. $\mathbb{C}[\texttt{iss}_\alpha\ Q] = \mathbb{C}[\texttt{rep}_\alpha\ Q]^{GL(\alpha)}$ *is* `commalg`*-smooth.*
2. $(Q_f, \alpha_f) \rightsquigarrow (Q_f, \alpha_f)$ *with* (Q_f, α_f) *one of the following quiver settings*



PROOF. (2) $\Rightarrow$ (1) : Follows from the foregoing theorem and the fact that the rings of invariants of the three quiver settings are resp. $\mathbb{C}$, $\mathbb{C}[tr(X), tr(X^2), \ldots, tr(X^k)]$ and $\mathbb{C}[tr(X), tr(Y), tr(X^2), tr(Y^2), tr(XY)]$.

(1) $\Rightarrow$ (2) : Take a final reduction $(Q, \alpha) \rightsquigarrow (Q_f, \alpha_f)$ and to avoid subscripts rename $(Q_f, \alpha_f) = (Q, \alpha)$ (observe that the condition of the theorem as well as (1) is preserved under the reduction steps by the foregoing theorem). That is, we will assume that (Q, α) is final whence, in particular as `b1` cannot be applied,

$$\chi_Q(\alpha, \epsilon_v) < 0 \qquad \chi_Q(\epsilon_v, \alpha) < 0$$

for all vertices v of Q. With $\mathbb{1}$ we denote the dimension vector $(1, \ldots, 1)$.

claim 1 : Either $(Q, \alpha) = \textcircled{k}$ or Q has loops. Assume neither, then if $\alpha \neq \mathbb{1}$ we can choose a vertex v with maximal α_v. By the above inequalities and theorem 85 we have that

$$\tau = (1, \alpha - \epsilon_v; 1, \epsilon_v) \in \mathtt{types}_\alpha Q$$

As there are no loops in v, we have

$$\begin{cases} \chi_Q(\alpha - \epsilon_v, \epsilon_v) & = \chi(\alpha, \epsilon_v) - 1 < -1 \\ \chi_Q(\epsilon_v, \alpha - \epsilon_v) & = \chi(\epsilon_v, \alpha) - 1 < -1 \end{cases}$$

and the local quiver setting (Q_τ, α_τ) contains the subquiver

$$\textcircled{1} \overset{k}{\underset{l}{\rightleftarrows}} \textcircled{1} \qquad \text{with } k, l \geq 2$$

The invariant ring of the local quiver setting cannot be a polynomial ring as it contains the subalgebra

$$\frac{\mathbb{C}[a, b, c, d]}{(ab - cd)}$$

where $a = x_1 y_1$, $b = x_2 y_2$, $c = x_1 y_2$ and $d = x_2 y_1$ are necklaces of length 2 with x_i arrows from w_1 to w_2 and y_i arrows from w_2 to w_1. This contradicts the assumption (1) by the étale local structure result.

Hence, $\alpha = \mathbb{1}$ and because (Q, α) is final, every vertex must have least have two incoming and two outgoing arrows. Because Q has no loops,

$$dim\ \mathtt{iss}_{\mathbb{1}}\ Q = 1 - \chi_Q(\mathbb{1}, \mathbb{1}) = \#\mathtt{arrows} - \#\mathtt{vertices} + 1$$

On the other hand, a minimal generating set for $\mathbb{C}[\mathtt{iss}_{\mathbb{1}}\ Q]$ is the set of *Eulerian necklaces* , that is, those necklaces in Q not re-entering any vertex. By (1) both numbers must be equal, so we will reach a contradiction by showing that $\#\mathtt{euler}$, the number of Eulerian necklaces is strictly larger than $\chi(Q) = \#\mathtt{arrows} - \#\mathtt{vertices} + 1$. We will do this by induction on the number of vertices.

If $\#\mathtt{vertices} = 2$, the statement is true because

$$Q := \textcircled{1} \overset{k}{\underset{l}{\rightleftarrows}} \textcircled{1} \qquad \text{whence } \#\mathtt{euler} = kl > \chi(Q) = k + l - 1$$

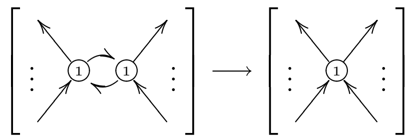
as both k and l are at least 2.

Assume $\#\mathtt{vertices} > 2$ and that there is a subquiver of the form

$$\mathtt{basic} = \textcircled{1} \overset{k}{\underset{l}{\rightleftarrows}} \textcircled{1}$$

If $k > 1$ and $l > 1$ we have seen before that this subquiver and hence Q cannot have a polynomial ring of invariants.

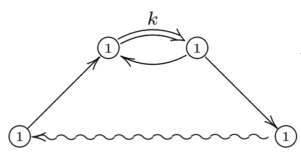
If $k = 1$ and $l = 1$ then substitute this subquiver by one vertex.



The new quiver Q' is again final without loops because there are at least four incoming arrows in the vertices of the subquiver and we only deleted two (the same holds for the outgoing arrows). Q' has one Eulerian necklace less than Q. By induction, we have that

$$\begin{aligned} \#\texttt{euler} &= \#\texttt{euler}' + 1 \\ &> \chi(Q') + 1 \\ &= \chi(Q). \end{aligned}$$

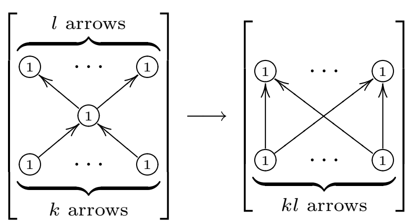
If $k > 1$ then one can look at the subquiver Q' of Q obtained by deleting $k - 1$ of these arrows. If Q' is final, we are in the previous situation and obtain the inequality as before. If Q' is not final, then Q contains a subquiver of the form



,

which cannot have a polynomial ring of invariants, as it is reducible to `basic` with both k and l at least equal to 2.

Finally, if $\#\texttt{vertices} > 2$ and there is no `basic`-subquiver, take an arbitrary vertex v. Construct a new quiver Q' bypassing v



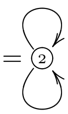
.

Q' is again final without loops and has the same number of Eulerian necklaces. By induction

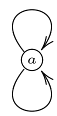
$$\begin{aligned} \#\texttt{euler} &= \#\texttt{euler}' \\ &> \#\texttt{arrows}' - \#\texttt{vertices}' + 1 \\ &= \#\texttt{arrows} + (kl - k - l) - \#\texttt{vertices} + 1 + 1 \\ &> \#\texttt{arrows} - \#\texttt{vertices} + 1. \end{aligned}$$

In all cases, we obtain a contradiction with (1) and hence have proved **claim1**. So we may assume from now on that Q has loops.

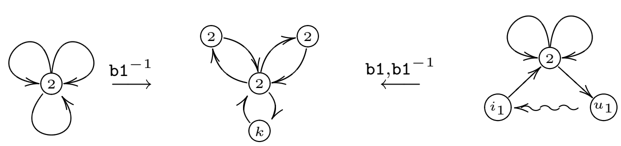
claim 2 : If Q has loops in v, then there is at most one loop in v or (Q, α) is

$\texttt{2twobytwo} =$ 

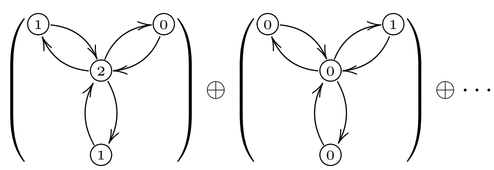
Because (Q,α) is final, we have $\alpha_v \geq 2$. If $\alpha_v = a \geq 3$ then there is only one loop in v. If not, there is a subquiver of the form



and its ring of invariants cannot be a polynomial algebra. Indeed, consider its representation type $\tau = (1, k-1; 1, 1)$ then the local quiver is of type `basic` with $k = l = a - 1 \geq 2$ and we know already that this cannot have a polynomial algebra as invariant ring. If $\alpha_v = 2$ then either we are in the `2twobytwo` case or there is at most one loop in v. If not, we either have at least three loops in v or two loops and a cyclic path through v, but then we can use the reductions



The middle quiver cannot have a polynomial ring as invariants because we consider the type



The number of arrows between the first and the second simple component equals

$$-\begin{pmatrix} 2 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 2$$

whence the corresponding local quiver contains `basic` with $k = l = 2$ as subquiver. This proves **claim 2**. From now on we will assume that the quiver setting (Q,α) is such that there is precisely one loop in v and that $k = \alpha_v \geq 2$. Let

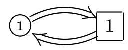
$$\tau = (1, \mathbb{1}; 1, \epsilon_v; \alpha_{v_1} - 1, \epsilon_{v_1}; \ldots; \ldots; \alpha_v - 2, \epsilon_v; \ldots; \alpha_{v_l} - 1, \epsilon_{v_l}) \in \texttt{types}_\alpha Q$$

Here, the second simple representation, concentrated in v has non-zero trace in the loop whereas the remaining $\alpha_v - 2$ simple representations concentrated in v have zero trace. Further, $\mathbb{1} \in \texttt{simp}\langle Q \rangle$ as Q is strongly connected by theorem 85. We work out the local quiver setting (Q_τ, α_τ). The number of arrows between the vertices in Q_τ corresponding to simple components concentrated in a vertex is equal to the number of arrows in Q between these vertices. We will denote the vertex (and multiplicity) in Q_τ corresponding to the simple component of dimension vector $\mathbb{1}$ by $\boxed{1}$.

The number of arrows between the vertex in Q_τ corresponding to a simple concentrated in vertex w in Q to $\boxed{1}$ is $-\chi_Q(\epsilon_w, \mathbb{1})$ and hence is one less than the number of outgoing arrows from w in Q. Similarly, the number of arrows from the vertex $\boxed{1}$ to that of the simple concentrated in w is $-\chi_Q(\mathtt{1}, \epsilon_w)$ and is equal to one less than the number of incoming arrows in w in Q. But then we must have for all vertices w in Q that

$$\chi_Q(\epsilon_w, \mathtt{1}) = -1 \qquad \text{or} \qquad \chi_Q(\mathtt{1}, \epsilon_w) = -1$$

Indeed, because (Q, α) is final we know that these numbers must be strictly negative, but they cannot be both ≤ -2 for then the local quiver Q_τ will contain a subquiver of type

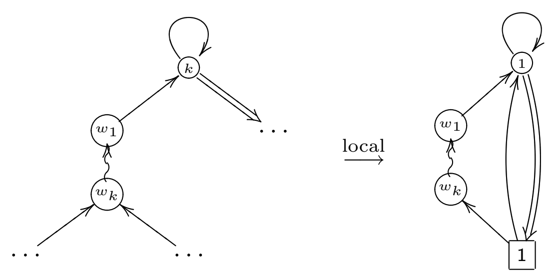


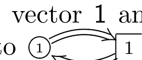
contradicting that the ring of invariants is a polynomial ring. Similarly, we must have

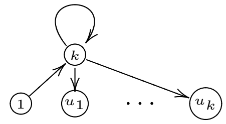
$$\chi_Q(\epsilon_w, \epsilon_v) \geq -1 \qquad \text{or} \qquad \chi_Q(\epsilon_v, \epsilon_v)$$

for all vertices w in Q for which $\alpha_w \geq 2$. Let us assume that $\chi_Q(\epsilon_v, \mathtt{1}) = -1$.

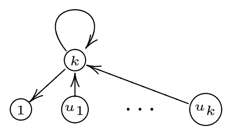
claim 3 : If w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$, then $\alpha_{w_1} = 1$. If this was not the case there is a vertex corresponding to a simple representation concentrated in w_1 in the local quiver Q_τ. If $\chi_Q(\mathtt{1}, \epsilon_{w_1}) = 0$ then the dimension of the unique vertex w_2 with an arrow to w_1 has strictly bigger dimension than w_1, otherwise $\chi_Q(\alpha, \epsilon_{w_1}) \geq 0$ contradicting finality of (Q, α). The vertex w_2 corresponds again to a vertex in the local quiver. If $\chi_Q(\mathtt{1}, \epsilon_{w_2}) = 0$, the unique vertex w_3 with an arrow to w_2 has strictly bigger dimension than w_2. Proceeding this way one can find a sequence of vertices with increasing dimension, which attains a maximum in vertex w_k. Therefore $\chi_Q(\mathtt{1}, \epsilon_{w_k}) \leq -1$. This last vertex is in the local quiver connected with W, so one has a path from $\mathtt{1}$ to ϵ_v.



The subquiver of the local quiver Q_τ consisting of the vertices corresponding to the simple representation of dimension vector $\mathtt{1}$ and the simples concentrated in vertex v resp. w_k is reducible via b1 to , at least if $\chi_Q(\mathtt{1}, \epsilon_v) \leq -2$, a contradiction finishing the proof of the claim. But then, the quiver setting (Q, α) has the following shape in the neighborhood of v



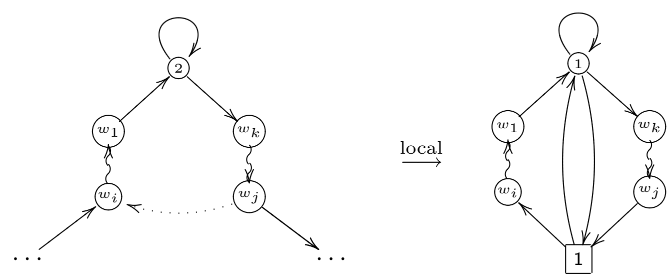
contradicting finality of (Q, α) for we can apply b3. In a similar way one proves that the quiver setting (Q, α) has the form

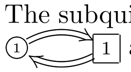


in a neighborhood of v if $\chi_Q(1, \epsilon_v) = -1$ and $\chi_Q(\epsilon_v, 1) \leq -2$, again contradicting finality.

There remains one case to consider : $\chi_Q(1, \epsilon_v) = -1$ *and* $\chi_Q(\epsilon_v, 1) = -1$. Suppose w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$ and w_k is the unique vertex in Q such that $\chi_Q(\epsilon_{w_k}, \epsilon_v) = -1$, then we claim :

claim 4 : Either $\alpha_{w_1} = 1$ or $\alpha_{w_k} = 1$. If not, consider the path connecting w_k and w_1 and call the intermediate vertices w_i, $1 < i < k$. Starting from w_1 we go back the path until α_{w_i} reaches a maximum. at that point we know that $\chi_Q(1, \epsilon_{w_k}) \leq -1$, otherwise $\chi_Q(\alpha, \epsilon_{w_k}) \geq 0$. In the local quiver there is a path from the vertex corresponding to the 1-dimensional simple over the ones corresponding to the simples concentrated in w_i to v. Repeating the argument, starting from w_k we also have a path from the vertex of the simple v-representation over the vertices of the w_j-simples to the vertex of the 1-dimensional simple.



The subquiver consisting of 1, ϵ_v and the two paths through the ϵ_{w_i} is reducible to  and we again obtain a contradiction.

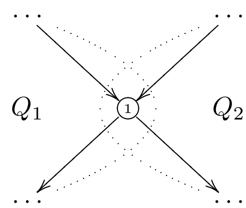
The only way out of these dilemmas is that the final quiver setting (Q, α) is of the form

finishing the proof. □

Definition 96. Let (Q, α) and (Q', α') be two quiver settings such that there is a vertex v in Q and a vertex v' in Q' with $\alpha_v = 1 = \alpha'_{v'}$. We define the *connected sum* of the two settings to be the quiver setting

$$(Q \underset{v'}{\overset{v}{\#}} Q' \ , \ \alpha \underset{v'}{\overset{v}{\#}} \alpha')$$

where $\underset{v'}{\overset{v}{Q\#Q}}$ is the quiver obtained by identifying the two vertices v and v'



and where $\underset{v'}{\overset{v}{\alpha\#\alpha'}}$ is the dimension vector which restricts to α (resp. α') on Q (resp. Q').

EXAMPLE 135. With this notation we have

$$\mathbb{C}[\mathtt{iss}_{\underset{v'}{\overset{v}{\alpha\#\alpha'}}} \underset{v'}{\overset{v}{Q\#Q'}}] \simeq \mathbb{C}[\mathtt{iss}_\alpha Q] \otimes \mathbb{C}[\mathtt{iss}_{\alpha'} Q']$$

Because traces of a necklaces passing more than once through a vertex where the dimension vector is equal to 1 can be split as a product of traces of necklaces which pass through this vertex only one time, we see that the invariant ring of the connected sum is generated by Eulerian necklaces fully contained in Q or in Q'.

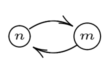
Theorem 99 gives a procedure to decide whether a given quiver setting (Q, α) has a regular ring of invariants. However, is is not feasible to give a graphtheoretic description of all such settings in general. Still, in the special (but important) case of *symmetric* quivers, there is a nice graphtheoretic characterization.

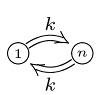
THEOREM 100 (Bocklandt). *Let (Q, α) be a symmetric quiver setting such that Q is connected and has no loops. Then, the ring of polynomial invariants*

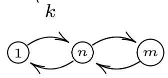
$$\mathbb{C}[\mathtt{iss}_\alpha Q] = \mathbb{C}[\mathtt{rep}_\alpha Q]^{GL(\alpha)}$$

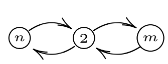
is a polynomial ring if and only if the following conditions are satisfied

(1) *Q is tree-like, that is, if we draw an edge between vertices of Q whenever there is at least one arrow between them in Q, the graph obtained in a tree.*
(2) *α is such that in every branching vertex v of the tree we have $\alpha_v = 1$.*
(3) *The quiver subsetting corresponding to branches of the tree are connected sums of the following atomic pieces :*

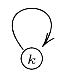
I 

II , $k \leq n$

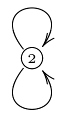
III 

IV ,

PROOF. Using theorem 99 any of the atomic quiver settings has a polynomial ring of invariants. Type I reduces via b1 to

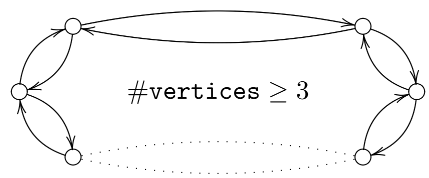


where $k = min(m, n)$, type `II` reduces via `b1` and `b2` to ①, type `III` reduces via `b1`, `b3`, `b1` and `b2` to ① and finally, type `IV` reduces via `b1` to

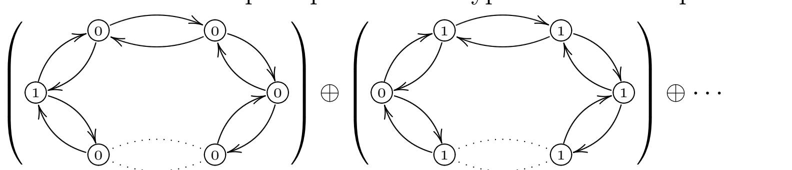


By the previous example, any connected sum constructed out of these atomic quiver settings has a regular ring of invariants. Observe that such connected sums satisfy the first two requirements. Therefore, any quiver setting satisfying the requirements has indeed a polynomial ring of invariants.

Conversely, assume that the ring of invariants $\mathbb{C}[\mathtt{iss}_\alpha Q]$ is a polynomial ring, then there can be no quiver subsetting of the form

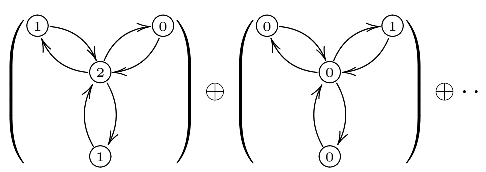


For we could look at a semisimple representation type τ with decomposition



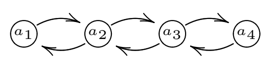
The local quiver contains a subquiver (corresponding to the first two components) of type `basic` with k and $l \geq 2$ whence cannot give a polynomial ring. That is, Q is tree-like.

Further, the dimension vector α cannot have components ≥ 2 at a branching vertex v. For we could consider the semisimple representation type with decomposition

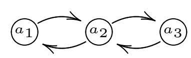


and again the local quiver contains a subquiver setting of type `basic` with $k = 2 = l$ (the one corresponding to the first two components). Hence, α satisfies the second requirement.

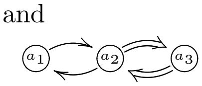
Remains to show that the branches do not contain other subquiver settings than those made of the atomic components. That is, we have to rule out the following subquiver settings :



with $a_2 \geq 2$ and $a_3 \geq 2$,

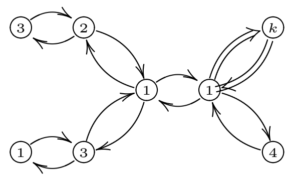


with $a_2 \geq 3$ and $a_1 \geq 2$, $a_3 \geq 2$ and



whenever $a_2 \geq 2$. These situations are easily ruled out by theorem 99 and we leave this as a pleasant exercise. $\square$

EXAMPLE 136. The quiver setting



has a polynomial ring of invariants if and only if $k \geq 2$.

EXAMPLE 137. Let $(Q^\bullet, \alpha)$ be a *marked* quiver setting and assume that $\{l_1, \ldots, l_u\}$ are the marked loops in $Q^\bullet$. If Q is the underlying quiver forgetting the markings we have by separating traces that

$$\mathbb{C}[\mathtt{iss}_\alpha Q] \simeq \mathbb{C}[\mathtt{iss}_\alpha Q^\bullet][tr(l_1), \ldots, tr(l_u)]$$

Hence, we do not have to do extra work in the case of marked quivers :

A marked quiver setting $(Q^\bullet, \alpha)$ *has a regular ring of invariants if and only if* (Q, α) *can be reduced to a one of the three final quiver settings of theorem 99.*

We relate this result to the necklace Lie algebra $\mathtt{neck}_Q$ of section 3.4. Recall that if Q is a symmetric quiver, we can define symplectic structures $*$ on it which is a partitioning of the arrows

$$Q_a = L \sqcup R \qquad \text{such that} \qquad L^* = R$$

Let Q_L be the subquiver on the arrows from L, then the *trace pairing* gives a natural identification

$$\mathtt{rep}_\alpha\, Q \leftrightarrow T^*\mathtt{rep}_\alpha\, Q_L$$

between the representation space of Q and the cotangent bundle on the representation space of Q_L. Therefore, $\mathtt{rep}_\alpha\, Q$ comes equipped with a canonical symplectic structure.

If $\alpha = (a_1, \ldots, a_k)$, then, for every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in L we have an $a_j \times a_i$ matrix of coordinate functions A_{uv} with $1 \leq u \leq a_j$ and $1 \leq v \leq a_i$ and for the adjoined arrow $\textcircled{j} \xrightarrow[a^*]{} \textcircled{i}$ in R an $a_i \times a_j$ matrix of coordinate functions A^*_{vu}. The canonical symplectic structure on $\mathtt{rep}_\alpha\, Q$ is induced by the closed 2-form

$$\omega = \sum_{\substack{\textcircled{j} \xleftarrow{a} \textcircled{i}}}^{\substack{1 \leq v \leq a_i \\ 1 \leq u \leq a_j}} dA_{uv} \wedge dA^*_{vu}$$

where the sum is taken over all $a \in L$. This symplectic structure induces a *Poisson bracket* on the coordinate ring $\mathbb{C}[\mathtt{rep}_\alpha\, Q]$ by the formula

$$\{f, g\} = \sum_{\substack{\textcircled{j} \xleftarrow{a} \textcircled{i}}}^{\substack{1 \leq v \leq a_i \\ 1 \leq u \leq a_j}} \left(\frac{\partial f}{\partial A_{uv}} \frac{\partial g}{\partial A^*_{vu}} - \frac{\partial f}{\partial A^*_{vu}} \frac{\partial g}{\partial A_{uv}} \right)$$

The basechange action of $GL(\alpha)$ on the representation space $\mathtt{rep}_\alpha\ Q$ is *symplectic* which means that for all *tangentvectors* $t, t' \in T\ \mathtt{rep}_\alpha\ Q$ we have for the induced $GL(\alpha)$ action that

$$\omega(t,t') = \omega(g.t, g.t')$$

for all $g \in GL(\alpha)$. The infinitesimal $GL(\alpha)$ action gives a Lie algebra homomorphism

$$\mathfrak{pgl}(\alpha) \longrightarrow Vect_\omega\ \mathtt{rep}_\alpha\ Q$$

which factorizes through a Lie algebra morphism H to the coordinate ring making the diagram below commute

$$\begin{array}{ccc} & \mathfrak{pgl}(\alpha) & \\ {\scriptstyle H=\mu_{\mathbb{C}}^*} \swarrow & & \searrow \\ \mathbb{C}[\mathtt{rep}_\alpha\ Q] & \xrightarrow[f \mapsto \xi_f]{} & Vect_\omega\ \mathtt{rep}_\alpha\ Q \end{array}$$

where $\mu_{\mathbb{C}}$ is the *complex moment map* . That is, the map

$$\mathtt{rep}_\alpha\ Q \xrightarrow{\mu_{\mathbb{C}}} \mathfrak{gl}(\alpha) \qquad \mu_\alpha(V)_i = \sum_{\circ \xleftarrow{a} (i)} V_a V_{a^*} - \sum_{(i) \xleftarrow{a} \circ} V_{a^*} V_a$$

We say that the action of $GL(\alpha)$ on $\mathtt{rep}_\alpha\ Q$ is *Hamiltonian* . This makes the ring of polynomial invariants $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha)}$ into a *Poisson algebra* and we will write

$$\mathtt{neck}_\alpha = (\mathbb{C}[rep_\alpha\ Q^d]^{GL(\alpha)}, \{-,-\})$$

for the corresponding abstract *infinite dimensional Lie algebra*. The dual space of this Lie algebra $\mathtt{neck}_\alpha^*$ is then a *Poisson manifold* equipped with the *Kirillov-Kostant bracket*. Evaluation at a point in the quotient variety $\mathtt{iss}_\alpha\ Q$ defines a linear function on $\mathtt{neck}_\alpha$ and therefore gives an embedding

$$\mathtt{iss}_\alpha\ Q \hookrightarrow \mathtt{neck}_\alpha^*$$

as Poisson varieties. That is, the induced map on the polynomial functions is a morphism of Poisson algebras. The following results are adaptations of an idea due to Victor Ginzburg [**20**].

THEOREM 101. *Let Q be a symmetric quiver and α a dimension vector such that $\mathtt{iss}_\alpha\ Q$ is smooth. Then, the Poisson embedding*

$$\mathtt{iss}_\alpha\ Q \hookrightarrow \mathtt{neck}_\alpha^*$$

makes $\mathtt{iss}_\alpha\ Q$ into a closed coadjoint orbit *of the infinite dimensional Lie algebra $\mathtt{neck}_\alpha$.*

PROOF. In general, if X is a smooth affine variety then the differentials of polynomial functions on X span the tangent spaces at all points $x \in X$. If X is in addition symplectic, the infinitesimal Hamiltonian action of the Lie algebra $\mathbb{C}[X]$ (with the natural Poisson bracket) on X is infinitesimally transitive. Therefore, the infinite dimensional group $\mathtt{Ham}$ generated by all *Hamiltonian flows* on X acts with open orbits. If X is in addition irreducible, it must be a single orbit. That is, X is a coadjoint orbit for $\mathbb{C}[X]$. By assumption, $\mathtt{iss}_\alpha\ Q$ is an affine smooth irreducible variety, so the argument applies. $\square$

Observe that the same argument applies to any affine smooth subvariety Y of $\mathtt{iss}_\alpha\ Q$ (regardless of α being such that the quotient variety is smooth) whenever the infinitesimal action of $\mathtt{neck}_\alpha$ on $\mathtt{neck}^*_\alpha$ preserves Y. In particular, this applies to (deformed) preprojective algebras and associated *quiver varieties* . We refer the interested reader to [**20**], [**3**] and [**39**].

We still have to explain the terminology $\mathtt{neck}_\alpha$. Recall that the necklace Lie algebra introduced in section 3.4

$$\mathtt{neck}_Q = \mathtt{DR}^0_{C_k}\ \langle Q \rangle = \frac{\langle Q \rangle}{[\langle Q \rangle, \langle Q \rangle]}$$

is the vectorspace with basis all the necklace words w in the quiver Q, that is, all equivalence classes of oriented cycles in the quiver Q^d, equipped with the Kontsevich bracket

$$\{w_1, w_2\}_K = \sum_{a \in L} \left(\frac{\partial w_1}{\partial a}\frac{\partial w_2}{\partial a^*} - \frac{\partial w_1}{\partial a^*}\frac{\partial w_2}{\partial a} \right) \quad \mathtt{mod}\ [\langle Q \rangle, \langle Q \rangle]$$

The algebra of polynomial quiver invariants $\mathbb{C}[\mathtt{iss}_\alpha\ Q]$ is generated by traces of necklace words. That is, we have a map

$$\mathtt{neck}_Q = \frac{\langle Q \rangle}{[\langle Q \rangle, \langle Q \rangle]} \xrightarrow{tr} \mathtt{neck}_\alpha = \mathbb{C}[\mathtt{iss}_\alpha\ Q]$$

From the definition of the Lie bracket on $\mathtt{neck}_\alpha$ we see that this map is actually a Lie algebra map, that is, for all necklace words w_1 and w_2 in Q we have the identity

$$tr\ \{w_1, w_2\}_K = \{tr(w_1), tr(w_2)\}$$

The image of tr contains a set of *algebra* generators of $\mathbb{C}[\mathtt{iss}_\alpha\ Q]$, so the elements $tr\ \mathtt{neck}_Q$ are enough to separate points in $\mathtt{iss}_\alpha\ Q$. Repeating the previous argument, we obtain :

THEOREM 102. *Let Q be a symmetric quiver and α a dimension vector such that $\mathtt{iss}_\alpha\ Q$ is a smooth variety. The embeddings*

$$\mathtt{iss}_\alpha\ Q \hookrightarrow \mathtt{neck}_\alpha \hookrightarrow \mathtt{neck}_Q$$

make $\mathtt{iss}_\alpha\ Q$ into a coadjoint orbit for the necklace Lie algebra $\mathtt{neck}_Q$.

One should not read too much into this statement. Recall from theorem 100 that we have a complete characterization of the quiver-settings (Q, α) satisfying the requirements of the theorem. Consider such a setting, that is an admissible tree, and consider any proper subtree setting (Q', α'). Then also $\mathtt{iss}_{\alpha'}\ Q'$ is a smooth variety and therefore also a 'coadjoint' orbit for $\mathtt{neck}_Q$. Therefore, these coadjoint orbits can have proper closed coadjoint suborbits.

CHAPTER 7

Nullcones

"There have been much talk about a theory of non-commutative algebraic geometry. It is not my intention here to add to this, but rather to point out that our preceding theory does give us a functor from rings to topological spaces which is a simple summary of the information on possible homomorphisms from the ring to simple artinian rings. It would be possible to equip this space with a sheaf of rings, and to represent modules over the ring as a sheaf of modules over this sheaf of rings; however, in the absence of any obvious use for this machinery, I shall leave it to future mathematicians of greater insight."

Aidan Schofield in [**60**].

In this chapter we will describe the nullcones of quiver representations. We give a representation theoretic interpretation of the Hesselink stratification. It turns out that non-emptiness of a potential stratum is decided by the existence of specific semistable representations for an associated quiver setting.

For this reason we investigate moduli spaces of semistable quiverrepresentations in some detail. In particular we show that a θ-stable representations becomes simple in a universal localization which allows for a local investigation of these moduli spaces (in particular the description of their singular locus) by means of local quivers.

These results extend to a large class of $\mathtt{alg}$-smooth algebras using the theory, due to Aidan Schofield, of localizations at Sylvester rank functions. One covers $\mathtt{ress}A$, the Abelian category of all finite dimensional semistable representations of A locally by $\mathtt{rep}A_\Sigma$ for certain universal localizations, thereby reducing to the theory developed in the foregoing chapter.

7.1. Stability structures.

Some algebras just have not enough simple finite dimensional representations for the previous reductions to have any effect. For example, if A is a finite dimensional $\mathtt{alg}$-algebra, then A is Morita equivalent to the path algebra $\langle Q \rangle$ of a quiver without oriented cycles. The only simple representations of $\langle Q \rangle$ are the vertex-simples and therefore $\mathtt{emp}\langle Q \rangle \simeq Q$. Clearly, all finite dimensional representations of $\langle Q \rangle$ are nilpotent, whence the statement $\mathtt{iso}(\mathtt{rep}A) \leftrightarrow \mathtt{iso}(\mathtt{nullemp}A)$ is a tautology.

If the $\mathtt{alg}$-smooth algebra A has a shortage of simples, it is better to extend the foregoing strategy as follows. Let $\mathtt{schur}A$ denote the subset of $\mathtt{comp}A$ consisting of those irreducible components of $\mathtt{rep}A$ containing a *Schur representation* . That is, a finite dimensional representation M such that $End_A(M) \simeq \mathbb{C}$. Next, define the *bigger empire* $\mathtt{Emp}A$ to be the (usually infinite) quiver with vertices v_α corresponding to the Schur roots $\alpha \in \mathtt{schur}A$ having $ext(\alpha, \beta)$ directed arrows from v_α to v_β.

In this section we will prove, using the work on *semistable representations* of Alexei Rudakov [**58**], that (with obvious notations) there is a natural one-to-one correspondence

$$\mathtt{iso}(\mathtt{ress}A) \leftrightarrow \mathtt{iso}(\mathtt{nullEmp}A)$$

where $\mathtt{ress}A$ are the finite dimensional representations which are semistable for *some* stability structure on $\mathtt{rep}A$. This again reduces the study to a purely combinatorial part, the description of orbits in nullcones of quiver representations, and the description of the Azumaya loci of sheaves of orders over the moduli spaces parametrizing direct sums of stable representations.

However, this study can often be reduced to the setting studied in the foregoing chapter by universal localization. In this section we will outline the reduction and we will give ample detail in the special (but important) case of finite dimensional algebras (or path algebras of quivers without oriented cycles) in the next section.

Definition 97 (Rudakov). For $A \in \mathtt{alg}$ let $\mathtt{rep}A$ denote the Abelian category of all finite dimensional representations of A. A *preorder* on $\mathtt{rep}A$ has the property that for any two non-zero representations V and W we have either $V < W$ or $W < V$ or $V \asymp W$ (V is confused with W). A preorder id said to be a *stability structure* on $\mathtt{rep}A$ if and only if every short exact sequence in $\mathtt{rep}A$

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

satisfies the *seesaw property* , that is,

$$\begin{array}{rl} \text{either} & U < V \Leftrightarrow U < W \Leftrightarrow V < W \\ \text{or} & U > V \Leftrightarrow U > W \Leftrightarrow V > W \\ \text{or} & U \asymp V \Leftrightarrow U \asymp W \Leftrightarrow V \asymp W \end{array}$$

A representation V is said to be *semistable* if $V \neq 0$ and for every non-trivial subrepresentation $U \subset V$ we have $U \leq V$ (that is, $U < V$ or $U \asymp V$). Equivalently, $V \leq W$ for every non-trivial factorrepresentation $V \twoheadrightarrow W$.

A representation V is said to be *stable* if $V \neq 0$ and for every non-trivial subrepresentation $U \subset V$ we have $U < V$. Equivalently, $V < W$ for every non-trivial factorrepresentation $V \twoheadrightarrow W$.

Example 138. We collect some easy consequences of the seesaw-property. For a short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

in $\mathtt{rep}A$ and any nonzero $M \in \mathtt{rep}A$ we have

$$\begin{array}{lcl} \text{if } U < M \text{ and } W < M & \Rightarrow & V < M \\ \text{if } U > M \text{ and } W > M & \Rightarrow & V > M \\ \text{if } U \asymp M \text{ and } W \asymp M & \Rightarrow & V \asymp M \end{array}$$

More generally, if V has a finite filtration by subrepresentations

$$0 = U_{n+1} \subset U_n \subset \ldots \subset U_1 \subset U_0 = V$$

with subsequent factorrepresentations $W_i = U_i/U_{i+1}$, then

$$\begin{array}{lcl} \text{if } W_i < M \text{ for all } 1 \leq i \leq n & \Rightarrow & V < M \\ \text{if } W_i > M \text{ for all } 1 \leq i \leq n & \Rightarrow & V > M \\ \text{if } W_i \asymp M \text{ for all } 1 \leq i \leq n & \Rightarrow & V \asymp M \end{array}$$

This property we call the *center of mass* property . If moreover we have that $W_0 < W_1 < \ldots < W_n$ and if we denote $W_i(j) = U_i/U_{i+j}$ then we have that

$$W_i(j) < W_k(l) \quad \Rightarrow \quad (i,j) <_{lex} (k,l)$$

where $<_{lex}$ is the lexicographic ordering.

Example 139. We claim that stable representations behave like simples. We have the following version of Schur's lemma. If $V \geq W$ are semistable representations with a non-zero morphism $V \xrightarrow{\phi} W$. We claim that the following properties hold

(1) $V \asymp W$
(2) if W is stable, then ϕ is onto
(3) if V is stable, then ϕ is mono
(4) if V and W are stable, then ϕ is an isomorphism.

Indeed, consider the exact sequences

$$0 \longrightarrow Ker\ \phi \longrightarrow V \longrightarrow Im\ \phi \longrightarrow 0 \quad 0 \longrightarrow Im\ \phi \longrightarrow W \longrightarrow Coker\ \phi \longrightarrow 0$$

As $\phi \neq 0$, $Im\ \phi$ a non-trivial factor- resp. subrepresentation whence

$$V \leq Im\ \phi \quad \text{and} \quad Im\ \phi \leq W \quad \text{whence} \quad V \leq W$$

and therefore $V \asymp Im\ \phi \asymp W$. For (2), if $Im\ \phi$ is a proper subrepresentation, then $Im\ \phi < W$ contradicting $Im\ \phi \asymp W$. (3) is proved similarly and (4) follows.

Example 140. If $0 \neq V \subset W$, then either V is semistable or there is a semistable subrepresentation $V' \subset V$ with $V' > V$. Indeed, assume V is not semistable, then there is a $V_1 \subset V$ with $V_1 > V$. Either V_1 is semistable and we are done or we can continue to find a subrepresentation $V_2 \subset V_1$ with $V_2 > V_1$. As all our representations are finite dimensional, this process must finish.

If $0 \neq V \subset W$ and there exists a semistable subrepresentation $U \subset W$ with $V > U$ then either $U \subset V$ or there is a subrepresentation $V \subset V' \subset W$ with $V' > V$. Consider the exact sequences

$$0 \longrightarrow V \cap U \longrightarrow U \longrightarrow X \longrightarrow 0 \quad 0 \longrightarrow V \subset V + U \longrightarrow X \longrightarrow 0$$

If $U \not\subseteq V$ then $X \neq 0$ If $V \cap U \neq 0$ then $V \cap U \leq U$ whence $U \leq X$. As $V < U$ we have $V < X$ and by the second sequence $V < V + U$.

Example 141. Call a subrepresentation $0 \neq V \subset W$ *greedy* if every semistable subrepresentation $U \subset W$ satisfying $V > U$ is contained in V. We claim that for any subrepresentation $0 \neq U \subset W$ there is a greedy subrepresentation $U \subset V \subset W$ with $U \leq V$.

Indeed if $V_1 = U$ does not satisfy the requirement, then there is a semistable $X > V_1$ not contained in V_1 whence by the foregoing example there is a $V_1 \subset V_2$ such that $V_1 < V_2$. If V_2 does not satisfy the requirement, we can repeat this process and obtain a properly increasing series $V_1 \subset V_2 \subset \ldots \subset W$ and by finite dimensionality of W it must stabilize proving the claim.

THEOREM 103 (Rudakov). *For every $W \in \mathtt{rep}A$ there is a unique semistable subrepresentation $\Delta(W)$ satisfying the following properties*

1. *For every $0 \neq V \subset W$ we have $V \leq \Delta(W)$.*
2. *For every $0 \neq V \subset W$ such that $V \asymp \Delta(W)$ we have $V \subset \Delta(W)$.*

For every $W \in \mathtt{rep}A$ there is a unique semistable factorrepresentation $\nabla(W)$ satisfying the following properties

1. *For every $W \twoheadrightarrow V \neq 0$ we have $V \geq \nabla(W)$.*
2. *For every $W \twoheadrightarrow V \neq 0$ such that $V \asymp \nabla(W)$ we have $\nabla(W) \twoheadrightarrow V$.*

PROOF. We prove the statement on $\Delta(W)$, that on $\nabla(W)$ follows by duality. There exists $W' \subset W$ satisfying (1). If W does not satisfy (1) there is a subrepresentation $V \subset W$ such that $V > W$ but then by the previous example there is a greedy subrepresentation $W_1 \subset W$ such that $W_1 \geq V > W$. If for all $U \subset W_1$ we have $U \leq W_1$, then W_1 satisfies (1). Indeed, let $V \subset W$ such that $V > W_1$, then either V is semistable or we can extend V to a semistable $V' \subset W$ such that $V' > V$ whence $V' > W_1$ but by greediness of W_1 we would have $V' \subset W_1$, a contradiction.

Otherwise, there is a $V \subset W$ such that $V > W_1$ and then we can extend V to a greedy $W_2 \subset W_1$ such that $W_2 \geq V > W_1$. We can proceed this way and obtain a strictly decreasing sequence of subrepresentation which must stabilize whence proving the existence of subrepresentations $W' \subset W$ satisfying (1).

Let $\mathcal{W}$ be the set of all these W' and take $W'_1 \in \mathcal{W}$. If W'_1 does not satisfy (2), there is $V \subset W$ with $V \asymp W'_1$ and $V \not\subseteq W'_1$ and we may assume that V is semistable (if not, extend V to semistable $V' > V$ but then $V' > W'_1$ contradicting that W'_1 satisfies (1)). Let $W'_2 = W'_1 + V$ then we claim that $W'_2 \geq W'_1$. We have the sequences

$$0 \longrightarrow V \cap W'_1 \longrightarrow V \longrightarrow U \longrightarrow 0 \quad 0 \longrightarrow W'_1 \longrightarrow W'_1 + V \longrightarrow U \longrightarrow 0$$

As V is semistable, $V \cap W'_1 \leq V$ whence $V \leq U$. If $W'_1 + V < W'_1$ then $U < W'_1$ whence $V \leq U < W'_1$ contradicting $V \asymp W'_1$. Therefore, $W'_1 \subset W'_2$ and $W'_2 \in \mathcal{W}$. Either W'_2 satisfies (2) or we can repeat the argument. This way we obtain a strictly increasing sequence of subrepresentations which must stabilize in order to give a subrepresentation Δ satisfying (1) and (2).

Remains to prove uniqueness. Assume Δ' also satisfies (1) and (2), then $\Delta \asymp \Delta'$ and by (2) both $\Delta \subset \Delta'$ and $\Delta' \subset \Delta$. □

THEOREM 104 (Rudakov). *There is a* Jordan-Hölder filtration *for semistable representations. That is, if W is a semistable representation, then there is a filtration*

$$0 = W_{n+1} \subset W_n \subset \ldots \subset W_1 \subset W_0 = W$$

such that the quotients $V_i = W_i/W_{i+1}$ are stable and $V_0 \asymp V_1 \asymp \ldots \asymp V_n$.

PROOF. Let W_n be the minimal subrepresentation such that $W_n \asymp W$, then W_n is obviously stable. We claim that W/W_n is semistable. By the seesaw property we have $W_n \asymp W \asymp W/W_n$. If W/W_n is not semistable, there is a $W_n \subset V \subset W$ with $V/W_n > W/W_n$. But then, $W \asymp W/W_n > W/V$ contradicting semistability of W as this implies $W \leq W/V$. Iterating this process finishes the proof. □

THEOREM 105 (Rudakov). *There is a Harder-Narasinham filtration for $W \in$ `rep`A. That is, there is a unique filtration*

$$0 = W_{m+1} \subset W_m \subset \ldots \subset W_1 \subset W_0 = W$$

such that the quotients $V_i = W_i/W_{i+1}$ are semistable and $V_0 < V_1 < \ldots < V_m$.

PROOF. Define $U_0 = 0$ and $U_{-1} = \Delta(W)$ and define for higher i the subrepresentation $U_{-(i+1)}$ by the property that

$$\frac{U_{-(i+1)}}{U_{-i}} = \Delta(\frac{W}{U_{-i}})$$

These quotients are semistable and it follows from the seesaw property and the sequences that

$$\frac{U_{-(i+2)}}{U_{-(i+1)}} < \frac{U_{-(i+1)}}{U_{-i}}$$

and hence the sequence $U_{-1} \subset U_{-2} \subset \ldots \subset U_{-(m+1)} = W$ satisfies the requirements of the theorem by example 138. Uniqueness follows from induction on the filtration length and the following

claim : If $0 = W_{m+1} \subset W_m \subset \ldots \subset W_1 \subset W_0 = W$ is such that the quotients $V_i = W_i/W_{i+1}$ are semistable and $V_0 < V_1 < \ldots < V_n$, then $W_m = \Delta(W)$.

We prove this by induction on m. If $m = 0$ then W is semistable. So assume by induction that $W_{m-1}/W_m = \Delta(W/W_m)$ then for any $V \subset W$ we have that

$$\frac{V}{W_m \cap V} \leq \frac{W_{m-1}}{W_m} = V_{m-1} < V_m = W_m$$

As W_m is semistable, $V \cap W_m \leq W_m$ but then from the center of mass property we deduce that $A \leq W_m$. That is, W_m satisfies (1) defining $\Delta(W)$. As for (2), assume that V is a subrepresentation such that $V \asymp W_m$, then

$$V \cap W_m \leq W_m \asymp V \quad \text{whence} \quad V \leq \frac{V}{V \cap W_m}$$

if this factor is nonzero. But then $V \asymp W_m = V_m > V_{m-1}$ whence $V/(V \cap W_m) > V_{m-1}$ a contradiction by induction. Therefore, the factor $V/(V \cap W_m) = 0$, that is, $V \subset W_m$ and so W_m satisfies (2) finishing the proof of the claim. □

Remains to prove the existence of stability structures on `rep`A.

EXAMPLE 142. Let c, r be two additive functions on `rep`A, that is, for every short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

we have $c(V) = c(U) + c(W)$ and $r(V) = r(U) + r(W)$. Assume that $r(V) > 0$ for all nonzero V (for example, let r be the dimension). The *slope* is defined to be

$$\mu(V) = \frac{c(V)}{r(V)}$$

We claim that the *slope order* defined by

$$\begin{aligned} V < W \quad &\Leftrightarrow \quad \mu(V) < \mu(W) \\ V \asymp W \quad &\Leftrightarrow \quad \mu(V) = \mu(W) \end{aligned}$$

is a stability structure on $\mathtt{rep}A$. Indeed, we have

$$\mu(V) - \mu(W) = \frac{1}{r(V)r(W)} \begin{vmatrix} r(W) & c(W) \\ r(V) & c(V) \end{vmatrix}$$

and because r is positive, the slope order is determined by the sign (or zeroing) of the determinant. Take an exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

then by additivity of c and r we have the following equalities between the determinants

$$\begin{aligned} \begin{vmatrix} r(V) & c(V) \\ r(U) & c(U) \end{vmatrix} &= \begin{vmatrix} r(U)+r(W) & c(U)+c(W) \\ r(U) & c(U) \end{vmatrix} = \begin{vmatrix} r(W) & c(W) \\ r(U) & c(U) \end{vmatrix} \\ &= \begin{vmatrix} r(W) & c(W) \\ r(U)+r(W) & c(U)+c(W) \end{vmatrix} = \begin{vmatrix} r(W) & c(W) \\ r(V) & c(V) \end{vmatrix} \end{aligned}$$

from which the claim follows.

Definition 98 (King). Let $A \in \mathtt{alg}$ and $K_0(A) \xrightarrow{\theta} \mathbb{R}$ be an additive function on the Grothendieck group.

A representation $V \in \mathtt{rep}A$ is said to be *θ-semistable* if $\theta(V) = 0$ and every subrepresentation $V' \subset V$ satisfies $\theta(V') \geq 0$.

A representation $V \in \mathtt{rep}A$ is said to be *θ-stable* if V is θ-semistable and the only subrepresentations V' with $\theta(V') = 0$ are 0 and V.

Example 143. Every slope stability structure as in example 142 determines for any $V \in \mathtt{rep}A$ an additive function θ on the Grothendieck group

$$\theta(W) = -c(W) + \frac{c(V)}{r(V)} r(W)$$

Observe that $\theta(V) = 0$ and M is (semi)stable for the slope stability structure if and only if M is θ-(semi)stable. For we have

$$\begin{aligned} \theta(V') \geq 0 \qquad &\Leftrightarrow \quad -c(V') + \tfrac{c(V)}{r(V)} r(V') \geq 0 \\ &\Leftrightarrow \quad \tfrac{c(V')}{r(V')} \leq \tfrac{c(V)}{r(V)} \end{aligned}$$

Example 144. Let Q be a finite quiver on k vertices, then $K_0\langle Q \rangle = \mathbb{Z}^k$ whence any additive function θ is determined by a k-tuple $(t_1, \ldots, t_k) \in \mathbb{R}^k$. If $M \in \mathtt{rep}_\alpha\, Q$, then we define

$$\theta(M) = \theta.\alpha = t_1 a_1 + \ldots + t_k a_k$$

Therefore, M is θ-semistable if for every subrepresentation $M' \subset M$ of dimension vector β we have $\theta.\beta \geq 0$ and is θ-stable if the only subrepresentations with $\theta(M') = 0$ are 0 or M.

On $\mathtt{rep}\langle Q \rangle$ slope stability structures can be defined by taking $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$ and $r = (r_1, \ldots, r_k) \in \mathbb{N}_0^k$ and defining for $M \in \mathtt{rep}_\alpha\, Q$

$$c(M) = c.\alpha \qquad \text{and} \qquad r(M) = r.\alpha$$

We will outline the interrelation between Schur roots, stability structures and the notion of *Sylvester rank function* introduced and studied by Aidan Schofield [**60**].

If $\alpha \in \mathtt{schur}A$, there is an open set in $\mathtt{rep}_\alpha A$ of representations such that the stabilizer subgroup is $\mathbb{C}^*$. This open set determines a principal PGL_n-fibration and therefore a central simple algebra Σ of dimension n^2 over its center. That is, we have an algebra morphism

$$A \longrightarrow \Sigma = M_r(D)$$

with D a central division algebra of dimension s^2 and $n = rs$. This defines an additive function

$$K_0(A) \xrightarrow{\rho} \frac{1}{r}\mathbb{Z} \qquad \text{such that} \qquad \rho([A]) = 1$$

by sending the class $[P]$ of the finitely generated projective P (or more generally, a finitely presented A-module) to $\rho([P])$ which is the rank over Σ of $\Sigma \otimes_A P$. This function has the following properties

(1) $\rho(A) = 1$
(2) $\rho(M \oplus M') = \rho(M) + \rho(M')$
(3) $\rho(M") \leq \rho(M') \leq \rho(M) + \rho(M")$ for every exact sequence

$$M \longrightarrow M' \longrightarrow M" \longrightarrow 0$$

of finitely presented A-modules.

More generally, let $G_0(A)$ be the Abelian group on the isomorphism classes of finitely presented A-modules with relations $[M \oplus M'] = [M] + [M']$ and define an ordering on $G_0(A)$ by specifying a positive cone $[M] > 0$ for all non-zero finitely presented modules M and if

$$M \longrightarrow M' \longrightarrow M" \longrightarrow 0$$

is an exact sequence, then we define $[M'] - [M"] \geq 0$ and $[M] - [M'] + [M"] \geq 0$. A *Sylvester module rank function* is defined to be an order preserving additive map

$$G_0(A) \xrightarrow{\rho} \mathbb{R} \qquad \text{such that} \qquad \rho(A) = 1$$

If $\rho_1, \ldots, \rho_k$ are Sylvester rank functions, then so is $q_1\rho_1 + \ldots q_k\rho_k$ with $q_i \in \mathbb{Q}, q_i > 0$ and $q_1 + \ldots + q_k = 1$. Hence the Sylvester rank functions form a $\mathbb{Q}$-convex subset in the space of all real valued order preserving morphisms on $G_0(A)$. In fact, Aidan Schofield proved [**60**, Thm.7.25] that they form an infinite dimensional $\mathbb{Q}$-simplex as any Sylvester rank function ρ can be written in a unique way as the weighted sum of *extremal points*, that is, functions that do not lie in the linear span of others. We call the set of all Sylvester rank functions on A the *Schofield fractal* $\mathtt{schof}A$ of the algebra A. It contains all information about algebra morphisms from A to simple Artinian algebras (not necessarily finite dimensional over their centers).

Returning to the study of $\mathtt{rep}A$ we can restrict Sylvester rank functions to $\mathtt{rep}A$ and then it turns out that the functions we constructed above from Schur roots $\alpha \in \mathtt{schur}A$ are extremal points in $\mathtt{schof}A$. Moreover, any Sylvester rank function ρ determines a universal localization A_ρ of A, see [**60**], and in most cases Schur representations in $\mathtt{rep}_\alpha A$ will become simple representations in an affine intermediate universal localization $A_\sigma \subset A_\rho$ for the Sylvester rank function determined by $\alpha \in \mathtt{schur}A$.

We will prove these claims in full detail in the next section for path algebras of quivers as they are important to us in the investigation of nullcones of quiver-representations. First, we prove Schofield's characterization of $\mathtt{schur}\langle Q \rangle$ based on his results on general subrepresentations [**61**] and the theory of *compartments* due to Harm Derksen and Jerzy Weyman [**13**] which can be seen as the part of the Schofield fractal $\mathtt{schof}\langle Q \rangle$ relevant for finite dimensional quiver-representations.

To enlarge our cultural luggage, we briefly recall some classical results on indecomposable representations of quivers due to Victor Kaĉ [**26**]. Any quiver-representation $V \in \mathtt{rep}_\alpha\ Q$ decomposes uniquely into a sum

$$V = W_1^{\oplus f_1} \oplus \ldots \oplus W_z^{\oplus f_z}$$

of *indecomposable* representations . This follows from the fact that $End_{\langle Q \rangle}(V)$ is finite dimensional. Recall also that a representation W of Q is indecomposable if and only if $End_{\langle Q \rangle}(W)$ is a finite dimensional *local algebra*, that is, the nilpotent endomorphisms in $End_{\langle Q \rangle}(W)$ form an ideal of codimension one. Equivalently, the maximal torus of the stabilizer subgroup $Stab_{GL(\alpha)}(W) = Aut_{\langle Q \rangle}(W)$ is one-dimensional, which means that every semisimple element of $Aut_{\langle Q \rangle}(W)$ lies in $\mathbb{C}^*(\mathbb{1}_{d_1}, \ldots, \mathbb{1}_{d_k})$.

In general, decomposing a representation V into indecomposables corresponds to choosing a maximal torus T in the stabilizer subgroup $Aut_{\langle Q \rangle}(V)$. Decompose the vertexspaces

$$V_i = \oplus_\chi V_i(\chi) \qquad \text{where} \qquad V_i(\chi) = \{ v \in V_i \mid t.v = \chi(t)v \ \forall t \in T \}$$

where χ runs over all characters of T. One verifies that each $V(\chi) = \oplus_i V_i(\chi)$ is a subrepresentation of V giving a decomposition $V = \oplus_\chi V(\chi)$. Because T acts by scalar multiplication on each component $V(\chi)$, we have that $\mathbb{C}^*$ is the maximal torus of $Aut_{\langle Q \rangle}(V(\chi))$, whence $V(\chi)$ is indecomposable.

Conversely, if $V = W_1 \oplus \ldots \oplus W_r$ is a decomposition with the W_i indecomposable, then the product of all the one-dimensional maximal tori in $Aut_{\langle Q \rangle}(W_i)$ is a maximal torus of $Aut_{\langle Q \rangle}(V)$.

Definition 99. The *Tits form* of a quiver Q is the symmetrization of its Euler form, that is,

$$T_Q(\alpha, \beta) = \chi_Q(\alpha, \beta) + \chi_Q(\beta, \alpha)$$

This symmetric bilinear form is described by the *Cartan matrix*

$$C_Q = \begin{bmatrix} c_{11} & \ldots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \ldots & c_{kk} \end{bmatrix} \quad \text{with} \quad c_{ij} = 2\delta_{ij} - \# \{ \text{ⓘ}\!\!-\!\!-\!\!-\!\!-\text{ⓘ} \}$$

where we count all arrows connecting v_i with v_j forgetting the orientation. The corresponding *quadratic form* $q_Q(\alpha) = \frac{1}{2}\chi_Q(\alpha, \alpha)$ on $\mathbb{Q}^k$ is

$$q_Q(x_1, \ldots, x_k) = \sum_{i=1}^k x_i^2 - \sum_{a \in Q_a} x_{t(a)} x_{h(a)}$$

Observe that $q_Q(\alpha) = dim\ GL(\alpha) - dim\ \mathtt{rep}_\alpha\ Q$. With Γ_Q we will denote the underlying graph of Q.

Definition 100. A quadratic form q on $\mathbb{Z}^k$ is said to be *positive definite* if $0 \neq \alpha \in \mathbb{Z}^k$ implies $q(\alpha) > 0$.

A quadratic form q on $\mathbb{Z}^k$ is called positive *semi-definite* if $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^k$. The *radical* of q is $\mathtt{rad}(q) = \{ \alpha \in \mathbb{Z}^k \mid T(\alpha, -) = 0 \}$. If q_Q is positive semi-definite, there exist a minimal $\delta_Q \geq \underline{0}$ with the property that $q_Q(\alpha) = 0$ if and only if $\alpha \in \mathbb{Q}\delta_Q$ if and only if $\alpha \in \mathtt{rad}(q_Q)$.

If the quadratic form q is neither positive definite nor semi-definite, it is called *indefinite* .

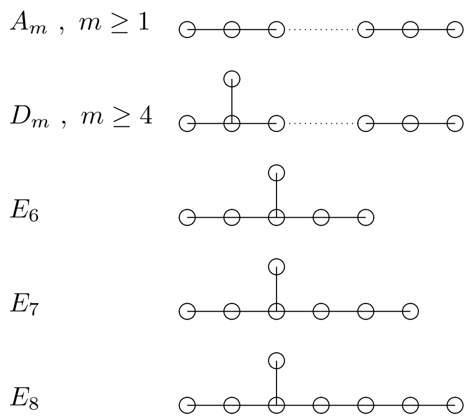


FIGURE 1. The Dynkin diagrams.

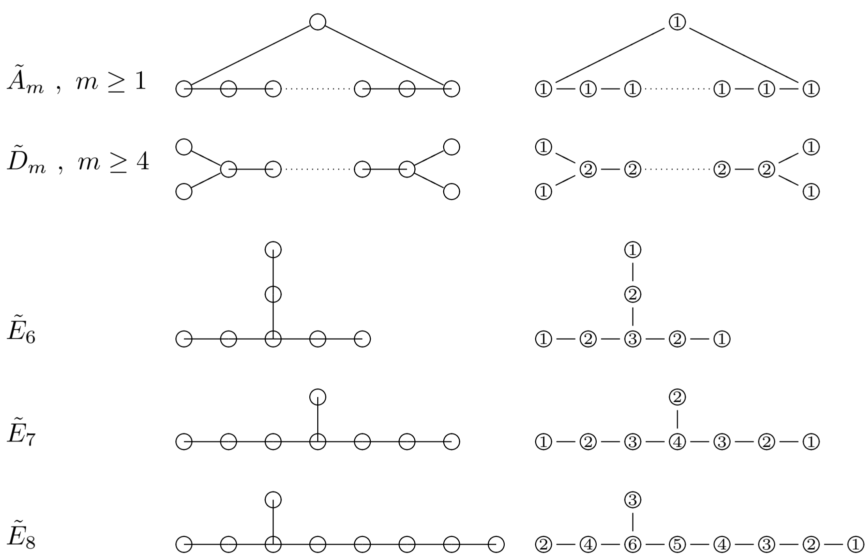


FIGURE 2. The extended Dynkin diagrams.

THEOREM 106. *Let Q be a connected quiver with Tits form q_Q, Cartan matrix C_Q and underlying graph Γ_Q. Then,*

(1) *q_Q is positive definite if and only if Γ_Q is a* Dynkin diagram *, that is one of the graphs of figure 1. The number of vertices is m.*
(2) *q_Q is semidefinite if and only if Γ_Q is an* extended Dynkin diagram*, that is one of the graphs of figure 2 and δ_Q is the indicated dimension vector. The number of vertices is $m + 1$.*

PROOF. Classical, see for example [**7**]. □

Definition 101. An *indecomposable root* of Q is the dimension vector of an indecomposable representation of Q. With $\mathtt{ind}Q$ we denote the set of all indecomposable roots of Q.

If $V \in \mathtt{rep}_\alpha\ Q$ has a decomposition into indecomposables

$$V = W_1^{\oplus f_1} \oplus \ldots \oplus W_z^{\oplus f_z}$$

with γ_i the dimension vector of W_i, we say that V is of type $\tau = (f_1, \gamma_1; \ldots; f_z, \gamma_z)$. With $\mathtt{itypes}_\alpha\ Q$ we denote the set of all decomposition types which do occur for α-dimensional representations.

Example 145. The canonical decomposition and generic representations. For a dimension vector α, we claim that there exists a unique type $\tau_{can} = (e_1, \beta_1; \ldots; e_l, \beta_l) \in \mathtt{itypes}_\alpha\ Q$ such that the set

$$\mathtt{rep}_\alpha(\tau_{can}) = \{V \in \mathtt{rep}_\alpha\ Q\ \mathtt{itype}(V) = \tau_{can}\}$$

contains a dense open set of $\mathtt{rep}_\alpha\ Q$. Indeed, by example 72 we know that for any dimension vector β the subset $\mathtt{rep}_\beta^{ind}\ Q$ of *indecomposable* representations of dimension β is constructible. For $\tau = (f_1, \gamma_1, ; \ldots; f_z, \gamma_z) \in \mathtt{itypes}_\alpha Q$ the subset

$$\mathtt{rep}_\alpha(\tau) = \{V \in \mathtt{rep}_\alpha\ Q \ \mid\ \mathtt{itype}(V) = \tau\}$$

is a constructible subset of $\mathtt{rep}_\alpha\ Q$ as it is the image of the constructible set

$$GL(\alpha) \times \mathtt{rep}_{\gamma_1}^{ind}\ Q \times \ldots \times \mathtt{rep}_{\gamma_z}^{ind}\ Q$$

under the map sending $(g, W_1, \ldots, W_z)$ to $g.(W_1^{\oplus f_1} \oplus \ldots \oplus W_z^{\oplus f_z})$. Because of the uniqueness of the decomposition into indecomposables we have a finite disjoint decomposition

$$\mathtt{rep}_\alpha\ Q = \bigsqcup_{\tau \in \mathtt{itypes}_\alpha Q} \mathtt{rep}_\alpha(\tau)$$

and by irreducibility of $\mathtt{rep}_\alpha\ Q$ precisely one of the $\mathtt{rep}_\alpha(\tau)$ contains a dense open set of $\mathtt{rep}_\alpha\ Q$. This unique type τ_{can} is said to be the *canonical decomposition* of α.

Consider the action morphisms $GL(\alpha) \times \mathtt{rep}_\alpha\ Q \xrightarrow{\phi} \mathtt{rep}_\alpha\ Q$. By Chevalley's theorem 48 we know that the function

$$V \mapsto dim\ Stab_{GL(\alpha)}(V)$$

is upper semi-continuous. Because $dim\ GL(\alpha) = dim\ Stab_{GL(\alpha)}(V) + dim\ \mathcal{O}(V)$ we conclude that for all m, the subset

$$\mathtt{rep}_\alpha(m) = \{V \in \mathtt{rep}_\alpha\ Q \ \mid\ dim\ \mathcal{O}(V) \geq m\}$$

is Zariski open. In particular, $\mathtt{rep}_\alpha(max)$, the union of all orbits of maximal dimension, is open and dense in $\mathtt{rep}_\alpha\ Q$.

A representation $V \in \mathtt{rep}_\alpha\ Q$ lying in the intersection

$$\mathtt{rep}_\alpha(\tau_{can}) \cap \mathtt{rep}_\alpha(max)$$

is called a *generic representation* of dimension vector α.

Example 146. Finite type quivers are Dynkin. Assume that Q is a connected quiver of *finite representation type* , that is, there are only a finite number of isomorphism classes of indecomposable representations. Let α be an arbitrary dimension vector. Since any representation of Q can be decomposed into a direct sum of indecomposables, $\mathtt{rep}_\alpha\ Q$ contains only finitely many orbits. Hence, one orbit $\mathcal{O}(V)$ must be dense and have the same dimension as $\mathtt{rep}_\alpha\ Q$, but then

$$dim\ \mathtt{rep}_\alpha\ Q = dim\ \mathcal{O}(V) \leq dim\ GL(\alpha) - 1$$

as any representation has $\mathbb{C}^*(\mathbb{1}_{a_1}, \ldots, \mathbb{1}_{a_k})$ in its stabilizer subgroup. That is, for every $\alpha \in \mathbb{N}^k$ we have $q_Q(\alpha) \geq 1$. Because all off-diagonal entries of the Cartan matrix C_Q are non-positive, it follows that q_Q is positive definite on $\mathbb{Z}^k$ whence Γ_Q must be a Dynkin diagram.

Definition 102. Let $\epsilon_i = (\delta_{1i}, \ldots, \delta_{ki})$ be the standard basis of $\mathbb{Q}^k$. The *fundamental set of roots* is defined to be the set of dimension vectors

$$F_Q = \{\alpha \in \mathbb{N}^k - \underline{0}\ \mid\ T_Q(\alpha, \epsilon_i) \leq 0 \text{ and } supp(\alpha) \text{ is connected }\}$$

Theorem 107. *Let* $\alpha = \beta_1 + \ldots + \beta_s \in F_Q$ *with* $\beta_i \in \mathbb{N}^k - \underline{0}$ *for* $1 \leq i \leq s \geq 2$. *If* $q_Q(\alpha) \geq q_Q(\beta_1) + \ldots + q_Q(\beta_s)$, *then* $supp(\alpha)$ *is a* tame quiver *(its underlying graph is an extended Dynkin diagram) and* $\alpha \in \mathbb{N}\delta_{supp(\alpha)}$.

Proof. (1) : Let $s = 2$, $\beta_1 = (c_1, \ldots, c_k)$ and $\beta_2 = (d_1, \ldots, d_k)$ and we may assume that $supp(\alpha) = Q$. By assumption $T_Q(\beta_1, \beta_2) = q_Q(\alpha) - q_Q(\beta_1) - q_Q(\beta_2) \geq 0$. Using that C_Q is symmetric and $\alpha = \beta_1 + \beta_2$ we have

$$\begin{aligned} 0 \leq T_Q(\beta_1, \beta_2) &= \sum_{i,j} c_{ij} c_i d_i \\ &= \sum_j \frac{c_j d_j}{a_j} \sum_i c_{ij} a_i + \frac{1}{2} \sum_{i \neq j} c_{ij} (\frac{c_i}{a_i} - \frac{c_j}{a_j})^2 a_i a_j \end{aligned}$$

and because $T_Q(\alpha, \epsilon_i) \leq 0$ and $c_{ij} \leq 0$ for all $i \neq j$, we deduce that

$$\frac{c_i}{a_i} = \frac{c_j}{a_j} \qquad \text{for all } i \neq j \text{ such that } c_{ij} \neq 0$$

Because Q is connected, α and β_1 are proportional. But then, $T_Q(\alpha, \epsilon_i) = 0$ and hence $C_Q \alpha = \underline{0}$. By the classification result, q_Q is semidefinite whence Γ_Q is an extended Dynkin diagram and $\alpha \in \mathbb{N}\delta_Q$. Finally, if $s > 2$, then

$$T_Q(\alpha, \alpha) = \sum_i T_Q(\alpha, \beta_i) \geq \sum_i T_Q(\beta_i, \beta_i)$$

whence $T_Q(\alpha - \beta_i, \beta_i) \geq 0$ for some i and then we can apply the foregoing argument to β_i and $\alpha - \beta_i$. □

Definition 103. If G is an algebraic group acting on a variety Y and if $X \subset Y$ is a G-stable subset, then we can decompose $X = \bigcup_d X_{(d)}$ where $X_{(d)}$ is the union of all orbits $\mathcal{O}(x)$ of dimension d. The *number of parameters* of X is

$$\mu(X) = \max_d\ (dim\ X_{(d)} - d)$$

where $dim\ X_{(d)}$ denotes the dimension of the Zariski closure of $X_{(d)}$.

In the special case of $GL(\alpha)$ acting on $\mathtt{rep}_\alpha\ Q$, we denote $\mu(\mathtt{rep}_\alpha(max)) = p_Q(\alpha)$ and call it the *number of parameters* of α. For example, if α is a simple root, then $p(\alpha) = dim\ \mathtt{rep}_\alpha\ Q - (dim\ GL(\alpha) - 1) = 1 - q_Q(\alpha)$.

Definition 104. Let v_i be a *source vertex* of Q and let $\alpha = (a_1, \ldots, a_k)$ a dimension vector satisfying $\sum_{t(a)=v_i} a_{h(a)} \geq a_i$. Consider the subset

$$\mathtt{rep}_\alpha^{mono}(i) = \{V \in \mathtt{rep}_\alpha\ Q \ \mid \ \oplus V_a : V_i \longrightarrow \ \oplus_{t(a)=v_i} V_{s(a)} \ \text{ is injective } \}$$

All indecomposable representations are contained in this subset.

The *reflected quiver* R_iQ is obtained from Q by reversing the direction of all arrows with tail v_i making v_i into a *sink vertex.*

The *reflected dimension vector* $R_i\alpha = (r1, \ldots, r_k)$ is defined to be

$$r_j = \begin{cases} a_j & \text{if } j \neq i \\ \sum_{t(a)=i} a_{s(a)} - a_i & \text{if } j = i \end{cases}$$

For the reflected quiver R_iQ we have that $\sum_{h(a)=i} r_{t(a)} \geq r_i$. Define the subset

$$\mathtt{rep}_{R_i\alpha}^{epi}(i) = \{V \in \mathtt{rep}_{R_i\alpha}\ R_iQ \ \mid \ \oplus V_a : \oplus_{s(a)=i} V_{t(a)} \longrightarrow \ V_i \text{ is surjective } \}$$

Theorem 108 (Bernstein-Gel'fand-Ponomarov). *Endowing both spaces with the quotient Zariski topology, there is an homeomorphism*

$$\mathtt{rep}_\alpha^{mono}(i)/GL(\alpha) \xrightarrow{\simeq} \mathtt{rep}_{R_i\alpha}^{epi}(i)/GL(R_i\alpha)$$

such that corresponding representations have isomorphic endomorphism rings.

In particular, the number of parameters as well as the number of irreducible components of maximal dimension are the same for $\mathtt{rep}_\alpha^{ind}\ Q_{(d)}$ *and* $\mathtt{rep}_{R_i\alpha}^{ind}\ R_iQ_{(d)}$ *for all dimensions d.*

Proof. Let us denote with $m = \sum_{t(a)=i} a_i$, $\overline{\mathtt{rep}} = \oplus_{t(a)\neq i} M_{a_{s(a)} \times a_{t(a)}}(\mathbb{C})$ and $\overline{GL} = \prod_{j\neq i} GL_{a_j}$. If $\mathtt{Grass}_k(l)$ denotes the *Grassmann manifold* of k-dimensional subspaces of $\mathbb{C}^l$, then there is a homeomorphism

$$\mathtt{rep}_\alpha^{mono}(i)/GL_{a_i} \xrightarrow{\simeq} \overline{\mathtt{rep}} \times \mathtt{Grass}_{a_i}(m)$$

sending a representation V to its restriction in $\overline{\mathtt{rep}}$ and the image of the map $\oplus V_a$ for all arrows leaving v_i.

Similarly, sending a representation V to its restriction in $\overline{\mathtt{rep}}$ and the kernel of the sum map $\oplus V_a$ for all arrows into v_i, we have an homeomorphism

$$\mathtt{rep}_{R_i\alpha}^{epi}(i)/GL_{r_i} \xrightarrow{\simeq} \overline{\mathtt{rep}} \times \mathtt{Grass}_{a_i}(m)$$

and the first claim follows from figure 3. If $V \in \mathtt{rep}_\alpha\ Q$ and $V' \in \mathtt{rep}_{R_i\alpha}\ R_iQ$ with images respectively v and v' in $\overline{rep} \times \mathtt{Grass}_{a_i}(m)$, we have isomorphisms

$$\begin{cases} Stab_{\overline{GL}\times GL_{a_i}}(V) & \xrightarrow{\simeq} Stab_{\overline{GL}}(v) \\ Stab_{\overline{GL}\times GL_{r_i}}(V') & \xrightarrow{\simeq} Stab_{\overline{GL}}(v') \end{cases}$$

from which the claim about endomorphisms follows. □

A similar results holds for *sink vertices*, hence we can apply these *Bernstein-Gelfand- Ponomarov reflection functors* iteratively using a sequence of *admissible vertices* (that is, either a source or a sink).

Definition 105. For a vertex v_i having no loops in Q, we define a *reflection* $\mathbb{Z}^k \xrightarrow{r_i} \mathbb{Z}^k$ by

$$r_i(\alpha) = \alpha - T_Q(\alpha, \epsilon_i)$$

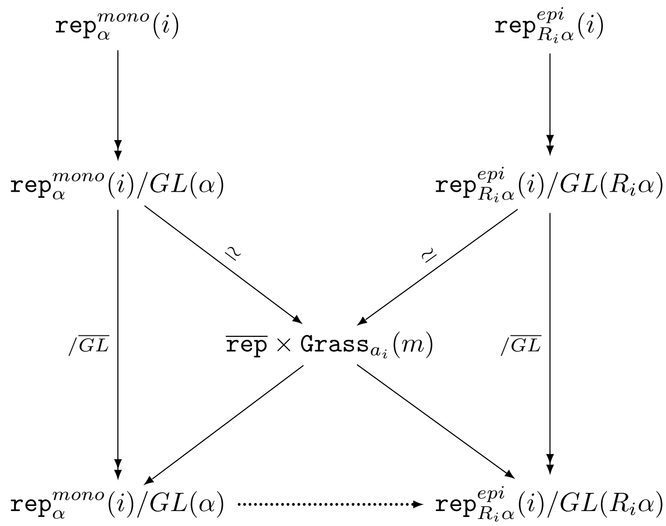


FIGURE 3. Reflection functor diagram.

The *Weyl group of the quiver* Q $\mathtt{Weyl}_Q$ is the subgroup of $GL_k(\mathbb{Z})$ generated by all reflections r_i.

A *root* of the quiver Q is a dimension vector $\alpha \in \mathbb{N}^k$ such that $\mathtt{rep}_\alpha\, Q$ contains indecomposable representations. All roots have connected support. A root is said to be

$$\begin{cases} \textit{real} & \text{if } \mu(\mathtt{rep}^{ind}_{\alpha}\; Q) = 0 \\ \textit{imaginary} & \text{if } \mu(\mathtt{rep}^{ind}_{\alpha}\; Q) \geq 1 \end{cases}$$

We denote the set of all roots, real roots and imaginary roots respectively by Δ, Δ_{re} and Δ_{im}. With Π we denote the set $\{\epsilon_i \mid v_i \text{ has no loops }\}$.

THEOREM 109 (Kaĉ). *With notations as before, we have*

(1) $\Delta_{re} = \mathtt{Weyl}_Q.\Pi \cap \mathbb{N}^k$ *and if* $\alpha \in \Delta_{re}$*, then* $\mathtt{rep}^{ind}_{\alpha}\; Q$ *is one orbit.*

(2) $\Delta_{im} = \mathtt{Weyl}.F_Q \cap \mathbb{N}^k$ *and if* $\alpha \in \Delta_{im}$ *then*

$$p_Q(\alpha) = \mu(\mathtt{rep}^{ind}_{\alpha}\; Q) = 1 - q_Q(\alpha)$$

PROOF. For a sketch of the proof we refer to [**19**, §7], full details can be found in the lecture notes [**37**]. □

Having a characterization of $\mathtt{ind}Q$ we will now determine the canonical decomposition. We first need a technical result.

THEOREM 110 (Happel-Ringel). (1) *If* $V = V' \oplus V" \in \mathtt{rep}_\alpha(max)$*, then* $Ext^1_{\langle Q\rangle}(V', V") = 0$.

(2) *If* W, W' *are indecomposables and* $Ext^1_{\langle Q\rangle}(W, W') = 0$*, then any non-zero map* $W' \xrightarrow{\phi} W$ *is an epimorphism or a monomorphism. In particular, if* W *is indecomposable with* $Ext^1_{\langle Q\rangle}(W, W) = 0$*, then* $End_{\langle Q\rangle}(W) \simeq \mathbb{C}$.

PROOF. (1) : Assume $Ext^1(V', V") \neq 0$, that is, there is a non-split exact sequence

$$0 \longrightarrow V" \longrightarrow W \longrightarrow V' \longrightarrow 0$$

It follows from section 4.4 that $\mathcal{O}(V) \subset \overline{\mathcal{O}(W)} - \mathcal{O}(W)$, whence $dim\ \mathcal{O}(W) > dim\ \mathcal{O}(V)$ contradicting the assumption that $V \in \mathtt{rep}_\alpha(max)$.

(2) : From the proof of theorem 70 we have the exact diagram

$$\begin{array}{ccccccc}
\underset{v_i \in Q_v}{\oplus} Hom_{\mathbb{C}}(V_i, W_i) & \xrightarrow{d^V_W} & \underset{a \in Q_a}{\oplus} Hom_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) & \longrightarrow & Ext^1_{\langle Q \rangle}(V, W) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \vdots & & \\
\underset{v_i \in Q_v}{\oplus} Hom_{\mathbb{C}}(V_i, W'_i) & \xrightarrow{d^V_{W'}} & \underset{a \in Q_a}{\oplus} Hom_{\mathbb{C}}(V_{s(a)}, W'_{t(a)}) & \longrightarrow & Ext^1_{\langle Q \rangle}(V, W') & \longrightarrow & 0
\end{array}$$

If $W \twoheadrightarrow W'$ then the dotted arrow is surjective. By a similar argument, if $W \hookrightarrow W'$ then the canonical map $Ext^1(W', V) \longrightarrow Ext^1(W, V)$ is surjective.

Assume ϕ is neither mono- nor epimorphism then decompose ϕ into

$$W' \overset{\epsilon}{\twoheadrightarrow} U \overset{\mu}{\hookrightarrow} W$$

As ϵ is epi, we have an epimorphism

$$Ext^1_{\langle Q \rangle}(W/U, W') \twoheadrightarrow Ext^1_{\langle Q \rangle}(W/U, U)$$

giving a representation V fitting into the exact diagram of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W' & \xrightarrow{\mu'} & V & \longrightarrow & W'/U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & U & \xrightarrow{\mu} & W & \longrightarrow & W'/U & \longrightarrow & 0
\end{array}$$

from which we construct an exact sequence of representations

$$0 \longrightarrow W' \xrightarrow{\begin{bmatrix} \epsilon \\ -\mu' \end{bmatrix}} U \oplus V \xrightarrow{\begin{bmatrix} \mu & \epsilon' \end{bmatrix}} W \longrightarrow 0$$

This sequence cannot split as otherwise we would have $W \oplus W' \simeq U \oplus V$ contradicting uniqueness of decompositions, whence $Ext^1_{\langle Q \rangle}(W, W') \neq 0$, a contradiction.

For the remaining part, as W is finite dimensional it follows that $End_{\langle Q \rangle}(W)$ is a (finite dimensional) division algebra whence it must be $\mathbb{C}$. □

Definition 106. Let $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$ be two dimension vectors. Consider the closed subvariety

$$\mathtt{Hom}_Q(\alpha, \beta) \hookrightarrow M_{a_1 \times b_1}(\mathbb{C}) \oplus \ldots \oplus M_{a_k \times b_k}(\mathbb{C}) \oplus \mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$$

consisting of triples (ϕ, V, W) where $\phi = (\phi_1, \ldots, \phi_k)$ is a morphism $V \longrightarrow W$. Projecting to the two last components we have an onto morphism between affine varieties

$$\mathtt{Hom}_Q(\alpha, \beta) \overset{h}{\twoheadrightarrow} \mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$$

The fiber dimension is upper-semicontinuous and as the target space $\mathtt{rep}_\alpha\ Q \oplus \mathtt{rep}_\beta\ Q$ is irreducible, it contains a non-empty open subset $\mathtt{hom}_{min}$ where the dimension of the fibers attains a minimal value. This minimal fiber dimension will be denoted by $hom(\alpha, \beta)$.

Similarly, there is an affine variety $\mathtt{Ext}_Q(\alpha,\beta)$ with fiber over a point $(V,W) \in \mathtt{rep}_\alpha\, Q \oplus \mathtt{rep}_\beta\, Q$ the extensions $Ext^1_{\langle Q\rangle}(V,W)$. Again, there is an open set $\mathtt{ext}_{min}$ where the dimension of $Ext^1(V,W)$ attains a minimum. This minimal value we denote by $ext(\alpha,\beta)$.

Because $\mathtt{hom}_{min} \cap \mathtt{ext}_{min}$ is a non-empty open subset we have the equality

$$hom(\alpha,\beta) - ext(\alpha,\beta) = \chi_Q(\alpha,\beta).$$

In particular, if $hom(\alpha,\alpha+\beta) > 0$, there will be an open subset where the morphism $V \xrightarrow{\phi} W$ is a monomorphism. Hence, there will be an open subset of $\mathtt{rep}_{\alpha+\beta}\, Q$ consisting of representations containing a subrepresentation of dimension vector α.

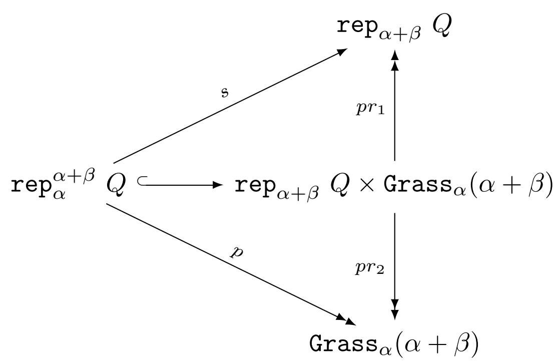
We say that α is a *general subrepresentation* of $\alpha+\beta$ and denote this with $\alpha \hookrightarrow \alpha+\beta$.

Similarly, α is a *general quotient* of $\alpha+\beta$, and we denote $\alpha+\beta \twoheadrightarrow \alpha$ if there is a Zariski open subset of $\mathtt{rep}_{\alpha+\beta}\, Q$ of representations having an α-dimensional quotient.

Example 147. The quiver Grassmannian is the projective manifold

$$\mathtt{Grass}_\alpha(\alpha+\beta) = \prod_{i=1}^k \mathtt{Grass}_{a_i}(a_i+b_i)$$

Consider the following diagram of morphisms of reduced varieties

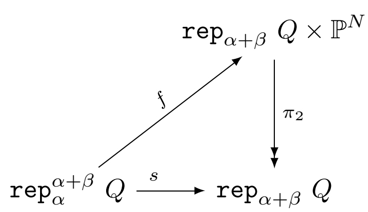


which satisfies the following properties :

$\mathtt{rep}_{\alpha+\beta}\, Q \times \mathtt{Grass}_\alpha(\alpha+\beta)$ is the trivial vectorbundle with fiber $\mathtt{rep}_{\alpha+\beta}\, Q$ over the projective smooth variety $\mathtt{Grass}_\alpha(\alpha+\beta)$ with structural morphism pr_2.

$\mathtt{rep}^{\alpha+\beta}_\alpha\, Q$ is the subvariety of $\mathtt{rep}_{\alpha+\beta}\, Q \times \mathtt{Grass}_\alpha(\alpha+\beta)$ consisting of couples (W,V) where V is a subrepresentation of W (observe that for fixed W this is a linear condition). Because $GL(\alpha+\beta)$ acts transitively on the Grassmannian $\mathtt{Grass}_\alpha(\alpha+\beta)$, $\mathtt{rep}^{\alpha+\beta}_\alpha\, Q$ is a sub-vectorbundle over $\mathtt{Grass}_\alpha(\alpha+\beta)$ with structural morphism p. In particular, $\mathtt{rep}^{\alpha+\beta}_\alpha\, Q$ is a reduced variety.

The morphism s is a projective morphism, that is, can be factored via the natural projection



Here, f is the composition of the inclusion $\texttt{rep}_{\alpha}^{\alpha+\beta}\ Q \hookrightarrow \texttt{rep}_{\alpha+\beta}\ Q \times \texttt{Grass}_{\alpha}(\alpha+\beta)$ with the natural embedding of Grassmannians in projective spaces $\texttt{Grass}_{\alpha}(\alpha+\beta) \hookrightarrow \prod_{i=1}^{k} \mathbb{P}^{n_i}$ with the Segre embedding $\prod_{i=1}^{k} \mathbb{P}^{n_i} \hookrightarrow \mathbb{P}^N$. In particular, s is proper by [**22**, Thm. II.4.9], that is, maps closed subsets to closed subsets.

Theorem 111 (Schofield). *For $W \in \texttt{rep}_{\alpha+\beta}\ Q$ in the image of the map s of the foregoing example, let $\texttt{Grass}_{\alpha}(W)$ denote the scheme-theoretic fiber $s^{-1}(W)$. Let $x = (W, V)$ be a geometric point of $\texttt{Grass}_{\alpha}(W)$, then*

$$T_x\ \texttt{Grass}_{\alpha}(W) = Hom_{\langle Q \rangle}(V, \frac{W}{V})$$

Proof. The geometric points of $\texttt{Grass}_{\alpha}(W)$ are couples (W, V) where V is an α-dimensional subrepresentation of W. Whereas $\texttt{Grass}_{\alpha}(W)$ is a projective scheme, it is in general neither smooth, nor irreducible nor even reduced. Therefore, in order to compute the tangent space in a point (W, V) of $\texttt{Grass}_{\alpha}(W)$ we have to clarify the functor it represents on the category $\texttt{commalg}$ of commutative $\mathbb{C}$-algebras.

Let C be a commutative $\mathbb{C}$-algebra, a representation $\mathcal{R}$ of the quiver Q over C consists of a collection $\mathcal{R}_i = P_i$ of projective C-modules of finite rank and a collection of C-module morphisms for every arrow a in Q

$$(j) \xleftarrow{a} (i) \qquad\qquad \mathcal{R}_j = P_j \xleftarrow{\mathcal{R}_a} P_i = \mathcal{R}_i$$

The dimension vector of the representation $\mathcal{R}$ is given by the k-tuple $(rk_C\ \mathcal{R}_1, \ldots, rk_C\ \mathcal{R}_k)$. A subrepresentation $\mathcal{S}$ of $\mathcal{R}$ is determined by a collection of projective sub-summands (and not merely sub-modules) $\mathcal{S}_i \triangleleft \mathcal{R}_i$. In particular, for $W \in \texttt{rep}_{\alpha+\beta}\ Q$ we define the representation $\mathcal{W}_C$ of Q over the commutative ring C by

$$\begin{cases} (\mathcal{W}_C)_i &= C \otimes_{\mathbb{C}} W_i \\ (\mathcal{W}_C)_a &= id_C \otimes_{\mathbb{C}} W_a \end{cases}$$

With these definitions, we can now define the functor represented by $\texttt{Grass}_{\alpha}(W)$ as the functor assigning to a commutative $\mathbb{C}$-algebra C the set of all subrepresentations of dimension vector α of the representation $\mathcal{W}_C$.

The tangent space in $x = (W, V)$ are the $\mathbb{C}[\epsilon]$-points of $\texttt{Grass}_{\alpha}(W)$ lying over (W, V). Let $V \xrightarrow{\psi} \frac{W}{V}$ be a homomorphism of representations of Q and consider a $\mathbb{C}$-linear lift of this map $\tilde{\psi} : V \longrightarrow W$. Consider the $\mathbb{C}$-linear subspace of $\mathcal{W}_{\mathbb{C}[\epsilon]} = \mathbb{C}[\epsilon] \otimes W$ spanned by the sets

$$\{v + \epsilon \otimes \tilde{\psi}(v)\ \mid\ v \in V\} \quad \text{and} \quad \epsilon \otimes V$$

This determines a $\mathbb{C}[\epsilon]$-subrepresentation of dimension vector α of $\mathcal{W}_{\mathbb{C}[\epsilon]}$ lying over (W, V) and is independent of the chosen linear lift $\tilde{\psi}$.

Conversely, if $\mathcal{S}$ is a $\mathbb{C}[\epsilon]$-subrepresentation of $\mathcal{W}_{\mathbb{C}[\epsilon]}$ lying over (W, V), then $\frac{\mathcal{S}}{\epsilon \mathcal{S}} = V \hookrightarrow W$. But then, a $\mathbb{C}$-linear complement of $\epsilon\mathcal{S}$ is spanned by elements of the form $v + \epsilon\psi(v)$ where $\psi(v) \in W$ and $\epsilon \otimes \psi$ is determined modulo an element of $\epsilon \otimes V$. But then, we have a $\mathbb{C}$-linear map $\tilde{\psi} : V \longrightarrow \frac{W}{V}$ and as $\mathcal{S}$ is a $\mathbb{C}[\epsilon]$-subrepresentation, $\tilde{\psi}$ must be a homomorphism of representations of Q. □

Theorem 112 (Schofield). *The following are equivalent*

(1) $\alpha \hookrightarrow \alpha + \beta$.

(2) *Every representation* $W \in \texttt{rep}_{\alpha+\beta}\ Q$ *has a subrepresentation* V *of dimension* α.
(3) $ext(\alpha, \beta) = 0$.

PROOF. (1) $\Rightarrow$ (2) : The image of the proper map $s : \texttt{rep}_{\alpha}^{\alpha+\beta}\ Q \longrightarrow \texttt{rep}_{\alpha+\beta}\ Q$ contains a Zariski open subset. Properness implies that the image of s is a closed subset of $\texttt{rep}_{\alpha+\beta}\ Q$ whence $Im\ s = \texttt{rep}_{\alpha+\beta}\ Q$. The implication (2) $\Rightarrow$ (1) is obvious.

We compute the dimension of the vectorbundle $\texttt{rep}_{\alpha}^{\alpha+\beta}\ Q$ over $\texttt{Grass}_{\alpha}(\alpha+\beta)$. The dimension of $\texttt{Grass}_k(l)$ is $k(l-k)$ and therefore the base has dimension $\sum_{i=1}^k a_i b_i$. Now, fix a point $V \hookrightarrow W$ in $\texttt{Grass}_{\alpha}(\alpha+\beta)$, then the fiber over it determines all possible ways in which this inclusion is a subrepresentation of quivers. That is, for every arrow in Q of the form $\textcircled{j} \xleftarrow{a} \textcircled{i}$ we need to have a commuting diagram

$$\begin{array}{ccc} V_i & \longrightarrow & V_j \\ \big\downarrow & & \big\downarrow \\ W_i & \longrightarrow & W_j \end{array}$$

Here, the vertical maps are fixed. If we modify $V \in \texttt{rep}_{\alpha}\ Q$, this gives us the $a_i a_j$ entries of the upper horizontal map as degrees of freedom, leaving only freedom for the lower horizontal map determined by a linear map $\frac{W_i}{V_i} \longrightarrow W_j$, that is, having $b_i(a_j + b_j)$ degrees of freedom. Hence, the dimension of the vectorspace-fibers is

$$\sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i a_j + b_i(a_j + b_j))$$

giving the total dimension of the reduced variety $\texttt{rep}_{\alpha}^{\alpha+\beta}\ Q$. But then,

$$\begin{aligned} dim\ \texttt{rep}_{\alpha}^{\alpha+\beta}\ Q - dim\ \texttt{rep}_{\alpha+\beta}\ Q &= \sum_{i=1}^k a_i b_i + \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i a_j + b_i(a_j + b_j)) \\ &\quad - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i + b_i)(a_j + b_j) \\ &= \sum_{i=1}^k a_i b_i - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i b_j = \chi_Q(\alpha, \beta) \end{aligned}$$

(2) $\Rightarrow$ (3) : The proper map $\texttt{rep}_{\alpha}^{\alpha+\beta} \xrightarrow{s} \texttt{rep}_{\alpha+\beta}\ Q$ is onto and as both varieties are reduced, the general fiber is a reduced variety of dimension $\chi_Q(\alpha, \beta)$, whence the general fiber contains points such that the tangentspace has dimension $\chi_Q(\alpha, \beta)$. By the previous theorem, the dimension of this tangentspace is $dim\ Hom_{\langle Q \rangle}(V, \frac{W}{V})$. But then, because

$$\chi_Q(\alpha, \beta) = dim_{\mathbb{C}}\ Hom_{\langle Q \rangle}(V, \frac{W}{V}) - dim_{\mathbb{C}}\ Ext^1_{\langle Q \rangle}(V, \frac{W}{V})$$

it follows that $Ext^1(V, \frac{W}{V}) = 0$ for some representation V of dimension vector α and $\frac{W}{V}$ of dimension vector β. But then, $ext(\alpha, \beta) = 0$.

(3) $\Rightarrow$ (2) : Assume $ext(\alpha, \beta) = 0$ then, for a general point $W \in \mathtt{rep}_{\alpha+\beta}\ Q$ in the image of s and for a general point in its fiber $(W, V) \in \mathtt{rep}_{\alpha}^{\alpha+\beta}\ Q$ we have $dim_{\mathbb{C}}\ Ext^1_{\langle Q \rangle}(V, \frac{W}{V}) = 0$ whence $dim_{\mathbb{C}}\ Hom_{\langle Q \rangle}(V, \frac{W}{V}) = \chi_Q(\alpha, \beta)$. But then, the general fiber of s has dimension $\chi_Q(\alpha, \beta)$ and as this is the difference in dimension between the two irreducible varieties, the map is generically onto. Finally, properness of s then implies that it is onto. $\square$

Definition 107. $\mathtt{Hom}_Q(\alpha, \beta)$ is the subvariety of the trivial vectorbundle

$$\begin{array}{ccc} \mathtt{Hom}_Q(\alpha,\beta) & \hookrightarrow & Hom(\alpha,\beta) \times \mathtt{rep}_\alpha\ Q \times \mathtt{rep}_\beta\ Q \\ & {\scriptstyle \Phi} \searrow & \downarrow {\scriptstyle pr} \\ & & \mathtt{rep}_\alpha\ Q \times \mathtt{rep}_\beta\ Q \end{array}$$

of triples (ϕ, V, W) such that $V \xrightarrow{\phi} W$ is a morphism of representations of Q. The fiber $\Phi^{-1}(V, W) = Hom_{\langle Q \rangle}(V, W)$ and as the fiber dimension is upper semi-continuous, there is an open subset $\mathtt{Hom}_{min}(\alpha, \beta)$ of $\mathtt{rep}_\alpha\ Q \times \mathtt{rep}_\beta\ Q$ consisting of points (V, W) where $dim_{\mathbb{C}}\ Hom_{\langle Q \rangle}(V, W)$ is minimal. For given dimension vector $\delta = (d_1, \ldots, d_k)$ consider the subset

$$\mathtt{Hom}_Q(\alpha, \beta, \delta) = \{(\phi, V, W) \in \mathtt{Hom}_Q(\alpha, \beta) \ \mid\ rk\ \phi = \delta\} \hookrightarrow \mathtt{Hom}_Q(\alpha, \beta)$$

which is a constructible subset of $\mathtt{Hom}_Q(\alpha, \beta)$. There is a unique dimension vector γ such that $\mathtt{Hom}_Q(\alpha, \beta, \gamma) \cap \Phi^{-1}(\mathtt{Hom}_{min}(\alpha, \beta))$ is constructible and dense in $\Phi^{-1}(\mathtt{Hom}_{min}(\alpha, \beta))$. This *gamma* is called the *generic rank* of morphisms from $\mathtt{rep}_\alpha\ Q$ to $\mathtt{rep}_\beta\ Q$ and will be denoted $\gamma = rk\ hom(\alpha, \beta)$.

$$\Phi(\mathtt{Hom}_Q(\alpha, \beta, \gamma) \cap \Phi^{-1}(\mathtt{Hom}_{min}(\alpha, \beta)))$$

is constructible and dense in $\mathtt{Hom}_{min}(V, W)$. Therefore it contains an open subset $Hom_m(V, W)$ consisting of couples (V, W) such that $dim_{\mathbb{C}} Hom_{\langle Q \rangle}(V, W)$ is minimal and such that $\{\phi \in Hom_{\langle Q \rangle}(V, W) \mid rk\phi = \gamma\}$ is a non-empty Zariski open subset of $Hom_{\langle Q \rangle}(V, W)$.

Theorem 113 (Schofield). *Let $\gamma = rk\ hom(\alpha, \beta)$, then*

(1) $\alpha - \gamma \hookrightarrow \alpha \twoheadrightarrow \gamma \hookrightarrow \beta \twoheadrightarrow \beta - \gamma$
(2) $ext(\alpha, \beta) = -\chi_Q(\alpha - \gamma, \beta - \gamma) = ext(\alpha - \gamma, \beta - \gamma)$

Proof. The first statement is obvious from the definitions, for if $\gamma = rk\ hom(\alpha, \beta)$, then a general representation of dimension α will have a quotient-representation of dimension γ (and hence a subrepresentation of dimension $\alpha - \gamma$) and a general representation of dimension β will have a subrepresentation of dimension γ (and hence a quotient-representation of dimension $\beta - \gamma$.

The strategy of the proof of the second statement is to compute the dimension of the subvariety $\mathtt{H}^{factor}$ of $Hom(\alpha,\beta) \times \mathtt{rep}_\alpha \times \mathtt{rep}_\beta \times \mathtt{rep}_\gamma$ defined by

$$\{(\phi, V, W, X) \ | \ \begin{array}{ccc} V & \xrightarrow{\phi} & W \\ & \searrow \qquad \nearrow & \\ & X = Im\ \phi & \end{array} \ \text{factors as representations } \}$$

in two different ways. Consider the intersection of the open set $Hom_m(\alpha,\beta)$ of the previous definition with the open set of couples (V,W) such that $dim\ Ext(V,W) = ext(\alpha,\beta)$ and let (V,W) be a point in this intersection. In theorem 111 we proved

$$dim\ \mathtt{Grass}_\gamma(W) = \chi_Q(\gamma, \beta - \gamma)$$

Let $\mathtt{H}$ be the subbundle of the trivial vectorbundle over $\mathtt{Grass}_\gamma(W)$

$$\begin{array}{ccc} \mathtt{H} & \hookrightarrow & Hom(\alpha, W) \times \mathtt{Grass}_\gamma(W) \\ & \searrow & \downarrow \\ & & \mathtt{Grass}_\gamma(W) \end{array}$$

consisting of triples (ϕ, W, U) with $\phi : \oplus_i \mathbb{C}^{\oplus a_i} \longrightarrow W$ a linear map such that $Im(\phi)$ is contained in the subrepresentation $U \hookrightarrow W$ of dimension γ. That is, the fiber over (W,U) is $Hom(\alpha, U)$ and therefore has dimension $\sum_{i=1}^k a_i c_i$. With $\mathtt{H}^{full}$ we consider the open subvariety of H of triples (ϕ, W, U) such that $Im\ \phi = U$. We have

$$dim\ \mathtt{H}^{full} = \sum_{i=1}^k a_i c_i + \chi_Q(\gamma, \beta - \gamma)$$

But then, $\mathtt{H}^{factor}$ is the subbundle of the trivial vectorbundle over $\mathtt{H}^{full}$

$$\begin{array}{ccc} \mathtt{H}^{factor} & \hookrightarrow & \mathtt{rep}_\alpha\ Q \times \mathtt{H}^{full} \\ & \searrow^{\pi} & \downarrow \\ & & \mathtt{H}^{full} \end{array}$$

consisting of quadruples (V, ϕ, W, X) such that $V \xrightarrow{\phi} W$ is a morphism of representations, with image the subrepresentation X of dimension γ. The fiber of π over a triple (ϕ, W, X) is determined by the property that for each arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ the following diagram must be commutative, where we decompose the vertex spaces

$V_i = X_i \oplus K_i$ for $K = Ker\ \phi$

$$\begin{array}{ccc} X_i \oplus K_i & \xrightarrow{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} & X_j \oplus K_j \\ {\scriptstyle \begin{bmatrix} \mathbb{1}_{c_i} & 0 \end{bmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{bmatrix} \mathbb{1}_{c_j} & 0 \end{bmatrix}} \\ X_i & \xrightarrow[A]{} & X_j \end{array}$$

where A is fixed, giving the condition $B = 0$ and hence the fiber has dimension equal to

$$\sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i - c_i)(a_j - c_j) + \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} c_i(a_j - c_j) = \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i(a_j - c_j)$$

This gives our first formula for the dimension of $\mathtt{H}^{factor}$

$$dim\ \mathtt{H}^{factor} = \sum_{i=1}^{k} a_i c_i + \chi_Q(\gamma, \beta - \gamma) + \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i(a_j - c_j)$$

On the other hand, we can consider the natural map $\mathtt{H}^{factor} \xrightarrow{\Phi} \mathtt{rep}_\alpha\ Q$ defined by sending a quadruple (V, ϕ, W, X) to V. the fiber in V is given by all quadruples (V, ϕ, W, X) such that $V \xrightarrow{\phi} W$ is a morphism of representations with $Im\ \phi = X$ a representation of dimension vector γ, or equivalently

$$\Phi^{-1}(V) = \{V \xrightarrow{\phi} W \ \mid \ rk\ \phi = \gamma\}$$

Now, recall our restriction on the couple (V, W) giving at the beginning of the proof. There is an open subset $\mathtt{max}$ of $\mathtt{rep}_\alpha\ Q$ of such V and by construction $\mathtt{max} \subset Im\ \Phi$, $\Phi^{-1}(\mathtt{max})$ is open and dense in $\mathtt{H}^{factor}$ and the fiber $\Phi^{-1}(V)$ is open and dense in $Hom_{\langle Q \rangle}(V, W)$. This provides us with the second formula for the dimension of $\mathtt{H}^{factor}$

$$dim\ \mathtt{H}^{factor} = dim\ \mathtt{rep}_\alpha\ Q + hom(\alpha, W) = \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i a_j + hom(\alpha, \beta)$$

Equating both formulas we obtain the equality

$$\chi_Q(\gamma, \beta - \gamma) + \sum_{i=1}^{k} a_i c_i - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i c_j = hom(\alpha, \beta)$$

which is equivalent to

$$\chi_Q(\gamma, \beta - \gamma) + \chi_Q(\alpha, \gamma) - \chi_Q(\alpha, \beta) = ext(\alpha, \beta)$$

Now, for our (V, W) we have that $Ext(V, W) = ext(\alpha, \beta)$ and we have exact sequences of representations

$$0 \longrightarrow S \longrightarrow V \longrightarrow X \longrightarrow 0 \qquad 0 \longrightarrow X \longrightarrow W \longrightarrow T \longrightarrow 0$$

and by theorem 110 this gives a surjection $Ext(V,W) \twoheadrightarrow Ext(S,T)$. On the other hand we always have from the homological interpretation of the Euler form the first inequality

$$\begin{aligned} dim_{\mathbb{C}}\ Ext(S,T) &\geq -\chi_Q(\alpha-\gamma,\beta-\gamma) = \chi_Q(\gamma,\beta-\gamma) - \chi_Q(\alpha,\beta) + \chi_Q(\alpha,\gamma) \\ &= ext(\alpha,\beta) \end{aligned}$$

As the last term is $dim_{\mathbb{C}}\ Ext(V,W)$, this implies that the above surjection must be an isomorphism and that

$$dim_{\mathbb{C}}\ Ext(S,T) = -\chi_Q(\alpha-\gamma,\beta-\gamma) \quad \text{whence} \quad dim_{\mathbb{C}}\ Hom(S,T) = 0$$

But this implies that $hom(\alpha-\gamma,\beta-\gamma) = 0$ and therefore $ext(\alpha-\gamma,\beta-\gamma) = -\chi_Q(\alpha-\gamma,\beta-\gamma)$. Finally,

$$ext(\alpha-\gamma,\beta-\gamma) = dim\ Ext(S,T) = dim\ Ext(V,W) = ext(\alpha,\beta)$$

finishing the proof. □

THEOREM 114 (Schofield). *For dimension vectors α and β we have*

$$\begin{aligned} ext(\alpha,\beta) &= \max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} -\chi_Q(\alpha',\beta') \\ &= \max_{\beta \twoheadrightarrow \beta"} -\chi_Q(\alpha,\beta") \\ &= \max_{\alpha" \hookrightarrow \alpha} -\chi_Q(\alpha",\beta) \end{aligned}$$

PROOF. Let V and W be representation of dimension vector α and β such that $dim\ Ext(V,W) = ext(\alpha,\beta)$. Let $S \hookrightarrow V$ be a subrepresentation of dimension α' and $W \twoheadrightarrow T$ a quotient representation of dimension vector β'. Then, we have

$$ext(\alpha,\beta) = dim_{\mathbb{C}}\ Ext(V,W) \geq dim_{\mathbb{C}}\ Ext(S,T) \geq -\chi_Q(\alpha',\beta')$$

where the first inequality follows from theorem 110 and the second follows from the interpretation of the Euler form. Therefore, $ext(\alpha,\beta)$ is greater or equal than all the terms in the statement of the theorem. The foregoing theorem asserts the first equality, as for $rk\ hom(\alpha,\beta) = \gamma$ we do have that $ext(\alpha,\beta) = -\chi_Q(\alpha-\gamma,\beta-\gamma)$.

In the proof of the previous theorem, we have found for sufficiently general V and W an exact sequence of representations

$$0 \longrightarrow S \longrightarrow V \longrightarrow W \longrightarrow T \longrightarrow 0$$

where S is of dimension $\alpha-\gamma$ and T of dimension $\beta-\gamma$. Moreover, we have a commuting diagram of surjections

$$\begin{array}{ccc} Ext(V,W) & \twoheadrightarrow & Ext(V,T) \\ \downarrow & \searrow & \downarrow \\ Ext(S,W) & \twoheadrightarrow & Ext(S,T) \end{array}$$

and the dashed map is an isomorphism, hence so are all the epimorphisms. Therefore, we have

$$\begin{cases} ext(\alpha,\beta-\gamma) & \leq dim\ Ext(V,T) = dim\ Ext(V,W) = ext(\alpha,\beta) \\ ext(\alpha-\gamma,\beta) & \leq dim\ Ext(S,W) = dim\ Ext(V,W) = ext(\alpha,\beta) \end{cases}$$

Further, let T' be a sufficiently general representation of dimension $\beta - \gamma$, then it follows from $Ext(V,T') \twoheadrightarrow Ext(S,T)$ that

$$ext(\alpha-\gamma,\beta-\gamma) \leq dim\ Ext(S,T') \leq dim\ Ext(V,T') = ext(\alpha,\beta-\gamma)$$

but the left term is equal to $ext(\alpha,\beta)$ by the above theorem. But then, we have $ext(\alpha,\beta) = ext(\alpha,\beta-\gamma)$. Now, we may assume by induction that the theorem holds for $\beta-\gamma$. That is, there exists $\beta-\gamma \twoheadrightarrow \beta"$ such that $ext(\alpha,\beta-\gamma) = -\chi_Q(\alpha,\beta")$. Whence, $\beta \twoheadrightarrow \beta"$ and $ext(\alpha,\beta) = -\chi_Q(\alpha,\beta")$ and the middle equality of the theorem holds. By a dual argument so does the last. □

EXAMPLE 148. This gives us the following inductive `algorithm` to find all the dimension vectors of general subrepresentations. Take a dimension vector α and assume by induction we know for all $\beta < \alpha$ the set of general subrepresentations $\beta' \hookrightarrow \beta$. Then, $\beta \hookrightarrow \alpha$ if and only if

$$0 = ext(\beta,\alpha-\beta) = \max_{\beta' \hookrightarrow \beta} -\chi_Q(\beta',\alpha-\beta)$$

where the first equality comes from theorem 112 and the last from the above theorem.

THEOREM 115. (1) *$\alpha \in \mathtt{schur}\langle Q\rangle$ if and only if $\mathtt{rep}_\alpha\ Q$ contains a Zariski open subset of indecomposable representations.*

(2) *If $\alpha \in F_Q$ and $\mathtt{supp}\alpha$ is not a tame quiver, then $\alpha \in \mathtt{schur}\langle Q\rangle$.*

(3) *If $\alpha \in \mathtt{schur}\langle Q\rangle$ and $\chi_Q(\alpha,\alpha) < 0$, then $n.\alpha \in \mathtt{schur}\langle Q\rangle$ for all integers n.*

PROOF. (1) : If $V \in \mathtt{rep}_\alpha\ Q$ is a Schur representation, $V \in \mathtt{rep}_\alpha(max)$ and therefore all representations in the dense open subset $\mathtt{rep}_\alpha(max)$ have endomorphism ring $\mathbb{C}$ and are therefore indecomposable.

Conversely, let $\mathtt{Ind} \subset \mathtt{rep}_\alpha\ Q$ be an open subset of indecomposable representations. Assume for $V \in \mathtt{Ind}$ we have $Stab_{GL(\alpha)}(V) \neq \mathbb{C}^*$ and consider $\phi_0 \in Stab_{GL(\alpha)}(V) - \mathbb{C}^*$. For any $g \in GL(\alpha)$ we define the *set of fixed elements*

$$\mathtt{rep}_\alpha(g) = \{W \in \mathtt{rep}_\alpha\ Q \ \mid\ g.W = W\}$$

and consider the subset of $GL(\alpha)$

$$S = \{g \in GL(\alpha)\ \mid\ dim\ \mathtt{rep}_\alpha(g) = dim\ \mathtt{rep}_\alpha(\phi_0)$$

which has no intersection with $\mathbb{C}^*(\mathbb{1}_{d_1},\ldots,\mathbb{1}_{d_k})$ as $\phi_0 \notin \mathbb{C}^*$. Consider the subbundle of the trivial vectorbundle over S

$$\mathcal{B} = \{(s,W) \in S \times \mathtt{rep}_\alpha\ Q\ \mid\ s.W = W\} \hookrightarrow S \times \mathtt{rep}_\alpha\ Q \xrightarrow{p} S$$

As all fibers have equal dimension, the restriction of p to $\mathcal{B}$ is a *flat morphism* whence *open* . In particular, the image of the open subset $\mathcal{B} \cap S \times \mathtt{Ind}$

$$S' = \{g \in S\ \mid\ \exists W \in \mathtt{Ind}\ :\ g.W = W\}$$

is an open subset of S. Now, S contains a dense set of semisimple elements, see for example [**37**, (2.5)], whence so does $S' = \cup_{W\in\mathtt{Ind}} End_{\langle Q\rangle}(W) \cap S$. But then one of the $W \in \mathtt{Ind}$ must have a torus of rank greater than one in its stabilizer subgroup contradicting indecomposability.

(2) : Let $\alpha = \beta_1 + \ldots + \beta_s$ be the canonical decomposition of α (some β_i may occur with higher multiplicity) and assume that $s \geq 2$. By definition, the image of

$$GL(\alpha) \times (\mathtt{rep}_{\beta_1}\ Q \times \ldots \times \mathtt{rep}_{\beta_s}\ Q) \xrightarrow{\phi} \mathtt{rep}_\alpha\ Q$$

is dense and ϕ is constant on orbits of the *free action* of $GL(\alpha)$ on the left hand side given by $h.(g,V) = (gh^{-1}, h.V)$. But then,

$$dim\ GL(\alpha) + \sum_i dim\ \mathtt{rep}_{\beta_i}\ Q - \sum_i dim\ GL(\beta_i) \geq dim\ \mathtt{rep}_\alpha\ Q$$

whence $q_Q(\alpha) \geq \sum_i q_Q(\beta_i)$ and theorem 107 finishes the proof.

(3) : There are infinitely many non-isomorphic Schur representations of dimension vector α. Pick n distinct of them $\{W_1, \ldots, W_n\}$ and from $\chi_Q(\alpha,\alpha) < 0$ we deduce

$$Hom_{\langle Q \rangle}(W_i, W_j) = \delta_{ij}\mathbb{C} \qquad \text{and} \qquad Ext^1_{\langle Q \rangle}(W_i, W_j) \neq 0$$

By the proof of theorem 110 we can construct a representation V_n having a filtration

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n \quad \text{with} \quad \frac{V_j}{V_{j-1}} \simeq W_j$$

and such that the short exact sequences $0 \longrightarrow V_{j-1} \longrightarrow V_j \longrightarrow W_j \longrightarrow 0$ do not split. By induction on n we may assume that $End_{\langle Q \rangle}(V_{n-1}) = \mathbb{C}$ and we have that $Hom_{\langle Q \rangle}(V_{n-1}, W_n) = 0$. But then, the restriction of any endomorphism ϕ of V_n to V_{n-1} must be an endomorphism of V_{n-1} and therefore a scalar $\lambda \mathbb{1}$. Hence, $\phi - \lambda\mathbb{1} \in End_{\langle Q \rangle}(V_n)$ is trivial on V_{n-1}. As $Hom_{\langle Q \rangle}(W_n, V_{n-1}) = 0$, $End_{\langle Q \rangle}(W_n) = \mathbb{C}$ and non-splitness of the sequence $0 \longrightarrow V_{n-1} \longrightarrow V_n \longrightarrow W_n \longrightarrow 0$ we must have $\phi - \lambda\mathbb{1} = 0$ whence $End_{\langle Q \rangle}(V_n) = \mathbb{C}$, that is, $n\alpha$ is a Schur root. □

EXAMPLE 149. Schur roots and Azumaya algebras. If $\alpha = (a_1, \ldots, a_k)$ is a Schur root, then there is a $GL(\alpha)$-stable affine open subvariety U_α of $\mathtt{rep}_\alpha\ Q$ such that generic orbits are closed in U. Indeed, let $T_k = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ the k-dimensional torus in $GL(\alpha)$. Consider the semisimple subgroup $SL(\alpha) = SL_{a_1} \times \ldots \times SL_{a_k}$ and consider the corresponding quotient map

$$\mathtt{rep}_\alpha\ Q \xrightarrow{\pi_s} \mathtt{rep}_\alpha\ Q/SL(\alpha)$$

As $GL(\alpha) = T_k SL(\alpha)$, T_k acts on $\mathtt{rep}_\alpha\ Q/SL(\alpha)$ and the generic stabilizer subgroup is trivial by the Schur assumption. Hence, there is a T_k-invariant open subset U_1 of $\mathtt{rep}_\alpha\ Q/SL(\alpha)$ such that T_k-orbits are closed. But then, according to [**28**, §2, Thm.5] there is a T_k-invariant *affine* open U_2 in U_1. Because the quotient map ψ_s is an affine map, $U = \psi_s^{-1}(U_2)$ is an affine $GL(\alpha)$-stable open subvariety of $\mathtt{rep}_\alpha\ Q$. Let x be a generic point in U, then its orbit

$$\mathcal{O}(x) = GL(\alpha).x = T_k SL(\alpha).x = T_k(\psi_s^{-1}(\psi_s(x))) = \psi_s^{-1}(T_k.\psi_s(x))$$

is the inverse image under the quotient map of a closed set, hence is itself closed.

Because U_α is affine, we can define the witness algebra $\Uparrow^\alpha U_\alpha$ to be the ring of $GL(\alpha)$-equivariant maps from U_α to $M_n(\mathbb{C})$ with $n = |\alpha|$. Over the Azumaya locus $\mathtt{azum}$ of the order $\Uparrow^\alpha U_\alpha$ the quotient map

$$\mathtt{rep}_\alpha \Uparrow^\alpha U_\alpha \longrightarrow\!\!\!\!\!\rightarrow \mathtt{iss}_\alpha \Uparrow^\alpha U_\alpha$$

is a principal $PGL(\alpha)$-fibration in the étale topology and so determines an element of $H^1_{et}(\mathtt{azu}, PGL(\alpha))$. This pointed set classifies Azumaya algebras over $\mathtt{azu}$ with a distinguished embedding of $C_k = \mathbb{C} \times \ldots \times \mathbb{C}$ which are split by am étale cover on which this embedding is conjugated to the standard embedding $C_k \subset M_n(\mathbb{C})$.

Definition 108. We say that a dimension vector α is *left orthogonal* to β and denote $\alpha \perp \beta$ if $hom(\alpha,\beta) = 0$ and $ext(\alpha,\beta) = 0$.

An ordered sequence $C = (\beta_1, \ldots, \beta_s)$ of dimension vectors is said to be a *compartment* for Q if and only if

(1) for all i, $\beta_i \in \mathtt{schur}\langle Q \rangle$,
(2) for al $i < j$, $\beta_i \perp \beta_j$,
(3) for all $i < j$ we have $\chi_Q(\beta_j, \beta_i) \geq 0$.

Theorem 116 (Derksen-Weyman). *Suppose that $C = (\beta_1, \ldots, \beta_s)$ is a compartment for Q and that there are non-negative integers $e_1, \ldots, e_s$ such that $\alpha = e_1\beta_1 + \ldots + e_s\beta_s$. Assume that $e_i = 1$ whenever $\chi_Q(\beta_i, \beta_i) < 0$. Then,*

$$\tau_{can} = (e_1, \beta_1; \ldots; e_s, \beta_s)$$

is the canonical decomposition of the dimension vector α.

Proof. Let V be a generic representation of dimension vector α with decomposition into indecomposables

$$V = W_1^{\oplus e_1} \oplus \ldots \oplus W_s^{\oplus e_s} \qquad \text{with} \quad dim(W_i) = \beta_i$$

we will show that (after possibly renumbering the factors $(\beta_1, \ldots, \beta_s)$ is a compartment for Q. To start, it follows from theorem 110 that for all $i \neq j$ we have $Ext^1_{\langle Q \rangle}(W_i, W_j) = 0$. From theorem 110 we deduce a partial ordering $i \rightarrow j$ on the indices whenever $Hom_{\langle Q \rangle}(W_i, W_j) \neq 0$. Indeed, any non-zero morphism $W_i \longrightarrow W_j$ is either a mono- or an epimorphism, assume $W_i \twoheadrightarrow W_j$ then there can be no monomorphism $W_j \hookrightarrow W_k$ as the composition $W_i \longrightarrow W_k$ would be neither mono nor epi. That is, all non-zero morphisms from W_j to factors must be (proper) epi and we cannot obtain cycles in this way by counting dimensions. If $W_i \hookrightarrow W_j$, a similar argument proves the claim. From now on we assume that the chosen index-ordering of the factors is (reverse) compatible with the partial ordering $i \rightarrow j$, that is $Hom(W_i, W_j) = 0$ whenever $i < j$, that is, β_i is left orthogonal to β_j whenever $i < j$. As $Ext^1_{\langle Q \rangle}(W_j, W_i) = 0$, it follows that $\chi_Q(\beta_j, \beta_i) \geq 0$. As generic representations are open it follows that all $\mathtt{rep}_{\beta_i}\, Q$ have an open subset of indecomposables, proving that the β_i are Schur roots. Finally, it follows from theorem 115 that a Schur root β_i with $\chi_Q(\beta_i, \beta_i)$ can occur only with multiplicity one in any canonical decomposition.

Conversely, assume that $(\beta_1, \ldots, \beta_s)$ is a compartment for Q, $\alpha = \sum_i e_i \beta_i$ satisfying the requirements on multiplicities. Choose Schur representations $W_i \in \mathtt{rep}_{\beta_i}\, Q$, then we have to prove that

$$V = W_1^{\oplus e_1} \oplus \ldots \oplus W_s^{\oplus e_s}$$

is a generic representation of dimension vector α. In view of the properties of the compartment we already know that $Ext^1_{\langle Q \rangle}(W_i, W_j) = 0$ for all $i < j$ and we need to show that $Ext^1_{\langle Q \rangle}(W_j, W_i) = 0$. Indeed, if this condition is satisfied we have

$$\begin{aligned} dim\ \mathtt{rep}_\alpha\, Q - dim\ \mathcal{O}(V) &= dim_{\mathbb{C}} Ext^1(V, V) \\ &= \sum_i e_i^2 dim_{\mathbb{C}} Ext^1(W_i, W_i) = \sum_i e_i^2 (1 - q_Q(\beta_i) \end{aligned}$$

We know that the Schur representations of dimension vector β_i depend on $1 - q_Q(\beta_i)$ parameters by Kac s theorem 109 and $e_i = 1$ unless $q_Q(\beta_i) = 1$. Therefore, the union of all orbits of representations with the same Schur-decomposition type as V contain a dense open set of $\mathtt{rep}_\alpha\, Q$ and so this must be the canonical decomposition.

If this extension space is nonzero, $Hom_{\langle Q \rangle}(W_j, W_i) \neq 0$ as $\chi_Q(\beta_j, \beta_i) \geq 0$. But then by theorem 110 any non-zero homomorphism from W_j to W_i must be either a mono or an epi. Assume it is a mono, so $\beta_j < \beta_i$, so in particular a general representation of dimension β_i contains a subrepresentation of dimension β_j and hence by theorem 112 we have $ext(\beta_j, \beta_i - \beta_j) = 0$. Suppose that β_j is a real Schur root, then $Ext^1_{\langle Q \rangle}(W_j, W_j) = 0$ and therefore also $ext(\beta_j, \beta_i) = 0$ as $Ext^1_{\langle Q \rangle}(W_j, W_j \oplus (W_j/W_i)) = 0$. If β is not a real root, then for a general representation $S \in \mathtt{rep}_{\beta_j}\, Q$ take a representation $R \in \mathtt{rep}_{\beta_i}\, Q$ in the open set where $Ext^1_{\langle Q \rangle}(S, R) = 0$, then there is a monomorphism $S \hookrightarrow R$. Because $Ext^1_{\langle Q \rangle}(S, S) \neq 0$ we deduce from the proof of theorem 110 that $Ext^1_{\langle Q \rangle}(R, S) \neq 0$ contradicting the fact that $ext(\beta_i, \beta_j) = 0$. If the nonzero morphism $W_j \longrightarrow W_i$ is epi one has a similar argument. □

Example 150. Algorithm to compute the canonical decomposition. Let Q be a quiver *without oriented cycles* then we can order the vertices $\{v_1, \ldots, v_k\}$ such that there are no oriented paths from v_i to v_j whenever $i < j$ (start with a sink of Q, drop it and continue recursively).

`input :` quiver Q, ordered set of vertices as above, dimension vector $\alpha = (a_1, \ldots, a_k)$ and type $\tau = (a_1, \vec{v_1}; \ldots; a_k, \vec{v_k})$ where $\vec{v_i} = (\delta_{ij})_j = dim\ v_i$ is the canonical basis. By the assumption on the ordering of vertices we have that τ is a *good type* for α. We say that a type $(f_1, \gamma_1; \ldots; f_s, \gamma_s)$ is a good type for α if $\alpha = \sum_i f_i \gamma_i$ and the following properties are satisfies

(1) $f_i \geq 0$ for all i,
(2) γ_i is a Schur root,
(3) for each $i < j$, γ_i is left orthogonal to γ_j,
(4) $f_i = 1$ whenever $\chi_Q(\gamma_i, \gamma_i) < 0$.

A type is said to be *excellent* provided that, in addition to the above, we also have that for all $i < j$, $\chi_Q(\alpha_j, \alpha_i) \geq 0$. In view of theorem 116 the purpose of the algorithm is to transform the good type τ into the excellent type τ_{can}. We will describe the main loop of the algorithm on a good type $(f_1, \gamma_1; \ldots; f_s, \gamma_s)$.

`step 1 :` Omit all couples (f_i, γ_i) with $f_i = 0$ and verify whether the remaining type is excellent. If it is, `stop and output this type`. If not, proceed.
`step 2 :` Reorder the type as follows, choose i and j such that $j - i$ is minimal and $\chi_Q(\beta_j, \beta_i) < 0$. Partition the intermediate entries $\{i+1, \ldots, j-1\}$ into the sets

- $\{k_1, \ldots, k_a\}$ such that $\chi_Q(\gamma_j, \gamma_{k_m}) = 0$,
- $\{l_1, \ldots, l_b\}$ such that $\chi_Q(\gamma_j, \gamma_{l_m}) > 0$.

Reorder the couples in the type in the sequence

$$(1, \ldots, i-1, k_1, \ldots, k_a, i, j, l_1, \ldots, l_b, j+1, \ldots, s)$$

`define` $\mu = \gamma_i$, $\nu = \gamma_j$, $p = f_i$, $q = f_j$, $\zeta = p\mu + q\nu$ and $t = -\chi_Q(\nu, \mu)$, then proceed.
`step 3 :` Change the part $(p, \mu; q, \nu)$ of the type according to the following scheme

- If μ and ν are real Schur roots, consider the subcases

(1) $\chi_Q(\zeta,\zeta) > 0$, `replace` (p,μ,q,ν) by $(p',\mu';q';\nu')$ where ν' and ν' are non-negative combinations of ν and μ such that μ' is left orthogonal to ν', $\chi_Q(\nu',\mu') = t \geq 0$ and $\zeta = p'\mu' + q'\nu'$ for non-negative integers p', q'.
(2) $\chi_Q(\zeta,\zeta) = 0$, `replace` $(p,\mu;q,\nu)$ by (k,ζ') with $\zeta = k\zeta'$, k positive integer, and ζ' an indivisible root.
(3) $\chi_Q(\zeta,\zeta) < 0$, `replace` $(p,\mu;q,\nu)$ by $(1,\zeta)$.

- If μ is a real root and ν is imaginary, consider the subcases
  (1) If $p + q\chi_Q(\nu,\mu) \geq 0$, `replace` $(p,\mu;q,\nu)$ by $(q,\nu - \chi_Q(\nu,\mu)\mu; p + q\chi_Q(\nu,\mu),\mu)$.
  (2) If $p + q\chi_Q(\nu,\mu) < 0$, `replace` $(p,\mu;q,\nu)$ by $(1,\zeta)$.
- If μ is an imaginary root and ν is real, consider the subcases
  (1) If $q + p\chi_Q(\nu,\mu) \geq 0$, `replace` $(p,\mu;q,\nu)$ by $(q + p\chi_Q(\nu,\mu),\nu;p,\mu - \chi_Q(\nu,\mu)\nu)$.
  (2) If $q + p\chi_Q(\nu,\mu) < 0$, `replace` $(p,\mu;q,\nu)$ by $(1,\zeta)$.
- If μ and ν are imaginary roots, `replace` $(p,\mu;q,\nu)$ by $(1,\zeta)$.

then go to `step 1`.

One can show that in every loop of the algorithm the number $\sum_i f_i$ decreases, so the algorithm must stop, giving the canonical decomposition of α. A consequence of this algorithm is that $r(\alpha) + 2i(\alpha) \leq k$ where $r(\alpha)$ is the number of real Schur roots occurring in the canonical decomposition of α, $i(\alpha)$ the number of imaginary Schur roots and k the number of vertices of Q. For more details we refer to [**13**].

EXAMPLE 151. Fortunately, one can reduce a general quiver setting (Q,α) to one of a quiver without oriented cycles using the *bipartite double* Q^b of Q. We double the vertex-set of Q in a left and right set of vertices, that is

$$Q^b_v = \{v^l_1,\ldots,v^l_k,v^r_1,\ldots,v^r_k\}$$

To every arrow $a \in Q_a$ from v_i to v_j we assign an arrow $\tilde{a} \in Q^b_a$ from v^l_i to v^r_j. In addition, we have for each $1 \leq i \leq k$ one extra arrow $\tilde{i}$ in Q^b_a from v^l_i to v^r_i. If $\alpha = (a_1,\ldots,a_k)$ is a dimension vector for Q, the associated dimension vector $\tilde{\alpha}$ for Q^b has components

$$\tilde{\alpha} = (a_1,\ldots,a_k,a_1,\ldots,a_k).$$

If the canonical decomposition of α for Q is $\tau_{can} = (e_1,\beta_1;\ldots;e_s,\beta_s)$, then the canonical decomposition of $\tilde{\alpha}$ for Q^b is $(e_1,\tilde{\beta}_1;\ldots;e_s,\tilde{\beta}_s)$ as for a general representation of Q^b of dimension vector $\tilde{\alpha}$ the morphisms corresponding to $\tilde{i}$ for $1 \leq i \leq k$ are all invertible matrices and can be used to identify the left and right vertex sets, that is, there is an equivalence of categories between representations of Q^b where all the maps $\tilde{i}$ are invertible and representations of the quiver Q. Using this reduction, the foregoing example can be used to compute the canonical decomposition of an arbitrary quiver-setting.

For some pretty pictures of the fractal nature of the compartment division on $\mathtt{schof}\langle Q\rangle$ we refer to [**13**].

7.2. Moduli spaces.

In this section we will study $\mathtt{moss}_\alpha(Q,\theta)$, the *moduli space* of θ-semistable α-dimensional quiver representations. Here, $\theta = (t_1,\ldots,t_k) \in \mathbb{R}^k$ and $M \in \mathtt{rep}_\alpha Q$

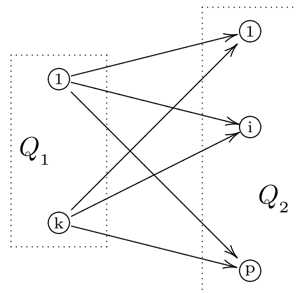


FIGURE 4. Free product $Q_1 * Q_2$ of quivers.

is θ-semistable if $\theta.\alpha = 0$ and for every proper subrepresentation $M' \subset M$ with $dim M' = \beta$ we have $\theta.\beta \geq 0$. From the general results of Rudakov it follows that points in $\mathtt{moss}_\alpha(Q,\theta)$ parametrize direct sum of θ-stable representations. Further we will prove that θ-stable representations become simple in a universal localization of $\langle Q \rangle$. As a consequence, $\mathtt{moss}_\alpha(Q,\theta)$ can be covered by open affine subsets of the form $\mathtt{iss}_\alpha \langle Q \rangle_\sigma$ and therefore the theory of local quivers, developed in the foregoing chapters, can be applies to study the local structure (in particular, the singular locus) of these moduli spaces.

We start with some examples illustrating that moduli spaces of quiver representations appear naturally (in disguise) in as different fields as representations of knot groups, linear dynamical systems and Brauer-Severi varieties. For the latter two examples we present the classical approach to the local study of these moduli spaces. Rephrased in quiver terms, it turns out that determinantal semi-invariants cover these moduli spaces.

EXAMPLE 152. (Free products of quivers) Let Q_1 and Q_2 are two finite quivers, then $\langle Q_1 \rangle * \langle Q_2 \rangle$ is an $\mathtt{alg}$-smooth algebra. We like to have a concrete description in quiver-terms of the finite dimensional representations of this algebra.

Let Q_1 be a quiver on k vertices $\{v_1,\ldots,v_k\}$ and Q_2 a quiver on p vertices $\{w_1,\ldots,w_p\}$ and consider the extended quiver $Q_1 * Q_2$ of figure 4. That is, we add one extra arrow from every vertex of Q_1 to every vertex of Q_2.

Consider the $p \times k$ matrix

$$M_\sigma = \begin{bmatrix} x_{11} & \ldots & x_{1k} \\ \vdots & & \vdots \\ x_{p1} & \ldots & x_{pk} \end{bmatrix}$$

where x_{ij} denotes the extra arrow from vertex v_j to vertex w_i. It follows from the definition of the algebra free product that every n-dimensional representation of $\langle Q_1 \rangle * \langle Q_2 \rangle$ is isomorphic to a representation V of the free-product quiver of dimension vector $(\alpha;\beta)$ (where we order the vertices $(v_1,\ldots,v_k;w_1,\ldots,w_p)$) with $|\alpha| = n = |\beta|$ such that $M_\sigma(M)$ is invertible. This defines a Zariski open subset of $\mathtt{rep}_{(\alpha;\beta)}\ Q_1 * Q_2$.

Define, with this ordering of vertices, $\theta = (-1,\ldots,-1;1,\ldots,1)$. We claim that any V in the open subset is θ-semistable. Indeed, $\theta(V) = 0$ because $|\alpha| = n = |\beta|$ and for a subrepresentation W of dimension vector $(\gamma;\delta)$ we have that $|\gamma| \leq |\delta|$ as otherwise the linear map $M_\sigma(W)$ would have a kernel contradicting invertibility of $M_\sigma(V)$. Moreover, the only subrepresentations $W \subset V$ which come from a representation of the algebra free product $\langle Q_1 \rangle * \langle Q_2 \rangle$ satisfy $\theta(W) = 0$. Therefore, V is a simple $\langle Q_1 \rangle * \langle Q_2 \rangle$-representation if and only if V lies in the open subset and is θ-stable.

In fact, the representations in the Zariski open subset determined by *det* $M_\sigma \neq 0$ are precisely the representations of a universal localization of $\langle Q_1 * Q_2 \rangle$. Let $\{P_1, \ldots, P_k\}$ be the projective left $\mathbb{C}Q_1 * Q_2$-modules corresponding to the vertices of Q_1 and $\{P'_1, \ldots, P'_p\}$ those corresponding to the vertices of Q_2 and consider the morphism

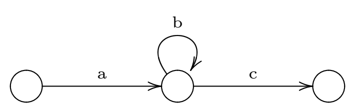
$$P'_1 \oplus \ldots \oplus P'_p \xrightarrow{\sigma} P_1 \oplus \ldots \oplus P_k$$

determined by the the matrix M_σ. The required universal localization is $\langle Q_1 * Q_2 \rangle_\sigma$. Later we will see that in general that θ-stable representations of a quiver Q become simple representations of a suitable universal localization of $\langle Q \rangle$ clarifying the similarity between stable representations and simples mentioned before. We have already seen in example 134 the concept of free products of quivers applied to the representation theory of torus knot groups.

Example 153. (Linear dynamical systems) A *linear time invariant dynamical system* Σ is determined by the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} & = Bx + Au \\ y & = Cx. \end{cases} \tag{7.1}$$

Here, $u(t) \in \mathbb{C}^m$ is the *input* or *control* of the system at time t, $x(t) \in \mathbb{C}^n$ the *state* of the system and $y(t) \in \mathbb{C}^p$ the *output* of the system Σ. *Time invariance* of Σ means that the matrices $A \in M_{n \times m}(\mathbb{C})$, $B \in M_n(\mathbb{C})$ and $C \in M_{p \times n}(\mathbb{C})$ are constant, that is $\Sigma = (A, B, C)$ is a representation of the quiver $\tilde{Q}$



of dimension vector $\alpha = (m, n, p)$. Recall that the *matrix exponential* e^{Bt} is the fundamental matrix for the homogeneous differential equation $\frac{dx}{dt} = Bx$. That is, the columns of e^{Bt} are a basis for the n-dimensional space of solutions of the equation $\frac{dx}{dt} = Bx$.

Motivated by this, we look for a solution to equation (7.1) as the form $x(t) = e^{Bt}g(t)$ for some function $g(t)$. Substitution gives the condition

$$\frac{dg}{dt} = e^{-Bt}Au \quad \text{whence} \quad g(\tau) = g(\tau_0) + \int_{\tau_0}^{\tau} e^{-Bt}Au(t)dt.$$

Observe that $x(\tau_0) = e^{B\tau_0}g(\tau_0)$ and we obtain the solution of the linear dynamical system $\Sigma = (A, B, C)$:

$$\begin{cases} x(\tau) & = e^{(\tau-\tau_0)B}x(\tau_0) + \int_{\tau_0}^{\tau} e^{(\tau-t)B}Au(t)dt \\ y(\tau) & = Ce^{B(\tau-\tau_0)}x(\tau_0) + \int_{\tau_0}^{\tau} Ce^{(\tau-t)B}Au(t)dt. \end{cases}$$

Differentiating we see that this is indeed a solution and it is the unique one having a prescribed starting state $x(\tau_0)$. Indeed, given another solution $x_1(\tau)$ we have that

$x_1(\tau) - x(\tau)$ is a solution to the homogeneous system $\frac{dx}{dt} = Bt$, but then

$$x_1(\tau) = x(\tau) + e^{\tau B} e^{-\tau_0 B}(x_1(\tau_0) - x(\tau_0)).$$

We call the system Σ *completely controllable* if we can steer any starting state $x(\tau_0)$ to the zero state by some control function $u(t)$ in a finite time span $[\tau_0, \tau]$. That is, the equation

$$0 = x(\tau_0) + \int_{\tau_0}^{\tau} e^{(\tau_0 - t)B} A u(t) dt$$

has a solution in τ and $u(t)$. As the system is time-invariant we may always assume that $\tau_0 = 0$ and have to satisfy the equation

$$0 = x_0 + \int_0^{\tau} e^{tB} A u(t) dt \quad \text{for every} \quad x_0 \in \mathbb{C}^n \tag{7.2}$$

Consider the *control matrix* $c(\Sigma)$ which is the $n \times mn$ matrix

$$c(\Sigma) = \boxed{\boxed{A}\;\boxed{BA}\;\boxed{B^2A}\quad \cdots \quad \boxed{B^{n-1}A}}$$

Assume that $rk\ c(\Sigma) < n$ then there is a non-zero state $s \in \mathbb{C}^n$ such that $s^{tr} c(\Sigma) = 0$, where s^{tr} denotes the transpose (row column) of s. Because B satisfies the characteristic polynomial $\chi_B(t)$, B^n and all higher powers B^m are linear combinations of $\{\mathbb{1}_n, B, B^2, \ldots, B^{n-1}\}$. Hence, $s^{tr} B^m A = 0$ for all m. Writing out the power series expansion of e^{tB} in equation (7.2) this leads to the contradiction that $0 = s^{tr} x_0$ for all $x_0 \in \mathbb{C}^n$. Hence, if $rk\ c(\Sigma) < n$, then Σ is not completely controllable.

Conversely, let $rk\ c(\Sigma) = n$ and assume that Σ is not completely controllable. That is, the space of all states

$$s(\tau, u) = \int_0^{\tau} e^{-tB} A u(t) dt$$

is a proper subspace of $\mathbb{C}^n$. But then, there is a non-zero state $s \in \mathbb{C}^n$ such that $s^{tr} s(\tau, u) = 0$ for all τ and all functions $u(t)$. Differentiating this with respect to τ we obtain

$$s^{tr} e^{-\tau B} A u(\tau) = 0 \quad \text{whence} \quad s^{tr} e^{-\tau B} A = 0 \tag{7.3}$$

for any τ as $u(\tau)$ can take on any vector. For $\tau = 0$ this gives $s^{tr} A = 0$. If we differentiate (7.3) with respect to τ we get $s^{tr} B e^{-\tau B} A = 0$ for all τ and for $\tau = 0$ this gives $s^{tr} BA = 0$. Iterating this process we show that $s^{tr} B^m A = 0$ for any m, whence

$$s^{tr} \begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A \end{bmatrix} = 0$$

contradicting the assumption that $rk\ c(\Sigma) = n$. That is,

A linear time-invariant dynamical system Σ determined by the matrices (A, B, C) is completely controllable if and only if $rk\ c(\Sigma)$ is maximal.

We say that a state $x(\tau)$ at time τ is *unobservable* if $Ce^{(\tau - t)B} x(\tau) = 0$ for all t. Intuitively this means that the state $x(\tau)$ cannot be detected uniquely from the output of the system Σ. Again, if we differentiate this condition a number of times and evaluate at $t = \tau$ we obtain the conditions

$$Cx(\tau) = CBx(\tau) = \ldots = CB^{n-1} x(\tau) = 0.$$

We say that Σ is *completely observable* if the zero state is the only unobservable state at any time τ. Consider the *observation matrix* $o(\Sigma)$ of the system Σ which is the $pn \times n$ matrix

$$o(\Sigma) = \begin{bmatrix} C^{tr} & (CB)^{tr} & \dots & (CB^{n-1})^{tr} \end{bmatrix}^{tr}$$

An analogous argument as before gives us that a linear time-invariant dynamical system Σ determined by the matrices (A,B,C) is completely observable if and only if $rk\ o(\Sigma)$ is maximal.

Assume we have two systems Σ and Σ', determined by matrix triples from $\mathtt{rep}_\alpha\ Q = M_{n\times m}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{p\times n}(\mathbb{C})$ producing the same output $y(t)$ when given the same input $u(t)$, for all possible input functions $u(t)$. We recall that the output function y for a system $\Sigma = (A,B,C)$ is determined by

$$y(\tau) = Ce^{B(\tau-\tau_0)}x(\tau_0) + \int_{\tau_0}^{\tau} Ce^{(\tau-t)B}Au(t)dt.$$

Differentiating this a number of times and evaluating at $\tau = \tau_0$ as before equality of input/output for Σ and Σ' gives the conditions

$$CB^iA = C'{B'}^iA' \quad \text{for all} \quad i.$$

But then, we have for any $v \in \mathbb{C}^{mn}$ that $c(\Sigma)(v) = 0 \Leftrightarrow c(\Sigma')(v) = 0$ and we can decompose $\mathbb{C}^{pn} = V \oplus W$ such that the restriction of $c(\Sigma)$ and $c(\Sigma')$ to V are the zero map and the restrictions to W give isomorphisms with $\mathbb{C}^n$. Hence, there is an invertible matrix $g \in GL_n$ such that $c(\Sigma') = gc(\Sigma)$ and from the commutative diagram

$$\begin{array}{ccccc} \mathbb{C}^{mn} & \xrightarrow{c(\Sigma)} & \mathbb{C}^n & \xrightarrow{o(\Sigma)} & \mathbb{C}^{pn} \\ \| & & \downarrow{\scriptstyle g} & & \| \\ \mathbb{C}^{mn} & \xrightarrow{c(\Sigma')} & \mathbb{C}^n & \xrightarrow{o(\Sigma')} & \mathbb{C}^{pn} \end{array}$$

we obtain that also $o(\Sigma') = o(\Sigma)g^{-1}$.

Consider the system $\Sigma_1 = (A_1, B_1, C_1)$ *equivalent* with Σ under the base-change matrix g. That is, $\Sigma_1 = g.\Sigma = (gA, gBg^{-1}, Cg^{-1})$. Then,

$$\begin{bmatrix} A_1, B_1A_1, \dots, B_1^{n-1}A_1 \end{bmatrix} = gc(\Sigma) = c(\Sigma') = \begin{bmatrix} A', B'A', \dots, {B'}^{n-1}A' \end{bmatrix}$$

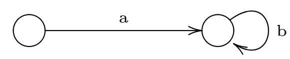
and so $A_1 = A'$. Further, as $B_1^{i+1}A_1 = {B'}^{i+1}A'$ we have by induction on i that the restriction of B_1 on the subspace of ${B'}^iIm(A')$ is equal to the restriction of B' on this space. Moreover, as $\sum_{i=0}^{n-1} {B'}^iIm(A') = \mathbb{C}^n$ it follows that $B_1 = B'$. Because $o(\Sigma') = o(\Sigma)g^{-1}$ we also have $C_1 = C'$. Therefore,

Let $\Sigma = (A,B,C)$ and $\Sigma' = (A',B',C')$ be two completely controllable and completely observable dynamical systems. The following are equivalent

1. *The input/output behavior of Σ and Σ' are equal.*
2. *The systems Σ and Σ' are equivalent, that is, there exists an invertible matrix $g \in GL_n$ such that*

$$A' = gA, \quad B' = gBg^{-1} \quad \text{and} \quad C' = Cg^{-1}.$$

Hence, in system identification it is important to classify completely controllable and observable systems $\Sigma \in \mathtt{rep}_\alpha\ \tilde{Q}$ under this restricted basechange action. We will concentrate on the input part and consider *completely controllable minisystems*, that is, representations $\Sigma = (A, B) \in \mathtt{rep}_\alpha\ Q$ for the quiver Q

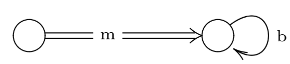


where $\alpha = (m, n)$ such that $c(\Sigma)$ is of maximal rank. Consider $\theta = (-n, m)$, then we can connect the notion of θ-semistability of quiver representations with system-theoretic notions :

If $\Sigma = (A, B) \in \mathtt{rep}_\alpha\ Q$ is θ-semistable, then Σ is completely controllable and $m \leq n$.

Indeed, if $m > n$ then $(Ker\ A, 0)$ is a proper subrepresentation of Σ of dimension vector $\beta = (dim\ Im\ A - m, 0)$ with $\theta(\beta) < 0$ so Σ cannot be θ-semistable. If Σ is not completely controllable then the subspace W of $\mathbb{C}^{\oplus n}$ spanned by the images of $A, BA, \ldots, B^{n-1}A$ has dimension $k < n$. But then, Σ has a proper subrepresentation of dimension vector $\beta = (m, k)$ with $\theta(\beta) < 0$, contradicting the θ-semistability assumption.

However, the restricted basechange action used in system-theory does not fit in well with the quiver setting. However, for a fixed dimension vector $\alpha = (m, n)$ we can remedy this by the *deframing trick* . Consider the quiver Q_m

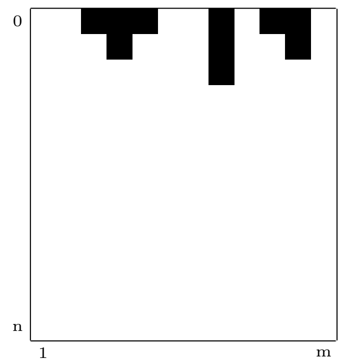


having m arrows from the first vertex to the second. If $\beta = (1, n)$ then there is a natural one-to-one correspondence

$$\mathtt{rep}_\alpha\ Q \quad \leftrightarrow \quad \mathtt{rep}_\beta\ Q_m$$

defined by splitting the $n \times m$ matrix $V(a)$ into its m columns. If $\theta' = (-n, 1)$ then under this correspondence θ-semistable representations in $\mathtt{rep}_\alpha\ Q$ correspond to θ'-semistable representations in $\mathtt{rep}_\beta\ Q_m$. More important, $GL(\beta)$-orbits in $\mathtt{rep}_\beta\ Q_m$ correspond to restricted base-change orbits in $\mathtt{rep}_\alpha\ Q$. To investigate the orbit space we introduce a combinatorial gadget : the *Kalman code* . It is an array consisting of $(n+1) \times m$ boxes each having a position label (i, j) where $0 \leq i \leq n$ and $1 \leq j \leq m$. These boxes are ordered *lexicographically* that is $(i', j') < (i, j)$ if and only if either $i' < i$ or $i' = i$ and $j' < j$. Exactly n of these boxes are painted black subject to the rule that if box (i, j) is black, then so is box (i', j) for all $i' < i$.

That is, a Kalman code looks like



We assign to a completely controllable couple $\Sigma = (A, B)$ its Kalman code $K(\Sigma)$ as follows : let $A = \begin{bmatrix} A_1 & A_2 & \dots & A_m \end{bmatrix}$, that is A_i is the i-th column of A. Paint the box (i, j) black if and only if the column vector $B^i A_j$ is linearly independent of the column vectors $B^k A_l$ for all $(k, l) < (i, j)$.

The painted array $K(\Sigma)$ is indeed a Kalman code. Assume that box (i, j) is black but box (i', j) white for $i' < i$, then

$$B^{i'} A_j = \sum_{(k,l)<(i',j)} \alpha_{kl} B^k A_l \quad \text{but then,} \quad B^i A_j = \sum_{(k,l)<(i',j)} \alpha_{kl} B^{k+i-i'} A_l$$

and all $(k + i - i', l) < (i, l)$, a contradiction. Moreover, $K(\Sigma)$ has exactly n black boxes as there are n linearly independent columns of the control matrix $c(\Sigma)$ when Σ is completely controllable.

The Kalman code is a discrete invariant of the orbit $\mathcal{O}(\Sigma)$ under the restricted basechange action by GL_n. This follows from the fact that $B^i A_j$ is linearly independent of the $B^k A_l$ for all $(k, l) < (i, j)$ if and only if $gB^i A_j$ is linearly independent of the $gB^k A_l$ for any $g \in GL_n$ and the observation that $gB^k A_l = (gBg^{-1})^k (gA)_l$.

With $\mathtt{rep}^c_\alpha\ Q$ we will denote the open subset of $\mathtt{rep}_\alpha\ Q$ of all completely controllable couples (A, B). We consider the map

$$\begin{array}{ccc} \mathtt{rep}_\alpha\ Q & \xrightarrow{\quad\psi\quad} & M_{n\times(n+1)m}(\mathbb{C}) \\ (A, B) & \mapsto & \begin{bmatrix} A & BA & B^2A & \dots & B^{n-1}A & B^nA \end{bmatrix} \end{array}$$

The matrix $\psi(A, B)$ determines a linear map $\psi_{(A,B)} : \mathbb{C}^{(n+1)m} \longrightarrow \mathbb{C}^n$ and (A, B) is a completely controllable couple if and only if the corresponding linear map $\psi_{(A,B)}$ is surjective. Moreover, there is a linear action of GL_n on $M_{n\times(n+1)m}(\mathbb{C})$ by left multiplication and the map ψ is GL_n-equivariant.

The Kalman code induces a *barcode* on $\psi(A, B)$, that is the $n \times n$ minor of $\psi(A, B)$ determined by the columns corresponding to black boxes in the Kalman code. By construction this minor is an invertible matrix $g^{-1} \in GL_n$. We can choose a canonical point in the orbit $\mathcal{O}(\Sigma)$: $g.(A, B)$. It does have the characteristic property that the $n \times n$ minor of its image under ψ, determined by the Kalman code is the identity matrix $\mathbb{1}_n$. The matrix $\psi(g.(A, B))$ will be denoted by $b(A, B)$ and is called barcode of the completely controllable pair $\Sigma = (A, B)$. We claim that the barcode determines the orbit uniquely. The map ψ is injective on the open set

$\mathtt{rep}_\alpha^c\ Q$. Indeed, if

$$\begin{bmatrix} A & BA & \dots & B^nA \end{bmatrix} = \begin{bmatrix} A' & B'A' & \dots & {B'}^nA' \end{bmatrix}$$

then $A = A'$, $B \mid Im(A) = B' \mid Im(A)$ and hence by induction also

$$B \mid B^i Im(A) = B' \mid {B'}^i Im(A') \qquad \text{for all } i \leq n-1.$$

But then, $B = B'$ as both couples (A,B) and (A',B') are completely controllable. Hence, the barcode $b(A,B)$ determines the orbit $\mathcal{O}(\Sigma)$ and is a point in the Grassmannian $\mathtt{Grass}_n(m(n+1))$. We have

$$\begin{array}{ccc} \mathtt{rep}_\alpha^c\ Q & \overset{\psi}{\hookrightarrow} & M^{max}_{n\times m(n+1)}(\mathbb{C}) \\ & \searrow^{b(.)} & \downarrow \chi \\ & & \mathtt{Grass}_n(m(n+1)) \end{array}$$

where ψ is a GL_n-equivariant embedding and χ the orbit map. Observe that the barcode matrix $b(A,B)$ shows that the stabilizer of (A,B) is trivial. Indeed, the minor of $g.b(A,B)$ determined by the Kalman code is equal to g. Moreover, continuity of b implies that the orbit $\mathcal{O}(\Sigma)$ is closed in $\mathtt{rep}_\alpha^c\ Q$.

Compute the differential of ψ. For all $(A,B) \in \mathtt{rep}_\alpha\ Q$ and for all $(X,Y) \in T_{(A,B)}\ \mathtt{rep}_\alpha\ Q$ we have

$$(B+\epsilon Y)^j(A+\epsilon X) = B^nA + \epsilon\ (B^nX + \sum_{i=0}^{j-1} B^iYB^{n-1-i}A).$$

Therefore the differential of ψ in (A,B), $d\psi_{(A,B)}(X,Y)$ is equal to

$$\begin{bmatrix} X & BX+YA & B^2X+BYA+YBA & \dots & B^nX+\sum_{i=0}^{n-1} B^iYB^{n-1-i}A \end{bmatrix}.$$

Assume $d\psi_{(A,B)}(X,Y)$ is the zero matrix, then $X = 0$ and substituting in the next term also $YA = 0$. Substituting in the third gives $YBA = 0$, then in the fourth $YB^2A = 0$ and so on until $YB^{n-1}A = 0$. But then,

$$Y \begin{bmatrix} A & BA & B^2A & \dots & B^{n-1}A \end{bmatrix} = 0.$$

If (A,B) is a completely controllable pair, this implies that $Y = 0$ and hence shows that $d\psi_{(A,B)}$ is injective for all $(A,B) \in \mathtt{rep}_\alpha^c\ Q$. Therefore, ψ is a GL_n-equivariant embedding of $\mathtt{rep}_\alpha^c\ Q$ with image a locally closed smooth subvariety of $M^{max}_{n\times(n+1)m}(\mathbb{C})$. The image of this subvariety under the orbit map χ is again smooth as all fibers are equal to GL_n. This concludes the difficult part of the *Kalman theorem* :

The orbit space $O_c = \mathtt{rep}_\alpha^c\ Q/GL_n$ *of equivalence classes of completely controllable couples is a locally closed smooth subvariety of dimension* $m.n$ *of the Grassmannian* $\mathtt{Grass}_n(m(n+1))$.

To prove the dimension statement, define $\mathtt{rep}_\alpha^c(K)$ the set of completely controllable pairs (A,B) having Kalman code K and let $O_c(K)$ be the image under the orbit map. After identifying $\mathtt{rep}_\alpha^c(K)$ with its image under ψ, the barcode matrix $b(A,B)$ gives a section $O_c(K) \overset{s}{\hookrightarrow} \mathtt{rep}_\alpha^c(K)$. In fact,

$$GL_n \times O_c(K) \longrightarrow \mathtt{rep}_\alpha^c(K) \qquad (g,x) \mapsto g.s(x)$$

is a GL_n-equivariant isomorphism because the $n \times n$ minor of $g, b(A,B)$ determined by K is g. Consider the *generic* Kalman code K^g obtained by painting the top boxes black from left to right until one has n black boxes. Clearly $\mathtt{rep}^c_\alpha(K^g)$ is open in $\mathtt{rep}^c_\alpha\ Q$ and one deduces

$$dim\ O_c = dim\ O_c(K^g) = dim\ \mathtt{rep}^c_\alpha(K^g) - dim\ GL_n = mn + n^2 - n^2 = mn.$$

EXAMPLE 154. (Brauer-Severi varieties) Let K be a field and $\Delta = (a,b)_K$ the quaternion algebra determined by $a, b \in K^*$. That is,

$$\Delta = K.1 \oplus K.i \oplus K.j \oplus K.ij \quad \text{with} \quad i^2 = a \quad j^2 = b \quad \text{and} \quad ji = -ij$$

The norm map on Δ defines a conic in $\mathbb{P}^2_K$ called the *Brauer-Severi variety* of Δ

$$BS(\Delta) = V(x^2 - ay^2 - bz^2) \hookrightarrow \mathbb{P}^2_K = \mathtt{proj}\ K[x,y,z].$$

Its characteristic property is that a fieldextension L of K admits an L-rational point on $BS(\Delta)$ if and only if $\Delta \otimes_K L$ admits zero-divisors and hence is isomorphic to $M_2(L)$. More generally, using the descent interpretation of étale (or Galois) cohomology we see that the cohomology pointed set

$$H^1_{et}(K, PGL_n)$$

classifies at the same time when $\mathbb{K}$ is the algebraic closure of K

- Brauer-Severi K-varieties BS, which are smooth projective K-varieties such that $BS_{\mathbb{K}} = BS \times_K \mathbb{K} \simeq \mathbb{P}^{n-1}_{\mathbb{K}}$.
- Central simple K-algebras Δ, which are K-algebras of dimension n^2 such that $\Delta \otimes_K \mathbb{K} \simeq M_n(\mathbb{K})$.

The one-to-one correspondence between these two sets is given by associating to a central simple K-algebra Δ its Brauer-Severi variety $BS(\Delta)$ which represents the functor associating to a fieldextension L of K the set of left ideals of $\Delta \otimes_K L$ which have L-dimension equal to n. In particular, $BS(\Delta)$ has an L-rational point if and only if $\Delta \otimes_K L \simeq M_n(L)$ and hence the geometric object $BS(\Delta)$ encodes the algebraic splitting behavior of Δ.

Brauer-Severi varieties (and schemes) were later defined for Azumaya algebras and even for arbitrary Cayley-Hamilton algebras. Historically, these concepts were introduced and studied by M. Nori [**49**] who called them noncommutative Hilbert schemes. We follow here the account of M. Van den Bergh in [**12**].

Let $(A, tr_A) \in \mathtt{alg@n}$ and consider the $GL_n(\mathbb{C})$ action on the product scheme $\mathtt{trep}_n A \times \mathbb{C}^n$ defined by

$$g.(M,v) = (g.M, gv)$$

where the action in the first factor is the basechange action on $\mathtt{trep}_n A$ and in the second factor is left multiplication. In this product we consider the set of *Brauer stable* points which are defined to be

$$\mathtt{brauer}A = \{(M,v) \mid \phi_M(A)v = \mathbb{C}^n\}$$

where $\phi_M : A \longrightarrow M_n(\mathbb{C})$ is the morphism defining M. This is also the subset of points with trivial stabilizer subgroup. Hence, every $GL_n(\mathbb{C})$-orbit in $\mathtt{brauer}A$ is closed and we can form the orbit space called the *Brauer-Severi scheme* of A

$$\mathtt{bs}A = \mathtt{brauer}/GL_n.$$

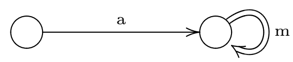
We will see in a moment that this is a projective space bundle over the quotient variety $\mathtt{trep}_n A/GL_n = \mathtt{tiss}_n A$. For arbitrary $(A, tr_A) \in \mathtt{alg@n}$ not much can be said about these Brauer-Severi schemes. However, if A is $\texttt{alg@n}$-smooth, we claim :

If $(A, tr_A) \in \texttt{alg@n}$ *is* $\texttt{alg@n}$*-smooth, then the Brauer-Severi scheme* $\texttt{bs}A$ *is a smooth variety.*

Indeed, as the action of GL_n on $\texttt{brauer}A$ is free, it suffices to prove that $\texttt{brauer}A$ is a smooth variety. But, $\texttt{brauer}A$ is a Zariski open subset of the smooth variety $\texttt{trep}_n A \times \mathbb{C}^n$. We will relate the study of the Brauer-Severi variety to that of θ-semistable points of a quiver setting. Consider the generic case, that is $A = \int_n \langle m \rangle$. In this case we have that

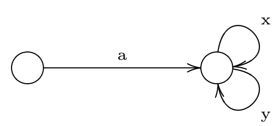
$$\texttt{trep} \int_n \langle m \rangle \times \mathbb{C}^n = \texttt{rep}_\alpha\ Q$$

where $\alpha = (1, n)$ and the quiver Q is

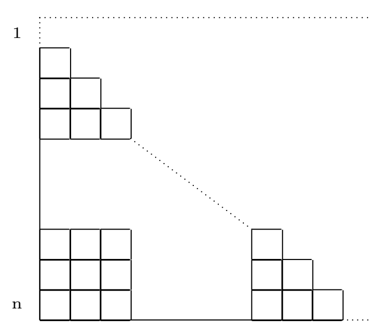


where the arrow a corresponds to the $\mathbb{C}^n$ component and the m loops give $M_n^m = \texttt{trep} \int_n \langle m \rangle$. Let $\theta = (-n, 1)$, then θ-semistable representations in $\texttt{rep}_\alpha\ Q$ are precisely the Brauer stable points $\texttt{brauer}\ \int_n \langle m \rangle$. Indeed, let $(A_1, \ldots, A_m, v) \in \texttt{rep}_\alpha\ Q$ be a Brauer stable point. This means that $\mathbb{C}^n$ is spanned by v and all vectors of the form $A_{i_1}^{m_1} \ldots A_{i_z}^{m_z} v$. But then there are no proper subrepresentations of dimension vector $\beta = (1, k)$ with $k < n$. Conversely, a θ-semistable representation is Brauer stable for assume that the subspace spanned by v and the above vectors is $k < n$ then there is a proper $\beta = (1, k)$-dimensional subrepresentation W with $\theta(W) = -n + k < 0$.

Let us present a concrete description of the Brauer-Severi variety in case $m = 2$, that is when Q is



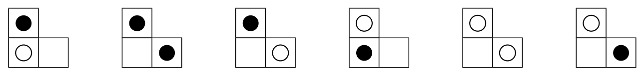
For the investigation of the GL_n-orbits on $\texttt{rep}_\alpha\ Q$ we introduce a combinatorial gadget : the *Hilbert n-stair* . This is the lower triangular part of a square $n \times n$ array of boxes



filled with *go-stones* according to the following two rules :

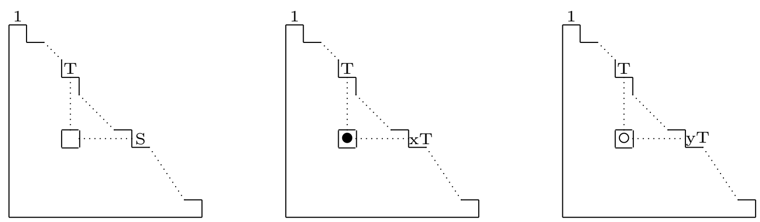
- each row contains exactly one stone, and
- each column contains at most one stone of each color.

For example, the set of all possible Hilbert 3-stairs is given below.

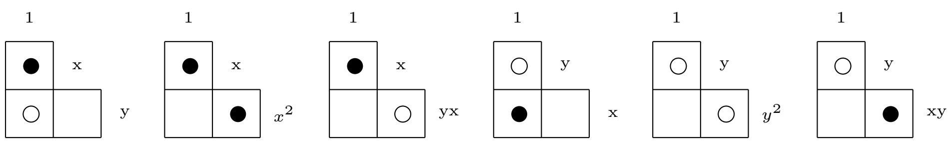


To every Hilbert stair σ we will associate a sequence of monomials $W(\sigma)$ in the free algebra $\langle 2 \rangle = \mathbb{C}\langle x, y \rangle$. At the top of the stairs we place the identity element 1. Then, we descend the stairs according to the following rule.

- Every go-stone has a *top word* T which we may assume we have constructed before and a *side word* S and they are related as indicated below



For example, for the Hilbert 3-stairs we have the following sequences of non-commutative words



We evaluate a Hilbert n-stair σ with associated sequence of non-commutative words $W(\sigma) = \{1, w_2(x,y), \ldots, w_n(x,y)\}$ on

$$\mathtt{rep}_\alpha\, Q = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n$$

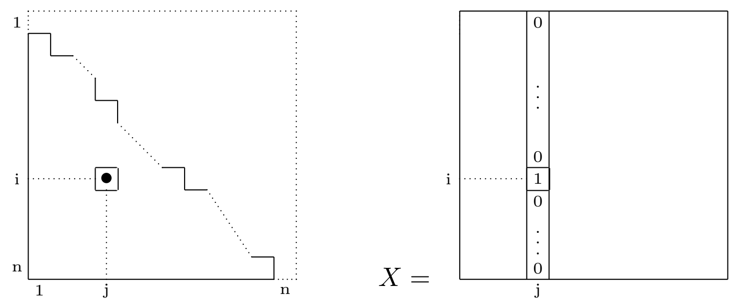
For a triple (X, Y, v) we replace every occurrence of x in the word $w_i(x,y)$ by X and every occurrence of y by Y to obtain an $n \times n$ matrix $w_i = w_i(X, Y) \in M_n(\mathbb{C})$ and by left multiplication on v a column vector $w_i.v$. The evaluation of σ on (X, Y, v) is the determinant of the $n \times n$ matrix

$$\sigma(X,Y,v) = det \begin{bmatrix} v & w_2.v & w_3.v & \ldots & w_n.v \end{bmatrix}$$

For a fixed Hilbert n-stair σ we denote with $\mathtt{rep}(\sigma)$ the subset of triples (X, Y, v) in $\mathtt{rep}_\alpha\, Q$ such that the evaluation $\sigma(X, Y, v) \neq 0$. We claim

For every Hilbert n-stair, rep $(\sigma) \neq \emptyset$

Let v be the first basic column vector e_1. Let every black stone in the Hilbert stair σ fix a column of X by the rule



That is, one replaces every black stone in σ by 1 at the same spot in X and fills the remaining spots in the same column by zeroes. The same rule applies to Y for white stones. We say that such a triple (X, Y, v) is in *σ-standard form*. With these conventions one easily verifies by induction that

$$w_i(X, Y)e_1 = e_i \quad \text{for all } 2 \leq i \leq n.$$

Hence, filling up the remaining spots in X and Y arbitrarily one has that $\sigma(X, Y, v) \neq 0$ proving the claim. Hence, $\texttt{rep}(\sigma)$ is an open subset of $\texttt{rep}_\alpha\ Q$ (consisting of θ-stable representations) for every Hilbert n-stair σ. Further, for every word (monomial) $w(x, y)$ and every $g \in GL_n(\mathbb{C})$ we have that

$$w(gXg^{-1}, gYg^{-1})gv = gw(X, Y)v$$

and therefore the open sets $\texttt{rep}(\sigma)$ are stable under the GL_n-action on $\texttt{rep}_\alpha\ Q$. We will give representatives of the orbits in $\texttt{rep}(\sigma)$.

Let $W_n = \{1, x, \ldots, x^n, xy, \ldots, y^n\}$ be the set of all words in the non-commuting variables x and y of length $\leq n$, ordered lexicographically. For every triple (X, Y, v) consider the $n \times m$ matrix

$$\psi(X, Y, v) = \begin{bmatrix} u & Xu & X^2u & \ldots & Y^nu \end{bmatrix}$$

where $m = 2^{n+1} - 1$ and the j-th column is the column vector $w(X, Y)v$ with $w(x, y)$ the j-th word in W_n. Hence, $(X, Y, v) \in \texttt{rep}(\sigma)$ if and only if the $n \times n$ minor of $\psi(X, Y, v)$ determined by the word-sequence $\{1, w_2, \ldots, w_n\}$ of σ is invertible. Moreover, as

$$\psi(gXg^{-1}, gYg^{-1}, gu, vg^{-1}) = g\psi(v, X, Y)$$

we deduce that the GL_n-orbit of (X, Y, v) contains a *unique* triple (X_1, Y_1, v_1) such that the corresponding minor of $\psi(X_1, Y_1, v_1) = \mathbb{1}_n$. Hence, each $GL_n(\mathbb{C})$-orbit in $\texttt{rep}(\sigma)$ contains a unique representant in σ-standard form. Therefore,

The action of GL_n on $\texttt{rep}(\sigma)$ is free and the orbit space is an affine space of dimension $n^2 + n$.

The dimension is equal to the number of non-forced entries in X, Y and v. As we fixed $n - 1$ columns in X or Y this dimension is equal to

$$k = 2n^2 - (n-1)n = n^2 + n.$$

The above argument shows that every GL_n-orbit contains a unique triple in σ-standard form so the orbit space is an affine space. We claim, *The Brauer-Severi variety* $\mathtt{bs}\int_n \langle 2 \rangle$ *is a smooth variety of dimension* n^2+n *and is covered (in the Zariski topology) by the affine spaces* $\mathtt{rep}(\sigma)$.

We still have to prove that any Brauer-stable triple $(X,Y,v) \in \mathtt{rep}_\alpha Q$ belongs to at least one of the open subsets $\mathtt{rep}(\sigma)$. Either $Xv \notin \mathbb{C}v$ or $Yv \notin \mathbb{C}v$. Fill the top box of the stairs with the corresponding stone and define the 2-dimensional subspace $V_2 = \mathbb{C}v_1 + \mathbb{C}v_2$ where $v_1 = v$ and $v_2 = w_2(X,Y)u$ with w_2 the corresponding word (either x or y). Assume by induction we have been able to fill the first i rows of the stairs with stones leading to the sequence of words $\{1, w_2(x,y), \ldots, w_i(x,y)\}$ such that the subspace $V_i = \mathbb{C}v_1 + \ldots + \mathbb{C}v_i$ with $v_i = w_i(X,Y)v$ has dimension i. Then, either $Xu_j \notin V_i$ for some j or $Yu_j \notin V_i$ for some j. Fill the j-th box in the $i+1$-th row of the stairs with the corresponding stone. Then, the top $i+1$ rows of the stairs form a Hilbert $i+1$-stair as there can be no stone of the same color lying in the same column. Define $w_{i+1}(x,y) = xw_i(x,y)$ (or $yw_i(x,y)$) and $v_{i+1} = w_{i+1}(X,Y)v$. Then, $V_{i+1} = \mathbb{C}v_1 + \ldots + \mathbb{C}v_{i+1}$ has dimension $i+1$ continuing we end up with a Hilbert n-stair σ such that $(X,Y,v) \in \mathtt{rep}(\sigma)$.

In the two previous examples we have seen that the varieties classifying closed orbits of semistable representations are covered by open sets defined by determinants. We will show that this is true in full generality. Closed orbits of representations were described by polynomial invariants, closed orbits of semistable representations will be described by *semi-invariants* .

DEFINITION 109. A *character* of $GL(\alpha)$ is an algebraic group morphism $\chi : GL(\alpha) \longrightarrow \mathbb{C}^*$. They correspond to integral k-tuples $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ by

$$GL(\alpha) \xrightarrow{\chi_\theta} \mathbb{C}^* \qquad (g_1, \ldots, g_k) \mapsto det(g_1)^{t_1}. \ \ldots \ .det(g_k)^{t_k}$$

For a fixed θ we can extend the $GL(\alpha)$-action to the space $\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}$ by

$$GL(\alpha) \times \mathtt{rep}_\alpha\ Q \oplus \mathbb{C} \longrightarrow \mathtt{rep}_\alpha\ Q \oplus \mathbb{C} \qquad g.(V,c) = (g.V, \chi_\theta^{-1}(g)c)$$

The coordinate ring $\mathbb{C}[\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}] = \mathbb{C}[\mathtt{rep}_\alpha][t]$ is graded by defining $deg(t) = 1$ and $deg(f) = 0$ for all $f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q]$. As action of $GL(\alpha)$ preserves this gradation, the ring of invariant polynomial maps $\mathbb{C}[\mathtt{rep}_\alpha\ Q][t]^{GL(\alpha)}$ is graded with homogeneous part of degree zero the ring of polynomial invariants $\mathbb{C}[\mathtt{rep}_\alpha]^{GL(\alpha)} = \mathbb{C}[\mathtt{iss}_\alpha\ Q]$.

An invariant of degree n, say ft^n with $f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q]$ satisfies

$$f(g.V) = \chi_\theta^n(g) f(V)$$

that is, f is a *semi-invariant* of weight χ_θ^n. That is, the graded decomposition of the invariant ring is

$$\mathbb{C}[\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = R_0 \oplus R_1 \oplus \ldots \quad \text{with} \quad R_i = \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha), \chi^n \theta}$$

The *moduli space of semi-stable quiver representations of dimension* α is the projective variety

$$\mathtt{moss}_\alpha(Q,\theta) = \mathtt{proj}\ \mathbb{C}[\mathtt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = \mathtt{proj}\ \oplus_{n=0}^\infty \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha),\chi^n \theta}$$

A representation $V \in \mathtt{rep}_\alpha\ Q$ is said to be χ_θ*-semistable* if and only if there is a semi-invariant $f \in \mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$ for some $n \geq 1$ such that $f(V) \neq 0$. The Zariski open subset of $\mathtt{rep}_\alpha\ Q$ consisting of all χ_θ-semistable representations will be denoted by $\mathtt{ress}_\alpha(Q,\theta)$.

THEOREM 117 (King). *The following are equivalent*

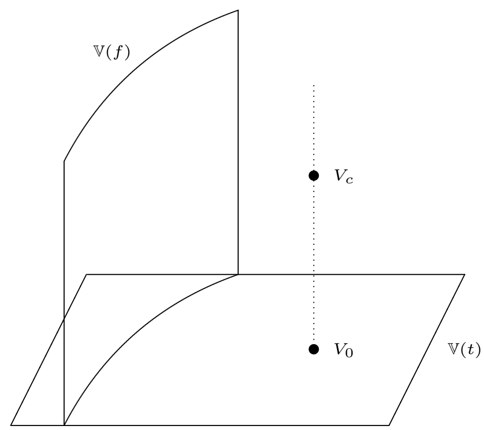
(1) *$V \in \texttt{rep}_\alpha\ Q$ is χ_θ-semistable.*
(2) *For $c \neq 0$, we have $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) = \emptyset$.*
(3) *For every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ we have $\underset{t \rightarrow 0}{lim}\ \lambda(t).V_c \notin \mathbb{V}(t) = \texttt{rep}_\alpha\ Q \times \{0\}$.*
(4) *For every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ such that $\underset{t \rightarrow 0}{lim}\ \lambda(t).V$ exists in $\texttt{rep}_\alpha\ Q$ we have $\theta(\lambda) \geq 0$.*

Moreover, this occurs only if $\theta(\alpha) = 0$. The moduli space of θ-semistable representations of $\texttt{rep}_\alpha\ Q$

$$\texttt{moss}_\alpha(Q, \theta)$$

classifies closed $GL(\alpha)$-orbits in the open subset $\texttt{ress}_\alpha(Q, \theta)$ of all χ_θ-semistable representations.

Proof. Lift a representation $V \in \texttt{rep}_\alpha\ Q$ to points $V_c = (V, c) \in \texttt{rep}_\alpha\ Q \oplus \mathbb{C}$ and use $GL(\alpha)$-invariant theory on this larger $GL(\alpha)$-module



Assume that the orbit closure $\overline{\mathcal{O}(V_c)}$ does not intersect $\mathbb{V}(t) = \texttt{rep}_\alpha\ Q \times \{0\}$. As both are $GL(\alpha)$-stable closed subsets of $\texttt{rep}_\alpha\ Q \oplus \mathbb{C}$ the separation property of invariant theory yields the existence of a $GL(\alpha)$-invariant function $g \in \mathbb{C}[\texttt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)}$ such that $g(\overline{\mathcal{O}(V_c)}) \neq 0$ but $g(\mathbb{V}(t)) = 0$. We may assume g to be homogeneous, that is, of the form $g = ft^n$ for some n. But then, f is a semi-invariant on $\texttt{rep}_\alpha\ Q$ of weight χ_θ^n and V must be χ_θ-semistable. Moreover, $\theta(\alpha) = \sum_{i=1}^k t_i a_i = 0$ as the one-dimensional central torus of $GL(\alpha)$

$$\mu(t) = (t\mathbb{1}_{a_1}, \ldots, t\mathbb{1}_{a_k}) \hookrightarrow GL(\alpha)$$

acts trivially on $\texttt{rep}_\alpha\ Q$ but acts on $\mathbb{C}$ via multiplication with $\prod_{i=1}^k t^{-a_i t_i}$. Hence, if $\theta(\alpha) \neq 0$ then $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) \neq \emptyset$.

It follows from the Hilbert criterium that $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) = \emptyset$ if and only if for every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ we have that $\underset{t \rightarrow 0}{lim}\ \lambda(t).V_c \notin \mathbb{V}(t)$. We can also formulate this in terms of the $GL(\alpha)$-action on $\texttt{rep}_\alpha\ Q$. The composition of a one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ with the character

$$\mathbb{C}^* \xrightarrow{\lambda(t)} GL(\alpha) \xrightarrow{\chi_\theta} \mathbb{C}^*$$

is an algebraic group morphism and is therefore of the form $t \longrightarrow t^m$ for some $m \in \mathbb{Z}$ and we denote this integer by $\theta(\lambda) = m$. Assume that $\lambda(t)$ is a one-parameter subgroup such that $\underset{t \to 0}{lim}\ \lambda(t).V = V'$ exists in $\texttt{rep}_\alpha\ Q$. Because $\lambda(t).(V,c) = (\lambda(t).V, t^{-m}c)$, $\theta(\lambda) \geq 0$ for the orbitclosure $\overline{\mathcal{O}(V_c)}$ not to intersect $\mathbb{V}(t)$.

As for the second assertion, let $g = ft^n$ be a homogeneous invariant function for the $GL(\alpha)$-action on $\texttt{rep}_\alpha\ Q \oplus \mathbb{C}$ and consider the affine open $GL(\alpha)$-stable subset $\mathbb{X}(g)$. The construction of the algebraic quotient and the fact that the invariant ring is graded asserts that the closed $GL(\alpha)$-orbits in $\mathbb{X}(g)$ are classified by the points of the graded localization at g which is of the form

$$(\mathbb{C}[\texttt{rep}_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)})_g = R_f[h, h^{-1}]$$

for some homogeneous invariant h where R_f is the coordinate ring of the affine open subset $\mathbb{X}(f)$ in $\texttt{moss}_\alpha(Q,\theta)$ determined by the semi-invariant f of weight χ_θ^n. The claim follows because the moduli space is covered by such open subsets. □

THEOREM 118 (King). *For $V \in \texttt{rep}_\alpha\ Q$ the following are equivalent*

1. *V is χ_θ-semistable.*
2. *V is θ-semistable.*

For a θ-semistable representation $V \in \texttt{rep}_\alpha\ Q$ equivalent are

1. *The orbit $\mathcal{O}(V)$ is closed in $\texttt{moss}_\alpha(Q,\alpha)$.*
2. *$V \simeq W_1^{\oplus e_1} \oplus \ldots \oplus W_l^{\oplus e_l}$ with W_i a θ-stable representation.*

The geometric points of the moduli space $\texttt{moss}_\alpha(Q,\theta)$ are in natural one-to-one correspondence with isomorphism classes of α-dimensional representations which are direct sums of θ-stable subrepresentations. The quotient map

$$\texttt{ress}_\alpha(Q,\theta) \twoheadrightarrow \texttt{moss}_\alpha(Q,\theta)$$

maps a θ-semistable representation V to the direct sum of its Jordan-Hölder factors in the Abelian category of semistable representations.

PROOF. For $\lambda : \mathbb{C}^* \longrightarrow GL(\alpha)$ a one-parameter subgroup and $V \in \texttt{rep}_\alpha\ Q$ we can decompose for every vertex v_i the vertex-space in weight spaces

$$V_i = \oplus_{n \in \mathbb{Z}} V_i^{(n)}$$

where $\lambda(t)$ acts on the weight space $V_i^{(n)}$ as multiplication by t^n. This decomposition allows us to define a filtration

$$V_i^{(\geq n)} = \oplus_{m \geq n} V_i^{(m)}$$

For every arrow $(j) \xleftarrow{a} (i)$, $\lambda(t)$ acts on the components of the arrow maps

$$V_i^{(n)} \xrightarrow{V_a^{m,n}} V_j^{(m)}$$

by multiplication with t^{m-n}. That is, a limit $\underset{t \to 0}{lim}\ V_a$ exists if and only if $V_a^{m,n} = 0$ for all $m < n$, that is, if V_a induces linear maps

$$V_i^{(\geq n)} \xrightarrow{V_a} V_j^{(\geq n)}$$

Hence, a limiting representation exists if and only if the vertex-filtration spaces $V_i^{(\geq n)}$ determine a subrepresentation $V_n \subset V$ for all n. A one-parameter subgroup λ such that $\underset{t \to}{lim}\ \lambda(t).V$ exists determines a decreasing filtration of V by subrepresentations

$$\ldots \supset V_n \supset V_{n+1} \supset \ldots$$

Further, the limiting representation is then the associated graded representation

$$\lim_{t\to 0} \lambda(t).V = \oplus_{n\in\mathbb{Z}} \frac{V_n}{V_{n+1}}$$

where of course only finitely many of these quotients can be nonzero. For the given character $\theta = (t_1, \ldots, t_k)$ and a representation $W \in \mathtt{rep}_\beta\ Q$ we denote

$$\theta(W) = t_1 b_1 + \ldots + t_k b_k \quad \text{where} \quad \beta = (b_1, \ldots, b_k)$$

Assume that $\theta(V) = 0$, then with the above notations, we have an interpretation of $\theta(\lambda)$ as

$$\theta(\lambda) = \sum_{i=1}^{k} t_i \sum_{n\in\mathbb{Z}} n\ dim_{\mathbb{C}}\ V_i^{(n)} = \sum_{n\in\mathbb{Z}} n\theta(\frac{V_n}{V_{n+1}}) = \sum_{n\in\mathbb{Z}} \theta(V_n)$$

(1) $\Rightarrow$ (2) : Let W be a subrepresentation of V and let λ be the one-parameter subgroup associated to the filtration $V \supset W \supset 0$, then $\lim_{t\to 0} \lambda(t).V$ exists whence by (4) of the previous theorem $\theta(\lambda) \geq 0$, but we have

$$\theta(\lambda) = \theta(V) + \theta(W) = \theta(W)$$

(2) $\Rightarrow$ (1) : Let λ be a one-parameter subgroup of $GL(\alpha)$ such that $\lim_{t\to 0} \lambda(t).V$ exists and consider the induced filtration by subrepresentations V_n defined above. By assumption all $\theta(V_n) \geq 0$, whence

$$\theta(\lambda) = \sum_{n\in Z} \theta(V_n) \geq 0$$

and the result follows from the foregoing theorem.

As for the second part. (1) $\Rightarrow$ (2) : Assume that $\mathcal{O}(V)$ is closed in $\mathtt{ress}_\alpha(Q,\theta)$ and consider the θ-semistable representation $W = gr\ V$, the direct sum of the Jordan-Hölder factors in the Abelian category of θ-semistable representations. As W is the associated graded representation of a filtration on V, there is a one-parameter subgroup λ of $GL(\alpha)$ such that $\lim_{t\to 0} \lambda(t).V \simeq W$, that is $\mathcal{O}(W) \subset \overline{\mathcal{O}(V)} = \mathcal{O}(V)$, whence $W \simeq V$. (2) $\Rightarrow$ (1) : Let $\mathcal{O}(W)$ be a closed orbit contained in $\overline{\mathcal{O}(V)}$ (one of minimal dimension). By the Hilbert criterium there is a one-parameter subgroup λ in $GL(\alpha)$ such that $\lim_{t\to 0} \lambda(t).V \simeq W$. Hence, there is a finite filtration of V with associated graded θ-semistable representation W. As none of the θ-stable components of V admits a proper quotient which is θ-semistable (being a direct summand of W), this shows that $V \simeq W$ and so $\mathcal{O}(V) = \mathcal{O}(W)$ is closed. The other statements are clear from this. □

EXAMPLE 155. Remains to determine the situations (α, θ) such that the corresponding moduli space $\mathtt{moss}_\alpha(Q,\theta)$ is non-empty, or equivalently, such that the Zariski open subset $\mathtt{ress}_\alpha(Q,\theta) \subset \mathtt{rep}_\alpha\ Q$ is non-empty. This follows from the results on general subrepresentations proved in section 7.1

Let α be a dimension vector such that $\theta(\alpha) = 0$. Then,

1. $\mathtt{ress}_\alpha(Q,\alpha)$ *is a non-empty Zariski open subset of* $\mathtt{rep}_\alpha\ Q$ *if and only if for every* $\beta \hookrightarrow \alpha$ *we have* $\theta(\beta) \geq 0$.
2. *The θ-stable representations form a non-empty Zariski open subset of* $\mathtt{rep}_\alpha\ Q$ *if and only if for every* $0 \neq \beta \hookrightarrow \alpha$ *we have* $\theta(\beta) > 0$

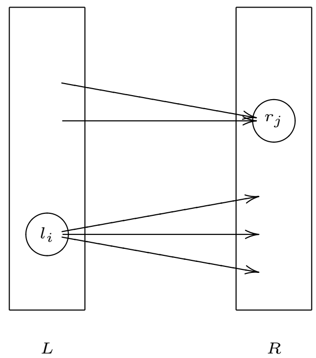


FIGURE 5. Left-right bipartite quiver.

We will study the moduli space $\mathtt{moss}_\alpha(Q,\theta)$ both in the Zariski- and the étale topology. To understand the first we have to determine the graded algebra of semi-invariants.

DEFINITION 110. (Determinantal semi-invariants) Let Q be a quiver on the vertices $\{v_1,\ldots,v_k\}$, fix a dimension vector $\alpha = (a_1,\ldots,a_k)$ and a character χ_θ where $\theta = (t_1,\ldots,t_k)$ such that $\theta(\alpha) = 0$. We will call a bipartite quiver Q' as in figure 5 on left vertex-set $L = \{l_1,\ldots,l_p\}$ and right vertex-set $R = \{r_1,\ldots,r_q\}$ and a dimension vector $\beta = (c_1,\ldots,c_p;d_1,\ldots,d_q)$ to be of type (Q,α,θ) if the following conditions are met

• All left and right vertices correspond to vertices of Q, that is, there are maps

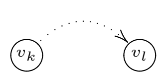
$$\begin{cases} L \xrightarrow{l} \{v_1,\ldots,v_k\} \\ R \xrightarrow{r} \{v_1,\ldots,v_k\} \end{cases}$$

possibly occurring with multiplicities, that is there is a map

$$L \cup R \xrightarrow{m} \mathbb{N}_+$$

such that $c_i = m(l_i)a_z$ if $l(l_i) = v_z$ and $d_j = m(r_j)a_z$ if $r(r_j) = v_z$.

• There can only be an arrow $\textcircled{l_i} \longrightarrow \textcircled{r_j}$ if for $v_k = l(l_i)$ and $v_l = r(r_i)$ there is an oriented path



in Q allowing the trivial path and loops if $v_k = v_l$.

• Every left vertex l_i is the source of exactly c_i arrows in Q' and every right-vertex r_j is the sink of precisely d_j arrows in Q'.

• Consider the $u \times u$ matrix where $u = \sum_i c_i = \sum_j d_j$ (both numbers are equal to the total number of arrows in Q') where the i-th row contains the entries of the i-th arrow in Q' with respect to the obvious left and right bases. Observe that this is a $GL(\beta)$ semi-invariant on $\mathtt{rep}_\beta\, Q'$ with weight determined by the integral $k+l$-tuple $(-1,\ldots,-1;1,\ldots,1)$. If we fix for every arrow a from l_i to r_j in Q' an $m(r_j) \times m(l_i)$ matrix p_a of linear combinations of paths in Q from $l(l_i)$ to $r(r_j)$, we obtain a morphism

$$\mathtt{rep}_\alpha\ Q \longrightarrow \mathtt{rep}_\beta\ Q'$$

sending a representation $V \in \mathtt{rep}_\alpha\ Q$ to the representation W of Q' defined by $W_a = p_a(V)$. Composing this map with the above semi-invariant we obtain a $GL(\alpha)$ semi-invariant of $\mathtt{rep}_\alpha\ Q$ with weight determined by the k-tuple $\theta = (t_1, \ldots, t_k)$ where

$$t_i = \sum_{j \in r^{-1}(v_i)} m(r_j) - \sum_{j \in l^{-1}(v_i)} m(l_j)$$

. We call such semi-invariants *standard determinantal* .

THEOREM 119 (Schofield-Van den Bergh). *The semi-invariants of the $GL(\alpha)$-action on $\mathtt{rep}_\alpha\ Q$ are generated by traces of oriented cycles and by standard determinantal semi-invariants.*

PROOF. See [**62**]. Observe that analogous descriptions of the semi-invariants were obtained in [**13**] and [**15**]. □

We will now clarify the relationship between θ-stable representations and simple representations.

THEOREM 120. *For $V \in \mathtt{rep}_\alpha\ Q$ the following are equivalent*

(1) *$V \in \mathtt{ress}_\alpha(Q, \theta)$, that is, V is θ-semistable.*
(2) *There is a universal localization $\langle Q \rangle_\sigma$ such that $\langle Q \rangle_\sigma \otimes V$ is a simple α-dimensional representation of $\langle Q \rangle_\sigma$.*

PROOF. Fix a character $\theta = (t_1, \ldots, t_k)$ and divide the set of vertex-indices into a left set $L = \{i_1, \ldots, i_u\}$ consisting of those $1 \leq i \leq k$ such that $t_i \leq 0$ and a right set $R = \{j_1, \ldots, j_v\}$ consisting of those $1 \leq j \leq k$ such that $t_j \geq 0$ (observe that $L \cap R$ may be non-empty). For every vertex v_i we consider the indecomposable projective module $P_i = \langle Q \rangle e_i$ spanned by all paths in the quiver Q starting at v_i. As a consequence we have that $Hom_{\langle Q \rangle}(P_i, P_j)$ is spanned by all paths $[j, i]$ in the quiver Q from vertex v_j to vertex i. For a fixed integer n we consider the set $\Sigma_\theta(n)$ of all $\langle Q \rangle$-module morphisms

$$\begin{array}{ccc} P_{i_1}^{\oplus -nt_{i_1}} \oplus \ldots \oplus P_{i_u}^{\oplus -nt_{i_u}} & \xrightarrow{\sigma} & P_{j_1}^{\oplus nt_{j_1}} \oplus \ldots \oplus P_{j_v}^{\oplus nt_{j_v}} \\ \|_{\text{notation}} & & \|_{\text{notation}} \\ P_{c_1} \oplus \ldots \oplus P_{c_p} & \xrightarrow{\quad\sigma\quad} & P_{d_1} \oplus \ldots \oplus P_{d_q} \end{array}$$

By the remark above, σ can be described by an $(p = n \sum t_{i_l}) \times (q = n \sum t_{j_m})$ matrix M_σ all entries of which are linear combinations p_{lm} of paths in the quiver Q from vertex v_{d_m} to vertex v_{c_l}. For $V \in \mathtt{rep}_\alpha\ Q$ we can substitute the arrow matrices $V(a)$ in the definition of p_{lm} and obtain a square matrix of size $a_{c_l} \times a_{d_m}$. If we do this for every entry of σ we obtain a square matrix as $\theta(V) = 0$ which we denote by $\sigma(V)$. But then, the function

$$d_\sigma(V) = det\ \sigma(V)\ :\ \mathtt{rep}_\alpha\ Q \longrightarrow \mathbb{C}$$

is a semi-invariant of weight $n\theta$. By the foregoing theorem, all semi-invariants in $\mathbb{C}[\mathtt{rep}_\alpha\ Q]^{GL(\alpha), \chi_\theta^n}$ are spanned by such determinantal semi-invariants. We define

$$X_\sigma(\alpha) = \{V \in \mathtt{rep}_\alpha\ Q \mid d_\sigma(V) \neq 0\}$$

Because d_σ is a semi-invariant of weight $n\theta$ it follows that $X_\sigma(\alpha)$ consists of θ-semistable representations. Also remark that $X_\sigma(\alpha)$ is the variety of α-dimensional representations of the universal localization $\langle Q \rangle_\sigma$.
(1) $\Rightarrow$ (2) : Let V be θ-stable and assume that $W \subset V$ is a proper sub $\langle Q \rangle_\sigma$-module of dimension vector β. Restricting W to a representation of Q we see that $W \in \mathtt{rep}_\beta\ Q$ and is a subrepresentation of V. Because $W \in \mathtt{rep}_\beta\ \langle Q \rangle_\sigma$ we have $\theta(\beta) = \theta(W) = 0$, which is impossible as V is θ-stable.
Let $\langle Q \rangle_\sigma \otimes V$ be a simple $\langle Q \rangle_\sigma$ representation and assume that $W \subset V$ is a proper subrepresentation of dimension vector $\beta = (b_1, \ldots, b_k)$ with $\theta(W) \leq 0$. If $\theta(W) < 0$ then $-\sum nt_{i_l} b_{i_l} > \sum nt_{j_m} b_{j_m}$ whence $\sigma(W)$ has a kernel but this contradicts the fact that $\sigma(V)$ is invertible. Hence, $\theta(W) = 0$ but then $\langle Q \rangle_\sigma \otimes W$ is a proper subrepresentation of $\langle Q \rangle_\sigma \otimes V$ contradicting simplicity. □

EXAMPLE 156. The quotient map π_θ is locally isomorphic to the quotient map

$$\begin{array}{ccc} \mathtt{rep}_\alpha\ \langle Q \rangle_\sigma = X_\sigma(\alpha) & \hookrightarrow & \mathtt{ress}_\alpha(Q,\theta) \\ \downarrow{\scriptstyle \pi_\sigma} & & \downarrow{\scriptstyle \pi_\theta} \\ \mathtt{iss}_\alpha\ \langle Q \rangle_\sigma = X_\sigma(\alpha)/GL(\alpha) & \hookrightarrow & \mathtt{moss}_\alpha(Q,\alpha) \end{array}$$

assigning to an α-dimensional $\langle Q \rangle_\sigma$-module its semi-simplification, that is, the direct sum of its Jordan-Hölder components. Because the affine open sets $X_\sigma(\alpha)$ cover $\mathtt{ress}_\alpha(Q,\theta)$, the moduli space of θ-semistable quiver representations $\mathtt{moss}_\alpha(Q,\alpha)$ is locally isomorphic to quotient varieties $\mathtt{iss}_\alpha\ \langle Q \rangle_\sigma$ for specific universal localizations of the path algebra, all of which are affine $\mathtt{alg}$-smooth algebras.

We now consider the étale local structure of the moduli spaces $\mathtt{moss}_\alpha(Q,\theta)$. As a consequence we will determine their singular loci.

DEFINITION 111. Let $\xi \in \mathtt{moss}_\alpha(Q,\alpha)$ be a geometric point of *semistable representation type* $\tau = (m_1, \beta_1; \ldots; m_l, \beta_l)$. That is, the unique closed orbit lying in the fiber $\pi_\theta^{-1}(\xi)$ is the isomorphism class of a direct sum

$$V_\xi = W_1^{\oplus m_1} \oplus \ldots \oplus W_l^{\oplus m_l}$$

with W_i a θ-stable representation of dimension vector β_i.
The *local quiver setting* (Q_ξ, α_ξ) associated to ξ is defined as follows :

- Q_ξ has l vertices $w_1, \ldots, w_l$ corresponding to the distinct θ-stable components of V_ξ, and
- the number of arrows from w_i to w_j is equal to

  $$\delta_{ij} - \chi_Q(\beta_i, \beta_j)$$

  where χ_Q is the Euler form of Q.
- the dimension vector $\alpha_\xi = (m_1, \ldots, m_l)$ gives the multiplicities of the stable summands.

Observe that the local quiver setting depends only on the semistable representation type.

THEOREM 121. *There is an étale isomorphism between*

1. *an affine neighborhood of ξ in the moduli space $\mathtt{moss}_\alpha(Q,\alpha)$, and*

(2) *an affine neighborhood of the image $\overline{0}$ of the zero representation in the quotient variety* $\mathtt{iss}_{\alpha_\xi}\ Q_\xi$ *corresponding to the local quiver setting* (Q_ξ, α_ξ).

Therefore, ξ is a smooth point of the moduli space $\mathtt{moss}_\alpha(Q, \theta)$ *if and only if the quiver setting* (Q_ξ, α_ξ) *satisfies the requirements of theorem 99.*

Proof. Let $\xi \in \mathtt{moss}_\alpha(Q, \alpha)$ with corresponding V_ξ having a decomposition into θ-stable representations W_i as above. We may assume that $V_\xi \in X_\sigma(\alpha)$ where $X_\sigma(\alpha)$ is the affine $GL(\alpha)$-invariant open subvariety of $\mathtt{ress}_\alpha(Q, \theta)$ defined by the determinantal semi-invariant d_σ. We have seen that $X_\sigma(\alpha) \simeq \mathtt{rep}_\alpha\ \langle Q \rangle_\sigma$ the variety of α-dimensional representations of the universal localization $\langle Q \rangle_\sigma$. Moreover, if we define $V'_\xi = \langle Q \rangle_\sigma \otimes V_\xi$ and $W'_i = \langle Q \rangle_\sigma \otimes W_i$ we have

$$V'_\xi = W_1'^{\oplus m_1} \oplus \ldots \oplus W_l'^{\oplus m_l}$$

is a decomposition of the semisimple $\mathbb{C}Q_\sigma$ representation V'_ξ into its simple components W'_i. Restricting to the affine smooth variety $X_\sigma(\alpha)$ we are in a situation to apply the Luna slice theorem to the representation scheme of the $\mathtt{alg}$-smooth algebra $\langle Q \rangle_\sigma$ as before.

The normal space to the orbit can be identified with the self-extensions

$$N_{V'_\xi} = Ext^1_{\langle Q \rangle_\sigma}(V'_\xi, V'_\xi) = \oplus_{i,j=1}^{l}\ Ext^1_{\langle Q \rangle_\sigma}(W'_i, W'_j)^{\oplus m_i m_j}$$

By Schur's lemma we know that the stabilizer subgroup of the semisimple module V'_ξ is equal to $GL(\alpha_\xi)$ and if we write out the action of this group on the self extensions we observe that it coincides with the action of the basechangegroup $GL(\alpha_\xi)$ on the representation space $\mathtt{rep}_{\alpha_\xi}\ \Gamma$ of a quiver Γ on l vertices such that the number of arrows from vertex w_i to vertex w_j is equal to the dimension of the extension group

$$Ext^1_{\langle Q \rangle_\sigma}(W'_i, W'_j)$$

Remains to prove that the quiver Γ is our local quiver Q_ξ. For this we apply a general homological result valid for universal localizations, [**60**, Thm 4.7]. If A_σ is a universal localization of an algebra A, then the category of left A_σ-modules is closed under extensions in the category of left A-modules. Therefore,

$$Ext^1_{A_\sigma}(M, N) = Ext^1_A(M, N)$$

for A_σ-modules M and N. Therefore,

$$Ext^1_{\langle Q \rangle_\sigma}(W'_i, W'_j) = Ext^1_{\langle Q \rangle}(W_i, W_j)$$

Further, as the W_i are θ-stable representations of the quiver Q we know that $Hom_{\langle Q \rangle}(W_i, W_j) = \delta_{ij}\mathbb{C}$. Finally, we use the homological interpretation of the Euler form

$$\chi_Q(\beta_i, \beta_j) = dim_{\mathbb{C}}\ Hom_{\langle Q \rangle}(W_i, W_j) - dim_{\mathbb{C}}\ Ext^1_{\langle Q \rangle}(W_i, W_j)$$

to deduce that $\Gamma = Q_\xi$. The last statement follows by étale descent. □

Example 157. The foregoing theorem can be used to determine the dimension vectors of θ-stable representations lying in the positive linear span of a set of dimension vectors of θ-stables.

Let $\beta_1, \ldots, \beta_l$ be dimension vectors of θ-stable representations of Q and assume there are integers $m_1, \ldots, m_l \geq 0$ such that

$$\alpha = m_1\beta_1 + \ldots + m_l\beta_l$$

Then, α is the dimension vector of a θ-stable representation of Q if and only if $\alpha' = (m_1, \ldots, m_l)$ is the dimension vector of a simple representation of the quiver Q' on l vertices $w_1, \ldots, w_l$ such that there are exactly

$$\delta_{ij} - \chi_Q(\beta_i, \beta_j)$$

arrows from w_i to w_j.

For, let W_i be a θ-stable representation of Q of dimension vector β_i and consider the α-dimensional representation

$$V = W_1^{\oplus m_1} \oplus \ldots \oplus W_l^{\oplus m_l}$$

It is clear from the definition that (Q', α') is the local quiver setting corresponding to V. As before, there is a semi-invariant d_σ such that $V \in X_\sigma(\alpha) = \mathtt{rep}_\alpha \langle Q \rangle_\sigma$ and $\langle Q \rangle_\sigma \otimes V$ is a semi-simple representation of the universal localization $\mathbb{C}Q_\sigma$. If there are θ-stable representations of dimension α, then there is an open subset of $X_\sigma(\alpha)$ consisting of θ-stable representations. But we have seen that they become simple representations of $\langle Q \rangle_\sigma$. This means that every Zariski neighborhood of $V \in \mathtt{rep}_\alpha \langle Q \rangle_\sigma$ contains simple α-dimensional representations. By the étale local isomorphism there are α'-dimensional simple representations of the quiver Q'.

Conversely, as any Zariski neighborhood of the zero representation in $\mathtt{rep}_{\alpha'} Q'$ contains simple representations, then so does any neighborhood of $V \in \mathtt{rep}_\alpha \langle Q \rangle_\sigma = X_\sigma(\alpha)$. We have seen before that $X_\sigma(\alpha)$ consists of θ-semistable representations and that the θ-stables correspond to the simple representations of $\langle Q \rangle_\sigma$, whence Q has θ-stable representations of dimension vector α. By theorem 85 the set of θ-stable dimension vectors can be described by a set of inequalities. For more results along similar lines we refer the reader to the recent preprint [**14**] of H. Derksen and J. Weyman.

7.3. Nullcones of quiverrepresentations.

In this last section we will conclude our approach to the study of $\mathtt{iso}(\mathtt{rep}A)$ for A an $\mathtt{alg}$-smooth algebra. Recall that if $\xi \in \mathtt{iss}_\alpha A$ corresponds to the semi-simple representation M_ξ, then the isomorphism classes of all representations $M \in \mathtt{rep}_\alpha A$ having Jordan-Hölder semisimplification M_ξ are the orbits in the fiber of the quotient map

$$\mathtt{rep}_\alpha A \xrightarrow{\pi} \mathtt{iss}_\alpha A$$

Using the results on the étale local structure, we know that as GL_n-varieties

$$\pi^{-1}(\xi) \simeq GL_n \times^{GL(\alpha_\xi)} \mathtt{null}_{\alpha_\xi} Q_\xi$$

Therefore, GL_n-orbits in the fiber correspond one-to-one to GL_n-orbits in the fiber bundle which, in turn, correspond one-to-one with $GL(\alpha_\xi)$-orbits in the nullcone $\mathtt{null}_{\alpha_\xi} Q_\xi$.

We will apply general results on nullcones due to Wim Hesselink [**23**] and Frances Kirwan [**30**] to give a representation theoretic description of nullcones of quiver-representations. First, we will outline the basic ideas in the case of the free algebra $\langle m \rangle$ after which the passage to the general case is merely a notational problem.

Example 158. (The generic case) We will outline the basic idea of the Hesselink stratification of the nullcone [**23**] in the generic case, that is, the action of GL_n by simultaneous conjugation on m-tuples of matrices $M_n^m = M_n \oplus \ldots \oplus M_n$. With $\mathtt{null}_n^m$ we denote the nullcone of this action

$$\mathtt{null}_n^m = \{x = (A_1, \ldots, A_m) \in M_n^m \quad | \quad \underline{0} = (0, \ldots, 0) \in \overline{\mathcal{O}(x)}\}$$

From the Hilbert criterium, theorem 51, we recall that $x = (A_1, \ldots, A_m)$ belongs to the nullcone if and only if there is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that

$$\lim_{t \to 0} \lambda(t).(A_1, \ldots, A_m) = (0, \ldots, 0).$$

Any one-parameter subgroup of GL_n is conjugated to one determined by an integral n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ and permuting the basis if necessary, we can conjugate this λ to one where the n-tuple if *dominant* , that is, $r_1 \geq r_2 \geq \ldots \geq r_n$. By applying *permutation Jordan-moves* , that is, by simultaneously interchanging certain rows and columns in all A_i, we may therefore assume that the limit-formula holds for a dominant one-parameter subgroup λ of the maximal torus

$$T_n \simeq \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{n} = \{ \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} \quad | \quad c_i \in \mathbb{C}^* \ \} \hookrightarrow GL_n$$

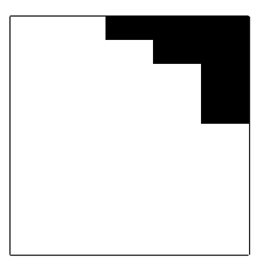
of GL_n. Computing its action on a $n \times n$ matrix A we obtain

$$\begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} t^{-r_1} & & 0 \\ & \ddots & \\ 0 & & t^{-r_n} \end{bmatrix} = \begin{bmatrix} t^{r_1 - r_1} a_{11} & \ldots & t^{r_1 - r_n} a_{1n} \\ \vdots & & \vdots \\ t^{r_n - r_1} a_{n1} & \ldots & t^{r_n - r_n} a_{nn} \end{bmatrix}$$

By dominance $r_i \leq r_j$ for $i \geq j$, the limit is defined only if $a_{ij} = 0$ for $i \geq j$, that is, when A is a strictly upper triangular matrix.

Any m-tuple $x = (A_1, \ldots, A_m) \in \mathtt{null}_n^m$ has a point in its orbit $\mathcal{O}(x)$, $x' = (A'_1, \ldots, A'_m)$ with all A'_i strictly upper triangular matrices. In fact permutation Jordan-moves suffice to arrive at x'.

For specific m-tuples $x = (A_1, \ldots, A_m)$ it might be possible to improve on this result. That is, we want to determine the smallest 'corner' C in the upper right hand corner of the matrix, such that all the component matrices A_i can be conjugated simultaneously to matrices A'_i having only non-zero entries in the corner C

$C =$ 

and no strictly smaller corner C' can be found with this property. We want to compile a list of the relevant corners and to define an order relation on this set.

Consider the *weight space decomposition* of M_n^m for the action by simultaneous conjugation of the maximal torus T_n,

$$M_n^m = \oplus_{1 \leq i,j \leq n} M_n^m(\pi_i - \pi_j) = \oplus_{1 \leq i,j \leq n} \mathbb{C}^{\oplus m}_{\pi_i - \pi_j}$$

where $c = diag(c_1, \ldots, c_n) \in T_m$ acts on any element of $M_n^m(\pi_i - \pi_j)$ by multiplication with $c_i c_j^{-1}$, that is, the eigenspace $M_n^m(\pi_i - \pi_j)$ is the space of the (i,j)-entries of the m-matrices. We call

$$\mathcal{W} = \{\pi_i - \pi_j \mid 1 \leq i, j \leq n\}$$

the set of T_n-*weights* of M_n^m. Let $x = (A_1, \ldots, A_m) \in \mathtt{null}_n^m$ and consider the subset $E_x \subset \mathcal{W}$ consisting of the elements $\pi_i - \pi_j$ such that for at least one of the matrix components A_k the (i,j)-entry is non-zero. Repeating the argument above, we see that if λ is a one-parameter subgroup of T_n determined by the integral n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ such that $lim\ \lambda(t).x = \underline{0}$ we have

$$\forall\ \pi_i - \pi_j \in E_x \quad \text{we have} \quad r_i - r_j \geq 1$$

Conversely, let $E \subset \mathcal{W}$ be a subset of weights, we want to determine the subset

$$\{s = (s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_i - s_j \geq 1\ \forall\ \pi_i - \pi_j \in E\ \}$$

and determine a point in this set, minimal with respect to the usual norm

$$\parallel s \parallel = \sqrt{s_1^2 + \ldots + s_n^2}$$

Let $s = (s_1, \ldots, s_n)$ attain such a minimum. We can partition the entries of s in a disjoint union of *strings*

$$\{p_i, p_i + 1, \ldots, p_i + k_i\}$$

with $k_i \in \mathbb{N}$ and subject to the condition that all the numbers $p_{ij} \overset{def}{=} p_i + j$ with $0 \leq j \leq k_i$ occur as components of s, possibly with a multiplicity that we denote by a_{ij}. We call a string $string_i = \{p_i, p_i + 1, \ldots, p_i + k_i\}$ of s *balanced* if and only if

$$\sum_{s_k \in string_i} s_j = \sum_{j=0}^{k_i} a_{ij}(p_i + j) = 0$$

In particular, all balanced strings consists entirely of rational numbers. We claim

Let $E \subset \mathcal{W}$, then the subset of $\mathbb{R}^n$ determined by

$$\mathbb{R}_E^n = \{\ (r_1, \ldots, r_n) \mid r_i - r_j \geq 1\ \forall\ \pi_i - \pi_j \in E\}$$

has a unique point $s_E = (s_1, \ldots, s_n)$ of minimal norm $\parallel s_E \parallel$. This point is determined by the characteristic feature that all its strings are balanced. In particular, $s_E \in \mathbb{Q}^n$.

Let s be a minimal point for the norm in $\mathbb{R}_E^n$ and consider a string of s and denote with S the indices $k \in \{1, \ldots, n\}$ such that $s_k \in string$. Let $\pi_i - \pi_j \in E$, then if only one of i or j belongs to S we have a strictly positive number a_{ij}

$$s_i - s_j = 1 + r_{ij} \quad \text{with} \quad r_{ij} > 0$$

Take $\epsilon_0 > 0$ smaller than all r_{ij} and consider the n-tuple

$$s_\epsilon = s + \epsilon(\delta_{1S}, \ldots, \delta_{nS}) \quad \text{with} \quad \delta_{kS} = 1 \text{ if } k \in S \text{ and } 0 \text{ otherwise}$$

with $\mid \epsilon \mid \leq \epsilon_0$. Then, $s_\epsilon \in \mathbb{R}_E^n$ for if $\pi_i - \pi_j \in E$ and i and j both belong to S or both do not belong to S then $(s_\epsilon)_i - (s_\epsilon)_j = s_i - s_j \geq 1$ and if one of i or j belong to S, then

$$(s_\epsilon)_i - (s_\epsilon)_j = 1 + r_{ij} \pm \epsilon \geq 1$$

by the choice of ϵ_0. However, the norm of s_ϵ is

$$\| s_\epsilon \| = \sqrt{\| s \| + 2\epsilon \sum_{k \in S} s_k + \epsilon^2 \# S}$$

Hence, if the string would not be balanced, $\sum_{k\in S} s_k \neq 0$ and we can choose ϵ small enough such that $\| s_\epsilon \| < \| s \|$, contradicting minimality of s and proving the claim.

For given n we have an `algorithm` to compile the list $\mathcal{S}_n$ of all dominant n-tuples $(s_1, \ldots, s_n)$ having all its strings balanced.

- List all Young-diagrams $\mathcal{Y}_n = \{Y_1, \ldots\}$ having $\leq n$ boxes.
- For every diagram Y_l fill the boxes with strictly positive integers subject to the rules : (1) the total sum is equal to n, (2) no two rows are filled identically and (3) at most one row has length 1. This gives a list $\mathcal{T}_n = \{T_1, \ldots\}$ of tableaux.
- For every tableau $T_l \in \mathcal{T}_n$, for each of its rows $(a_1, a_2, \ldots, a_k)$ find a solution p to the linear equation

$$a_1 x + a_2(x+1) + \ldots + a_k(x+k) = 0$$

and define the $\sum a_i$-tuple of rational numbers

$$(\underbrace{p, \ldots, p}_{a_1}, \underbrace{p+1, \ldots, p+1}_{a_2}, \ldots \underbrace{p+k, \ldots, p+k}_{a_k})$$

Repeating this process for every row of T_l we obtain an n-tuple, which we then order.

The list $\mathcal{S}_n$ will be the combinatorial object underlying the relevant corners and the stratification of the nullcone. To every $s = (s_1, \ldots, s_n) \in \mathcal{S}_n$ we associate the following data

- The *corner* C_s is the subspace of M_n^m consisting of those m tuples of $n \times n$ matrices with zero entries except perhaps at position (i,j) where $s_i - s_j \geq 1$. A partial ordering is defined on these corners by the rule

$$C_{s'} < C_s \Leftrightarrow \| s' \| < \| s \|$$

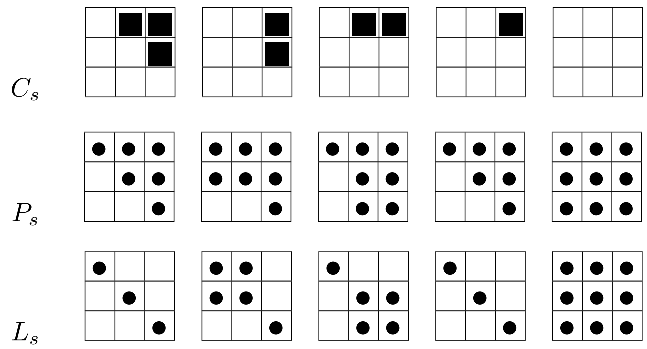
- The *parabolic subgroup* P_s which is the subgroup of GL_n consisting of matrices with zero entries except perhaps at entry (i,j) when $s_i - s_j \geq 0$.
- The *Levi subgroup* L_s which is the subgroup of GL_n consisting of matrices with zero entries except perhaps at entry (i,j) when $s_i - s_j = 0$. Observe that $L_s = \prod GL_{a_{ij}}$ where the a_{ij} are the multiplicities of $p_i + j$.

For example, $\mathcal{S}_3$ has five types described by

$\mathcal{S}_3 =$

tableau	s_1	s_2	s_3	$\| s \|^2$
[1][1][1]	1	0	-1	2
[1][2]	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
[2][1]	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
[1][1] / [1]	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
[3]	0	0	0	0

The corresponding corners, parabolic and Levi subgroups are respectively,



For $x = (A_1, \ldots, A_m) \in \mathtt{null}_n^m$, $E_x \subset \mathcal{W}$ determines a unique $s_{E_x} \in \mathbb{Q}^n$ which up to permuting the entries an element s of $\mathcal{S}_n$. Therefore,

Every $x = (A_1, \ldots, A_m) \in \mathtt{null}_n^m$ can be brought by permutation Jordan-moves to an m-tuple $x' = (A'_1, \ldots, A'_m) \in C_s$. Here, s is the dominant reordering of s_{E_x} with $E_x \subset \mathcal{W}$ the subset $\pi_i - \pi_j$ determined by the non-zero entries at place (i,j) of one of the components A_k. The permutation of rows and columns is determined by the dominant reordering.

The m-tuple s (or s_{E_x}) determines a one-parameter subgroup λ_s of T_n where λ corresponds to the unique n-tuple of integers

$$(r_1, \ldots, r_n) \in \mathbb{N}_+ s \cap \mathbb{Z}^n \quad \text{with} \quad gcd(r_i) = 1$$

For any one-parameter subgroup μ of T_n determined by an integral n-tuple $\mu = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and any $x = (A_1, \ldots, A_n) \in \mathtt{null}_n^m$ we define the integer

$$m(x, \mu) = min \; \{ a_i - a_j \;\mid\; x \text{ contains a non-zero entry in } M_n^m(\pi_i - \pi_j) \; \}$$

From the definition of $\mathbb{R}_E^n$ it follows that the minimal value s_E and λ_{s_E} is

$$s_{E_x} = \frac{\lambda_{s_{E_x}}}{m(x, \lambda_{s_{E_x}})} \quad \text{and} \quad s = \frac{\lambda_s}{m(x, \lambda_s)}$$

We claim :

Let $x = (A_1, \ldots, A_m) \in \mathtt{null}_n^m$ and let μ be a one-parameter subgroup contained in T_n such that $\lim_{t \rightarrow 0} \lambda(t).x = \underline{0}$, then

$$\frac{\parallel \lambda_{s_{E_x}} \parallel}{m(x, \lambda_{s_{E_x}})} \leq \frac{\parallel \mu \parallel}{m(x, \mu)}$$

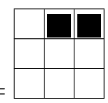
This follows immediately from the observation that $\frac{\mu}{m(x,\mu)} \in \mathbb{R}_{E_x}^n$ and the minimality of s_{E_x}. Phrased differently, there is no simultaneous reordering of rows and columns that admit an m-tuple $x" = (A"_1, \ldots, A"_m) \in C_{s'}$ for a corner $C_{s'} < C_s$.

EXAMPLE 159. It is possible that another point in the orbit $\mathcal{O}(x)$ say $y = g.x = (B_1, \ldots, B_m)$ can be transformed by permutation Jordan moves in a strictly smaller corner.

Consider one 3×3 nilpotent matrix of the form

$$x = \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad ab \neq 0$$

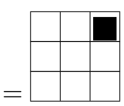
Then, $E_x = \{\pi_1 - \pi_2, \pi_1 - \pi_3\}$ and the corresponding $s = s_{E_x} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ so x is clearly of corner type

$C_s =$ 

However, x is a nilpotent matrix of rank 1 and by the Jordan-normalform we can conjugate it in standard form, that is, there is some $g \in GL_3$ such that

$$y = g.x = gxg^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For this y we have $E_y = \{\pi_1 - \pi_2\}$ and the corresponding $s_{E_y} = (\frac{1}{2}, -\frac{1}{2}, 0)$, which can be brought into standard dominant form $s' = (\frac{1}{2}, 0, -\frac{1}{2})$ by interchanging the two last entries. Hence, by interchanging the last two rows and columns, y is indeed of corner type

$C_{s'} =$ 

and we have that $C_{s'} < C_s$. Observe that we used the Jordan-normalform to produce this example. As there are no known canonical forms for m tuples of $n \times n$ matrices, it is a more difficult to determine the optimal corner type of an element in $\texttt{null}_n^m$.

Definition 112. Let $s \in \mathcal{S}_n$ be determined by the tableau T_s. The *associated quiver-setting* (Q_s, α_s) and character θ_s are defined as follows.

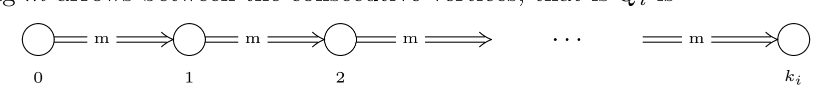
The quiver Q_s has as many connected components as there are rows in the tableau T_s. If the i-th row in T_s is

$$(a_{i0}, a_{i1}, \ldots, a_{ik_i})$$

then the corresponding string of entries in s is of the form

$$\{\underbrace{p_i, \ldots, p_i}_{a_{i0}}, \underbrace{p_i + 1, \ldots, p_i + 1}_{a_{i1}}, \ldots, \underbrace{p_i + k_i, \ldots, p_i + k_i}_{a_{ik_i}}\}$$

and the i-th component of Q_s is defined to be the quiver Q_i on $k_i + 1$ vertices having m arrows between the consecutive vertices, that is Q_i is



The dimension vector α_i for the i-th component quiver Q_i is equal to the i-th row of the tableau T_s, that is

$$\alpha_i = (a_{i0}, a_{i1}, \ldots, a_{ik_i})$$

and the total dimension vector α_s is the collection of these component dimension vectors.

The character $GL(\alpha_s) \xrightarrow{\chi_s} \mathbb{C}^*$ is determined by the integral n-tuple $\theta_s = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ where if entry k corresponds to the j-th vertex of the i-th component of Q_s we have

$$t_k = n_{ij} \overset{\text{def}}{=} d.(p_i + j)$$

where d is the least common multiple of the numerators of the p_i's for all i. Equivalently, the n_{ij} are the integers appearing in the description of the one-parameter subgroup $\lambda_s = (r_1, \ldots, r_n)$ grouped together according to the ordering of vertices in the quiver Q_s. Recall that the character χ_s is then defined to be

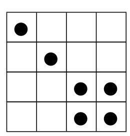
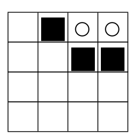
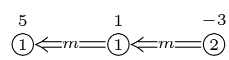
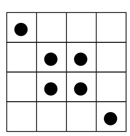
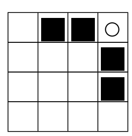
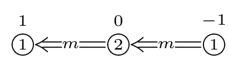
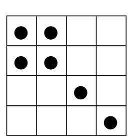
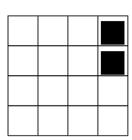
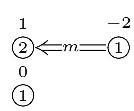
$$\chi_s(g_1 \ldots, g_n) = \prod_{i=1}^{n} \; det(g_i)^{t_i}$$

or in terms of $GL(\alpha_s)$ it sends an element $g_{ij} \in GL(\alpha_s)$ to $\prod_{i,j} \; det(g_{ij})^{n_{ij}}$.

Example 160. Define the *border* B_s to be the subspace of C_s consisting of those m-tuples of $n \times n$ matrices with zero entries except perhaps at entries (i,j) where $s_i - s_j = 1$. Observe that the action of the Levi-subgroup $L_s = \prod_{i,j} GL_{a_{ij}}$ on the border B_s coincides with the base-change action of $GL(\alpha_s)$ on the representation space $\mathtt{rep}_{\alpha_s} \; Q_s$. The isomorphism

$$B_s \longrightarrow \mathtt{rep}_{\alpha_s} \; Q_s$$

is given by sending an m-tuple of border B_s-matrices $(A_1, \ldots, A_m)$ to the representation in $\mathtt{rep}_{\alpha_s} \; Q_s$ where the j-th arrow between the vertices v_a and v_{a+1} of the i-th component quiver Q_i is given by the relevant block in the matrix A_j. We illustrate this with a few examples from 4×4 matrices.

tableau	L_s	B_s	θ_s	$(Q_s, \alpha_s, \theta_s)$
2 1 1			$(5, 1, -3, -3)$	
1 2 1			$(1, 0, 0, -1)$	
1 2 1			$(1, 1, 0, -2)$	

Theorem 122. *Let $x = (A_1, \ldots, A_m) \in \mathtt{null}_n^m$ be of corner type C_s. Then, x is of optimal corner type C_s, that is, there is no point $y = g.x \in \mathcal{O}(x)$ having corner type $C_{s'}$ with $C_{s'} < C_s$, if and only if under the natural maps*

$$C_s \longrightarrow\!\!\!\!\!\rightarrow B_s \xrightarrow{\simeq} \mathtt{rep}_{\alpha_s} \; Q_s$$

(the first map forgets the non-border entries) x is mapped to a θ_s-semistable representation in $\mathtt{rep}_{\alpha_s} \; Q_s$.

PROOF. This is a specialization of the description due to Kirwan [**30**]. □

If $N \subset M_n$ is the subspace of strictly upper triangular matrices, then the action map determines a surjection

$$GL_n \times N^m \xrightarrow{ac} \mathtt{null}_n^m$$

Recall that the standard *Borel subgroup* B is the subgroup of GL_n consisting of all upper triangular matrices and consider the action of B on $GL_n \times M_n^m$ determined by

$$b.(g,x) = (gb^{-1}, b.x)$$

Then, B-orbits in $GL_n \times N^m$ are mapped under the action map ac to the same point in the nullcone $\mathtt{null}_n^m$. Consider the morphisms

$$GL_n \times M_n^m \xrightarrow{\pi} GL_n/B \times M_n^m$$

which sends a point (g,x) to $(gB, g.x)$. The quotient GL_n/B is called a *flag variety* and is a projective manifold. Its points are easily seen to correspond to complete *flags*

$$\mathcal{F} \;:\; 0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n \quad \text{with} \quad dim_{\mathbb{C}}\ F_i = i$$

of subspaces of $\mathbb{C}^n$. Consider the fiber π^{-1} of a point $(\overline{g}, (B_1, \ldots, B_m)) \in GL_n/B \times M_n^m$. These are the points

$$(h, (A_1, \ldots, A_m)) \quad \text{such that} \quad \begin{cases} g^{-1}h & = b \in B \\ bA_ib^{-1} & = g^{-1}B_ig \quad \text{for all } 1 \leq i \leq m. \end{cases}$$

Therefore, the fibers of π are precisely the B-orbits in $GL_n \times M_n^m$. That is, there exists a quotient variety for the B-action on $GL_n \times M_n^m$ which is the trivial vectorbundle of rank mn^2

$$\mathcal{T} = GL_n/B \times M_n^m \xrightarrow{p} GL_n/B$$

over the flag variety GL_n/B. We will denote with $GL_n \times^B N^m$ the image of the subvariety $GL_n \times N^m$ of $GL_n \times M_n^m$ under this quotient map. That is, we have a commuting diagram

$$\begin{array}{ccc} GL_n \times N^m & \hookrightarrow & GL_n \times M_n^m \\ \downarrow & & \downarrow \\ GL_n \times^B N^m & \hookrightarrow & GL_n/B \times M_n^m \end{array}$$

Hence, $\mathcal{V} = GL_n \times^B N^m$ is a sub-bundle of rank $m.\frac{n(n-1)}{2}$ of the trivial bundle $\mathcal{T}$ over the flag variety. Note however that $\mathcal{V}$ itself is not trivial as the action of GL_n does not map N^m to itself.

THEOREM 123. *Let U be the open subvariety of m-tuples of strictly upper triangular matrices N^m consisting of those tuples such that one of the component matrices has rank $n-1$. The action map ac induces the commuting diagram of figure 6. The upper map is an isomorphism of GL_n-varieties for the action on fiber bundles to be left multiplication in the first component.*

$$\begin{array}{ccc} GL_n \times^B U & \xrightarrow{\simeq} & GL_n.U \\ \big\downarrow & & \big\downarrow \\ GL_n \times^B N^m & \xrightarrow{ac} & \mathtt{null}_n^m \end{array}$$

FIGURE 6. Resolution of the nullcone.

Therefore, there is a natural one-to-one correspondence between GL_n-orbits in $GL_n.U$ and B-orbits in U. Further, ac is a desingularization of the nullcone and $\mathtt{null}_n^m$ is irreducible of dimension

$$(m+1)\frac{n(n-1)}{2}.$$

PROOF. Let $A \in N$ be a strictly upper triangular matrix of rank $n-1$ and $g \in GL_n$ such that $gAg^{-1} \in N$, then $g \in B$ as one verifies by first bringing A into Jordan-normal form $J_n(0)$. This implies that over a point $x = (A_1, \ldots, A_m) \in U$ the fiber of the action map

$$GL_n \times N^m \xrightarrow{ac} \mathtt{null}_n^m$$

has dimension $\frac{n(n-1)}{2} = dim\ B$. Over all other points the fiber has at least dimension $\frac{n(n-1)}{2}$.But then, by the dimension formula we have

$$dim\ \mathtt{null}_n^m = dim\ GL_n + dim\ N^m - dim\ B = (m+1)\frac{n(n-1)}{2}$$

Over $GL_n.U$ this map is an isomorphism of GL_n-varieties. Irreducibility of $\mathtt{null}_n^m$ follows from surjectivity of ac as $\mathbb{C}[\mathtt{null}_n^m] \subset \mathbb{C}[GL_n] \otimes \mathbb{C}[N^m]$ and the latter is a domain. These facts imply that the induced action map

$$GL_n \times^B N^m \xrightarrow{ac} \mathtt{null}_n^m$$

is birational and as the former is a smooth variety (being a vectorbundle over the flag manifold), this is a desingularization. □

This result gives us a complexity-reduction, both in the dimension of the acting group and in the dimension of the space acted upon, from GL_n-orbits in the nullcone $\mathtt{null}_n^m$, to B-orbits in N^m at least on the stratum $GL_n.U$ described before. The aim of the *Hesselink stratification* of the nullcone is to extend this reduction also to the complement.

DEFINITION 113. Let $s \in \mathcal{S}_n$ and let C_s be the vectorspace of all m-tuples in M_n^m which are of corner-type C_s. We have seen that there is a Zariski open subset (but, possibly empty) U_s of C_s consisting of m-tuples of optimal corner type C_s. Observe that the action of conjugation of GL_n on M_n^m induces an action of the associated parabolic subgroup P_s on C_s.

The *Hesselink stratum* S_s associated to s is the subvariety $GL_n.U_s$ where U_s is the open subset of C_s consisting of the optimal C_s-type tuples.

THEOREM 124 (Hesselink). *With notations as before we have a commuting diagram*

$$\begin{array}{ccc} GL_n \times^{P_s} U_s & \xrightarrow{\simeq} & S_s \\ \cap\downarrow & & \cap\downarrow \\ GL_n \times^{P_s} C_s & \xrightarrow{ac} & \overline{S_s} \end{array}$$

where ac is the action map, $\overline{S_s}$ is the Zariski closure of S_s in $\mathtt{null}_n^m$ and the upper map is an isomorphism of GL_n-varieties.

Here, GL_n/P_s is the flag variety associated to the parabolic subgroup P_s and is a projective manifold. The variety $GL_n \times^{P_s} C_s$ is a vectorbundle over the flag variety GL_n/P_s and is a subbundle of the trivial bundle $GL_n \times^{P_s} M_n^m$.

Therefore, the Hesselink stratum S_s is an irreducible smooth variety of dimension

$$\begin{aligned} dim\ S_s &= dim\ GL_n/P_s + rk\ GL_n \times^{P_s} C_s \\ &= n^2 - dim\ P_s + dim_{\mathbb{C}}\ C_s \end{aligned}$$

and there is a natural one-to-one correspondence between the GL_n-orbits in S_s and the P_s-orbits in U_s.

Moreover, the vectorbundle $GL_n \times^{P_s} C_s$ is a desingularization of $\overline{S_s}$ hence 'feels' the gluing of S_s to the remaining strata. Finally, the ordering of corners has the geometric interpretation

$$\overline{S_s} \subset \bigcup_{\|s'\| \leq \|s\|} S_{s'}$$

PROOF. A similar argument as in the proof of theorem 123 using the facts we collected in previous examples. □

We have seen that $U_s = p^{-1}\ \mathtt{ress}_\alpha(Q_s, \theta_s)$ with $C_s \overset{p}{\twoheadrightarrow} B_s$ the canonical projection forgetting the non-border entries. As the action of the parabolic subgroup P_s restricts to the action of its Levi-part L_s on $B_s = \mathtt{rep}_{\alpha_s}\ Q$ there is a canonical projection

$$U_s/P_s \overset{p}{\twoheadrightarrow} \mathtt{moss}_{\alpha_s}(Q_s, \theta_s)$$

to the moduli space of θ_s-semistable representations in $\mathtt{rep}_{\alpha_s}\ Q_s$. As none of the components of Q_s admits cycles, these moduli spaces are projective varieties. For small values of m and n these moduli spaces give good approximations to the study of the orbits in the nullcone.

EXAMPLE 161. (The nullcone $\mathtt{null}_3^2$) Hanspeter Kraft described the orbits in $\mathtt{null}_3^2$ in [**35**, p. 202] by brute force. The orbit space decomposes as a disjoint union of tori is depicted in figure 7 Here, each node corresponds to a torus of dimension the right-hand side number in the bottom row. A point in this torus represents an orbit with dimension the left-hand side number. The top letter is included for classification purposes. That is, every orbit has a unique representant in the list of couples of 3×3 matrices (A, B) given in figure 8. The top letter gives the torus, the first 2 rows give the first two rows of A and the last two rows give the first two rows of B, $x, y \in \mathbb{C}^*$. We will derive this result from the above description of the Hesselink stratification. To begin, the relevant data concerning

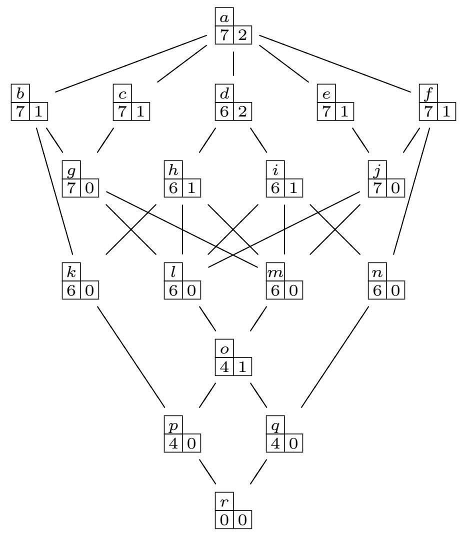


FIGURE 7. Kraft's diamond for $\texttt{null}_3^2$.

a		
0	1	0
0	0	1
0	x	0
0	0	y

b		
0	1	0
0	0	1
0	0	0
0	0	x

c		
0	1	0
0	0	1
0	x	0
0	0	0

d		
0	1	0
0	0	1
0	x	y
0	0	x

e		
0	1	0
0	0	1
0	x	0
0	0	0

f		
0	0	0
0	0	1
0	1	0
0	0	x

g		
0	1	0
0	0	0
0	0	0
0	0	1

h		
0	1	0
0	0	1
0	0	x
0	0	0

i		
0	0	x
0	0	0
0	1	0
0	0	1

j		
0	0	0
0	0	1
0	1	0
0	0	0

k		
0	0	1
0	0	0
0	1	0
0	0	0

l		
0	0	0
0	0	1
0	0	1
0	0	0

m		
0	0	1
0	0	0
0	1	0
0	0	0

n		
0	0	0
0	0	0
0	1	0
0	0	1

o		
0	1	0
0	0	0
0	x	0
0	0	0

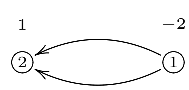
p		
0	1	0
0	0	0
0	0	0
0	0	0

q		
0	0	0
0	0	0
0	1	0
0	0	0

r		
0	0	0
0	0	0
0	0	0
0	0	0

FIGURE 8. Orbit representants in $\texttt{null}_3^2$.

$\mathcal{S}_3$ is summarized in the table of figure 9 For the last four corner types, $B_s = C_s$ whence the orbit space U_s/P_s is isomorphic to the moduli space $\texttt{moss}^{ss}_{\alpha_s}(Q_s, \theta_s)$. Consider the quiver-setting



If the two arrows are not linearly independent, then the representation contains a proper subrepresentation of dimension-vector $\beta = (1,1)$ or $(1,0)$ and in both cases $\theta_s(\beta) < 0$ whence the representation is not θ_s-semistable. If the two arrows are linearly independent, we can use the GL_2-component to bring them in the form $(\begin{bmatrix} 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \end{bmatrix})$, whence $\texttt{moss}^{ss}_{\alpha_s}(Q_s, \alpha_s)$ is reduced to one point, corresponding to the

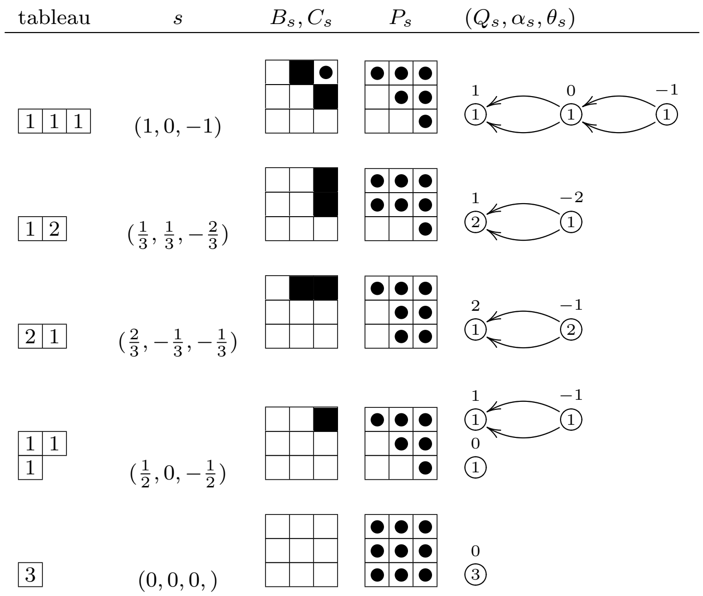
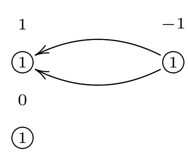


FIGURE 9. Hesselink strata for $\mathtt{null}_3^2$.

matrix-couple of type l

$$(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

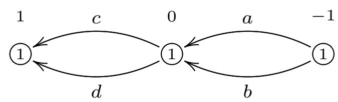
A similar argument, replacing linear independence by common zero-vector shows that also the quiver-setting corresponding to the tableau [2|1] has one point as its moduli space, the matrix-tuple of type k. Next, consider the quiver setting



A representation in $\mathtt{rep}_{\alpha_s}\ Q_s$ is θ_s-semistable if and only if the two maps are not both zero (otherwise, there is a subrepresentation of dimension $\beta = (1, 0)$ with $\theta_s(\beta) < 0$). The action of $GL(\alpha_s) = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2 - \underline{0}$ has a s orbit space $\mathbb{P}^1$ and they are represented by matrix-couples

$$(\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

with $[a:b] \in \mathbb{P}^1$ giving the types o,p and q. Clearly, the stratum $\boxed{3}$ consists just of the zero-matrix, which is type r. Remains to investigate the quiver-setting



Again, one verifies that a representation in $\mathtt{rep}_{\alpha_s}\ Q_s$ is θ_s-semistable if and only if $(a,b) \neq (0,0) \neq (c,d)$ (for otherwise one would have subrepresentations of dimensions $(1,1,0)$ or $(1,0,0)$). The corresponding $GL(\alpha_s)$-orbits are classified by

$$\mathtt{moss}^{ss}_{\alpha_s}(Q_s.\theta_s) \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

corresponding to the matrix-couples of types a,b,c,e,f,g,j,k and n

$$(\begin{bmatrix} 0 & c & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix})$$

where $[a:b]$ and $[c:d]$ are points in $\mathbb{P}^1$. In this case, however, $C_s \neq B_s$ and we need to investigate the fibers of the projection

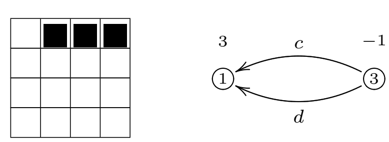
$$U_s/P_s \overset{p}{\twoheadrightarrow} \mathtt{moss}^{ss}_{\alpha_s}(Q_s,\alpha_s)$$

Now, P_s is the Borel subgroup of upper triangular matrices and one verifies that the following two couples

$$(\begin{bmatrix} 0 & c & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}) \quad \text{and} \quad (\begin{bmatrix} 0 & c & x \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & y \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix})$$

lie in the same B-orbit if and only if $det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \neq 0$, that is, if and only if $[a:b] \neq [c:d]$ in $\mathbb{P}^1$. Hence, away from the diagonal p is an isomorphism. On the diagonal one can again verify by direct computation that the fibers of p are isomorphic to $\mathbb{C}$, giving rise to the cases d,h and i in the classification.
The connection between this approach and Kraft's result is depicted in figure 10. The picture on the left is Kraft's toric degeneration picture where we enclosed all orbits belonging to the same Hesselink strata, that is, having the same optimal corner type. The dashed region enclosed the orbits which do not come from the moduli spaces $\mathtt{moss}^{ss}_{\alpha_s}(Q_s,\theta_s)$, that is, those coming from the projection $U_s/P_s \twoheadrightarrow \mathtt{moss}^{ss}_{\alpha_s}(Q_s,\theta_s))$. The picture on the right gives the ordering of the relevant corners.

Example 162. We see that we get most orbits in the nullcone from the moduli spaces $\mathtt{moss}^{ss}_{\alpha_s}(Q_s,\theta_s)$. The reader is invited to work out the orbits in $\mathtt{null}^2_4$. We list the moduli spaces of the relevant corners in figure 11 Observe that two potential corners are missing in this list. This is because we have the following quiver setting for the corner



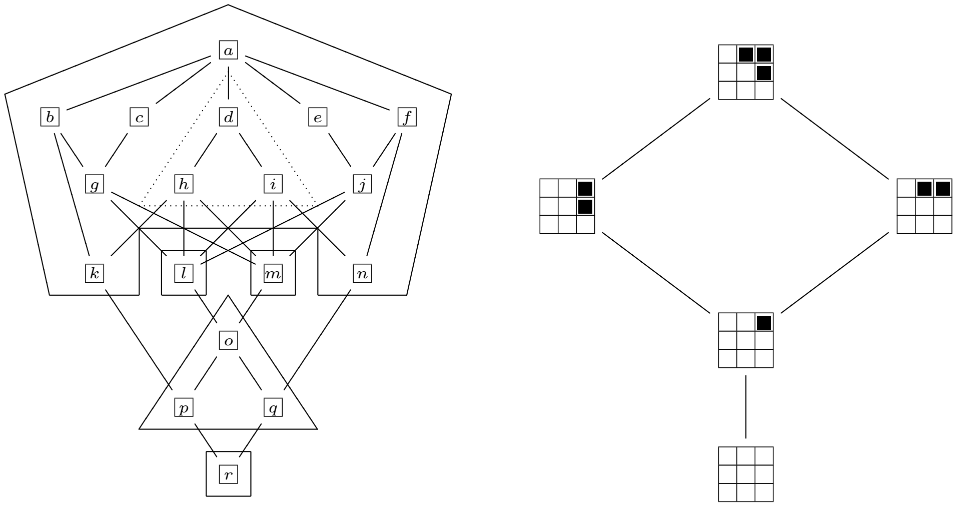


FIGURE 10. Nullcone of couples of 3 × 3 matrices.

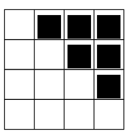
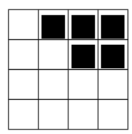
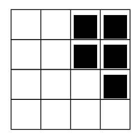
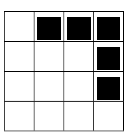
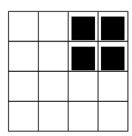
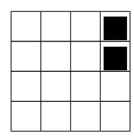
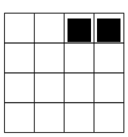
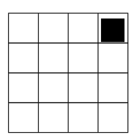
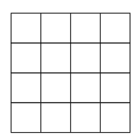
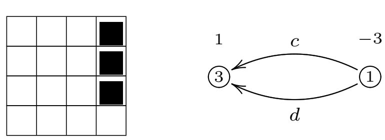
corner	$\mathtt{moss}^{ss}_{\alpha_s}(Q_s, \theta_s)$	corner	$\mathtt{moss}^{ss}_{\alpha_s}(Q_s, \theta_s)$	corner	$\mathtt{moss}^{ss}_{\alpha_s}(Q_s, \theta_s)$
	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$		$\mathbb{P}^1$		$\mathbb{P}^1$
	$\mathbb{P}^3 \sqcup \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1$		$\mathbb{P}^1 \sqcup S^2(\mathbb{P}^1)$		$\mathbb{P}^0$
	$\mathbb{P}^1$		$\mathbb{P}^1$		$\mathbb{P}^0$

FIGURE 11. Moduli spaces appearing in $\mathtt{null}^2_4$.

and there are no θ_s-semistable representations as the two maps have a common kernel, whence a subrepresentation of dimension $\beta = (1, 0)$ and $\theta_s(\beta) < 0$. A similar argument holds for the other missing corner and quiver setting



For general n, a similar argument proves that the corners associated to the tableaux $\boxed{1}\boxed{n}$ and $\boxed{n}\boxed{1}$ are not optimal for tuples in $\mathtt{null}_{n+1}^m$ unless $m \geq n$. It is also easy to see that with $m \geq n$ all relevant corners appear in $\mathtt{null}_{n+1}^m$, that is all potential Hesselink strata actually appear for large m.

After this lengthy description of the nullcone in the generic case we now describe $\mathtt{null}_\alpha\ Q$, the nullcone for the basechange action of $GL(\alpha)$ in $\mathtt{rep}_\alpha\ Q$. Fortunately, the only remaining difficulty is a notational one.

Definition 114. Up to conjugation, any one-parameter subgroup λ of $GL(\alpha)$ lies in the maximal torus T_a with $a = |\alpha|$ and can be represented by an integral a-tuple $(r_1, \ldots, r_a) \in \mathbb{Z}^a$. We have to take the quiver-vertices into account, so we decompose the integer interval $[1, 2, \ldots, a]$ into *vertex intervals* I_{v_i} such that

$$[1, 2, \ldots, a] = \sqcup_{i=1}^k I_{v_i} \quad \text{with} \quad I_{v_i} = [\sum_{j=1}^{i-1} a_j + 1, \ldots, \sum_{j=1}^{i} a_j]$$

The weights of T_a are isomorphic to $\mathbb{Z}^a$ having canonical generators π_p for $1 \leq p \leq a$. Decompose the representation space into weight spaces

$$\mathtt{rep}_\alpha\ Q = \bigoplus_{\pi_{pq} = \pi_q - \pi_p} \mathtt{rep}_\alpha\ Q(\pi_{pq})$$

where the eigenspace of π_{pq} is non-zero if and only if for $p \in I_{v_i}$ and $q \in I_{v_j}$, there is an arrow

$$\textcircled{j} \longleftarrow \textcircled{i}$$

in the quiver Q. Call $\pi_\alpha\ Q$ the set of weights π_{pq} which have non-zero eigenspace in $\mathtt{rep}_\alpha\ Q$. We can write every representation as $V = \sum_{p,q} V_{pq}$ where V_{pq} is a vector of the (p,q)-entries of the maps $V(a)$ for all arrows a in Q from v_i to v_j. The action of T_a on $\mathtt{rep}_\alpha\ Q$ is induced by conjugation, hence for λ determined by $(r_1, \ldots, r_a)$

$$\underset{t \rightarrow 0}{lim}\ \lambda(t).V = \underline{0} \ \Leftrightarrow\ r_q - r_p \geq 1 \text{ whenever } V_{pq} \neq 0$$

Again, we define the *corner type* C of the representation V by defining the subset of real a-tuples

$$E_V = \{(x_1, \ldots, x_a) \in \mathbb{R}^a \ \mid\ x_q - x_p \geq 1\ \forall\ V_{pq} \neq 0\}$$

and determine a minimal element s_V in it, minimal with respect to the usual norm on $\mathbb{R}^a$. Again, s_V is a uniquely determined point in $\mathbb{Q}^a$, having the characteristic property that its entries can be partitioned into strings

$$\{\underbrace{p_l, \ldots, p_l}_{a_{l0}}, \underbrace{p_l + 1, \ldots, p_l + 1}_{a_{l1}}, \ldots, \underbrace{p_l + k_l, \ldots, p_l + k_l}_{a_{lk_l}}\} \quad \text{with all } a_{lm} \geq 1$$

which are balanced, that is $\sum_{m=0}^{k_l} a_{lm}(p_l + m) = 0$. We cannot bring s_V into dominant form, as we can only permute base-vectors of the vertex-spaces. That is, we can only use the action of the *vertex-symmetric groups*

$$S_{a_1} \times \ldots \times S_{a_k} \hookrightarrow S_a$$

to bring s_V into *vertex dominant form* , that is if $s_V = (s_1, \ldots, s_a)$ then

$$s_q \leq s_p \quad \text{whenever} \quad p, q \in I_{v_i} \text{ for some } i \text{ and } p < q$$

EXAMPLE 163. We compile a list $\mathcal{S}_\alpha$ of such rational a-tuples by the following `algorithm`

• Start with the list $\mathcal{S}_a$ of matrix corner types.
• For every $s \in \mathcal{S}_a$ consider all permutations $\sigma \in S_a/(S_{a_1} \times \ldots \times S_{a_k})$ such that $\sigma.s = (s_{\sigma(1)}, \ldots, s_{\sigma(a)})$ is vertex dominant.
• Take $\mathcal{H}_\alpha$ to be the list of the distinct a-tuples $\sigma.s$ which are vertex dominant.
• Remove $s \in \mathcal{H}_\alpha$ whenever there is an $s' \in \mathcal{H}_\alpha$ such that

$$\pi_s\ Q = \{\pi_{pq} \in \pi_\alpha\ Q \mid s_q - s_p \geq 1\} \subset \pi_{s'}\ Q = \{\pi_{pq} \in \pi_\alpha\ Q\ \mid\ s'_q - s'_p \geq 1\}$$

and $\parallel s \parallel > \parallel s' \parallel$.
• The list $\mathcal{S}_\alpha$ are the remaining entries s from $\mathcal{H}_\alpha$.

For $s \in \mathcal{S}_\alpha$, we define *associated quiver data* similar to the case of matrices

• The *corner* C_s is the subspace of $\mathtt{rep}_\alpha\ Q$ such that all arrow matrices V_b, when viewed as $a \times a$ matrices using the partitioning in vertex-entries, have only non-zero entries at spot (p,q) when $s_q - s_p \geq 1$.
• The *border* B_s is the subspace of $\mathtt{rep}_\alpha\ Q$ such that all arrow matrices V_b, when viewed as $a \times a$ matrices using the partitioning in vertex-entries, have only non-zero entries at spot (p,q) when $s_q - s_p = 1$.
• The *parabolic subgroup* $P_s(\alpha)$ is the intersection of $P_s \subset GL_a$ with $GL(\alpha)$ embedded along the diagonal. $P_s(\alpha)$ is a parabolic subgroup of $GL(\alpha)$, that is, contains the product of the Borels $B(\alpha) = B_{a_1} \times \ldots \times B_{a_k}$.
•The *Levi-subgroup* $L_s(\alpha)$ is the intersection of $L_s \subset GL_a$ with $GL(\alpha)$ embedded along the diagonal.

We say that a representation $V \in \mathtt{rep}_\alpha\ Q$ is of *corner type* C_s whenever $V \in C_s$. By permuting the vertex-bases, every representation $V \in \mathtt{rep}_\alpha\ Q$ can be brought to a corner type C_s for a uniquely determined s which is a vertex-dominant reordering of s_V.

We solve the problem of *optimal corner representations* by introducing a new quiver setting. Fix a type $s \in \mathcal{S}_\alpha\ Q$ and let $J_1, \ldots, J_u$ be the distinct strings partitioning the entries of s, say with

$$J_l = \{\underbrace{p_l, \ldots, p_l}_{\sum_{i=1}^k b_{i,l0}}, \underbrace{p_l + 1, \ldots, p_l + 1}_{\sum_{i=1}^k b_{i,l1}}, \ldots, \underbrace{p_l + k_l, \ldots, p_l + k_l}_{\sum_{i=1}^k b_{i,lk_l}}\}$$

where $b_{i,lm}$ is the number of entries $p \in I_{v_i}$ such that $s_p = p_l + m$. To every string l we will associate a quiver $Q_{s,l}$ and dimension vector $\alpha_{s,l}$ as follows

• The quiver $Q_{s,l}$ has $k.(k_l+1)$ vertices labeled (v_i, m) with $1 \leq i \leq k$ and $0 \leq m \leq k_l$. In $Q_{s,l}$ there are as many arrows from vertex (v_i, m) to vertex $(v_j, m+1)$ as there are arrows in Q from vertex v_i to vertex v_j. There are no arrows between (v_i, m) and (v_j, m') if $m' - m \neq 1$.
• The dimension-component of $\alpha_{s,l}$ in vertex (v_i, m) is equal to $b_{i,lm}$.

The quiver-setting (Q_s, α_s) associated to a type $s \in \mathcal{S}_\alpha\ Q$ will be the disjoint union of the string quiver-settings $(Q_{s,l}, \alpha_{s,l})$ for $1 \leq l \leq u$. Again, there are natural isomorphisms

$$\begin{cases} B_s & \simeq \mathtt{rep}_{\alpha_s}\ Q_s \\ L_s(\alpha) & \simeq GL(\alpha_s) \end{cases}$$

Moreover, the base-change action of $GL(\alpha_s)$ on $\mathtt{rep}_{\alpha_s}\ Q_s$ coincides under the isomorphisms with the action of the Levi-subgroup $L_s(\alpha)$ on the border B_s.

In order to determine the representations in $\texttt{rep}_{\alpha_s}\ Q_s$ which have *optimal corner type* C_s we define the following character on the Levi-subgroup

$$L_s(\alpha) = \prod_{l=1}^{u} \times_{i=1}^{k} \times_{m=0}^{k_l} GL_{b_{i,lm}} \xrightarrow{\chi\theta_s} \mathbb{C}^*$$

determined by sending a tuple $(g_{i,lm})_{ilm} \longrightarrow \prod_{ilm} det\ g_{i,lm}^{m_{i,lm}}$ where the exponents are determined by

$$\theta_s = (m_{i,lm})_{ilm} \quad \text{where} \quad m_{i,lm} = d(p_l + m)$$

with d the least common multiple of the numerators of the rational numbers p_l for all $1 \leq l \leq u$.

THEOREM 125. *Let* $V \in \texttt{null}_\alpha\ Q$ *of corner type* C_s*. Then,* V *is of optimal corner type* C_s *if and only if under the natural maps*

$$C_s \xrightarrow{\pi} B_s \xrightarrow{\simeq} \texttt{rep}_{\alpha_s}\ Q_s$$

V *is mapped to a* θ_s*-semistable representation in* $\texttt{rep}_{\alpha_s}\ Q_s$*. If* U_s *is the open subvariety of* C_s *consisting of all representations of optimal corner type* C_s*, then*

$$U_s = \pi^{-1}\ \texttt{ress}_{\alpha_s}(Q_s, \theta_s)$$

For the corresponding Hesselink stratum $S_s = GL(\alpha).U_s$ *we have the commuting diagram*

$$\begin{array}{ccc} GL(\alpha) \times^{P_s(\alpha)} U_s & \xrightarrow{\simeq} & S_s \\ \downarrow & & \downarrow \\ GL(\alpha) \times^{P_s(\alpha)} C_s & \xrightarrow{ac} & \overline{S_s} \end{array}$$

where ac is the action map, $\overline{S_s}$ *is the Zariski closure of* S_s *in* $\texttt{null}_\alpha\ Q$ *and the upper map is an isomorphism as* $GL(\alpha)$*-varieties.*

Here, $GL(\alpha)/P_s(\alpha)$ *is the flag variety associated to the parabolic subgroup* $P_s(\alpha)$ *and is a projective manifold. The variety* $GL(\alpha) \times^{P_s(\alpha)} C_s$ *is a vectorbundle over the flag variety* $GL(\alpha)/P_s(\alpha)$ *and is a subbundle of the trivial bundle* $GL(\alpha) \times^{P_s(\alpha)} \texttt{rep}_\alpha\ Q$*.*

Hence, the Hesselink stratum S_s *is an irreducible smooth variety of dimension*

$$\begin{aligned} dim\ S_s &= dim\ GL(\alpha)/P_s(\alpha) + rk\ GL(\alpha) \times^{P_s(\alpha)} C_s \\ &= \sum_{i=1}^{k} a_i^2 - dim\ P_s(\alpha) + dim_{\mathbb{C}}\ C_s \end{aligned}$$

and there is a natural one-to-one correspondence between the $GL(\alpha)$*-orbits in* S_s *and the* $P_s(\alpha)$*-orbits in* U_s*.*

Moreover, the vectorbundle $GL(\alpha) \times^{P_s(\alpha)} C_s$ *is a desingularization of* $\overline{S_s}$ *hence 'feels' the gluing of* S_s *to the remaining strata. The ordering of corners has the geometric interpretation*

$$\overline{S_s} \subset \bigcup_{\|s'\| \leq \|s\|} S_{s'}$$

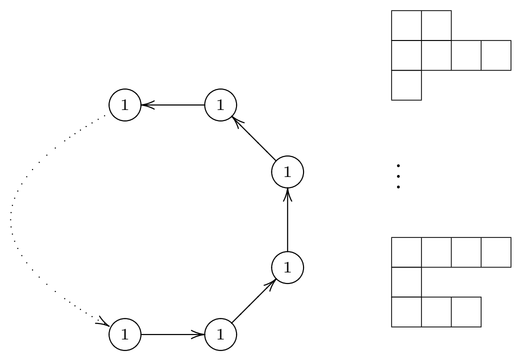


FIGURE 12. Local quiver settings for curve orders.

Finally, because $P_s(\alpha)$ acts on B_s by the restriction to its subgroup $L_s(\alpha) = GL(\alpha_s)$ we have a projection from the orbit space

$$U_s/P_s \xrightarrow{p} \mathtt{moss}_{\alpha_s}(Q_s, \theta_s)$$

to the moduli space of θ_s-semistable quiver representations.

EXAMPLE 164. Let $(A, tr_A) \in \mathtt{alg@n}$ over an affine curve $X = \mathtt{tiss}_n A$ and $\xi \in \mathtt{smooth}A$, then the local quiver setting (Q, α) is determined by an oriented cycle Q on k vertices with $k \leq n$ being the number of distinct simple components of M_ξ, the dimension vector $\alpha = (1, \ldots, 1)$ as in figure 12 and an unordered partition $p = (d_1, \ldots, d_k)$ having precisely k parts such that $\sum_i d_i = n$, determining the dimensions of the simple components of M_ξ.

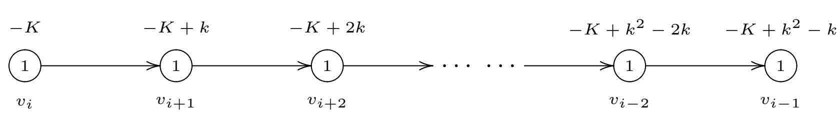
Fix a cyclic ordering of the k-vertices $\{v_1, \ldots, v_k\}$, then the set of weights of the maximal torus $T_k = \mathbb{C}^* \times \ldots \times \mathbb{C}^* = GL(\alpha)$ occurring in $\mathtt{rep}_\alpha\ Q$ is the set

$$\pi_\alpha\ Q = \{\pi_{k1}, \pi_{12}, \pi_{23}, \ldots, \pi_{k-1k}\}$$

Denote $K = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}$ and consider the one string vector

$$s = (\ \ldots, k-2-\frac{K}{k}, k-1-\frac{K}{k}, \underbrace{-\frac{K}{k}}_{i}, 1-\frac{K}{k}, 2-\frac{K}{k}, \ldots\ )$$

then s is balanced and vertex-dominant, $s \in \mathcal{S}_\alpha\ Q$ and $\pi_s\ Q = \Pi$. To check whether the corresponding Hesselink strata in $\mathtt{null}_\alpha\ Q$ is nonempty we have to consider the associated quiver-setting $(Q_s, \alpha_s, \theta_s)$ which is



It is well known and easy to verify that $\mathtt{rep}_{\alpha_s}\ Q_s$ has an open orbit with representative all arrows equal to 1. For this representation all proper subrepresentations have dimension vector $\beta = (0, \ldots, 0, 1, \ldots, 1)$ and hence $\theta_s(\beta) > 0$. That is, the representation is θ_s-stable and hence the corresponding Hesselink stratum $S_s \neq \emptyset$. Finally, because the dimension of $\mathtt{rep}_{\alpha_s}\ Q_s$ is $k-1$ we have that the dimension of

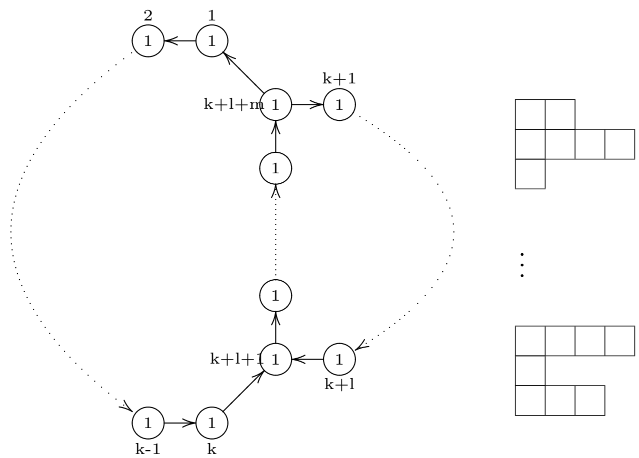


FIGURE 13. Local quiver settings for surface orders.

this component in the representation fiber $\pi^{-1}(x)$ is equal to

$$dim\ GL_n - dim\ GL(\alpha) + dim\ \mathtt{rep}_{\alpha_s}\ Q_s = n^2 - k + k - 1 = n^2 - 1$$

and we obtain the following characterization of the representation fiber

The representation fiber $\pi^{-1}(\xi)$ has exactly k irreducible components of dimension $n^2 - 1$, each the closure of one orbit. In particular, if A is alg@n*-smooth, the quotient map*

$$\mathtt{trep}_n A \xrightarrow{\pi} \mathtt{tiss}_n A = X$$

is flat (all fibers have the same dimension $n^2 - 1$).

EXAMPLE 165. Let $(A, tr_A) \in \mathtt{alg@n}$ over an affine surface $S = \mathtt{tiss}_n A$ and let $\xi \in \mathtt{smooth}A$. The local structure of A is determined by a quiver setting (Q, α) where $\alpha = (1, \ldots, 1)$ and Q is a two-circuit quiver on $k + l + m \leq n$ vertices, corresponding to the distinct simple components of M_ξ as in figure 13 and an unordered partition $p = (d_1, \ldots, d_{k+l+m})$ of n with $k + l + m$ non-zero parts determined by the dimensions of the simple components of M_ξ. With the indicated ordering of the vertices we have that

$$\pi_\alpha\ Q = \{\pi_{i\ i+1} \mid \begin{cases} 1 & \leq i \leq k-1 \\ k+1 & \leq i \leq k+l-1 \\ k+l+1 & \leq i \leq k+l+m-1 \end{cases} \}$$
$$\cup \{\pi_{k\ k+l+1}, \pi_{k+l\ k+l+1}, \pi_{k+l+m\ 1}, \pi_{k+l+m\ k+1}\}$$

As the weights of a corner cannot contain all weights of an oriented cycle in Q we have to consider the following two types of potential corner-weights Π of maximal cardinality

- (outer type) : $\Pi = \pi_\alpha\ Q - \{\pi_a, \pi_b\}$ where a is an edge in the interval $[v_1, \ldots, v_k]$ and b is an edge in the interval $[v_{k+1}, \ldots, v_{k+l}]$.
- (inner type) : $\Pi = \pi_\alpha\ Q - \{\pi_c\}$ where c is an edge in the interval $[v_{k+l+1}, v_{k+l+m}]$.

A (lengthy) investigation of all the different cases results in the following result which we leave as an exercise :

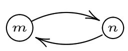
Let $\xi \in$ `smooth`A be of local type (A_{klm}, α). Then, the representation fiber $\pi^{-1}(\xi)$ has exactly $2+(k-1)(l-1)+(m-1)$ irreducible components of which $2+(k-1)(l-1)$ are of dimension n^2-1 and are closure of one orbit and the remaining $m-1$ have dimension n^2 and are closures of a one-dimensional family of orbits. In particular, if A is `alg@n`*-smooth, then the algebraic quotient map*

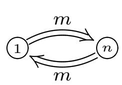
$$\mathtt{trep}_n A \xrightarrow{\pi} \mathtt{tiss}_n A = S$$

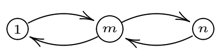
is flat if and only if all local quiver settings of A have quiver A_{klm} with $m = 1$.

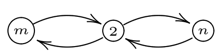
The final example will determine the fibers over *smooth* points in the quotient varieties (or moduli spaces) provided the local quiver is *symmetric*. This computation is due to Geert Van de Weyer [**11**].

Example 166. (Smooth symmetric settings) Recall from theorem 100 that a smooth symmetric quiver setting (`sss`) if and only if it is a tree constructed as a connected sum of three different types of quivers:

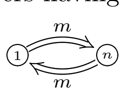
- 

- 

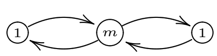
, with $m \leq n$
- 

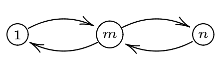
- 

where the connected sum is taken in the vertex with dimension 1. We call the vertices where the connected sum is taken *connecting vertices* and graphically depict them by a square vertex $\square$. We want to study the nullcone of connected sums composed of more than one of these quivers so we will focus on instances of these four quivers having at least one vertex with dimension 1:

I 

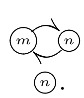
, with $m \leq n$

II(1) 

II(2) 

We will call the quiver settings of type **I** and **II** forming an `sss` (Q, α) the *terms* of Q.

claim 1 : *Let (Q, α) be an* `sss` *and Q_μ a type quiver for Q, then any string quiver of Q_μ is either a connected sum of string quivers of type quivers for terms of Q or a string quiver of type quivers of*

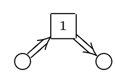


Consider a string quiver $Q_{\mu(i)}$ of Q_μ. By definition vertices in a type quiver are only connected if they originate from the same term in Q. This means we may divide the string quiver $Q_{\mu(i)}$ into segments, each segment either a string quiver of a type quiver of a term of Q (if it contains the connecting vertex) or a level quiver of a type quiver of the quivers listed above (if it does not contain the connecting vertex).

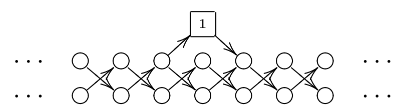
The only vertices these segments may have in common are instances of the connecting vertices. Now note that there is only one instance of each connecting vertex in Q_μ because the dimension of each connecting vertex is 1. Moreover, two segments cannot have more than one connecting vertex in common as this would mean that in the original quiver there is a cycle, proving the claim.

Hence, constructing a type quiver for an `sss` boils down to patching together string quivers of its terms. These string quivers are subquivers of the following two quivers:

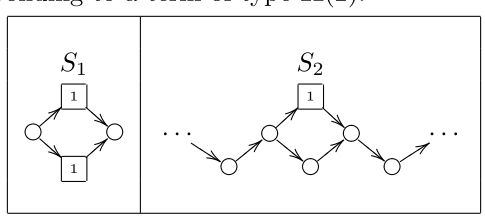
I:



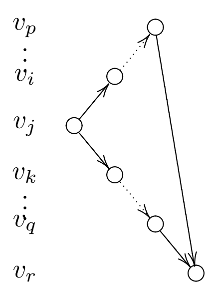
II:



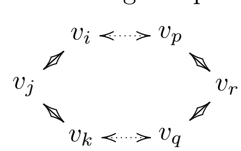
Observe that the second quiver has two components. So a string quiver will either be a tree (possible from all components) or a quiver containing a square. We will distinguish two different types of squares; S_1 corresponding to a term of type **II**(1) and S_2 corresponding to a term of type **II**(2).



These squares are the only polygons that can appear in our type quiver. Indeed, consider a possible polygon



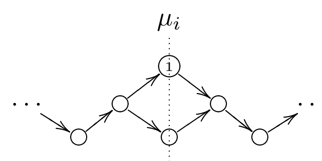
This polygon corresponds to the following subquiver of Q:



But Q is a tree, so this is only a subquiver if it collapses to $v_i \longleftrightarrow v_j \longleftrightarrow v_k$.

claim 2 : *Let (Q, α) be an* `sss` *and Q_μ a type quiver containing (connected) squares. If Q_μ determines a non-empty Hesselink stratum then*

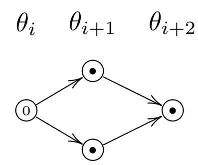
(i) *the 0-axis in Q_μ lies between the axes containing the outer vertices of the squares of type S_1;*
(ii) *squares of type S_1 are connected through paths of maximum length 2;*
(iii) *squares of type S_1 that are connected through a path of length 2 are connected to other quivers in top and bottom vertex (and hence originate from type **II**(1) terms that are connected to other terms in both their connecting vertices);*
(iv) *the string $\mu(i)$ containing squares of type S_1 connected through a path of length two equals $(\dots, -2, -1, 0, 1, 2, \dots)$.*
(v) *for a square of type S_2:*



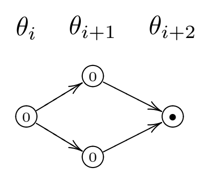
with p vertices on its left branch and q vertices on its right branch we have

$$-\frac{q}{2} \leq \mu_i \leq \frac{p}{2}$$

Let us call the string quiver of Q_μ containing the squares $Q_{\mu(i)}$ and let $\theta \in \mu(i)\mathbb{N}_0$ be the character determining this string quiver. Consider the subrepresentation



This subrepresentation has character $\theta(\alpha_{\mu(i)}) - \alpha_{\mu(i)}(\mathbf{v})\theta_i \geq 0$ where $\mathbf{v}$ is the vertex which dimension we reduced to 0, so $\theta_i \leq 0$. But then the subrepresentation



gives $\theta_{i+2} \geq 0$, whence (i). Note that the left vertex of one square can never lie on an axis right of the right vertex of another square. At most it can lie on the same axis as the right vertex, in which case this axis is the 0-axis and the squares are

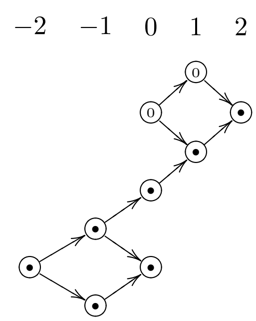
$$\alpha_1 = \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} & 1 & \\ 1 & & 2 \\ & 1 & \end{pmatrix}$$

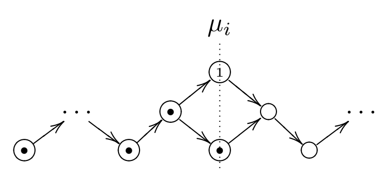
$$\alpha_3 = \begin{pmatrix} & 1 & \\ 2 & & 2 \\ & 1 & \end{pmatrix}$$

FIGURE 14. Possible dimension vectors for squares.

connected by a path of length 2. In order to prove (iii) look at the subrepresentation



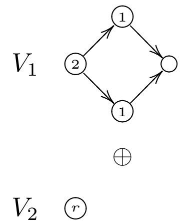
This subrepresentation has negative character and hence the original representation was not semistable. Finally, for (v) we look at the subrepresentation obtained by reducing the dimension of all dotted vertices by 1:



having character $-((p+1)\mu_i - \sum_{j=1}^{p} j) \geq 0$. So $\mu_i \leq \frac{p}{2}$. Mirroring this argument yields the other inequality $\mu_i \geq -\frac{q}{2}$.

claim 3 : *Let (Q, α) be an* `sss` *and Q_μ be a type quiver determining a non-empty stratum and let $Q_{\mu(i)}$ be a string quiver determined by a segment $\mu(i)$ not containing 0. Then the only possible dimension vectors for squares of type S_1 in $Q_{\mu(i)}$ are those of figure 14.*

Top and bottom vertex of the square are constructed from the connecting vertices so can only be one-dimensional. Left and right vertex of the square are constructed from a vertex of dimension n. Claim 2 asserts that the leftmost vertex lies on a negative axis while the rightmost vertex lies on a positive axis. If the left dimension is > 2 then the representation splits



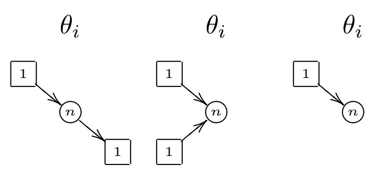
with $r = m - 2$. By semistability the character of V_2 must be zero. A similar argument applies to the right vertex.

claim 4 : *Let μ be a type determining a non-empty stratum.*

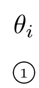
(i) *When a vertex (v, i) in Q_μ determined by a term of type **II**(1) has $\alpha(v, i) > 2$ then $\mu_i = 0$.*

(ii) *When a vertex (v, i) in Q_μ determined by a term of type **I** with m arrows has $\alpha(v, i) > m$ then $\mu_i = 0$.*

Suppose we have a vertex v with dimension $\alpha_{\mu(i)}(v) > 2$, then the number of paths running through this vertex is at most 2: would there be at least three paths arriving or departing in the vertex, it would be a connecting vertex which is not possible because of its dimension. Are there two paths arriving and at least one path departing, it must be a central vertex of a type **II**(2) term. But then the only possible subtrees generated from type **II**(1) terms with vertices of dimension at least three are (modulo reversing all arrows)



In the last tree there are no other arrows from the vertex with dimension n. For each of these trees we have a subrepresentation

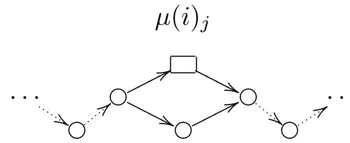


whence $\theta_i \geq 0$. But if $\theta_i > 0$, reducing the dimension of the vertex with dimension ≥ 3 gives a subrepresentation with negative character, so $\theta_i = 0$. The second part is proved similarly.

Summarizing these results we obtain the description of the nullcone of a smooth symmetric quiver-setting.

Let (Q, α) be an `sss` *and μ a type determining a non-empty stratum in* $\texttt{null}_\alpha\ Q$. *Let Q_μ be the corresponding type quiver and α_μ the corresponding dimension vector, then*

(i) *every connected component $Q_{\mu(i)}$ of Q_μ is a connected sum of string quivers of either terms of Q or quivers generated from terms of Q by removing the connecting vertex. The connected sum is taken in the instances of the connecting vertices and results in a connected sum of trees and quivers of* *the form*



(ii) *For a square of type S_1 we have $\mu(i)_{j-1} \leq 0 \leq \mu(i)_{j+1}$. Moreover, such squares cannot be connected by paths longer than two arrows and can only be connected by paths of this length if $\mu(i)_{j+1} = 0$.*

(iii) *For vertices (v, j) constructed from type **II**(1) terms we have $\alpha_{\mu_i}(v, j) \leq 2$ when $\mu_i \neq 0$.*

(iv) *For a vertex (v, j) constructed from a type **I** term with m arrows we have $\alpha_{\mu_i}(v, j) \leq m$ when $\mu_i \neq 0$.*

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