

Noncommutative Smooth Models

Lieven Le Bruyn

*Departement Wiskunde en Informatica, Universiteit Antwerpen (UIA)
B-2610 Antwerp (Belgium)
E-mail : lebruyn@uia.ua.ac.be*

ABSTRACT: We determine the central simple algebras Σ over a functionfield K of transcendence degree two which admit a model of smooth Cayley-Hamilton algebras. This happens if and only if there is a smooth model S of K such that the ramification divisor of a maximal $\mathcal{O}_S$ -order in Σ is a disjoint union of smooth curves. Further, we prove that the Brauer-Severi fibration of smooth models which are maximal orders is a flat morphism.

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One can define noncommutative smooth orders either by homological or by geometrical properties. The first approach has the advantage that there are enough regular algebras around to have smooth models at least in central simple algebras over surfaces, see [8]. The second approach, based on smooth Cayley-Hamilton orders, is already tedious in dimension two but has the advantage of having an étale local description which makes it possible (at least in principle) to generalize the results in this paper to arbitrary dimensions.

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1. Smooth Cayley-Hamilton models

Let K be a functionfield of transcendence degree d over an algebraically closed field of characteristic zero which we denote by $\mathbb{C}$. Let Σ be a central simple K -algebra of dimension n^2 and let tr be the reduced trace of Σ . Let C be an affine normal domain with function field K and let A be a C -order in Σ , that is, A is an affine algebra with center C such that $A.K = \Sigma$. Because C is integrally closed it follows that the reduced trace tr on Σ makes A into a Cayley-Hamilton algebra of degree n , see [17] or [12], such that $tr(A) = C$.

Let $\text{trep}_n A$ be the affine scheme of all n -dimensional *trace preserving* representations of A , that is, all $\mathbb{C}$ -algebra morphisms $\phi : A \longrightarrow M_n(\mathbb{C})$ compatible with the trace maps

$$\begin{array}{ccc} A & \xrightarrow{\phi} & M_n(\mathbb{C}) \\ \text{tr} \downarrow & & \downarrow Tr \\ A & \xrightarrow{\phi} & M_n(\mathbb{C}) \end{array}$$

where Tr is the usual trace on $M_n(\mathbb{C})$. Clearly, the scheme $\text{trep}_n A$ comes equipped with a natural GL_n (actually PGL_n) action and it follows from [1] and [17] that the geometric points of the quotient scheme

$$\text{triss}_n A = \text{trep}_n A // GL_n$$

are in one-to-one correspondence with the isomorphism classes of n -dimensional semisimple trace preserving representations of A . Moreover, we can recover the algebras C and A from the GL_n -action

$$C = \mathbb{C}[\text{triss}_n A] \quad A = M_n(\mathbb{C}[\text{trep}_n A])^{GL_n}$$

as respectively the algebra of polynomial GL_n -invariants on $\text{trep}_n A$ and the algebra of GL_n -equivariant maps from $\text{trep}_n A$ to $M_n(\mathbb{C})$, see [17].

Definition 1 With notation as before, A is said to be a *smooth* Cayley-Hamilton order if and only if $\text{trep}_n A$ is a smooth GL_n -scheme (in particular reduced).

In [17] it is shown that this implies that A satisfies the extension of Grothendieck's characterization of commutative regular algebras (by the universal lifting property modulo nilpotent ideals) to the category of all algebras satisfying the n -th formal Cayley-Hamilton equation.

Definition 2 With notations as before, Σ is said to possess a *noncommutative smooth model* if there is a projective normal variety X with functionfield $\mathbb{C}(X) = K$ and a sheaf $\mathcal{A}$ of $\mathcal{O}_X$ -orders in Σ such that for an affine cover $\{U_i\}$ of X the sections

$$A_i = \Gamma(U_i, \mathcal{A})$$

are smooth Cayley-Hamilton orders. That is, there is a smooth GL_n -variety $\text{trep}_n \mathcal{A}$ with algebraic quotient $\text{trep}_n \mathcal{A} / GL_n = X$.

If X is a smooth projective model of K and if Σ contains a sheaf of Azumaya $\mathcal{O}_X$ -algebras $\mathcal{A}$, then Σ has a noncommutative smooth model as $\text{trep}_n \mathcal{A}$ is a principal PGL_n -bundle over X . Hence, the problem reduces to determining which ramified classes of the Brauer group contain a noncommutative smooth model.

A natural strategy to prove that every central simple K -algebra Σ possesses a noncommutative smooth model would be the following. Let X be a smooth projective model of K and let $\mathcal{A}$ be a sheaf of maximal $\mathcal{O}_X$ -orders in Σ and construct the scheme $\text{trep}_n \mathcal{A}$. Usually, this scheme will have singularities but we can construct a GL_n -equivariant desingularization of it

$$\begin{array}{ccc} \widetilde{\text{trep}_n \mathcal{A}} & \longrightarrow & \text{trep}_n \mathcal{A} \\ \downarrow \tilde{\pi} & & \downarrow \\ \tilde{X} & \longrightarrow & X \end{array}$$

If the GL_n -quotient variety $\tilde{X}$ of this desingularization has an affine open cover $\{V_i\}$ such that there are $\Gamma(V_i, \mathcal{O}_{\tilde{X}})$ -orders A_i such that

$$\tilde{\pi}^{-1}(V_i) \simeq \text{trep}_n A_i$$

as GL_n -varieties, then Σ would have a noncommutative smooth model. Unfortunately, this approach is far too optimistic.

Example 1 Let A be the *quantum plane* of order two,

$$A = \frac{\mathbb{C}\langle x, y \rangle}{(xy + yx)}$$

One verifies that $u = x^2$ and $v = y^2$ are central elements of A and that A is a free module of rank 4 over $\mathbb{C}[u, v]$. In fact, A is a $\mathbb{C}[u, v]$ -order in the quaternion division algebra

$$\Sigma = \begin{pmatrix} u & v \\ \mathbb{C}(u, v) & \end{pmatrix}$$

The induced reduced trace tr is the linear map on A such that

$$\begin{cases} tr(x^i y^j) = 0 & \text{if either } i \text{ or } j \text{ are odd, and} \\ tr(x^i y^j) = 2x^i y^j & \text{if } i \text{ and } j \text{ are even.} \end{cases}$$

In particular, a trace preserving 2-dimensional representation is determined by a couple of 2×2 matrices

$$\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix}, \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix} \right) \quad \text{with} \quad tr \left(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \cdot \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix} \right) = 0$$

That is, $\text{trep}_2 A$ is the hypersurface in $\mathbb{C}^6$ determined by the equation

$$\text{trep}_2 A = \mathbb{V}(2x_1x_4 + x_2x_6 + x_3x_5) \hookrightarrow \mathbb{C}^6$$

and is therefore irreducible of dimension 5 with an isolated singularity at $p = (0, \dots, 0)$.

Consider the blow-up of $\mathbb{C}^6$ at p which is the closed subvariety of $\mathbb{C}^6 \times \mathbb{P}^5$ defined by

$$\tilde{\mathbb{C}}^6 = \mathbb{V}(x_i X_j - x_j X_i)$$

with the X_i the projective coordinates of $\mathbb{P}^5$. The strict transform of $\text{trep}_2 A$ is the subvariety

$$\widetilde{\text{trep}_2 A} = \mathbb{V}(x_i X_j - x_j X_i, 2X_1X_4 + X_2X_6 + X_3X_5) \hookrightarrow \mathbb{C}^6 \times \mathbb{P}^5$$

which is a smooth variety. Moreover, there is a natural GL_2 -action on it induced by simultaneous conjugation on the fourtuple of 2×2 matrices

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \quad \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix} \quad \begin{bmatrix} X_1 & X_2 \\ X_3 & -X_1 \end{bmatrix} \quad \begin{bmatrix} X_4 & X_5 \\ X_6 & -X_4 \end{bmatrix}$$

As the projection $\widetilde{\text{trep}_2 A} \longrightarrow \text{trep}_2 A$ is a GL_2 -isomorphism outside the exceptional fiber, we only need to investigate the semi-stable points over p . Take the particular point x

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

which is semi-stable and has as stabilizer

$$\text{Stab}(x) = \boldsymbol{\mu}_2 = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \hookrightarrow PGL_2$$

There is no affine GL_2 -stable open of $\widetilde{\text{trep}_2 A}$ containing x of the form $\text{trep}_2 B$ for some order B as this would contradict the fact that the stabilizer subgroup of any finite dimensional representation is connected.

Therefore, we need a subtler strategy to investigate the obstruction.

2. The strategy

The class $[\Sigma] \in Br_n(K) = H_{et}^2(K, \boldsymbol{\mu}_n)$ in the (n -torsion part of the) Brauer group of K can be found using the *coniveau spectral sequence*

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_{et}^{q-p}(\mathbb{C}(x), \boldsymbol{\mu}_n^{\otimes l-p}) \Rightarrow H_{et}^{p+q}(X, \boldsymbol{\mu}_n^{\otimes l})$$

where X is a smooth projective model for K and $X^{(p)}$ is the set of irreducible subvarieties x of X of codimension p with function field $\mathbb{C}(x)$, see [9]. In low dimensions, the resulting sequence for the Brauer group can be expressed in terms of ramification data of maximal orders in the central simple algebra, see for example the Artin-Mumford exact sequence for the Brauer group of a smooth surface [5]. In case X has singularities one can extend this using the Bloch-Ogus coniveau spectral sequence, [6].

To apply this result to our problem we need to have some control on the central singularities of smooth Cayley-Hamilton orders and on the étale local structure of the smooth order and thereby on its ramification locus. Both problems can be solved (at least in small dimensions) and we refer for more details to the papers [12] and [7]. Here, we merely state the results.

Let $\mathcal{A}$ be a sheaf of smooth Cayley-Hamilton $\mathcal{O}_X$ -orders in Σ where X is a normal projective variety with affine open cover $\{U_a\}$ and corresponding sections $A_a = \Gamma(U_a, \mathcal{A})$ and $C_a = \Gamma(U_a, \mathcal{O}_X)$. A point $p \in U_a$ determines an isomorphism class of a semisimple n -dimensional representation of A_a say

$$V_p = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

where S_i is a simple representation of dimension d_i and occurring in V_p with multiplicity e_i , that is $n = \sum_i d_i e_i$. The space of self-extensions

$$\begin{aligned} N_p &= \text{Ext}_{A_a}^1(V_p, V_p) = \bigoplus_{i,j} \text{Ext}_{A_a}^1(S_i, S_j)^{\oplus e_i e_j} \\ &= \text{rep}_\alpha Q \end{aligned}$$

is the representation space of α -dimensional representations with $\alpha = (e_1, \dots, e_k)$ of the quiver Q on k vertices $\{v_1, \dots, v_k\}$ such that the number of directed arrows between v_i and v_j is equal to $\dim_{\mathbb{C}} \text{Ext}_{A_a}^1(S_i, S_j) - \delta_{ij}$.

The normal space N_p to the orbit of V_p in $\text{trep}_n \mathcal{A}$ is the space of trace-preserving self-extensions and is therefore a linear subspace of $\text{rep}_\alpha Q$ which can be described as the representation space of α -dimensional representations $\text{rep}_\alpha Q^\bullet$ of a *marked* subquiver $Q^\bullet$ of Q . That is,

some loops of Q may acquire a marking and the matrices corresponding to these loops have to be of trace zero in $\text{rep}_\alpha Q^\bullet$. The stabilizer subgroup of V_p is the group $GL(\alpha) = GL_{e_1} \times \dots \times GL_{e_k}$ and the action of it on the normalspace N_p coincides with the basechange action on $\text{rep}_\alpha Q^\bullet$. The relevance of this description is that by the Luna slice theorems [16] the étale local structure of X near p is isomorphic to that of the marked quiver quotient variety

$$\text{iss}_\alpha Q^\bullet = \text{rep}_\alpha Q^\bullet // GL(\alpha)$$

near the trivial representation $\bar{0}$. Also, the GL_n -structure of $\text{trep}_n \mathcal{A}$ near the orbit of V_p is étale isomorphic to that of the associated fiber bundle

$$GL_n \times^{GL(\alpha)} \text{rep}_\alpha Q^\bullet$$

near the orbit of $(\bar{1}_n, 0)$ where the embedding of $GL(\alpha) \hookrightarrow GL_n$ is determined by the k -tuple $\gamma = (d_1, \dots, d_k)$ of the dimensions of the simple components of V_p .

These facts allow us to describe the completion $\hat{\mathcal{A}}_p$ of the stalk of $\mathcal{A}$ in p as well as $\widehat{\mathcal{O}_{X,p}}$ by walking through the marked quiver setting $(Q^\bullet, \alpha)$ using the results of [14] and [12]. The ring of polynomial invariants $\mathbb{C}[\text{iss}_\alpha Q^\bullet]$ is generated by taking traces of oriented cycles in the quiver Q and $\widehat{\mathcal{O}_{X,p}}$ is the completion of $\mathbb{C}[\text{iss}_\alpha Q^\bullet]$ at the maximal graded ideal. If M_{ij} is the $\mathbb{C}[\text{iss}_\alpha Q^\bullet]$ -module of oriented paths from v_i to v_j in (Q, α) , then we have a block-decomposition

$$\hat{\mathcal{A}}_p \simeq \begin{bmatrix} M_{d_1}(M_{11}) & M_{d_1 \times d_2}(M_{12}) & \dots & M_{d_1 \times d_k}(M_{1k}) \\ M_{d_2 \times d_1}(M_{21}) & M_{d_2}(M_{22}) & \dots & M_{d_2 \times d_k}(M_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_k \times d_1}(M_{k1}) & M_{d_k \times d_2}(M_{k2}) & \dots & M_{d_k}(M_{kk}) \end{bmatrix} \otimes \widehat{\mathcal{O}_{X,p}}$$

The fact that $\mathcal{A}$ is an order in a central simple algebra translates to the fact that α must be the dimension vector of a simple representation of Q . If χ_Q is the Euler form of the quiver Q (that is $\chi_{ij} = \delta_{ij} - \#\{\text{arrows from } v_i \text{ to } v_j\}$) then by [14] β is the dimension vector of a simple representation of Q iff Q is strongly connected and for all vertex dimension δ_i we have

$$\chi_Q(\beta, \delta_i) \leq 0 \quad \chi_Q(\delta_i, \beta) \leq 0$$

(unless $Q = \tilde{A}_k$ in which case $\beta = (1, \dots, 1)$). Under this condition the dimension of the quotient variety $\text{iss}_\alpha Q^\bullet$ (which must be equal to d , the dimension of X) is given by

$$d(\alpha) = 1 - \chi_Q(\alpha, \alpha) - \#\{\text{marked loops}\}$$

We can also read off the étale local structure of the ramification locus of $\mathcal{A}$ in p . If $\beta_1, \dots, \beta_l$ are dimension vectors of simple representations of Q and if $m_1, \dots, m_l \in \mathbb{N}$ such that $\alpha = m_1\beta_1 + \dots + m_l\beta_l$ then there is a component of the ramification locus in p of dimension $d(\beta_1) + \dots + d(\beta_l)$ and locally isomorphic to

$$\text{iss}_{\beta_1} Q^\bullet \times \dots \times \text{iss}_{\beta_l} Q^\bullet$$

Working backwards, if we fix the central dimension d and the index n of the central simple algebra, then we have to determine all marked quiver settings $(Q^\bullet, \alpha)$ such that

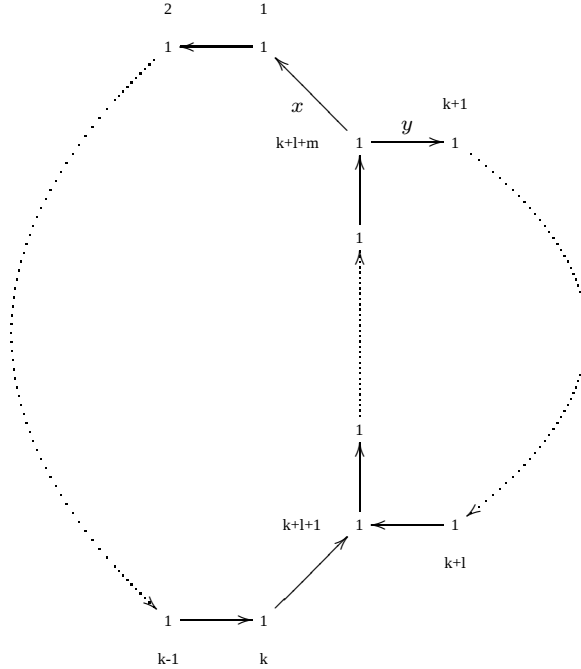
1. $d = d(\alpha)$ and α is the dimension vector of a simple representation of $Q^\bullet$.
2. For $\alpha = (e_1, \dots, e_k)$ determine all $\gamma = (d_1, \dots, d_k) \in \mathbb{N}_+^k$ such that $e_1 d_1 + \dots + e_k d_k = n$.

Then $\hat{A}_p$ must be of the form described above corresponding to such a triple $(Q^\bullet, \alpha, \gamma)$, its ramification locus in p only depends on $(Q^\bullet, \alpha)$ as is the étale type of X in p . In [12] a method is given to classify all relevant triples $(Q^\bullet, \alpha, \gamma)$ in low dimensions (≤ 4) and in [7] reduction steps were given to classify all possible central singularities in low dimensions (≤ 6). As for the singularities, there are none in dimension ≤ 2 and in dimensions 3 (resp. 4, 5, 6) there are precisely 1 (resp. 3, 10, 53) types of possible singularities for smooth Cayley-Hamilton orders.

These explicit étale local data combined with the Bloch-Ogus spectral sequence description of the n part of the Brauer group should be enough information to determine those central simple algebras Σ over K allowing a noncommutative smooth model in low central dimensions. In the next section we will perform the required calculations when the central dimension $d = 2$.

3. The case of surfaces

First, we recall the classification of all admissible triples $(Q^\bullet, \alpha, \gamma)$ for $d = 2$ given in [12, §6]. All admissible (marked) quiver settings $(Q^\bullet, \alpha)$ are of the form A_{klm} , $k, l \geq 0, m > 0$



where we make the obvious changes whenever k and/or l is zero. If $p = k + l + m$ then the admissible vectors γ are the unordered partitions $\gamma = (d_1, \dots, d_p)$ of n having exactly p parts. This allows us to determine all possible étale local structures of smooth Cayley-Hamilton $\mathcal{O}_S$ -orders $\mathcal{A}$ in Σ where S is a normal projective surface with function field K . If $p \in S$ has corresponding data

$$(A_{klm}, (1, \dots, 1), (d_1, \dots, d_p))$$

then using [12, Prop. 6.4] with the indicated arrows x and y (one can always reduce to this case using the $GL(\alpha)$ -action) we have

$$\widehat{\mathcal{O}_{S,p}} \simeq \mathbb{C}[[x, y]]$$

$$\hat{A}_p \simeq \begin{array}{|c|c|c|} \hline & \begin{array}{c} (1) \\ (x) \end{array} & \\ \hline & (y) & (1) \\ \hline \begin{array}{c} (x) \\ (x) \end{array} & \begin{array}{c} (1) \\ (y) \end{array} & (1) \\ \hline \begin{array}{c} (x) \\ (y) \end{array} & & \begin{array}{c} (1) \\ (x, y) \end{array} \\ \hline \end{array} \xrightarrow{\quad} M_n(\mathbb{C}[[x, y]])$$

$\underbrace{\hspace{1.5cm}}_k \quad \underbrace{\hspace{1.5cm}}_l \quad \underbrace{\hspace{1.5cm}}_m$

where at place (i, j) for $1 \leq i, j \leq p$ there is a block of dimension $d_i \times d_j$ with entries in the indicated ideal of $\mathbb{C}[[x, y]]$. Some immediate consequences can be drawn from these descriptions :

1. p is a smooth point of S . Hence, the central variety S must be a smooth projective surface.
2. $\mathcal{A}$ is étale splittable in p as $\hat{A}_p$ is an order in $M_n(\mathbb{C}[[x, y]])$.
3. For A_{001} , $\mathcal{A}_p$ is an Azumaya algebra. In all other cases, p is a point of the ramification locus. There are three possible subcases :
 - p is an isolated point of the ramification locus in case A_{00m} with $m > 1$.
 - p is a smooth point of a one-dimensional branch of the ramification locus through p in case A_{k01} or A_{0l1} .
 - The ramification locus has a normal crossing at p in all other cases.

Therefore we may assume that the central variety S of a smooth Cayley-Hamilton $\mathcal{O}_S$ -order in Σ is a smooth projective variety. In this case the coniveau spectral sequence describing the Brauer group $Br(K)$ is known as the Artin-Mumford exact sequence [5]. Consider the sequence of Abelian groups

$$0 \longrightarrow Br(S) \xrightarrow{i} Br(K) \xrightarrow{a} \bigoplus_C H^1(\mathbb{C}(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_p \mu^{-1} \xrightarrow{s} \mu^{-1} \longrightarrow 0$$

where the first sum runs over all irreducible curves C in S and the second over all points $p \in S$ and where the maps have the following ringtheoretic interpretation (see [5] or [18] for more details). The inclusion i is induced by assigning to an Azumaya $\mathcal{O}_S$ -order $\mathcal{A}$ its central simple ring of

central quotients. We have already seen that central simple algebras living in this image do have noncommutative smooth models.

The map a is determined by taking the ramification divisor of a maximal $\mathcal{O}_S$ -order $\mathcal{A}$ in Σ . Let C be an irreducible curve in S then it determines a discrete valuation ring R in K with residue field the functionfield $\mathbb{C}(C)$ which has a trivial Brauer group by Tsen's theorem. But then,

$$\frac{\mathcal{A} \otimes R}{\text{rad } \mathcal{A} \otimes R} \simeq M_s(L)$$

where L is a cyclic field extension of $\mathbb{C}(C)$. The map a sends the class $[\Sigma]$ to the class $[L]$ in $H^1(\mathbb{C}(C), \mathbb{Q}/\mathbb{Z})$ which classifies all cyclic extensions of $\mathbb{C}(C)$. The union of all curves C for which the extension is non-trivial is said to be the ramification divisor of $\mathcal{A}$ and because $\mathcal{A}$ is a maximal order it coincides with the non-Azumaya locus of $\mathcal{A}$.

The group $\mu^{-1} = \cup_n \text{Hom}(\mu_n, \mathbb{Q}/\mathbb{Z})$ and the map r measures the ramification of the cyclic extension L of $\mathbb{C}(C)$ in points $c \in C$, or equivalently, of the ramified cyclic covering of the normalization $\tilde{C}$ of C in all points $\tilde{c}$ of $\tilde{C}$ lying over c , see [18, p.110]. Finally, s is the sum map.

The Artin-Mumford theorem asserts that the maps i and a form an exact sequence whenever S is a smooth projective surface and the full sequence is exact whenever S is in addition simply connected. We will use a local version of the Artin-Mumford sequence, thus avoiding the issue of simple connectivity.

It will turn out that the requirement of being étale splittable at all points is the crucial obstruction to having a noncommutative smooth model. Therefore, we may restrict attention to maximal orders for if $\mathcal{A}$ is a smooth Cayley-Hamilton $\mathcal{O}_S$ -order in Σ which is everywhere étale splittable, so is every maximal order containing it.

Start with a smooth model S of K and take $\mathcal{A}_0$ to be a maximal $\mathcal{O}_S$ -order in Σ having D_0 as ramification divisor which may be highly singular. Using embedded resolution of singularities (see for example [10]) we can construct another smooth model $\tilde{S}$ of K such that

$$\begin{array}{ccc} & \tilde{S} & \\ & \downarrow \pi & \\ D_0 & \hookrightarrow & S \end{array}$$

the inverse image $\pi^{-1}(D_0) = \tilde{D}_0 \cup E$ (where $\tilde{D}_0$ is the strict transform and E is the exceptional fiber) has at worst normal crossings.

Next, take a maximal $\mathcal{O}_{\tilde{S}}$ -order $\mathcal{A}$ in Σ . Its ramification divisor D is a subdivisor of $\pi^{-1}(D_0)$ which (as we will see below) may contain components of the exceptional divisor E . Still, D has at worst normal crossings as singularities and we will determine the étale local structure of $\mathcal{A}$ in all points $p \in D$ to verify where it is a smooth Cayley-Hamilton order. We separate two cases :

case 1 : Let p be a smooth point of D . For $U = \text{spec } \mathbb{C}[[x, y]] - V(x)$ we have that $Br_n U = 0$ and as this is the étale local structure of $\tilde{S}$ and D near p it follows that $\mathcal{A}$ is étale splittable in p . Moreover, M. Artin determined in [2] the étale local structure of maximal orders over surfaces in a

smooth point of the ramification divisor. His results assert that $\hat{\mathcal{A}}_p$ has the block-decomposition

$$\hat{\mathcal{A}}_p \simeq \begin{bmatrix} (1) & \dots & \dots & (1) \\ (x) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ (x) & \dots & (x) & (1) \end{bmatrix}$$

where $n = a.b$ and all b^2 blocks have sizes $a \times a$ and entries in the indicated ideal of $\mathbb{C}[[x, y]]$. But this étale local structure corresponds to the triple

$$(A_{b-101}, (1, \dots, 1), (a, \dots, a))$$

whence $\mathcal{A}$ is a smooth Cayley-Hamilton order in p .

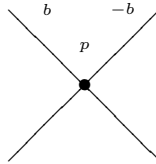
case 2 : Let p be a normal crossing of the ramification divisor D of $\mathcal{A}$. Let $U = \text{spec } \mathbb{C}[[x, y]] - V(xy)$ then the coniveau spectral sequence (or see [3]) gives the exact sequence

$$0 \longrightarrow Br_n U \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_n \xrightarrow{+} \mathbb{Z}_n \longrightarrow 0$$

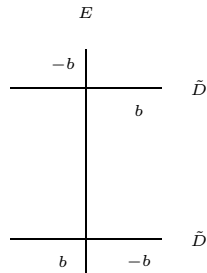
and all classes are determined by quantum planes $\mathbb{C}_q[[u, v]]$ where $vu = quv$ where q is an n -th root of one. As this is the étale local structure of $\tilde{S}$ and D near p , a combination with the results from [2] (or see [8]) asserts that

$$\hat{A}_p \simeq \begin{bmatrix} M_c(\mathbb{C}_q[[u, v]]) & \dots & \dots & M_c(\mathbb{C}_q[[u, v]]) \\ M_c(u\mathbb{C}_q[[u, v]]) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ M_c(u\mathbb{C}_q[[u, v]]) & \dots & M_c(u\mathbb{C}_q[[u, v]]) & M_c(\mathbb{C}_q[[u, v]]) \end{bmatrix} \hookrightarrow M_{ac}(\mathbb{C}_q[[u, v]])$$

where q is a primitive b -th root of one and $n = a.b.c$. This class corresponds to the local ramification data



where the two branches are given classes b and $-b$ in $\mathbb{Z}_n$. As a consequence, $\mathcal{A}$ is not étale splittable in p whenever $b \neq 0$. This problem cannot be resolved by blowing up the point p as there is only one possibility to satisfy the local sum zero condition for the divisor



which means that the exceptional divisor E will be part of the ramification divisor of a maximal order in Σ over the blow-up.

This concludes the proof of our main theorem :

Theorem 1 *Let K be a functionfield of transcendence degree two over an algebraically closed field of characteristic zero and let Σ be an n^2 -dimensional central simple K -algebra. Then, Σ contains a noncommutative smooth model if and only if there exists a smooth projective surface S with $\mathbb{C}(S) = K$ such that the ramification divisor of a maximal $\mathcal{O}_S$ -order A in Σ consists of a disjoint union of smooth irreducible curves.*

If $K = \mathbb{C}(x, y)$ then the 3-dimensional Sklyanin algebra (see for example [4]) defines a maximal $\mathcal{O}_{\mathbb{P}^2}$ -orders having as ramification divisor a smooth elliptic curve. As a consequence it is a noncommutative smooth model of the corresponding division algebra. On the contrary, the triangle 3-dimensional Auslander regular algebra defines a maximal $\mathcal{O}_{\mathbb{P}^2}$ -order with ramification divisor three projective lines intersecting transversally but with non-trivial ramification data. Therefore, it cannot be a noncommutative smooth model for the division algebra of the quantum plane. In fact, this division algebra cannot contain any noncommutative smooth model.

Another class of division algebras over $\mathbb{C}(x, y)$ possessing a noncommutative smooth model are the quaternion algebras from [5] in their construction of unirational non-rational threefolds. These threefolds are the Brauer-Severi varieties of the corresponding maximal orders over a blow-up.

4. Brauer-Severi varieties

Assume we are in the setting of section 1, that is, C is a normal affine domain with field of fractions K and A is a C -order in a central simple K -algebra Σ of dimension n^2 . We will define the *Brauer-Severi scheme* $\text{BSev } A$ following the account of M. Van den Bergh in [19].

There is a natural GL_n -action on the product $\text{trep}_n A \times \mathbb{C}^n$ defined by

$$g \cdot (\phi, v) = (g\phi g^{-1}, gv)$$

With $\text{brauer } A$ we will denote the open subset of *Brauer stable* points of this action

$$\text{brauer } A = \{(\phi, v) \mid \phi(A)v = \mathbb{C}^n\}$$

which is also the set of points with trivial stabilizer subgroup. Hence, every GL_n -orbit in $\text{brauer } A$ is closed and we can form the orbit space

$$\text{BSev } A = \text{brauer } A / GL_n \longrightarrow \text{triss}_n A$$

which is known to be a projective space bundle over the quotient variety $\text{triss}_n A$ (which has coordinate ring C). In general not much can be said about these Brauer-Severi schemes.

Lemma 1 *If A is a smooth Cayley-Hamilton C -order in Σ , then its Brauer-Severi scheme $\text{BSev } A$ is a smooth variety.*

Proof. Because brauer $A \longrightarrow \text{BSev } A$ is a principal GL_n -bundle it suffices to show that the total space brauer A is smooth. This is clear as it is an open subscheme of the smooth variety $\text{trep}_n A \times \mathbb{C}^n$.

If X is a normal projective variety with function field $\mathbb{C}(X) = K$ and $\mathcal{A}$ is a $\mathcal{O}_X$ -order in Σ we can sheafify the previous construction to obtain a projective space bundle

$$\text{BSev } \mathcal{A} \xrightarrow{\pi} X$$

with generic fibers isomorphic to $\mathbb{P}^{n-1}$ (in the Azumaya points) and if $\mathcal{A}$ is a noncommutative smooth model, $\text{BSev } \mathcal{A}$ is a smooth variety. An important problem is to determine in which cases π is a flat morphism and to determine the structure of all the fibers $\pi^{-1}(p)$ for $p \in X$.

If $\mathcal{A}$ is a noncommutative smooth model of Σ , then the étale local structure of $\mathcal{A}$ and of X near p is fully determined by a triple $(Q^\bullet, \alpha, \gamma)$. This triple also contains enough information to describe the étale local structure of the fibration $\text{BSev } A \longrightarrow X$ near p as the moduli space of a certain quiver setting which we now recall, more information can be found in [15].

If $Q^\bullet$ is a marked quiver on the vertices $\{v_1, \dots, v_p\}$ then we denote with $\tilde{Q}$ the extended marked quiver obtained from $Q^\bullet$ by adding an additional vertex v_0 and if $\gamma = (d_1, \dots, d_p)$ by adding for each $1 \leq i \leq p$ exactly d_i directed arrows from v_0 to v_i . If $\alpha = (e_1, \dots, e_p)$ then we will denote with $\tilde{\alpha}$ the extended dimension vector $(1, e_1, \dots, e_p)$. Any character of $GL(\tilde{\alpha}) = \mathbb{C}^* \times GL(\alpha) \longrightarrow \mathbb{C}^*$ is of the form $(g_0, g_1, \dots, g_p) \mapsto \det(g_0)^{t_0} \det(g_1)^{t_1} \dots \det(g_p)^{t_p}$ and is therefore fully determined by the $p+1$ -tuple of integers $\theta = (t_0, t_1, \dots, t_p)$. Consider the character determined by

$$\theta = (-n, d_1, \dots, d_p)$$

then clearly $\theta \cdot \tilde{\alpha} = 0$. In the quiver representation space $\text{rep}_{\tilde{\alpha}} \tilde{Q}$ consider the open subset rep^θ of θ -semistable representations as defined in [11], that is, all representations V such that for every proper subrepresentation W of V we have that $\theta \cdot \dim(W) \geq 0$. In this particular case we have that all θ -semistable representations are actually θ -stable, that is, for all proper subrepresentations W of V we have $\theta \cdot \dim(W) > 0$. The corresponding moduli space of quiver representations

$$\text{rep}^\theta / GL(\tilde{\alpha}) = \text{moduli}_{\tilde{\alpha}}^\theta \tilde{Q} \longrightarrow \text{iss}_{\tilde{\alpha}} \tilde{Q} = \text{iss}_\alpha Q$$

is a projective space bundle over $\text{iss}_\alpha Q$ and this bundle is étale isomorphic to the Brauer-Severi fibration in a neighborhood of p .

The upshot of this description is that it allows us to calculate the fibers $\pi^{-1}(p)$ of the Brauer-Severi fibration using the isomorphism (in the Zariski topology)

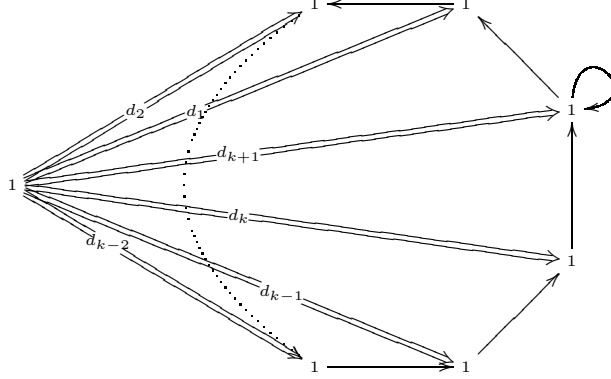
$$\pi^{-1}(p) = (\text{null}_{\tilde{\alpha}} \tilde{Q} \cap \text{rep}^\theta) / GL(\tilde{\alpha})$$

where $\text{null}_{\tilde{\alpha}} \tilde{Q}$ is the nullcone of quiver representation which can be analyzed by means of the Hesselink stratification as described in [13].

As an example, let us perform the necessary computations in the case of surfaces and a point $p \in S$ where the triple is of the form

$$(Q^\bullet = A_{k01}, \alpha = (1, \dots, 1), \gamma = (d_1, \dots, d_{k+1}))$$

and γ is a unordered partition of n having precisely $k + 1$ parts. Observe that the étale local description of M. Artin of maximal orders over surfaces in smooth points of the ramification divisor is a special case of such a setting having the additional restriction that $d_1 = \dots = d_{k+1}$. The extended quiver setting $(\tilde{Q}, \tilde{\alpha})$ is of the form



and the character corresponds to $\theta = (-n, d_1, \dots, d_{k+1})$. The weights of the maximal torus $T = GL(\tilde{\alpha})$ occurring in the representation space $\text{rep}_{\tilde{\alpha}} \tilde{Q}$ form the set

$$\{\pi_{0i}, \pi_{ii+1} \mid 1 \leq i \leq k + 1\}$$

and we know from [13] that strata in the nullcone $\text{null}_{\tilde{\alpha}} \tilde{Q}$ are determined by saturated subsets of this set of weights. Potential maximal strata correspond to maximal saturated sets which in this case are exactly the following subsets

$$S_i = \{\pi_{ii+1}, \pi_{i+1i+2}, \dots, \pi_{i-2i-1}\} \cup \{\pi_{0j} \mid 1 \leq j \leq k + 1\}$$

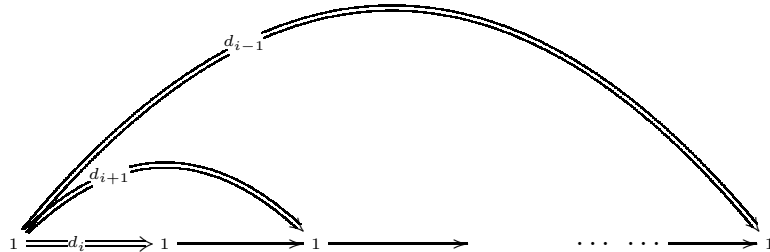
for every $1 \leq i \leq k + 1$. To determine whether the corresponding Hesselink strata of the nullcone is non-empty we have to determine whether the associated level quiver has θ_i -semistable representations for a specific character θ_i (we refer for all these notions to the paper [13]). Allow us to state that the data associated to S_i are : the level quiver data (Q_i, α_i) is of the form

$$\begin{array}{ccccccc} -k-1 & & -k+1 & & -k+3 & & & & k-1 & & k+1 \\ 1 \rightrightarrows_{d_i} 1 & \longrightarrow & 1 & \longrightarrow & \dots & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$

where the superscripts indicate the entries of the character θ_i . Clearly, there are θ_i -semistable representations in $\text{rep}_{\alpha_i} Q_i$ (for example, all arrows equal to 1) and in fact the corresponding moduli space is

$$\text{moduli}_{\alpha_i}^{\theta_i} Q_i \simeq \mathbb{P}^{d_i-1}$$

Hence, the Hesselink strata determined by S_i is non-empty and using the description of [13] the full stratum consists of all representations rep_i^{ss} of the extended quiver $\tilde{Q}_i$ which is of the form



such that the part of the representation in the level quiver Q_i is θ_i -semistable. That is, there is an irreducible component of $\text{null}_{\tilde{\alpha}} \tilde{Q}$ corresponding to each $1 \leq i \leq k+1$ each being the closure rep_i^{ss} which is of dimension $n+k$. As every point in the (non-empty) open subset

$$\text{rep}_i^{ss} \cap \text{rep}^\theta$$

is θ -stable and hence has stabilizer subgroup $\mathbb{C}^*$ in $GL(\tilde{\alpha})$ which is a $k+2$ -dimensional torus we have that the dimension of the irreducible component in the Brauer-Severi fiber $\pi^{-1}(p)$ determined by S_i has dimension $n+k-(k+1)=n-1$ for each of the $k+1$ choices of i . Clearly, the fibers of the Brauer-Severi fibration $\text{BSev } A \xrightarrow{\pi} S$ over an Azumaya point $p \in S$ are isomorphic to $\mathbb{P}^{n-1}$ and hence are also of dimension $n-1$. This completes the proof of

Theorem 2 *Let $\mathcal{A}$ be a noncommutative smooth model of Σ over a smooth surface S such that the ramification locus of $\mathcal{A}$ consists (at worst) of a disjoint union of smooth irreducible curves in S . Then, the Brauer-Severi fibration*

$$\text{BSev } \mathcal{A} \xrightarrow{\pi} S$$

is a flat morphism and the fiber over a ramified point with corresponding marked quiver A_{k01} has exactly $k+1$ irreducible components (each of dimension $n-1$).

Recall that the condition is automatic for $\mathcal{A}$ a smooth maximal $\mathcal{O}_S$ -order hence this result extends the calculation of Artin and Mumford in [5] for the Brauer-Severi scheme of smooth maximal orders in specific quaternion algebras.

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