universiteit antwerpen • uia • universiteitsplein 1 • 2610 antwerpen (wilrijk) • tel (03)820 20 20

I

$$\begin{tikzcd}
 & \mathrm{Hilb}_n \mathbb{C}^2 \arrow[dl] \arrow[dr, dash, "C^\infty \text{ diffeo}"] & \\
\cdots\ S^n \mathbb{C}^2 \ \cdots & & \mathrm{Calo}_n \ \cdots
\end{tikzcd}$$

(Wilson) $\bigsqcup_n \mathrm{Calo}_n = Gr^{ad}$

(Ginzburg) $\mathrm{Calo}_n \hookrightarrow g^*$

g Poisson-Lie algebra of ftions of n.c. symplectic manif.

$$\bigsqcup_n \mathrm{Calo}_n \hookrightarrow g^*$$

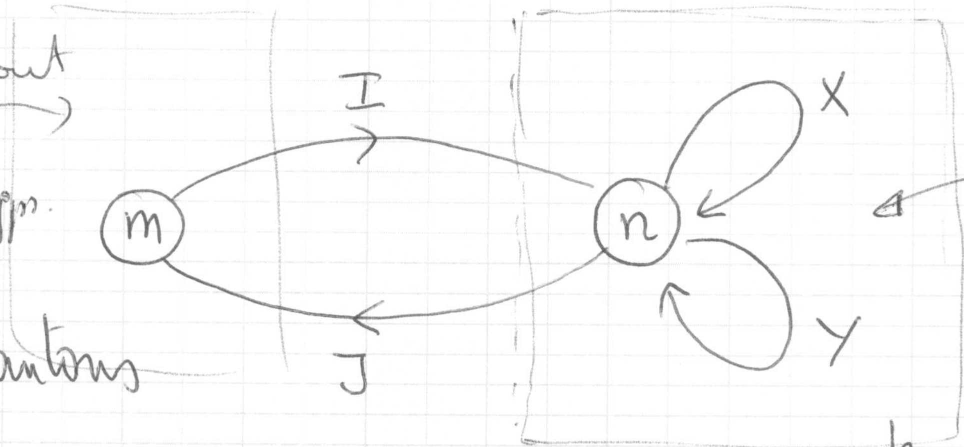
? how general ?

Thm: Every $\underbrace{\text{quiver variety}}_{\text{today}}$ is $\underbrace{\text{diffeomorphic}}_{\text{hyper-Kähler}}$ to a $\underbrace{\text{coadj. orbit}}_{\text{n.c. symplectic}}$

$$\mathrm{Calo}_n = \{ (X,Y) \in M_n \oplus M_n \mid \mathrm{rk}([X,Y] - \mathbb{1}_n) \leq 1 \} // GL_n$$

info about rk's symmetry grps

rk $\leq m$



info about changes

basechange

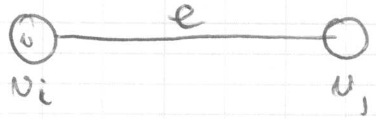
m : Donaldson : instantons

typical relations : $[X,Y] + IJ = \mathbb{1}_n$

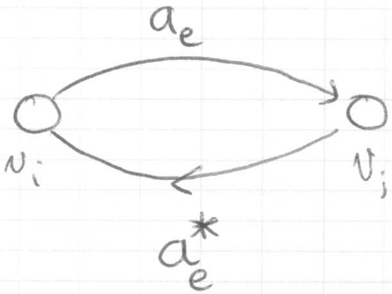
universiteit antwerpen • uia • universiteitsplein 1 • 2610 antwerpen (wilrijk) • tel (03)820 20 20

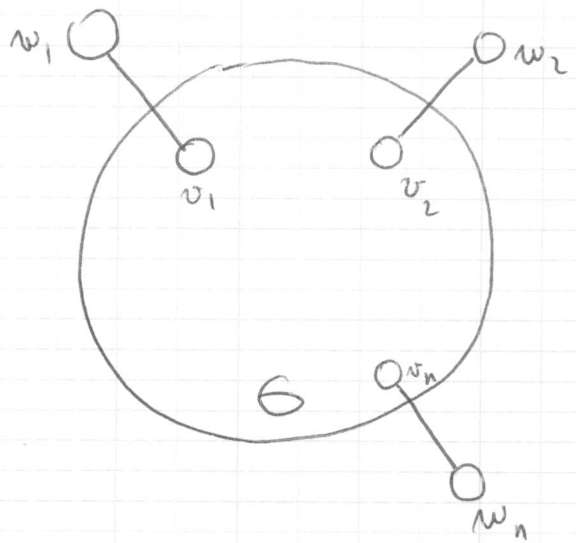
NOTA'S NOTA'S

$\boxed{G}$ fin. graph on n vertices $\{v_1, \dots, v_n\}$
allow loops + multiple edges

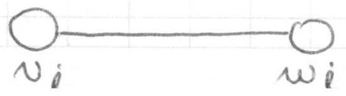


$\leftrightsquigarrow$





add n external vertices
$\{w_1, \dots, w_n\}$
n external edges.



$\leftrightsquigarrow$

$\underline{a} = (a_1, \ldots, a_n)$
$\underline{b} = (b_1, \ldots, b_n)$ $\in \mathbb{N}^n$ dimensions of vertexsp at $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_n\}$

$$R(\underline{a}, \underline{b}) \xrightarrow{\mu_{\mathbb{C}}} M_{a_1}(\mathbb{C}) \oplus \cdots \oplus M_{a_n}(\mathbb{C}) = \mathrm{Lie}\, GL(\underline{a})$$

$$\underbrace{(A_e, A_e^*, I_e, J_e)}_{GL(\underline{a}) \text{ acts via basechange}} \longmapsto \Big(\quad , \ \overbrace{\sum A_e A_e^* - \sum A_e^* A_e + I_k J_k}^{k} \ , \quad \Big)$$

$\xrightarrow{a_e} v_k$ $\qquad$ $v_k \xrightarrow{a_e}$

$v_l \xrightarrow{a_e} v_k$

$g \cdot A_e \rightsquigarrow g_k A_e g_l^{-1}$

$g \cdot I_k \rightsquigarrow g_k I_k \ldots$

Zenter $\mathrm{Lie}\, GL(\underline{a}) = \{\lambda \mid (\lambda_1 \mathbb{1}_{a_1}, \ldots, \lambda_n \mathbb{1}_{a_n})$
$\lambda_i \in \mathbb{C}\}$

λ- Nakajima variety

$$\boxed{N_\lambda(\underline{a},\underline{b}) = \underbrace{\mu_{\mathbb{C}}^{-1}(\lambda) /\!/ GL(\underline{a})}_{\text{closed orbits in } \mu_{\mathbb{C}}^{-1}(\lambda)}}$$

Ex:
$$\begin{tikzcd} 1 \arrow[r, bend left] & n \arrow[l, bend left] \arrow[loop above] \arrow[loop below] \end{tikzcd}$$
$N_0 = S^n \mathbb{C}^2$, $N_1 = Calo_n$

Stability condition on pts in $\mu_{\mathbb{C}}^{-1}(0)$: $\mu_{\mathbb{C}}^{-1}(0)^s$

$\forall k$: $V'_k \subset \mathbb{C}^{a_k}$ s.t. ① $\begin{tikzcd} v_k \arrow[r, "A_e^{(*)}"] & v_l \end{tikzcd}$

$A_e^*(V'_k) \subset V'_l$

② $I_k V'_k = 0$

$\Longrightarrow$ all $V'_k = 0$

Quiver-variety

$$M_0(\underline{a},\underline{b}) = \mu_{\mathbb{C}}^{-1}(0)^s /\!/ GL(a)$$

$$M_0(\underline{a}, b)$$

$M_0(\underline{a},b) \longrightarrow N_0(a,b)$; $M_0(\underline{a},b)$ — C^∞ diffeo — $N_\lambda(a,b)$

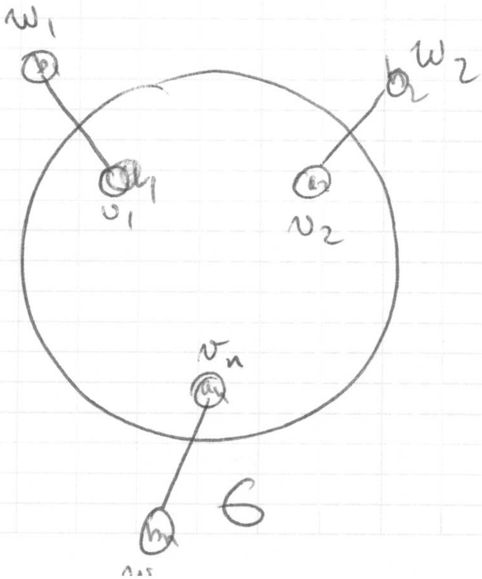
- - - $N_0(a,b)$ - - - - - - $N_\lambda(a,b)$ - - -

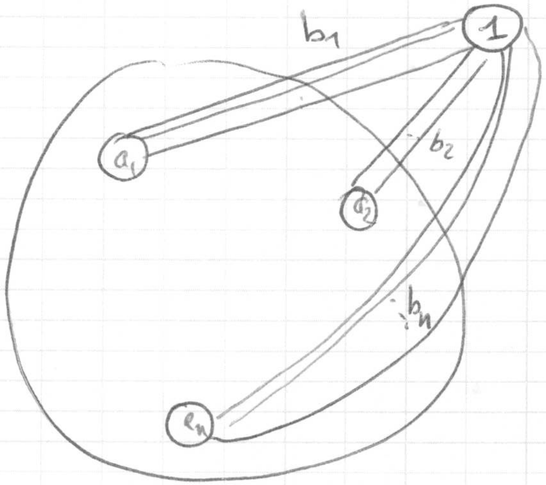
sing   smooth

same $\underline{b}$, diff $\underline{a}$ $\Rightarrow$ same Γ

same $\underline{a}$, diff $\underline{b}$ $\Rightarrow$ diff Γ

<u>deframing</u> (Crawley-Boevey)





Γ corresp. quiver

$\alpha = (a_1, \dots, a_n, 1)$

decomp in rows/colums.

$$\begin{tikzcd}
R(\underline{a},\underline{b}) \arrow[d, "\mu_{\mathbb{C}}"'] \arrow[r, leftrightarrow] & \mathrm{rep}_\alpha \Gamma \arrow[d, "\mu_{\mathbb{C}}"] & (A, A^*) \arrow[d, mapsto] \\
GL(\underline{a}) \arrow[r, leftrightarrow] & PGL(\alpha) = GL(\alpha)/\mathbb{C}^*(1, \dots, 1) & \\
\mathrm{Lie}\ GL(\underline{a}) \arrow[r, leftrightarrow] & \mathrm{Lie}\ PGL(\alpha) = GL(\alpha)/(\Sigma\, t_i = 0) & (\dots , \sum AA^* - \sum A^*A , \dots)
\end{tikzcd}$$

$\underline{\lambda\text{- moment algebra}}$

$$\Pi_\lambda = \frac{\mathbb{C}\Gamma}{(\sum [a, a^*]) - \lambda_1 e_1 - \dots - \lambda_\infty e_\infty)} \longleftarrow \text{path algebra.}$$

$$\mathrm{rep}_\alpha \Pi_\lambda = \mu_{\mathbb{C}}^{-1}(\lambda)$$

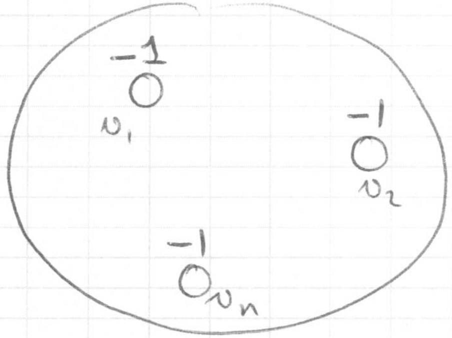
$$\mu_{\mathbb{C}}^{-1}(\lambda) = \mathrm{rep}_\alpha\, \Pi_\lambda \qquad \boxed{N_\lambda(\underline{a},\underline{b}) = \mathrm{iss}_\alpha\, \Pi_\lambda}$$

isomorphy classes of α-dim'l semi-simple repn of Π_λ

$$\mu_{\mathbb{C}}^{-1}(0)^s = \underbrace{\mathrm{rep}_\alpha^{\theta\text{-st}}\, \Pi_0}$$

Σa_i

$\bigcirc_{v_\infty}$



$\theta = (-1, -1, \dots, -1, \Sigma a_i)$

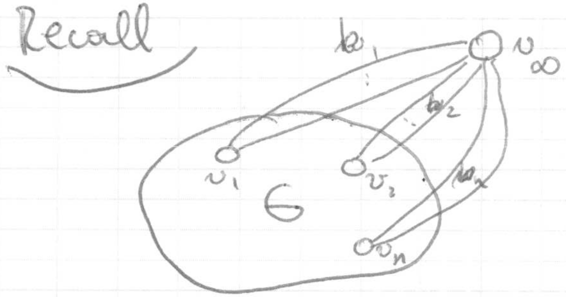
$\theta . \alpha = 0$

$$\|$$

$$\{ V \in \mathrm{rep}_\alpha\, \Pi_0 \mid \forall\ 0 \neq W \subsetneq V \ :\ \theta . \dim(W) > 0 \}$$

isomorphy classes of θ-stable α-dim'l ... of Π $= \boxed{M_\lambda(\underline{a},\underline{b}) = M^s_\alpha(\Pi_0, \theta)}$

Recall



Γ is corresp quiver

$\alpha = (a_1, \ldots, a_n, 1)$

$\lambda = (\lambda_1, \ldots, \lambda_n, \lambda_\infty) \in \mathbb{C}^{*n+1}$

$\theta = (-1, \ldots, -1, \Sigma a_i)$

$$\mathbb{C}\Gamma \text{ path alg} \longrightarrow \Pi_\lambda = \frac{\mathbb{C}\Gamma}{(\Sigma[a,a^*] - \lambda)}$$

edge $\circ\!\!-\!\!\circ$ is $\circ \underset{a^*}{\overset{a}{\rightleftarrows}} \circ$

quiver var. $M_\alpha^{\theta\text{-}st}(\Pi_0, \theta)$

will prove : (1) $\swarrow$ (2) diffeo $\searrow$

$$\ldots \; \mathrm{im}_\alpha \Pi_0 \; \text{-------} \; \mathrm{im}_\alpha \Pi_\lambda \; \ldots$$

Nakajima var.

nilp — smooth

to do

$\boxed{A}$ when varieties $\neq \emptyset$
i.e. when do equation have solution

$\boxed{B}$ varieties come from invar. theory. Recall
- alg geom. descript
- diff geom. descript.

Holland + Crawley-Boevey

$\mathrm{rep}_\alpha \Pi \xrightarrow{\mu_C} \mathrm{Lie}\, PGL(\alpha)$ ②

$X \longmapsto \sum_a [X_a, X_{a^*}]$

$\boxed{A}$ First application of connection with NC alg: when is $\mu_\alpha(x) \neq \emptyset$?

Choose orientation (a, a^*) on graph ~~double~~ quiver Q on a's

$V \in \mathrm{rep}_\alpha Q$ have exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{C}Q}(V,V) \longrightarrow M_\alpha(\mathbb{C}) \xrightarrow{f} \mathrm{rep}_\alpha Q \longrightarrow \mathrm{Ext}^1_{\mathbb{C}Q}(V,V) \to 0$$

$$(M_1, \dots, M_n, M_\infty) \mapsto \begin{array}{ccc} \circ & \xrightarrow{V_a} & \circ \\ {\scriptstyle M_{s(a)}}\downarrow & & \downarrow{\scriptstyle M_{t(a)}} \\ \circ & \xrightarrow[V_a]{} & \circ \end{array}$$

$$M_{t(a)} V_a - V_a M_{s(a)}$$

dualize as $\mathbb{C}$-spaces

$$0 \to \mathrm{Ext}^1(V,V)^* \longrightarrow \mathrm{rep}_\alpha Q^{op} \xrightarrow{c} M_\alpha(\mathbb{C})^* \xrightarrow{t} \mathrm{Hom}(V,V)^* \longrightarrow 0$$

(Q^{op} = on a^* arrow)

|| trace pair

$M_\alpha(\mathbb{C})$

$$W \longmapsto \sum_a [V_a, W_{a^*}]$$

$$(M_1, \dots, M_n, M_\infty) \mapsto \left((N_1, \dots, N_n, N_\infty) \mapsto \sum_{i}^{\infty} \mathrm{tr}(M_i N_i) \right)$$

V can be extended to $X \in \mu_\alpha^{-1}(\lambda)$ iff $\forall\ V' \lhd V$: $\lambda . \dim V' = 0$

$\Rightarrow$ the $\exists W \in rep_\alpha Q^{op}$ s.t. $\lambda \in \operatorname{Im}(c)$ so because exact $t(\lambda) = 0$ so for each $N \in Hom(V,V)$: $\lambda . tr(N) = 0$ $V' \lhd V$ take $N: V \twoheadrightarrow V' \hookrightarrow V$ then $tr(N) = \dim V'$

$\Leftarrow$ suffices to prove lifting for all indec. s.t. $\lambda . \dim = 0$ any $N \in Hom(V,V)$ local to $N =$ nilp + scalar so $\sum \lambda_i\, tr(N_i) = 0$ but then λ must be in image of c $\square$

From now on : always assume $\mu_\alpha^{-1}(\lambda) \neq 0$

$$\begin{array}{ccccc}
\mathrm{rep}_\alpha \Pi_\lambda = \mu_{\mathbb{C}}^{-1}(x) & \hookrightarrow & \mathrm{rep}_\alpha \Gamma & \xrightarrow{\mu_{\mathbb{C}}} & \mathrm{Lie}\, PGL(\alpha) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
\mathrm{iss}_\alpha \Pi_\lambda & \hookrightarrow & \mathrm{rep}_\alpha \Gamma /\!/ GL(\alpha) & & \\
 & & \parallel & & \\
 & & \mathrm{iss}_\alpha \Gamma & &
\end{array}$$

closed orbits = (Artin) semisimple repr.

Ringtheoret. descr. $\underbrace{\mathbb{C}[\mathrm{iss}_\alpha \Gamma] = \mathbb{C}[\mathrm{rep}_\alpha \Gamma]^{GL(\alpha)}}_{\text{if we know generators}} \twoheadrightarrow \mathbb{C}[\mathrm{iss}_\alpha \Pi_\lambda]$

$\Rightarrow$ know generators J

<u>Example</u> Γ (one vertex with a loop) $\mathrm{rep}_n \Gamma = M_n(\mathbb{C})$ $GL(n)$ acts via ~~sim~~. conjugation

orbits : Jordan normal form | closed orbits = diag. matrices

invariants = coeff. of char. polynomial $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & \varepsilon \\ 0 & c \end{pmatrix}$

$= Tr(X^k)$

$\downarrow$

$\mathbb{C}^n \simeq \mathbb{C}[Tr(X), \ldots, Tr(X^n)]$

$(V_b V_b^\dagger - V_b^\dagger V_b)^\dagger$

$= V_b V_b^\dagger - V_b \dots$

Thm (LB + Procesi) $\mathbb{C}[\mathrm{iss}_\alpha \Gamma]$ is generated by traces along oriented cycles in Γ.

diff geom description

$$\mathrm{rep}_\alpha \Gamma \xrightarrow{\mu_{\mathbb{R}}} \mathrm{Lie}\Big(U(\alpha) = U(a_1) \times \dots \times U(a_k) \times U(1) / U(1)\ (\dots)$$

Lie $U(n)$ = skew Hermit. matrices

$$V \longmapsto \frac{i}{2} \sum_{b \in \Gamma_{arr}} [V_b, V_b^\dagger]$$

Example 

$$M_n(\mathbb{C}) \xrightarrow{\mu_{\mathbb{R}}} i\,\mathrm{Herm}_n$$

$$A \longmapsto \frac{i}{2}[A, A^\dagger]$$

$\mu_{\mathbb{R}}^{-1}(0) = \{A \mid AA^\dagger = A^\dagger A\}$ normal matrices = diagonaliz. matr

every $U(n)$-orbit contains ~~every U(n)-orbit~~ and diag. mat are normal and permut. $\hookrightarrow U(n)$.

have $M_n(\mathbb{C}) // GL(n) \longleftrightarrow \mu_{\mathbb{R}}^{-1}(0)/U(n)$

Thm (Kempf-Ness) $\mathrm{iss}_\alpha \Gamma = \mu_{\mathbb{R}}^{-1}(0)/U(\alpha)$ as a consequence: have for

Nakajima varieties

$$\boxed{N = \mathrm{iss}_\alpha \Pi_\lambda = \left(\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda)\right) / U(\alpha)}$$

take

$\theta = (\theta_1, \dots, \theta_\infty) \in \mathbb{Z}^{n+1}$ with $\theta . \alpha = 0$

$$\text{(semi)}\quad rep_\alpha^{\theta\text{-}st}\,\Gamma \hookrightarrow rep_\alpha \Gamma$$

reps having no proper subm s.t. $\theta . \alpha_{(i)} \leq 0$

the $GL(\alpha)$ has max closed orbits on this open set

$$rep_\alpha^{\theta\text{-}ss}\,\Gamma // GL(\alpha) = M_\alpha^n(\Gamma, \theta)$$

Ex: Γ: $\textcircled{1} \xrightarrow{x_0} \textcircled{1}$, $\vdots$, $\xrightarrow{x_n}$

θ: $-1 \qquad 1$

$\mathbb{C}^{n+1} - \{(0, \dots, 0)\} = rep^\theta \Gamma \subset rep \Gamma = \mathbb{C}^{n+1}$

$PGL(\alpha) = (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{C}^*$ act via

$(\mu \lambda^{-1} x_0, \dots, \mu \lambda^{-1} x_n)$ so

$$rep^\theta \Gamma // GL(\alpha) = \mathbb{P}^n$$

$(x_0, \dots, x_n)$ are all θ-semi's.

$SI = \mathbb{C}[x_0, \dots, x_n]$

$\mathbb{P}^n = \text{proj } \mathbb{C}[x_0, \dots, x_n]$

(ringtheoretical) $\quad GL(\alpha) \xrightarrow{\theta} \mathbb{C}^*$

$$(g_1, \dots, g_\infty) \mapsto \prod g_i^{\theta_i}$$

$$SI = \bigoplus_{n=0}^{\infty} \{ f \in \mathbb{C}[rep \Gamma] \mid g . f = \hat{\theta}(g) f \}$$

<u>Th</u> (King) $\quad M_\alpha^{\theta\text{-}ss}(\Gamma, \theta) = \text{proj}(SI)$

and $SI_0 = \mathbb{C}[rep \Gamma]^{GL(\alpha)}$

so gives projection $M \twoheadrightarrow iss_\alpha$ diagram

(geometrical)

$$rep_\alpha \Gamma = \mathbb{C}^{n+1} \xrightarrow{\mu_{\mathbb{R}}}$$

$(x_0, \dots, x_n) \mapsto i(-\bar{x}_0 x_0 - \dots - \bar{x}_n x_n, \; x_0 \bar{x}_0 + \dots + x_n \bar{x}_n)$

$x_n = y_n + i z_n$

$$\mu_{\mathbb{R}}^{-1}\left(\tfrac{i}{2}\theta\right) = \mu_{\mathbb{R}}^{-1}\left(-\tfrac{i}{2}, \tfrac{i}{2}\right) = \{(x_0, \dots, x_n) \mid x_0\bar{x}_0 + \dots + x_n\bar{x}_n = 1\} = S^{2n+1} \hookrightarrow \mathbb{R}^{2n+2}$$

$$\downarrow \text{Hopf fibration}$$

$$S^{2n+1}/U(1) = \mathbb{P}^n$$

$U(\alpha) = U(1) \times U(1)/U(1) = U(1)$

<u>Thm</u> (Kingy) $M_\alpha^{\theta}(\Gamma, \theta) = \mu_{\mathbb{R}}^{-1}(\tfrac{i\theta}{2}) / U(\alpha)$ so in particular for

Quiver varieties

$\theta = (-1, \dots, -1, \Sigma a_i)$

$$\boxed{M_\xi = M_\alpha^n(\Pi_0, \theta) = \left(\mu_{\mathbb{R}}^{-1}(\tfrac{i\theta}{2}) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/U(\alpha)}$$

Nakajima's hyper-Kähler trick

$\mathbb{H} = \mathbb{R}.1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$

$ij = -ji = k$
$i^2 = j^2 = k^2 = -1$

action of $\mathbb{H}$ on rep Γ

(check satisfies)

$$\begin{cases} i \cdot V_b = iV_b & \\ j \cdot V_a = -V_{a^*}^{+} & j \cdot V_{a^*} = V_a^{+} \\ k \cdot V_a = -iV_{a^*}^{+} & k \cdot V_{a^*} = iV_a^{+} \end{cases}$$

$$h = \frac{i + k}{\sqrt{2}}$$

check: $\begin{cases} \mu_{\mathbb{C}}(h.V) = \frac{1}{2}\left(\mu_{\mathbb{C}}(V)^{+} - \mu_{\mathbb{C}}(V)\right) - i\mu_{\mathbb{R}}(V) \\ \mu_{\mathbb{R}}(h.V) = \frac{i}{2}\left(\mu_{\mathbb{C}}(V)^{+} + \mu_{\mathbb{C}}(V)\right) \end{cases}$

For $\frac{1}{2}\theta = \frac{1}{2}(-1, \ldots, -1, \sum a_i) \in \mathbb{Z}^{n+1}$

$$\begin{tikzcd}
\mu_{\mathbb{C}}^{-1}(\frac{1}{2}\theta) \cap \mu_{\mathbb{R}}^{-1}(0) \arrow[r, "h."', ""] \arrow[r, "\text{diffeo}"'] \arrow[d] & \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\frac{i}{2}\theta) \arrow[d] \\
\mu_{\mathbb{C}}^{-1}(\frac{1}{2}\theta) \cap \mu_{\mathbb{R}}^{-1}(0)/U(\alpha) \arrow[r, "h."] \arrow[r, "\text{diffeo}"'] & \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\frac{i\theta}{2})
\end{tikzcd}$$

and h. commutes with $U(\alpha)$-action

$$\boxed{\mathrm{iss}_{\alpha}\, \Pi_{\frac{1}{2}\theta} \underset{\text{diffeo}}{\sim} M_{\alpha}^{n}(\Pi_0, \theta)}$$

III

$a \leftrightarrow a^*$ on arrows

Γ symmetric quiver on $n+1$ vertices ; $\begin{cases} \alpha = (a_1, \dots, a_n, 1) \\ \lambda = (-1, \dots, -1, \sum a_i) \end{cases}$

$$rep_\alpha \Gamma \xrightarrow{\mu_{\mathbb{C}}} M(\alpha) = M_{a_1} \oplus \dots \oplus M_{a_n} \oplus \mathbb{C}$$

$$V = \{(A, A^*)\} \longmapsto \sum_a [A, A^*]$$

$$\mu_{\mathbb{C}}^{-1}(\lambda) = rep_\alpha \Pi_\lambda \qquad \text{where} \qquad \Pi_\lambda = \frac{\mathbb{C}\Gamma}{(\sum [a, a^*] - \lambda)}$$

We want to show that $\qquad$ is a coadjoint orbit

$$N_\lambda = rep_\alpha \Pi_\lambda /\!/ GL(\alpha) = iss_\alpha \Pi_\lambda$$ (pts $\leftrightarrow$ isoclasses of s.s. α-diml representations)

what do we mean by this and
is there any reason to expect this to be true?

universiteit antwerpen • uia • universiteitsplein 1 • 2610 antwerpen (wilrijk) • tel (03)820 20 20

Recall: only representations V of Π_λ are such that λ-$\underline{\dim}\, V = 0$
so $V \in rep_\alpha \Pi_\lambda$ is always a simple Π_λ-module.

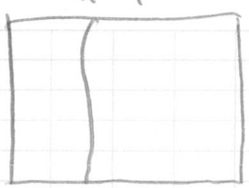
Pf: If not go to JH-series and $gr\, V = S_1 \oplus \dots \oplus S_e$
$\sum di\, S_i = (a_1, \dots, a_n, 1)$ so $\exists!\, S_i$ has $di\, S_i = (\dots, 1)$
but $\lambda . di\, S_i = 0$ so $di\, S_i = (a_1, \dots, a_n, 1)$.

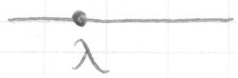
so quotient variety is really an orbit space $rep_\alpha \Pi_\lambda // GL(\alpha) = rep_\alpha \Pi_\lambda / GL(\alpha)$
and $rep_\alpha \Pi_\lambda$ is principal $PGL(\alpha)$-bundle so corresponds to an

$$\begin{array}{c} rep_\alpha \Pi_\lambda \\ \downarrow \\ iss_\alpha \Pi_\lambda \end{array}$$

Azumaya algebra

Computing differentials one shows that $\mu_{\mathbb{C}}^{-1}(\lambda)$ is smooth in any pt with trivial stabilizer, so here $rep_\alpha \Pi_\lambda$ is smooth and hence also $N_\lambda = iss_\alpha \Pi_\lambda$ is smooth. All pts are similar (all simple)





? $\exists$ action of a ∞-dim'l alg gp acting transitively

? a natural choice might be

$$G = \{\sigma \in Aut\, \mathbb{C}\Gamma : \sigma(\sum[a,a^*]) = \sum[a,a^*]\}$$

because this preserves the fibers. or an extension $\tilde{G} \to G$ of it.

what we will show is that this holds at least at the Lie level, so we will prove

$$\underbrace{N_\lambda = iss_\alpha \pi_\lambda \hookrightarrow \mathfrak{g}^*}_{\text{comes from a Lie algebra map}} \quad \text{where} \quad \overset{\text{Lie alg}}{\mathfrak{g}} \twoheadrightarrow \overset{\text{derivations preserving } \sum[a,a^*]}{Der_\omega}\, \mathbb{C}\Gamma$$

$$\mathfrak{g} \longrightarrow \mathbb{C}[iss_\alpha \pi_\lambda] = \mathbb{C}[rep_\alpha \pi_\lambda]^{GL(\alpha)}$$

before we construct $\mathfrak{g}$ we will look at Lie structure on $\mathbb{C}[iss_\alpha \pi_\lambda]$

$\mathbb{C}[x_1,y_1;\ \dots\ ;x_n,y_n]$

f,g

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i^*} - \frac{\partial f}{\partial x_i^*}\frac{\partial g}{\partial x_i}\right)$$

makes algebra into a Poisson algebra

$\mathbb{C}^{2n}$ with anti-symm. bilinear form

$$\begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ & & \ddots \end{bmatrix}$$

have pairing $x_i \longleftrightarrow x_i^* = y_i$

$rep_\alpha \Gamma$ has also natural anti-symmetric bilin. form

$$(V,W) = \sum_a tr(V_a W_{a^*} - W_{a^*} W_a)$$

on $\mathbb{C}[rep_\alpha \Gamma]$ then gives pairing of coordinate ftions

$$\begin{tikzcd} k \arrow[r, bend left, "a"] & l \arrow[l, bend left, "a^*"] \end{tikzcd}$$

$$\begin{array}{cc} & k \\ l & \begin{bmatrix} x_{ij}(a) \end{bmatrix} \end{array} \qquad \begin{array}{cc} k & \begin{bmatrix} y_{ij}(a^*) \end{bmatrix} \\ & l \end{array}$$

$$x_{ij}(a) \longleftrightarrow x_{ij}(a)^* = y_{ji}(a^*)$$

this pairing gives rise to Lie bracket on $\mathbb{C}[rep_\alpha \Gamma]$ as before.

$$\mathbb{C}[rep_\alpha \Gamma] \ ; \ \{\ \}$$

compatible with action of $GL(\alpha)$ so induces Lie algebra structure on $\mathbb{C}[rep_\alpha \Gamma]^{GL(\alpha)} = \mathbb{C}[iss_\alpha \Gamma]$

moreover, because of form also induces $\{,\}$ on $\mathbb{C}[rep_\alpha \Pi_\lambda]$ and hence on $\mathbb{C}[N_\lambda]$

universiteit antwerpen • uia • universiteitsplein 1 • 2610 antwerpen (wilrijk) • tel (03)820 20 20

NOTA'S NOTA'S

$\mathbb{C}[\mathrm{iss}_\alpha \Gamma] \twoheadrightarrow \mathbb{C}[\mathrm{iss}_\alpha \Gamma_\lambda] = N_\lambda$ is Lie morphism

NOW: want to find common Lie algebra $\mathfrak{g}$ for all α and Lie algebra map $\mathfrak{g} \twoheadrightarrow \mathbb{C}[\mathrm{iss}_\alpha \Gamma]$

$\mathbb{C}\Gamma$ is a quasi-free algebra and there is a natural pairing $a \leftrightarrow a^*$ between generators so expect $\mathfrak{g}$ to be a Poisson Lie algebra on n.c. ftions

? what are n.c. ftions

? define $\frac{\partial}{\partial a}$, $\frac{\partial}{\partial a^*}$ on the

idea is to end up with a unique description of the for

exterior diff of ftion $\rightarrow$ $df = \sum_a g_a \, da + \sum_{a^*} h_{a^*} \, da^*$

and can then define

$g_a = \frac{\partial f}{\partial a}$ and $h_{a^*} = \frac{\partial f}{\partial a^*}$

so will start off with dfn of Cuntz-Quillen of n.c. differentialforms and do some reductions until we get what we want.

Cuntz - Quillen   n.c. differential forms

$$\bar{A} = A / \mathbb{C}.1$$

$\Omega^n A = A \otimes \underbrace{\bar{A} \otimes \cdots \otimes \bar{A}}_{n}$ $(a_0, a_1, \ldots, a_n) \approx a_0 da_1 \ldots da_n$

$\Omega^\bullet A$ is differential graded algebra

complex

$$\Omega^0 A \xrightarrow{d} \Omega^1 A \xrightarrow{d} \Omega^2 A \xrightarrow{d} \cdots$$

multipl. : $(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_m) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_m)$

d $d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n)$ $d \circ d = 0$.

ex: $(a_0, a_1)(a_2, a_3) = -(a_0 a_1, a_2, a_3) + (a_0, a_1 a_2, a_3)$

$(a_0 da_1)(a_2 da_3)$ | $(1, a_1)(1, a_3) = -(a_1, 1, a_3) + (1, a_1, a_3)$

C-Q

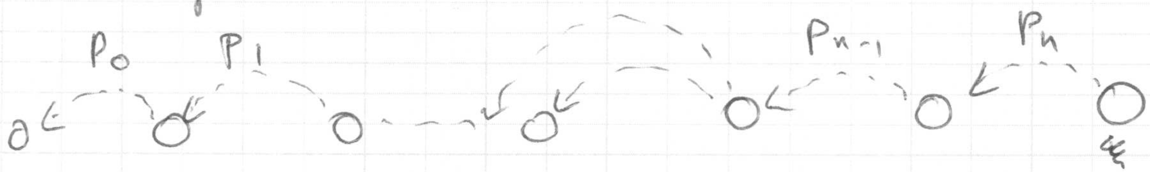
(1) relative diff forms

Bring down to "manageable" size for $A = \mathbb{C}Q$

$V \hookrightarrow \mathbb{C}Q = A$ $\qquad \bar{A}_V = A/V$ = cokernel of V-bimodule emb $V \hookrightarrow \mathbb{C}Q$

$$\Omega^n_V A = A \otimes_V (\bar{A}_V \otimes_V \cdots \otimes_V \bar{A}_V)$$

$\Omega^n_V \mathbb{C}Q$ = vectorspace spanned on



Multiplication

alternating sum over bridges

d

Ok.

$p_0 \, dp_1 \cdots dp_n$ $\qquad p_i$ paths in Q of length ≥ 1 for $i \geq 1$

connecting

and length ≥ 0 for p_0

θ V-derivation of $\mathbb{C}Q$ i.e. $\begin{cases} \theta(ab) = \theta(a)b + a\,\theta(b) \\ \theta(\text{vert}) = 0 \end{cases}$

$$\begin{tikzcd}
\Omega^{n-1}_V \mathbb{C}Q \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^{n}_V \mathbb{C}Q \arrow[l, bend left, "i_\theta"] \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^{n+1}_V \mathbb{C}Q \arrow[l, bend left, "i_\theta"] \arrow[loop below, "L_\theta"]
\end{tikzcd}$$

Cartan homotopy formula

$$\begin{cases} i_\theta(a) = 0 \\ i_\theta(da) = \theta(a) \end{cases} \qquad \begin{cases} L_\theta(a) = \theta(a) \\ L_\theta(da) = d\,\theta(a) \end{cases}$$

!

$$\boxed{\begin{aligned} L_\theta &= i_\theta \circ d + d \circ i_\theta \\ [L_\theta, i_\gamma] &= i_{[\theta,\gamma]} \\ [L_\theta, L_\gamma] &= L_{[\theta,\gamma]} \end{aligned}}$$

44 - 1 0

$$H^n_V \mathbb{C}Q = \frac{\mathrm{Ker}(\Omega^n_V \mathbb{C}Q \xrightarrow{d} \Omega^{n+1}_V \mathbb{C}Q)}{\mathrm{Im}(\Omega^{n-1}_V \mathbb{C}Q \xrightarrow{d} \Omega^n_V \mathbb{C}Q)} = \begin{cases} V & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

"$\mathbb{C}Q$ contractible to vertices"

pf E Euler derivation

$E(\text{vertex}) = 0$

$E(\text{arrow}) = \text{arrow}.$

$E(\text{path}) = \text{length}(\text{path}) . \text{path}$

$E(ab) = E(a)\,b + a\,E(b) = ab + ab = 2ab$ etc.

$\Rightarrow L_E(p_0\, dp_1 \ldots dp_n) = (l(p_0) + \cdots + l(p_n))\, p_0\, dp_1 \ldots dp_n$ so

linear iso on $\Omega^n_V \mathbb{C}Q$ if $n>0$

$+ \; \boxed{L_E = d \circ i_E + i_E \circ d} \Rightarrow$ result

Karoubi complex

nonc diff n-forms $\longrightarrow$ $dR^n_V\, \mathbb{C}Q = \dfrac{\Omega^n_V\, \mathbb{C}Q}{\underbrace{\sum_{i=0}^{n} [\Omega^i_V\, \mathbb{C}Q, \Omega^{n-i}_V\, \mathbb{C}Q]}_{\text{supercommutators.}}}$ subspace space.

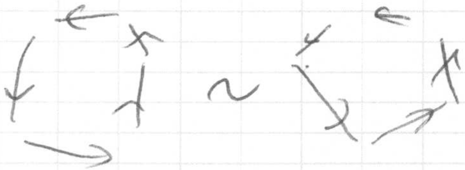
functions

$dR^0_V\, \mathbb{C}Q = \dfrac{\mathbb{C}Q}{[\mathbb{C}Q, \mathbb{C}Q]}$ i.e. $[\omega, \omega'] = \omega.\omega' + (-1)^i \omega'\omega$

$\omega \in \Omega^i$

= space on necklace words in Q

i.e. oriented cycles up to cyclic order



1-form

$dR^1_V\, \mathbb{C}Q = \oplus$ ① $\xrightarrow{a}$ ② ... ① path ① d ② ← ③