SIMPLE ROOTS OF DEFORMED PREPROJECTIVE ALGEBRAS

LIEVEN LE BRUYN

For Idun Reiten on her 60th birthday.

ABSTRACT. In [1] W. Crawley-Boevey gave a description of the set Σ_λ consisting of the dimension vectors of simple representations of the deformed preprojective algebra Π_λ. In this note we present alternative descriptions of Σ_λ.

1. Reduction to Π_0

Recall that a quiver $\vec{Q}$ is a finite directed graph on a set of vertices $Q_v = \{v_1, \ldots, v_k\}$, having a finite set of arrows $Q_a = \{a_1, \ldots, a_l\}$ where we allow both multiple arrows between vertices and loops in vertices. The Euler form of $\vec{Q}$ is the bilinear form on $\mathbb{Z}^k$ determined by the integral $k \times k$ matrix having as its (i,j)-entry $\chi_{ij} = \delta_{ij} - \#\{\text{arrows from } v_i \text{ to } v_j\}$. The double quiver $\bar{Q}$ of the quiver $\vec{Q}$ is the quiver obtained by adjoining to every arrow $a \in Q_a$ an arrow a^* in the opposite direction. The path algebra $\mathbb{C}\bar{Q}$ has as $\mathbb{C}$-basis the set of all oriented paths $p = a_{i_u} \ldots a_{i_1}$ of length $u \geq 1$ together with the vertex-idempotents e_i considered as paths of length zero. Multiplication in $\mathbb{C}\bar{Q}$ is induced by concatenation (on the left) of paths. For rational numbers λ_i, the deformed preprojective algebra is the quotient algebra

$$\Pi_\lambda = \Pi_\lambda(\bar{Q}) = \frac{\mathbb{C}\bar{Q}}{(\sum_{a \in Q_a} [a, a^*] - \sum_{v_i \in Q_v} \lambda_i e_i)}$$

The (difficult) problem of describing the set Σ_λ of all dimension vectors of simple representations of Π_λ was solved by W. Crawley-Boevey in [1]. He proved that for α a positive root of $\bar{Q}$, $\alpha \in \Sigma_\lambda$ if and only if

$$p(\alpha) > p(\beta_1) + \ldots + p(\beta_r)$$

for every decomposition $\alpha = \beta_1 + \ldots + \beta_r$ with $r \geq 2$ all all β_i positive roots of $\bar{Q}$ such that $\lambda.\beta_i = 0$ and where $p(\beta) = 1 - \chi(\beta, \beta)$.

For a given dimension vector $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}^k$ one defines the affine scheme $\mathtt{rep}_\alpha\ \Pi_\lambda$ of α-dimensional representations of Π_λ. There is a natural action of the basechange group $GL(\alpha) = \prod_{i=1}^k GL_{a_i}$ on this scheme and the corresponding quotient morphism

$$\mathtt{rep}_\alpha\ \Pi_\lambda \xrightarrow{\pi} \mathtt{iss}_\alpha\ \Pi_\lambda$$

sends a representation V to the isomorphism class of the direct sum of its Jordan-Hölder factors. Let ξ be a geometric point of $\mathtt{iss}_\alpha\ \Pi_\lambda$, then ξ determines the isomorphism class of a semisimple α-dimensional representation say with decomposition

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_l^{\oplus e_l}$$

with the S_i distinct simple representations of Π_λ with dimension vector β_i which occurs in M_ξ with multiplicity e_i. We say that ξ is of representation type $\tau = (e_1, \beta_1; \ldots; e_l, \beta_l)$. Construct a graph G_B depending on the set of simple dimension vectors $B = \{\beta_1, \ldots, \beta_l\}$ having l vertices $\{w_1, \ldots, x_l\}$ having $2p(\beta_i) = 2(1 - \chi(\beta_i, \beta_i)$ loops in vertex w_i and $-\chi(\beta_i, \beta_j) - \chi(\beta_j, \beta_i)$ edges between w_i and w_j.

Let $\bar{Q}_B$ be the (double) quiver obtained from G_B by replacing each solid edge by a pair of directed arrows with opposite ordering. In [3, §4] W. Crawley-Boevey proved that there is an étale isomorphism between a neighborhood of ξ in $\mathtt{iss}_\alpha\ \Pi_\lambda$ and a neighborhood of the trivial representation $\overline{0}$ in $\mathtt{iss}_{\alpha_\tau}\ \Pi_0(\bar{Q}_B)$ where $\alpha_\tau = (e_1, \ldots, e_l)$ determined by the multiplicities of the simple factors of M_ξ.

The arguments in [3, §4] actually prove that there is a $GL(\alpha)$-equivariant étale isomorphism between a neighborhood of the orbit of M_ξ in $\mathtt{rep}_\alpha\ \Pi_\lambda(\bar{Q})$ and a neighborhood of the orbit of $\overline{(1,0)}$ in the principal fiber bundle

$$GL(\alpha) \times^{GL(\alpha_\tau)} \mathtt{rep}_{\alpha_\tau}\ \Pi_0(\bar{Q}_\tau)$$

Using the description of Σ_λ it was proved in [1] that $\mathtt{iss}_\alpha\ \Pi_\lambda$ is irreducible whenever $\alpha \in \Sigma$.

In this note we will give two alternative descriptions of the set Σ_λ stressing the fundamental role of the extended Dynkin quivers in the study of deformed preprojective algebras. Both descriptions rely on the above irreducibility result so they do *not* give a short proof of Crawley-Boevey's result unless an independent proof of irreducibility of $\mathtt{iss}_\alpha\ \Pi_\lambda$ for all $\alpha \in \Sigma_\lambda$ is found. In the statement of the results we have therefore separated the parts that depend on the irreducibility statement.

Proposition 1.1. *Let ξ be a geometric point of $\mathtt{iss}_\alpha\ \Pi_\lambda$ of representation type $\tau = (e_1, \beta_1; \ldots; e_l, \beta_l)$. The following are equivalent*

1. *Any neighborhood of ξ in $\mathtt{iss}_\alpha\ \Pi_\lambda$ contains a point of representation type $(1, \alpha)$ (whence, in particular, $\alpha \in \Sigma_\lambda$).*
2. *$\alpha_\tau = (e_1, \ldots, e_l)$ is the dimension vector of a simple representation of $\Pi_0(\bar{Q}_B)$.*
3. *Any neighborhood of $\overline{0}$ in $\mathtt{iss}_{\alpha_\tau}\ \Pi_0(\bar{Q}_B)$ contains a point of representation type $(1, \alpha_\tau)$ (whence, in particular, α_τ is the dimension vector of a simple representation of $\Pi_0(\bar{Q}_B)$.*

If moreover $\mathtt{iss}_\alpha\ \Pi_\lambda$ is irreducible these statements are equivalent to

- *$\alpha \in \Sigma_\lambda$.*

Proof. By comparing the stabilizer subgroups of the closed orbits determined by corresponding points under the étale isomorphism it follows that (1) $\Leftrightarrow$ (3) and clearly (3) $\Rightarrow$ (2). Because the equations of $\Pi_0(\bar{Q}_B)$ are homogeneous there is a $\mathbb{C}^*$-action on $\mathtt{rep}_{\alpha_\tau}\ \Pi_0(\bar{Q}_B)$ (multiplying all matrices by $t \in \mathbb{C}^*$). The limit point $t \to 0$ of any representation is the trivial representation. Starting from a simple representation V, any neighborhood of $\overline{0}$ contains a point determined by $t.V$ for suitable t proving (2) $\Rightarrow$ (3). To prove that $\bullet \Rightarrow$ (1) observe that the set of all points of representation type $(1, \alpha)$ form an open subset of $\mathtt{iss}_\alpha\ \Pi_\lambda$ (follows from the étale local description), whence if $\mathtt{iss}_\alpha\ \Pi_\lambda$ is irreducible this set is dense. $\square$

This result allows us to describe Σ_λ inductively if we can determine the sets of simple dimension vectors for preprojective algebras. The induction starts off by

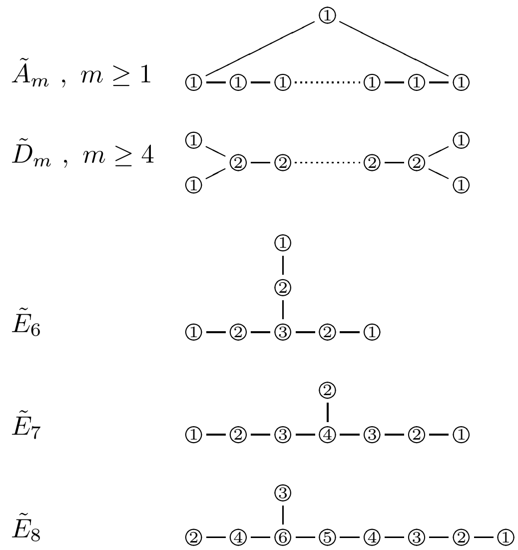

FIGURE 1. The tame settings.

taking the positive roots α for $\vec{Q}$ minimal w.r.t. $\lambda.\alpha = 0$. It follows from the easier part of [1] that these $\alpha \in \Sigma_\lambda$.

2. GENETIC DESCRIPTION OF Σ_0

In this section we start with the quiver $\vec{Q}$ and will give an inductive procedure to determine Σ_0, the set of simple dimension vectors of $\Pi_0 = \Pi(\bar{Q})$.

Assume we have constructed a set $B = \{\beta_1, \ldots, \beta_l\}$ with $\beta_i \in \Sigma_0$ (we can take $\beta = \beta_i = \beta_j$ for $i \neq j$ provided $p(\beta) > 0$). We want to determine the *minimal* linear combinations

$$\alpha = e_1\beta_1 + \ldots + e_l\beta_l$$

such that $\alpha \in \Sigma_0$. We will do this in terms of the graph G_B constructed in the previous section and the dimension vector $\alpha_\tau = (e_1, \ldots, e_l)$.

The tame settings are the couples (D, δ) where D is an extended Dynkin diagram and δ the corresponding imaginary root. The list of tame settings is given in figure 1. We say that a tame setting (D, δ) is contained in $(G_B.\alpha_\tau)$ if D is a subgraph of G_B and if $\delta \leq \alpha_\tau$.

Recall from [4] that all polynomial invariants of quivers are generated by taking traces along oriented cycles in the quiver. As a consequence, the coordinate algebra $\mathbb{C}[\mathtt{iss}_\alpha\ \Pi_0] = \mathbb{C}[\mathtt{rep}_\alpha\ \Pi_0]^{GL(\alpha)}$ is generated by traces in the quiver $\bar{Q}$. Note that non-trivial invariants exist whenever $\alpha \in \Sigma_0$ and α is not a real root of $\vec{Q}$. The crucial ingredient in our descriptions is the following technical result.

Proposition 2.1. *For $\alpha \in \Sigma_0$, if α is not a real root of $\vec{Q}$ and $\vec{Q}$ has only loops at vertices where α is one, then there is a non-loop tame setting $(\bar{D}, \delta)$ contained in $(\bar{Q}, \alpha)$.*

Proof. Assume $\bar{Q}$ is a counterexample with a minimal number of vertices. There are at most two directed arrows between two vertices $((\tilde{A}_1, (1,1))$ is not contained) so we can define the graph G replacing a pair of directed arrows by a solid edge. Then, G is a tree $((\tilde{A}_m, (1, \ldots, 1))$ is not contained).

We claim that the component of α for every internal (not a leaf) vertex is at least two. Assume v in internal and has dimension one, then any non-zero trace $tr(c)$ along a circuit in Γ passing through v (which must be the case by minimality of the counterexample) can be decomposed as

$$0 \neq tr(c) = tr(t_1)tr(t_2)\ldots tr(t_m)$$

where t_i is part of the circuit along a subtree rooted at v. But then $tr(t_i) \neq 0$ when evaluated at representations of the preprojective algebra of the corresponding subtree, contradicting minimality of the counterexample.

Hence, G is a binary tree $((\tilde{D}_4, (2,1,1,1))$ is not contained) and even a star with at most three arms $((\tilde{D}_m, (2, \ldots, 2, 1, 1, 1, 1))$ is not contained). If G does not contain $\tilde{E}_i$ for $6 \leq i \leq 8$ as subgraph, then $\bar{Q}$ is a Dynkin quiver and one knows that in this case there are no nontrivial invariants, a contradiction.

If δ_v is the vertex-simple concentrated in vertex v, we claim that

$$\chi(\alpha, \delta_v) + \chi(\delta_v, \alpha) \leq 0$$

for every vertex v. Indeed, it follows from [2] that for any non-isomorphic simple Π_0-representations V and W of dimension vectors β and γ we have

$$dim\ Ext^1_{\Pi_0}(V, W) = -\chi(\beta, \gamma) - \chi(\gamma, \beta)$$

Therefore, twice the dimension of α at v is smaller or equal to the sum of the dimensions of α in the two (maximum three) neighboring vertices. Fill up the arm of G corresponding to the longest arm of $\tilde{E}_i$ with dimensions starting with 1 at the leaf and proceeding by the rule that twice the dimension is equal to the sum of the neighboring dimensions, then we obtain a dimension vector β such that

$$\delta_i \leq \beta \leq \alpha$$

where δ_i is the imaginary root of $\tilde{E}_i$, a contradiction. □

Theorem 2.2. *With notations as above, we have*

1. $\alpha = e_1\beta_1 + \ldots + e_l\beta_l \in \Sigma_0$ *whenever* $\delta = (e_1, \ldots, e_l)$ *is the imaginary root of an extended Dynkin subgraph* D *of* G_B.
2. *If moreover* $\mathtt{iss}_\alpha\ \Pi_0$ *is irreducible for all* $\alpha \in \Sigma_0$, *the set* Σ_0 *is obtained by iterating the procedure in* (1) *starting from the set of all real roots of* $\vec{Q}$.

Proof. (1) : There is a point $\xi \in \mathtt{iss}_\alpha\ \Pi_0$ determined by a semi-simple representation M_ξ of representation type $\tau = (e_1, \beta_1; \ldots; e_l, \beta_l)$. A neighborhood of ξ is étale isomorphic to a neighborhood of $\overline{0}$ in $\mathtt{iss}_\delta\ \Pi_0(\bar{Q}_B)$. It is well known that $\mathtt{iss}_\delta\ \Pi_0(\bar{D})$ contains points of representation type $(1, \delta)$ whence δ is a dimension vector of a simple representation of $\Pi_0(\bar{Q}_B)$ (take a simple of $\Pi_0(\bar{D})$ and add zero matrices for the remaining arrows). By proposition 1.1 it follows that $\alpha \in \Sigma_0$.

(2) : Let $\alpha \in \Sigma_0$ and take a decomposition (representation type)

$$\alpha = d_1\beta_1 + \ldots + d_l\beta_l$$

with all $\beta_i \in \Sigma_0$, $\beta_i < \alpha$ and $d = \sum_i d_i$ minimal. Note that we can take all $d_i = 1$ whenever $p(\beta_i) > 0$ (as then there are infinitely many non-isomorphic simples of dimension vector β_i). As a consequence G_B only has loops at vertices where α_τ is equal to one and α_τ is a simple root for $\Pi_0(\vec{G_B})$ (here we used irreducibility of

$\mathtt{iss}_\alpha \; \Pi_0$ in order to apply proposition 1.1. By proposition 2.1 there is a non-loop tame subsetting (D, δ) contained in (G_B, α_τ) and if $\delta = (e_1, \ldots, e_l)$ then we have a decomposition

$$\alpha = (d_1 - e_1)\beta_1 + \ldots + (d_l - e_l)\beta_l + 1.(\delta.\beta)$$

which has strictly smaller total number of multiplicities unless $\alpha = \delta.\beta$. Induction on the total dimension finishes the proof. □

3. Another description of Σ_λ

In this section we reformulate the previous arguments in a more manageable statement.

Take a non-trivial representation type $\tau = (d_1, \beta_1; \ldots; d_l, \beta_l)$ of α with all $\beta_i \in \Sigma_\lambda$. Let τ' be the representation type obtained from τ by replacing each (d_i, β_i) by $(1, \beta_i; \ldots; 1, \beta_i)$ whenever $p(\beta_i) > 1$ (see the proof of theorem 2.2) and let B' be the corresponding set os simple root (some occurring more than once).

Theorem 3.1. *The following are equivalent*

1. $\alpha \in \Sigma_\lambda$ *and* $\mathtt{iss}_\alpha \; \Pi_\lambda$ *is irreducible.*
2. *For all non-trivial representation types* τ *of* α *there is a non-loop tame setting contained in* $(G_{B'}, \alpha_{\tau'})$.

Proof. $(2) \Rightarrow (1)$: We claim that $(1, \alpha)$ is the unique maximal representation type in the ordering of inclusion in Zariski-closures. Assume not and let τ be another maximal type, then $\tau = \tau'$ and by proposition 2.1 there is a tame setting contained in (G_B, α_τ) but then there are non-loop polynomial invariants, whence τ is not maximal.

$(1) \Rightarrow (2)$: Follows from proposition 1.1 and proposition 2.1. □

Hence, the dimension vectors obtained from the genetic construction of theorem 2.2 are exactly those $\alpha \in \Sigma_0$ such that $\mathtt{iss}_\alpha \; \Pi_0$ is irreducible.

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Universiteit Antwerpen (UIA), B-2610 Antwerp (Belgium)
E-mail address: `lebruyn@uia.ua.ac.be`
URL: `http://win-www.uia.ac.be/u/lebruyn/`