

FORMAL STRUCTURES AND REPRESENTATION SPACES

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ABSTRACT. M. Kapranov introduced and studied in math.AG/9802041 the noncommutative formal structure of a smooth affine variety. In this note we show that his construction is a particular case of micro-localization and extend the construction functorially to representation schemes of affine algebras. We describe explicitly the formal completions in the case of path algebras of quivers and initiate the study of their finite dimensional representations.

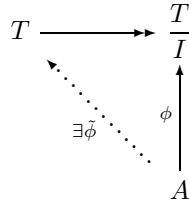
1. INTRODUCTION.

Let R be an associative \mathbb{C} -algebra, R^{Lie} its Lie structure and R_m^{Lie} the subspace spanned by the expressions $[r_1, [r_2, \dots, [r_{m-1}, r_m] \dots]]$ containing $m - 1$ instances of Lie brackets. The *commutator filtration* of R is the (increasing) filtration by ideals $(F^k R)_{k \in \mathbb{Z}}$ with $F^k R = R$ for $k \in \mathbb{N}$ and

$$F^{-k} R = \sum_m \sum_{i_1 + \dots + i_m = k} RR_{i_1}^{Lie} R \dots RR_{i_m}^{Lie} R$$

Observe that all \mathbb{C} -algebra morphisms preserve the commutator filtration. The associated graded $gr_F R$ is a (negatively) graded commutative Poisson algebra with part of degree zero $R_{ab} = \frac{R}{[R, R]}$.

Denote with nil_k the category of associative \mathbb{C} -algebras R such that $F^{-k} R = 0$ (in particular, $\text{nil}_1 = \text{commalg}$ the category of commutative \mathbb{C} -algebras). An algebra $A \in \text{Ob}(\text{nil}_k)$ is said to be k -smooth if and only if for all $T \in \text{Ob}(\text{nil}_k)$, all nilpotent twosided ideals $I \triangleleft T$ and all \mathbb{C} -algebra morphisms $A \xrightarrow{\phi} \frac{T}{I}$ there exist a lifted \mathbb{C} -algebra morphism



making the diagram commutative.

Kapranov proves [4, Thm 1.6.1] that an affine commutative *smooth* algebra C has a *unique* (upto \mathbb{C} -algebra isomorphism identical on C) k -smooth *thickening* $C^{(k)}$ with $C_{ab}^{(k)} \simeq C$. The inverse limit (connecting morphisms are given by the uniqueness result)

$$C^f = \lim_{\leftarrow} C^{(k)}$$

is then called the *formal completion* of C . Clearly, one has $C_{ab}^f = C$.

Example 1.1. Consider the affine space \mathbb{A}^d with coordinate ring $\mathbb{C}[x_1, \dots, x_d]$ and order the coordinate functions $x_1 < x_2 < \dots < x_d$. Let \mathfrak{f}_d be the free Lie algebra on $\mathbb{C}x_1 \oplus \dots \oplus \mathbb{C}x_d$ which has an ordered basis $B = \cup_{k \geq 1} B_k$ defined as follows. B_1 is the ordered set $\{x_1, \dots, x_d\}$ and $B_2 = \{[x_i, x_j] \mid j < i\}$, ordered such that $B_1 < B_2$

and $[x_i, x_j] < [x_k, x_l]$ iff $j < l$ or $j = l$ and $i < k$. Having constructed the ordered sets B_l for $l < k$ we define

$$B_k = \{[t, w] \mid t = [u, v] \in B_l, w \in B_{k-l} \text{ such that } v \leq w < t \text{ for } l < k\}.$$

For $l < k$ we let $B_l < B_k$ and B_k is ordered by $[t, w] < [t', w']$ iff $w < w'$ or $w = w'$ and $t < t'$. It is well known (see for example [10, Ex. 5.6.10]) that B is an ordered \mathbb{C} -basis of the Lie algebra \mathfrak{f}_d and that its enveloping algebra

$$U(\mathfrak{f}_d) = \mathbb{C}\langle x_1, \dots, x_d \rangle$$

is the free associative algebra on the x_i . We number the elements of $\cup_{k \geq 2} B_k$ according to the order $\{b_1, b_2, \dots\}$ and for $b_i \in B_k$ we define $ord(b_i) = k - 1$ (the number of brackets needed to define b_i). Let Λ be the set of all functions with finite support $\lambda : \cup_{k \geq 2} B_k \longrightarrow \mathbb{N}$ and define $ord(\lambda) = \sum \lambda(b_i) ord(b_i)$. Rephrasing the Poincaré-Birkhoff-Witt result for $U(\mathfrak{f}_d)$ we have that any noncommutative polynomial $p \in \mathbb{C}\langle x_1, \dots, x_d \rangle$ can be written uniquely as a finite sum

$$p = \sum_{\lambda \in \Lambda} \llbracket f_\lambda \rrbracket M_\lambda$$

where $\llbracket f_\lambda \rrbracket \in \mathbb{C}\langle x_1, \dots, x_d \rangle = S(B_1)$ and $M_\lambda = \prod_i b_i^{\lambda(b_i)}$. In particular, for every $\lambda, \mu, \nu \in \Lambda$, there is a unique bilinear differential operator with polynomial coefficients

$$C_{\lambda\mu}^\nu : \mathbb{C}\langle x_1, \dots, x_d \rangle \otimes_{\mathbb{C}} \mathbb{C}\langle x_1, \dots, x_d \rangle \longrightarrow \mathbb{C}\langle x_1, \dots, x_d \rangle$$

defined by expressing the product $\llbracket f \rrbracket M_\lambda \cdot \llbracket g \rrbracket M_\mu$ in $\mathbb{C}\langle x_1, \dots, x_d \rangle$ uniquely as $\sum_{\nu \in \Lambda} \llbracket C_{\lambda\mu}^\nu(f, g) \rrbracket M_\nu$, see [4, Prop. 3.4.3]. By associativity of $\mathbb{C}\langle x_1, \dots, x_d \rangle$ the $C_{\lambda\mu}^\nu$ satisfy the associativity constraint, that is, we have equality of the trilinear differential operators

$$\sum_{\mu_1} C_{\mu_1 \lambda_3}^\nu \circ (C_{\lambda_1 \lambda_2}^{\mu_1} \otimes \text{id}) = \sum_{\mu_2} C_{\lambda_1 \mu_2}^\nu \circ (\text{id} \otimes C_{\lambda_2 \lambda_3}^{\mu_2})$$

for all $\lambda_1, \lambda_2, \lambda_3, \nu \in \Lambda$. Kapranov defines the algebra $\mathbb{C}\langle x_1, \dots, x_d \rangle_{[[\text{ob}]]}$ to be the \mathbb{C} -vectorspace of possibly *infinite formal sums* $\sum_{\lambda \in \Lambda} \llbracket f_\lambda \rrbracket M_\lambda$ with multiplication defined by the operators $C_{\lambda\mu}^\nu$.

To prove that this algebra is the formal completion of $\mathbb{C}\langle x_1, \dots, x_d \rangle$ observe that the commutator filtration on $\mathbb{C}\langle x_1, \dots, x_d \rangle$ has components

$$F^{-k} \mathbb{C}\langle x_1, \dots, x_d \rangle = \left\{ \sum_{\lambda} \llbracket f_\lambda \rrbracket M_\lambda, \forall \lambda : ord(\lambda) \geq k \right\}$$

Moreover, the quotient $\frac{\mathbb{C}\langle x_1, \dots, x_d \rangle}{F^{-k} \mathbb{C}\langle x_1, \dots, x_d \rangle}$ is k -smooth using the lifting property of free algebras and the fact that algebra morphisms preserve the commutator filtration. Therefore,

$$\mathbb{C}\langle x_1, \dots, x_d \rangle^f = \lim_{\leftarrow} \frac{\mathbb{C}\langle x_1, \dots, x_d \rangle}{F^{-k} \mathbb{C}\langle x_1, \dots, x_d \rangle} \simeq \mathbb{C}\langle x_1, \dots, x_d \rangle_{[[\text{ob}]]}.$$

Let X be a (commutative) smooth affine variety (both assumptions are crucial!), then Kapranov uses the formal completion of the algebra of functions, $\mathbb{C}[X]^f$, to define a sheaf of noncommutative algebras on X , \mathcal{O}_X^f the Kapranov formal structure on X .

From [4, §4] it follows that it is not possible in general to extend a Kapranov formal structure on an arbitrary smooth variety X from that on the affine open pieces. In fact, the obstruction gives important new invariants of smooth varieties, related to Atiyah classes.

When X is affine, smoothness is essential to construct and prove uniqueness of the Kapranov formal structure \mathcal{O}_X^f . At present there is no natural functorial extension

of formal structures to arbitrary affine varieties. One of the major goals of this note is to construct such an extension for representation spaces of affine non-commutative algebras.

Example 1.2. Let us recall the construction of $\mathcal{O}_{\mathbb{A}^d}^f$ from [4]. Let $A_d(\mathbb{C})$ be the d -th Weyl algebra, that is, the ring of differential operators with polynomial coefficients on \mathbb{A}^d . Let $\mathcal{O}_{\mathbb{A}^d}$ be the structure sheaf on \mathbb{A}^d then it is well-known that the ring of sections $\mathcal{O}_{\mathbb{A}^d}(U)$ on any Zariski open subset $U \hookrightarrow \mathbb{A}^d$ is a left $A_d(\mathbb{C})$ -module. Kapranov defines in [4, Thm. 3.5.3] a sheaf $\mathcal{O}_{\mathbb{A}^d}^f$ of noncommutative algebras on \mathbb{A}^d by taking as its sections over U the algebra

$$\mathcal{O}_{\mathbb{A}^d}^f(U) = \mathbb{C}\langle x_1, \dots, x_d \rangle_{[[\text{ab}]]} \otimes_{\mathbb{C}[x_1, \dots, x_d]} \mathcal{O}_{\mathbb{A}^d}(U)$$

that is the \mathbb{C} -vectorspace of possibly infinite formal sums $\sum_{\lambda \in \Lambda} \llbracket f_\lambda \rrbracket M_\lambda$ with $f_\lambda \in \mathcal{O}_{\mathbb{A}^d}(U)$ and the multiplication is given as before by the action of the bilinear differential operators $C_{\lambda\mu}^\nu$ on the left $A_d(\mathbb{C})$ -module $\mathcal{O}_{\mathbb{A}^d}(U)$, that is, for all $f, g \in \mathcal{O}_{\mathbb{A}^d}(U)$ we have

$$\llbracket f \rrbracket M_\lambda \cdot \llbracket g \rrbracket M_\mu = \sum_{\nu} \llbracket C_{\lambda\mu}^\nu(f, g) \rrbracket M_\nu$$

The sheaf of noncommutative algebras $\mathcal{O}_{\mathbb{A}^d}^f$ is called *Kapranov's formal structure* on \mathbb{A}^d .

In the next section we will prove that the construction of Kapranov's formal structure \mathcal{O}_X^f is a special case of micro-localization. In section three we will use this strategy to extend the construction of formal structures in a functorial way to affine commutative schemes $X = \mathbf{rep}_n A$, the scheme of all n -dimensional representations of an affine \mathbb{C} -algebra A . In section four we will work out the special important case when A is the path algebra of a quiver and in the final section we will make some comments on the representation theory of these formal completions.

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2. MICROLOCAL INTERPRETATION.

We recall briefly the algebraic construction of microlocalization. For more details we refer to the monographs [9] and [12]. Let R be a filtered algebra with a separated filtration $\{F_n\}_{n \in \mathbb{Z}}$ and let S be a multiplicatively closed subset of R containing 1 but not 0. For any $r \in F_n - F_{n-1}$ we denote its principal character $\sigma(r)$ to be the image of r in the associated graded algebra $gr(R)$. We assume that the set $\sigma(S)$ is a multiplicatively closed subset of $gr(R)$ (always not containing 0 whence σ is multiplicative on S). We define the *Rees ring* \tilde{R} to be the graded algebra

$$\tilde{R} = \bigoplus_{n \in \mathbb{Z}} F_n t^n \hookrightarrow R[t, t^{-1}]$$

where t is an extra central variable. If $\sigma(s) \in gr(R)_n$ then we define the element $\tilde{s} = st^n \in \tilde{R}_n$. The set $\tilde{S} = \{\tilde{s}, s \in S\}$ is a multiplicatively closed subset of homogeneous elements in \tilde{R} .

Assume that $\sigma(S)$ is an Ore set in $gr(R) = \frac{\tilde{R}}{(t)}$, then for every $n \in \mathbb{N}_0$ the image $\pi_n(\tilde{S})$ is an Ore set in $\frac{\tilde{R}}{(t^n)}$ where $\tilde{R} \twoheadrightarrow \frac{\tilde{R}}{(t^n)}$ is the quotient morphism. Hence, we have an inverse system of graded localizations and can form the inverse limit in the graded sense

$$Q_S^\mu(\tilde{R}) = \varprojlim^g \pi_n(\tilde{S})^{-1} \frac{\tilde{R}}{(t^n)}$$

The element t acts torsionfree on this limit and hence we can form the filtered algebra

$$Q_S^\mu(R) = \frac{Q_{\tilde{S}}^\mu(\tilde{R})}{(t-1)Q_{\tilde{S}}^\mu(\tilde{R})}$$

which is the *micro-localization* of R at the multiplicatively closed subset S . We recall that the associated graded algebra of the microlocalization can be identified as

$$gr(Q_S^\mu(R)) = \sigma(S)^{-1}gr(R).$$

Let R be a \mathbb{C} -algebra with $R_{ab} = \frac{R}{[R, R]} = C$. We assume that the commutator filtration $(F^k)_{k \in \mathbb{Z}}$ introduced in (1.3) is a separated filtration on R . Observe that this is not always the case (for example consider $U(\mathfrak{g})$ for \mathfrak{g} a semi-simple Lie algebra) but often one can repeat the argument below replacing R with $\frac{R}{\cap F^n}$.

Observe that $gr(R)$ is a negatively graded commutative algebra with part of degree zero C . Take a multiplicatively closed subset S_c of C , then $S = S_c + [R, R]$ is a multiplicatively closed subset of R with the property that $\sigma(S) = S_c$ and clearly S_c is an Ore set in $gr(R)$. Therefore, \tilde{S} is a multiplicatively closed set of the Rees ring \tilde{R} consisting of homogeneous elements of degree zero. Observing that $(t^n)_0 = F^{-n}t^n$ for all $n \in \mathbb{N}_0$ we see that

$$Q_S^\mu(R) = \varprojlim \pi_n(S)^{-1} \frac{R}{F^{-n}}$$

where $R \xrightarrow{\pi_n} \frac{R}{F^{-n}}$ is the quotient morphism and Q_S^μ is filtered again by the commutator filtration and has as associated graded algebra

$$gr(Q_S^\mu(R)) = S_c^{-1}gr(R).$$

That is, the rings constructed in [4, §2] are just microlocalizations.

As explained in more detail in [12] one can define a *microstructure sheaf* \mathcal{O}_R^μ on the affine scheme X of C by taking as its sections over the affine Zariski open set $X(f)$

$$\Gamma(X(f), \mathcal{O}_R^\mu) = Q_{S_f}^\mu(R)$$

where $S = \{1, f, f^2, \dots\} + [R, R]$. Comparing with [4] we have proved.

Theorem 2.1. *Let $X = \text{spec } R_{ab}$ be smooth, then the microstructure sheaf \mathcal{O}_R^μ coincides with Kapranov's formal structure \mathcal{O}_X^f .*

An important remark to make is that one really needs microlocalization to construct a sheaf of noncommutative algebras on X . If by some fluke we would have that all the S_f are already Ore sets in R , we might optimistically assume that taking as sections over $X(f)$ the Ore localization $S_f^{-1}R$ we would define a sheaf \mathcal{O}_R over X . This is in general *not* the case as the Ore set S_g need no longer be Ore in a localization $S_f^{-1}R$!

3. REPRESENTATION SCHEMES.

We will first show that representation schemes of noncommutative formally smooth algebras give smooth affine varieties and are therefore endowed with a Kapranov formal structure. A \mathbb{C} -algebra A is said to be *formally smooth* if and only if it has the lifting property with respect to *test-objects* in \mathbf{alg} . That is, let $I \triangleleft T$ be a nilpotent twosided ideal, then one can complete any \mathbb{C} -algebra morphism

diagram

$$\begin{array}{ccc}
 T & \longrightarrow & \frac{T}{I} \\
 & \swarrow \text{dotted} & \uparrow \phi \\
 & & A
 \end{array}$$

$\exists \tilde{\phi}$

If we restrict both A and the testobjects (T, I) to $\mathbf{commalg}$ we get Grothendieck's notion of formally smooth commutative algebras. Grothendieck showed that if A is essentially of finite type, then this formal smoothness notion coincides with the usual geometric notion of smoothness.

Motivated by this analogy, Quillen and Cunz [3] have suggested to take formally smooth algebras as coordinate rings of noncommutative smooth varieties.

Let $A = \frac{\mathbb{C}\langle x_1, \dots, x_d \rangle}{I_A}$ be an affine \mathbb{C} -algebra. The representation space $\mathbf{rep}_n A$ is the affine scheme representing the functor

$$\mathbf{commalg} \xrightarrow{\text{Hom}_{\mathbf{alg}}(A, M_n(-))} \mathbf{sets}.$$

Its coordinate ring $\mathbb{C}[\mathbf{rep}_n A]$ can be described as follows. For each x_k consider an $n \times n$ matrix of indeterminates $X_k = (x_{ij,k})_{i,j}$. For every relation $f(x_1, \dots, x_d) \in I_A$ consider the $n \times n$ matrix in $M_n(\mathbb{C}[x_{ij,k}, \forall i, j, k])$ given by $f(X_1, \dots, X_d) = (f_{ij})_{i,j}$. Each of the f_{ij} is a polynomial in $\mathbb{C}[x_{ij,k}, \forall i, j, k]$. Then we have

$$\mathbb{C}[\mathbf{rep}_n A] = \frac{\mathbb{C}[x_{ij,k}, \forall i, j, k]}{(f_{ij}, \forall f \in I_A)}$$

Representability of the above functor equips us with a universal \mathbb{C} -algebra morphism $A \xrightarrow{j_A} M_n(\mathbb{C}[\mathbf{rep}_n A])$ such that for every \mathbb{C} -algebra morphism $A \xrightarrow{\phi} M_n(C)$ with C commutative

$$\begin{array}{ccc}
 A & \xrightarrow{j_A} & M_n(\mathbb{C}[\mathbf{rep}_n A]) \\
 \downarrow \phi & \searrow \text{dotted} & \\
 M_n(C) & & \exists! M_n(\psi)
 \end{array}$$

there is a unique morphism $\mathbb{C}[\mathbf{rep}_n A] \xrightarrow{\psi} C$, making the above diagram commute. For example, if $I_A = 0$, that is $A \simeq \mathbb{C}\langle x_1, \dots, x_d \rangle$, then $\mathbb{C}[\mathbf{rep}_n A] = \mathbb{C}[x_{ij,k}, \forall i, j, k]$, whence

$$\mathbf{rep}_n \mathbb{C}\langle x_1, \dots, x_d \rangle = \mathbb{A}^{dn^2} = \underbrace{M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})}_d.$$

The universal map j_A maps x_k to the *generic matrix* $X_k = (x_{ij,k})_{i,j}$ and the morphism ψ is determined by sending the variable $x_{ij,k}$ to the (i, j) -entry of the matrix $\phi(x_i)$.

More generally, if A is an affine formally smooth algebra, then all the representation spaces $\mathbf{rep}_n A$ are affine smooth (commutative) varieties. Indeed, as $\mathbb{C}[\mathbf{rep}_n A]$ is affine, it suffices to verify Grothendieck's formal smoothness property

$$\begin{array}{ccc}
 T_c & \longrightarrow & \frac{T_c}{I} \\
 & \swarrow \text{dotted} & \uparrow \phi \\
 & & \mathbb{C}[\mathbf{rep}_n A]
 \end{array}$$

$? \tilde{\phi}$

for T_c a commutative algebra with nilpotent ideal I . Using formal smoothness of A we have the existence of ψ

$$\begin{array}{ccc}
 M_n(T_c) & \longrightarrow & M_n\left(\frac{T_c}{I}\right) \\
 \uparrow \exists \psi & \nearrow \exists M_n(\tilde{\phi}) & \uparrow M_n(\phi) \\
 A & \xrightarrow{j_A} & M_n(\mathbb{C}[\text{rep}_n A])
 \end{array}$$

and universality of j_A provides us with the required lift $\tilde{\phi}$, proving

Proposition 3.1. *If A is a formally smooth algebra then $\text{rep}_n A$ is an affine smooth variety for all n .*

For an arbitrary algebra A , however, the representation space $\text{rep}_n A$ is in general *not* smooth nor even reduced so as mentioned before it is not immediately clear how to define a canonical and sufficiently functorial formal structure on it. We will now show how this can be done.

The starting point is that for every associative algebra A the functor

$$\text{alg} \xrightarrow{\text{Hom}_{\text{alg}}(A, M_n(-))} \text{sets}$$

is *representable* in alg . That is, there exists an associative \mathbb{C} -algebra $\sqrt[n]{A}$ such that there is a natural equivalence between the functors

$$\text{Hom}_{\text{alg}}(A, M_n(-)) \underset{n.e.}{\sim} \text{Hom}_{\text{alg}}(\sqrt[n]{A}, -).$$

In other words, for every associative \mathbb{C} -algebra B , there is a functorial one-to-one correspondence between the sets

$$\begin{cases} \text{algebra maps } A \longrightarrow M_n(B) \\ \text{algebra maps } \sqrt[n]{A} \longrightarrow B \end{cases}$$

Example 3.2. If $A = \mathbb{C}\langle x_1, \dots, x_d \rangle$, then it is easy to see that $\sqrt[n]{A} = \mathbb{C}\langle x_{11,1}, \dots, x_{nn,d} \rangle$. For, given an algebra map $A \xrightarrow{\phi} M_n(B)$ we obtain an algebra map $\sqrt[n]{A} \longrightarrow B$ by sending the free variable $x_{ij,k}$ to the (i, j) -entry of the matrix $\phi(x_k) \in M_n(B)$. Conversely, to an algebra map $\sqrt[n]{A} \xrightarrow{\psi} B$ we assign the algebra map $A \longrightarrow M_n(B)$ by sending x_k to the matrix $(\psi(x_{ij,k}))_{i,j} \in M_n(B)$. Clearly, these operations are each others inverses.

To define $\sqrt[n]{A}$ in general, consider the free algebra product $A * M_n(\mathbb{C})$ and consider the subalgebra

$$\sqrt[n]{A} = A * M_n(\mathbb{C})^{M_n(\mathbb{C})} = \{p \in A * M_n(\mathbb{C}) \mid p.(1 * m) = (1 * m).p \ \forall m \in M_n(\mathbb{C})\}$$

Before we can prove the universal property of $\sqrt[n]{A}$ we need to recall a property that $M_n(\mathbb{C})$ shares with any Azumaya algebra : if $M_n(\mathbb{C}) \xrightarrow{\phi} R$ is an algebra morphism and if $R^{M_n(\mathbb{C})} = \{r \in R \mid r.\phi(m) = \phi(m).r \ \forall m \in M_n(\mathbb{C})\}$, then we have $R \simeq M_n(\mathbb{C}) \otimes_{\mathbb{C}} R^{M_n(\mathbb{C})}$.

In particular, if we apply this to $R = A * M_n(\mathbb{C})$ and the canonical map $M_n(\mathbb{C}) \xrightarrow{\phi} A * M_n(\mathbb{C})$ where $\phi(m) = 1 * m$ we obtain that $M_n(\sqrt[n]{A}) = M_n(\mathbb{C}) \otimes_{\mathbb{C}} \sqrt[n]{A} = A * M_n(\mathbb{C})$.

Hence, if $\sqrt[n]{A} \xrightarrow{f} B$ is an algebra map we can consider the composition

$$A \xrightarrow{id_A * 1} A * M_n(\mathbb{C}) \simeq M_n(\sqrt[n]{A}) \xrightarrow{M_n(f)} M_n(B)$$

to obtain an algebra map $A \longrightarrow M_n(B)$. Conversely, consider an algebra map $A \xrightarrow{g} M_n(B)$ and the canonical map $M_n(\mathbb{C}) \xrightarrow{i} M_n(B)$ which centralizes B in $M_n(B)$. Then, by the universal property of free algebra products we have an algebra map $A * M_n(\mathbb{C}) \xrightarrow{g*i} M_n(B)$ and restricting to $\sqrt[n]{A}$ we see that this maps factors

$$\begin{array}{ccc} A * M_n(\mathbb{C}) & \xrightarrow{g*i} & M_n(B) \\ \uparrow & & \uparrow \\ \sqrt[n]{A} & \dashrightarrow & B \end{array}$$

and one verifies that these two operations are each others inverses.

It follows from the functoriality of the $\sqrt[n]{\cdot}$ construction that $\mathbb{C}\langle x_1, \dots, x_d \rangle \longrightarrow A$ implies that $\sqrt[n]{\mathbb{C}\langle x_1, \dots, x_d \rangle} \longrightarrow \sqrt[n]{A}$. Therefore, if A is affine and generated by $\leq d$ elements, then $\sqrt[n]{A}$ is also affine and generated by $\leq dn^2$ elements.

Next, we define a formal completion of $\mathbb{C}[\text{rep}_n A]$ in a functorial way for any associative algebra A . Equip $\sqrt[n]{A}$ with the commutator filtration

$$\dots \hookrightarrow F_{-2} \sqrt[n]{A} \hookrightarrow F_{-1} \sqrt[n]{A} \hookrightarrow \sqrt[n]{A} = \sqrt[n]{A} = \dots$$

Because algebra morphisms are commutator filtration preserving, it follows from the universal property of $\sqrt[n]{A}$ that $\frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}}$ is the object in nil_k representing the functor

$$\text{nil}_k \xrightarrow{\text{Hom}_{\text{alg}}(A, M_n(-))} \text{sets}.$$

In particular, because the categories commalg and nil_1 are naturally equivalent, we deduce that

$$\sqrt[n]{A}_{ab} = \frac{\sqrt[n]{A}}{[\sqrt[n]{A}, \sqrt[n]{A}]} = \frac{\sqrt[n]{A}}{F_{-1} \sqrt[n]{A}} \simeq \mathbb{C}[\text{rep}_n A]$$

because both algebras represent the same functor. We now define

$$\sqrt[n]{A}_{[[\text{ab}]]} = \lim_{\leftarrow} \frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}}.$$

Assume now that A is formally smooth, then so is $\sqrt[n]{A}$ because we have seen before that

$$M_n(\sqrt[n]{A}) \simeq A * M_n(\mathbb{C})$$

and the class of formally smooth algebras is easily seen to be closed under free products and matrix algebras. Alternatively, one can apply Bergman's coproduct theorems [2] or [11, Thm. 2.20] for a strong version.

As a consequence, we have for every $k \in \mathbb{N}$ that the quotient $\frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}}$ is k -smooth. Moreover, we have that

$$\left(\frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}} \right)_{ab} \simeq \frac{\sqrt[n]{A}}{[\sqrt[n]{A}, \sqrt[n]{A}]} \simeq \mathbb{C}[\text{rep}_n A].$$

Because $\mathbb{C}[\text{rep}_n A]$ is an affine commutative smooth algebra, we deduce from Kapranov's uniqueness result of k -smooth thickenings that

$$\mathbb{C}[\text{rep}_n A]^{(k)} \simeq \frac{\sqrt[n]{A}}{F_{-k} \sqrt[n]{A}}$$

and consequently that the formal completion of $\mathbb{C}[\text{rep}_n A]$ can be identified with

$$\mathbb{C}[\text{rep}_n A]^f \simeq \sqrt[n]{A}_{[[\text{ab}]]}.$$

Theorem 3.3. *Defining for an arbitrary \mathbb{C} -algebra A the formal completion of $\mathbb{C}[\text{rep}_n A]$ to be $\sqrt[n]{A}_{[\text{cb}]}$ gives a canonical extension of Kapranov's formal structure on affine smooth commutative algebras to the class the coordinate rings of representation spaces on which it is functorial in the algebras.*

There is a natural action of GL_n by algebra automorphisms on $\sqrt[n]{A}$. Let u_A denote the universal morphism $A \xrightarrow{u_A} M_n(\sqrt[n]{A})$ corresponding to the identity map on $\sqrt[n]{A}$. For $g \in GL_n$ we can consider the composed algebra map

$$\begin{array}{ccc} A & \xrightarrow{u_A} & M_n(\sqrt[n]{A}) \\ & \searrow \psi_g & \downarrow g \cdot g^{-1} \\ & & M_n(\sqrt[n]{A}) \end{array}$$

Then g acts on $\sqrt[n]{A}$ via the automorphism $\sqrt[n]{A} \xrightarrow{\phi_g} \sqrt[n]{A}$ corresponding to the composition ψ_g . It is easy to verify that this defines indeed a GL_n -action on $\sqrt[n]{A}$. The formal structure sheaf $\mathcal{O}_{\text{rep}_n A}^f$ defined over $\text{rep}_n A$, constructed from $\sqrt[n]{A}$ by microlocalization as in the foregoing section, will be denoted from now on by $\mathcal{O}_{\sqrt[n]{A}}^f$. We see that it actually has a GL_n -structure which is compatible with the GL_n -action on $\text{rep}_n A$.

Finally we should clarify what representation theoretic information is contained in $\sqrt[n]{A}_{[\text{cb}]}$. The reduced variety of $\text{rep}_n A$ gives information about the \mathbb{C} -algebra maps $A \longrightarrow M_n(\mathbb{C})$. The scheme structure of $\text{rep}_n A$ gives us the \mathbb{C} -algebra maps $A \longrightarrow M_n(C)$ where C is a finite dimensional commutative \mathbb{C} -algebra. The formal structure now gives us the \mathbb{C} -algebra maps $A \longrightarrow M_n(B)$ where B is a finite dimensional noncommutative but basic \mathbb{C} -algebra. Recall that an algebra is said to be basic if all its simple representations have dimension one.

4. PATH ALGEBRAS OF QUIVERS.

Even in the case when A is formally smooth it is by no means easy to describe and manipulate the n -th root algebra $\sqrt[n]{A}$ and its corresponding formal completion $\sqrt[n]{A}_{[\text{cb}]}$. In this and the next section we will discuss these facts in the special (but important) case of path algebras of quivers.

Let Q be a *quiver*, that is a directed graph on a finite set $Q_v = \{v_1, \dots, v_k\}$ of vertices having a finite set $Q_a = \{a_1, \dots, a_l\}$ of arrows where we allow multiple arrows between vertices and loops in vertices. We will depict vertex v_i by \textcircled{i} and an arrow a from vertex v_i to v_j by $\textcircled{j} \xleftarrow{a} \textcircled{i}$.

The path algebra $\mathbb{C}Q$ has as underlying \mathbb{C} -vectorspace basis the set of all oriented paths in Q , including those of length zero corresponding to the vertices v_i . Multiplication in $\mathbb{C}Q$ is induced by (left) concatenation of paths. More precisely, $1 = v_1 + \dots + v_k$ is a decomposition of 1 into mutually orthogonal idempotents and further we define

- $v_j \cdot a$ is always zero unless $\textcircled{j} \xleftarrow{a} \textcircled{}$ in which case it is the path a ,
- $a \cdot v_i$ is always zero unless $\textcircled{} \xleftarrow{a} \textcircled{i}$ in which case it is the path a ,
- $a_i \cdot a_j$ is always zero unless $\textcircled{} \xleftarrow{a_i} \textcircled{} \xleftarrow{a_j} \textcircled{}$ in which case it is the path $a_i a_j$.

In order to see that $\mathbb{C}Q$ is formally smooth, take an algebra T with a nilpotent twosided ideal $I \triangleleft T$ and consider

$$\begin{array}{ccc} T & \longrightarrow & \frac{T}{I} \\ & \swarrow \text{?}\tilde{\phi} & \uparrow \phi \\ & & \mathbb{C}Q \end{array}$$

The decomposition $1 = \phi(v_1) + \dots + \phi(v_k)$ into mutually orthogonal idempotents in $\frac{T}{I}$ can be lifted up the nilpotent ideal I to a decomposition $1 = \tilde{\phi}(v_1) + \dots + \tilde{\phi}(v_k)$ into mutually orthogonal idempotents in T . But then, taking for every arrow a

$$\textcircled{1} \xleftarrow{a} \textcircled{1} \quad \text{an arbitrary element} \quad \tilde{\phi}(a) \in \tilde{\phi}(v_j)(\phi(a) + I)\tilde{\phi}(v_i)$$

gives a required lifted algebra morphism $\mathbb{C}Q \xrightarrow{\tilde{\phi}} T$.

Next, we will describe the smooth affine schemes $\text{rep}_n \mathbb{C}Q$. Consider the semisimple subalgebra $V = \underbrace{\mathbb{C} \times \dots \times \mathbb{C}}_k$ generated by the vertex-idempotents $\{v_1, \dots, v_k\}$.

Every n -dimensional representation of V is semi-simple and determined by the multiplicities by which the factors occur. That is, we have a decomposition

$$\text{rep}_n V = \bigsqcup_{\sum a_i = n} GL_n / (GL_{a_1} \times \dots \times GL_{a_k}) = \bigsqcup_{\alpha} \text{rep}_{\alpha} V$$

into homogeneous spaces where α runs over the *dimension vectors* $\alpha = (a_1, \dots, a_k)$ such that $\sum_i a_i = n$. The inclusion $V \hookrightarrow \mathbb{C}Q$ induces a map $\text{rep}_n \mathbb{C}Q \xrightarrow{\psi} \text{rep}_n V$ and we have the decomposition of $\text{rep}_n \mathbb{C}Q$ into associated fiber bundles

$$\psi^{-1}(\text{rep}_{\alpha} V) = GL_n \times^{GL(\alpha)} \text{rep}_{\alpha} Q.$$

Here, $GL(\alpha) = GL_{a_1} \times \dots \times GL_{a_k}$ embedded along the diagonal in GL_n and $\text{rep}_{\alpha} Q$ is the affine space of α -dimensional representations of the quiver Q . That is,

$$\text{rep}_{\alpha} Q = \bigoplus_{\textcircled{1} \xleftarrow{a} \textcircled{1}} M_{a_j \times a_i}(\mathbb{C})$$

and $GL(\alpha)$ acts on this space via base-change in the vertex-spaces. That is, $\text{rep}_n \mathbb{C}Q$ is the disjoint union of smooth affine components depending on the dimension vectors $\alpha = (a_1, \dots, a_k)$ such that $\sum a_i = n$.

The importance of the class of path algebras comes from the fact that if A is an arbitrary affine formally smooth algebra, the *étale local* GL_n -structure of the representation space $\text{rep}_n A$ is determined by a $\text{rep}_{\alpha} Q$ for a certain quiver Q and dimension vector α . We refer to [6] or [7] for more details.

We have seen before that $\sqrt[\nu]{\mathbb{C}Q}$ is formally smooth, hence we would like to determine the quiver settings relevant for the study of the representation space $\text{rep}_m \sqrt[\nu]{\mathbb{C}Q}$ for arbitrary $m \in \mathbb{N}$. Before we can do this we need to recall the notion of *universal localization*. We refer to [11, Chp. 4] for full details.

Let A be a \mathbb{C} -algebra and $\text{projmod } A$ the category of finitely generated projective left A -modules. Let Σ be some class of maps in this category (that is some left A -module morphisms between certain projective modules). In [11, Chp. 4] it is shown that there exists an algebra map $A \xrightarrow{j_{\Sigma}} A_{\Sigma}$ with the universal property that the maps $A_{\Sigma} \otimes_A \sigma$ have an inverse for all $\sigma \in \Sigma$. A_{Σ} is called the universal localization of A with respect to the set of maps Σ .

When A is formally smooth, so is A_Σ . Indeed, consider a test-object (T, I) in \mathbf{alg} , then we have the following diagram

$$\begin{array}{ccc} T & \xrightarrow{\quad} & T \\ \vdots \uparrow \psi & \nearrow \tilde{\phi} & \uparrow \phi \\ A & \xrightarrow{j_\Sigma} & A_\Sigma \end{array}$$

where ψ exists by smoothness of A . By Nakayama's lemma all maps $\sigma \in \Sigma$ become isomorphisms under tensoring with ψ . Then, $\tilde{\phi}$ exists by the universal property of A_Σ .

Consider the special case when A is the path algebra $\mathbb{C}Q$ of a quiver on k vertices. Then, we can identify the isomorphism classes in $\mathbf{projmod} \mathbb{C}Q$ with \mathbb{N}^k . To each vertex v_i corresponds an *indecomposable* projective left $\mathbb{C}Q$ -ideal P_i having as \mathbb{C} -vector space basis all paths in Q starting at v_i (similarly, there is an indecomposable projective right $\mathbb{C}Q$ -ideal P_i^r with basis the paths ending in v_i). We can also determine the space of homomorphisms

$$\mathrm{Hom}_{\mathbb{C}Q}(P_i, P_j) = \bigoplus_{\textcircled{i} \xrightarrow{p} \textcircled{j}} \mathbb{C}p$$

where p is an oriented path in Q starting at v_j and ending at v_i . Therefore, any A -module morphism σ between two projective left modules

$$P_{i_1} \oplus \dots \oplus P_{i_u} \xrightarrow{\sigma} P_{j_1} \oplus \dots \oplus P_{j_v}$$

can be represented by an $u \times v$ matrix M_σ whose (p, q) -entry m_{pq} is a linear combination of oriented paths in Q starting at v_{j_q} and ending at v_{i_p} .

Now, form an $v \times u$ matrix N_σ of free variables y_{pq} and consider the algebra $\mathbb{C}Q_\sigma$ which is the quotient of the free product $\mathbb{C}Q * \mathbb{C}\langle y_{11}, \dots, y_{uv} \rangle$ modulo the ideal of relations determined by the matrix equations

$$M_\sigma \cdot N_\sigma = \begin{bmatrix} v_{i_1} & & 0 \\ & \ddots & \\ 0 & & v_{i_u} \end{bmatrix} \quad N_\sigma \cdot M_\sigma = \begin{bmatrix} v_{j_1} & & 0 \\ & \ddots & \\ 0 & & v_{j_v} \end{bmatrix}$$

Equivalently, $\mathbb{C}Q_\sigma$ is the path algebra of a quiver *with relations* where the quiver is Q extended with arrows y_{pq} from v_{i_p} to v_{j_q} for all $1 \leq p \leq u$ and $1 \leq q \leq v$ and the relations are the above matrix entry relations.

Repeating this procedure for every $\sigma \in \Sigma$ we obtain the universal localization $\mathbb{C}Q_\Sigma$. Observe that if Σ is a finite set of maps, then the universal localization $\mathbb{C}Q_\Sigma$ is an affine algebra.

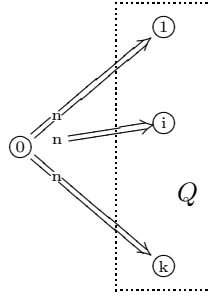
It is easy to see that the representation space $\mathbf{rep}_n \mathbb{C}Q_\sigma$ is an affine Zariski open subscheme (but possibly empty) of $\mathbf{rep}_n \mathbb{C}Q$. Indeed, if $m = (m_a)_a \in \mathbf{rep}_\alpha Q$, then m determines a point in $\mathbf{rep}_n \mathbb{C}Q_\Sigma$ if and only if the matrices $M_\sigma(m)$ in which the arrows are all replaced by the matrices m_a are invertible for all $\sigma \in \Sigma$. In particular, this induces numerical conditions on the dimension vectors α such that $\mathbf{rep}_\alpha \mathbb{C}Q_\Sigma \neq \emptyset$. Let $\alpha = (a_1, \dots, a_k)$ be a dimension vector such that $\sum a_i = n$ then every $\sigma \in \Sigma$ say with

$$P_1^{\oplus e_1} \oplus \dots \oplus P_k^{\oplus e_k} \xrightarrow{\sigma} P_1^{\oplus f_1} \oplus \dots \oplus P_k^{\oplus f_k}$$

gives the numerical condition

$$e_1 a_1 + \dots + e_k a_k = f_1 a_1 + \dots + f_k a_k.$$

Let Q be a quiver on k vertices and consider the extended quiver \tilde{Q}_n



That is, we add to the vertices and arrows of Q one extra vertex v_0 and for every vertex v_i in Q we add n directed arrows from v_0 to v_i . We will denote the j -th arrow $1 \leq j \leq n$ from v_0 to v_i by x_{ij} . From now on we will depict a bundle of n arrows between two fixed vertices v_k and v_l by $\textcircled{1} \xrightarrow{n} \textcircled{i}$. Consider the morphism between projective left $\mathbb{C}\tilde{Q}_n$ -modules

$$P_1 \oplus P_2 \oplus \dots \oplus P_k \xrightarrow{\sigma} \underbrace{P_0 \oplus \dots \oplus P_0}_n$$

determined by the matrix

$$M_\sigma = \begin{bmatrix} x_{11} & \dots & \dots & x_{1n} \\ \vdots & & & \vdots \\ x_{k1} & \dots & \dots & x_{kn} \end{bmatrix}.$$

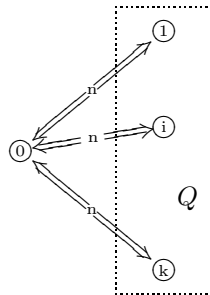
We consider the universal localization $\mathbb{C}\tilde{Q}_{n\sigma}$, that is, we add for each vertex v_i in Q another n arrows y_{ij} with $1 \leq j \leq n$ from v_i to v_0 . With these arrows y_{ij} one forms the $n \times k$ matrix

$$N_\sigma = \begin{bmatrix} y_{11} & \dots & y_{k1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ y_{1n} & \dots & y_{kn} \end{bmatrix}$$

and the universal localization $\mathbb{C}\tilde{Q}_{n\sigma}$ is described by the above quiver with relations

$$M_\sigma \cdot N_\sigma = \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_k \end{bmatrix} \quad \text{and} \quad N_\sigma \cdot M_\sigma = \begin{bmatrix} v_0 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & v_1 \end{bmatrix}.$$

We will depict this quiver with relations by the picture $\tilde{Q}_{n\sigma}$



From the discussion above we conclude :

Theorem 4.1. *With notations as before, we have an isomorphism of \mathbb{C} -algebras*

$$\sqrt[n]{\mathbb{C}Q} \simeq v_0 \mathbb{C}\tilde{Q}_{n\sigma} v_0.$$

Proof. Indeed, the right hand side is generated by all the oriented cycles in $\tilde{Q}_{n\sigma}$ starting and ending at v_0 and is therefore generated by the $y_{ip}x_{iq}$ and the $y_{ip}ax_{jq}$ where a is an arrow in Q starting in v_j and ending in v_i . If we have an algebra morphism

$$\mathbb{C}Q \xrightarrow{\phi} M_n(B)$$

then we have an associated algebra morphism

$$v_0 \mathbb{C}\tilde{Q}_{n\sigma} v_0 \xrightarrow{\psi} B$$

defined by sending $y_{ip}ax_{jq}$ to the (p, q) -entry of the $n \times n$ matrix $\phi(a)$ and $y_{ip}x_{iq}$ to the (p, q) -entry of $\phi(v_i)$. The defining relations among the x_{ip} and y_{iq} introduced before imply that ψ is indeed an algebra morphism. \square

Example 4.2. Let us specialize to the case when $A = \mathbb{C}\langle a, b \rangle$, that is A is the path algebra of the quiver



In order to describe $\sqrt[n]{A}$ we consider the quiver with relations

$$\textcircled{0} \xleftarrow{n} \textcircled{1} \xrightarrow{n} \textcircled{1} \quad : \quad y_i x_j = \delta_{ij} v_0, \quad \sum_i x_i y_i = v_1.$$

We see that the algebra of oriented cycles in v_0 in this quiver with relations is isomorphic to the free algebra in $2n^2$ free variables

$$\mathbb{C}\langle y_1 a x_1, \dots, y_n a x_n, y_1 b x_1, \dots, y_n b x_n \rangle$$

which coincides with our knowledge of $\sqrt[n]{\mathbb{C}\langle a, b \rangle}$.

5. REPRESENTATIONS OF $\sqrt[n]{\mathbb{C}Q_{[[ab]]}}$.

In this section we initiate the study of the finite dimensional representations of $\sqrt[n]{\mathbb{C}Q_{[[ab]]}}$. The key observation is to observe that all simple representations of this algebra are one-dimensional and that these one-dimensional representations are identified with $\text{rep}_n \mathbb{C}Q$ by definition of the root algebra. In order to describe all m -dimensional representations of $\sqrt[n]{\mathbb{C}Q_{[[ab]]}}$ we look at the quotient map

$$\text{rep}_m \sqrt[n]{\mathbb{C}Q_{[[ab]]}} \xrightarrow{\pi} \text{iss}_m \sqrt[n]{\mathbb{C}Q_{[[ab]]}}$$

where iss_m is the variety of isomorphism classes of m -dimensional semi-simple representations (which is identified with the quotient variety of rep_m under the action of GL_m). First we will stratify this quotient variety by smooth subvarieties.

Any point ξ of $\text{iss}_m \sqrt[n]{\mathbb{C}Q_{[[ab]]}}$ corresponds to an m -dimensional semisimple representation of the form

$$M_\xi = S_1^{\oplus e_1} \oplus \dots \oplus S_z^{\oplus e_z}$$

where the S_i are distinct one-dimensional simple representations occurring with multiplicity e_i and we choose the ordering of the components such that $e_1 \geq e_2 \geq \dots \geq e_z > 0$. That is, to ξ we can associate a partition $\lambda(\xi) = (e_1, \dots, e_z)$ of m .

Every partition λ of m determines a stratum $\text{iss}_m(\lambda)$ consisting of all points ξ such that $\lambda(\xi) = \lambda$. Using the decomposition

$$\text{rep}_n \mathbb{C}Q = \bigsqcup_{\alpha} GL_n \times^{GL(\alpha)} \text{rep}_{\alpha} Q$$

(where α is a dimension vector of total dimension n) we conclude :

Theorem 5.1. $\text{iss}_m \sqrt[n]{\mathbb{C}Q}_{[[\text{ob}]]}$ has a stratification in locally closed smooth subvarieties

$$\text{iss}_m(\lambda)$$

where λ runs over the partition of m and each stratum is the disjoint union of smooth substrata

$$\text{iss}_m(\lambda)(\alpha_1, \dots, \alpha_z)$$

where α_i is a dimension vector for the quiver Q of total dimension n and λ has z nonzero parts.

Smoothness follows from the fact that if $\lambda = \lambda_1^{l_1} \dots \lambda_y^{l_y}$, then $\text{iss}_m(\lambda)$ is the product of y smooth varieties, the i -th component being isomorphic to the l_i -th symmetric power of $\text{rep}_n Q$ with all the diagonals removed.

Next we claim that the fibers of the quotient map π are constant along these substrata $\text{iss}_m(\lambda)(\alpha_1, \dots, \alpha_k)$ can be identified with the nullcone of a specified quiver setting. To prove this we relate $\text{rep}_n \mathbb{C}Q$ to semistable representations of the extended quiver \tilde{Q}_n .

If $S_i \in GL_n \times^{GL(\alpha_i)} \text{rep}_{\alpha_i} Q$ (with $\alpha_i = (a_1, \dots, a_k)$) we can use the GL_n -action to assign to S_i the representation \tilde{S}_i of \tilde{Q}_n of dimension vector $\tilde{\alpha}_i = (1, \alpha_i)$ such that the restriction to the subquiver Q is S_i , we put a one-dimension space at vertex v_0 and take the arrows from v_0 to v_j such that

- the first $\sum_{l=1}^{j-1} a_l$ arrows map $\tilde{S}(v_0)$ to the zero vector.
- the arrows $\sum_{l=1}^{j-1} a_l + 1$ through $\sum_{l=1}^j a_l$ map $\tilde{S}(v_0)$ to a standard basis of the vertexspace $S(v_j)$.
- the remaining arrows all map $\tilde{S}(v_0)$ to the zero vector.

If we take the character of $GL(\tilde{\alpha}_i)$ determined by $\theta = (-m, 1, \dots, 1)$ it follows immediately that \tilde{S}_i is a θ -stable representation (that is, it contains no proper subrepresentations of dimension vector $\tilde{\beta}$ such that $\theta(\tilde{\beta}) \leq 0$, see [5] for more details).

Hence, M_{ξ} determines a θ -semistable representation of \tilde{Q}_n . In [1] it is proved that the étale local description of the corresponding moduli space is determined by a local quiver setting. In our case the relevant quiver for the substratum determined by the partition $\lambda = (e_1, \dots, e_z)$ and the dimension vectors α_i is the quiver setting (Γ, γ) where Γ is the quiver on z vertices, say w_1, \dots, w_z such that the number of directed arrows from w_i to w_j is equal to

$$\delta_{ij} - \chi_{\tilde{Q}_n}(\tilde{\alpha}_i, \tilde{\alpha}_j)$$

where $\chi_{\tilde{Q}_n}$ is the Euler form of the extended quiver, which is related to the Euler form of the original quiver Q by

$$\chi_{\tilde{Q}_n} = \begin{pmatrix} 1 & -n & \cdots & -n \\ 0 & \lceil \quad \quad \quad \rceil \\ \cdots & \lceil \quad \quad \quad \rceil & \chi_Q & \lceil \quad \quad \quad \rceil \\ 0 & \lceil \quad \quad \quad \rceil & \lceil \quad \quad \quad \rceil & \lceil \quad \quad \quad \rceil \end{pmatrix}$$

and the new dimension vector $\gamma = (e_1, \dots, e_z)$ is determined by the multiplicities.

As we know that the Jordan-Hölder components of any representation in $\pi^{-1}(\xi)$ must be the S_i , one deduces

Theorem 5.2. *With notations as before, the fiber $\pi^{-1}(\xi)$ of the quotient morphism $\text{rep}_m \sqrt[n]{\mathbb{C}Q_{[[\text{ob}]}} \xrightarrow{\pi} \text{iss}_m \sqrt[n]{\mathbb{C}Q_{[[\text{ob}]}}$ in a point ξ belonging to the substratum $\text{iss}_m(\lambda)(\alpha_1, \dots, \alpha_z)$ is determined by the nullcone of $\text{rep}_\gamma \Gamma$.*

These observations are of particular importance in case the partition $\lambda = (m)$, that is if the corresponding representation to ξ is $M_\xi = S^{\oplus m}$. We then have an embedding

$$\text{rep}_\alpha Q \xrightarrow{i_m} M_\theta^{ss}(\tilde{Q}_n, m\tilde{\alpha})$$

into the moduli space of θ -semistable representations of \tilde{Q}_n of dimension vector $m\tilde{\alpha}$. The collection of these embeddings i_m for $m \in \mathbb{N}$ is essentially equivalent to defining the formal structure on $\text{rep}_\alpha Q$. For more details we refer to [8].

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