X a smooth projective curve of genus g. $\pi_1(X)$ fundamental group

$$\frac{\langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle}{[x_1, y_1] \ldots [x_g, y_g]}$$

$M = M_X^{ss}(0, n)$ the moduli space of semi-stable vectorbundles over X of rank n and degree 0.

$$\pi_1(X) \longrightarrow U_n(\mathbb{C})$$

stable bundle $\leftrightarrow$ simple representation. Determines smooth open subset M^s of M.

$x \in M - M^s \leftrightarrow$ semi-simple representation. Unless $(g, n) = (2, 2)$, x is a singularity of M (C.S. Seshadri).

Question : natural desingularization $N \longrightarrow M$

(Seshadri) : semi-stable bundles V of rank n^2 and degree 0 such that $End(V)$ is a degeneration of $M_n(\mathbb{C})$ in alg_{n^2}.

$N = M_X^{ps}(0, n^2)$ moduli space of such 'parabolic stable' bundles. If $V \in N$

$$gr(V) = \underbrace{W \oplus \ldots \oplus W}_{n}$$

with W a direct sum of stable bundles of total rank n.

Birational projective map $N \dashrightarrow M$ by $V \mapsto W$.

$N \dashrightarrow M$ is desingularization iff $\overline{\mathcal{O}(M_n(\mathbb{C}))}$ is smooth in alg_{n^2}. This is true for $n = 2$.

(Le Bruyn-Reichstein) $\overline{\mathcal{O}(A)}$ is smooth in alg_m iff A is m-dimensional 2-nilpotent or generalized Kronecker or $m = 3$ or 4 and

$$A = \mathbb{C} \times \mathbb{C} \times \mathbb{C} \quad \text{or} \quad A = M_2(\mathbb{C})$$

Paradigm shift : In the commalg-world, M is hopelessly singular. However, in the assoc-world, M is a smooth (noncommutative) manifold.

today : What are worlds ? world-manifolds ?

tomorrow : Local study and classification of world-manifolds.

Also global study of (nice)world-manifolds : differential forms, necklace Lie algebras etc. (cfr. V. Ginzburg).

world = cyclic quadratic operad. (V. Ginzburg, M. Kapranov, M. Kontsevich a. many o.)

$$E = \underbrace{\Box\!\Box \oplus \ldots \oplus \Box\!\Box}_{k} \oplus \underbrace{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix} \oplus \ldots \oplus \begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}_{l}$$

k commutative products $(a,b)_i = (b,a)_i$, $1 \leq i \leq k$. l alternating products $[a,b]_j = -[b,a]_j$, $1 \leq j \leq l$.

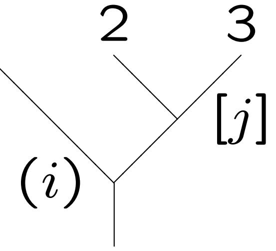
$E_{(3)}$ vectorspace on all different 2-brackets in variables x_1, x_2, x_3. For example, $([x_1,x_2]_i, x_3)_j$

$$= (x_3, [x_1,x_2]_i)_j = -(x_3,[x_2,x_1]_i)_j = \ldots$$

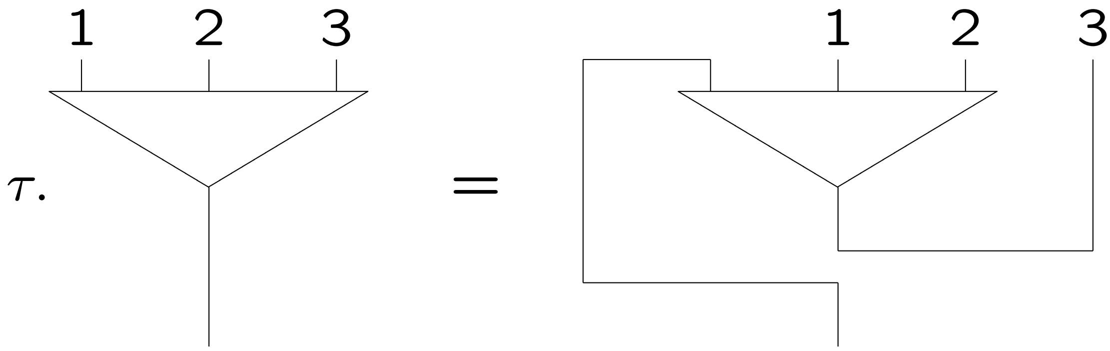
Permuting the x_i gives S_3-action

$$E_{(3)} = Ind_{S_2}^{S_3}(E)^{\oplus k+l}$$

2-brackets $\leftrightarrow$ labeled binary 3-trees

$(x_1, [x_2, x_3]_j)_i \leftrightarrow$ 

Have S_4-action on $E_{(3)}$. $\tau = (0123)$, define



for example

$$\tau.([x_3, x_1], x_2) = [(x_1, x_3), x_2]$$

world = (E, R) with $R \triangleleft_{S_4} E_{(3)}$.

world-algebra = $\mathbb{C}$-vectorspace V with maps

$$S^2\ V \xrightarrow{(i)} V \quad \text{and}\ \wedge^2\ V \xrightarrow{[j]} V$$

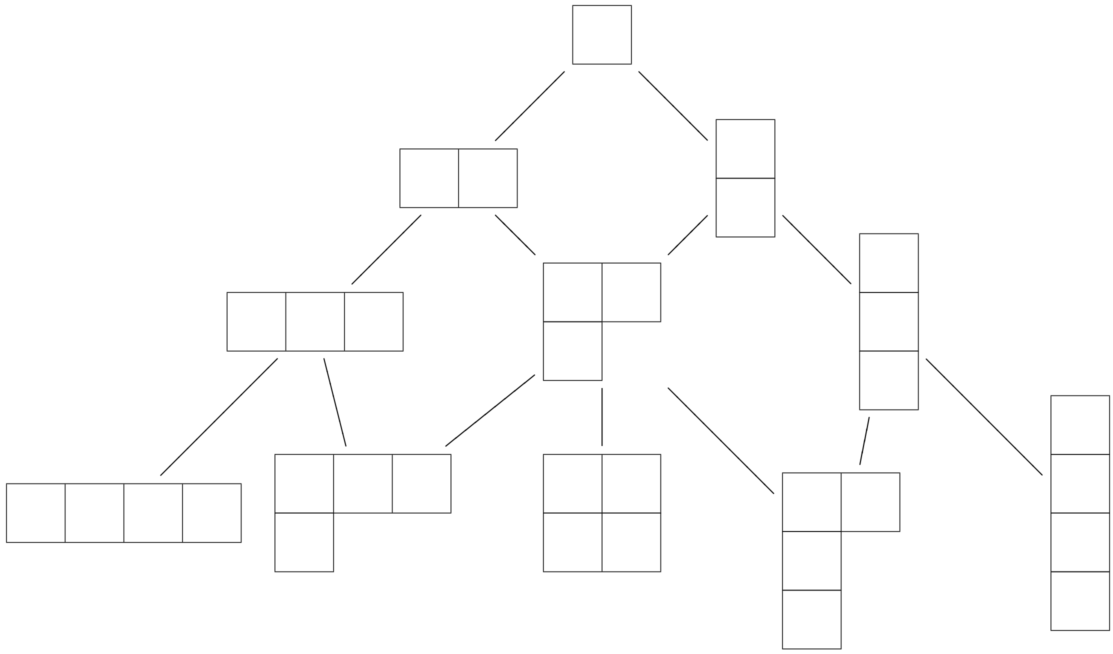
satisfying all relations in R.

Example : $\mathfrak{assoc}$-algebras are determined by a map

$$V \otimes V = S^2\ V \oplus \wedge^2\ V \xrightarrow{()+[]} V$$

whence corresponding $E = \begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$.

induction-restriction diagram



$$E = \ydiagram{2} \oplus \ydiagram{1,1} \qquad E_{(3)} = \ydiagram{3}^{2} \oplus \ydiagram{2,1}^{4} \oplus \ydiagram{1,1,1}^{2}$$

among the options one calculates that the S_4-action on $E_{(3)}$ splits

$$\ydiagram{4} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2}^{2} \oplus \ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1}$$

rescaling products gives torus action with two point orbits $0, \infty$ and one open orbit 1 on the $\mathbb{P}^1$ of S_4-submodules of

$$\ydiagram{2,2} \oplus \ydiagram{2,2}$$

(Getzler-Kapranov) : there are precisely 80 different worlds for $E = \ydiagram{2} \oplus \ydiagram{1,1}$.

Example : assoc is the E-world with relations

$$R = \ydiagram{3,1} \oplus \underbrace{\ydiagram{2,2}}_{''1''} \oplus \ydiagram{1,1,1,1}$$

world-algebra V **with unit** $1 \in V$ satisfying

$$[1, v]_j = 0 \quad \text{and} \quad (1, v)_i = v$$

substituting 1 for x_1, x_2, x_3 gives maps

$$E_{(3)} \xrightarrow{s_1, s_2, s_3} E$$

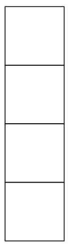
unital world (E, R) such that $s_i(R) = 0$ for $1 \leq i \leq 3$

Example : for $E = \yng(2) \oplus \yng(1,1)$, unital

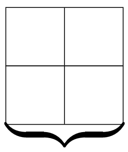
$$R \lhd_{S_4} \yng(1,1,1,1) \oplus \yng(2,2)^2 \oplus \yng(3,1)$$

there are precisely 20 unital E-worlds.

relations for unital E-worlds.

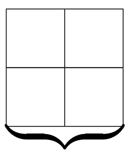


$$\leftrightarrow \begin{cases} [[1,2],3]+[[2,3],1]+[[3,1],2] \\ [,]\ \text{is Lie algebra} \end{cases}$$



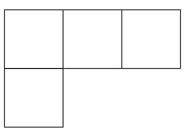
''0''

$$\leftrightarrow \begin{cases} ((1,2),3)-((2,3),1) \\ ((1,2),3)-((3,1),2) \\ (,)\ \text{is associative} \end{cases}$$

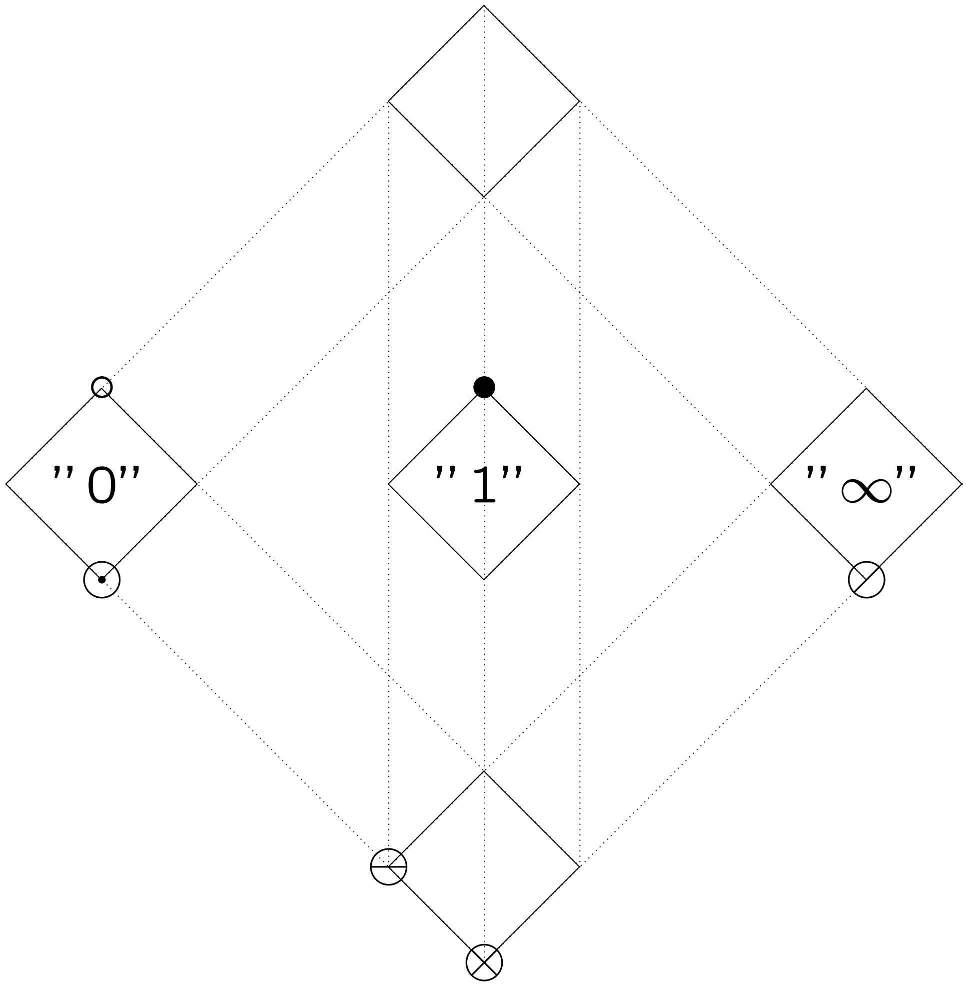


∞

$$\leftrightarrow \begin{cases} [[1,2],3]-[[2,3],1] \\ [[1,2],3]-[[3,1],2] \\ [,]\ \text{is anti-associative} \end{cases}$$



$$\leftrightarrow \begin{cases} ([2,3],1)-[(1,2),3]+([3,1],2) \\ ([2,3],1)-[(3,1),2]-([1,2],3) \\ ([3,1],2)-[(2,3),1]+([1,2],3) \\ [x,-]\ \text{is derivation for product}\ (,) \end{cases}$$



$\bullet$ = assoc, $\circ$ = poisson, $\odot$ = commalg, $\ominus$ = lie, $\otimes$ = inproduct, $\oslash$ = mock-comm.

Question : Study $w - alg_n$ (n-dim world-algebras) and classify the smooth orbit closures.

T **target algebra** iff T fin. dml. w-alg with

$$T \otimes T \xrightarrow{B} \mathbb{C}$$

which is **non-degenerate** and **invariant**

$$\begin{cases} B((x,y)_i, z) = B(x, (y,z)_i) \\ B([x,y]_j, z) = B(x, [y,z]_j) \end{cases}$$

T unital then trace $tr(x) = B(x, 1)$. AuT all B-invariant w-alg autos of T (T simple, often reductive).

Example : $T = M_n(\mathbb{C})$ in assoc$_1$, then $B(x,y) = tr(xy)$ and $AuT = PGL_n$.

A f.g. world-algebra gen. $a_1, \ldots, a_k$.

T-representations of A

$$rep_T(A) = w - alg(A,T) \hookrightarrow T \times \ldots \times T$$

have AuT-action with quotient space

$$iss_T(A) = rep_T(A)/AuT$$

the semisimple reps in T.

(d'apres V. Popov) if $E = \begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ and T simple generated by $\leq k$ elements, then 'traces' separate generic AuT-orbits in $rep_T(A)$.

Question : General result ? Generators of invariants ?

A is T-**smooth** iff $rep_T(A)$ is smooth. A **world-smooth** iff T-smooth for all targets T (possibly wrt. base B).

Example : assoc if A is formally smooth (Cuntz-Quillen), then assoc-smooth. e.g. $A = \mathbb{C}Q$ or $\mathbb{C}[X]$, X smooth affine curve, free products etc. $T = M_n(\mathbb{C})$ and B vertices of Q, then

$$\begin{cases} rep_T(\mathbb{C}Q) = rep_n\ Q = \bigsqcup\ GL_n \times^{GL(\alpha)} rep_\alpha\ Q \\ rep_{T,B}(\mathbb{C}Q) = rep_\alpha\ Q \end{cases}$$

when $B \hookrightarrow M_n(\mathbb{C})$ have ranks = dim vector.

Moduli problem $X \dashrightarrow M$ is **world-manifold** iff locally isomorphic to quotients

$$rep_{T,B}(A) \dashrightarrow iss_{T,B}(A)$$

for A a world-smooth algebra (or T-smooth). good points $\leftrightarrow$ simple T-reps.

Example : Q cycle-free quiver having θ-stables of dim α, then

$$M_\alpha^{ss}(Q,\theta)$$

is a projective assoc-manifold.

determinental semi-invariants σ determine universal localizations $\mathbb{C}Q_\sigma$ of $\mathbb{C}Q$ such that

$$\theta-\text{stable} \leftrightarrow \text{simple repr. of} \mathbb{C}Q_\sigma$$

Example : X projective smooth curve of genus g, then

$$M_X^{ss}(0,n)$$

is a projective $\mathfrak{assoc}$-manifold.

(A. Schofield) semistable bundles of degree 0 and rank n are locally determined by representations of dimension vector (n, gn) of the formally smooth algebra

$$\begin{bmatrix} \mathbb{C} & A \\ 0 & A \end{bmatrix}$$

where A is affine piece of X. A universal localization does the job.

world : (E,R) with $R \triangleleft_{S_4} E_{(3)}$.

target : $T \in w-alg$ with invariant nondegenerate bilinear form (trace). AuT invariant automorphisms (reductive).

smooth algebra : $A \in w{-}alg$ such that $rep_T(A) = w-alg(A,T)$ is smooth (for all targets).

world manifold : moduli problem $X \longrightarrow\!\!\!\!\!\!\!\!\rightarrow M$ which is locally iso to

$$rep_T(A) \longrightarrow iss_T(A) = rep_T(A)/AuT$$

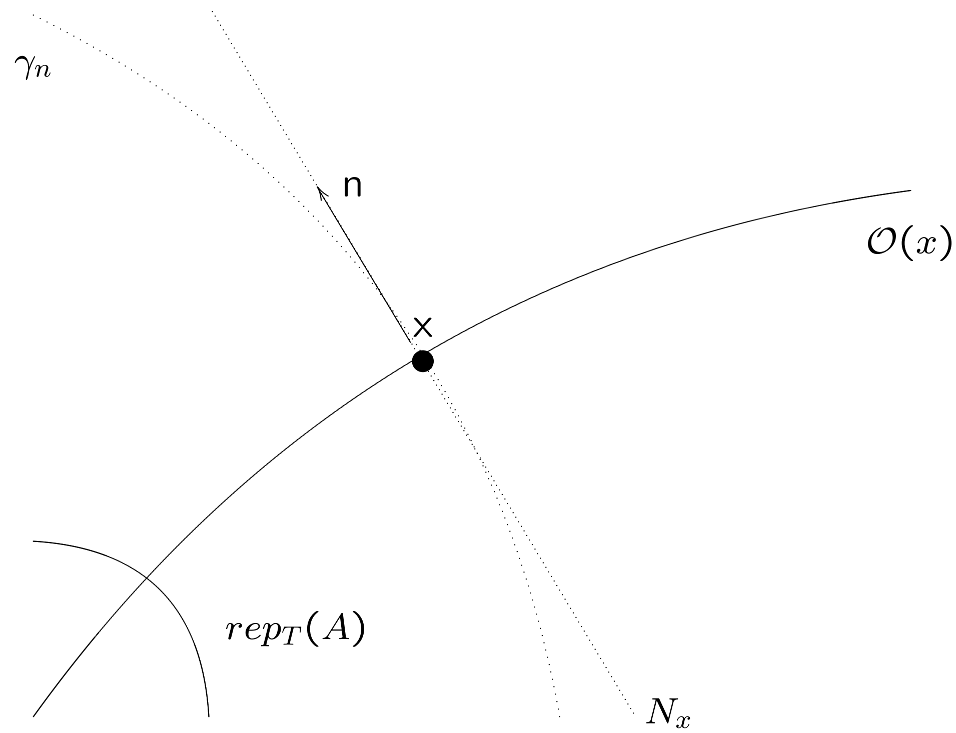
and good orbits correspond to simple T-reps.

today : local classification and study of world-manifolds.

A is a T-smooth world-alg, $rep_T(A)$ smooth with AuT-action.

$\xi \in iss_T(A)$, want to describe analytic local structure of $iss_T(A)$ near ξ.

ξ determines a unique **closed** orbit $\mathcal{O}(x)$ in $rep_T(A)$. $G_\xi = Stab_x(AuT)$ is reductive.



Luna's slice theorem :

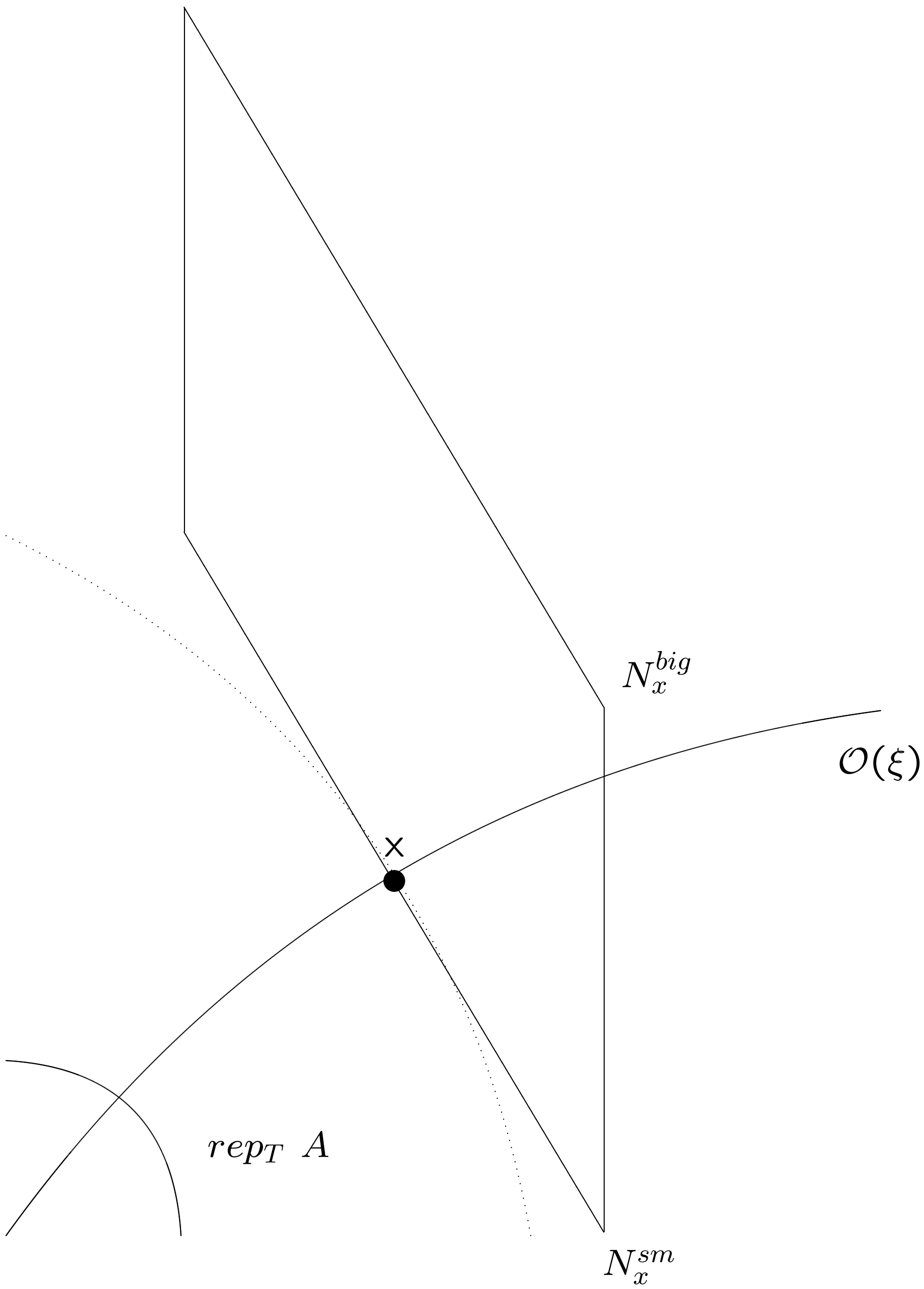
There is an analytic (étale) local isomorphism between

- a neighborhood of ξ in $iss_T(A)$ and
- a neighborhood of $\overline{0}$ in the quotient variety N_x/G_ξ.

aim : Have to classify all pairs (G_ξ, N_x) which can appear for arbitrary T-smooth world algebras A. **problem :** no control on T-smooth world-algebras.

A generated by k elements then

$$rep_T(A) \hookrightarrow \underbrace{T \times \ldots \times T}_{k}$$



$G = Stab_{AuT}(x)$ is reductive so

$$N_x^{sm} \triangleleft_G N_x^{big}$$

general point in N_x^{sm} must have trivial G-stabilizer.

method :

- describe all types τ of semi-simples in $T \times \ldots \times T$ under simultaneous action of AuT and their stabilizers $G(\tau)$.

- describe the isotypical decomposition of the $G(\tau)$ representations $N(\tau) = N_x^{big}$.

- describe the subsummands S which have generic trivial $G(\tau)$ stabilizer.

- classify all obtained $(S, G(\tau))$ according to the dimension of $S/G(\tau)$.

classification for world manifolds : For given dimension d and target T there are finitely many classes of (analytic/étale) local behavior for world manifolds.

compare with comm-manifolds : for every dimension d just 1 analytic type $\mathbb{A}^d$.

now : apply method to $\mathtt{assoc}$. will show ubiquity of quiver-representations for noncommutative geometry.

assoc-**classification.**

A formally smooth algebra, $T = M_n(\mathbb{C})$, $AuT = PGL_n$, $rep_T(A) = rep_n(A)$ is smooth affine PGL_n-variety.

Artin-Voigt : $\xi \in iss_n(A) = rep_n(A)/PGL_n$ is an isoclass of n-dml semisimple A-representation,

$$M_x = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

S_i simple A-representation of dimension d_i occurring with multiplicity e_i, $n = \sum_i d_i e_i$.

Stabilizer : $Stab_x(PGL_n) = PGL(\alpha)$ with $\alpha = (e_1, \ldots, e_k)$ and embedding coming from

$$\begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes 1_{d_1}) & 0 & \ldots & 0 \\ 0 & GL_{e_2}(\mathbb{C} \otimes 1_{d_2}) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & GL_{e_z}(\mathbb{C} \otimes 1_{d_z}) \end{bmatrix}$$

in GL_n.

Gabriel : normal space to the orbit is

$$N_x = Ext^1_A(M_x, M_x) = \oplus_{i,j=1}^{k} Ext^1_A(S_i, S_j)^{\oplus e_i e_j}$$

As a $Stab_x = PGL(\alpha)$-representation N_x is a quiver-situation

$$N_x = rep_\alpha\ Q_\xi$$

with the **local quiver** Q_ξ on k vertices $v_1, \ldots, v_k$ (corresponding to distinct simple components $S_1, \ldots, S_k$) and

$$\#\ j \longleftarrow i \quad = \quad dim_{\mathbb{C}}\ Ext^1_A(S_i, S_j)$$

when is α dimension vector of simple representation ? what is dimension of quotient variety $rep_\alpha\ Q_\xi/PGL(\alpha) = iss_\alpha\ Q_\xi$?

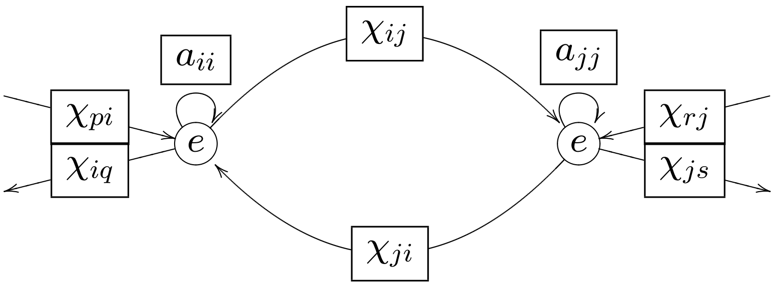
(Le Bruyn-Procesi) χ Euler-form of Q_ξ, then α is dim-vector of simple representation iff Q_ξ is strongly connected and

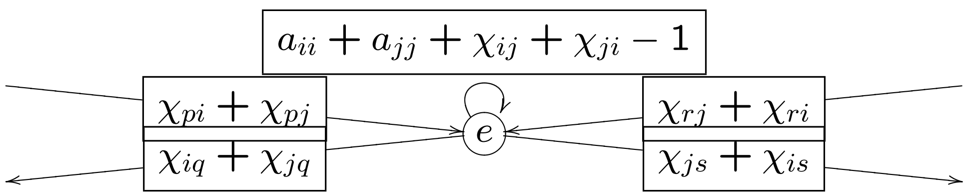
$$\chi(\alpha, \epsilon_i) \leq 0 \quad \text{and} \quad \chi(\epsilon_i, \alpha) \leq 0$$

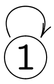
for all $1 \leq i \leq k$ unless $Q_\xi = \tilde{A}_k$ then $\alpha = (1, \ldots, 1)$. In this case

$$dim\ iss_\alpha\ Q_\xi = 1 - \chi(\alpha, \alpha)$$

Shrinking to simplest form







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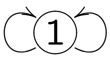

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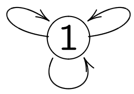

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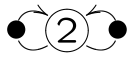

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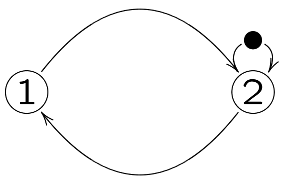

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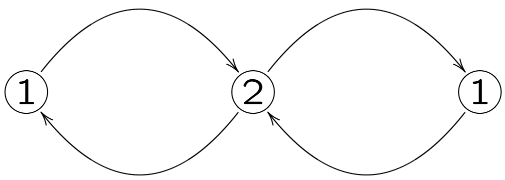

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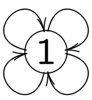

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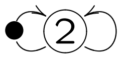

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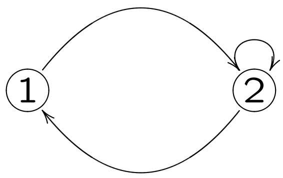

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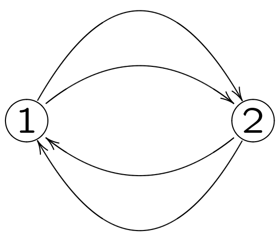

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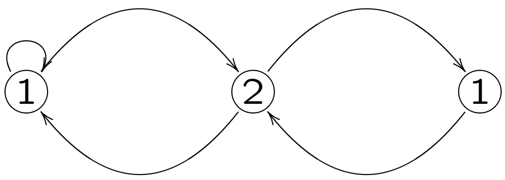
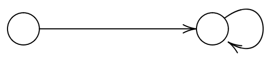


$\boxed{4}$

moduli space $M_X^{ss}(0,n)$.

(n, gn)-dml representations of $\begin{bmatrix} \mathbb{C} & A \\ 0 & A \end{bmatrix}$



$$\chi = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

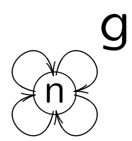
local quiver-situation Q_V in point

$$V = W_1^{\oplus e_1} \oplus \ldots \oplus W_k^{\oplus e_k}$$

W_i stable of rank d_i, so $\alpha_i = (d_i, gd_i)$.

$$\# \; (j) \leftarrow (i) = \delta_{ij} - \chi(\alpha_i, \alpha_j) = \delta_{ij} + (g-1)d_i d_j$$

and $\alpha_V = (e_1, \ldots, e_k)$.

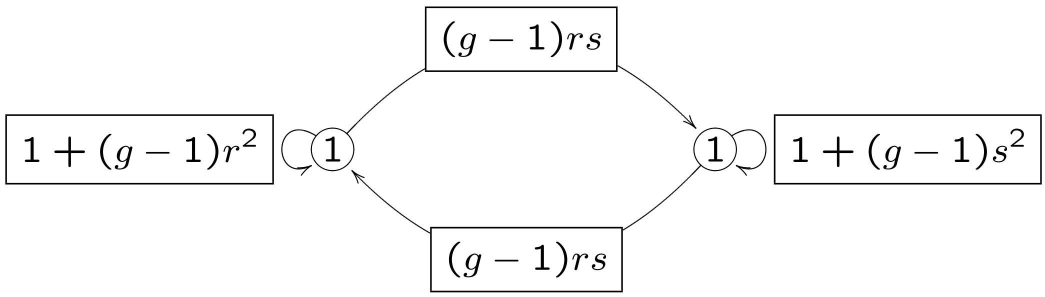
Example : $V = \mathcal{O} \oplus \ldots \oplus \mathcal{O}$, then $Q_V =$ 

hence local isomorphic to

$$\underbrace{M_n \oplus \ldots \oplus M_n}_{g} / PGL_n$$

(C.S. Seshadri).

Example : $V = W \oplus W'$, $rk(W) = r, rk(W') = s$



singular unless invariants (traces along oriented cycles) form polynomial algebra : $(g-1)rs = 1$, i.e. $g = 2$, $r + s = n = 2$.

C.S. Seshadri : $M^{sing} = M - M^s$ unless $(g, n) = (2, 2)$ in which case M is smooth.

isoclasses of semistable W s.t. $gr(W) = V$ = orbit-structure of **nullcone** of Q_V.

Example : 2 copies of $\mathbb{P}^{(g-1)rs-1}$ together with one point.

preprojective algebra $\Pi_0(Q)$.

Crawley-Boevey : S_i, S_j two simple Π_0-reps of dimension α_i, α_j

$$dim\ Ext^1_{\Pi_0}(S_i, S_j) = 2\delta_{ij} - T_Q(\alpha_i, \alpha_j)$$

and $dim\ iss_\alpha(\Pi_0) = 2(1 - \chi_Q(\alpha, \alpha))$ (when $\alpha \in \Sigma_0$).

$V = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$ has local quiver-situation of dimension type $dim\ iss_\alpha(\Pi_0)$ iff $k = 1$ and $e = 1$ i.e. when V is simple.

$$\alpha - Smoothlocus(\Pi_0) = iss^s_\alpha(\Pi_0)$$

Example : Q extended Dynkin, $\alpha = \delta_Q$. Kleinian singularity remains singular in assoc.

deformed preprojective $\Pi_\lambda(Q)$.

Equivalent are :

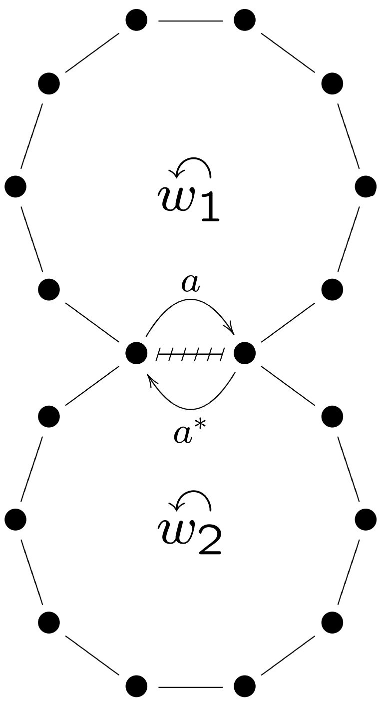
- Π_λ is α-smooth.
- α is minimal element of Σ_λ.
- $iss_\alpha(\Pi_\lambda)$ is coadjoint orbit for neclace Lie algebra $\mathbb{N}_Q$.

noncommutative functions $\mathbb{N}_Q = \frac{\mathbb{C}\tilde{Q}}{[\mathbb{C}\tilde{Q},\mathbb{C}\tilde{Q}]}$ has basis the necklaces in $\tilde{Q}$

$$0 \longrightarrow V \longrightarrow \mathbb{N}_Q \longrightarrow Der_\omega\ \mathbb{C}\tilde{Q} \longrightarrow 0$$

exact as Lie algebras, with Lie bracket on $\mathbb{N}_Q$ the noncommutative Poisson bracket.

$$[w_1, w_2] = \sum_{a \in Q_a}$$



$$-$$