

# LOCAL QUIVERS AND STABLE REPRESENTATIONS

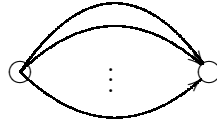
JAN ADRIAENSSENS AND LIEVEN LE BRUYN

ABSTRACT. In this paper we introduce and study the local quiver as a tool to investigate the étale local structure of moduli spaces of  $\theta$ -semistable representations of quivers. As an application we determine the dimension vectors associated to irreducible representations of the torus knot groups  $G_{p,q} = \langle a, b \mid a^p = b^q \rangle$ .

## 1. INTRODUCTION.

Polynomial invariants of quiver representations are generated by taking traces along oriented cycles in the quiver, see [13]. Hence, if we have a quiver  $Q$  without oriented cycles on  $k$  vertices, the quotient variety of  $\alpha$ -dimensional representations  $\text{rep}_\alpha Q$  under the basechange group  $GL(\alpha)$  consists of one point corresponding to the unique  $\alpha$ -dimensional semi-simple representation which is the direct sum of the vertex spaces. Still, there may be lots of semi-invariants on  $\text{rep}_\alpha Q$  with respect to a character  $\chi = \prod \det(g_i)^{t_i}$  of  $GL(\alpha)$ , determined by an integral vector  $\theta = (t_1, \dots, t_k) \in \mathbb{Z}^k$ . If we collect all semi-invariants with weight  $n\chi$  for all  $n \in \mathbb{N}$  we obtain a positively graded connected algebra with associated projective variety  $M_\alpha^{ss}(Q, \theta)$  the moduli space of  $\theta$ -semistable  $\alpha$ -dimensional representations of  $Q$ , see [11]. The points of  $M_\alpha^{ss}(Q, \theta)$  correspond to isomorphism classes of direct sums of  $\theta$ -stable representations of  $Q$ . Recall that an  $\alpha$ -dimensional representation  $V$  is  $\theta$ -stable (resp.  $\theta$ -semistable) if for all subrepresentation  $W \hookrightarrow V$  we have  $\theta(W) = \sum_i t_i b_i > 0$  (resp.  $\geq 0$ ) where  $\beta = (b_1, \dots, b_k)$  is the dimension vector of  $W$ . Not much is known on the geometry of these moduli spaces in particular of their singularities.

As an illustrative example, consider the extended  $n$ -arrow Kronecker quiver  $K_n$



Take the dimension vector  $\alpha = (1, r)$  with  $r \leq n$  and the character on  $GL(\alpha) = \mathbb{C}^* \times GL_r$  determined by  $\theta = (-r, 1)$ . An  $\alpha$ -dimensional representation  $V \in \text{rep}_\alpha K_n$  is fully determined by the  $r \times n$  matrix whose  $i$ -th column is the linear map corresponding to the  $i$ -th arrow. Clearly, a  $\theta$ -stable representation  $V \in \text{rep}_\alpha K_n$  must have an  $r \times n$  matrix of maximal rank for if the rank were  $s < r$ , then  $V$  has a subrepresentation of dimension vector  $\beta = (1, s)$  and  $\theta(W) = -r \cdot 1 + 1 \cdot s < 0$ . The basechange action is given by left multiplication by  $GL_r$  and scalar multiplication by  $\lambda^{-1}$  for  $\lambda \in \mathbb{C}^*$ . Therefore, in this case we have that the moduli space of  $\theta$ -stable quiver representations

$$M_\alpha^{ss}(K_n, \theta) \simeq \text{Grass}(r, n)$$

the Grassmannian of  $r$ -dimensional subspaces in  $\mathbb{C}^n$ . Smoothness of this moduli space follows from the fact that  $\alpha$  is minimal among the dimension vectors having  $\theta$ -stable representations. Things become more complicated for multiples  $k\alpha$  because

---

Research Director of the FWO (Belgium).

in this case the moduli spaced  $M_{k\alpha}^{ss}(K_n, \theta)$  contains points corresponding to direct sums of  $\theta$ -stable representation such as

$$V = W^{\oplus k} \quad \text{for } W \in \text{rep}_\alpha K_n \text{ } \theta\text{-stable.}$$

We will show in this paper that the étale local structure of  $M_{k\alpha}^{ss}(K_n, \theta)$  near the point determined by  $V$  is that of the quotient variety of  $r(n-r)$ -tuples of  $k \times k$  matrices under simultaneous conjugation near the point determined by the tuple of zero matrices. It is well known that this quotient variety is smooth only if  $k=1$  or  $r(n-r)=2=k$ . Therefore, the only moduli spaces  $M_{k\alpha}^{ss}(K_n, \theta)$  which are smooth are the Grassmannians described before and

$$M_{(2,2)}^{ss}(K_3, \theta) \text{ for } \theta = (-1, 1) \quad \text{and} \quad M_{(2,4)}^{ss}(K_3, \theta) \text{ for } \theta = (-2, 2)$$

Both these varieties are isomorphic to  $M_{\mathbb{P}^2}(0, 2) \simeq \mathbb{P}^5$ , the moduli space of semistable rank 2 bundles on  $\mathbb{P}^2$  with Chern classes  $c_1 = 0$  and  $c_2 = 2$  by results of Barth and Hulek [3], [10] (and reflexion functors).

The general result is as follows. Let  $\xi \in M_\alpha^{ss}(Q, \theta)$  be a point corresponding to a representation

$$V_\xi = W_1^{\oplus m_1} \oplus \dots \oplus W_l^{\oplus m_l}$$

where  $W_i$  is a  $\theta$ -stable representation of  $Q$  of dimension vector  $\beta_i$ . Consider the local quiver  $Q'$  on  $l$  vertices  $w_1, \dots, w_l$  (corresponding to the distinct  $\theta$ -stable components) with  $\delta_{ij} - \chi_Q(\beta_i, \beta_j)$  arrows from  $w_i$  to  $w_j$  where  $\chi_Q$  is the Euler form of the quiver  $Q$ . Consider the dimension vector  $\alpha' = (m_1, \dots, m_l)$  for  $Q'$  determined by the multiplicities of the  $\theta$ -stable components.

**Theorem 1.1.** *With notations as before, there is an étale isomorphism between a neighborhood of  $\xi$  in the moduli space  $M_\alpha^{ss}(Q, \theta)$  and a neighborhood of the trivial representation  $\bar{0}$  in the quotient variety  $\text{iss}_{\alpha'} Q'$  of semi-simple  $\alpha'$ -dimensional representations of the local quiver  $Q'$ .*

The proof uses two ingredients : first we use the theory of universal localizations to reduce the Zariski local structure of  $M_\alpha^{ss}(Q, \theta)$  near  $\xi$  to that of  $\alpha$ -dimensional semi-simple representations of a universal localization  $\mathbb{C}Q_\sigma$  of the path algebra. Next, we apply the Luna slice machinery to this quotient variety to obtain the result.

The theorem can also be used to obtain an inductive method to determine all dimension vectors of  $\theta$ -stable representations.

**Theorem 1.2.** *With notations as above,  $\alpha$  is the dimension vector of a  $\theta$ -stable representation if and only if  $\alpha'$  is the dimension vector of a simple representation of the local quiver  $Q'$ .*

The utility of this result follows from the easy description of the dimension vectors of simple quiver representations given in [13] by a set of inequalities, rather than an inductive procedure. Finally, we apply this result to determine the dimension vectors associated to irreducible representations of the torus knot groups, among which is the braid group on three strings.

## 2. MODULI SPACES OF QUIVER REPRESENTATIONS.

In this section we recall briefly the basic setting. Recall that a *quiver*  $Q$  is a finite directed graph on a set of vertices  $Q_v = \{v_1, \dots, v_k\}$ , having a finite set  $Q_a = \{a_1, \dots, a_l\}$  of arrows, where we allow loops as well as multiple arrows between vertices. An arrow  $a$  with starting vertex  $s(a) = v_i$  and terminating vertex  $t(a) = v_j$  will be depicted as  $\textcircled{i} \xleftarrow{a} \textcircled{j}$ . The quiver information is encoded in the *Euler form* which is the bilinear form on  $\mathbb{Z}^k$  determined by the matrix  $\chi_Q \in M_k(\mathbb{Z})$  with

$$\chi_{ij} = \delta_{ij} - \# \{ a \in Q_a \mid \textcircled{i} \xleftarrow{a} \textcircled{j} \}$$

The *path algebra*  $\mathbb{C}Q$  of a quiver  $Q$  has as basis the set of all oriented paths  $p = a_{i_u} \dots a_{i_1}$  of length  $u \geq 1$  in the quiver, that is  $s(a_{i_{j+1}}) = t(a_{i_j})$  together with the vertex-idempotents  $e_i$  of length zero. Multiplication in  $\mathbb{C}Q$  is induced by (left) concatenation of paths. More precisely,  $1 = e_1 + \dots + e_k$  is a decomposition of 1 into mutually orthogonal idempotents and further we define

- $e_j.a$  is always zero unless  $\textcircled{j} \xleftarrow{a} \textcircled{i}$  in which case it is the path  $a$ ,
- $a.e_i$  is always zero unless  $\textcircled{i} \xleftarrow{a} \textcircled{j}$  in which case it is the path  $a$ ,
- $a_i.a_j$  is always zero unless  $\textcircled{i} \xleftarrow{a_i} \textcircled{k} \xleftarrow{a_j} \textcircled{j}$  in which case it is the path  $a_i a_j$ .

Path algebras of quivers are the archetypical examples of *formally smooth algebras* as introduced and studied in [5].

For a dimension vector  $\alpha = (a_1, \dots, a_k) \in \mathbb{N}^k$  we define the representation space

$$\text{rep}_\alpha Q = \bigoplus_{\textcircled{j} \xleftarrow{\quad} \textcircled{i}} M_{a_j \times a_i}(\mathbb{C})$$

which is an affine space with natural action of the basechange group  $GL(\alpha) = GL_{a_1} \times \dots \times GL_{a_k}$ . The points of  $\text{rep}_\alpha Q$  correspond to left  $\mathbb{C}Q$ -modules  $V$  such that  $\dim_{\mathbb{C}} e_i.V = a_i$ . The  $GL(\alpha)$ -orbits correspond to isomorphism classes as  $\mathbb{C}Q$ -modules. By abuse of notation we speak of morphisms (in particular subrepresentations) and extensions of representations for the corresponding module theoretic notions.

Let  $\theta = (t_1, \dots, t_k) \in \mathbb{Z}^k$  and define for a  $Q$ -representation  $W$  of dimension vector  $\beta = (b_1, \dots, b_k)$  the number  $\theta(W) = t_1 b_1 + \dots + t_k b_k$ . Following A. King in [11] we say that a representation  $V \in \text{rep}_\alpha Q$  is  $\theta$ -semistable if  $\theta(V) = 0$  and for every subrepresentation  $W \subset V$  we have  $\theta(W) \geq 0$ . A  $\theta$ -semistable representation  $V$  is said to be  $\theta$ -stable if the only subrepresentations  $W \subset V$  with  $\theta(W) = 0$  are 0 and  $V$  itself.

$\theta$  defines a character on  $GL(\alpha)$  by sending a  $k$ -tuple  $(g_1, \dots, g_k) \in GL(\alpha)$  to  $\det(g_1)^{t_1} \dots \det(g_k)^{t_k} \in \mathbb{C}^*$ . With  $SI(Q, \theta)$  we denote the space of all semi-invariant functions  $f$  on  $\text{rep}_\alpha Q$  of weight  $\theta$ , that is  $f \in \mathbb{C}[\text{rep}_\alpha Q]$  such that

$$f(g.V) = \theta(g)f(V) \quad \text{for all } g \in GL(\alpha) \text{ and } V \in \text{rep}_\alpha Q.$$

In [11, Prop.3.1] it is shown that  $V \in \text{rep}_\alpha Q$  is  $\theta$ -semistable if and only if there is a semi-invariant  $f \in SI(Q, n\theta)$  for some  $n \geq 0$  such that  $f(V) \neq 0$ . For this reason we consider the positively graded algebra

$$R_\theta = \bigoplus_{n=0}^{\infty} SI(Q, n\theta)$$

with part of degree zero  $SI(Q, 0) = \mathbb{C}[\text{rep}_\alpha Q]^{GL(\alpha)}$  the ring of polynomial quiver-invariants which is, by a result of [13] generated by traces along oriented cycles in the quiver  $Q$ . In particular, if  $Q$  is a quiver without oriented cycles, then  $R_\theta(0) = \mathbb{C}$  and we obtain a projective variety

$$\text{proj } R_\theta = M_\theta^{ss}(Q, \alpha)$$

called the moduli space of  $\theta$ -semistable representations of the quiver  $Q$  of dimension vector  $\alpha$ .

By [11, Prop. 3.2] a point  $\xi \in M_\theta^{ss}(Q, \alpha)$  determines uniquely the orbit of a representation  $V_\xi \in \text{rep}_\alpha Q$  such that

$$V = W_1^{\oplus m_1} \oplus \dots \oplus W_l^{\oplus m_l}$$

where the  $W_i$  are  $\theta$ -stable representations of  $Q$  of dimension vector  $\beta_i$  which occur in  $V_\xi$  with multiplicity  $m_i$ . In particular, the *semistable representation type*  $\tau = (m_1, \beta_1; \dots; m_l, \beta_l)$  of  $\xi$  or  $V_\xi$  satisfies

$$\alpha = m_1 \beta_1 + \dots + m_l \beta_l$$

In this paper we will introduce a local quiver  $Q_\xi$  (depending only on the semistable representation type) as a tool to investigate the étale (that is, analytic) local structure of the moduli space  $M_\theta^{ss}(Q, \theta)$  in a neighborhood of  $\xi$ .

### 3. UNIVERSAL LOCALIZATIONS.

In this section we will show that one can cover the projective moduli space  $M_\theta^{ss}(Q, \alpha)$  by affine open sets, each of which isomorphic to the quotient variety of  $rep_\alpha \mathbb{C}Q_\sigma$  where  $\mathbb{C}Q_\sigma$  is a certain universal localization of the path algebra  $\mathbb{C}Q$ . First, we need to recall a result on the generation of semi-invariants of quiver representations, see for example [17], [6] and [8]. Throughout, we will assume that  $Q$  is a quiver without oriented cycles.

Fix a character  $\theta = (t_1, \dots, t_k)$  and divide the set of vertex-indices into a left set  $L = \{i_1, \dots, i_u\}$  consisting of those  $1 \leq i \leq k$  such that  $t_i \leq 0$  and a right set  $R = \{j_1, \dots, j_v\}$  consisting of those  $1 \leq j \leq k$  such that  $t_j \geq 0$  (observe that  $L \cap R$  may be non-empty).

For every vertex  $v_i$  we define the indecomposable projective module  $P_i = \mathbb{C}Qe_i$  spanned by all paths in the quiver  $Q$  starting at  $v_i$ . As a consequence we have that  $Hom_Q(P_i, P_j)$  is spanned by all paths  $[j, i]$  in the quiver  $Q$  from vertex  $v_j$  to vertex  $v_i$ . Because  $Q$  has no oriented cycles, these spaces are finite dimensional.

For a fixed integer  $n$  we consider the set  $\Sigma_\theta(n)$  of all  $\mathbb{C}Q$ -module morphisms

$$\begin{array}{ccc}
 P_{i_1}^{\oplus -nt_{i_1}} \oplus \dots \oplus P_{i_u}^{\oplus -nt_{i_u}} & \xrightarrow{\sigma} & P_{j_1}^{\oplus nt_{j_1}} \oplus \dots \oplus P_{j_v}^{\oplus nt_{j_v}} \\
 \parallel \text{notation} & & \parallel \text{notation} \\
 P_{c_1} \oplus \dots \oplus P_{c_p} & \xrightarrow{\sigma} & P_{d_1} \oplus \dots \oplus P_{d_q}
 \end{array}$$

By the remark above,  $\sigma$  can be described by an  $(p = n \sum t_{i_i}) \times (q = n \sum t_{j_m})$  matrix  $M_\sigma$  all entries of which are linear combinations  $p_{lm}$  of paths in the quiver  $Q$  from vertex  $v_{d_m}$  to vertex  $v_{c_l}$ . For  $V \in rep_\alpha Q$  we can substitute the arrow matrices  $V(a)$  in the definition of  $p_{lm}$  and obtain a square matrix of size  $a_{c_l} \times a_{d_m}$ . If we do this for every entry of  $\sigma$  we obtain a square matrix as  $\theta(V) = 0$  which we denote by  $\sigma(V)$ . But then, the function

$$d_\sigma(V) = \det \sigma(V) : rep_\alpha Q \longrightarrow \mathbb{C}$$

is a semi-invariant of weight  $n\theta$ . The semi-invariants  $SI(Q, n\theta)$  are spanned by such determinantal semi-invariants.

The universal localization  $\mathbb{C}Q_\sigma$  of the path algebra  $\mathbb{C}Q$  with respect to the morphism  $\sigma$  is the algebra extension  $\mathbb{C}Q \xrightarrow{j_\sigma} \mathbb{C}Q_\sigma$  such that the extended morphism  $\mathbb{C}Q_\sigma \otimes_{\mathbb{C}Q} \sigma$  becomes invertible and  $j_\sigma$  has the universal property with respect to this condition.

It is easy to see that  $\mathbb{C}Q_\sigma$  is an affine  $\mathbb{C}$ -algebra. For, add an additional  $q \times p$  arrows  $n_{lm}$  to the quiver  $Q$  where  $n_{lm}$  is an arrow from vertex  $v_{c_l}$  to vertex  $v_{d_m}$  and consider the  $q \times p$  matrix

$$N_\sigma = \begin{bmatrix} n_{11} & \dots & n_{1p} \\ \vdots & & \vdots \\ n_{q1} & \dots & n_{qp} \end{bmatrix}$$

Then, the universal localization  $\mathbb{C}Q_\sigma$  is the quotient of the path algebra of this extended quiver modulo the ideal of relations given by the entries of the following

matrix identities

$$M_\sigma \cdot N_\sigma = \begin{bmatrix} v_{c_1} & & 0 \\ & \ddots & \\ 0 & & v_{c_p} \end{bmatrix} \quad N_\sigma \cdot M_\sigma = \begin{bmatrix} v_{d_1} & & 0 \\ & \ddots & \\ 0 & & v_{d_q} \end{bmatrix}$$

It follows from this description that for a dimension vector  $\beta = (b_1, \dots, b_k)$  the variety of  $\beta$ -dimensional representations of  $\mathbb{C}Q_\sigma$  is the open (but possibly empty) subvariety of  $\text{rep}_\beta Q$  consisting of those representations  $W$  such that  $\sigma(W)$  is a square invertible matrix. In particular, it follows that  $\text{rep}_\beta \mathbb{C}Q_\sigma = \emptyset$  unless  $\theta(\beta) = \sum t_i b_i = 0$ .

Returning to the study of  $\theta$ -semistable quiver representations and their moduli spaces, we define

$$(\text{rep}_\alpha \mathbb{C}Q_\sigma) X_\sigma(\alpha) = \{V \in \text{rep}_\alpha Q \mid d_\sigma(V) \neq 0\}$$

Because  $d_\sigma$  is a semi-invariant of weight  $n\theta$  it follows that  $X_\sigma(\alpha)$  consists of  $\theta$ -semistable representations.

**Lemma 3.1.** *For  $V \in X_\sigma(\alpha)$ , the following are equivalent*

1.  $V$  is a  $\theta$ -stable representation.
2.  $\mathbb{C}Q_\sigma \otimes V$  is a simple  $\alpha$ -dimensional representation of  $\mathbb{C}Q_\sigma$ .

*Proof.* Let  $V$  be  $\theta$ -stable and assume that  $W \subset V$  is a proper sub  $\mathbb{C}Q_\sigma$ -module of dimension vector  $\beta$ . Restricting  $W$  to a representation of  $Q$  we see that  $W \in \text{rep}_\beta Q$  and is a subrepresentation of  $V$ . Because  $W \in \text{rep}_\beta \mathbb{C}Q_\sigma$  we have  $\theta(\beta) = \theta(W) = 0$ , which is impossible as  $V$  is  $\theta$ -stable.

Let  $\mathbb{C}Q_\sigma \otimes V$  be a simple  $\mathbb{C}Q_\sigma$  representation and assume that  $W \subset V$  is a proper subrepresentation of dimension vector  $\beta = (b_1, \dots, b_k)$  with  $\theta(W) \leq 0$ . If  $\theta(W) < 0$  then  $-\sum nt_{i_l} b_{i_l} > \sum nt_{j_m} b_{j_m}$  whence  $\sigma(W)$  has a kernel but this contradicts the fact that  $\sigma(V)$  is invertible. Hence,  $\theta(W) = 0$  but then  $\mathbb{C}Q_\sigma \otimes W$  is a proper subrepresentation of  $\mathbb{C}Q_\sigma \otimes V$  contradicting simplicity.  $\square$

Let  $\text{rep}_\alpha^{\theta\text{-ss}} Q$  be the open set of  $\theta$ -semistable representations in  $\text{rep}_\alpha Q$ . Let  $\xi \in M_\theta^{\text{ss}}(Q, \alpha)$  of semistable representation type  $\tau = (m_1, \beta_1; \dots; m_l, \beta_l)$ , then the fiber of the quotient map

$$\text{rep}_\alpha^{\theta\text{-ss}} Q \xrightarrow{\pi_\theta} M_\theta^{\text{ss}}(Q, \alpha)$$

contains a unique closed orbit corresponding to

$$V_\xi = W_1^{\oplus m_1} \oplus \dots \oplus W_l^{\oplus m_l}$$

with the  $W_i$  all  $\theta$ -stable. There is a determinantal semi-invariant  $d_\sigma \in SI(Q, n\theta)$  for some  $n$  such that  $d_\sigma(V_\xi) \neq 0$ , hence  $V_\xi$  belongs to the affine open subset  $X_\sigma(\alpha)$  of  $\text{rep}_\alpha Q$  (note that  $X_\sigma(\alpha)$  is also an affine open  $GL(\alpha)$ -invariant subset of  $\text{rep}_\alpha^{\theta\text{-ss}} Q$ ). Every  $\theta$ -semistable representation in the fiber  $\pi_\theta^{-1}(\xi)$  has a filtration with factors the  $W_i$ . After tensoring with the universal localization  $\mathbb{C}Q_\sigma$  we have that the  $W_i$  become simple representations,  $V_\xi$  is a semi-simple representation and the above mentioned filtration becomes a Jordan-Hölder filtration.

**Theorem 3.2.** *The quotient map  $\pi_\theta$  is locally isomorphic to the quotient map*

$$\begin{array}{ccc} \text{rep}_\alpha \mathbb{C}Q_\sigma = X_\sigma(\alpha) & \hookrightarrow & \text{rep}_\alpha^{\theta\text{-ss}} Q \\ \pi_\sigma \downarrow & & \downarrow \pi_\theta \\ \text{iss}_\alpha \mathbb{C}Q_\sigma = X_\sigma(\alpha)/GL(\alpha) & \hookrightarrow & M_\theta^{\text{ss}}(Q, \alpha) \end{array}$$

*assigning to an  $\alpha$ -dimensional  $\mathbb{C}Q_\sigma$ -module its semi-simplification, that is, the direct sum of its Jordan-Hölder components. Because the affine open sets  $X_\sigma(\alpha)$  cover*

$\text{rep}_\alpha^{\theta-ss} Q$ , the moduli space of  $\theta$ -semistable quiver representations  $M_\theta^{ss}(Q, \alpha)$  is locally isomorphic to quotient varieties  $\text{iss}_\alpha \mathbb{C}Q_\sigma$  for specific universal localizations  $\mathbb{C}Q_\sigma$  of the path algebra.

#### 4. THE LOCAL QUIVER.

In this section we will prove that the local structure of the moduli space  $M_\theta^{ss}(Q, \alpha)$  is determined by a new quiver setting which we will now describe. Let  $\xi \in M_\theta^{ss}(Q, \alpha)$  of semistable representation type  $\tau = (m_1, \beta_1; \dots; m_l, \beta_l)$ , that is the unique closed orbit lying in the fiber  $\pi_\theta^{-1}(\xi)$  is the isomorphism class of a direct sum

$$V_\xi = W_1^{\oplus m_1} \oplus \dots \oplus W_l^{\oplus m_l}$$

with  $W_i$  a  $\theta$ -stable representation of dimension vector  $\beta_i$ . We introduce a new quiver setting  $(Q_\xi, \alpha_\xi)$  as follows :

- $Q_\xi$  has  $l$  vertices  $w_1, \dots, w_l$  corresponding to the distinct  $\theta$ -stable components of  $V_\xi$ , and
- the number of arrows from  $w_i$  to  $w_j$  is equal to

$$\delta_{ij} - \chi_Q(\beta_i, \beta_j)$$

where  $\chi_Q$  is the Euler form of the original quiver  $Q$ .

- the dimension vector  $\alpha_\xi = (m_1, \dots, m_l)$  gives the multiplicities of the stable summands.

Observe that the local quiver setting depends only on the semistable representation type.

**Theorem 4.1.** *There is an étale isomorphism between an affine neighborhood of  $\xi$  in the moduli space  $M_\theta^{ss}(Q, \alpha)$  and an affine neighborhood of the image  $\bar{0}$  of the zero representation in the quotient variety  $\text{iss}_{\alpha_\xi} Q_\xi = \text{rep}_{\alpha_\xi} Q_\xi / GL(\alpha_\xi)$  corresponding to the local quiver setting  $(Q_\xi, \alpha_\xi)$ .*

Let  $\xi \in M_\theta^{ss}(Q, \alpha)$  with corresponding  $V_\xi$  having a decomposition into  $\theta$ -stable representations  $W_i$  as above. By the results of the preceding section we may assume that  $V_\xi \in X_\sigma(\alpha)$  where  $X_\sigma(\alpha)$  is the affine  $GL(\alpha)$ -invariant open subvariety of  $\text{rep}_\alpha^{ss} Q$  defined by the determinantal semi-invariant  $d_\sigma$ .

We have seen that  $X_\sigma(\alpha) \simeq \text{rep}_\alpha \mathbb{C}Q_\sigma$  the variety of  $\alpha$ -dimensional representations of the universal localization  $\mathbb{C}Q_\sigma$ . Moreover, if we define  $V'_\xi = \mathbb{C}Q_\sigma \otimes V_\xi$  and  $W'_i = \mathbb{C}Q_\sigma \otimes W_i$  we have seen that

$$V'_\xi = W'_1{}^{\oplus m_1} \oplus \dots \oplus W'_l{}^{\oplus m_l}$$

is a decomposition of the semisimple  $\mathbb{C}Q_\sigma$  representation  $V'_\xi$  into its simple components  $W'_i$ . Restricting to the affine smooth variety  $X_\sigma(\alpha)$  we are in a situation to apply the strong form of the Luna slice theorem, [14] or [18], which we now recall.

Let  $X$  be a smooth affine variety on which a reductive group  $G$  acts. Let  $x \in X$  be a point in a closed  $G$ -orbit, then the stabilizer subgroup  $G_x = \{g \in G \mid g.x = x\}$  is again a reductive group. Let  $N_x$  be the normal space to the orbit  $G.x$  of  $x$ , that is,  $N_x = \frac{T_x X}{T_x G.x}$ , then  $N_x$  is a finite dimensional  $G_x$ -representation. The Luna slice theorem asserts the existence of a locally closed subvariety (the slice)  $S \hookrightarrow X$  satisfying the following properties :

- $S$  is a smooth affine variety with a  $G_x$ -action.
- The action map  $G \times S \longrightarrow X$  sending  $(g, s)$  to  $g.s$  induces a  $G$ -equivariant étale morphism from the associated fiber bundle

$$G \times^{G_x} S \xrightarrow{\psi} X$$

with an affine image and such that the induced quotient map

$$(G \times^{G_x} S)/G_x \simeq N_x/G_x \xrightarrow{\psi/G} X/G$$

is étale.

- There is a  $G_x$ -equivariant morphism  $S \xrightarrow{\phi} T_x S = N_x$  with  $\phi(x) = 0$ , having an affine image and such that the induced quotient map

$$S/G_x \xrightarrow{\phi/G_x} N_x/G_x$$

is an étale map.

That is, we have a commutative diagram of varieties

$$\begin{array}{ccccc} G \times^{G_x} N_x & \xleftarrow{G \times^{G_x} \phi} & G \times^{G_x} S & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow & & \downarrow \pi \\ N_x/G_x & \xleftarrow{\phi/G_x} & S/G_x & \xrightarrow{\psi/G} & X/G \end{array}$$

where the upper horizontal maps are all  $G$ -equivariant étale maps and the lower horizontal maps are étale. If  $\bar{x} = \pi(x)$ , we deduce that the étale local structure of the quotient variety  $X/G$  in a neighborhood of  $\bar{x}$  is étale isomorphic to the local structure of the quotient variety  $N_x/G_x$  in a neighborhood of  $\bar{0}$  the point corresponding to the fixed  $G_x$ -orbit of the zero vector in  $N_x$ .

We will now make this general result explicit in the case of interest to us.  $rep_\alpha \mathbb{C}Q_\sigma$  is a smooth affine variety with action of  $GL(\alpha)$  by basechange. The point  $V'_\xi$  lies in a closed  $GL(\alpha)$ -orbit as closed orbits correspond to semisimple representations by the result of Artin [2] and Voigt [9]. By a result of Gabriel's [9] we know that the normal space to the orbit  $GL(\alpha).V'_\xi$  can be identified with the self-extensions

$$N_{V'_\xi} = \frac{T_{V'_\xi} rep_\alpha \mathbb{C}Q_\sigma}{T_{V'_\xi} GL(\alpha).V'_\xi} = Ext_{\mathbb{C}Q_\sigma}^1(V'_\xi, V'_\xi) = \bigoplus_{i,j=1}^l Ext_{\mathbb{C}Q_\sigma}^1(W'_i, W'_j)^{\oplus m_i m_j}$$

By Schur's lemma we know that the stabilizer subgroup of the semisimple module  $V'_\xi$  is equal to

$$G_{V'_\xi} \simeq GL(\alpha_\xi) = GL_{m_1} \times \dots \times GL_{m_l} \hookrightarrow GL(\alpha)$$

and if we write out the action of this group on the self extensions we observe that it coincides with the action of the basechange group  $GL(\alpha_\xi)$  on the representation space  $rep_{\alpha_\xi} \Gamma$  of a quiver  $\Gamma$  on  $l$  vertices such that the number of arrows from vertex  $w_i$  to vertex  $w_j$  is equal to the dimension of the extension group

$$Ext_{\mathbb{C}Q_\sigma}^1(W'_i, W'_j)$$

Remains to prove that the quiver  $\Gamma$  is our local quiver  $Q_\xi$ . For this we apply a general homological result valid for universal localizations, [15, Thm 4.7]. If  $A_\sigma$  is a universal localization of an algebra  $A$ , then the category of left  $A_\sigma$ -modules is closed under extensions in the category of left  $A$ -modules. Therefore,

$$Ext_{A_\sigma}^1(M, N) = Ext_A^1(M, N)$$

for  $A_\sigma$ -modules  $M$  and  $N$ . We apply this general result to get

$$Ext_{\mathbb{C}Q_\sigma}^1(W'_i, W'_j) = Ext_Q^1(W_i, W_j)$$

Further, as the  $W_i$  are  $\theta$ -stable representations of the quiver  $Q$  we know that  $Hom_Q(W_i, W_j) = \delta_{ij} \mathbb{C}$ . Finally, we use the homological interpretation of the Euler form valid for all representations of quivers, see for example [12]

$$\chi_Q(\beta_i, \beta_j) = \dim_{\mathbb{C}} Hom_Q(W_i, W_j) - \dim_{\mathbb{C}} Ext_Q^1(W_i, W_j)$$

to deduce that  $\Gamma = Q_\xi$ , which finishes the proof of the theorem.

An important application of this local description is to determine the smooth locus of the moduli spaces  $M_\theta^{ss}(Q, \alpha)$ .

**Proposition 4.2.**  *$\xi \in M_\theta^{ss}(Q, \alpha)$  is a smooth point if and only if the ring of polynomial quiver invariants of the corresponding local quiver setting*

$$\mathbb{C}[\text{rep}_{\alpha_\xi} Q_\xi]^{GL(\alpha_\xi)}$$

*is a polynomial ring.*

As we know from [13] that the ring of invariants is generated by taking traces along oriented cycles in the quiver  $Q_\xi$  this criterium is rather effective. The classification of all coregular quiver settings  $(Q, \alpha)$  (that is such that the ring of polynomial invariants is a polynomial algebra) seems to be a difficult problem though recently significant progress has been made by R. Bocklandt [4].

## 5. $\theta$ -STABLE REPRESENTATIONS.

In this section we will give an algorithm to determine the set of dimension vectors of  $\theta$ -stable representations of a quiver  $Q$ . In the following sections we will give an application of this general result.

An elegant, though inefficient, algorithm to determine the dimension vectors of  $\theta$ -(semi)stable representations is based on the general quiver representation results of A. Schofield [16]. If  $\alpha$  is the dimension vector of a  $\theta$ -(semi)stable representation, then there is a Zariski open subset of  $\theta$ -(semi)stable representations in  $\text{rep}_\alpha Q$ .

For dimension vectors  $\alpha$  and  $\beta$  we denote  $\beta \curvearrowright \alpha$  if there is an open subset of  $\text{rep}_\alpha Q$  of representations containing a subrepresentation of dimension vector  $\beta$ . Assume by induction we know for all  $\gamma < \alpha$  the set of dimension vectors  $\delta$  such that  $\delta \curvearrowright \alpha$ , then Schofield shows that

$$\beta \curvearrowright \alpha \quad \text{iff} \quad 0 = \max_{\gamma \curvearrowright \beta} -\chi_Q(\gamma, \alpha - \beta)$$

For given  $\alpha$  let  $S_\alpha$  be the finite set of dimension vectors  $\beta$  such that  $\beta \curvearrowright \alpha$ , then

- $\alpha$  is the dimension vector of a  $\theta$ -semistable representation of  $Q$  if and only if  $\theta(\beta) \geq 0$  for all  $\beta \in S_\alpha$ .
- $\alpha$  is the dimension vector of a  $\theta$ -stable representation of  $Q$  if and only if  $\theta(\beta) > 0$  for all  $\beta \in S_\alpha$ .

In many important applications it is easy to prove that certain low-dimensional dimension vectors have  $\theta$ -stable representations. The following result can then be used to construct all dimension vectors of  $\theta$ -stables lying in the positive integral span of these vectors.

**Theorem 5.1.** *Let  $\beta_1, \dots, \beta_l$  be dimension vectors of  $\theta$ -stable representations of  $Q$  and assume there are integers  $m_1, \dots, m_l \geq 0$  such that*

$$\alpha = m_1\beta_1 + \dots + m_l\beta_l$$

*Then,  $\alpha$  is the dimension vector of a  $\theta$ -stable representation of  $Q$  if and only if  $\alpha' = (m_1, \dots, m_l)$  is the dimension vector of a simple representation of the quiver  $Q'$  on  $l$  vertices  $w_1, \dots, w_l$  such that there are exactly*

$$\delta_{ij} - \chi_Q(\beta_i, \beta_j)$$

*arrows from  $w_i$  to  $w_j$ .*

*Proof.* Let  $W_i$  be a  $\theta$ -stable representation of  $Q$  of dimension vector  $\beta_i$  and consider the  $\alpha$ -dimensional representation

$$V = W_1^{\oplus m_1} \oplus \dots \oplus W_l^{\oplus m_l}$$



It is clear from the definition that  $(Q', \alpha')$  is the local quiver setting corresponding to  $V$ .

As before, there is a semi-invariant  $d_\sigma$  such that  $V \in X_\sigma(\alpha) = \text{rep}_\alpha \mathbb{C}Q_\sigma$  and  $\mathbb{C}Q_\sigma \otimes V$  is a semi-simple representation of the universal localization  $\mathbb{C}Q_\sigma$ .

If there are  $\theta$ -stable representations of dimension  $\alpha$ , then there is an open subset of  $X_\sigma(\alpha)$  consisting of  $\theta$ -stable representations. But we have seen that they become simple representations of  $\mathbb{C}Q_\sigma$ . This means that every Zariski neighborhood of  $V \in \text{rep}_\alpha \mathbb{C}Q_\sigma$  contains simple  $\alpha$ -dimensional representations. By the étale local isomorphism there are  $\alpha'$ -dimensional simple representations of the quiver  $Q'$ .

Conversely, as any Zariski neighborhood of the zero representation in  $\text{rep}_{\alpha'} Q'$  contains simple representations, then so does any neighborhood of  $V \in \text{rep}_\alpha \mathbb{C}Q_\sigma = X_\sigma(\alpha)$ . We have seen before that  $X_\sigma(\alpha)$  consists of  $\theta$ -semistable representations and that the  $\theta$ -stables correspond to the simple representations of  $\mathbb{C}Q_\sigma$ , whence  $Q$  has  $\theta$ -stable representations of dimension vector  $\alpha$ .  $\square$

We recall from [13] the classification of dimension vectors of simple representations of quivers.  $\alpha' = (m_1, \dots, m_l)$  is the dimension vector of a simple representation of  $Q'$  if and only if the support of  $\alpha'$  is a strongly connected quiver (that is, every pair of vertices belongs to an oriented cycle) and we are in one of the following two cases

1. Either  $\text{supp}(\alpha') = \tilde{A}_n$  the extended Dynkin diagram with cyclic orientation for some  $n$  and  $\alpha' \mid \text{supp}(\alpha') = (1, \dots, 1)$ .
2. Or  $\text{supp}(\alpha') \neq \tilde{A}_n$  and we have the following inequalities

$$\chi_{Q'}(\epsilon_i, \alpha') \leq 0 \quad \text{and} \quad \chi_{Q'}(\alpha', \epsilon_i) \leq 0$$

for all vertices  $w_i \in \text{supp}(\alpha')$  where  $\epsilon_i = (\delta_{1i}, \dots, \delta_{li})$ .

The above theorem asserts that the set of  $\theta$ -stable dimension vectors can be described by a set of inequalities. For more results along similar lines we refer the reader to the recent preprint [7] of H. Derksen and J. Weyman.

In the following section we will apply these results to the classification of dimension vectors corresponding to simple representations of the free product groups  $\mathbb{Z}_p * \mathbb{Z}_q$ .

## 6. TORUS KNOT GROUPS.

In this section we translate the problem of describing all irreducible representations of a torus knot group to a quiver setting. Our method is based on earlier work of B. Westbury [19]. In the next section we will apply the foregoing results to determine the relevant dimension vectors.

Consider a solid cylinder  $C$  with  $q$  line segments on its curved face, equally spaced and parallel to the axis. If the ends of  $C$  are identified with a twist of  $2\pi \frac{p}{q}$  where  $p$  is an integer relatively prime to  $q$ , we obtain a single curve  $K_{p,q}$  on the surface of a solid torus  $T$ . If we assume that  $T$  lies in  $\mathbb{R}^3$  in the standard way, the curve  $K_{p,q}$  is called the  $(p, q)$  torus knot.

The fundamental group of the complement  $\mathbb{R}^3 - K_{p,q}$  is called the  $(p, q)$ -torus knot group  $G_{p,q}$  which has a presentation

$$G_{p,q} = \pi_1(\mathbb{R}^3 - K_{p,q}) \simeq \langle a, b \mid a^p = b^q \rangle$$

An important special case is  $(p, q) = (2, 3)$  in which case we obtain the three string braid group,  $G_{2,3} \simeq B_3$ .

Recall that the center of  $G_{p,q}$  is generated by  $a^p$  and that the quotient group is the free product of cyclic groups of order  $p$  and  $q$

$$\overline{G_{p,q}} = \frac{G_{p,q}}{\langle a^p \rangle} \simeq \mathbb{Z}_p * \mathbb{Z}_q$$

As the center acts by scalar multiplication on any irreducible representation, the representation theory of  $G_{p,q}$  essentially reduces to that of  $\mathbb{Z}_p * \mathbb{Z}_q$ . Observe that in the special case  $(p, q) = (2, 3)$  considered above, the quotient group is the modular group  $PSL_2(\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$ .

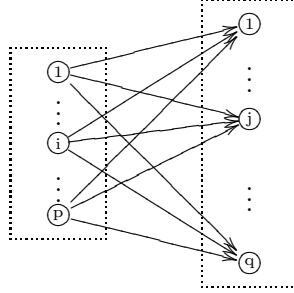
Let  $V$  be an  $n$ -dimensional representation of  $\mathbb{Z}_p * \mathbb{Z}_q$ , then the restriction of  $V$  to the cyclic subgroups  $\mathbb{Z}_p$  and  $\mathbb{Z}_q$  decomposes into eigenspaces

$$\begin{cases} V \downarrow_{\mathbb{Z}_p} & \simeq S_1^{\oplus a_1} \oplus S_\zeta^{\oplus a_2} \oplus \dots \oplus S_{\zeta^{p-1}}^{\oplus a_p} \\ V \downarrow_{\mathbb{Z}_q} & \simeq T_1^{\oplus b_1} \oplus T_\xi^{\oplus b_2} \oplus \dots \oplus T_{\xi^{q-1}}^{\oplus b_q} \end{cases}$$

where  $\zeta$  (resp.  $\xi$ ) is a primitive  $q$ -th (resp.  $p$ -th) root of unity and where  $S_{\zeta^i}$  (resp.  $T_{\xi^i}$ ) is the one-dimensional space  $\mathbb{C}v$  with action  $a.v = \zeta^i v$  (resp.  $b.v = \xi^i v$ ). Using these decompositions we define linear maps  $\phi_{ij}$  as follows

$$\begin{array}{ccc} S_{\zeta^{i-1}}^{\oplus a_i} & \xrightarrow{\phi_{ij}} & T_{\xi^{j-1}}^{\oplus b_j} \\ \downarrow & & \uparrow \\ S_1^{\oplus a_1} \oplus S_\zeta^{\oplus a_2} \oplus \dots \oplus S_{\zeta^{p-1}}^{\oplus a_p} & = & V = T_1^{\oplus b_1} \oplus T_\xi^{\oplus b_2} \oplus \dots \oplus T_{\xi^{q-1}}^{\oplus b_q} \end{array}$$

This means that we can associate to an  $n$ -dimensional representation  $V$  of  $\mathbb{Z}_p * \mathbb{Z}_q$  a representation of the full bipartite quiver on  $p + q$  vertices



where we put at the left  $i$ -th vertex the space  $S_{\zeta^{i-1}}^{\oplus a_i}$ , on the right  $j$ -th vertex the space  $T_{\xi^{j-1}}^{\oplus b_j}$  and the morphism connecting the  $i$ -th left vertex to the right  $j$ -vertex is the map  $\phi_{ij}$ . That is, to  $V$  we associate a representation  $V_Q$  of dimension vector  $\alpha = (a_1, \dots, a_p; b_1, \dots, b_q)$  and of course we have that  $a_1 + \dots + a_p = n = b_1 + \dots + b_q$ .

If  $V$  and  $W$  are isomorphic as  $\mathbb{Z}_p * \mathbb{Z}_q$  representation, they have isomorphic weight space decompositions for the restrictions to  $\mathbb{Z}_p$  and  $\mathbb{Z}_q$  and fixing bases in these weight spaces gives isomorphic quiver representations  $V_Q \simeq W_Q$ . Further, observe that if  $V$  is a representation of  $\mathbb{Z}_p * \mathbb{Z}_q$  then the matrix

$$m(V_Q) = \begin{bmatrix} \phi_{11}(V_Q) & \dots & \phi_{p1}(V_Q) \\ \vdots & & \vdots \\ \phi_{1q}(V_Q) & \dots & \phi_{pq}(V_Q) \end{bmatrix}$$

is invertible and the determinant  $d = \det(m)$  is a semi-invariant for  $\text{rep}_\alpha Q$  of weight  $\theta = (\underbrace{-1, \dots, -1}_p; \underbrace{1, \dots, 1}_q)$ . That is, all representations of the form  $V_Q$  are

$\theta$ -semistable representations of  $Q$ .

**Lemma 6.1.** *If  $V$  is an irreducible  $\mathbb{Z}_p * \mathbb{Z}_q$  representation, then  $V_Q$  is a  $\theta$ -stable quiver representation. Conversely, if  $W \in \text{rep}_\alpha Q$  is  $\theta$ -semistable such that  $d(V) \neq 0$ , then  $W = V_Q$  for a representation  $V$  of  $\mathbb{Z}_p * \mathbb{Z}_q$  and if  $W$  is  $\theta$ -stable then  $V$  is irreducible.*

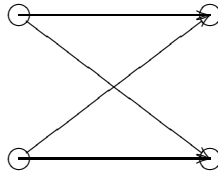
*Proof.* Let  $W \in \text{rep}_\alpha Q$  such that  $d(W) \neq 0$  and  $U \hookrightarrow W$  a subrepresentation of dimension vector  $\beta = (c_1, \dots, c_p; d_1, \dots, d_q)$ . Then,  $\theta(U) \geq 0$  for if  $\theta(W) < 0$  then  $\sum_i c_i > \sum_j d_j$  and the composed map  $U_L \xrightarrow{\phi(U)} U_R$  must have a kernel contradicting the fact that  $\phi(V)$  is an isomorphism. If  $\theta(U) = 0$ , then  $\sum_i c_i = \sum_j d_j = m$  and the restriction of  $\phi(V)$  to  $U$  gives an linear isomorphism  $U_L \simeq U_R$ . Finally, the decomposition  $U_L$  determines the action of  $\mathbb{Z}_p$  on an  $m$ -dimensional space  $U'$  and  $U_R$  the action of  $\mathbb{Z}_q$  on  $U'$ , that is,  $U \simeq U'_Q$  for a representation  $U'$  of  $\mathbb{Z}_p * \mathbb{Z}_q$ .  $\square$

That is, the variety of semi-simple  $n$ -dimensional representations of  $\mathbb{Z}_p * \mathbb{Z}_q$  decomposes into components

$$\text{iss}_n \mathbb{Z}_p * \mathbb{Z}_q = \bigsqcup_{\sum a_i = \sum b_j = n} \text{iss}_\alpha \mathbb{C}Q_\sigma \hookrightarrow \bigsqcup_{\sum a_i = \sum b_j = n} M_\alpha^{ss}(Q, \theta)$$

(where  $\sigma$  is the map  $\phi$  described before) which are affine open subsets of the projective moduli spaces of  $\theta$ -semistable representations of  $Q$ . In the next section we will determine the dimension vectors  $\alpha$  corresponding to irreducible representations.

As an example, consider the infinite dihedral group  $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ , then the corresponding quiver is  $Q = \tilde{A}_4$  with orientation



For  $\alpha = (1, 1; 1, 1)$  we have  $M_\alpha^{ss}(Q, \theta) \simeq \mathbb{P}^1$  and  $\text{iss}_\alpha D_\infty = \mathbb{A}^1$ , the one point missing is the representation  $V$  where all maps are the identity because in that case  $d(V) = 0$ .

## 7. DIMENSION VECTORS OF IRREDUCIBLE REPRESENTATIONS.

In this section we will characterize the dimension vectors  $\alpha$  such that  $\mathbb{Z}_p * \mathbb{Z}_q$  has irreducible representations  $V$  such that  $V_Q$  has dimension vector  $\alpha$ . In view of the above we have to determine the dimension vectors of  $\theta$ -stable representations of  $Q$ . There are some obvious 1-dimensional irreducible representations of  $\mathbb{Z}_p * \mathbb{Z}_q$

$$V_{ij} = \mathbb{C}v \quad \text{with } a.v = \zeta^{i-1}v \text{ and } b.v = \zeta^{j-1}v.$$

which have dimension vector  $\alpha_{ij} = (\delta_{1i}, \dots, \delta_{pi}; \delta_{1j}, \dots, \delta_{qj})$ .

We consider the quiver  $\Gamma$  on  $i, j$  vertices  $v_{ij}$  (corresponding to the  $\theta$ -stable representations  $V_{ij}$ ) such that the number of arrows from  $v_{ij}$  to  $v_{kl}$  is equal to

$$\delta_{ij,kl} - \chi_Q(\alpha_{ij}, \alpha_{kl})$$

Given the special form of the full bipartite quiver  $Q$  it is easy to verify that

$$\# \{ a \in \Gamma_a \mid \textcircled{v_{kl}} \xleftarrow{a} \textcircled{v_{ij}} \} = \begin{cases} 1 & \text{if } i \neq k \text{ and } j \neq l \\ 0 & \text{otherwise.} \end{cases}$$

For example, in the modular case  $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$  the quiver  $\Gamma$  has the form of a large cycle with subsequent vertices  $v_{21}, v_{12}, v_{23}, v_{11}, v_{22}, v_{13}$  and only one arrow in each direction between two consecutive vertices. We consider the direct sum of  $\theta$ -stable representations

$$V = \bigoplus_{i,j} V_{ij}^{\oplus m_{ij}}$$

that is, the dimension vector of  $V$  is  $\alpha = \sum_{i,j} m_{ij} \alpha_{ij}$ . In view of theorem 5.1 to verify that there are  $\theta$ -stable representations of dimension vector  $\alpha$  is equivalent to verifying that  $\gamma = (m_{11}, \dots, m_{pq})$  is the dimension vector of a simple representation of  $\Gamma$ .

**Theorem 7.1.**  $\alpha = (a_1, \dots, a_p; b_1, \dots, b_q)$  with  $\sum_i a_i = n = \sum_j b_j$  is the dimension vector of a  $\theta$ -stable representation of  $Q$  if and only if

$$n = 1 \quad \text{or} \quad a_i + b_j \leq n$$

for all  $1 \leq i \leq p$  and  $1 \leq j \leq q$ .

*Proof.* Assume that  $\alpha$  is the dimension vector of a  $\theta$ -stable representation of  $Q$ , then the set of irreducible representations of  $\mathbb{Z}_p * \mathbb{Z}_q$  is an open subset of  $M_\alpha^{ss}(Q, \theta)$ . Assume that  $a_i + b_j > n$  and consider the vertex spaces (the eigenspaces)  $S_i^{\oplus a_i}$  and  $T_j^{\oplus b_j}$  as subspaces of the  $n$ -dimensional representation  $V' = \mathbb{C}^n$  (via the identification given by the matrix  $m$ ). Then  $W = S_i^{\oplus a_i} \cap T_j^{\oplus b_j} \neq 0$  is a subrepresentation of  $V$  of dimension vector  $a\alpha_{ij}$  for some  $a \in \mathbb{N}_+$ . But  $\theta(W) = 0$  so  $V'$  cannot be  $\theta$ -stable, a contradiction.

Conversely, assume the numerical condition is satisfied and consider the direct sum of  $\theta$ -stable representations

$$V = \bigoplus_{i,j} V_{ij}^{\oplus m_{ij}}$$

We note that

$$a_i = \sum_{j=1}^q m_{ij} \quad \text{and} \quad b_l = \sum_{i=1}^p m_{il}.$$

Let  $I_p \in M_p(\mathbb{C})$  be the  $p \times p$  identity matrix and let  $A_p \in M_p(\mathbb{C})$  be the  $p \times p$  matrix of the form

$$A_p = \begin{pmatrix} 0 & -1 & \dots & -1 & -1 \\ -1 & 0 & \dots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & 0 & -1 \\ -1 & -1 & \dots & -1 & 0 \end{pmatrix}.$$

Then the Euler form of the local quiver  $\Gamma$  is the symmetric matrix

$$\chi_\Gamma = \begin{pmatrix} I_p & A_p & \dots & A_p & A_p \\ A_p & I_p & \dots & A_p & A_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_p & A_p & \dots & I_p & A_p \\ A_p & A_p & \dots & A_p & I_p \end{pmatrix} \in M_q(M_p(\mathbb{C})).$$

When  $n = 1$ , we have that  $V = V_{11}$  is obviously a  $\theta$ -stable representation. When  $n = 2$ , we notice that  $\gamma$  is the dimension vector of a simple representation if and only if

$$\gamma|_{\text{supp}(\gamma)} = (1, 1; 1, 1) \quad \text{because} \quad \text{supp}(\gamma) = \tilde{A}_2$$

Now, consider  $n \geq 3$  and consider the dimension vector  $\gamma = (m_{11}, \dots, m_{pq})$ . We have to verify that  $\gamma$  is the dimension vector of a simple representation of  $\Gamma$  which, by symmetry of  $\chi_\Gamma$ , amounts to checking that

$$\chi_\Gamma(\gamma, \epsilon_{kl}) = \chi_\Gamma(\epsilon_{kl}, \gamma) \leq 0$$

for all  $1 \leq k \leq p$  and  $1 \leq l \leq q$  where  $\epsilon_{kl} = (\delta_{ij,kl})$  are the standard base vectors. Computing the left hand term this is equivalent to

$$m_{kl} + \sum_{k \neq i=1}^p \sum_{l \neq j=1}^q -m_{ij} \leq 0.$$

or

$$m_{kl} + \sum_{k \neq i=1}^p m_{il} \leq \sum_{k \neq i=1}^p \sum_{l \neq j=1}^q m_{ij} + \sum_{k \neq i=1}^p m_{il}$$

Resubstituting the values of  $a_i$  and  $b_l$  in this expression we see that this is equivalent to

$$b_l \leq \sum_{k \neq i=1}^p a_i = n - a_k$$

Therefore, the condition is satisfied if for all  $1 \leq k \leq p$  and  $1 \leq l \leq q$  we have

$$a_k + b_l \leq n.$$

□

In the special case of  $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ , our condition on the dimension vector  $\alpha = (a_1, a_2; b_1, b_2, b_3)$  is equivalent to

$$a_i + b_j \leq n = a_1 + a_2 \quad \text{whence} \quad b_j \leq a_i$$

for all  $1 \leq i \leq 2$  and  $1 \leq j \leq 3$  which was the criterium found by B. Westbury in [19].

Using the results of the present paper, a complete characterization is given in [1] of the dimension vectors  $\alpha$  for which the moduli space  $M_\alpha^{ss}(Q, \theta)$  is smooth.

## REFERENCES

- [1] J. Adriaenssens, R. Bocklandt and G. Van de Weyer, *Smooth character varieties for torus knot groups*, to appear
- [2] M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J.Alg. 11 (1969) 523-563
- [3] W. Barth, *Moduli of vector bundles on the projective plane*, Invent. math. 42 (1977) 63-91
- [4] R. Bocklandt, *Coregular symmetric quiver representations*, to appear
- [5] J. Cuntz and D. Quillen *Algebra extensions and nonsingularity* Journal AMS 8 (1995) 251-289
- [6] H. Derksen and J. Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, to appear in Journal AMS
- [7] H. Derksen and J. Weyman, *On the  $\sigma$ -stable decomposition of quiver representations*, to appear
- [8] M. Domokos and A. Zubkov, *Semi-invariants of quivers as determinants*, Transf. Groups (2000)
- [9] P. Gabriel, *Finite representation type is open*, in "Representations of algebras", Lect. notes Math. 488 (1974) 132-155
- [10] K. Hulek, *On the classification of stable rank  $r$  vector bundles on the projective plane*, in "Vector bundles and differential equations", Progress in Math. 7 (1980) 113-144
- [11] A. King *Moduli of representations of finite dimensional algebras* Quat. J. Math. Oxford 45 (1994) 515-530
- [12] H.P. Kraft and Ch. Riedtmann, *Geometry of representations of quivers*, Representations of Algebras, London MS Lecture Notes 116 (1986) 109-145
- [13] L. Le Bruyn and C. Procesi, *Semisimple representations of quivers* Trans. AMS 317 (1990) 585-598
- [14] D. Luna, *Slices etales* Bull.Soc.Math. France Mem 33 (1973) 81-105
- [15] A. Schofield, *Representations of rings over skew fields*, London MS Lecture Notes 92 (1985)
- [16] A. Schofield, *General representations of quivers*, Proc. LMS 65 (1992) 46-64
- [17] A. Schofield and M. Van den Bergh, *Semi-invariants of quivers for arbitrary dimension vectors* math.RA/9907174 (1999)
- [18] P. Slodowy, *Der Scheibensatz für Algebraische Transformationsgruppen*, DMV Seminar, vol 13 (1989) 89-114
- [19] B. W. Westbury, *On the character varieties of the modular group*, preprint, University of Nottingham 95.28 (1995)

UNIVERSITEIT ANTWERPEN (RUCA), B-2020 ANTWERP (BELGIUM)  
*E-mail address:* adri@ruca.ua.ac.be  
*URL:* <http://math.ruca.ua.ac.be/~adri/>

UNIVERSITEIT ANTWERPEN (UIA), B-2610 ANTWERP (BELGIUM)  
*E-mail address:* lebrun@uia.ua.ac.be  
*URL:* <http://win-www.uia.ac.be/u/lebrun/>