life after Kaplansky 10 ?

~~Dear colleagues~~,

25 years ago Irving Kaplansky conjectured that over an alg. closed field (say of char 0) there are only finitely many iso classes of Hopf algebras of any fixed dimension.

As you know this conjecture was disproved last november in an hilarious sequence of emails.

These examples are ~~seen~~ considered by some to be evidence that the classification problem of f.d. Hopf algebras is hopeless.

The aim of this talk is to outline three directions where i think ~~advance~~ it is possible and desirable to make progress.

The emphasis of this talk will be on questions and conjectures rather than on results. I will only include some easy proofs and examples to give a feeling for the techniques in these three areas.

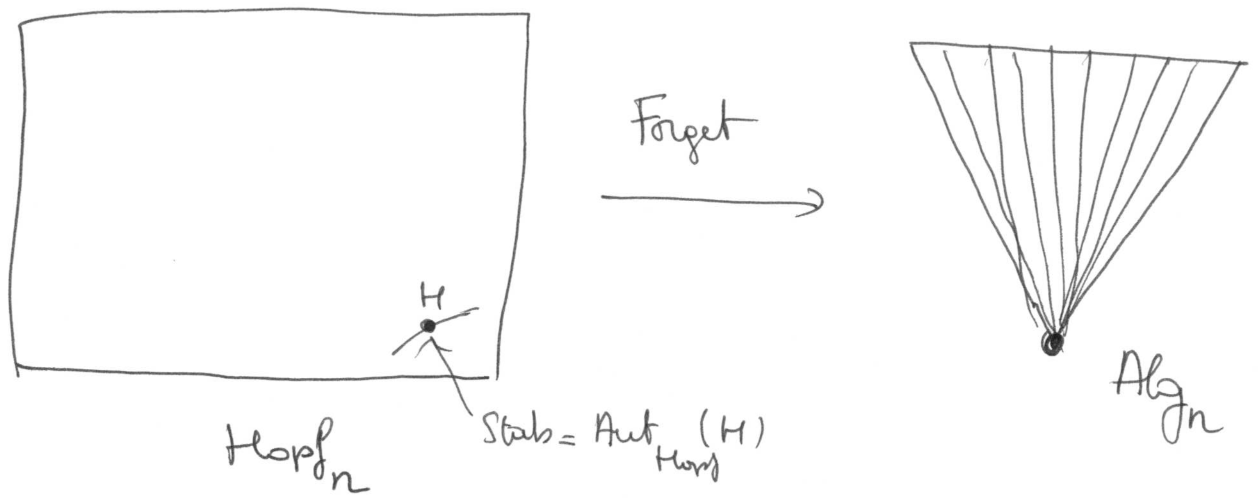
The first will be the geometric study of Hopf algebras and aim to construct more infinite families of Hopf algebras.

Then i want to make some advertisement for some recent papers of Etingof and Gelaki because they contain the first ideas of a coherent approach of the most important classification project: that of semisimple Hopf algebras.

And lastly i'll recall how one can use Galois descent to study classes of Hopf algebras over arbitrary fields once one has a classification in the algebraically closed case. These observations are some work in progress together with Stef Caenepeel and Sorin Dascalescu.

I: Geometry

Precisely as in the case of finite dml algebras, the existence of infinite families calls for the study of the variety of all n-dml Hopfalgebras, the points of which describe Hopf



structures on an n-dml vectorspace V. As the Hopf structure is determined by a finite number of linear maps satisfying certain polynomial equations it is trivial to see that $Hopf_n$ is an affine variety. Basechange in the underlying vectorspace V gives an action of the group GL_n on this variety and the orbits under this action correspond to isomorphism classes of Hopf algebras. Moreover, the stabilizer gp in a point is the group of Hopfalgebra automorphisms.

If we forget the costructure we have a morphism from this variety to the variety of n-dml algebras which has been studied for over a century. Geometrically, Alg_n is not very nice. It has an exponentially growing number of irreducible components and also the singularities are very bad.

Last year I classified together with Zinovy Reichstein all smooth closed subvarieties of Alg_n which are left invariant under the action of the basechange group.

For example we showed that if $n \geq 5$ there is precisely one smooth irreducible component. As a consequence we solved a question of Seshadri asking to classify semi-simple algebras A having a smooth orbit closure

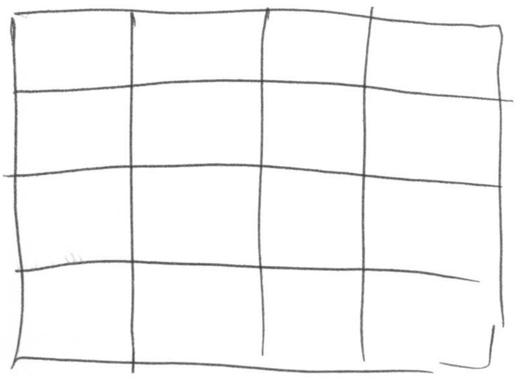
$$\begin{array}{l} \overline{\mathcal{O}_A} \text{ smooth} \\ A \text{ semi-simple} \end{array} \iff A = \begin{cases} \mathbb{C}, \quad \mathbb{C}\oplus\mathbb{C} \text{ or } \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C} \\ M_2(\mathbb{C}) \end{cases}$$

The reason for all this is that Alg_n has a unique closed orbit, so every algebra degenerates to the ~~one~~ radical square zero alg.

$$\mathbb{C}[x_1, \ldots, x_{n-1}]/(x_1, \ldots, x_{n-1})^2$$

and then one can compute the tangentspace to subvarieties in this point to derive non smoothness.

Returning to Hop_n, clearly it has also several irreducible components



$Hopf_n$

but as far as i know nothing is known how the # of components grows with n.

Moreover, there is <u>no</u> Hopf algebra to which every other degenerates giving some hope for a nicer geometrical structure.

As an indication of this, let us give the analogous result for semi-simple Hopfs (this observation was also known to Dragos Stefan and possibly others)

$$H \text{ s.s. Hopf} \Rightarrow \overline{\mathcal{O}_H} \text{ is smooth}$$

The proof is a one-liner. Consider a Hopf alg in the closure

Then because the $\mathcal{O}_{H'}$ must lie in the closure it must have strictly smaller dimension than the orbit of H, hence

$$\dim \mathrm{Aut}_{Hopf} H' > \dim \mathrm{Aut}_{Hopf} H = 0 \qquad (\text{Radford})$$

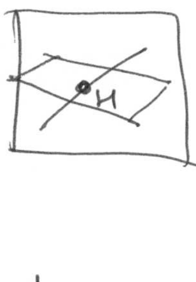
On the other hand $S^2 = id$ on $\mathcal{O}_H$ and this is a closed condition so also $\underline{S^2_{H'} = id \text{ but then } H' \text{ should be s.s.}}$ check contradicting the fact on Hopf autos. So $\overline{\mathcal{O}}_H = \mathcal{O}_H$ and hence smooth.

By an important result of Drapos, every s.s. Hopf determines an irreducible component of Hopf_n so there are at least many smooth irreducible components. This leads to a rather wicked conjecture

<u>CONJECTURE</u>: Hopf_n is smooth.

$\boxed{\text{eerst nog tegenvb}}$

Let me briefly indicate why a positive solution would be relevant to the classification problem. Assume have a component ~~smooth compon~~ and Hopf algebra s.t. $\mathcal{O}_H$ is closed and components is smooth in H.



Then we can ~~fo~~ compute the normalspace to the orbit. This normal space will be a subspace of $H^1_{bial}(H)$ invariant under the reductive gp $\mathrm{Aut}_{Hopf}(H)$. Then, one can show that there are many other closed orbits in this component parametrized by the quotient variety $N // \mathrm{Aut}_{Hopf}$. Moreover, one can also construct Hopf algebras degenerating to H by studying the nullcone of this situation. So, if one can compute

$H^1_{Incl}(H)$ of a closed orbit, then one will usually construct/discover infinite families of Hopf algebras (assuming the conjecture is true). (5)

How to find counterexamples to the conjecture? As in the case of algebraic varieties, the chances that something is smooth is connected to how many degenerations a component has. So we need components containing lots of non-closed orbits degenerating one to another. Surprisingly, it is not so easy to construct degenerations of Hopf algebras. (cf. case of s.s. Hopf). A test that an orbit is not closed and hence that there should be degenerations is given by the Aut_{Hopf} not reductive. However, most Hopf algebras we know have reductive automorphism grps (e.g. finite grps, or GL_m's or so).

Let me give an example of interesting degenerations which one ~~will~~ ~~be~~ studied together with Stef and Sorin

Example (Hopf-Clifford algebras)

$$H^n_m : \quad c^4 = 1 \qquad c \text{ gplike}$$

$$x_i c + c x_i = 0 \qquad 1 \leq i \leq n \qquad\qquad \Delta x_i = c \otimes x_i + x_i \otimes 1$$

$$\varepsilon(x_i) = 0 \quad S(x_i) = -x_i c^{-1}$$

$$x_i x_j + x_j x_i = 0 \qquad i \neq j$$

$$x_i^2 = c^2 - 1 \quad \text{for} \quad 0 \leq i \leq m$$

$$x_i^2 = 0 \quad \text{for} \quad m < i \leq n$$

is pointed Hopf alg of dim 2^{n+2}

One can show

$$\mathrm{Aut}_{\mathrm{Hopf}}(H^n_m) = \left[\begin{array}{c|c} O_m & * \\ \hline 0 & GL_{n-m} \end{array}\right] \hookrightarrow GL_n$$

so is non-reductive for all $0 < m < n$

and we have degenerations

$$\begin{array}{ll} H^n_n & \leftarrow \text{auto } O_n \\ | & \\ \left.\begin{array}{l} H^n_{n-1} \\ | \\ H^n_{n-2} \\ \vdots \end{array}\right\} & \text{non reductive autos.} \\ | & \\ H^n_0 & \leftarrow \text{auto } GL_n \end{array}$$

So there is no bound on the number of successive degenerations of Hopf algebras.

<u>Question</u>: Construct more such families.

The moral of this part is: if we have a smooth component, then find closed orbit, compute bialgebra cohomology and reduce the classification problem to invariant theory.

But: classification of closed orbits has as a subproblem the classification of s.s. Hopfs. Fortunately very recently there is a coherent approach to this problem.

II HopfBots

If you surf on the net you may have encountered pages with JavaBots that is ~~pages~~ collections of colourfull dots moving about on the screen and when they meet they multiply, change color or eat each other according to some preset interaction rules.

HopfBots (or as some people prefer 2-dml topological quantum field theories) are similar. We have a collection of particles living on a surface and interacting in such a way that when we compute the expectation value of a certain process it only depends on the topology of ~~the~~ its world-history, that is the link in 3 space (2dml manifspace x time).

~~Only 2 weeks ago i came to know reading Susan Montgomery's survey paper~~ a strikingly beautifull result due to

<u>Etingof - Gelaki</u>: H s.s Hopf $\Rightarrow$ Rep $D(H)$ is HopfBot, i.e. is a modular semi-simple ribbon category

Let me briefly explain these terms starting with the one best known to you

- <u>ribbon category</u>: we all know that if you have a quasi-triangular Hopf algebra such as the Drinfeld-double the modules form a braided category. To turn it into a ribbon category one usually has to take a quadratic extension but in the semi-simple case (as the square of the antipode is the identity) it is automatically ribbon.

<u>semi-simple</u>: is what you expect. There are a finite # of simple objects

$$\underset{\text{triv}}{\mathbb{1}} = V_1, \ldots, V_t$$

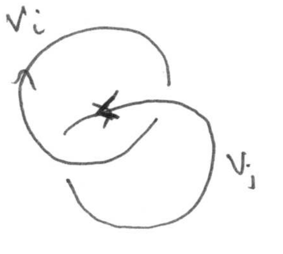
and every other object is a certain direct sum of these. In particular, we have an involution

$$V_i^* = V_{i^*}$$

and multiplicity rules

$$V_i \otimes V_j = \bigoplus_k V_k^{\oplus N_{ij}^k}$$

so far nothing quite unexpected. The surprising thing is that it is <u>modular</u>. This is defined by the fact that if we take the $t \times t$-matrix consisting of expectation values (certain traces)

$$\Big\langle$$  $$\Big\rangle = (S_{ij})_{i,j}$$

the the matrix must be invertible.

innocent as it may look this leads to all sorts of powerfull equations and restrictions because it determines a h-dim. representation of the modular gp

$$SL_2(\mathbb{Z}) = \left\langle \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

given by the matrices

$$\underbrace{s = \left(\frac{S_{ij}}{n}\right)_{i,j}}_{\text{symmetric matrix}} \qquad \underbrace{t = \left(\frac{\theta_i \delta_{ij}}{\eta}\right)}_{\text{diagonal matrix}}$$

where  $= \theta_i$ 

In fact one can reconstruct Rep $D(H)$ from this representation.

- simples $\longleftrightarrow$ colons / rows of s
- $\dim V_i = n\, s_{1i}$
- $s^2 = (\delta_{ii^*})$ we get a permutation matrix giving us the duality

$$V_i \longleftrightarrow V_i^* = V_{i^*}$$

and even the multiplicities we can recover from s

- $$N_{ij}^k = \sum_r \frac{s_{ir}\, s_{jr}\, s_{k^*r}}{s_{1r}}$$

Moreover, this matrix is unitary in a certain sense giving analogon to the orthogonality relations of characters

$$(s_{ji})\cdot(s_{ij^*}) = I_k$$

and it also determines the Verlinde algebra.

$$\mathbb{V} = \mathbb{C}b_1 + \dots + \mathbb{C}b_k \qquad \text{with} \quad b_i \cdot b_j = \sum_k N_{ij}^k\, b_k$$

$$\cong \mathbb{C} \times \dots \times \mathbb{C} \qquad \text{is comm semisimple alg.}$$

and we can give a basis of idempotents

$$b^j = \sum s_{jk^*}\, b_k$$

and computing these we find

$$b_i \cdot b^j = \frac{s_{ji}}{s_{j1}}\, b^j$$

All of these things are explained for arb. modular categories in the book of Turaev.

A surprising immediate consequence of this theory is

Thm (Etingof - Gelaki) $D(H)$ satisfies Kapl 6 in particular even stronger: $\dim V_j \mid n$

Pf From the unitary condition we have

$$\sum_i s_{ji}\, s_{ij^*} = 1 \qquad \text{and} \qquad s_{j1} = s_{1j^*} = \frac{\dim V_j}{n}$$

so
$$\sum_i \left(\frac{s_{ji}}{s_{j1}}\right)\left(\frac{s_{ij^*}}{s_{1j^*}}\right) = \frac{n^2}{(\dim V_j)^2}$$

Now look at the action of multiplication of b_i on the two descriptions of the Verlinde algebra. Then

$$\frac{s_{ji}}{s_{j1}} \quad \text{is eigenvalue of the matrix} \quad \left(N_{ij}^{k}\right)_{kj} \in M_k(\mathbb{Z})$$

so is an algebraic integer. Similar for $\dfrac{s_{ij^*}}{s_{1j^*}}$. Hence
$\dfrac{n^2}{(\dim V_j)^2}$ must be an algebraic integer in $\mathbb{Q}$ so $\in \mathbb{Z}$ ■

So we have a good knowledge about $D(H)$ of H ss. but how to get to H itself? That is another idea of Etingof and Gelaki. They show that to any sub Hopf Bot so to every subfamily of simples closed under $\otimes$ there exists a Hopf quotient

$$\begin{tikzcd}
H \arrow[dr, hook] \arrow[drr, bend left] & & \\
& D(H) \arrow[r, two heads] & A \\
H^* \arrow[ur, hook] \arrow[urr, bend right] & &
\end{tikzcd}$$

quasi Hopf

with corresponding Hopf Bot this subfamily

and H^* are sub Hopf alg of $D(H)$ and we can consider their images in A. In good situations H, H^* will be sub Hopf of A and

the dimension of A will be small compared to that of $D(H)$. (11)
Then one can use restriction / induction arguments to get to the structure of H. This is the method they use to prove that ss Hopfs of dimension $p.q$ are trivial.

III Galois descent

Will there be life after Kaplansky 6? Is everything solved once we have a classification for algebraically closed fields.
The corresponding situation in noncommutative algebra would be that everything is known once we proved that simple algebras over alg. closed fields are matrix rings. Of course, in algebra the it has been shown that the study of simple algebras over arbitrary fields provide interesting invariants for these fields.
Similarly, a study of Hopf algebras over arbitrary fields may lead to new invariants. Surprisingly few papers i know addressed the question of Galois descent for Hopf algebras.
The only papers known to me are an old paper by Radford – Taft and Wilson and some slightly weird looking paper by Pareigis.
So what is the problem: consider a Hopf algebra over k

$$\begin{array}{ccc} h & \text{———} & h \otimes \bar{k} \cong H \\ | & & | \\ k & \text{———} & \bar{k} \end{array}$$

and consider it over the algebraic closure. We want to classify all k-Hopf algebras which become H over the algebraic closure. Since the work of Serre we know that the answer is given by Galois descent. We have the absolute Galois grp

$$\mathrm{Gal}(\bar{k}/k)$$

which usually is a continuous profinite gp. There is a natural action of this group on $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ by writing $H = h \otimes \bar{k}$

and letting $Gal(\bar{k}/k)$ act on the second factor. This makes $Aut_{Hopf}(H)$ a ~~Continuous~~ continuous $Gal(\bar{k}/k)$-module via conjugation.

that is $\quad \sigma . f = (1 \otimes \sigma) \circ f \circ (1 \otimes \sigma^{-1}) \quad$ for all $\sigma \in Gal(\bar{k}/k)$

A special case of Serre's general theory is

$$\left.\begin{array}{l}\text{isoclasses of}\\ k\text{-Hopf } h\\ \text{with}\\ h \otimes \bar{k} \cong H\end{array}\right\} \longleftrightarrow \begin{array}{c}\text{elements of (pointed)}\\ \text{set}\\ H^1(Gal, Aut_{Hopf} H)\end{array} = \frac{\left\{ \begin{array}{l} c : Gal \to Aut \\ c(gg') = c(g)\, g.c(g') \end{array} \right\}}{\left\{ \begin{array}{l} c : Gal \to Aut \quad \exists a \text{ s.t.} \\ c(g) = a^{-1} g.a \text{ for} \end{array} \right\}}$$

Let us compute an easy case: we want to classify all twisted k-forms of a group algebra.

$$Aut_{Hopf}\ \bar{k} G = Aut_{gp}\ G$$

Moreover, there is at least one twisted form k-Hopf algebra: kG and we see that the Galois gp acts trivially on the automorphism gp. That is

$$\begin{array}{c}\text{isoclasses of}\\ k\text{-forms of}\\ \bar{k} G\end{array} \xleftrightarrow{1-1} Hom_{gp}(Gal(\bar{k}/k), Aut_{gp} G)$$

$$\Big\updownarrow 1\text{-}1 \qquad \underline{\text{clear}}$$

Finite Galois ext $\ell \supset k$ with $Gal(\ell/k) \hookrightarrow Aut_{gp} G$

the proof is obvious

$$\mathrm{Ker}\,\varphi \longrightarrow \mathrm{Gal}(\bar{k}/k) \xrightarrow{\varphi} \mathrm{Im}\,\varphi \hookrightarrow \mathrm{Aut}_{gp}\, G$$

Then

$$l = \bar{k}^{\mathrm{Ker}\,\varphi} \text{ is Galois with fin. Gal gp } \mathrm{Im}\,\varphi$$

$\mathrm{Im}\,\varphi$ acts as Hopf automorphisms on lG and the invar

$$(lG)^{\mathrm{Im}\,\varphi} \text{ is a } k\text{-Hopf algebra which is}$$

a twisted form of $\bar{k}G$.

as there is only one Hopf of dim p over $\bar{k}$ namely $\bar{k}C_p$ we see that

In particular;

$$\text{k-Hopf algs of dim } p \longleftrightarrow \text{cyclic Galois extn of } k \text{ of degree } \mid (p-1)$$

and for small values of p one can be very specific. Stef calculated an afternoon to show that the twisted forms for $p = 3$ are of the form

$$H = \frac{k[w]}{(w^3+3aw)} \longleftrightarrow l = \frac{k[x]}{(x^2-a)}$$

$$\Delta w = 1 \otimes w + w \otimes 1 + \frac{1}{2a}(w \otimes w^2 + w^2 \otimes w)$$

$$\varepsilon w = 0$$

$$S w = -w$$

If we want to obtain similar results for arbitrary semisimple Hopf algebras we need to answer the following two important questions :

H s.s. over $\bar{k}$

$\boxed{Q1}$ What is the minimal field of defn of H. That is what is $k \subset \bar{k}$ with s-Hopf h s.t. $h \otimes \bar{k} = H$.

Conjecture: $k = \mathbb{Q}(\xi)$? cyclotomic field

$= \mathbb{Q}$?

$\boxed{Q2}$ For H having a k-form. What finite $Gal(\bar{k}/k)$-modules can occur? So, in general the Gal structure will not be trivial but i do not have even a plausible conjecture which sets can occur.

I think it will be interesting to compute these gps and cohomology sets for other classes of s.s. Hopf algebras and relate these ~~to~~ arithmetic information of the field.

One may even use this to attack Kaplansky 6. For example consider the n-th Hopf-Brauer gp of k

$$HB_n(k) = \{[A] : A \text{ c.s.}/k \text{ component of } H \text{ of dim } n\} \hookrightarrow Br(k)$$

H s.s. Hopf / k

Then it is trivial to see that

$$\text{Kapl } 6 \Rightarrow HB_n(k) \text{ has index } n$$

So if one can find a cohomological description of these gps (and for these one needs to know splitting behaviour, e.g. that every s.s. Hopf is split by a cyclotomic). And several approaches to Brauer gps having natural extentions to these cases

Of course one does not have to restrict to s.s. Hopf to compute twisted forms. For example, as an application of Hilbert 90 in Galois theory one finds that there is at most one twisted form of Taft's Hopf algebra. And for the pointed Hopf-Clifford algebras i introduced before one computes via some exact sequences that

$$H^1(\mathrm{Gal}, \mathrm{Aut}_{\mathrm{Hopf}}\, H^n_m) = H^1(\mathrm{Gal}, O_m(\bar{k}))$$

$$\Big\updownarrow \text{ Serre}$$

eq. classes of m-ary quadratic forms over k.

In this case the twisted forms are easy to write down

$$\langle a_1, \to a_m \rangle \longleftrightarrow H^n_m(\underline{a}) \quad \text{as before with}$$

$$a_i \in k^* / (k^*)^2 \qquad x_i^2 = a_i (c^2 - 1)$$

for $1 \le i \le m$