Bonn talk

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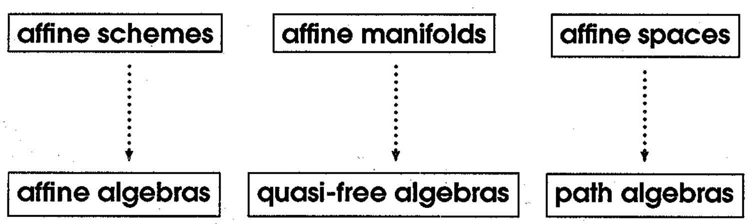
These are notes of a talk given at the 'Workshop in noncommutative geometry' held at the Max-Planck Institute in Bonn from June 25th till July 3rd 1999. In contains a proposal to construct global noncommutative manifolds using the setting of [3] which in turn is based on ideas of Kapranov [1] and of Kontsevich and Rosenberg [2].

1 noncommutative geometry

In developing **a** noncommutative (algebraic) geometry one should first fix ones points of interest. It is conceivable that if you are interested in graded quadratic algebras you will end up with another 'geometry' than someone interested in finite dimensional algebras or in enveloping algebras. So let me get my priorities straight. I am interested in a geometry in which the manifolds are pieced together locally from quasi free algebras and are equipped with generalizations of the formal structures Kapranov puts on commutative manifolds. Let me briefly recall these notions.

1.1 Quasi-free algebras

In the naive setup where we declare noncommutative affine schemes to be things associated to affine associative $\mathbb{C}$-algebras (always with a unit), there is some agreement based on work of Cuntz and Quillen that the affine smooth varieties ought to correspond to *quasi-free algebras*.



Recall that an algebra is quasi-free if it has the lifting property for algebra morphisms modulo nilpotent ideals. Just as a commutative n-dimensional manifold is locally affine n-space we would like to have a manageable analytic local description of these quasi-free algebras. I will argue that these local forms are given by the subclass of *path algebras of quivers*.

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The paradigmatic example is the noncommutative space X of a quasi-free algebra A. In that case we have

$$\begin{array}{ccc} & & \boxed{\begin{array}{c} \mathcal{O}^{NC}_{\mathsf{rep}_n A} \\ \vdots \\ \boxed{\mathcal{O}_{\mathsf{rep}_n A}} \end{array}} \\ & & \vdots \\ X_n & = & \mathsf{rep}_n A \end{array}$$

where $\mathsf{rep}_n A$ is the affine scheme of n-dimensional representations of A which is verified to be smooth. As such we can equip $\mathsf{rep}_n A$ with the canonical Kapranov formal structure. The connecting morphisms $c_{\underline{n}}$ are given by taking direct sums of representations

$$\mathsf{rep}_{n_1} A \times \ldots \times \mathsf{rep}_{n_k} A \xrightarrow{c_{\underline{n}}} \mathsf{rep}_{\sum n_i} A.$$

which is clearly compatible for sequence refinements and compatibility of the formal structures will be given below.

2 noncommutative geometry@$_{\mathbf{n}}$

Before formalizing what level n geometric objects are we will indicate to what extend path algebras of quivers can be seen as affine spaces describing the local structure of quasi-free algebras, we will describe how to calculate with the formal structures in the case of path algebras and extend these formal structures to arbitrary algebras.

2.1 The baby version

Let A be an affine $\mathbb{C}$-algebra, then $\mathsf{rep}_n A$ is the affine scheme representing the functor

$$\textbf{commalg} \longrightarrow \textbf{sets} \qquad B \mapsto Hom_{\mathsf{alg}}(A, M_n(B)).$$

There is a natural action of GL_n on this scheme by conjugation in the target space. In fact we have the following situation (most results here are due to C. Procesi)

$$\begin{array}{ccc} \boxed{M_n(\mathcal{O}_{\mathsf{rep}_n A})} & & \boxed{\mathcal{O}_{A@_n}} \\ \vdots & & \vdots \\ \mathsf{rep}_n A & \twoheadrightarrow & \mathsf{fac}_n A \end{array}$$

Here, $\mathsf{fac}_n A$ is the GL_n-quotient scheme, the points of which correspond to isomorphism classes of *semi-simple* n-dimensional representations. The universal map

$$A \xrightarrow{j_A} M_n(\mathbb{C}[\mathsf{rep}_n A])$$

If A is quasi-free, then so is $\sqrt[n]{A}$ as $M_n(\sqrt[n]{A}) = A * M_n(\mathbb{C})$ and by the essential uniqueness result of formal structures on affine smooth varieties we have in this case

$$\mathcal{O}^{NC}_{\mathsf{rep}_n A} = \mathcal{O}^{\mu}_{\sqrt[n]{A}}$$

and therefore can extend the formal structure in a functorial way to all affine representation schemes $\mathsf{rep}_n A$.
For arbitrary A we have by the universal property canonical algebra morphisms

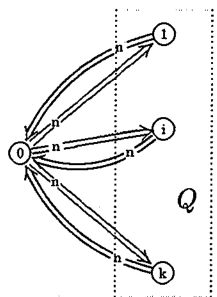
$$\sqrt[\Pi n_i]{A} \longrightarrow \sqrt[n_1]{\sqrt[n_2]{\ldots \sqrt[n_k]{A}}}$$

$$\sqrt[\Sigma n_i]{A} \longrightarrow \sqrt[n_1]{A} * \sqrt[n_2]{A} * \ldots * \sqrt[n_k]{A}$$

Taking one dimensional representations with respect to the latter one gives us the required connecting morphisms

$$\mathsf{spec} A@_{\mathbf{n_1}} \times \ldots \times \mathsf{spec} A@_{\mathbf{n_k}} \xrightarrow{c_{\underline{n}}} \mathsf{spec} A@_{\sum \mathbf{n_i}}.$$

Universal constructions such as $\sqrt[n]{A}$ are only as useful as one can compute with them. In particular we must have a description of $\sqrt[n]{\mathbb{C}Q}$ when Q is a quiver on k vertices. Consider the extended quiver



where we add to the vertices and arrows of Q one extra vertex 0 and for every vertex i in Q we add n directed arrows x_{ij} from 0 to i and n directed arrows y_{ij} for i to 0. We then consider the matrices

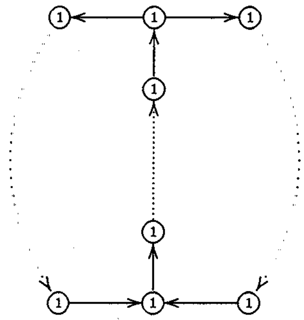
$$M_\sigma = \begin{bmatrix} x_{11} & \ldots & \ldots & x_{1n} \\ \vdots & & & \vdots \\ x_{k1} & \ldots & \ldots & x_{kn} \end{bmatrix} \quad \text{and} \quad N_\sigma = \begin{bmatrix} y_{11} & \ldots & y_{k1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ y_{1n} & \ldots & y_{kn} \end{bmatrix}$$

and impose the conditions

$$M_\sigma . N_\sigma = \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_k \end{bmatrix} \quad \text{and} \quad N_\sigma . M_\sigma = \begin{bmatrix} v_0 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & v_0 \end{bmatrix}$$

where v_j is the idempotent in the path algebra corresponding to vertex $0 \leq j \leq k$.
Using these conventions we can then identify $\sqrt[n]{\mathbb{C}Q}$ as the algebra of oriented *loops* based at vertex 0 in this quiver with relations.

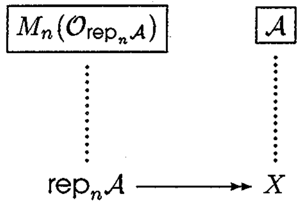
For example if X is a surface, then the only quiver-data that can occur are



possibly with one or both of the circuits reduced to a loop. Invariant theory of quivers and étale descent then implies the following restrictions on X, Δ and $\mathcal{A}$

- X must be smooth,
- the ramification divisor of $\mathcal{A}$ has only normal crossings as singularities,
- $\mathcal{A}$ is étale locally splittable.

This information can then be combined with the Artin-Mumford sequence describing the Brauer group $Br(\mathbb{C}(X))$ to determine the classes of Δ admitting an order $\mathcal{A}$ such that $\mathsf{rep}_n\mathcal{A}$ is a smooth GL_n-scheme over X. The étale local structure of maximal orders over surfaces due to Artin then actually provides the orders $\mathcal{A}$. The strategy in higher dimension is similar (but substantially harder). For more details we refer to [4].
This provides us with a huge class of settings



which can be seen as a level n extensions of smooth projective varieties. However, as in the commutative case not all of these will carry a global formal structure. I have not worked out the obstruction yet but they are expected to be GL_n-equivariant variations of the obstructions found by Kapranov. Still, it is reasonable to expect that many of the above setting actually give a manifold in **geometry@$_\mathbf{n}$** (for example if $\mathcal{A}$ is an Azumaya algebra over a smooth manifold X having a formal structure).

Assume we have a collection of integers n_i and manifolds in **geometry@$_{\mathbf{n_i}}$** as above