noncommutative geometry@n

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(1999)

Motivation.

This book is all about smooth noncommutative algebras and combinatorial tools to study them. It is an old result, due to A. Grothendieck, that when A is a commutative affine $\mathbb{C}$-algebra, A is the coordinate ring of a smooth affine variety if and only if A satisfies the following lifting property. For every test-object (B, I) where B is a commutative $\mathbb{C}$-algebra and $I \triangleleft R$ is a nilpotent ideal and any $\mathbb{C}$-algebra morphism $\phi : A \longrightarrow \frac{B}{I}$ there is a $\mathbb{C}$-algebra lift $\tilde{\phi}$

$$\begin{tikzcd} A \arrow[r, dotted, "\exists\ \tilde{\phi}"] \arrow[dr, "\phi"'] & B \arrow[d] \\ & \frac{B}{I} \end{tikzcd}$$

making the diagram commute. As this is a purely categorical characterization of smooth affine commutative algebras, it can be extended to more general settings where we restrict the test-objects and morphisms to a specified category of $\mathbb{C}$-algebras. In this book we focuss on two such settings.

If we take the category alg with objects all (not necessarily commutative) $\mathbb{C}$-algebras and as morphisms all $\mathbb{C}$-algebra morphisms, then algebras satisfying the above lifting property with respect to test-objects in alg are called (formally) smooth algebras or Quillen-smooth algebras. The importance of this class of algebras is that they are often associated to natural families of moduli problems. Examples of Quillen-smooth algebras include : semi-simple $\mathbb{C}$-algebras (as we can lift idempotents modulo nilpotent ideals), free algebras $\mathbb{C}\langle x_1, \ldots, x_k \rangle$, path algebras $\mathbb{C}Q$ of quivers (oriented graphs) as well as more exotic algebras constructed from these by universal constructions such as the free product $A_1 * A_2$, the n-th root $\sqrt[n]{A}$ and so on. However, the coordinate ring of an affine smooth variety is Quillen-smooth if and only if the variety is a curve. To see at least one implication of this equivalence it suffices by reasoning locally to verify the lifting property for the formal power series ring $\mathbb{C}[[x_1, \ldots, x_k]]$. Take the 4-dimensional noncommutative algebra

$$T = \frac{\mathbb{C}\langle x, y \rangle}{(x^2, y^2, xy + yx)} = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy$$

then the quotient modulo the nilpotent ideal $I = (xy - yx)$ is a 3-dimensional commutative ring and we have an algebra map $x_1 \mapsto x$, $x_2 \mapsto y$ and $x_i \mapsto 0$ for $i \geq 3$ which is verified not to allow a lift unless $k = 1$, the curve case.

The second category we will consider is $CH(n)$, the category of all Cayley-Hamilton algebras of degree n. Its objects are $\mathbb{C}$-algebras A equipped with a linear trace map $tr : A \longrightarrow A$ satisfying $tr(a)b = btr(a), tr(ab) = tr(ba)$ and $tr(tr(a)b) = tr(a)tr(b)$ for all $a, b \in A$. The formal Cayley-Hamilton polynomial of degree n of an element $a \in A$ is defined by expressing $\prod_{i=1}^n (t - x_i)$ as a polynomial in t with coefficients which (being symmetric functions) can be expressed as polynomial functions in the Newton functions $\sum x_i^m$. Replacing $\sum x_i^m$ by $tr(a^m)$ we obtain the Cayley-Hamilton polynomial $\chi_a^{(n)}(t) \in A[t]$. A is said to be a Cayley-Hamilton algebra of degree n provided $tr(1) = n$ and $\chi_a^{(n)}(a) = 0$ for all $a \in A$. Naturally, morphisms in $CH(n)$ must be trace preserving. A $\mathbb{C}$-algebra $A \in CH(n)$ with the above lifting property for test-objects (B, I), where $B \in CH(n)$ and $I \triangleleft B$ nilpotent such that $tr(I) \subset I$ (making $\frac{B}{I} \in CH(n)$) and where $\phi : A \longrightarrow \frac{B}{I}$ as well as its lift $\tilde{\phi} : A \longrightarrow B$ must be trace preserving, is called a Cayley-smooth algebra (of degree n). Examples include semi-simple algebras $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ with $n = \sum_i n_i$ (equipped with the sum of the natural traces) as well as many nice $\mathbb{C}[X]$-orders in central simple $\mathbb{C}(X)$-algebras of dimension n^2 where X is an affine normal variety. An important subclass are the so called Azumaya algebras over X, such as $M_n(\mathbb{C}[X])$, when X is smooth. The more interesting examples include ramified orders where the central variety X is allowed to have singularities. We will give a complete étale local classifications of Cayley-smooth orders and of the central singularities.

In the next two chapters we motivate the study of these noncommutative smooth algebras. In chapter 1 we study Calogero particles which is a classical n-particle system in $\mathbb{C}$ with positions $x_i \in \mathbb{C}$ and velocities $y_i \in \mathbb{C}$ and Hamiltonian

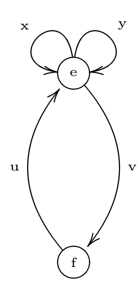
$$H = \frac{1}{2}\sum_i y_i^2 - \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

This is a completely integrable dynamical system via the equations of motion. If the n particles are distinct (that is $x_i \neq x_j$ for $i \neq j$) the corresponding point in the phase space is the $2n$-tuple $(x_1, y_1; \ldots; x_n, y_n)$ under the proviso that two such tuples are the same when we permute the n couples (x_i, y_i). As the system is attractive, collisions will occur and one wants to extend the phase space analytically. This was done by G. Wilson in [33] who showed that the extended phase space $Calo_n$ is the $2n$-dimensional connected manifold of GL_n-orbits of quadruples

$$(X, Y, u, v) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*} \quad \text{such that} \quad [X, Y] + u.v = \mathbb{1}_n$$

where the action is given by $g.(X, Y, u, v) = (gXg^{-1}, gYg^{-1}, gu, vg^{-1})$. This phase space $Calo_n$ should be compared with the $2n$-dimensional Hilbert scheme $Hilb_n$ of n points in $\mathbb{C}^2$. Whereas $Hilb_n$ decomposes in strata according to the multiplicities of the points, $Calo_n$ was shown by V. Ginzburg [8] to be a coadjoint orbit for an infinite dimensional Lie algebra, which is independent of n.

The Lie algebra is naturally associated to a Quillen-smooth algebra $\mathbb{M}$ which is the path algebra of the quiver



We will define noncommutative functions, differential forms and symplectic structures on Quillen-smooth algebras. They have the characteristic property of inducing corresponding GL_m-invariant classical structures on the representation varieties $rep_m\ A$ of these algebras. In the case of path algebras such as $\mathbb{M}$, these representation spaces decompose according to the dimensions of the vertex spaces. In the special case of dimension vector $\alpha = (n, 1)$ we have

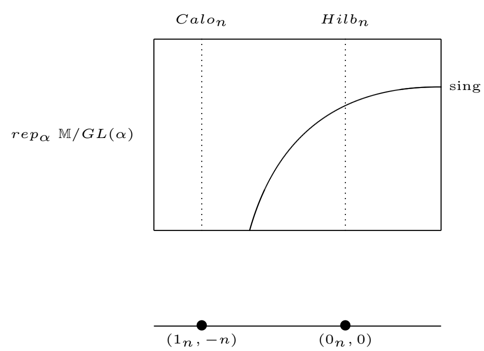
$$rep_\alpha\ \mathbb{M} = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$$

with the $GL(\alpha) = GL_n \times \mathbb{C}^*$-action given by $(g,\lambda).(X,Y,u,v) = (gXg^{-1}, gYg^{-1}, gu\lambda^{-1}, \lambda vg^{-1})$. To this action we associate the moment map

$$rep_\alpha\ \mathbb{M} \xrightarrow{\mu} M_n(\mathbb{C}) \oplus \mathbb{C} \qquad (X,Y,u,v) \mapsto ([X,Y] + u.v, -v.u)$$

and one proves that $Calo_n$ is equal to $\mu^{-1}(\mathbb{1}_n, -n)/GL(\alpha)$ and that $Hilb_n$ is an open subvariety of $\mu^{-1}(0_n, 0)/GL(\alpha)$. The infinite dimensional Lie algebra mentioned before is associated to the group of $\mathbb{C}$-algebra morphisms of $\mathbb{M}$ preserving the moment-map element $[x,y] + [u,v]$.

In analogy with the commutative case, one would expect that the difference in behaviour between $Calo_n$ and $Hilb_n$ is caused by the fact that the fiber-algebra associated to $Calo_n$ is Quillen-smooth whereas that associated to $Hilb_n$ is not (contains noncommutative singularities).



A natural definition of these noncommutative fiber algebras are the deformed preprojective algebras introduced and studied by W. Crawley-Boevey and M.P. Holland in [6]. In the case under consideration, the fiber algebra of $Calo_n$ (resp. $Hilb_n$) is $\mathbb{M}_1$ (resp. $\mathbb{M}_0$) where for any $\lambda \in \mathbb{C}$ we define

$$\mathbb{M}_\lambda = \frac{\mathbb{M}}{([x,y] + [u,v] - \lambda(e - nf))}$$

One verifies that for every $\lambda \in \mathbb{C}$ the fiber-algebra $\mathbb{M}_\lambda$ is not Quillen-smooth, so apparently there is no difference between $\mathbb{M}_1$ and $\mathbb{M}_0$.

However, if one focusses on $Calo_n$ or $Hilb_n$ for a specific value of n, these fiber-algebras are a bit too big and it is more natural to consider algebras associated to $\mathbb{M}_\lambda$ and the dimension vector $\alpha = (n, 1)$. First, we define the algebra $\mathbb{M}(n)$ to be the $\mathbb{C}$-subalgebra of $M_{n+1}(\mathbb{C}[rep_\alpha\ \mathbb{M}])$ generated by the polynomial invariants $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$ (embedded as scalar matrices) and the following matrices

$$e_n = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad f_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad x_n = \begin{bmatrix} x_{11} & \dots & x_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \dots & x_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$y_n = \begin{bmatrix} y_{11} & \dots & y_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ y_{n1} & \dots & y_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad u_n = \begin{bmatrix} 0 & \dots & 0 & u_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & u_n \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad v_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ v_1 & \dots & v_n & 0 \end{bmatrix}$$

The usual trace map on $M_{n+1}(\mathbb{C}[rep_\alpha\ \mathbb{M}])$ makes $\mathbb{M}(n)$ a Cayley-Hamilton algebra of degree $n+1$ and it is even a Cayley-smooth algebra of degree $n+1$. The closed affine subscheme $\pi^{-1}(\lambda \mathbb{1}_n, -n\lambda)$ has as its defining ideal of relations I_λ the entries in the $n+1 \times n+1$ matrix

$$[x_n, y_n] + [u_n, v_n] - \begin{bmatrix} \lambda \mathbb{1}_n & 0 \\ 0 & -n\lambda \end{bmatrix}$$

By invariant theory we have that the defining ideal of the quotient scheme $\mu^{-1}(\lambda \mathbb{1}_n, -n\lambda)/GL(\alpha)$ is $J_\lambda = I_\lambda \cap \mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$. The restricted fiber-algebra we are interested in is

$$\mathbb{M}_\lambda(n) = \frac{\mathbb{M}(n)}{\mathbb{M}(n) J_\lambda \mathbb{M}(n)}$$

Again, $\mathbb{M}_\lambda(n)$ is a Cayley-Hamilton algebra of degree $n+1$. The distinction between $Calo_n$ and $Hilb_n$ is a consequence of the fact that

$$\begin{cases} \mathbb{M}_1(n) & \text{is Cayley-smooth of degree } n+1, \\ \mathbb{M}_0(n) & \text{is not Cayley-smooth of degree } n+1 \end{cases}$$

and the last fact holds even when we restrict to the open subvariety $Hilb_n$ of $\mu^{-1}(0_n, 0)$. The homogeneous character of $Calo_n$ follows from the fact that $\mathbb{M}_1(n)$ is an Azumaya algebra over $Calo_n$. That is, to every point in the extended phase space corresponds a simple $n+1$-dimensional representation. In chapter 12 we will generalize these results to fiber-algebras for the moment map of arbitrary quiver settings.

In chapter 1 we will give an outline of the main ingredients going into the proof of Ginzburg's result on coadjointness of $Calo_n$, in particular the acyclicity of the Karoubi complex and noncommutative symplectic geometry identifying tangent vectors (derivations) with noncommutative 1-forms. More details will be given in chapters 9,10 and 12.

The investigation of Cayley-smooth algebras has also a more classical motivation as we will recall in chapter 2. Let X be a projective normal variety with function field $\mathbb{C}(X)$. An important birational invariant of X is the Brauer group $Br\ \mathbb{C}(X)$. The elements of $Br\ \mathbb{C}(X)$ are equivalence classes of central simple $\mathbb{C}(X)$-algebras Δ. That is, the center of Δ is $\mathbb{C}(X)$, Δ has no proper twosided ideals and $dim_{\mathbb{C}(X)}\ \Delta = n^2$ for some integer n. Two such algebras Δ and Δ' are equivalent if $M_k(\Delta) \simeq M_l(\Delta')$ for certain k, l and the isomorphism is as $\mathbb{C}(X)$-algebras. Then, tensor product over $\mathbb{C}(X)$ induces a group structure on $Br\ \mathbb{C}(X)$. In chapter 2 we will recall the necessary ingredients from étale cohomology to outline the proof of the coniveau spectral sequence which gives us a handle on the n-torsion part of $Br\ \mathbb{C}(X)$. This result also shows that the collection of all central simple $\mathbb{C}(X)$-algebras of degree n is huge.

For example, a subset of the n-torsion part of $Br\ \mathbb{C}(x, y)$ is given by the following geometrical data. Let C and C' be two smooth irreducible projective curves in $\mathbb{P}^2$, intersecting each other transversally in the points $\{P_1, \ldots, P_k\}$. Let $a_i \in \mathbb{Z}/n\mathbb{Z}$ for every $1 \leq i \leq k$ such that $\sum_{i=1}^k a_i = 0$. Now, take a cyclic $\mathbb{Z}/n\mathbb{Z}$-cover of smooth curves

$$D \twoheadrightarrow C \quad \text{and} \quad D' \twoheadrightarrow C'$$

such that D (resp. D') are ramified only in the points P_i with ramification determined by the class $a_i \in \mathbb{Z}/n\mathbb{Z}$ (resp. $-a_i \in \mathbb{Z}/n\mathbb{Z}$). One can control such coverings using the fundamental group of the Riemann surfaces C (and C') with k punctures. For example, if C has genus g, then the collection of such covers is in one-to-one correspondence with group morphisms

$$\pi_1(C - \{P_1, \ldots, P_k\}) = \frac{\langle u_1, v_1, \ldots, u_g, v_g, x_1, \ldots, x_k \rangle}{(u_1 v_1 u_1^{-1} v_1^{-1} \ldots u_g v_g u_g^{-1} v_g^{-1} x_1 \ldots x_k)} \longrightarrow Aut\ \mathbb{Z}/n\mathbb{Z}$$

mapping x_i to the multiplication by a_i (and similarly for C' mapping the x_i to multiplication by $-a_i$). To every such data corresponds a class of order n in $Br\ \mathbb{C}(x,y)$ which often corresponds to a central simple $\mathbb{C}(x,y)$-algebra of degree n, that is, of dimension n^2 over $\mathbb{C}(x,y)$.

Returning to the general setting of a projective normal variety X, let Δ be a central simple $\mathbb{C}(X)$-algebra of degree n. If $\mathbb{K}$ is the algebraic closure of $\mathbb{C}(X)$, then $\Delta \otimes_{\mathbb{C}(X)} \mathbb{K} \simeq M_n(\mathbb{K})$ and by Galois descent the usual trace on $M_n(\mathbb{K})$ induces a trace map on Δ making it a Cayley-Hamilton algebra of degree n. An important class of Cayley-Hamilton algebras of degree n is given by section algebras of $\mathcal{O}_X$-orders in Δ. That is, let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras over X such that for every affine open subset $U \hookrightarrow X$ we have

$$\Gamma(U,\mathcal{A})\Gamma(U,\mathcal{O}_X)^{-1} = \Delta$$

Because $\Gamma(U,\mathcal{O}_X)$ is integrally closed, the trace map on Δ determines a trace map on the section algebra $\Gamma(U,\mathcal{A})$ making it a Cayley-Hamilton algebra of degree n. We call a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-orders in Δ to be a noncommutative smooth model for Δ if all affine section algebras are Cayley-smooth of degree n.

The archetypical example of such a noncommutative smooth model is given by the Artin-Mumford counterexamples to the Lüroth problem [3] , that is the construction of certain unirational non-rational threefolds. Let C and C' be two smooth elliptic curves in $\mathbb{P}^2$ intersecting transversally in $\{P_1,\ldots,P_9\}$, let all $a_i = 0$ and consider two unramified $\mathbb{Z}/2\mathbb{Z}$-covers $D \twoheadrightarrow C$ and $D' \twoheadrightarrow C'$. Then, D and D' are again elliptic curves and the covers are given by dividing out a point of order two. Let Δ be the corresponding central simple algebra over $\mathbb{C}(x,y)$ which is a quaternion algebra. Next, let $X \twoheadrightarrow \mathbb{P}^2$ be the rational projective surface obtained by blowing up the P_i and let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-order in Δ. M. Artin and D.Mumford are able to calculate the local description of $\mathcal{A}$. If $x \in X$ not lying on $\tilde{C} \cup \tilde{C}'$ then $\mathcal{A}_x$ is an Azumaya algebra and if $x \in \tilde{C} \cup \tilde{C}'$ (the strict transforms), then

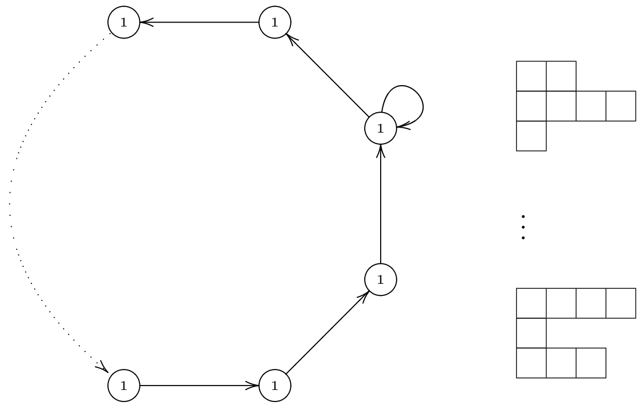
$$\mathcal{A}_x = \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x} i \oplus \mathcal{O}_{X,x} j \oplus \mathcal{O}_{X,x} ij \quad \text{with} \quad \begin{cases} i^2 &= a \\ j^2 &= bt \\ ji &= -ij \end{cases}$$

where a and b are units in $\mathcal{O}_{X,x}$ and $t = 0$ is a local equation for $\tilde{C} \cup \tilde{C}'$ near x. Extending the classical notion of Brauer-Severi varieties of central simple algebras to these orders they define $BS(\mathcal{A})$ which is a projective space bundle over X

$$BS(\mathcal{A}) \xrightarrow{\pi} X$$

Using the local description of $\mathcal{A}$ they show that $BS(\mathcal{A})$ is a smooth variety, π is a flat morphisms and the geometric fibers are isomorphic to $\mathbb{P}^1$ (resp. to $\mathbb{P}^1 \vee \mathbb{P}^1$) whenever $x \notin \tilde{C} \cup \tilde{C}'$ (resp. $x \in \tilde{C} \cup \tilde{C}'$). For specific starting configurations they then show that the threefold $BS(\mathcal{A})$ is unirational but non-rational. With hindsight, the characteristic property of $\mathcal{A}$ allowing a local description, a smooth Brauer-Severi scheme and a description of the fibers is that $\mathcal{A}$ is a noncommutative smooth model for Δ. In this book we will generalize these computations both to higher degree central simple algebras and higher dimensional base varieties.

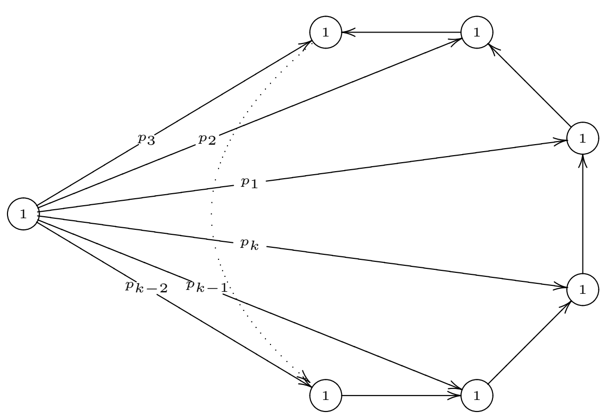
In chapter 6 we will give a complete characterization of the central simple algebras Δ over $\mathbb{C}(S)$ where S is a projective smooth surface such that Δ allows a noncommutative smooth model. For example, among the subclass of $Br_n\ \mathbb{C}(x,y)$ described before by two curves and ramified covers those allowing a smooth model are precisely the configurations where all $a_i = 0$, that is, such that the covers are unramified. In fact, for such a Δ an explicit smooth model is obtained by taking a maximal order $\mathcal{A}$ in Δ where X is the surface obtained after blowing up all the intersection points P_i. In this generality we will be able in chapter 5 and 6 to determine the étale local structure of $\mathcal{A}_z$ in $z \in \tilde{C} \cup \tilde{C'}$ (in all other points it is an Azumaya algebra). If Δ is of degree n it is determined by combinatorial data consisting of a circuit on $k \leq n$ vertices ordered starting in the vertex having an extra loop



and an unordered partition $p = (p_1, \ldots, p_k)$ of n having exactly k parts. The $\mathfrak{m}_z$-adic completion of $\mathcal{A}_z$ is then isomorphic to the subalgebra of $M_n(\mathbb{C}[[x,y]])$ for local coordinates (x,y) near z having the following block form

$$\hat{\mathcal{A}}_z \simeq \begin{bmatrix} M_{p_1}(\mathbb{C}[[x,y]]) & M_{p_1 \times p_2}((x)) & \ldots & M_{p_1 \times p_k}((x)) \\ M_{p_2 \times p_1}(\mathbb{C}[[x,y]]) & M_{p_2}(\mathbb{C}[[x,y]]) & \ldots & M_{p_2 \times p_k}((x)) \\ \vdots & & \ddots & \vdots \\ M_{p_k \times p_1}(\mathbb{C}[[x,y]]) & m_{p_k \times p_2}(\mathbb{C}[[x,y]]) & \ldots & M_{p_k}(\mathbb{C}[[x,y]]) \end{bmatrix}$$

The combinatorial data is constant along $\tilde{C}$ and a possibly different data is constant along $\tilde{C'}$. We will show in chapter 2 that for such orders the Brauer-Severi scheme $BS(\mathcal{A})$ is a smooth variety. In chapter 8 we will give a combinatorial method to describe the fibers of the structural morphism $BS(\mathcal{A}) \twoheadrightarrow X$. Consider the quiver-data



That is, we add a vertex v_0 and connect it to vertex v_i with p_i arrows. We will prove in chapter 8 that the fiber is the moduli space of θ-semistable representations in the nullcone of this quiver with dimension vector $(1,1,\ldots,1)$ and where $\theta = (n, -p_1, -p_2, \ldots, -p_k)$. That is, we have to classify isomorphism classes of representations such that at least one of the arrows in the circuit is zero and such that the representation contains no proper subrepresentation of dimension vector

$(1, n_1, n_2, \ldots, n_k)$ with all $n_i = 0$ or 1 such that $n - \sum_i p_i n_i > 0$. In chapter 7 we will give more details on such moduli spaces and the combinatorial aspects of θ-semistable representations.

Chapter 1

Calogero Systems.

One motivation to study noncommutative geometry comes from physics. One wants to understand the behaviour of n-particle systems when $n \longrightarrow \infty$. In this chapter we will give an illustrative example : collisions of Calogero particles.

We will first describe the phase space of collisions of n Calogero particles $Calo_n$ using invariant theory. Its description is closely related to that of the Hilbert scheme of n points in the plane, $Hilb_n$. In fact, Nakajima [21] and G. Wilson [33] have shown that there is a diffeomorphism of $\mathcal{C}^\infty$ (real) manifolds between the two spaces. However, this diffeomorphism does not respect the complex structure and, in fact, both spaces have fundamentally different properties. Recent work of Y. Berest and G. Wilson [4] relating the phase space of Calogero particles to the study of isomorphism classes of right ideals in the Weyl algebra suggests that $Calo_n$ is a coadjoint orbit of some infinite dimensional algebraic group. This conjecture was recently proved by V. Ginzburg [8] using noncommutative symplectic geometry.

We will briefly indicate the main steps in Ginzburg's proof (more details will be given in chapter 12). The phase space $Calo_n$ turns out to be a fiber of the moment map on the representation space of a noncommutative smooth algebra. Following the lead of M. Karoubi [11] and J. Cuntz and D. Quillen [7] one defines differential forms and de Rham cohomology for noncommutative algebras. In the case of the smooth algebra $\mathbb{M}$ under consideration the de Rham complex is shown to be acyclic, a result first proved by M. Kontsevich [13] for the free algebra. Moreover, there is a natural symplectic structure on $\mathbb{M}$ giving a natural one-to-one correspondence between 1-forms and derivations (vector fields). Ginzburg's result then follows from a noncommutative version of the classical exact sequence in symplectic geometry describing Hamiltonian vector fields.

Similarly, the Hilbert scheme $Hilb_n$ is (part of) a fiber of the moment map but one knows that this space cannot be a coadjoint orbit. The distinction between the two cases comes from investigating noncommutative algebras, finite modules over their centers, associated to these fibers. It turns out that the algebra corresponding to $Calo_n$ is a smooth order (in fact it is even an Azumaya algebra over $Calo_n$), whereas that corresponding to $Hilb_n$ is not. The description and study of such smooth orders will be one of the main goals of this book. In chapter 12, we will be able to extend the above results to general quiver varieties.

1.1 Calogero particles.

The *Calogero system* is a classical particle system of n particles on the real line with inverse square potential.

That is, if the i-th particle has position x_i and velocity (momentum) y_i, then the Hamiltonian is equal to

$$H = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

The Hamiltonian *equations of motions* is the system of $2n$ differential equations

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial y_i} \\[2ex] \dfrac{dy_i}{dt} = -\dfrac{\partial H}{\partial x_i} \end{cases}$$

This defines a dynamical system which is *integrable*.

A convenient way to study this system is as follows. Assign to a position defined by the $2n$ vector $(x_1, y_1; \dots, x_n, y_n)$ the couple of *Hermitian or self-adjoint* $n \times n$ matrices

$$X = \begin{bmatrix} y_1 & \frac{i}{x_1-x_2} & \cdots & \cdots & \frac{i}{x_1-x_n} \\ \frac{i}{x_2-x_1} & y_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{i}{x_{n-1}-x_n} \\ \frac{i}{x_n-x_1} & \cdots & \cdots & \frac{i}{x_n-x_{n-1}} & y_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}$$

Physical quantities are given by invariant polynomial functions under the action of the unitary group $U_n(\mathbb{C})$ under simultaneous conjugation. In particular one considers the functions

$$F_j = tr\ \frac{X^j}{j}$$

For example,

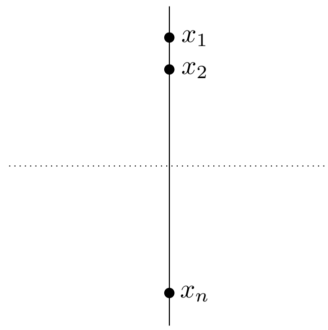
$$\begin{cases} tr(X) = \sum y_i & \text{the total momentum} \\ \frac{1}{2}tr(X^2) = \frac{1}{2}\sum y_i^2 - \sum_{i<j}\frac{1}{(x_i-x_j)^2} & \text{the Hamiltonian} \end{cases}$$

We can now consider the $U_n(\mathbb{C})$-translates of these matrix couples. This is shown to be a manifold with a free action of $U_n(\mathbb{C})$ such that the orbits are in one-to-one correspondence with points $(x_1, y_1; \dots; x_n, y_n)$ in the phase space (that is, we agree that two such $2n$ tuples are determined only up to permuting the couples (x_i, y_i). The n-functions F_j give a completely integrable system on the phase space via Liouville's theorem, see for example [1].

In the classical case, all points are assumed to lie on the real axis and the potential is repulsive so that collisions do not appear. G. Wilson [33] considered an alternative where the points are assumed to lie in the complex numbers and such that the potential is attractive (to allow for collisions), that is, the Hamiltonian is of the form

$$H = \frac{1}{2}\sum_i y_i^2 - \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

giving again rise to a dynamical system via the equations of motion. One recovers the classical situation back if the particles are assumed only to move on the imaginary axis.



In general, we want to extend the phase space of n distinct points analytically to allow for collisions. When all the points are distinct, that is, if all eigenvalues of Y are distinct we will see in a moment that there is a unique $GL_n(\mathbb{C})$-orbit of couples of $n \times n$ matrices (up to permuting the n couples (x_i, y_i)).

$$X = \begin{bmatrix} y_1 & \frac{1}{x_1-x_2} & \dots & \dots & \frac{1}{x_1-x_n} \\ \frac{1}{x_2-x_1} & y_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{x_{n-1}-x_n} \\ \frac{1}{x_n-x_1} & \dots & \dots & \frac{1}{x_n-x_{n-1}} & y_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}$$

For matrix couples in this standard form one verifies that

$$[X, Y] + \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} = \mathbb{1}_n$$

This equality suggests an approach to extend the phase space of n distinct complex Calogero particles to allow for collisions.

Consider the $2n^2 + 2n$-dimensional vectorspace (the notation will be explained later)

$$rep_\alpha\ \mathbb{M} = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$$

(where $\mathbb{C}^{n*}$ is the space of row-vectors). Consider the subvariety $CALO_n$ of quadruples (X, Y, u, v) such that

$$[X, Y] + u.v = \mathbb{1}_n$$

There is an action of $GL_n(\mathbb{C})$ on the vectorspace defined by

$$g.(X, Y, u, v) = (gXg^{-1}, gYg^{-1}, gu, vg^{-1})$$

which preserves $CALO_n$ and which we will show to be free.

We can define the *phase space for Calogero collisions* of n particles to be the orbit space

$$Calo_n = CALO_n / GL_n(\mathbb{C})$$

the space of orbits of $GL_n(\mathbb{C})$ on $CALO_n$. In a moment we will show :

Theorem 1.1 *The phase space $Calo_n$ of Calogero collisions of n-particles is a connected complex manifold of dimension $2n$.*

1.2 The moment map.

The *moment map* for the $GL_n(\mathbb{C})$-action on $M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$ is defined to be

$$\begin{array}{ccc} M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*} & \xrightarrow{\mu} & M_n(\mathbb{C}) \\ (X, Y, u, v) & \mapsto & [X, Y] + u.v \end{array}$$

We will be interested in the *differential* $d\mu$ of this map which we can compute by the ϵ*-method* : $[(X + \epsilon A), (Y + \epsilon B)] + (u + \epsilon c).(v + \epsilon d)$ is equal to

$$([X, Y] + u.v) + \epsilon([X, B] + [A.Y] + u.d + c.v)$$

whence the differential $d\mu$ in the point (X, Y, u, v) is equal to

$$d\mu_{(X,Y,u,v)}\ (A, B, c, d) = [X, B] + [A, Y] + u.d + c.v.$$

We say that u is a *cyclic vector* for the matrix-couple $(X, Y) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ if there is no proper subspace of $\mathbb{C}^n$ containing u which is stable under left multiplication by X and Y.

Lemma 1.2 *The differential $d\mu$ is surjective in (X, Y, u, v) if u is a cyclic vector for (X, Y).*

Proof. Consider the *nondegenerate* symmetric bilinear form on $M_n(\mathbb{C})$

$$\begin{array}{ccc} M_n(\mathbb{C}) \times M_n(\mathbb{C}) & \longrightarrow & \mathbb{C} \\ (M, N) & \mapsto & tr(MN) \end{array}$$

Nondegeneracy means that $tr(MN) = 0, \forall N \in M_n(\mathbb{C})$ is equivalent to $M = 0$.

With respect to this inproduct on $M_n(\mathbb{C})$ the space orthogonal to the image of $d\mu_{(X,Y,u,v)}$ is equal to

$$\{M \in M_n(\mathbb{C}) \mid\ tr([X, B]M + [A, Y]M + u.dM + c.vM) = 0, \forall(A, B, c, d)\}$$

Because the trace does not change under cyclic permutations and is nondegenerate we see that this space is equal to

$$\{M \in M_n(\mathbb{C}) \mid\ [M, X] = 0 \quad [Y, M] = 0 \quad Mu = 0 \quad \text{and} \quad vM = 0\}$$

But then, the kernel $ker\ M$ is a subspace of $\mathbb{C}^n$ containing u and stable under left multiplication by X and Y. By the cyclicity assumption this implies that $ker\ M = \mathbb{C}^n$ or equivalently that $M = 0$.

As $d\mu^{\perp}_{(X,Y,u,v)} = 0$ and tr is nondegenerate, this implies that the differential is surjective. □

It follows from the *implicit function theorem* that the image of the moment map is open in $M_n(\mathbb{C})$. If we denote by $rep^s_\alpha\ \mathbb{M}$ (again, we will explain the terminology later) the open submanifold of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$ consisting of quadruples (X, Y, u, v) such that u is a cyclic vector for (X, Y) then we obtain

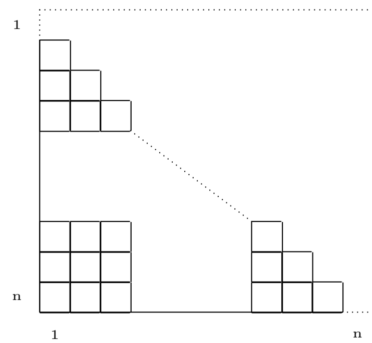
Proposition 1.3 *For every matrix $M \in M_n(\mathbb{C})$ in the image of the map*

$$rep_\alpha^s \ \mathbb{M} \xrightarrow{\mu} M_n(\mathbb{C})$$

the inverse image $\mu^{-1}(M)$ is a submanifold of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$ of dimension $n^2 + 2n$.

1.3 Hilbert stairs.

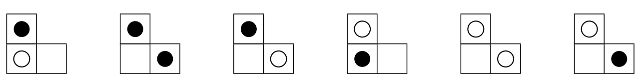
For the investigation of the $GL_n(\mathbb{C})$-orbits on $rep_\alpha^s \ \mathbb{M}$ we introduce a combinatorial gadget : the *Hilbert n-stair*. This is the lower triangular part of a square $n \times n$ array of boxes



filled with *go-stones* according to the following two rules :

- each row contains exactly one stone, and
- each column contains at most one stone of each color.

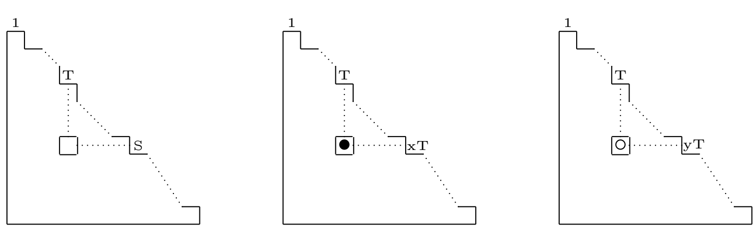
For example, the set of all possible Hilbert 3-stairs is given below.



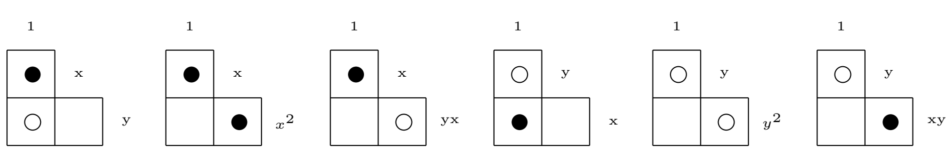
To every Hilbert stair σ we will associate a sequence of monomials $W(\sigma)$ in the free noncommutative algebra $\mathbb{C}\langle x, y\rangle$, that is $W(\sigma)$ is a sequence of words in x and y.

At the top of the stairs we place the identity element 1. Then, we descend the stairs according to the following rule.

- Every go-stone has a *top word T* which we may assume we have constructed before and a *side word S* and they are related as indicated below



For example, for the Hilbert 3-stairs we have the following sequences of noncommutative words



We will evaluate a Hilbert n-stair σ with associated sequence of non-commutative words $W(\sigma) = \{1, w_2(x,y), \dots, w_n(x,y)\}$ on

$$rep_\alpha\ \mathbb{M} = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$$

For a quadruple (X,Y,u,v) we replace every occurrence of x in the word $w_i(x,y)$ by X and every occurrence of y by Y to obtain an $n \times n$ matrix $w_i = w_i(X,Y) \in M_n(\mathbb{C})$ and by left multiplication on u a column vector $w_i.v$. The *evaluation of σ on* (X,Y,u,v) is the determinant of the $n \times n$ matrix

$$\sigma(X,Y,u,v) = det \begin{bmatrix} u & w_2.u & w_3.u & \cdots & w_n.u \end{bmatrix}$$

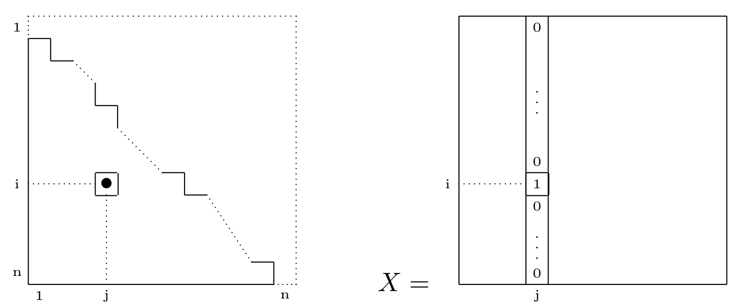
For a fixed Hilbert n-stair σ we denote with $rep(\sigma)$ the subset of quadruples (X,Y,u,v) in $rep_\alpha\ \mathbb{M}$ such that the evaluation $\sigma(v,X,Y) \neq 0$.

Theorem 1.4 *For every Hilbert n-stair, $rep\ (\sigma) \neq \emptyset$*

Proof. Let u be the basic column vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let every black stone in the Hilbert stair σ fix a column of X by the rule



That is, one replaces every black stone in σ by 1 at the same spot in X and fills the remaining spots in the same column by zeroes. The same rule applies to Y for white stones. We say that such a quadruple (X,Y,u,v) is in *σ-standard form.*

With these conventions one easily verifies by induction that

$$w_i(X,Y)e_1 = e_i \quad \text{for all } 2 \leq i \leq n.$$

Hence, filling up the remaining spots in X and Y arbitrarily one has that $\sigma(X,Y,u,v) \neq 0$ proving the claim. □

Hence, $rep\ (\sigma)$ is an open subset of $rep_\alpha\ \mathbb{M}$ (and even of $rep^s_\alpha\ \mathbb{M}$) for every Hilbert n-stair σ. Further, for every word (monomial) $w(x,y)$ and every $g \in GL_n(\mathbb{C})$ we have that

$$w(gXg^{-1}, gYg^{-1})gv = gw(X,Y)v$$

and therefore the open sets $rep\ (\sigma)$ are stable under the $GL_n(\mathbb{C})$-action on $rep_\alpha\ \mathbb{M}$. We will give representatives of the orbits in $rep\ (\sigma)$.

Let $W_n = \{1, x, \ldots, x^n, xy, \ldots, y^n\}$ be the set of all words in the non-commuting variables x and y of length $\leq n$, ordered lexicographically.

For every quadruple $(X, Y, u, v) \in rep_\alpha\ \mathbb{M}$ consider the $n \times m$ matrix

$$\psi(X, Y, u, v) = \begin{bmatrix} u & Xu & X^2u & \ldots & Y^nu \end{bmatrix}$$

where $m = 2^{n+1} - 1$ and the j-th column is the column vector $w(X, Y)v$ with $w(x, y)$ the j-th word in W_n.

Hence, $(X, Y, u, v) \in rep\ (\sigma)$ if and only if the $n \times n$ minor of $\psi(X, Y, u, v)$ determined by the word-sequence $\{1, w_2, \ldots, w_n\}$ of σ is invertible. Moreover, as

$$\psi(gXg^{-1}, gYg^{-1}, gu, vg^{-1}) = g\psi(v, X, Y)$$

we deduce that the $GL_n(\mathbb{C})$-orbit of $(X, Y, u, v) \in rep_\alpha\ \mathbb{M}$ contains a *unique* quadruple (X_1, Y_1, u_1, v_1) such that the corresponding minor of $\psi(X_1, Y_1, u_1, v_1) = \mathbb{1}_n$.

Hence, each $GL_n(\mathbb{C})$-orbit in $rep\ (\sigma)$ contains a unique representant in σ-standard form. Therefore,

Proposition 1.5 *The action of $GL_n(\mathbb{C})$ on rep (σ) is free and the orbit space*

$$rep\ (\sigma)/GL_n(\mathbb{C})$$

is an affine space of dimension $n^2 + 2n$.

Proof. The dimension is equal to the number of non-forced entries in X, Y and v. As we fixed $n - 1$ columns in X or Y this dimension is equal to

$$k = 2n^2 - (n-1)n + n = n^2 + 2n.$$

The argument above shows that every $GL_n(\mathbb{C})$-orbit contains a unique quadruple in σ-standard form so the orbit space is an affine space. □

Theorem 1.6 *The orbit space*

$$rep^s_\alpha\ \mathbb{M}/GL_n(\mathbb{C})$$

is a complex manifold of dimension $n^2 + 2n$ and is covered by the affine spaces rep (σ).

Proof. Recall that $rep^s_\alpha\ \mathbb{M}$ is the open submanifold consisting of quadruples (x, Y, u, v) such that u is a cyclic vector of (X, Y) or equivalently such that

$$\mathbb{C}\langle X, Y\rangle u = \mathbb{C}^n$$

where $\mathbb{C}\langle X, Y\rangle$ is the not necessarily commutative subalgebra of $M_n(\mathbb{C})$ generated by the matrices X and Y.

Hence, clearly $rep\ (\sigma) \subset rep_n\ \mathbb{M}$ for any Hilbert n-stair σ. Conversely, we claim that a quadruple $(X, Y, u, v) \in rep^s_\alpha\ \mathbb{M}$ belongs to at least one of the open subsets $rep\ (\sigma)$.

Indeed, either $Xu \notin \mathbb{C}u$ or $Yu \notin \mathbb{C}u$ as otherwise the subspace $W = \mathbb{C}u$ would contradict the cyclicity assumption. Fill the top box of the stairs with the corresponding stone and define the 2-dimensional subspace $V_2 = \mathbb{C}u_1 + \mathbb{C}u_2$ where $u_1 = u$ and $u_2 = w_2(X, Y)u$ with w_2 the corresponding word (either x or y).

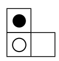
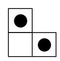
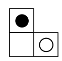
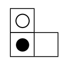
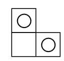
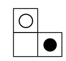
Assume by induction we have been able to fill the first i rows of the stairs with stones leading to the sequence of words $\{1, w_2(x, y), \ldots, w_i(x, y)\}$ such that the subspace $V_i = \mathbb{C}u_1 + \ldots + \mathbb{C}u_i$ with $u_i = w_i(X, Y)v$ has dimension i.

Then, either $Xu_j \notin V_i$ for some j or $Yu_j \notin V_i$ (if not, V_i would contradict cyclicity). Then, fill the j-th box in the $i+1$-th row of the stairs with the corresponding stone. Then, the top $i+1$ rows of the stairs form a Hilbert $i+1$-stair as there can be no stone of the same color lying in the same column. Define $w_{i+1}(x,y) = xw_i(x,y)$ (or $yw_i(x,y)$) and $u_{i+1} = w_{i+1}(X,Y)u$. Then, $V_{i+1} = \mathbb{C}u_1 + \ldots + \mathbb{C}u_{i+1}$ has dimension $i+1$.

Continuing we end up with a Hilbert n-stair σ such that $(X,Y,u,v) \in rep\ (\sigma)$. This concludes the proof. $\square$

Example 1.7 Orbits when $n = 3$.

Representatives for the $GL_3(\mathbb{C})$-orbits in $rep\ (\sigma)$ are given by the following quadruples for σ a Hilbert 3-stair :

						
X	$\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$	$\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}$	$\begin{bmatrix} 0 & a & b \\ 0 & c & d \\ 1 & e & f \end{bmatrix}$	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} a & 0 & b \\ c & 0 & d \\ e & 1 & f \end{bmatrix}$
Y	$\begin{bmatrix} 0 & g & h \\ 0 & i & j \\ 1 & k & l \end{bmatrix}$	$\begin{bmatrix} d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}$	$\begin{bmatrix} g & 0 & h \\ i & 0 & j \\ k & 1 & l \end{bmatrix}$	$\begin{bmatrix} 0 & g & h \\ 1 & i & j \\ 0 & k & l \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & j \\ 1 & 0 & k \\ 0 & 1 & l \end{bmatrix}$	$\begin{bmatrix} 0 & g & h \\ 1 & i & j \\ 0 & k & l \end{bmatrix}$
u	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
v	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$	$\begin{bmatrix} m & n & o \end{bmatrix}$

Let $\lambda = \lambda \mathbb{1}_n$ be a scalar matrix in $M_n(\mathbb{C})$ and hence fixed under the action by conjugation of $GL_n(\mathbb{C})$. Then, the subvariety $\mu^{-1}(\lambda)$ of $rep_\alpha\ \mathbb{M}$ is GL_n-stable. Because the $GL_n(\mathbb{C})$-action is free on $rep_\alpha^s\ \mathbb{M}$ we have the following situation

$$\begin{tikzcd}
\mu^{-1}(\lambda) \cap rep_\alpha^s\ \mathbb{M} \arrow[r, hook] \arrow[d, two heads] & rep_\alpha^s\ \mathbb{M} \arrow[d, two heads] \\
(\mu^{-1}(\lambda) \cap rep_\alpha^s\ \mathbb{M})/GL_n(\mathbb{C}) \arrow[r, hook] & rep_\alpha^s\ \mathbb{M}/GL_n(\mathbb{C})
\end{tikzcd}$$

and we obtain :

Theorem 1.8 *For a scalar matrix $\lambda \in M_n(\mathbb{C})$ lying in the image of μ, the orbit space*

$$(\mu^{-1}(\lambda) \cap rep_\alpha^s\ \mathbb{M})/GL_n(\mathbb{C})$$

is a submanifold of $rep_\alpha^s\ \mathbb{M}/GL_n(\mathbb{C})$ of dimension $2n$.

We will now investigate two of these manifolds : the *Hilbert scheme* of n points in the plane and the phase space of collisions of n Calogero particles.

1.4 The Hilbert scheme $Hilb_n$.

Consider n distinct points in the complex plane $\mathbb{C}^2$. Identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ we can view these points as particles in (flat) space-time. Particles have a tendency to collide with each other and we want to construct a manifold describing all possible collisions.

To a point $p = (a,b) \in \mathbb{C}^2$ corresponds a maximal ideal $\mathfrak{m} = (x-a, y-b)$ of the polynomial algebra $\mathbb{C}[x,y]$. To a set of n distinct points $P = \{p_1, \ldots, p_n\}$ we can associate a codimension n ideal

$$\mathfrak{i}_P = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n \triangleleft \mathbb{C}[x,y]$$

Now, consider an arbitrary codimension n ideal $\mathfrak{i} \triangleleft \mathbb{C}[x,y]$ and fix a basis $\{v_1, \ldots, v_n\}$ in the quotient space

$$V_{\mathfrak{i}} = \frac{\mathbb{C}[x,y]}{\mathfrak{i}} = \mathbb{C}v_1 + \ldots + \mathbb{C}v_n.$$

Multiplication by x on $\mathbb{C}[x,y]$ induces a linear operator on the quotient $V_{\mathfrak{i}}$ and hence determines a matrix $X_{\mathfrak{i}} \in M_n(\mathbb{C})$ with respect to the chosen basis $\{v_1, \ldots, v_n\}$. Similarly, multiplication by y determines a matrix $Y_{\mathfrak{i}} \in M_n(\mathbb{C})$.

Moreover, the image of the unit element $1 \in \mathbb{C}[x,y]$ in $V_{\mathfrak{i}}$ determines with respect to the basis $\{v_1, \ldots, v_n\}$ a column vector $u \in \mathbb{C}^n = V_{\mathfrak{i}}$. Clearly, this vector and matrices satisfy :

$$[X_{\mathfrak{i}}, Y_{\mathfrak{i}}] = 0 \quad \text{and} \quad \mathbb{C}[X_{\mathfrak{i}}, Y_{\mathfrak{i}}]u = \mathbb{C}^n.$$

Here, $\mathbb{C}[X_{\mathfrak{i}}, Y_{\mathfrak{i}}]$ is the n-dimensional subalgebra of $\mathsf{M}_n(\mathbb{C})$ generated by the two matrices $X_{\mathfrak{i}}$ and $Y_{\mathfrak{i}}$. In particular, u is a cyclic vector for the matrix-couple (X,Y).

We have seen that to a codimension n ideal $\mathfrak{i}$ corresponds a cyclic triple $(u_{\mathfrak{i}}, X_{\mathfrak{i}}, Y_{\mathfrak{i}})$ such that $X_{\mathfrak{i}}$ and $Y_{\mathfrak{i}}$ commute with each other.

Conversely, if $(X,Y,u) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n$ is a cyclic triple such that $[X,Y] = 0$, then $\mathbb{C}\langle X,Y\rangle = \mathbb{C}[X,Y]$ is an n-dimensional commutative subalgebra of $M_n(\mathbb{C})$ and the kernel of the natural epimorphism

$$\mathbb{C}[x,y] \twoheadrightarrow \mathbb{C}[X,Y] \qquad x \mapsto X \quad y \mapsto Y$$

is a codimension n ideal $\mathfrak{i}$ of $\mathbb{C}[x,y]$. Indeed, the linear map

$$\mathbb{C}[x,y] \xrightarrow{\phi} \mathbb{C}^n$$

defined by sending a polynomial $f(x,y) \in \mathbb{C}[x,y]$ to the vector $f(X,Y).u$ is surjective by the cyclicity assumption.

However, there is some redundancy in the assignment $\mathfrak{i} \longrightarrow (X_{\mathfrak{i}}, Y_{\mathfrak{i}}, u_{\mathfrak{i}})$ as it depends on the choice of basis of $V_{\mathfrak{i}}$. If we choose a different basis $\{v_1', \ldots, v_n'\}$ with basechange matrix $g \in GL_n(\mathbb{C})$, then the corresponding triple is

$$(X_{\mathfrak{i}}', Y_{\mathfrak{i}}', u_{\mathfrak{i}}') = (g.X_{\mathfrak{i}}.g^{-1}, g.Y_{\mathfrak{i}}.g^{-1}, gu_{\mathfrak{i}})$$

That is, we have an action of $GL_n(\mathbb{C})$ on the space of all triples

$$\begin{array}{rcl} GL_n(\mathbb{C}) \times (M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n) & \longrightarrow & M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \\ (g,(X,Y,u)) & \mapsto & (g.X.g^{-1}, g.Y.g^{-1}, gu) \end{array}$$

The above discussion shows that there is a one-to-one correspondence between

- codimension n ideals $\mathfrak{i}$ of $\mathbb{C}[x,y]$, and
- $GL_n(\mathbb{C})$-orbits of cyclic triples (X,Y,u) in $M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n$ such that $[X,Y] = 0$.

Example 1.9 The Hilbert scheme $Hilb_1$.

The space of all triples when $n = 1$ is $(X, Y, u) \in \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Clearly, they are all commuting and u is a cyclic vector for (X, Y) if and only if $u \neq 0$.

The basechange group in this case is $GL_1(\mathbb{C}) = \mathbb{C}^*$ and it acts on the space of triples via

$$g.(X, Y, u) = (X, Y, gu)$$

Therefore, the $GL_1(\mathbb{C})$ orbits of the cyclic (commuting) triples are parameterized by the points

$$\{(X, Y, 1) \in \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\} \simeq \mathbb{C}^2$$

The ideal $\mathfrak{i}$ of codimension one corresponding to $(X, Y, 1)$ are the polynomials vanishing in the point $p = (X, Y) \in \mathbb{C}^2$. Hence, $Hilb_1 \simeq \mathbb{C}^2$.

Example 1.10 The Hilbert scheme $Hilb_2$.

Consider a triple $(X, Y, u) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C}^2$ and assume that either X or Y has distinct eigenvalues (type a). As

$$[\begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}] = \begin{bmatrix} 0 & (\nu_1 - \nu_2)b \\ (\nu_2 - \nu_1)c & 0 \end{bmatrix}$$

we have a representant in the orbit of the form

$$(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})$$

where cyclicity of the column vector implies that $u_1 u_2 \neq 0$.

The stabilizer subgroup of the matrix-pair is the group of diagonal matrices $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow GL_2(\mathbb{C})$, hence the orbit has a unique representant of the form

$$(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

The corresponding ideal $\mathfrak{i} \triangleleft \mathbb{C}[x, y]$ is then

$$\mathfrak{i} = \{f(x, y) \in \mathbb{C}[x, y] \mid f(\lambda_1, \mu_1) = 0 = f(\lambda_2, \mu_2)\}$$

hence these orbits correspond to sets of two *distinct* points in $\mathbb{C}^2$.

The situation is slightly more complicated when X and Y have only one eigenvalue (type b). If (X, Y, u) is a cyclic commuting triple, then either X or Y is not diagonalizable. But then, as

$$[\begin{bmatrix} \nu & 1 \\ 0 & \nu \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}] = \begin{bmatrix} c & d - a \\ 0 & c \end{bmatrix}$$

we have a representant in the orbit of the form

$$(\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \mu & \beta \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})$$

with $[\alpha : \beta] \in \mathbb{P}^1$ and $u_2 \neq 0$. The stabilizer of the matrixpair is the subgroup

$$\{\begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \mid c \neq 0\} \hookrightarrow GL_2(\mathbb{C})$$

and hence we have a unique representant of the form

$$(\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \mu & \beta \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

The corresponding ideal $\mathfrak{i} \triangleleft \mathbb{C}[x, y]$ is

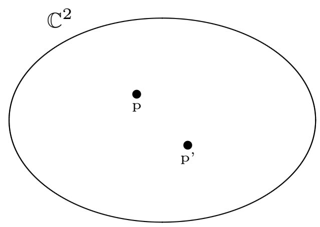
$$\mathfrak{i} = \{f(x, y) \in \mathbb{C}[x, y] \mid f(\lambda, \mu) = 0 \text{ and } \alpha \frac{\partial f}{\partial x}(\lambda, \mu) + \beta \frac{\partial f}{\partial y}(\lambda, \mu) = 0\}$$

as one proves by verification on monomials because

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}^k \begin{bmatrix} \mu & \beta \\ 0 & \mu \end{bmatrix}^l \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k\alpha\lambda^{k-1}\mu^l + l\beta\lambda^k\mu^{l-1} \\ \lambda^k\mu^l \end{bmatrix}$$

Therefore, $\mathfrak{i}$ corresponds to the set of two points at $(\lambda, \mu) \in \mathbb{C}^2$ infinitesimally attached to each other in the direction $\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$. For each point in $\mathbb{C}^2$ there is a $\mathbb{P}^1$ family of such *fat points*.

Thus, points of $Hilb_2$ correspond to either of the following two situations :



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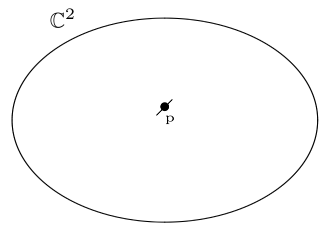


type b

The Hilbert-Chow map $Hilb_2 \xrightarrow{\pi} S^2\ \mathbb{C}^2$ (where $S^2\ \mathbb{C}^2$ is the symmetric power of $\mathbb{C}^2$, that is $S_2 = \mathbb{Z}/2\mathbb{Z}$ orbits of couples of points from $\mathbb{C}^2$) sends a point of type a to the formal sum $[p]+[p']$ and a point of type b to $2[p]$. Over the complement of (the image of) the diagonal, this map is a one-to-one correspondence.

However, over points on the diagonal the fibers are $\mathbb{P}^1$ corresponding to the directions in which two points can approach each other in $\mathbb{C}^2$. As a matter of fact, the symmetric power $S^2\ \mathbb{C}^2$ has singularities and the Hilbert-Chow map $Hilb_2 \overset{\pi}{\twoheadrightarrow} S^2\ \mathbb{C}^2$ is a *resolution of singularities.*

Theorem 1.11 *Let* $rep_\alpha\ \mathbb{M} \xrightarrow{\mu} M_n(\mathbb{C})$ *be the moment map, then*

$$Hilb_n \simeq (\mu^{-1}(0) \cap rep^s_\alpha\ \mathbb{M})/GL_n(\mathbb{C})$$

and is therefore a complex manifold of dimension $2n$.

Proof. We identify the triples $(X, Y, u) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n$ such that u is a cyclic vector of (X, Y) and $[X, Y] = 0$ with the subspace

$$\{(X, Y, u, \underline{0})\ \mid\ [X, Y] = 0 \quad \text{and } u \text{ is cyclic }\} \hookrightarrow \mathsf{rep}^s_\alpha\ \mathbb{M}$$

which is clearly contained in $\mu^{-1}(0)$. To prove the converse inclusion assume that (X, Y, u, v) is a cyclic quadruple such that

$$[X, Y] + uv = 0.$$

Let $m(x, y)$ be any word in the noncommuting variables x and y. We claim that

$$v.m(X, Y).u = 0.$$

We will prove this by induction on the length $l(m)$ of the word $m(x, y)$. When $l(m) = 0$ then $l(x, y) = 1$ and we have

$$v.l(X, Y).u = v.u = tr(u.v) = tr([X, Y]) = 0.$$

Assume we proved the claim for all words of length $< l$ and take a word of the form $m(x, y) = m_1(x, y)yxm_2(x, y)$ with $l(m_1) + l(m_2) + 2 = l$. Then, we have

$$\begin{aligned} wm(X,Y) &= wm_1(X,Y)YXm_2(X,Y) \\ &= wm_1(X,Y)([Y,X]+XY)m_2(X,Y) \\ &= (wm_1(X,Y)v).wm_2(X,Y) + wm_1(X,Y)XYm_2(X,Y) \\ &= wm_1(X,Y)XYm_2(X,Y) \end{aligned}$$

where we used the induction hypotheses in the last equality (the bracketed term vanishes).

Hence we can reorder the terms in $m(x,y)$ if necessary and have that $wm(X,Y) = wX^{l_1}Y^{l_2}$ with $l_1 + l_2 = l$ and l_1 the number of occurrences of x in $m(x,y)$. Hence, we have to prove the claim for $X^{l_1}Y^{l_2}$.

$$\begin{aligned}
wX^{l_1}Y^{l_2}v = & \quad tr(X^{l_1}Y^{l_2}vw) \\
= & \quad -tr(X^{l_1}Y^{l_2}[X,Y]) \\
= & \quad -tr([X^{l_1}Y^{l_2},X]Y) \\
= & \quad -tr(X^{l_1}[Y^{l_2},X]Y) \\
= & \quad -\sum_{i=0}^{l_2-1} tr(X^{l_1}Y^i[Y,X]Y^{l_2-i}) \\
= & \quad -\sum_{i=0}^{l_2-1} tr(Y^{l_2-i}X^{l_1}Y^i[Y,X] \\
= & \quad -\sum_{i=0}^{l_2-1} tr(Y^{l_2-i}X^{l_1}Y^i v.w \\
= & \quad -\sum_{i=0}^{l_2-1} wY^{m_2-i}X^{l_1}Y^i v
\end{aligned}$$

But we have seen that $wY^{l_2-i}X^{l_1}Y^i = wX^{l_1}Y^{l_2}$ hence the above implies that $wX^{l_1}Y^{l_2}v = -l_2 wX^{l_1}Y^{l_2}v$. But then $wX^{l_1}Y^{l_2}v = 0$, proving the claim.

Consequently, $w.\mathbb{C}\langle X,Y\rangle.v = 0$ and by the cyclicity condition we have $w.\mathbb{C}^n = 0$ hence $w = 0$. Finally, as $v.w + [X,Y] = 0$ this implies that $[X,Y] = 0$ and we can identify the fiber $\mu^{-1}(0)$ with the indicated subspace. From this the result follows. $\square$

We can cover the subset $(X,Y,u,\underline{0})$ such that $[X,Y] = 0$ and u a cyclic vector by their intersections with the *rep* (σ) for σ a Hilbert n-stair. In particular, we can cover $Hilb_n$ by open subsets

$$Hilb_n\ (\sigma) = \{(X,Y,u,\underline{0}) \quad \text{in } \sigma\text{-standard form such that } [X,Y] = 0\}.$$

Example 1.12 The Hilbert scheme $Hilb_2$.

Consider $Hilb_2$ (). Because

$$[\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}, \begin{bmatrix} c & d \\ e & f \end{bmatrix}] = \begin{bmatrix} ae-d & af-ac-bd \\ c+be-f & d-ae \end{bmatrix}$$

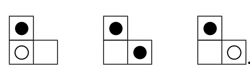
this subset can be identified with $\mathbb{C}^4$ using the equalities

$$d = ar \quad \text{and} f = c + be.$$

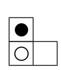
Similarly, $Hilb_2$ () $\simeq \mathbb{C}^4$.

Example 1.13 The Hilbert scheme $Hilb_3$.

Up to change of colors there are three 3-stairs to consider



We claim that

$Hilb_3$ () $\simeq \mathbb{C}^6$.

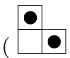
For consider the commutator

$$[\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}, \begin{bmatrix} 0 & g & h \\ 0 & i & j \\ 1 & k & k \end{bmatrix}] = \begin{bmatrix} b-g & ai+bk-cg-eh & aj+bl-dg-fh \\ d-i & g+dk-ej & h+cj+dl-di-fj \\ f-k & -a-ck-el+ei+fk & -b-dk+ej \end{bmatrix}$$

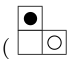
Taking the Groebner basis of these relations one finds the following relations

$$\begin{cases} f & = k \\ g & = ej - ik \\ d & = i \\ h & = i^2 - cj + jk - il \\ b & = g \\ a & = ei - ck + k^2 - el \end{cases}$$

from which the claim follows. In a similar manner one proves that

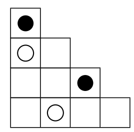
$Hilb_3$ () $\simeq \mathbb{C}^6$.

However, the situation for

$Hilb_3$ ()

is more complicated.

Observe that some of these intersections may be empty. For example, for the Hilbert 5-stair

$Hilb_5$ () $= \emptyset$

Indeed, the associated series of words is

$$\{1, x, y, xy, yx\}$$

whence $\sigma(X, Y, u, \underline{0}) = 0$ whenever $[X, Y] = 0$. Hence all Hilbert stairs σ containing this stair (that is, if we recover the 5-stair after removing certain rows and columns) satisfy $Hilb_n\ (\sigma) = \emptyset$.

We have shown that $Hilb_n$ is a manifold of dimension $2n$. A priori it may have many connected components (all of dimension $2n$). We will now show that $Hilb_n$ is *connected.*

Theorem 1.14 *The Hilbert scheme $Hilb_n$ of n points in $\mathbb{C}^2$ is a complex connected manifold of dimension $2n$.*

Proof. The symmetric power $S^n\ \mathbb{C}^1$ parametrizes *sets* of n-points on the line $\mathbb{C}^1$ and can be identified with $\mathbb{C}^n$. Consider the map

$$Hilb_n \xrightarrow{\pi} S^n\ \mathbb{C}^1$$

defined by mapping a cyclic triple (X, Y, u) with $[X, Y] = 0$ in the orbit corresponding to the point of $Hilb_n$ to the set $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues of X. Observe that this map does not depend on the point chosen in the orbit.

Let Δ be the *big diagonal* in $S^n\ \mathbb{C}^1$, that is, $S^n\ \mathbb{C}^1 - \Delta$ is the space of all sets of n distinct points from $\mathbb{C}^1$. Clearly, $S^n\ \mathbb{C}^1 - \Delta$ is a connected n-dimensional manifold. We claim that

$$\pi^{-1}(S^n\ \mathbb{C}^1 - \Delta) \simeq (S^n\ \mathbb{C}^1 - \Delta) \times \mathbb{C}^n$$

and hence is connected.

Indeed, take a matrix X with n distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. We may diagonalize X. But then, as

$$[\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \begin{bmatrix} y_{11} & \ldots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \ldots & y_{nn} \end{bmatrix}] = \begin{bmatrix} (\lambda_1 - \lambda_1)y_{11} & \ldots & (\lambda_1 - \lambda_n)y_{1n} \\ \vdots & & \vdots \\ (\lambda_n - \lambda_1)y_{n1} & \ldots & (\lambda_n - \lambda_n)y_{nn} \end{bmatrix}$$

we see that also Y must be a diagonal matrix with entries $(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ where $\mu_i = y_{ii}$. But then the cyclicity condition implies that all coordinates of v must be non-zero.

Now, the stabilizer subgroup of the commuting (diagonal) matrix-pair (X, Y) is the *maximal torus* $T_n = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ of diagonal invertible $n \times n$ matrices.

Using its action we may assume that all coordinates of v are equal to 1. That is, the points in $\pi^{-1}(\{\lambda_1,\ldots,\lambda_n\})$ with $\lambda_i \neq \lambda_j$ have unique (up to permutation as before) representatives of the form

$$\left(\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}, \begin{bmatrix}\mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n\end{bmatrix}, \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\right)$$

that is $\pi^{-1}(\{\lambda_1,\ldots,\lambda_n\}$ can be identified with $\mathbb{C}^n$, proving the claim.

Next, we claim that *all* the fibers of π have dimension at most n. Let $\{\lambda_1,\ldots,\lambda_n\} \in S^n\ \mathbb{C}^1$ then there are only finitely many X in Jordan normalform with eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$. Fix such an X, then the subset $T(X)$ of cyclic triples (X,Y,u) with $[X,Y]=0$ has dimension at most $n + dim\ C(X)$ where $C(X)$ is the centralizer of X in $M_n(\mathbb{C})$, that is,

$$C(X) = \{Y \in M_n(\mathbb{C}) \mid XY = YX\}.$$

The stabilizer subgroup $Stab(X) = \{g \in GL_n(\mathbb{C}) \mid gXg^{-1} = X\}$ is an open subset of the vectorspace $C(X)$ and acts freely on the subset $T(X)$ because the action of $GL_n(\mathbb{C})$ on $\mu^{-1}(0) \cap rep^s_\alpha\ \mathbb{M}$ has trivial stabilizers.

But then, the orbitspace for the $Stab(X)$-action on $T(X)$ has dimension at most

$$n + dim\ C(X) - dim\ Stab(X) = n.$$

As we only have to consider finitely many X this proves the claim. The diagonal Δ has dimension $n-1$ in $S^n\ \mathbb{C}^1$ and hence by the foregoing we know that the dimension of $\pi^{-1}(\Delta)$ is at most $2n-1$. Let H be the connected component of $Hilb_n$ containing the connected subset $\pi^{-1}(S^n\ \mathbb{C}^1 - \Delta)$. If $\pi^{-1}(\Delta)$ were not entirely contained in H, then $Hilb_n$ would have a component of dimension less than $2n$, which we proved not to be the case. This finishes the proof. □

1.5 The phase space $Calo_n$.

Recall that Calogero quadruples were defined to be

$$CALO_n = \{(X,Y,u,v) \mid [X,Y] + u.v = \mathbb{1}_n\}$$

and that the phase space of collisions of n Calogero particles is the orbit space $Calo_n = CALO_n/GL_n(\mathbb{C})$.

Theorem 1.15 *Let* $\mathsf{rep}_\alpha\ \mathbb{M} \xrightarrow{\mu} M_n(\mathbb{C})$ *be the moment map, then*

$$\mathsf{Calo}_n \simeq \mu^{-1}(\mathbb{1}_n)/GL_n(\mathbb{C}) = (\mu^{-1}(\mathbb{1}_n) \cap rep^s_\alpha\ \mathbb{M})/GL_n(\mathbb{C})$$

and is therefore a complex manifold of dimension $2n$.

Proof. The result will follow if we can prove that any Calogero quadruple (X,Y,u,v) has the property that u is a cyclic vector, that is, lies in $rep^s_\alpha\ \mathbb{M}$.

Assume that U is a subspace of $\mathbb{C}^n$ stable under X and Y and containing u. U is then also stable under left multiplication with the matrix

$$A = [Y,X] + \mathbb{1}_n$$

and we have that $tr(A \mid U) = tr(\mathbb{1}_n \mid U) = dim\ U$. On the other hand, $A = u.v$ and therefore

$$A. \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} . \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} . \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = (\sum_{i=1}^{n} v_i c_i) \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Hence, if we take a basis for U containing u, then we have that

$$tr(A \mid U) = a$$

where $A.u = au$, that is $a = \sum u_i v_i$.

But then, $tr(A \mid U) = dim\ U$ is independent of the choice of U. Now, $\mathbb{C}^n$ is clearly a subspace stable under X and Y and containing u, so we must have that $a = n$ and so the only subspace U possible is $\mathbb{C}^n$ proving cyclicity of u with respect to the matrix-couple (X, Y). $\square$

Again, it follows that we can cover the phase space $Calo_n$ by open subsets

$$Calo_n\ (\sigma) = \{(X, Y, u, v) \quad \text{in } \sigma\text{-standard form such that } [X.Y] + u.v = \mathbb{1}_n\ \}$$

where σ runs over the Hilbert n-stairs.

Example 1.16 The phase space $Calo_2$.

Consider $Calo_2$ ($\boxed{\bullet}$). Because

$$[\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}, \begin{bmatrix} c & d \\ e & f \end{bmatrix}] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \begin{bmatrix} g & h \end{bmatrix} - \mathbb{1}_2 = \begin{bmatrix} g - d + ae - 1 & h + af - ac - bd \\ c - f + be & d - ae - 1 \end{bmatrix}$$

We obtain after taking Groebner bases that the defining equations are

$$\begin{cases} g & = 2 \\ h & = b \\ f & = c + eh \\ d & = 1 + ae \end{cases}$$

In particular we find

$$Calo_2\ (\boxed{\bullet}) = \{(\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}, \begin{bmatrix} c & 1 + ae \\ e & c + be \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & b \end{bmatrix}) \mid a, b, c, e \in \mathbb{C}\} \simeq \mathbb{C}^4$$

and a similar description holds for $Calo_2$ ($\boxed{\circ}$).

Example 1.17 The phase space $Calo_3$.

We claim that

$$Calo_3\ (\begin{array}{|c|c} \hline \bullet & \\ \hline \circ & \multicolumn{1}{c|}{} \\ \hline \end{array}) \simeq \mathbb{C}^6$$

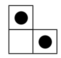
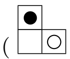
For, if we compute the 3×3 matrix

$$[\begin{bmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{bmatrix}, \begin{bmatrix} 0 & g & h \\ 0 & i & j \\ 1 & k & l \end{bmatrix}] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} . \begin{bmatrix} m & n & o \end{bmatrix} - \mathbb{1}_3$$

then the Groebner basis for its entries gives the following defining equations

$$\begin{cases} m & = 3 \\ n & = c + k \\ o & = i + l \\ f & = k \\ d & = o - l \\ g & = 2 + b \\ l & = g - ej - kl + ko \\ h & = 2jk + 2l^2 - jn - 3lo + o^2 \\ a & = 2k^2 - 2el - kn + eo \end{cases}$$

In a similar manner one can show that

$Calo_3$ () $\simeq \mathbb{C}^6$ but $Calo_3$ ()

is again more difficult to describe.

We will prove that $Calo_n$ is connected by a strategy similar to that used for $Hilb_n$.

Proposition 1.18 *Let $(X, Y, u, v) \in CALO_n$ and suppose that X is diagonalizable. Then*

1. *all eigenvalues of X are distinct, and*
2. *the $GL_n(\mathbb{C})$-orbit contains a representative such that*

$$X = \begin{bmatrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_n \end{bmatrix} \qquad Y = \begin{bmatrix} \alpha_1 & \frac{1}{\lambda_1-\lambda_2} & \cdots & \cdots & \frac{1}{\lambda_1-\lambda_n} \\ \frac{1}{\lambda_2-\lambda_1} & \alpha_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{\lambda_{n-1}-\lambda_n} \\ \frac{1}{\lambda_n-\lambda_1} & \cdots & \cdots & \frac{1}{\lambda_n-\lambda_{n-1}} & \alpha_n \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$$

and this representative is unique up to permutation of the n couples (λ_i, α_i).

Proof. Choose a representative with X a diagonal matrix as indicated. Equating the diagonal entries in $[X, Y] + u.v = \mathbb{1}_n$ we obtain that for all $1 \leq i \leq n$ we have $u_i v_i = 1$. Hence, none of the entries of

$$[Y, X] + \mathbb{1}_n = u.v$$

is zero. Consequently, by equating the (i, j)-entry it follows that $\lambda_i \neq \lambda_j$ for $i \neq j$.

The representative with X a diagonal matrix is therefore unique up to the action of a diagonal matrix D and of a permutation. The freedom in D allows us to normalize u and v as indicated, the effect of the permutation is described in the last sentence.

Finally, the precise form of Y can be calculated from the normalized forms of X, u and v and the equation $[X, Y] + u.v = \mathbb{1}_n$. $\square$

Consider the map

$$Calo_n \xrightarrow{\pi} S^n\ \mathbb{C}^1$$

by mapping a point in $CALO_n$ to the set of the eigenvalues of X (and as this does not depend on the point in the orbit this map factors through $Calo_n$).

The foregoing proposition describe $\pi^{-1}(S^n\ \mathbb{C}^1 - \Delta)$ where Δ is the big diagonal and hence the subset of $Calo_n$ with X diagonalizable is connected as it coincides with $(S^n\ \mathbb{C}^1 - \Delta) \times \mathbb{C}^n$. The identification is made through the parameters λ_i and α_i of the proposition.

Theorem 1.19 *The phase space $Calo_n$ is a complex connected manifold of dimension $2n$.*

Proof. Reasoning as in the proof for the Hilbert scheme, the result will follow once we prove that *all* the fibers of π have dimension at most n.

Consider the projection to the last two factors

$$CALO_n \xrightarrow{p} \mathbb{C}^{2n}$$

For a fixed matrix $X \in M_n(\mathbb{C})$ we claim that the subset $CALO_n(X)$ (Calogero quadruples with fixed X) maps under p into an n-dimensional subvariety of $\mathbb{C}^{2n}$.

Because p is $GL_n(\mathbb{C})$-equivariant we may assume that X is in *Jordan normal form*. Consider first the case of one block, that is

$$X = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

Then, in $[Y, X]$ every diagonal below the main diagonal has entries that add up to 0 as one verifies. On the other hand, for a rank one matrix, the lowest nonvanishing diagonal can have just one non-zero entry.

Therefore, the rank one matrix $[Y, X] + \mathbb{1}_n = u.v$ is upper triangular and there is just one non-zero entry (which must be equal to n by taking traces) on the main diagonal. If the first non-zero entry of v occurs at place i then the last non-zero entry of u must also occur in place i and the product of both must be equal to n. In this way, the possible pairs (u, v) fall into n families (indexed by i) each of dimension n. This proves the claim in the case of one Jordan block.

For the general case $X = \oplus_j X_j$ one writes Y, u and v in the corresponding block forms, and one sees that

$$[Y_{jj}, X_j] + \mathbb{1}_n = u_j.v_j$$

and one repeats the above argument for each of the blocks, proving the claim.

Now, define $Calo_n(\mathcal{O})$ to be the subset of $Calo_n$ represented by quadruples (X, Y, u, v) such that X belongs to a fixed conjugacy class $\mathcal{O}$ in $M_n(\mathbb{C})$. We claim that $Calo_n(\mathcal{O})$ has dimension at most n.

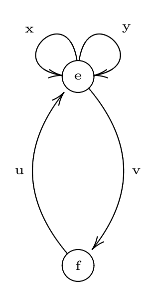
Fix a matrix $X \in \mathcal{O}$, then $Calo_n(\mathcal{O}) = CALO_n(X)/G$ where G is the centralizer of X in $GL_n(\mathbb{C})$. Now, the part of $CALO_n(X)$ lying over a fixed $(u, v) \in \mathbb{C}^{2n}$ is parametrized by the Lie algebra $Lie(G)$ and so by the foregoing claim

$$dim\ CALO_n(X) \leq n + dim\ G$$

Finally, the action of G is free and we have proved that all the fibers of π have dimension at most n. $\square$

1.6 The noncommutative smooth algebra $\mathbb{M}$.

We will now bring in some noncommutative algebras. Consider the following *quiver* (that is, directed graph) on two vertices



We define $\mathbb{M}$ to be the *path algebra* of this quiver. That is, as a $\mathbb{C}$-vectorspace $\mathbb{M}$ has as basis the oriented paths in the quiver, including those of length zero corresponding to the two vertices. We agree that we write paths from right to left (as we do with compositions of morphisms). To each vertex there is a path of length zero. An associative algebra structure on $\mathbb{M}$ is induced by concatenation of paths when possible and zero otherwise.

That is, $\mathbb{M}$ is the algebra on 6 noncommuting generators

$$\begin{cases} e, f & \text{the paths of length zero} \\ x, y, u, v & \text{the paths of length one} \end{cases}$$

Concatenation of paths induces the following defining relations for $\mathbb{M}$

$$\begin{cases} e^2 = e \quad f^2 = f \quad e + f = 1 \\ e.x = x \quad e.y = y \quad e.u = u \quad e.v = 0 \\ x.e = x \quad y.e = y \quad u.e = 0 \quad v.e = v \\ f.x = 0 \quad f.y = 0 \quad f.u = 0 \quad f.v = v \\ x.f = 0 \quad y.f = 0 \quad u.f = u \quad v.f = 0 \\ x.v = 0 \quad u.x = 0 \\ y.v = 0 \quad u.y = 0 \\ u.u = 0 \quad v.v = 0 \end{cases}$$

Horrible as these relations may seem, the algebra $\mathbb{M}$ has one important property, it is *(formally) smooth*. That is, if A is any $\mathbb{C}$-algebra having a twosided ideal I with $I^2 = 0$, then we can lift $\mathbb{C}$-algebra morphisms

$$\begin{tikzcd} \mathbb{M} \arrow[r, dotted, "\exists\tilde{\phi}"] \arrow[dr, "\phi"'] & A \arrow[d, "\pi"] \\ & \frac{A}{I} \end{tikzcd}$$

from the quotient to A. The relevance of this notion will become clear when we study differential forms and connections on noncommutative manifolds.

This lifting property can be seen as follows. Let $a \in A$ be such that $\pi(a) = \phi(e)$, then one verifies that

$$E = (2 - a)^2 a^2$$

is an idempotent of A such that $\pi(E) = \phi(e)$. Define a lift $\tilde{\phi}$ by sending $e \mapsto E$, $f \mapsto F = 1 - E$ and

$$\begin{cases} x \mapsto \text{any element of } E.(\phi(x) + I).E, \\ y \mapsto \text{any element of } E.(\phi(y) + I).E, \\ u \mapsto \text{any element of } E.(\phi(u) + I).F, \\ v \mapsto \text{any element of } F.(\phi(v) + I).E. \end{cases}$$

and one immediately verifies that all the relations holding in $\mathbb{M}$ are preserved under $\tilde{\phi}$ whence $\tilde{\phi}$ is an algebra morphism lifting ϕ.

A *representation* of a quiver assigns to each vertex a finite dimensional vectorspace and to each arrow a linear map between the corresponding vertex-spaces. The *dimension-vector* of a representation is then the integral vector containing the dimensions of the vertex-spaces. Fixing a dimension-vector and a basis in each vertex-space we see that the representations form an affine space.

For the quiver described above, the set of representations of dimension vector $\alpha = (m, n)$ can be identified with the affine space

$$rep_\alpha(\mathbb{M}) = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_{n\times m}(\mathbb{C}) \oplus M_{m\times n}(\mathbb{C})$$

the factors corresponding respectively to the arrows x, y, u and v. On this affine space there is a natural action of the algebraic group

$$GL(\alpha) = GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$$

given by base-change in the vertex-spaces. That is, $(g, h) \in GL(\alpha)$ acts as

$$(g, h).(X, Y, U, V) = (gXg^{-1}, gYg^{-1}, gUh^{-1}, hVg^{-1}).$$

Two representations are said to be *isomorphic* if and only if they belong to the same $GL(\alpha)$-orbit.

We will now relate the vectorspaces $rep_\alpha\ \mathbb{M}$ and the action of $GL(\alpha)$ on it to the study of finite dimensional *representations* of $\mathbb{M}$. For an integer $\overline{n} \in \mathbb{N}$, an $\overline{n}$-dimensional representation of $\mathbb{M}$ is a $\mathbb{C}$-algebra morphism

$$\mathbb{M} \xrightarrow{\phi} M_{\overline{n}}(\mathbb{C})$$

and two $\overline{n}$-dimensional representations ϕ_1 and ϕ_2 are said to be isomorphic (or equivalent) iff there is a $g \in GL_{\overline{n}}(\mathbb{C})$ such that the diagram below commutes

$$\begin{tikzcd} & \mathbb{M} \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ M_{\overline{n}}(\mathbb{C}) \arrow[rr, "c_g"] & & M_{\overline{n}}(\mathbb{C}) \end{tikzcd}$$

where c_g is conjugation by g on $M_{\overline{n}}(\mathbb{C})$. Because $\mathbb{M}$ is an affine $\mathbb{C}$-algebra, the set of all $\overline{n}$-dimensional representations is an *affine* variety $rep_{\overline{n}}\ \mathbb{M}$. Indeed, any representation is determined by the images of the generators e, f, u, v, x and y. That is,

$$rep_{\overline{n}}\ \mathbb{M} \hookrightarrow M_{\overline{n}}(\mathbb{C})^{\oplus 6}$$

is the closed subvariety where the ideal of relations is generated by the entries of the matrix identities determined by the defining relations for $\mathbb{M}$.

Clearly, conjugation on $M_{\overline{n}}(\mathbb{C})$ defines an action of $GL_{\overline{n}}(\mathbb{C})$ on the affine variety $rep_{\overline{n}}\ \mathbb{M}$. For $\alpha = (n, m)$ let $\overline{n} = n + m$ and consider the diagonal embedding of $GL(\alpha)$ in $GL_{\overline{n}}(\mathbb{C})$

$$\begin{bmatrix} GL_n(\mathbb{C}) & 0 \\ 0 & GL_m(\mathbb{C}) \end{bmatrix} \hookrightarrow GL_{\overline{n}}(\mathbb{C}).$$

Using this embedding there is a natural $GL(\alpha)$ action on the product $GL_{\overline{n}}(\mathbb{C}) \times rep_\alpha\ \mathbb{M}$ given by

$$(g, h).(G, X, Y, U, V) = (G \begin{bmatrix} g^{-1} & 0 \\ 0 & h^{-1} \end{bmatrix}, gXg^{-1}, gYg^{-1}, gUh^{-1}, hVg^{-1})$$

and the space of $GL(\alpha)$-orbits is called the *associated fiber bundle* and denoted by

$$GL_{\overline{n}}(\mathbb{C}) \times^{GL(\alpha)} rep_\alpha\ \mathbb{M}$$

Left multiplication defines a $GL_{\overline{n}}(\mathbb{C})$-action on the product $GL_{\overline{n}}(\mathbb{C}) \times rep_\alpha\ \mathbb{M}$ which commutes with the action of $GL(\alpha)$ and hence induces an action on the associated fiber bundle.

Lemma 1.20 *The representation spaces $rep_{\overline{n}}\ \mathbb{M}$ are affine manifolds and*

$$rep_{\overline{n}}\ \mathbb{M} \simeq \bigsqcup_{\substack{\alpha=(n,m)\\ n+m=\overline{n}}} GL_{\overline{n}}(\mathbb{C}) \times^{GL(\alpha)} rep_\alpha\ \mathbb{M}$$

as manifolds with $GL_{\overline{n}}(\mathbb{C})$-action.

Proof. Given a representation $\mathbb{M} \xrightarrow{\phi} M_{\overline{n}}(\mathbb{C})$, the images $\phi(e)$ and $\phi(f)$ give an orthogonal decomposition of $\mathbb{1}_{\overline{n}}$ into idempotents. If the rank of $\phi(e)$ is n, then the rank of $\phi(f)$ is $m = \overline{n} - n$ and under simultaneous conjugation by an element $G \in GL_{\overline{n}}(\mathbb{C})$ we reduce to the case that

$$\phi(e) = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \phi(f) = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1}_m \end{bmatrix}$$

But then using equations such as $exe = x$ and $euf = u$ we deduce that the images of the other generators are of the following matrix shape in $M_{\overline{n}}(\mathbb{C})$

$$\phi(x) = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \quad \phi(y) = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \quad \phi(u) = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix} \quad \phi(v) = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}$$

and hence correspond to a point in $rep_\alpha\ \mathbb{M}$ for $\alpha = (n, m)$. The assertion is now easy to verify. $\square$

In chapter 5 we will prove that the representation spaces $rep_{\overline{n}}\ A$ of any formally smooth algebra are affine manifolds and even have an analytic local description by quiver representations.

For applications to Calogero particles and Hilbert schemes, we specialize to the case $m = 1$, that is, $\alpha = (n, 1)$ and $\overline{n} = n + 1$ and recover the vectorspace $rep_\alpha\ \mathbb{M}$ of the previous section.

Lemma 1.21 *There is a natural one-to-one correspondence between*

- *$GL_n \times \mathbb{C}^*$-orbits in $rep_\alpha\ \mathbb{M} = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$*
- *GL_n-orbits in $M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$ as in the previous section*

Proof. One implication is obvious by taking $(g, 1) \in GL(\alpha) = GL_n \times \mathbb{C}^*$. Conversely, assume that the quadruples (X, Y, u, v) and (X', Y', u', v') belong to the same $GL(\alpha)$-orbit in $rep_\alpha\ \mathbb{M}$. Then there is a $g \in GL_n$ and $\lambda \in \mathbb{C}^*$ such that

$$gXg^{-1} = X' \quad gYg^{-1} = Y' \quad gu\lambda^{-1} = u' \quad \text{and} \quad \lambda vg^{-1} = v'$$

Then, taking the matrix $g' = g.(\lambda^{-1}\mathbb{1}_n) \in GL_n$ we see that these quadruples also belong to the same GL_n-orbit. $\square$

Therefore, we would like to construct an orbit space for the $GL(\alpha)$-action on $rep_\alpha\ \mathbb{M}$. However, such an orbit-space would have horrible topological properties (such as being non-Hausdorff) as there are $GL(\alpha)$-orbits which are not closed. For example, if $\alpha = (2, 1)$ consider the representation

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

and consider the elements $G_t = (\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}, 1) \in GL(\alpha)$, then the action on the above quadruple gives the representation

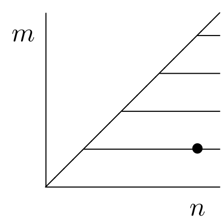
$$X = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

whence if $t \longrightarrow 0$ we see that the zero representation lies in the closure of the $GL(\alpha)$-orbit.

1.7 Invariant theory and rep_α $\mathbb{M}$.

Invariant theory provides us with the best Hausdorff approximation to the orbit space problem, that is a classifying space for the closed orbits. We will prove in chapter 4 that closed $GL_{\overline{n}}(\mathbb{C})$-orbits in the representation space $rep_{\overline{n}}$ A for an arbitrary affine $\mathbb{C}$-algebra A are in one-to-one correspondence with isomorphism classes of *semi-simple* $\overline{n}$-dimensional representations of A. Recall that a representation is *simple* if it has no proper (non-zero) subrepresentations and is *semi-simple* if it is the direct sum of simple representations.

Our first job is to find a criterium on the dimension vector $\alpha = (n, m)$ to ensure that rep_α $\mathbb{M}$ contains simple representations. A necessary condition is $m \leq n$



Indeed, if $m > n$ then any representation (X, Y, U, V) contains as non-trivial subrepresentation the trivial representation (all matrices zero-matrices) of dimension vector $(dim\ Ker\ U, 0)$. In chapter 6 we will give a combinatorial description of the dimension vectors of simple representations of quivers which implies that for the quiver under consideration this necessary condition is also sufficient. In particular, for $\alpha = (1, n)$ there is an open submanifold of rep_α $\mathbb{M}$ consisting of simple representations.

Invariant theory learns us that closed orbits can be separated by *invariant polynomial functions*. We will focuss here on the special case of interest $\alpha = (n, 1)$ although the arguments hold more generally as we will prove in chapter 4. The *coordinate ring* of rep_α $\mathbb{M}$ is the polynomial algebra

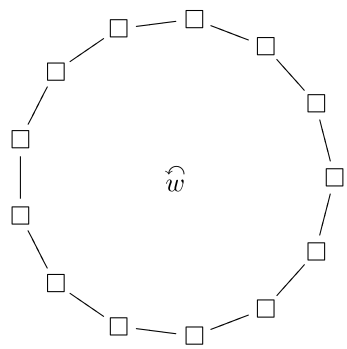
$$\mathbb{C}[rep_\alpha\ \mathbb{M}] = \mathbb{C}[x_{11}, \ldots, x_{nn}, y_{11}, \ldots, y_{nn}, u_1, \ldots, u_n, v_1, \ldots, v_n]$$

where the x_{ij}, y_{ij}, u_i and v_j with $1 \leq i, j \leq n$ are the coordinate functions of the matrices (X, Y, u, v). The group $GL(\alpha)$ acts on this algebra by automorphisms as follows, let $G = (g, \lambda) \in GL(\alpha) = GL_n \times \mathbb{C}^*$ then ψ_G is the automorphism sending

- x_{ij} to the (i, j) entry of the matrix $g \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix} g^{-1}$,
- y_{ij} to the (i, j) entry of the matrix $g \begin{bmatrix} y_{11} & \ldots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \ldots & y_{nn} \end{bmatrix} g^{-1}$,

- u_i to the $(i,1)$ entry of the matrix $g \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \lambda^{-1}$,
- v_j to the $(1,j)$ entry of the matrix $\lambda \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} g^{-1}$.

The ring of polynomial invariants $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$ is the subalgebra consisting of polynomials P such that $\psi_G(P) = P$ for all $G \in GL(\alpha)$. We will prove in chapter 5 that this algebra is generated by *traces along oriented cycles in the quiver*. That is, consider all *necklace words* w



where each bead is one of the following $n \times n$ matrices

$$\boxed{\bullet} = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \quad \boxed{\circ} = \begin{bmatrix} y_{11} & \dots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \dots & y_{nn} \end{bmatrix} \quad \boxed{\blacktriangledown} = \begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ \vdots & & \vdots \\ u_nv_1 & \dots & u_nv_n \end{bmatrix}$$

Multiplying these bead-matrices cyclicly in the indicated orientation and taking the trace of the $n \times n$ matrix obtained gives a polynomial $tr(w)$ of $\mathbb{C}[rep_\alpha\ \mathbb{M}]$ which is clearly left invariant under the $GL(\alpha)$-action. The assertion is that these invariants generate all the invariant functions. We will even show that it suffices to take necklace words having at most $(n+1)^2+1$ beads.

Assume there are s distinct necklace words of length $\leq (n+1)^2+1$, then we can evaluate $tr(w)$ at a representation $(X, Y, u, v) \in rep_\alpha\ \mathbb{M}$ by substituting the entries for the coordinate functions and obtain a map

$$rep_\alpha\ \mathbb{M} \xrightarrow{\pi} \mathbb{C}^s$$

The image of π will be shown to be the affine variety corresponding to the ring of invariant polynomials. It is called the *quotient variety* and is denoted $rep_\alpha\ \mathbb{M}/GL(\alpha)$. If $\xi \in Im\ \pi$ then $\pi^{-1}(\xi)$ contains a unique closed orbit. In particular, if two semi-simple representations (X_1, Y_1, u_1, v_1) and (X_2, Y_2, u_2, v_2) have all their necklace-invariants equal then they belong to the same orbit.

These quotients varieties are in general not manifolds. In fact, we will give in chapter 6 combinatorial tools to determine the singularities and to describe the analytic local structure of the quotient variety near these singularities. Applying these results we will see that $rep_\alpha\ \mathbb{M}/GL(\alpha)$ always has singularities except in the trivial case $\alpha = (1,1)$ where the quotient variety is easily seen to be $\mathbb{C}^3$.

When studying $rep_\alpha\ \mathbb{M}$ for a specific dimension vector $\alpha = (n, 1)$, working with $\mathbb{M}$ is overdoing things. We will construct another noncommutative algebra $\mathbb{M}(n)$ which is a finite module over the ring of polynomial invariants $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$. We have a canonical morphism $\mathbb{M} \xrightarrow{t_n} \mathbb{M}(n)$ with the property that α-dimensional representation of $\mathbb{M}$ (with the images of e and f fixed) factors through t_n. One way to define $\mathbb{M}(n)$ is as the ring of *equivariant* maps from $rep_\alpha\ \mathbb{M}$ to $M_{n+1}(\mathbb{C})$.

Embed $GL(\alpha)$ diagonally in $GL_{n+1}(\mathbb{C})$, that is

$$GL(\alpha) = \begin{bmatrix} GL_n(\mathbb{C}) & 0 \\ 0 & \mathbb{C}^* \end{bmatrix} \hookrightarrow GL_{n+1}(\mathbb{C})$$

then $GL(\alpha)$ acts on $M_{n+1}(\mathbb{C})$ via the conjugation action of $GL_{n+1}(\mathbb{C})$. A polynomial map

$$rep_\alpha\ \mathbb{M} \xrightarrow{p} M_{n+1}(\mathbb{C})$$

is said to be $GL(\alpha)$-equivariant if it is compatible with the $GL(\alpha)$-action on source and target space. That is, for all $(g, \lambda) \in GL(\alpha)$ and all $(X, Y, u, v) \in rep_\alpha(\mathbb{C})$ we have

$$p(gXg^{-1}, gYg^{-1}, gu\lambda^{-1}, \lambda vg^{-1}) = \begin{bmatrix} g & 0 \\ 0 & \lambda \end{bmatrix} p(X, Y, u, v) \begin{bmatrix} g^{-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

Addition and multiplication in the target space $M_{n+1}(\mathbb{C})$ define a $\mathbb{C}$-algebra structure on the equivariant maps.

A concrete realization of this ring $\mathbb{M}(n)$ can be given as follows. Consider the matrix algebra $M_{n+1}(\mathbb{C}[rep_\alpha\ \mathbb{M}])$, then $\mathbb{M}(n)$ is the $\mathbb{C}$-subalgebra generated by the polynomial invariants $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$ (embedded as scalar matrices) and the following matrices

$$e_n = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad f_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad x_n = \begin{bmatrix} x_{11} & \dots & x_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \dots & x_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$y_n = \begin{bmatrix} y_{11} & \dots & y_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ y_{n1} & \dots & y_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad u_n = \begin{bmatrix} 0 & \dots & 0 & u_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & u_n \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad v_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ v_1 & \dots & v_n & 0 \end{bmatrix}$$

Clearly, $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$ is a central subring of $\mathbb{M}(n)$ and from the generation of the polynomial invariants by traces of necklace words we see that the restriction of the usual trace tr on $M_{n+1}(\mathbb{C}[rep_\alpha\ \mathbb{M}])$ to the subring $\mathbb{M}(n)$ defines a *trace map* on $\mathbb{M}(n)$, that is a map t

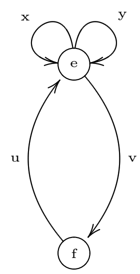
$$\begin{tikzcd} \mathbb{M}(n) \arrow[rr, "t"] \arrow[dr, "tr"'] & & \mathbb{M}(n) \\ & \mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)} \arrow[ur, hook] & \end{tikzcd}$$

satisfying $t(ab) = t(ba), t(a)b = bt(a), t(t(a)b) = t(a)t(b)$ and $t(1) = n$. It is a rather straightforward consequence of the Cayley-Hamilton equation that $\mathbb{M}(n)$ is a finite module over its center which is equal to $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$. The factorizing property for representations in $rep_\alpha\ \mathbb{M}$ mentioned above follows.

1.8 de Rham cohomology for $\mathbb{M}$.

Formally smooth algebras should be viewed as the coordinate rings of affine noncommutative manifolds. Associated to them is a well developed theory of differential forms, de Rham cohomology groups, connections and so on. We will introduce and study all these concepts in chapter 9 in great detail. Here, we merely sketch these notions for the formally smooth algebra $\mathbb{M}$ as they are crucial in the proof of Ginzburg's result.

Recall that $\mathbb{M}$ (resp. $\mathbb{C} \times \mathbb{C}$) is the path algebra of the quiver



resp.

As the images of $S = \mathbb{C} \times \mathbb{C}$ is fixed in any representation of $rep_\alpha\ \mathbb{M}$, we are interested in *relative* differential forms of $\mathbb{M}$ with respect to the subalgebra $\mathbb{C} \times \mathbb{C}$. Let $\overline{\mathbb{M}}$ be the S-bimodule cokernel of the inclusion $S \hookrightarrow \mathbb{M}$, then we define the space of relative differential forms of degree n to be

$$\Omega^n_{rel}\ \mathbb{M} = \mathbb{M} \otimes_S \underbrace{\overline{\mathbb{M}} \otimes_S \ldots \otimes_S \overline{\mathbb{M}}}_{n}$$

We will denote a tensor $a_0 \otimes \overline{a}_1 \otimes \ldots \otimes \overline{a}_n \in \Omega^n_{rel}\ \mathbb{M}$ with all $a_i \in \mathbb{M}$ by $a_0 da_1 \ldots da_n$. It is easy to see that a basis for $\Omega^n_{rel}\ \mathbb{M}$ is given by the elements

$$p_0 dp_1 \ldots dp_n$$

where p_i is an oriented path in the quiver such that *length* $p_0 \geq 0$ and *length* $p_i \geq 1$ for $1 \leq i \leq n$ and such that the starting point of p_i is the endpoint of p_{i+1} for all $1 \leq i \leq n-1$. We define an algebra structure on $\Omega_{rel}\ \mathbb{M} = \oplus_n\ \Omega^n_{rel}\ \mathbb{M}$ by the product rule

$$\begin{gathered}(a_0 da_1 \ldots da_n)(a_{n+1} da_{n+2} \ldots da_m) = \\ \sum_{i=0}^n (-1)^{n-1} a_0 da_1 \ldots da_{i-1} d(a_i a_{i+1}) da_{i+2} \ldots da_m\end{gathered}$$

and we make this into a *differential graded* $\mathbb{C}$-*algebra* by defining a differential of degree one

$$\ldots \xrightarrow{d} \Omega^{n-1}_{rel}\ \mathbb{M} \xrightarrow{d} \Omega^n_{rel}\ \mathbb{M} \xrightarrow{d} \Omega^{n+1}_{rel}\ \mathbb{M} \xrightarrow{d} \ldots$$

by the rule that $d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n$. Clearly $d \circ d = 0$ and d is a super-derivation meaning that

$$d(rs) = (dr)s + (-1)^i r(ds)$$

when $r \in \Omega^i_{rel}\ \mathbb{M}$. For $\omega \in \Omega^i_{rel}\ \mathbb{M}$ and $\omega' \in \Omega^j_{rel}\ \mathbb{M}$ we define the *super-commutator* to be

$$[\omega, \omega'] = \omega\omega' - (-1)^{ij} \omega'\omega$$

and following M. Karoubi [11] we define the space of *noncommutative differential forms* of degree i on $\mathbb{M}$ to be the quotient space

$$dR^i_{rel}\ \mathbb{M} = \frac{\Omega^i_{rel}\ \mathbb{M}}{\sum_{j=0}^i\ [\ \Omega^j_{rel}\ \mathbb{M}, \Omega^{i-j}_{rel}\ \mathbb{M}\ ]}$$

and one verifies that the differential d induces a differential on the *Karoubi complex* of $\mathbb{M}$

$$dR^0_{rel}\ \mathbb{M} \xrightarrow{d} dR^1_{rel}\ \mathbb{M} \xrightarrow{d} dR^2_{rel}\ \mathbb{M} \xrightarrow{d} \dots$$

An important result we will prove in chapter 9 is that this complex is acyclic, that is, if we define the i-th *de Rham cohomology group* of $\mathbb{M}$ to be the i-th homology of the Karoubi complex

$$H^i_{dR}\ \mathbb{M} = \frac{Ker\ dR^i_{rel}\ \mathbb{M} \xrightarrow{d} dR^{i+1}_{rel}\ \mathbb{M}}{Im\ dR^{i-1}_{rel}\ \mathbb{M} \xrightarrow[d]{} dR^i_{rel}\ \mathbb{M}}$$

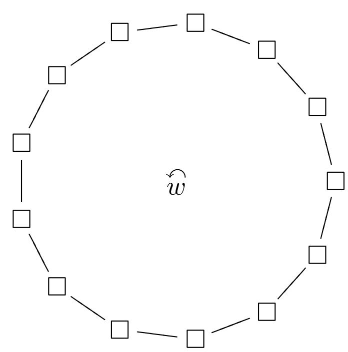
then we will prove that

$$H^i_{dR}\ \mathbb{M} = \begin{cases} \mathbb{C} \times \mathbb{C} & \text{when } i = 0, \\ 0 & \text{when } i \geq 1. \end{cases}$$

We will compute the first few terms in the Karoubi complex. Noncommutative *functions* on $\mathbb{M}$ are the 0-forms, which is by definition the quotient space

$$dR^0_{rel}\ \mathbb{M} = \frac{\mathbb{M}}{[\ \mathbb{M}, \mathbb{M}\ ]}$$

If p is an oriented path of length ≥ 1 in the quiver with different begin- and endpoint, then we can write p as a concatenation $p = p_1 p_2$ with p_i an oriented path of length ≥ 0 such that $p_2 p_1 = 0$ in $\mathbb{M}$. As $[p_1, p_2] = p_1 p_2 - p_2 p_1 = 0$ in $dR^0_{rel}\ \mathbb{M}$ we deduce that the space of noncommutative functions on $\mathbb{M}$ has as $\mathbb{C}$-basis the *necklace words* w



where each bead is this time one of the elements

$$\boxed{\bullet} = x \qquad \boxed{\circ} = y \quad \text{and} \quad \boxed{\blacktriangledown} = uv$$

together with the necklace words of length zero e and f. Each necklace word w corresponds to the equivalence class of the words in $\mathbb{M}$ obtained from multiplying the beads in the indicated orientation and and two words in $\{x, y, u, v\}$ in $\mathbb{M}$ are said to be equivalent if they are identical up to cyclic permutation of the terms.

Substituting each bead with the $n \times n$ matrices specified before and taking traces we get a map

$$dR^0_{rel}\ \mathbb{M} = \frac{\mathbb{M}}{[\ \mathbb{M}, \mathbb{M}\ ]} \xrightarrow{tr} \mathbb{C}[rep_\alpha\ \mathbb{M}]$$

Hence, noncommutative functions on $\mathbb{M}$ induce ordinary functions on *all* the representation spaces $rep_\alpha\ \mathbb{M}$ and these functions are $GL(\alpha)$-invariant. Moreover, the image of this map generates the ring of polynomial invariants as we mentioned before.

Next, we consider *noncommutative 1-forms* on $\mathbb{M}$ which are by definition elements of the space

$$dR^1_{rel}\ \mathbb{M} = \frac{\Omega^1_{rel}\ \mathbb{M}}{[\ \mathbb{M}, \Omega^1_{rel}\ \mathbb{M}\ ]}$$

Recall that $\Omega^1_{rel}\ \mathbb{M}$ is spanned by the expressions $p_0 dp_1$ with p_0 resp. p_1 oriented paths in the quiver of length ≥ 0 resp. ≥ 1 and such that the starting point of p_0 is the end point of p_1. To form $dR^1_{rel}\ \mathbb{M}$ we have to divide out expressions such as

$$[\ p, p_0 dp_1\ ] = pp_0 dp_1 + p_0 p_1 dp - p_0 d(p_1 p)$$

That is, if we have connecting oriented paths p_2 and p_1 both of length ≥ 1 we have in $dR^1_{rel}\ \mathbb{M}$

$$p_0 d(p_1 p_2) = p_2 p_0 dp_1 + p_0 p_1 dp_2$$

and by iterating this procedure whenever the differential term is a path of length ≥ 2 we can represent each class in $dR^1_{rel}\ \mathbb{M}$ as a combination from

$$\mathbb{M}e\ dx + \mathbb{M}e\ dy + \mathbb{M}e\ du + \mathbb{M}f\ dv$$

Now, $\mathbb{M}e = e\mathbb{M}e + f\mathbb{M}e$ and let $p \in f\mathbb{M}e$. Then, we have in $dR^1_{rel}\ \mathbb{M}$

$$d(xp) = p\ dx + x\ dp$$

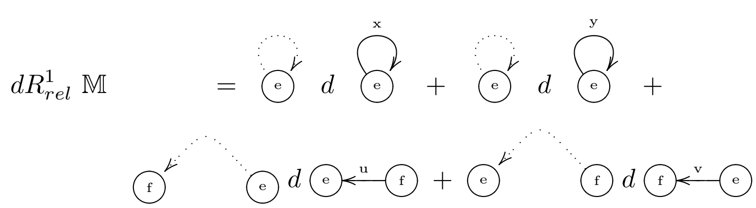
but by our description of $\Omega^1\ \mathbb{M}$ the left hand term is zero as is the second term on the right, whence $p\ dx = 0$. A similar argument holds replacing x by y. As for u, let $p \in e\mathbb{M}e$, then we have in $dR^1_{rel}\ \mathbb{M}$

$$d(up) = p\ du + u\ dp$$

and again the left-hand and the second term on the right are zero whence $p\ du = 0$. An analogous result holds for v and $p \in f\mathbb{M}f$. Therefore, we have the description of noncommutative 1-forms on $\mathbb{M}$

$$dR^1_{rel}\ \mathbb{M} = e\mathbb{M}e\ dx + e\mathbb{M}e\ dy + f\mathbb{M}e\ du + e\mathbb{M}f\ dv$$

That is, in graphical terms



1.9 Symplectic geometry on $\mathbb{M}$.

Recall that a *symplectic structure* on a (commutative) manifold M is given by a closed differential 2-form. The non-degenerate 2-form ω gives a canonical isomorphism

$$T\ M \simeq T^*\ M$$

that is, between vector fields on M and differential 1-forms. Further, there is a unique $\mathbb{C}$-linear map from functions f on M to vectorfields ξ_f by the requirement that $-df = i_{\xi_f}\omega$ where i_ξ is the contraction of n-forms to $n-1$-forms using the vectorfield ξ. We can make the functions on M into a *Poisson algebra* by defining

$$\{f, g\} = \omega(\xi_f, \xi_g)$$

and one verifies that this bracket satisfies the Jacobi and Leibnitz identities.

The *Lie derivative* L_ξ with respect to ξ is defined by the Cartan homotopy formula

$$L_\xi \ \varphi = i_\xi d\varphi + d i_\xi \varphi$$

for any differential form φ. A vectorfield ξ is said to be *symplectic* if it preserves the symplectic form, that is, $L_\xi \omega = 0$. In particular, for any function f on M we have that ξ_f is symplectic. Moreover the assignment

$$f \longrightarrow \xi_f$$

defines a Lie algebra morphism from the functions $\mathcal{O}(M)$ on M equipped with the Poisson bracket to the Lie algebra of symplectic vectorfields, $Vect_\omega \ M$. Moreover, this map fits into the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(M) \longrightarrow Vect_\omega \ M \longrightarrow H^1_{dR} \ M \longrightarrow 0$$

It is this sequence that we will generalize to the noncommutative algebra $\mathbb{M}$.

We say that a *noncommutative symplectic structure* on $\mathbb{M}$ is given by a 2-form

$$\omega \in dR^2_{rel} \ \mathbb{M} \quad \text{ such that } \quad d \ \omega = 0 \in dR^3_{rel} \ \mathbb{M}$$

Given the shape of the defining quiver, a natural choice of symplectic structure is taking

$$\omega = dx \ dy + du \ dv$$

By a (relative) vectorfield on $\mathbb{M}$ we understand a $\mathbb{C} \times \mathbb{C}$-derivation on $\mathbb{M}$. That is, a linear map $\mathbb{M} \xrightarrow{\theta} \mathbb{M}$ such that

$$\theta(ab) = \theta(a)b + a\theta(b) \quad \text{and} \quad \theta(e) = 0 = \theta(f).$$

For a given θ we define a degree preserving derivation L_θ and a degree -1 super-derivation i_θ on $\Omega \ \mathbb{M}$

$$\begin{tikzcd}
\Omega^{n-1} \ \mathbb{M} \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^n \ \mathbb{M} \arrow[l, bend left, "i_\theta"] \arrow[r, bend left, "d"] \arrow[loop below, "L_\theta"] & \Omega^{n+1} \ \mathbb{M} \arrow[l, bend left, "i_\theta"] \arrow[loop below, "L_\theta"]
\end{tikzcd}$$

defined by the rules

$$\begin{cases} L_\theta(a) = \theta(a) & L_\theta(da) = d \ \theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

for all $a \in \mathbb{M}$. We will prove in chapter 10 an analog for the Cartan homotopy formula

$$L_\theta = i_\theta \circ d + d \circ i_\theta$$

and that these operators induce operators on the Karoubi complex $dR \ \mathbb{M}$. The analog of the isomorphism $T \ M \simeq T^* \ M$ is the isomorphism

$$Der_{\mathbb{C} \times \mathbb{C}} \ \mathbb{M} \xrightarrow{i.\omega} dR^1_{rel} \ \mathbb{M}$$

as for any $\mathbb{C} \times \mathbb{C}$-derivation θ we have

$$\begin{aligned} i_\theta \ \omega &= i_\theta(dx)dy - dx i_\theta(dy) + i_\theta(du)dv - du i_\theta(dv) \\ &= \theta(x)dy - dx\theta(y) + \theta(u)dv - du\theta(v) \\ &\equiv \theta(x)dy - \theta(y)dx + \theta(u)dv - \theta(v)du \end{aligned}$$

and using the relations in $\mathbb{M}$ we can easily prove that any $\mathbb{C} \times \mathbb{C}$ derivation on $\mathbb{M}$ must satisfy

$$\theta(x) \in e\mathbb{M}e \quad \theta(y) \in e\mathbb{M}e \quad \theta(u) \in e\mathbb{M}f \quad \theta(v) \in f\mathbb{M}e$$

so the above expression belongs to $dR^1_{rel}\ \mathbb{M}$. Conversely, any θ defined by its images on the generators x, y, u and v by

$$-\theta(y)dx + \theta(x)dy - \theta(v)du + \theta(u)dv \in dR^1_{rel}\ \mathbb{M}$$

induces a derivation on $\mathbb{M}$.

In analogy with the classical case we define a derivation θ to be *symplectic* if and only if $L_\theta \omega = 0$ in $dR^2_{rel}\ \mathbb{M}$. We denote these derivations by $Der_\omega\ \mathbb{M}$. From the homotopy formula it follows that

$$\theta \in Der_\omega\ \mathbb{M} \Longleftrightarrow d(i_\theta \omega) = 0 \quad \text{in} \quad dR^2_{rel}\ \mathbb{M}$$

But then, using the above identification $Der_{\mathbb{C}\times\mathbb{C}}\ \mathbb{M} \simeq dR^1_{rel}\ \mathbb{M}$ and the fact that $H^1_{dR}\ \mathbb{M} = 0$ we obtain an analog of the map $f \longrightarrow \xi_f$ from functions to symplectic vectorfields in the classical case

$$\frac{\mathbb{M}}{[\ \mathbb{M}, \mathbb{M}\ ]} = dR^0_{rel}\ \mathbb{M} \xrightarrow{d} (dR^1_{rel}\ \mathbb{M})_{closed} \xrightarrow{i_{\cdot}\omega^{-1}} Der_\omega\ \mathbb{M}$$

which fits into the exact sequence, using our knowledge of the de Rham cohomology of $\mathbb{M}$

$$0 \longrightarrow \mathbb{C} \times \mathbb{C} \longrightarrow \frac{\mathbb{M}}{[\ \mathbb{M}, \mathbb{M}\ ]} \longrightarrow Der_\omega\ \mathbb{M} \longrightarrow 0$$

which we claim to be an exact sequence of Lie algebras. Hence we need to define a Poisson bracket on the noncommutative functions $\frac{\mathbb{M}}{[\ \mathbb{M},\mathbb{M}\ ]}$. We want to mimic the Poisson bracket on $\mathbb{C}[x, y, u, v]$ determined by $dx \wedge dy + du \wedge dv$ which is

$$\{f, g\} = (\tfrac{\partial f}{\partial x}.\tfrac{\partial g}{\partial y} - \tfrac{\partial f}{\partial y}.\tfrac{\partial g}{\partial x}) + (\tfrac{\partial f}{\partial u}.\tfrac{\partial g}{\partial v} - \tfrac{\partial f}{\partial v}.\tfrac{\partial g}{\partial u})$$

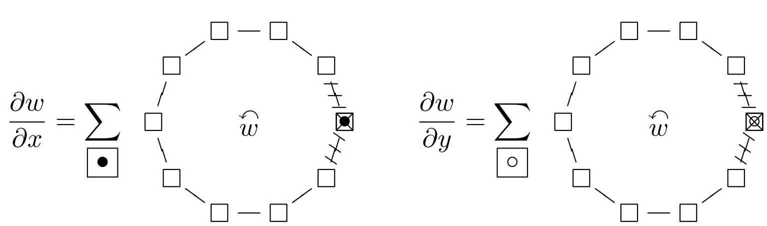
but then we need a substitute for these partial derivatives. Using our description of $dR^1_{rel}\ \mathbb{M}$ we have for any $f \in dR^0_{rel}\ \mathbb{M} == \frac{\mathbb{M}}{[\mathbb{M},\mathbb{M}]}$ uniquely defined partial derivatives

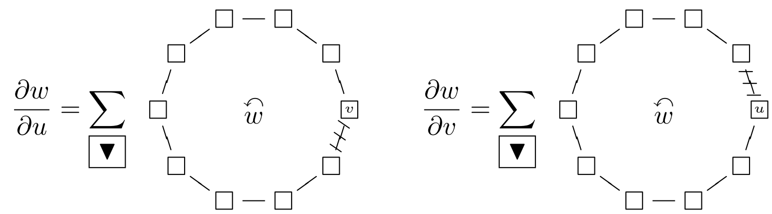
$$\tfrac{\partial a}{\partial x}, \tfrac{\partial a}{\partial y}, \tfrac{\partial a}{\partial u}, \tfrac{\partial a}{\partial v} : \ \frac{\mathbb{M}}{[\ \mathbb{M}, \mathbb{M}\ ]} \longrightarrow \mathbb{M}$$

by the formula

$$da = \tfrac{\partial a}{\partial x} \otimes x + \tfrac{\partial a}{\partial y} \otimes y + \tfrac{\partial a}{\partial u} \otimes u + \tfrac{\partial a}{\partial v} \otimes v.$$

We have to specify these on necklace words w. Using the calculation rules in $dR^1_{rel}\ \mathbb{M}$ one verifies that the partial derivatives of w are the sums of the oriented paths in the quiver one obtains by cyclicly running through the remaining path by deleting occurrences of the differentiating arrow. That is,

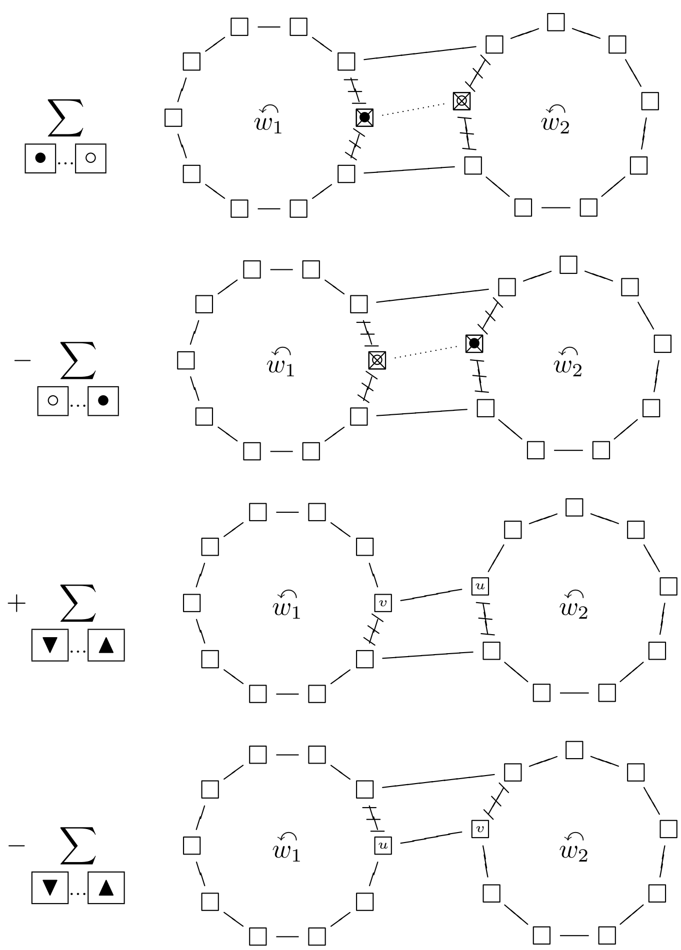




Using these partial derivatives one can then define a Poisson product on $dR^0_{rel}\ \mathbb{M}$ by

$$\{w_1, w_2\}_K \equiv \left(\tfrac{\partial w_1}{\partial x}.\tfrac{\partial w_2}{\partial y} - \tfrac{\partial w_1}{\partial y}.\tfrac{\partial w_2}{\partial x}\right) + \left(\tfrac{\partial w_1}{\partial u}.\tfrac{\partial w_2}{\partial v} - \tfrac{\partial w_1}{\partial v}.\tfrac{\partial w_2}{\partial u}\right) \text{ modulo } [\ \mathbb{M}, \mathbb{M}\ ]$$

In particular, if the w_i are necklace words, the Poisson bracket $\{w_1, w_2\}_K$ is a sum of necklace words. Using the above graphical description we have that $\{w_1, w_2\}$ is equal to



1.10 Fibers of the moment map.

After this excursion to the symplectic geometry of the noncommutative manifold $\mathbb{M}$ it is time to return to the study of the phase space $Calo_n$. Observe that the center $\mathbb{C}^* = (\lambda \mathbb{1}_n, \lambda) \hookrightarrow GL(\alpha)$ acts trivially on $rep_\alpha\ \mathbb{M}$ so the relevant acting group is rather

$$PGL(\alpha) = \frac{GL(\alpha)}{\mathbb{C}^*(\mathbb{1}_n, 1)}.$$

There is an open subset in $rep_\alpha\ \mathbb{M}$ of representations such that the stabilizer subgroup of $PGL(\alpha)$ is trivial. This is the case for all simple representations by Schur's lemma. However, there are others. For example any representation $(X, Y, u, \underline{0}) \in Hilb_n$ has trivial stabilizer, but none of them are simple representations of $\mathbb{M}$ as they always have a nontrivial subrepresentation of dimension vector $(1, 0)$ determined by a common eigenvector of X and Y (which exists because X and Y commute with each other). We will denote the open set of representations with trivial stabilizer by $rep_\alpha^s\ \mathbb{M}$ and call any $(X, Y, u, v) \in rep_\alpha^s\ \mathbb{M}$ a *Schur representation* of the quiver.

Remark that the *Lie algebra* of $PGL(\alpha)$ is the vectorspace

$$Lie\ PGL(\alpha) = M_\alpha^0(\mathbb{C}) = \{\ (M, c) \in M_n(\mathbb{C}) \oplus \mathbb{C}\ \mid\ tr(M) + c = 0\ \}$$

The relevant moment map for the action of $(P)GL(\alpha)$ on the representation space $rep_\alpha\ \mathbb{M}$ is

$$rep_\alpha\ \mathbb{M} \xrightarrow{\quad\mu\quad} Lie\ PGL(\alpha)$$

$$(X, Y, u, v) \qquad \mapsto \qquad ([X, Y] + uv\ ,\ -vu)$$

Fix $\lambda \in \mathbb{C}$, then $\underline{\lambda} = (\lambda \mathbb{1}_n, -n\lambda) \in M_\alpha^0(\mathbb{C})$ and is fixed under conjugation by $GL(\alpha)$. Therefore, its preimage

$$\pi^{-1}(\underline{\lambda}) = \{(X, Y, u, v) \in rep_\alpha\ \mathbb{M}\ \mid\ [X, Y] + uv = \lambda \mathbb{1}_n \text{ and } vu = n\lambda\ \}$$

is a closed affine subscheme of $rep_\alpha\ \mathbb{M}$ stable under the action of $GL(\alpha)$. In particular we have that

$$CALO_n = \pi^{-1}(\underline{1}).$$

We will now associate noncommutative algebras to the fibers $\pi^{-1}(\underline{\lambda})$. These are special examples of the *deformed preprojective algebras* introduced by W. Crawley-Boevey and M.P. Holland in [6]. For $\lambda \in \mathbb{C}$ we define

$$\mathbb{M}_\lambda = \frac{\mathbb{M}}{([x, y] + [u, v] - \lambda(e - nf))}$$

and by an argument similar to that of $\mathbb{M}$ we see that $\mu^{-1}(\underline{\lambda})$ is the space of $n+1$-dimensional representations $\mathbb{M}_\lambda \xrightarrow{\phi} M_{n+1}(\mathbb{C})$ such that

$$\phi(e) = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \phi(f) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The closed affine subscheme $\pi^{-1}(\underline{\lambda})$ has as its defining ideal of relations I_λ the entries in the $n+1 \times n+1$ matrix

$$[x_n, y_n] + [u_n, v_n] - \begin{bmatrix} \lambda \mathbb{1}_n & 0 \\ 0 & -n\lambda \end{bmatrix}$$

We consider the ring of polynomial invariants $\mathbb{C}[\mu^{-1}(\underline{\lambda})]^{GL(\alpha)}$ and its corresponding affine scheme $\mu^{-1}(\underline{\lambda})/GL(\alpha)$. The natural algebra morphisms give the following geometric picture

$$\begin{tikzcd} \mu^{-1}(\underline{\lambda}) \arrow[r, hook] \arrow[d, "pi"] & rep_\alpha\ \mathbb{M} \arrow[d] \\ \mu^{-1}(\underline{\lambda})/GL(\alpha) \arrow[r, hook] & rep_\alpha\ \mathbb{M}/GL(\alpha) \end{tikzcd}$$

Invariant theory tells us that the defining ideal of the closed subscheme $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ is $J_\lambda = I_\lambda \cap \mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$. Again. points in $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ are in one-to-one correspondence with isomorphism classes of $n+1$-dimensional semi-simple representations of the algebra M_λ with specified ranks of the images of e and f.

Analogous to the construction of the algebra $\mathbb{M}(n)$ we construct an algebra $M_\lambda(n)$ which is a finite module over its center by dividing out the centrally generated ideal of $\mathbb{M}$ determined by J_λ

$$\mathbb{M}_\lambda(n) = \frac{\mathbb{M}(n)}{\mathbb{M}(n)\ J_\lambda\ \mathbb{M}(n)}$$

Again, one verifies that the α-dimensional representations of $\mathbb{M}_\lambda$ factor through $\mathbb{M}_\lambda(n)$. The trace map t on $\mathbb{M}(n)$ introduced before defines a trace map on $\mathbb{M}_\lambda(n)$. Clearly, this trace satisfies the Cayley-Hamilton identities for $n+1 \times n+1$ matrices and $t(1) = n+1$.

We define a category $CH(n+1)$ of all $\mathbb{C}$-algebras A equipped with a trace map $A \xrightarrow{t_A} A$ such that $t_A(1) = n+1$ and t_A satisfies all Cayley-Hamilton identities holding for $n+1 \times n+1$ matrices. Morphisms $A \xrightarrow{\phi} B$ in $CH(n+1)$ are $\mathbb{C}$-algebra morphisms compatible with the trace maps, that is, making the diagram below commute

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "t_A"'] & B \arrow[d, "t_B"] \\ A \arrow[r, "\phi"] & B \end{tikzcd}$$

We say that an algebra S in $CH(n+1)$ is *smooth* in $CH(n+1)$ if it has the lifting property with respect to nilpotent ideals in $CH(n+1)$. That is, for all $A \in CH(n+1)$, $I \triangleleft A$ a nilpotent ideal in A such that $t_A(I) \subset I$ and a morphism ψ in $CH(n+1)$ we can complete the diagram below by a morphism $\tilde{\psi}$ in $CH(n+1)$

$$\begin{tikzcd} S \arrow[r, dotted, "\tilde{\psi}"] \arrow[dr, "\psi"'] & A \arrow[d, two heads] \\ & \frac{A}{I} \end{tikzcd}$$

We will prove in chapter 5 that this property is equivalent to the geometric formulation that the space of *trace preserving* $n+1$-*dimensional representations* of $\mathbb{S}$ is a smooth variety. Here, a trace preserving representation is a morphism $\mathbb{S} \xrightarrow{\phi} M_{n+1}(\mathbb{C})$ in $CH(n+1)$ where $M_{n+1}(\mathbb{C})$ is equipped with the usual trace.

For example, the algebra $\mathbb{M}(n)$ is smooth in $CH(n+1)$ as its space of trace preserving $n+1$-dimensional representations is $GL_{n+1}(\mathbb{C}) \times^{GL(\alpha)} rep_\alpha\ \mathbb{M}$. Similarly, the space of trace preserving $n+1$-dimensional representations of the fiber algebra $\mathbb{M}_\lambda(n)$ is equal to the fiber bundle

$$GL_{n+1}(\mathbb{C}) \times^{GL(\alpha)} \mu^{-1}(\underline{\lambda})$$

and is therefore smooth in $CH(n+1)$ if and only if the fiber $\mu^{-1}(\underline{\lambda})$ is a smooth submanifold of $rep_\alpha\ \mathbb{M}$. In particular, because $CALO_n = \mu^{-1}(\underline{1})$ we deduce

Theorem 1.22 *The fiber algebra $\mathbb{M}_1(n)$ corresponding to the phase space $Calo_n$ is a smooth algebra in $CH(n+1)$. In fact, it is an Azumaya algebra over $Calo_n = \mu^{-1}(\underline{1})/GL(\alpha)$.*

Proof. The first statement follows from the fact that $CALO_n$ is a submanifold of $rep_\alpha\ \mathbb{M}$. Further, we know that the GL_n-action on $CALO_n$ is free, hence the $GL(\alpha)$-action has all its stabilizer subgroups equal to the center $\mathbb{C}^*$. But then, $CALO_n \twoheadrightarrow Calo_n$ is a principal $PGL(\alpha)$-fibration. We will see in chapter 5 that then the ring of $PGL(\alpha)$-equivariant maps

$$CALO_n \longrightarrow M_{n+1}(\mathbb{C})$$

(which is equal to the fiber algebra $\mathbb{M}_1(n)$ is an Azumaya algebra with a specified embedding of the idempotents. □

This result gives a ringtheoretical interpretation of the uniformity of the phase space $Calo_n$. Each point in the phase space corresponds to a simple $n+1$-dimensional representation of $\mathbb{M}_1(n)$.

In contrast, the fiber algebra $\mathbb{M}_0(n)$ corresponding to $Hilb_n$ is not smooth in $CH(n+1)$. Indeed, the fiber $\mu^{-1}(\underline{0})$ is not even irreducible but even the irreducible component determined by $Hilb_n$ contains singularities.

1.11 Ginzburg's theorem for $Calo_n$.

We have now all the necessary ingredients to sketch the proof of Ginzburg's result that $Calo_n$ i s the coadjoint orbit of some infinite dimensional Lie algebra. To begin, we equip $rep_\alpha\ \mathbb{M}$ with the symplectic structure induced by the 2-form

$$\omega_\alpha = \sum_{1 \leq i,j \leq n} dx_{ij} \wedge dy_{ij} + \sum_{i=1}^{n} du_i \wedge dv_i$$

The induced Poisson bracket $\{-.-\}_\alpha$ on the ring of polynomial functions $\mathbb{C}[rep_\alpha\ \mathbb{M}]$ is defined to be

$$\{f,g\}_\alpha = \sum_{1 \leq i,j \leq n} \left(\frac{\partial f}{\partial x_{ij}} \frac{\partial g}{\partial y_{ij}} - \frac{\partial f}{\partial y_{ij}} \frac{\partial g}{\partial x_{ij}}\right) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i} \frac{\partial g}{\partial v_i} - \frac{\partial f}{\partial v_i} \frac{\partial g}{\partial u_i}\right)$$

The action of $GL(\alpha)$ on $rep_\alpha\ \mathbb{M}$ is *symplectic* meaning that

$$\omega_\alpha(t,t') = \omega_\alpha(gt, gt')$$

for all $t, t' \in T_{(X,Y,u,v)}\ rep_\alpha\ \mathbb{M}$ in all points (X,Y,u,v) and for all $g \in GL(\alpha)$. The infinitesimal $GL(\alpha)$ action gives a Lie algebra homomorphism

$$Lie\ PGL(\alpha) \longrightarrow Vect_{\omega_\alpha}\ rep_\alpha\ \mathbb{M}$$

which factorizes through a Lie algebra morphism H to the coordinate ring making the diagram below commute

$$\begin{tikzcd} & Lie\ PGL(\alpha) \arrow[dl, "H=\mu^*"'] \arrow[dr] & \\ \mathbb{C}[rep_\alpha\ \mathbb{M}] \arrow[rr, "f \mapsto \xi_f"'] & & Vect_{\omega_\alpha}\ rep_\alpha\ \mathbb{M} \end{tikzcd}$$

We say that the action of $GL(\alpha)$ on $rep_\alpha\ \mathbb{M}$ is *Hamiltonian*.

This makes the ring of polynomial invariants $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$ into a Poisson algebra and we will write

$$Lie_1 = (\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}, \{-,-\}_\alpha)$$

for the corresponding abstract infinite dimensional Lie algebra. The dual space of this Lie algebra Lie_1^* is then a Poisson manifold equipped with the Kirillov-Kostant bracket. Evaluation at a point in the quotient variety $rep_\alpha\ \mathbb{M}/GL(\alpha)$ defines a linear function on Lie_1 and therefore evaluation gives an embedding

$$rep_\alpha\ \mathbb{M}/GL(\alpha) \hookrightarrow Lie_1^*$$

as Poisson varieties. That is, the induced map on the polynomial functions is a morphism of Poisson algebras.

Let $\lambda \in \mathbb{C}$ and let $\underline{\lambda} = (\lambda\mathbb{1}_n, -n\lambda) \in Lie\ PGL(\alpha)$. Then, we have

Theorem 1.23 *Assume that $\mu^{-1}(\underline{\lambda})$ is a smooth variety on which $PGL(\alpha)$ acts freely. Then, the quotient variety $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ is an affine symplectic manifold and the Poisson embeddings*

$$\mu^{-1}(\underline{\lambda})/GL(\alpha) \hookrightarrow rep_\alpha\ \mathbb{M}/GL(\alpha) \hookrightarrow Lie_1^*$$

makes each connected component of $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ a closed coadjoint orbit of Lie_1^*.

Proof. (sketch) Because the action of the reductive group $PGL(\alpha)$ is free on the smooth affine variety $\pi^{-1}(\underline{\lambda})$, the quotient variety $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ is smooth and affine. Moreover, the infinitesimal coadjoint action of Lie_1 on Lie_1^* preserves $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ and factors through the quotient Lie algebra $\frac{Lie_1}{J_\lambda}$.

In general, if X is a smooth affine variety, then the differentials of polynomial functions on X span the tangent spaces at all points x of X. Therefore, if X is in addition symplectic, the infinitesimal Hamiltonian action of the Lie algebra $\mathbb{C}[X]$ (with the natural Poisson bracket) on X is infinitesimally transitive. But then, the evaluation map makes X a coadjoint orbit of the dual Lie algebra $\mathbb{C}[X]^*$.

Hence, $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ is a coadjoint orbit in $\mathbb{C}[\mu^{-1}(\underline{\lambda})/GL(\alpha)]^*$. Therefore, the infinite dimensional group Ham generated by all Hamiltonian flows on $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ acts with open orbits. Being connected, each irreducible component of $\mu^{-1}(\underline{\lambda})/GL(\alpha)$ is a single Ham-orbit finishing the proof. $\square$

Observe that the conditions hold for $\lambda = 1$, that is, $Calo_n$ is a coadjoint orbit in Lie_1^*. In contrast, the fiber corresponding to $Hilb_n$, that is, $\lambda = 0$ does not satisfy the requirements.

The drawback is that the Lie algebra Lie_1 still depends on n and we want a similar result holding for all n. We will now show that all $Calo_n$ are coadjoint orbits in the dual of a central extension of the Lie algebra $Der_\omega\ \mathbb{M}$.

The central extension in question is given by the exact sequence of Lie algebras we found when investigating the noncommutative deRham cohomology of $\mathbb{M}$

$$0 \longrightarrow \mathbb{C}\times\mathbb{C} \longrightarrow \frac{\mathbb{M}}{[\ \mathbb{M},\mathbb{M}\ ]} \longrightarrow Der_\omega\ \mathbb{M} \longrightarrow 0$$

Moreover, we have seen that both $\frac{\mathbb{M}}{[\ \mathbb{M},\mathbb{M}\ ]}$ and Lie_1 are generated by necklace words. the crucial point to note is now that

$$Lie = \frac{\mathbb{M}}{[\ \mathbb{M},\mathbb{M}\ ]} \xrightarrow{tr} \mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)} = Lie_1$$

obtained by mapping a necklace word w to its trace $tr(w)$ using the $n+1 \times n+1$ matrices x_n, y_n, u_n and v_n introduced before, is a Lie algebra morphism where we equip Lie with the Poisson bracket determined by the partial derivations determined from our description of $dR^1\ \mathbb{M}$ and Lie_1 is equipped with the Poisson bracket $\{-,-\}_\alpha$. That is, we have for all necklace words w_1 and w_2 the identity

$$\{tr\ w_1, tr\ w_2\}_\alpha = tr\ \{w_1, w_2\}_K$$

Because the $tr\ w$ generate the ring of polynomial invariants $\mathbb{C}[rep_\alpha\ \mathbb{M}]^{GL(\alpha)}$ it follows that the elements $tr\ \frac{\mathbb{M}}{[\ \mathbb{M},\mathbb{M}\ ]}$ separate points of $Calo_n = \mu^{-1}(\underline{1})/GL(\alpha)$ as subset of $rep_\alpha\ \mathbb{M}/GL(\alpha)$. That is, the composition

$$Calo_n \hookrightarrow rep_\alpha\ \mathbb{M}/GL(\alpha) \xrightarrow{tr^*} \left(\frac{\mathbb{M}}{[\ \mathbb{M},\mathbb{M}\ ]}\right)^*$$

is injective. Analogously, the differentials of functions on $Calo_n$ obtained by restricting traces of necklace words viewed as linear functions on Lie_1^* span the cotangent spaces at all points of $Calo_n$, concluding the proof of

Theorem 1.24 *For all n, the phase space $Calo_n$ is a coadjoint orbit in the dual of the Lie algebra $\frac{\mathbb{M}}{[\ \mathbb{M},\mathbb{M}\ ]}$ which is a central extension of the Lie algebra $Der_\omega\ \mathbb{M}$.*

Chapter 2

Brauer-Severi Varieties.

Let K be a field and $\Delta = (a,b)_K$ the quaternion algebra determined by $a, b \in K^*$. That is,

$$\Delta = K.1 \oplus K.i \oplus K.j \oplus K.ij \quad \text{with} \quad i^2 = a \quad j^2 = b \quad \text{and} \quad ji = -ij$$

The norm map on Δ defines a conic in $\mathbb{P}^2_K$ called the Brauer-Severi variety of Δ

$$BS(\Delta) = V(x^2 - ay^2 - bz^2) \hookrightarrow \mathbb{P}^2_K = Proj\ K[x,y,z].$$

Its characteristic property is that a fieldextension L of K admits an L-rational point on $BS(\Delta)$ if and only if $\Delta \otimes_K L$ admits zero-divisors and hence is isomorphic to $M_2(L)$.

More generally, let $\mathbb{K}$ be the algebraic closure of K. We will see that the Galois cohomology pointed set

$$H^1(Gal(\mathbb{K}/K), PGL_n(\mathbb{K}))$$

classifies at the same time the isomorphism classes of the following geometric and algebraic objects

- Brauer-Severi K-varieties BS, which are smooth projective K-varieties such that $BS_{\mathbb{K}} \simeq \mathbb{P}^{n-1}_{\mathbb{K}}$.
- Central simple K-algebras Δ, which are K-algebras of dimension n^2 such that $\Delta \otimes_K \mathbb{K} \simeq M_n(\mathbb{K})$.

The one-to-one correspondence between these two sets is given by associating to a central simple K-algebra Δ its Brauer-Severi variety $BS(\Delta)$ which represents the functor associating to a fieldextension L of K the set of left ideals of $\Delta \otimes_K L$ which have L-dimension equal to n. In particular, $BS(\Delta)$ has an L-rational point if and only if $\Delta \otimes_K L \simeq M_n(L)$ and hence the geometric object $BS(\Delta)$ encodes the algebraic splitting behaviour of Δ.

Now restrict to the case when K is the functionfield $\mathbb{C}(X)$ of a projective variety X and let Δ be a central simple $\mathbb{C}(X)$-algebra of dimension n^2. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-orders in Δ then we will see that there is a Brauer-Severi scheme $BS(\mathcal{A})$ which is a projective space bundle over X with general fiber isomorphic to $\mathbb{P}^{n-1}(\mathbb{C})$ embedded in $\mathbb{P}^N(\mathbb{C})$ where $N = \binom{n+k-1}{k} - 1$. Over an arbitrary point of x the fiber may degenerate, for example if $n = 2$ the $\mathbb{P}^1(\mathbb{C})$ embedded as a conic in $\mathbb{P}^2(\mathbb{C})$ can degenerate into a pair of $\mathbb{P}^1(\mathbb{C})$'s. The special case when $BS(\mathcal{A})$ is a $\mathbb{P}^{n-1}(\mathbb{C})$-bundle corresponds to the case when $\mathcal{A}$ is a sheaf of Azumaya algebras over X.

For arbitrary orders, the geometric structure of $BS(\mathcal{A})$ can be fairly complicated. However, when $\mathcal{A}$ is a sheaf of smooth orders we will prove in chapter 8 that $BS(\mathcal{A})$

is a smooth variety and indicate how one can compute the fibers over X explicitly. The motivating class of such examples are the unirational non-rational threefolds constructed by M. Artin and D. Mumford [3] in 1972 as Brauer-Severi varieties of maximal orders in specified quaternion algebras (the restrictions impose that these maximal orders are indeed smooth orders). A major step in their construction is the description of the Brauer group of a simply connected projective surface using étale cohomology. We will outline this result in some detail as it gives us the opportunity to introduce some basic results on étale extentions, étale cohomology and étale descent. Roughly speaking, étale extensions give us an algebraic alternative for the implicit function theorem in differential geometry. In this book we will give several applications of étale descent. For example, we will give an étale local description of smooth orders which will allow us to deduce from the Artin-Mumford exact sequence which central simple algebras over a smooth projective surface allow a noncommutative smooth model, see chapter 6.

2.1 Unirational non-rational threefolds.

In this section we will outline the major steps in the Artin-Mumford construction of unirational non-rational threefolds. We use this class of examples as motivation for introducing étale cohomology and smooth orders. For more details we refer to the original paper [3].

Consider $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$. We want to describe all central simple algebras Δ over the functionfield $\mathbb{C}(x,y)$. In this chapter we will prove that this is a huge collection. The Artin-Mumford result describes them by a certain geo-combinatorial package which we call a $\mathbb{Z}_n$-*wrinkle* over $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$. A $\mathbb{Z}_n$-wrinkle is determined by

- A finite collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of *irreducible curves* in $\mathbb{P}^2$, that is, $C_i = V(F_i)$ for an irreducible form in $\mathbb{C}[X,Y,Z]$ of degree d_i.

- A finite collection $\mathcal{P} = \{P_1, \ldots, P_l\}$ of *points* of $\mathbb{P}^2$ where each P_i is either an intersection point of two or more C_i or a singular point of some C_i.

- For each $P \in \mathcal{P}$ the *branch-data* $b_P = (b_1, \ldots, b_{i_P})$ with $b_i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\{1, \ldots, i_P\}$ the different branches of $\mathcal{C}$ in P. These numbers must satisfy the admissibility condition

$$\sum_i b_i = 0 \in \mathbb{Z}_n$$

for every $P \in \mathcal{P}$

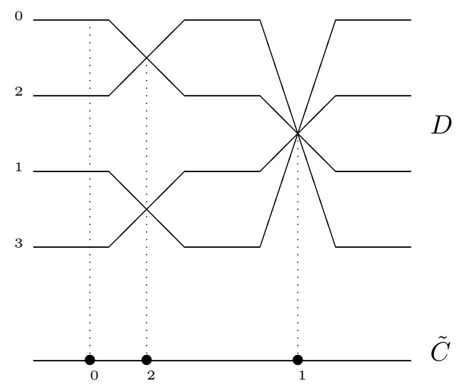
- for each $C \in \mathcal{C}$ we fix a cyclic $\mathbb{Z}_n$-cover of smooth curves

$$D \twoheadrightarrow \tilde{C}$$

of the desingularization $\tilde{C}$ of C which is compatible with the branch-data. That is, if $Q \in \tilde{C}$ corresponds to a $\mathcal{C}$-branch b_i in P, then D is ramified in Q with stabilizer subgroup

$$Stab_Q = \langle b_i \rangle \subset \mathbb{Z}_n$$

For example, a portion of a $\mathbb{Z}_4$-wrinkle can have the following picture



It is clear that the cover-data is the most untractable part of a $\mathbb{Z}_n$-wrinkle, so we want to have some control on the covers $D \twoheadrightarrow \tilde{C}$. Let $\{Q_1, \ldots, Q_z\}$ be the points of $\tilde{C}$ where the cover ramifies with branch numbers $\{b_1, \ldots, b_z\}$, then D is determined by a continuous module structure (that is, a cofinite subgroup acts trivially) of

$$\pi_1(\tilde{C} - \{Q_1, \ldots, Q_z\}) \text{ on } \mathbb{Z}_n$$

where the fundamental group of the Riemann surface $\tilde{C}$ with z punctures is known (topologically) to be equal to the group

$$\langle u_1, v_1, \ldots, u_g, v_g, x_1, \ldots, x_z \rangle / ([u_1, v_1] \ldots [u_g, v_g] x_1 \ldots x_z)$$

where g is the genus of $\tilde{C}$. The action of x_i on $\mathbb{Z}_n$ is determined by multiplication with b_i. In fact, we need to use the étale fundamental group, see [20], but this group has the same finite continuous modules as the topological fundamental group.

Example 2.1 Covers of $\mathbb{P}^1$ and elliptic curves.

1. If $\tilde{C} = \mathbb{P}^1$ then $g = 0$ and hence $\pi_1(\mathbb{P}^1 - \{Q_1, \ldots, Q_z\}$ is zero if $z \leq 1$ (whence no covers exist) and is $\mathbb{Z}$ if $z = 2$. Hence, there exists a unique cover $D \twoheadrightarrow \mathbb{P}^1$ with branch-data $(1, -1)$ in say $(0, \infty)$ namely with D the normalization of $\mathbb{P}^1$ in $\mathbb{C}(\sqrt[n]{x})$.

2. If $\tilde{C} = E$ an elliptic curve, then $g = 1$. Hence, $\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$ and there exist unramified $\mathbb{Z}_n$-covers. They are given by the isogenies

$$E' \twoheadrightarrow E$$

where E' is another elliptic curve and $E = E'/\langle \tau \rangle$ where τ is an n-torsion point on E'.

We will show that any n-fold cover $D \twoheadrightarrow \tilde{C}$ is determined by a function $f \in \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$. This allows us to put a group-structure on the equivalence classes of $\mathbb{Z}_n$-wrinkles. In particular, we call a wrinkle *trivial* provided all coverings $D_i \twoheadrightarrow \tilde{C}_i$ are trivial (that is, D_i is the disjoint union of n copies of $\tilde{C}_i$).

One of the main results we will prove in this chapter is the Artin-Mumford exact sequence for Brauer groups of simply connected surfaces. In the case of $\mathbb{C}(x, y)$ this result can be phrased as

Theorem 2.2 *If Δ is a central simple $\mathbb{C}(x, y)$-algebra of dimension n^2, then Δ determines uniquely a $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$. Conversely, any $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$ determines a unique division $\mathbb{C}(x, y)$- algebra whose class in the Brauer group has order n.*

Specialize to the case of quaternion algebras, that is $n = 2$. Consider E_1 and E_2 two elliptic curves in $\mathbb{P}^2$ and take $\mathcal{C} = \{E_1, E_2\}$ and $\mathcal{P} = \{P_1, \dots P_9\}$ the intersection points and all the branch data zero. Let E_i' be a twofold unramified cover of E_i as in the example above given by modding our a 2-torsion point from E_i'. By the Artin-Mumford result there is a quaternion algebra Δ corresponding to this $\mathbb{Z}_2$-wrinkle.

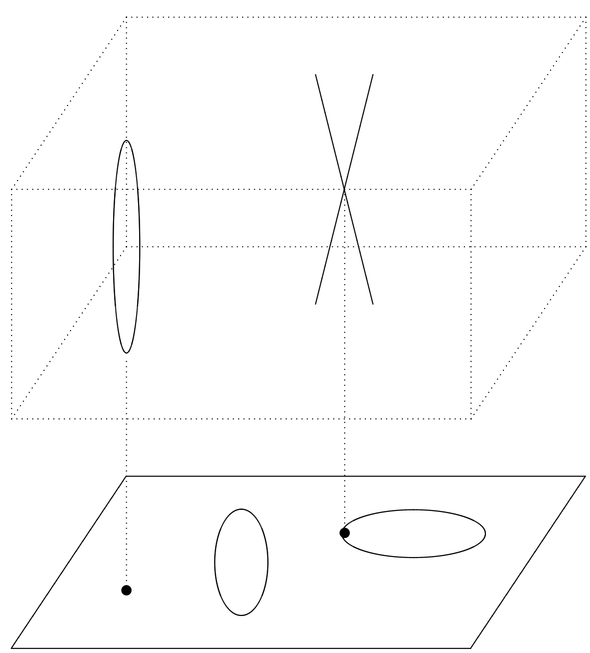
Next, blow up the intersection points to get a surface S with disjoint elliptic curves C_1 and C_2. Now take a maximal $\mathcal{O}_S$ order in Δ then we will see that the relevance of the curves C_i is that they are the locus of the points $s \in S$ where $\overline{\mathcal{A}}_s \not\simeq M_2(\mathbb{C})$, the so called *ramification locus* of the order $\mathcal{A}$. The local structure of $\mathcal{A}$ in a point $s \in S$ is

- when $s \notin C_1 \cup C_2$, then $\mathcal{A}_s$ is an Azumaya $\mathcal{O}_{S,s}$-algebra in Δ,
- when $s \in C_i$, then $\mathcal{A}_s = \mathcal{O}_{S,s}.1 \oplus \mathcal{O}_{S,s}.i \oplus \mathcal{O}_{S,s}.j \oplus \mathcal{O}_{S,s}.ij$ with

$$\begin{cases} i^2 &= a \\ j^2 &= bt \\ ji &= -ij \end{cases}$$

  where $t = 0$ is a local equation for C_i and a and b are units in $\mathcal{O}_{S,s}$.

In chapter 6 we will see that this is the local description of a smooth order over a smooth surface in a quaternion algebra. Artin and Mumford then define the Brauer-Severi scheme of $\mathcal{A}$ as representing the functor which assigns to an S-scheme S' the set of left ideals of $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ which are locally free of rank 2. Using the local description of $\mathcal{A}$ they show that $BS(\mathcal{A})$ is a projective space bundle over S



with the properties that $BS(\mathcal{A})$ is a smooth variety and the projection morphism $BS(\mathcal{A}) \xrightarrow{\pi} S$ is flat, all of the geometric fibers being isomorphic to $\mathbb{P}^1$ (resp. to $\mathbb{P}^1 \vee \mathbb{P}^1$) whenever $s \notin C_1 \cup C_2$ (resp. $s \in C_1 \cup C_2$).

Finally, for specific starting configurations E_1 and E_2, they prove that the obtained Brauer-Severi variety $BS(\mathcal{A})$ cannot be rational because there is torsion in $H^4(BS(\mathcal{A}), \mathbb{Z}_2)$, whereas $BS(\mathcal{A})$ can be shown to be unirational for these specific configurations.

2.2 Etale morphisms and sheaves.

In the next few sections we will introduce the basic tools from étale topology leading to a sketch of the Artin-Mumford exact sequence.

Definition 2.3 A finite morphism $A \xrightarrow{f} B$ of commutative $\mathbb{C}$-algebras is said to be *étale* if and only if

$$B = A[t_1, \ldots, t_k]/(f_1, \ldots, f_k) \text{ such that } det\ (\frac{\partial f_i}{\partial x_j}\ )_{i,j} \in B^*$$

Example 2.4 Consider the morphism $\mathbb{C}[x, x^{-1}] \hookrightarrow \mathbb{C}[x, x^{-1}][\sqrt[n]{x}]$ and the induced map on the affine varieties

$$Var\ \mathbb{C}[x, x^{-1}][\sqrt[n]{x}] \xrightarrow{\psi} Var\ \mathbb{C}[x, x^{-1}] = \mathbb{C} - \{0\}.$$

Clearly, every point $\lambda \in \mathbb{C} - \{0\}$ has exactly n preimages $\lambda_i = \zeta^i \sqrt[n]{\lambda}$. Moreover, in a neighborhood of λ_i, the map ψ is a diffeomorphism. Still, we do not have an inverse map in algebraic geometry as $\sqrt[n]{x}$ is not a polynomial map. However, $\mathbb{C}[x, x^{-1}][\sqrt[n]{x}]$ is an étale extension of $\mathbb{C}[x, x^{-1}]$. That is, étale morphisms can be seen as an algebraic substitute for the nonexistence of an inverse function theorem in algebraic geometry.

Proposition 2.5 *Etale morphisms satisfy 'sorite', that is*

$$\begin{tikzcd} A' \arrow[r, dotted, "et"] & A' \otimes_A B \\ A \arrow[u] \arrow[r, "et"] & B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, "et"] & \\ A \arrow[ur, "et"] \arrow[rr, dotted, "et"] & & C \end{tikzcd}$$

$(basechange)$ $(composition)$

$$\begin{tikzcd} A' \arrow[r, "et"] & A' \otimes_A B \\ A \arrow[u, "f.f."] \arrow[r, dotted, "et"] & B \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "et"'] \arrow[dr, "et"] & \\ B \arrow[rr, dotted, "et", "A-alg"'] & & B' \end{tikzcd}$$

$(descent)$ $(morphisms)$

Here et means an étale morphism and f.f. stands for a faithfully flat morphism.

Definition 2.6 The étale site of A, which we will denote by A_{et} is the category with

- objects : the étale extensions $A \xrightarrow{f} B$ of A
- morphisms : compatible A-algebra morphisms

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ B_1 \arrow[rr, "\phi"] & & B_2 \end{tikzcd}$$

Observe that by the foregoing proposition all morphisms in A_{et} are étale. We can put on A_{et} a (Grothendieck) topology by defining

- cover : a collection $\mathcal{C} = \{B \xrightarrow{f_i} B_i\}$ in A_{et} such that

$$Spec\ B = \cup_i\ Im\ (Spec\ B_i \xrightarrow{f} Spec\ B\ )$$

  where *Spec* is the prime spectrum of a commutative algebra, that is the collection of all its prime ideals equipped with the Zariski topology.

An étale presheaf of groups on A_{et} is a functor

$$\mathbb{G} : A_{et} \longrightarrow \text{Groups}$$

In analogy with usual (pre)sheaf notation we denote for each

- object $B \in A_{et}$: $\Gamma(B, \mathbb{G}) = \mathbb{G}(B)$
- morphism $B \xrightarrow{\phi} C$ in A_{et} : $Res^B_C = \mathbb{G}(\phi) : \mathbb{G}(B) \longrightarrow \mathbb{G}(C)$ and $g \mid C = \mathbb{G}(\phi)(g)$.

A presheaf $\mathbb{G}$ is a sheaf provided for every $B \in A_{et}$ and every cover $\{B \longrightarrow B_i\}$ we have exactness of the equalizer diagram

$$0 \longrightarrow \mathbb{G}(B) \longrightarrow \prod_i \mathbb{G}(B_i) \rightrightarrows \prod_{i,j} \mathbb{G}(B_i \otimes_B B_j)$$

Example 2.7 *Constant sheaf.*
If G is a group, then

$$\mathbb{G} : A_{et} \longrightarrow \text{Groups} \quad B \mapsto G^{\oplus \pi_0(B)}$$

is a sheaf where $\pi_0(B)$ is the number of connected components of *Spec B*.

Example 2.8 *Multiplicative group $\mathbb{G}_m$.*
The functor

$$\mathbb{G}_m : A_{et} \longrightarrow \text{Ab} \quad B \mapsto B^*$$

is a sheaf on A_{et}.

A sequence of sheaves of Abelian groups on A_{et} is said to be exact

$$\mathbb{G}' \xrightarrow{f} \mathbb{G} \xrightarrow{g} \mathbb{G}"$$

if for every $B \in A_{et}$ and $s \in \mathbb{G}(B)$ such that $g(s) = 0 \in \mathbb{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in A_{et} and sections $t_i \in \mathbb{G}'(B_i)$ such that $f(t_i) = s \mid B_i$.

Example 2.9 *Roots of unity μ_n.*
We have a sheaf morphism

$$\mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m$$

and we denote the kernel with μ_n. As A is a $\mathbb{C}$-algebra we can identify μ_n with the constant sheaf $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ via the isomorphism $\zeta^i \mapsto i$ after choosing a primitive n-th root of unity $\zeta \in \mathbb{C}$.

Lemma 2.10 *The (Kummer)-sequence of sheaves of Abelian groups*

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 0$$

is exact on A_{et} (but not necessarily on A_{Zar}).

Proof. We only need to verify surjectivity. Let $B \in A_{et}$ and $b \in \mathbb{G}_m(B) = B^*$. Consider the étale extension $B' = B[t]/(t^n - b)$ of B, then b has an n-th root over in $\mathbb{G}_m(B')$. Observe that this n-th root does not have to belong to $\mathbb{G}_m(B)$. $\square$

If $\mathfrak{p}$ is a prime ideal of A we will denote with $\mathbf{k}_\mathfrak{p}$ the algebraic closure of the field of fractions of $A/\mathfrak{p}$. An étale neighborhood of $\mathfrak{p}$ is an étale extension $B \in A_{et}$ such that the diagram below is commutative

$$\begin{tikzcd} A \arrow[r, "nat"] \arrow[d, "et"'] & \mathbf{k}_\mathfrak{p} \\ B \arrow[ur] & \end{tikzcd}$$

The analogue of the localization $A_\mathfrak{p}$ for the étale topology is the strict Henselization

$$A^{sh}_\mathfrak{p} = \varinjlim B$$

where the limit is taken over all étale neighborhoods of $\mathfrak{p}$.

Recall that a local algebra L with maximal ideal m and residue map $\pi : L \twoheadrightarrow L/m = k$ is said to be Henselian if the following condition holds. Let $f \in L[t]$ be a monic polynomial such that $\pi(f)$ factors as $g_0.h_0$ in $k[t]$, then f factors as $g.h$ with $\pi(g) = g_0$ and $\pi(h) = h_0$. If L is Henselian then tensoring with k induces an equivalence of categories between the étale A-algebras and the étale k-algebras.

An Henselian local algebra is said to be strict Henselian if and only if its residue field is algebraically closed. Thus, a strict Henselian ring has no proper finite étale extensions and can be viewed as a local algebra for the étale topology.

Example 2.11 The algebraic functions $\mathbb{C}\{x_1,\ldots,x_d\}$

Consider the local algebra of $\mathbb{C}[x_1,\ldots,x_d]$ in the maximal ideal $(x_1,\ldots,x_d)$, then the Henselization and strict Henselization are both equal to

$$\mathbb{C}\{x_1,\ldots,x_d\}$$

the ring of algebraic functions. That is, the subalgebra of $\mathbb{C}[[x_1,\ldots,x_d]]$ of formal power-series consisting of those series $\phi(x_1,\ldots,x_d)$ which are algebraically dependent on the coordinate functions x_i over $\mathbb{C}$. In other words, those ϕ for which there exists a non-zero polynomial $f(x_i,y) \in \mathbb{C}[x_1,\ldots,x_d,y]$ with $f(x_1,\ldots,x_d,\phi(x_1,\ldots,x_d)) = 0$.

These algebraic functions may be defined implicitly by polynomial equations. Consider a system of equations

$$f_i(x_1,\ldots,x_d;y_1,\ldots,y_m) = 0 \text{ for } f_i \in \mathbb{C}[x_i,y_j] \text{ and } 1 \leq i \leq m$$

Suppose there is a solution in $\mathbb{C}$ with

$$x_i = 0 \text{ and } y_j = y^o_j$$

such that the Jacobian matrix is non-zero

$$det\ (\frac{\partial f_i}{\partial y_j}(0,\ldots,0;y_1^o,\ldots,y_m^0)) \neq 0$$

Then, the system can be solved uniquely for power series $y_j(x_1,\ldots,x_d)$ with $y_j(0,\ldots,0) = y_j^o$ by solving inductively for the coefficients of the series. One can show that such implicitly defined series $y_j(x_1,\ldots,x_d)$ are algebraic functions and that, conversely, any algebraic function can be obtained in this way.

If $\mathbb{G}$ is a sheaf on A_{et} and $\mathfrak{p}$ is a prime ideal of A, we define the stalk of $\mathbb{G}$ in $\mathfrak{p}$ to be

$$\mathbb{G}_{\mathfrak{p}} = \underrightarrow{lim}\ \mathbb{G}(B)$$

where the limit is taken over all étale neighborhoods of $\mathfrak{p}$. One can verify mono- epi- or isomorphisms of sheaves by checking it in all the stalks.

If A is an affine algebra defined over an algebraically closed field, then it suffices to verify in the maximal ideals of A.

2.3 Etale cohomology

Before we define cohomology of sheaves on A_{et} let us recall the definition of derived functors. Let $\mathcal{A}$ be an Abelian category. An object I of $\mathcal{A}$ is said to be injective if the functor

$$\mathcal{A} \longrightarrow Ab \quad M \mapsto Hom_{\mathcal{A}}(M,I)$$

is exact. We say that $\mathcal{A}$ has enough injectives if, for every object M in $\mathcal{A}$, there is a monomorphism $M \hookrightarrow I$ into an injective object.

If $\mathcal{A}$ has enough injectives and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor from $\mathcal{A}$ into a second Abelian category $\mathcal{B}$, then there is an essentially unique sequence of functors

$$R^i\ f : \mathcal{A} \longrightarrow \mathcal{B} \quad i \geq 0$$

called the right derived functors of f having the following properties

- $R^0\ f = f$
- $R^i\ I = 0$ for I injective and $i > 0$
- For every short exact sequence in $\mathcal{A}$

  $$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

  there are connecting morphisms $\delta^i : R^i\ f(M") \longrightarrow R^{i+1}\ f(M')$ for $i \geq 0$ such that we have a long exact sequence

  $$\ldots \longrightarrow R^i\ f(M) \longrightarrow R^i\ f(M") \xrightarrow{\delta^i} R^{i+1}\ f(M') \longrightarrow R^{i+1}\ f(M) \longrightarrow \ldots$$

- For any morphism $M \longrightarrow N$ there are morphisms $R^i\ f(M) \longrightarrow R^i\ f(N)$ for $i \geq 0$

In order to compute the objects $R^i\ f(M)$ define an object N in $\mathcal{A}$ to be f-acyclic if $R^i\ f(M) = 0$ for all $i > 0$. If we have a resolution of M

$$0 \longrightarrow M \longrightarrow N_0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \ldots$$

by f-acyclic object N_i, then the objects $R^i\ f(M)$ are canonically isomorphic to the cohomology objects of the complex

$$0 \longrightarrow f(N_0) \longrightarrow f(N_1) \longrightarrow f(N_2) \longrightarrow \ldots$$

One can show that all injectives are f-acyclic and hence that derived objects of M can be computed from an injective resolution of M.

Now, let $\mathbf{S}^{ab}(A_{et})$ be the category of all sheaves of Abelian groups on A_{et}. This is an Abelian category having enough injectives whence we can form right derived functors of left exact functors. In particular, consider the global section functor

$$\Gamma : \mathbf{S}^{ab}(A_{et}) \longrightarrow \text{Ab} \quad \mathbb{G} \mapsto \mathbb{G}(A)$$

which is left exact. The right derived functors of Γ will be called the étale cohomology functors and we denote

$$R^i\ \Gamma(\mathbb{G}) = H^i_{et}(A, \mathbb{G})$$

In particular, if we have an exact sequence of sheaves of Abelian groups $0 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 0$, then we have a long exact cohomology sequence

$$\ldots \longrightarrow H^i_{et}(A, \mathbb{G}) \longrightarrow H^i_{et}(A, \mathbb{G}") \longrightarrow H^{i+1}_{et}(A, \mathbb{G}') \longrightarrow \ldots$$

If $\mathbb{G}$ is a sheaf of non-Abelian groups (written multiplicatively), we cannot define cohomology groups. Still, one can define a pointed set $H^1_{et}(A, \mathbb{G})$ as follows. Take an étale cover $\mathcal{C} = \{A \longrightarrow A_i\}$ of A and define a 1-cocycle for $\mathcal{C}$ with values in $\mathbb{G}$ to be a family

$$g_{ij} \in \mathbb{G}(A_{ij}) \text{ with } A_{ij} = A_i \otimes_A A_j$$

satisfying the cocycle condition

$$(g_{ij} \mid A_{ijk})(g_{jk} \mid A_{ijk}) = (g_{ik} \mid A_{ijk})$$

where $A_{ijk} = A_i \otimes_A A_j \otimes_A A_k$.

Two cocycles g and g' for $\mathcal{C}$ are said to be cohomologous if there is a family $h_i \in \mathbb{G}(A_i)$ such that for all $i, j \in I$ we have

$$g'_{ij} = (h_i \mid A_{ij}) g_{ij} (h_j \mid A_{ij})^{-1}$$

This is an equivalence relation and the set of cohomology classes is written as $H^1_{et}(\mathcal{C}, \mathbb{G})$. It is a pointed set having as its distinguished element the cohomology class of $g_{ij} = 1 \in \mathbb{G}(A_{ij})$ for all $i, j \in I$.

We then define the non-Abelian first cohomology pointed set as

$$H^1_{et}(A, \mathbb{G}) = \underset{\longrightarrow}{lim}\ H^1_{et}(\mathcal{C}, \mathbb{G})$$

where the limit is taken over all étale coverings of A. It coincides with the previous definition in case $\mathbb{G}$ is Abelian.

A sequence $1 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 1$ of sheaves of groups on A_{et} is said to be exact if for every $B \in A_{et}$ we have

- $\mathbb{G}'(B) = Ker\ \mathbb{G}(B) \longrightarrow \mathbb{G}"(B)$
- For every $g" \in \mathbb{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in A_{et} and sections $g_i \in \mathbb{G}(B_i)$ such that g_i maps to $g" \mid B$.

Proposition 2.12 *For an exact sequence of groups on A_{et}*

$$1 \longrightarrow \mathbb{G}' \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}" \longrightarrow 1$$

there is associated an exact sequence of pointed sets

$$\begin{array}{c} 1 \longrightarrow \mathbb{G}'(A) \longrightarrow \mathbb{G}(A) \longrightarrow \mathbb{G}"(A) \xrightarrow{\delta} H^1_{et}(A,\mathbb{G}') \longrightarrow \\ \longrightarrow H^1_{et}(A,\mathbb{G}) \longrightarrow H^1_{et}(A,\mathbb{G}") \dashrightarrow H^2_{et}(A,\mathbb{G}') \end{array}$$

where the last map exists when $\mathbb{G}'$ is contained in the center of $\mathbb{G}$ (and therefore is Abelian whence H^2 is defined).

Proof. The connecting map δ is defined as follows. Let $g" \in \mathbb{G}"(A)$ and let $\mathcal{C} = \{A \longrightarrow A_i\}$ be an étale covering of A such that there are $g_i \in \mathbb{G}(A_i)$ that map to $g \mid A_i$ under the map $\mathbb{G}(A_i) \longrightarrow \mathbb{G}"(A_i)$. Then, $\delta(g)$ is the class determined by the one cocycle

$$g_{ij} = (g_i \mid A_{ij})^{-1}(g_j \mid A_{ij})$$

with values in $\mathbb{G}'$. The last map can be defined in a similar manner, the other maps are natural and one verifies exactness. $\square$

The main applications of this non-Abelian cohomology to non-commutative algebra is as follows. Let Λ be a not necessarily commutative A-algebra and M an A-module. Consider the sheaves of groups $\mathbf{Aut}(\Lambda)$ resp. $\mathbf{Aut}(M)$ on A_{et} associated to the presheaves

$$B \mapsto Aut_{B-alg}(\Lambda \otimes_A B) \text{ resp. } B \mapsto Aut_{B-mod}(M \otimes_A B)$$

for all $B \in A_{et}$. A twisted form of Λ (resp. M) is an A-algebra Λ' (resp. an A-module M') such that there is an étale cover $\mathcal{C} = \{A \longrightarrow A_i\}$ of A such that there are isomorphisms

$$\Lambda \otimes_A A_i \xrightarrow{\phi_i} \Lambda' \otimes_A A_i \text{ resp. } M \otimes_A A_i \xrightarrow{\psi_i} M' \otimes_A A_i$$

of A_i-algebras (resp. A_i-modules). The set of A-algebra isomorphism classes (resp. A-module isomorphism classes) of twisted forms of Λ (resp. M) is denoted by $Tw_A(\Lambda)$ (resp. $Tw_A(M)$). To a twisted form Λ' one associates a cocycle on $\mathcal{C}$

$$\alpha_{\Lambda'} = \alpha_{ij} = \phi_i^{-1} \circ \phi_j$$

with values in $\mathrm{Aut}(\Lambda)$. Moreover, one verifies that two twisted forms are isomorphic as A-algebras if their cocycles are cohomologous. That is, there is an embedding

$$Tw_A(\Lambda) \hookrightarrow H^1_{et}(A, \mathbf{Aut}(\Lambda)) \text{ and similarly } Tw_A(M) \hookrightarrow H^1_{et}(A, \mathbf{Aut}(M))$$

In favorable situations one can even show bijectivity. In particular, this is the case if the automorphisms group is a smooth affine algebraic group-scheme.

For example, consider $\Lambda = M_n(A)$, then the automorphism group is PGL_n and twisted forms of Λ are classified by elements of the cohomology group

$$H^1_{et}(A, PGL_n)$$

These twisted forms are precisely the Azumaya algebras of rank n^2 with center A. When A is an affine commutative $\mathbb{C}$-algebra and B is an A-algebra with center A, then B is an Azumaya algebra of rank n^2 if and only if

$$\frac{B}{B\mathfrak{m}B} \simeq M_n(\mathbb{C})$$

for every maximal ideal $\mathfrak{m}$ of A. For example, the fiber algebra $\mathbb{M}_1(n)$ introduced in the foregoing chapter is an Azumaya algebra of rank $(n+1)^2$ over its center which is $\mathbb{C}[Calo_n]$.

2.4 Central simple algebras

Let K be a field of characteristic zero, choose an algebraic closure $\mathbb{K}$ with absolute Galois group $G_K = Gal(\mathbb{K}/K)$.

Lemma 2.13 *The following are equivalent*

1. $K \longrightarrow A$ *is étale*
2. $A \otimes_K \mathbb{K} \simeq \mathbb{K} \times \ldots \times \mathbb{K}$
3. $A = \prod L_i$ *where* L_i/K *is a finite field extension*

Proof. Assume (1), then $A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ where f_i have invertible Jacobian matrix. Then $A \otimes \mathbb{K}$ is a smooth algebra (hence reduced) of dimension 0 so (2) holds.

Assume (2), then

$$Hom_{K-alg}(A, \mathbb{K}) \simeq Hom_{\mathbb{K}-alg}(A \otimes \mathbb{K}, \mathbb{K})$$

has $dim_{\mathbb{K}}(A \otimes \mathbb{K})$ elements. On the other hand we have by the Chinese remainder theorem that

$$A/Jac\ A = \prod_i L_i$$

with L_i a finite field extension of K. However,

$$dim_{\mathbb{K}}(A \otimes \mathbb{K}) = \sum_i dim_K(L_i) = dim_K(A/Jac\ A) \leq dim_K(A)$$

and as both ends are equal A is reduced and hence $A = \prod_i L_i$ whence (3).

Assume (3), then each $L_i = K[x_i]/(f_i)$ with $\partial f_i/\partial x_i$ invertible in L_i. But then $A = \prod L_i$ is étale over K whence (1). $\square$

To every finite étale extension $A = \prod L_i$ we can associate the finite set $rts(A) = Hom_{K-alg}(A, \mathbb{K})$ on which the Galois group G_K acts via a finite quotient group. If we write $A = K[t]/(f)$, then $rts(A)$ is the set of roots in $\mathbb{K}$ of the polynomial f with obvious action by G_K. Galois theory, in the interpretation of Grothendieck can now be stated as

Proposition 2.14 *The functor*

$$K_{et} \xrightarrow{rts(-)} \textit{finite } G_K - \textit{sets}$$

is an anti-equivalence of categories.

We will now give a similar interpretation of the Abelian sheaves on K_{et}. Let $\mathbb{G}$ be a presheaf on K_{et}. Define

$$M_{\mathbb{G}} = \varinjlim \ \mathbb{G}(L)$$

where the limit is taken over all subfields $L \hookrightarrow \mathbb{K}$ that are finite over K. The Galois group G_K acts on $\mathbb{G}(L)$ on the left through its action on L whenever L/K is Galois. Hence, G_K acts an $M_{\mathbb{G}}$ and $M_{\mathbb{G}} = \cup M_{\mathbb{G}}^H$ where H runs through the open subgroups of G_K. That is, $M_{\mathbb{G}}$ is a continuous G_K-module.

Conversely, given a continuous G_K-module M we can define a presheaf $\mathbb{G}_M$ on K_{et} such that

- $\mathbb{G}_M(L) = M^H$ where $H = G_L = Gal(\mathbb{K}/L)$.
- $\mathbb{G}_M(\prod L_i) = \prod \mathbb{G}_M(L_i)$.

One verifies that $\mathbb{G}_M$ is a sheaf of Abelian groups on K_{et}.

Theorem 2.15 *There is an equivalence of categories*

$$\mathbf{S}(K_{et}) \rightleftarrows G_K - \mathbf{mod}$$

induced by the correspondences $\mathbb{G} \mapsto M_{\mathbb{G}}$ and $M \mapsto \mathbb{G}_M$. Here, $G_K - \mathbf{mod}$ is the category of continuous G_K-modules.

Proof. A G_K-morphism $M \longrightarrow M'$ induces a morphism of sheaves $\mathbb{G}_M \longrightarrow \mathbb{G}_{M'}$. Conversely, if H is an open subgroup of G_K with $L = \mathbb{K}^H$, then if $\mathbb{G} \xrightarrow{\phi} \mathbb{G}'$ is a sheafmorphism, $\phi(L) : \mathbb{G}(L) \longrightarrow \mathbb{G}'(L)$ commutes with the action of G_K by functoriality of ϕ. Therefore, $\underrightarrow{lim}\ \phi(L)$ is a G_K-morphism $M_{\mathbb{G}} \longrightarrow M_{\mathbb{G}'}$.

One verifies easily that $Hom_{G_K}(M, M') \longrightarrow Hom(\mathbb{G}_M, \mathbb{G}_{M'})$ is an isomorphism and that the canonical map $\mathbb{G} \longrightarrow \mathbb{G}_{M_{\mathbb{G}}}$ is an isomorphism. □

In particular, we have that $\mathbb{G}(K) = \mathbb{G}(\mathbb{K})^{G_K}$ for every sheaf $\mathbb{G}$ of Abelian groups on K_{et} and where $\mathbb{G}(\mathbb{K}) = M_{\mathbb{G}}$. Hence, the right derived functors of Γ and $(-)^G$ coincide for Abelian sheaves.

The category $G_K - \mathbf{mod}$ of continuous G_K-modules is Abelian having enough injectives. Therefore, the left exact functor

$$(-)^G : G_K - \mathbf{mod} \longrightarrow \mathbf{Ab}$$

admits right derived functors. They are called the Galois cohomology groups and denoted

$$R^i\ M^G = H^i(G_K, M)$$

Therefore, we have.

Proposition 2.16 *For any sheaf of Abelian groups $\mathbb{G}$ on K_{et} we have a group isomorphism*

$$H^i_{et}(K, \mathbb{G}) \simeq H^i(G_K, \mathbb{G}(\mathbb{K}))$$

Therefore, étale cohomology is a natural extension of Galois cohomology to arbitrary algebras.

The following definition-characterization of central simple algebras is classical

Proposition 2.17 *Let A be a finite dimensional K-algebra. The following are equivalent :*

1. *A has no proper twosided ideals and the center of A is K.*
2. *$A_{\mathbb{K}} = A \otimes_K \mathbb{K} \simeq M_n(\mathbb{K})$ for some n.*
3. *$A_L = A \otimes_K L \simeq M_n(L)$ for some n and some finite Galois extension L/K.*
4. *$A \simeq M_k(D)$ for some k where D is a division algebra of dimension l^2 with center K.*

The last part of this result suggests the following definition. Call two central simple algebras A and A' equivalent if and only if $A \simeq M_k(\Delta)$ and $A' \simeq M_l(\Delta)$ with Δ a division algebra. From the second characterization it follows that the tensorproduct of two central simple K-algebras is again central simple. Therefore, we can equip the set of equivalence classes of central simple algebras with a product induced from the tensorproduct. This product has the class $[K]$ as unit element and $[\Delta]^{-1} = [\Delta^{opp}]$, the opposite algebra as $\Delta \otimes_K \Delta^{opp} \simeq End_K(\Delta) = M_{l^2}(K)$. This group is called the *Brauer group* and is denoted $Br(K)$. We will quickly recall its cohomological description, all of which is classical.

GL_r is an affine smooth algebraic group defined over K and is the automorphism group of a vectorspace of dimension r. It defines a sheaf of groups on K_{et} that we will denote by $\mathbf{GL_r}$. Using the general results on twisted forms of the foregoing chapter we have

Lemma 2.18

$$H^1_{et}(K, \mathbf{GL_r}) = H^1(G_K, GL_r(\mathbb{K})) = 0$$

In particular, we have 'Hilbert's theorem 90'

$$H^1_{et}(K, \mathbb{G}_m) = H^1(G_K, \mathbb{K}^*) = 0$$

Proof. The cohomology group classifies K-module isomorphism classes of twisted forms of r-dimensional vectorspaces over K. There is just one such class. □

PGL_n is an affine smooth algebraic group defined over K and it is the automorphism group of the K-algebra $M_n(K)$. It defines a sheaf of groups on K_{et} denoted by $\mathbf{PGL_n}$. By the proposition we know that any central simple K-algebra Δ of dimension n^2 is a twisted form of $M_n(K)$. Therefore,

Lemma 2.19 *The pointed set of K-algebra isomorphism classes of central simple algebras of dimension n^2 over K coincides with the cohomology set*

$$H^1_{et}(K, \mathbf{PGL_n}) = H^1(G_K, PGL_n(\mathbb{K}))$$

Theorem 2.20 *There is a natural inclusion*

$$H^1_{et}(K, \mathbf{PGL_n}) \hookrightarrow H^2_{et}(K, \mu_n) = Br_n(K)$$

where $Br_n(K)$ is the n-torsion part of the Brauer group of K. Moreover,

$$Br(K) = H^2_{et}(K, \mathbb{G}_m)$$

is a torsion group.

Proof. Consider the exact commutative diagram of sheaves of groups on K_{et}.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r, "(-)^n"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{SL_n} \arrow[r] \arrow[d] & \mathbf{GL_n} \arrow[r, "det"] \arrow[d] & \mathbb{G}_m \arrow[r] & 1 \\
 & \mathbf{PGL_n} \arrow[r, equal] \arrow[d] & \mathbf{PGL_n} \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

Taking cohomology of the second exact sequence we obtain

$$GL_n(K) \xrightarrow{det} K^* \longrightarrow H^1_{et}(K, \mathbf{SL_n}) \longrightarrow H^1_{et}(K, \mathbf{GL_n})$$

where the first map is surjective and the last term is zero, whence

$$H^1_{et}(K, \mathbf{SL_n}) = 0$$

Taking cohomology of the first vertical exact sequence we get

$$H^1_{et}(K, \mathbf{SL_n}) \longrightarrow H^1_{et}(K, \mathbf{PGL_n}) \longrightarrow H^2_{et}(K, \mu_n)$$

from which the first claim follows.

As for the second, taking cohomology of the first exact sequence we get

$$H^1_{et}(K, \mathbb{G}_m) \longrightarrow H^2_{et}(K, \mu_n) \longrightarrow H^2_{et}(K, \mathbb{G}_m) \xrightarrow{n.} H^2_{et}(K, \mathbb{G}_m)$$

By Hilbert 90, the first term vanishes and hence $H^2_{et}(K, \mu_n)$ is equal to the n-torsion of the group

$$H^2_{et}(K, \mathbb{G}_m) = H^2(G_K, \mathbb{K}^*) = Br(K)$$

where the last equality follows from the crossed product result, see for example [23]. $\square$

So far, the field K was arbitrary. If K is of transcendence degree d, this will put restrictions on the 'size' of the Galois group G_K. In particular this will enable us to show that $H^i(G_K, \mu_n) = 0$ for $i > d$. Before we can prove this we need to refresh our memory on spectral sequences.

2.5 Spectral sequences

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be Abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and consider left exact functors

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

Let the functors be such that f maps injectives of $\mathcal{A}$ to g-acyclic objects in $\mathcal{B}$, that is $R^i\ g(f\ I) = 0$ for all $i > 0$. Then, there are connections between the objects

$$R^p\ g(R^q\ f(A)) \text{ and } R^n\ gf(A)$$

for all objects $A \in \mathcal{A}$. These connections can be summarized by giving a spectral sequence

Theorem 2.21 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories with $\mathcal{A}, \mathcal{B}$ having enough injectives and left exact functors*

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

such that f takes injectives to g-acyclics.

Then, for any object $A \in \mathcal{A}$ there is a spectral sequence

$$E_2^{p,q} = R^p\ g(R^q\ f(A)) \Longrightarrow R^n\ gf(A)$$

In particular, there is an exact sequence

$$0 \longrightarrow R^1\ g(f(A)) \longrightarrow R^1\ gf(A) \longrightarrow g(R^1\ f(A)) \longrightarrow R^2\ g(f(A)) \longrightarrow \ldots$$

Moreover, if f is an exact functor, then we have

$$R^p\ gf(A) \simeq R^p\ g(f(A))$$

A spectral sequence $E_2^{p,q} \Longrightarrow E^n$ (or $E_1^{p,q} \Longrightarrow E^n$) consists of the following data

1. A family of objects $E_r^{p,q}$ in an Abelian category for $p, q, r \in \mathbb{Z}$ such that $p, q \geq 0$ and $r \geq 2$ (or $r \geq 1$).

2. A family of morphisms in the Abelian category

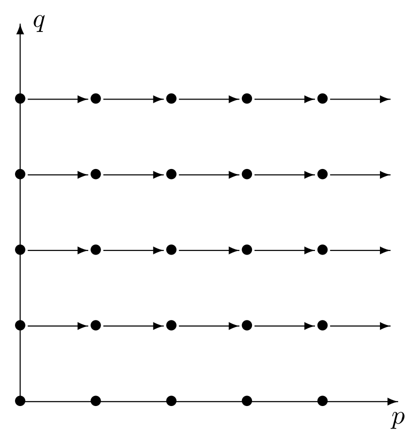
   $$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

   satisfying the complex condition

   $$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$$

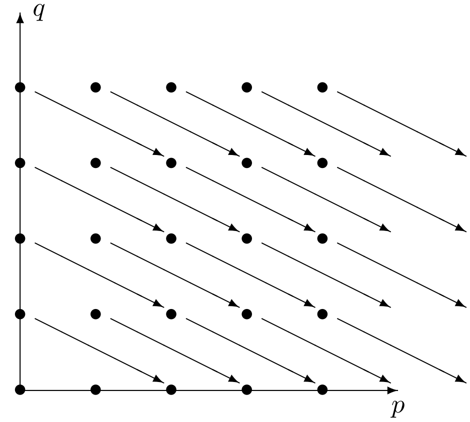
   and where we assume that $d_r^{p,q} = 0$ if any of the numbers $p, q, p+r$ or $q-r+1$ is < 1. At level one we have the following

$E_1^{p,q} \quad =$



At level two we have the following

$E_2^{p,q} \quad =$



3. The objects $E_{r+1}^{p,q}$ on level $r+1$ are derived from those on level r by taking the cohomology objects of the complexes, that is,

$$E^p_{r+1} = Ker\ d_r^{p,q} \ / \ Im\ d_r^{p-r,q+r-1}$$

At each place (p,q) this process converges as there is an integer r_0 depending on (p,q) such that for all $r \geq r_0$ we have $d_r^{p.q} = 0 = d_r^{p-r,q+r-1}$. We then define

$$E_\infty^{p,q} = E_{r_0}^{p,q} (= E_{r_0+1}^{p,q} = \ldots)$$

Observe that there are injective maps $E_\infty^{0,q} \hookrightarrow E_2^{0,q}$.

4. A family of objects E^n for integers $n \geq 0$ and for each we have a filtration

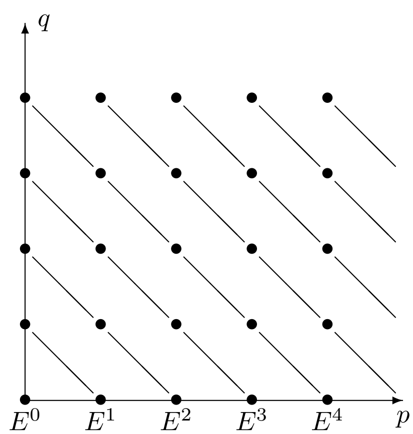
$$0 \subset E_n^n \subset E_{n-1}^n \subset \ldots \subset E_1^n \subset E_0^n = E^n$$

such that the successive quotients are given by

$$E_p^n \ / \ E_{p+1}^n = E_\infty^{p,n-p}$$

That is, the terms $E_\infty^{p,q}$ are the composition terms of the limiting terms E^{p+q}. Pictorially,

$E_\infty^{p,q} \quad =$



For small n one can make the relation between E^n and the terms $E_2^{p,q}$ explicit. First note that

$$E_2^{0,0} = E_\infty^{0,0} = E^0$$

Also, $E_1^1 = E_\infty^{1,0} = E_2^{1,0}$ and $E^1/E_1^1 = E_\infty^{0,1} = Ker\ d_2^{0,1}$. This gives an exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow E^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0}$$

Further, $E^2 \supset E_1^2 \supset E_2^2$ where

$$E_2^2 = E_\infty^{2,0} = E_2^{2,0} \;/\; Im\ d_2^{0,1}$$

and $E_1^2/E_2^2 = E_\infty^{1,1} = Ker\ d_2^{1,1}$ whence we can extend the above sequence to

$$\dots \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow E_1^2 \longrightarrow E_2^{1,1} \xrightarrow{d_2^{1,1}} E_2^{3,0}$$

as $E^2/E_1^2 = E_\infty^{0,2} \hookrightarrow E_2^{0,2}$ we have that $E_1^2 = Ker\ (E^2 \longrightarrow E_2^{0,2})$. If we specialize to the spectral sequence $E_2^{p,q} = R^p\ g(R^q\ f(A)) \Longrightarrow R^n\ gf(A)$ we obtain the exact sequence

$$0 \longrightarrow R^1\ g(f(A)) \longrightarrow R^1\ gf(A) \longrightarrow g(R^1\ f(A)) \longrightarrow R^2\ g(f(A)) \longrightarrow$$

$$\longrightarrow E_1^2 \longrightarrow R^1\ g(R^1\ f(A)) \longrightarrow R^3\ g(f(A))$$

where $E_1^2 = Ker\ (R^2\ gf(A) \longrightarrow g(R^2\ f(A)))$.

An important example of a spectral sequence is the Leray spectral sequence. Assume we have an algebra morphism $A \xrightarrow{f} A'$ and a sheaf of groups $\mathbb{G}$ on A'_{et}. We define the *direct image* of $\mathbb{G}$ under f to be the sheaf of groups $f_*\ \mathbb{G}$ on A_{et} defined by

$$f_*\ \mathbb{G}(B) = \mathbb{G}(B \otimes_A A')$$

for all $B \in A_{et}$ (recall that $B \otimes_A A' \in A'_{et}$ so the right hand side is well defined).

This gives us a left exact functor

$$f_* : \mathbf{S}^{ab}(A'_{et}) \longrightarrow \mathbf{S}^{ab}(A_{et})$$

and therefore we have right derived functors of it $R^i\ f_*$. If $\mathbb{G}$ is an Abelian sheaf on A'_{et}, then $R^i\ f_*\mathbb{G}$ is a sheaf on A_{et}. One verifies that its stalk in a prime ideal $\mathfrak{p}$ is equal to

$$(R^i\ f_*\mathbb{G})_{\mathfrak{p}} = H^i_{et}(A^{sh}_{\mathfrak{p}} \otimes_A A', \mathbb{G})$$

where the right hand side is the direct limit of cohomology groups taken over all étale neighborhoods of $\mathfrak{p}$. We can relate cohomology of $\mathbb{G}$ and $f_*\mathbb{G}$ by the following

Theorem 2.22 *(Leray spectral sequence) If $\mathbb{G}$ is a sheaf of Abelian groups on A'_{et} and $A \xrightarrow{f} A'$ an algebra morphism, then there is a spectral sequence*

$$E_2^{p,q} = H^p_{et}(A, R^q\ f_*\mathbb{G}) \Longrightarrow H^n_{et}(A, \mathbb{G})$$

In particular, if $R^j\ f_*\mathbb{G} = 0$ for all $j > 0$, then for all $i \geq 0$ we have isomorphisms

$$H^i_{et}(A, f_*\mathbb{G}) \simeq H^i_{et}(A', \mathbb{G})$$

2.6 Tsen and Tate fields

Definition 2.23 A field K is said to be a $Tsen^d$-field if every homogeneous form of degree deg with coefficients in K and $n > deg^d$ variables has a non-trivial zero in K.

For example, an algebraically closed field $\mathbb{K}$ is a $Tsen^0$-field as any form in n-variables defines a hypersurface in $\mathbb{P}^{n-1}_{\mathbb{K}}$. In fact, algebraic geometry tells us a stronger story

Lemma 2.24 *Let $\mathbb{K}$ be algebraically closed. If $f_1, \dots, f_r$ are forms in n variables over $\mathbb{K}$ and $n > r$, then these forms have a common non-trivial zero in $\mathbb{K}$.*

Proof. Each f_i defines a hypersurface $V(f_i) \hookrightarrow \mathbb{P}^{n-1}_{\mathbb{K}}$. The intersection of r hypersurfaces has dimension $\geq n-1-r$ from which the claim follows. $\square$

We want to extend this fact to higher Tsen-fields. The proof of the following result is technical unenlightening inequality manipulation, see for example [30].

Proposition 2.25 *Let K be a $Tsen^d$-field and $f_1, \ldots, f_r$ forms in n variables of degree deg. If $n > r deg^d$, then they have a non-trivial common zero in K.*

For our purposes the main interest in Tsen-fields comes from :

Theorem 2.26 *Let K be of transcendence degree d over an algebraically closed field $\mathbb{C}$, then K is a $Tsen^d$-field.*

Proof. First we claim that the purely transcendental field $\mathbb{C}(t_1, \ldots, t_d)$ is a $Tsen^d$-field. By induction we have to show that if L is $Tsen^k$, then $L(t)$ is $Tsen^{k+1}$.

By homogeneity we may assume that $f(x_1, \ldots, x_n)$ is a form of degree deg with coefficients in $L[t]$ and $n > deg^{k+1}$. For fixed s we introduce new variables $y_{ij}^{(s)}$ with $i \leq n$ and $0 \leq j \leq s$ such that

$$x_i = y_{i0}^{(s)} + y_{i1}^{(s)} t + \ldots + y_{is}^{(s)} t^s$$

If r is the maximal degree of the coefficients occurring in f, then we can write

$$f(x_i) = f_0(y_{ij}^{(s)}) + f_1(y_{ij}^{(s)})t + \ldots + f_{deg.s+r}(y_{ij}^{(s)})t^{deg.s+r}$$

where each f_j is a form of degree deg in $n(s+1)$-variables. By the proposition above, these forms have a common zero in L provided

$$n(s+1) > deg^k(ds + r + 1) \Longleftrightarrow (n - deg^{i+1})s > deg^i(r+1) - n$$

which can be satisfied by taking s large enough. the common non-trivial zero in L of the f_j, gives a non-trivial zero of f in $L[t]$.

By assumption, K is an algebraic extension of $\mathbb{C}(t_1, \ldots, t_d)$ which by the above argument is $Tsen^d$. As the coefficients of any form over K lie in a finite extension E of $\mathbb{C}(t_1, \ldots, t_d)$ it suffices to prove that E is $Tsen^d$.

Let $f(x_1, \ldots, x_n)$ be a form of degree deg in E with $n > deg^d$. Introduce new variables y_{ij} with

$$x_i = y_{i1}e_1 + \ldots y_{ik}e_k$$

where e_i is a basis of E over $\mathbb{C}(t_1, \ldots, t_d)$. Then,

$$f(x_i) = f_1(y_{ij})e_1 + \ldots + f_k(y_{ij})e_k$$

where the f_i are forms of degree deg in $k.n$ variables over $\mathbb{C}(t_1, \ldots, t_d)$. Because $\mathbb{C}(t_1, \ldots, t_d)$ is $Tsen^d$, these forms have a common zero as $k.n > k.deg^d$. Finding a non-trivial zero of f in E is equivalent to finding a common non-trivial zero to the $f_1, \ldots, f_k$ in $\mathbb{C}(t_1, \ldots, t_d)$, done. $\square$

A direct application of this result is Tsen's theorem :

Theorem 2.27 *Let K be the functionfield of a curve C defined over an algebraically closed field. Then, the only central simple K-algebras are $M_n(K)$. That is, $Br(K) = 1$.*

Proof. Assume there exists a central division algebra Δ of dimension n^2 over K. There is a finite Galois extension L/K such that $\Delta \otimes L = M_n(L)$. If $x_1, \ldots, x_{n^2}$ is a K-basis for Δ, then the reduced norm of any $x \in \Delta$,

$$N(x) = det(x \otimes 1)$$

is a form in n^2 variables of degree n. Moreover, as $x \otimes 1$ is invariant under the action of $Gal(L/K)$ the coefficients of this form actually lie in K.

By the main result, K is a $Tsen^1$-field and $N(x)$ has a non-trivial zero whenever $n^2 > n$. As the reduced norm is multiplicative, this contradicts $N(x)N(x^{-1}) = 1$. Hence, $n = 1$ and the only central division algebra is K itself. □

If K is the functionfield of a surface, we also have an immediate application :

Proposition 2.28 *Let K be the functionfield of a surface defined over an algebraically closed field. If Δ is a central simple K-algebra of dimension n^2, then the reduced norm map*

$$N \ : \ \Delta \longrightarrow K$$

is surjective.

Proof. Let $e_1, \ldots, e_{n^2}$ be a K-basis of Δ and $k \in K$, then

$$N(\sum x_i e_i) - k x_{n^2+1}^n$$

is a form of degree n in n^2+1 variables. Since K is a $Tsen^2$ field, it has a non-trivial solution (x_i^0), but then, $\delta = (\sum x_i^0 e_i) x_{n^2+1}^{-1}$ has reduced norm equal to k. □

From the cohomological description of the Brauer group it is clear that we need to have some control on the absolute Galois group $G_K = Gal(\mathbb{K}/K)$. We will see that finite transcendence degree forces some cohomology groups to vanish.

Definition 2.29 The cohomological dimension of a group G, $cd(G) \leq d$ if and only if $H^r(G, A) = 0$ for all $r > d$ and all torsion modules $A \in G$-mod.

Definition 2.30 A field K is said to be a $Tate^d$-field if the absolute Galois group $G_K = Gal(\mathbb{K}/K)$ satisfies $cd(G) \leq d$.

First, we will reduce the condition $cd(G) \leq d$ to a more manageable one. To start, one can show that a profinite group G (that is, a projective limit of finite groups, see [30] for more details) has $cd(G) \leq d$ if and only if

$$H^{d+1}(G, A) = 0 \text{ for all torsion } G\text{-modules } A$$

Further, as all Galois cohomology groups of profinite groups are torsion, we can decompose the cohomology in its p-primary parts and relate their vanishing to the cohomological dimension of the p-Sylow subgroups G_p of G. This problem can then be verified by computing cohomology of finite simple G_p-modules of p-power order, but for a profinite p-group there is just one such module namely $\mathbb{Z}/p\mathbb{Z}$ with the trivial action.

Combining these facts we have the following manageable criterium on cohomological dimension.

Proposition 2.31 $cd(G) \leq d$ *if* $H^{d+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$ *for the simple G-modules with trivial action $\mathbb{Z}/p\mathbb{Z}$.*

We will need the following spectral sequence in Galois cohomology

Proposition 2.32 *(Hochschild-Serre spectral sequence) If N is a closed normal subgroup of a profinite group G, then*

$$E_2^{p,q} = H^p(G/N, H^q(N, A)) \Longrightarrow H^n(G, A)$$

holds for every continuous G-module A.

Now, we are in a position to state and prove Tate's theorem

Theorem 2.33 *Let K be of transcendence degree d over an algebraically closed field, then K is a $Tate^d$-field.*

Proof. Let $\mathbb{C}$ denote the algebraically closed basefield, then K is algebraic over $\mathbb{C}(t_1, \ldots, t_d)$ and therefore

$$G_K \hookrightarrow G_{\mathbb{C}(t_1,\ldots,t_d)}$$

Thus, K is $Tate^d$ if $\mathbb{C}(t_1, \ldots, t_d)$ is $Tate^d$. By induction it suffices to prove

$$\text{If } cd(G_L) \leq k \text{ then } cd(G_{L(t)}) \leq k+1$$

Let $\mathbb{L}$ be the algebraic closure of L and $\mathbb{M}$ the algebraic closure of $L(t)$. As $L(t)$ and $\mathbb{L}$ are linearly disjoint over L we have the following diagram of extensions and Galois groups

$$\begin{tikzcd}
\mathbb{L} \arrow[r, hook] & \mathbb{L}(t) \arrow[r, hook, "G_{\mathbb{L}(t)}"] & \mathbb{M} \\
L \arrow[u, hook, "G_L"] \arrow[r, hook] & L(t) \arrow[u, hook, "G_L"] \arrow[ur, hook, "G_{L(t)}"'] &
\end{tikzcd}$$

where $G_{L(t)}/G_{\mathbb{L}(t)} \simeq G_L$.

We claim that $cd(G_{\mathbb{L}(t)}) \leq 1$. Consider the exact sequence of $G_{L(t)}$-modules

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{M}^* \xrightarrow{(-)^p} \mathbb{M}^* \longrightarrow 0$$

where μ_p is the subgroup (of $\mathbb{C}^*$) of p-roots of unity. As $G_{L(t)}$ acts trivially on μ_p it is after a choice of primitive p-th root of one isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Taking cohomology with respect to the subgroup $G_{\mathbb{L}(t)}$ we obtain

$$0 = H^1(G_{\mathbb{L}(t)}, \mathbb{M}^*) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{M}^*) = Br(\mathbb{L}(t))$$

But the last term vanishes by Tsen's theorem as $\mathbb{L}(t)$ is the functionfield of a curve defined over the algebraically closed field $\mathbb{L}$. Therefore, $H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all simple modules $\mathbb{Z}/p\mathbb{Z}$, whence $cd(G_{\mathbb{L}(t)}) \leq 1$.

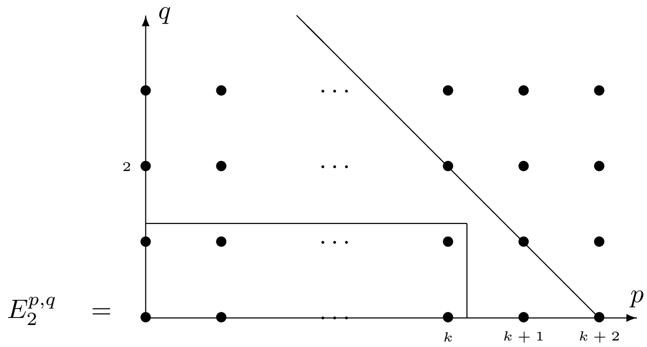
By the inductive assumption we have $cd(G_L) \leq k$ and now we are going to use exactness of the sequence

$$0 \longrightarrow G_L \longrightarrow G_{L(t)} \longrightarrow G_{\mathbb{L}(t)} \longrightarrow 0$$

to prove that $cd(G_{L(t)}) \leq k+1$. For, let A be a torsion $G_{L(t)}$-module and consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_L, H^q(G_{\mathbb{L}(t)}, A)) \Longrightarrow H^n(G_{L(t)}, A)$$

By the restrictions on the cohomological dimensions of G_L and $G_{\mathbb{L}(t)}$ the level two term has following shape



where the only non-zero groups are lying in the lower rectangular region. Therefore, all $E_\infty^{p,q} = 0$ for $p+q > k+1$. Now, all the composition factors of $H^{k+2}(G_{L(t)}, A)$ are lying on the indicated diagonal line and hence are zero. Thus, $H^{k+2}(G_{L(t)}, A) = 0$ for all torsion $G_{L(t)}$-modules A and hence $cd(G_{L(t)}) \leq k+1$. □

Theorem 2.34 *If* $\mathbf{A}$ *is a constant sheaf of an Abelian torsion group* A *on* K_{et}*, then*

$$H^i_{et}(K, \mathbf{A}) = 0$$

whenever $i > trdeg_{\mathbb{C}}(K)$.

2.7 Coniveau spectral sequence

Consider the setting

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\pi"] & K \\ k & \end{tikzcd}$$

where A is a discrete valuation ring in K with residue field $A/m = k$. As always, we will assume that A is a $\mathbb{C}$-algebra. By now we have a grip on the Galois cohomology groups

$$H^i_{et}(K, \mu_n^{\otimes l}) \text{ and } H^i_{et}(k, \mu_n^{\otimes l})$$

and we will use this information to compute the étale cohomology groups

$$H^i_{et}(A, \mu_n^{\otimes l})$$

Here, $\mu_n^{\otimes l} = \underbrace{\mu_n \otimes \ldots \otimes \mu_n}_{l}$ where the tensorproduct is as sheafs of invertible $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$-modules.

We will consider the Leray spectral sequence for i and hence have to compute the derived sheaves of the direct image

Lemma 2.35 *1.* $R^0\ i_*\mu_n^{\otimes l} \simeq \mu_n^{\otimes l}$ *on* A_{et}.

2. $R^1\ i_*\mu_n^{\otimes l} \simeq \mu_n^{\otimes l-1}$ *concentrated in* m.

3. $R^j\ i_*\mu_n^{\otimes l} \simeq 0$ *whenever* $j \geq 2$.

Proof. The strict Henselizations of A at the two primes $\{0, m\}$ are resp.

$$A_0^{sh} \simeq \mathbb{K} \text{ and } A_m^{sh} \simeq \mathbf{k}\{t\}$$

where $\mathbb{K}$ (resp. $\mathbf{k}$) is the algebraic closure of K (resp. k). Therefore,

$$(R^j\ i_* \mu_n^{\otimes l})_0 = H_{et}^j(\mathbb{K}, \mu_n^{\otimes l})$$

which is zero for $i \geq 1$ and $\mu_n^{\otimes l}$ for $j = 0$. Further, $A_m^{sh} \otimes_A K$ is the field of fractions of $\mathbf{k}\{t\}$ and hence is of transcendence degree one over the algebraically closed field $\mathbf{k}$, whence

$$(R^j\ i_* \mu_n^{\otimes l})_m = H_{et}^j(L, \mu_n^{\otimes l})$$

which is zero for $j \geq 2$ because L is $Tate$[1].

For the field-tower $K \subset L \subset \mathbb{K}$ we have that $G_L = \hat{\mathbb{Z}} = \underleftarrow{lim}\ \mu_m$ because the only Galois extensions of L are the Kummer extensions obtained by adjoining $\sqrt[m]{t}$. But then,

$$H_{et}^1(L, \mu_n^{\otimes l}) = H^1(\hat{Z}, \mu_n^{\otimes l}(\mathbb{K})) = Hom(\hat{Z}, \mu_n^{\otimes l}(\mathbb{K})) = \mu_n^{\otimes l-1}$$

from which the claims follow. $\square$

Theorem 2.36 *We have a long exact sequence*

$$0 \longrightarrow H^1(A, \mu_n^{\otimes l}) \longrightarrow H^1(K, \mu_n^{\otimes l}) \longrightarrow H^0(k, \mu_n^{\otimes l-1}) \longrightarrow$$

$$H^2(A, \mu_n^{\otimes l}) \longrightarrow H^2(K, \mu_n^{\otimes l}) \longrightarrow H^1(k, \mu_n^{\otimes l-1}) \longrightarrow \dots$$

Proof. By the foregoing lemma, the second term of the Leray spectral sequence for $i_* \mu_n^{\otimes l}$ looks like

0	0	0	$\dots$
$H^0(k, \mu_n^{\otimes l-1})$	$H^1(k, \mu_n^{\otimes l-1})$	$H^2(k, \mu_n^{\otimes l-1})$	$\dots$
$H^0(A, \mu_n^{\otimes l})$	$H^1(A, \mu_n^{\otimes l})$	$H^2(A, \mu_n^{\otimes l})$	$\dots$

with connecting morphisms

$$H_{et}^{i-1}(k, \mu_n^{\otimes l-1}) \xrightarrow{\alpha_i} H_{et}^{i+1}(A, \mu_n^{\otimes l})$$

The spectral sequences converges to its limiting term which looks like

0	0	0	$\dots$
$Ker\ \alpha_1$	$Ker\ \alpha_2$	$Ker\ \alpha_3$	$\dots$
$H^0(A,\mu_n^{\otimes l})$	$H^1(A,\mu_n^{\otimes l})$	$Coker\ \alpha_1$	$\dots$

and the Leray sequence gives the short exact sequences

$$0 \longrightarrow H^1_{et}(A,\mu_n^{\otimes l}) \longrightarrow H^1_{et}(K,\mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_1 \longrightarrow 0$$
$$0 \longrightarrow Coker\ \alpha_1 \longrightarrow H^2_{et}(K,\mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_2 \longrightarrow 0$$
$$0 \longrightarrow Coker\ \alpha_{i-1} \longrightarrow H^i_{et}(K,\mu_n^{\otimes l}) \longrightarrow Ker\ \alpha_i \longrightarrow 0$$

and gluing these sequences gives us the required result. □

In particular, if A is a discrete valuation ring of K with residue field k we have for each i a connecting morphism

$$H^i_{et}(K,\mu_n^{\otimes l}) \xrightarrow{\partial_{i,A}} H^{i-1}_{et}(k,\mu_n^{\otimes l-1})$$

Like any other topology, the étale topology can be defined locally on any scheme X. That is, we call a morphism of schemes

$$Y \xrightarrow{f} X$$

an étale extension (resp. cover) if locally f has the form

$$f^a \mid U_i : A_i = \Gamma(U_i,\mathcal{O}_X) \longrightarrow B_i = \Gamma(f^{-1}(U_i),\mathcal{O}_Y)$$

with $A_i \longrightarrow B_i$ an étale extension (resp. cover) of algebras.

Again, we can construct the étale site of X locally and denote it with X_{et}. Presheaves and sheaves of groups on X_{et} are defined similarly and the right derived functors of the left exact global sections functor

$$\Gamma : \mathbf{S}^{ab}(X_{et}) \longrightarrow \mathbf{Ab}$$

will be called the cohomology functors and we denote

$$R^i\ \Gamma(\mathbb{G}) = H^i_{et}(X,\mathbb{G})$$

From now on we restrict to the case when X is a smooth, irreducible projective variety of dimension d over $\mathbb{C}$. In this case, we can initiate the computation of the cohomology groups $H^i_{et}(X,\mu_n^{\otimes l})$ via Galois cohomology of functionfields of subvarieties using the coniveau spectral sequence

Theorem 2.37 *Let X be a smooth irreducible variety over $\mathbb{C}$. Let $X^{(p)}$ denote the set of irreducible subvarieties x of X of codimension p with functionfield $\mathbb{C}(x)$, then there exists a coniveau spectral sequence*

$$E_1^{p.q} = \bigoplus_{x\in X^{(p)}} H^{q-p}_{et}(\mathbb{C}(x),\mu_n^{\otimes l-p}) \Longrightarrow H^{p+q}_{et}(X,\mu_n^{\otimes l})$$

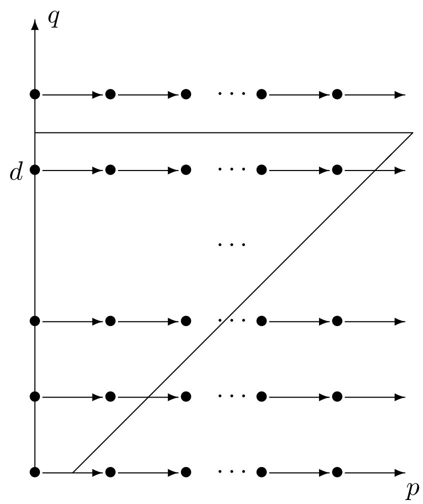
In contrast to the spectral sequences used before, the existence of the coniveau spectral sequence by no means follows from general principles. In it, a lot of heavy machinery on étale cohomology of schemes is encoded. In particular,

- cohomology groups with support of a closed subscheme, see for example [20, p. 91-94], and
- cohomological purity and duality, see [20, p. 241-252]

a detailed exposition of which would take us too far afield. For more details we refer the reader to [5].

Using the results on cohomological dimension and vanishing of Galois cohomology of $\mu_n^{\otimes k}$ when the index is larger than the transcendence degree, we see that the coniveau spectral sequence has the following shape

$E_1^{p,q} \quad =$



where the only non-zero terms are in the indicated region.

Let us understand the connecting morphisms at the first level, a typical instance of which is

$$\bigoplus_{x\in X^{(p)}} H^i(\mathbb{C}(x),\mu_n^{\oplus l-p}) \longrightarrow \bigoplus_{y\in X^{(p+1)}} H^{i-1}(\mathbb{C}(y),\mu_n^{\oplus l-p-1})$$

and consider one of the closed irreducible subvarieties x of X of codimension p and one of those y of codimension $p+1$. Then, either y is not contained in x in which case the component map

$$H^i(\mathbb{C}(x),\mu_n^{\oplus l-p}) \longrightarrow H^{i-1}(\mathbb{C}(y),\mu_n^{\oplus l-p-1})$$

is the zero map. Or, y is contained in x and hence defines a codimension one subvariety of x. That is, y defines a discrete valuation on $\mathbb{C}(x)$ with residue field $\mathbb{C}(y)$. In this case, the above component map is the connecting morphism defined above.

In particular, let K be the functionfield of X. Then we can define the unramified cohomology groups

$$F_n^{i,l}(K/\mathbb{C}) = Ker\ H^i(K,\mu_n^{\otimes l}) \xrightarrow{\oplus\partial_{i,A}} \oplus H^{i-1}(k_A,\mu_n^{\otimes l-1})$$

where the sum is taken over all discrete valuation rings A of K (or equivalently, the irreducible codimension one subvarieties of X) with residue field k_A. By definition, this is a (stable) birational invariant of X. In particular, if X is (stably) rational over $\mathbb{C}$, then

$$F_n^{i,l}(K/\mathbb{C}) = 0 \text{ for all } i,l \geq 0$$

2.8 The Artin-Mumford exact sequence

In this section S will be a smooth irreducible projective surface.

Definition 2.38 S is called simply connected if every étale cover $Y \longrightarrow S$ is trivial, that is, Y is isomorphic to a finite disjoint union of copies of S.

The first term of the coniveau spectral sequence of S has following shape

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	0	0	0	$\cdots$
$H^2(\mathbb{C}(S), \mu_n)$	$\oplus_C H^1(\mathbb{C}(S), \mathbb{Z}_n)$	$\oplus_P \ \mu_n^{-1}$	0	$\cdots$
$H^1(\mathbb{C}(S), \mu_n)$	$\oplus_C \ \mathbb{Z}_n$	0	0	$\cdots$
μ_n	0	0	0	$\cdots$

where C runs over all irreducible curves on S and P over all points of S.

Lemma 2.39 *For any smooth S we have $H^1(\mathbb{C}(S), \mu_n) \twoheadrightarrow \oplus_C \mathbb{Z}_n$. If S is simply connected, $H^1_{et}(S, \mu_n) = 0$.*

Proof. Using the Kummer sequence $1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)} \mathbb{G}_m \longrightarrow 1$ and Hilbert 90 we obtain that

$$H^1_{et}(\mathbb{C}(S), \mu_n) = \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$$

The first claim follows from the exact diagram describing divisors of rational functions

$$\begin{tikzcd}
 & \mu_n \arrow[d] \arrow[r, phantom, "\simeq"] & \mu_n \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d] & \mathbb{C}(S)^* \arrow[r, "div"] \arrow[d, "(-)^n"'] & \oplus_C \mathbb{Z} \arrow[r] \arrow[d, "n."'] & 0 \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d] & \mathbb{C}(S)^* \arrow[r, "div"] \arrow[d] & \oplus_C \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & \oplus_C \mathbb{Z}_n \arrow[r, phantom, "\simeq"] & \oplus_C \mathbb{Z}_n &
\end{tikzcd}$$

By the coniveau spectral sequence we have that $H^1_{et}(S, \mu_n)$ is equal to the kernel of the morphism

$$H^1_{et}(\mathbb{C}(S), \mu_n) \xrightarrow{\gamma} \oplus_C \mathbb{Z}_n$$

and in particular, $H^1(S, \mu_n) \hookrightarrow H^1(\mathbb{C}(S), \mu_n)$.

As for the second claim, an element in $H^1(S,\mu_n)$ determines a cyclic extension $L = \mathbb{C}(S)\sqrt[n]{f}$ with $f \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$ such that in each fieldcomponent L_i of L there is an étale cover $T_i \longrightarrow S$ with $\mathbb{C}(T_i) = L_i$. By assumption no non-trivial étale covers exist whence $f = 1 \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$. □

If we invoke another major tool in étale cohomology of schemes, Poincaré duality, see for example [20, VI,§11], we obtain the following information on the cohomology groups for S.

Proposition 2.40 *(Poincaré duality for S) If S is simply connected, then*

1. $H^0_{et}(S,\mu_n) = \mu_n$
2. $H^1_{et}(S,\mu_n) = 0$
3. $H^3_{et}(S,\mu_n) = 0$
4. $H^4_{et}(S,\mu_n) = \mu_n^{-1}$

Proof. The third claim follows from the second as both groups are dual to each other. The last claim follows from the fact that for any smooth irreducible projective variety X of dimension d one has that

$$H^{2d}_{et}(X,\mu_n) \simeq \mu_n^{\otimes 1-d}$$

□

We are now in a position to state and prove the important

Theorem 2.41 *(Artin-Mumford exact sequence) If S is a simply connected smooth projective surface, then the sequence*

$$0 \longrightarrow Br_n(S) \longrightarrow Br_n(\mathbb{C}(S)) \longrightarrow \oplus_C\, \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow$$
$$\longrightarrow \oplus_P\, \mu_n^{-1} \longrightarrow \mu_n^{-1} \longrightarrow 0$$

is exact.

Proof. The top complex in the first term of the coniveau spectral sequence for S was

$$H^2(\mathbb{C}(S),\mu_n) \xrightarrow{\alpha} \oplus_C\, H^1(\mathbb{C}(C),\mathbb{Z}_n) \xrightarrow{\beta} \oplus_P\, \mu_n$$

The second term of the spectral sequence (which is also the limiting term) has the following form

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	0	0	0	$\cdots$
$Ker\ \alpha$	$Ker\ \beta/Im\ \alpha$	$Coker\ \beta$	0	$\cdots$
$Ker\ \gamma$	$Coker\ \gamma$	0	0	$\cdots$
μ_n	0	0	0	$\cdots$

By the foregoing lemma we know that $Coker\ \gamma = 0$. By Poincare duality we know that $Ker\ \beta = Im\ \alpha$ and $Coker\ \beta = \mu_n^{-1}$. Hence, the top complex was exact in its middle term and can be extended to an exact sequence

$$0 \longrightarrow H^2(S, \mu_n) \longrightarrow H^2(\mathbb{C}(S), \mu_n) \longrightarrow \oplus_C H^1(\mathbb{C}(C), \mathbb{Z}_n) \longrightarrow$$

$$\oplus_P \mu_n^{-1} \longrightarrow \mu_n^{-1} \longrightarrow 0$$

As $\mathbb{Z}_n \simeq \mu_n$ the third term is equal to $\oplus_C \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$ by the argument given before and the second term we remember to be $Br_n(\mathbb{C}(S)$. The identification of $Br_n(S)$ with $H^2(S, \mu_n)$ will be explained below. □

Some immediate consequences can be drawn from this :

- For a smooth simply connected surface S, $Br_n(S)$ is a birational invariant (it is the birational invariant $F_n^{2,1}(\mathbb{C}(S)/\mathbb{C})$ of the foregoing section.
- In particular, if $S = \mathbb{P}^2$ we have that $Br_n(\mathbb{P}^2) = 0$ and we obtain the description of $Br_n(\mathbb{C}(x,y))$ by $\mathbb{Z}_n$-wrinkles as

$$0 \longrightarrow Br_n\ \mathbb{C}(x,y) \longrightarrow \oplus_C\ \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow \oplus_P\ \mu_n^{-1} \longrightarrow \mu_n \longrightarrow 0$$

Example 2.42 If S is not necessarily simply connected, show that any class in $Br(\mathbb{C}(S))_n$ determines a $\mathbb{Z}_n$-wrinkle.

Example 2.43 If X is a smooth irreducible rational projective variety of dimension d, show that the obstruction to classifying $Br(\mathbb{C}(X))_n$ by $\mathbb{Z}_n$-wrinkles is given by $H^3_{et}(X, \mu_n)$.

We will give a ringtheoretical interpretation of the maps in the Artin-Mumford sequence. Observe that nearly all maps are those of the top complex of the first term in the coniveau spectral sequence for S. We gave an explicit description of them using discrete valuation rings. The statements below follow from this description.

Let us consider a discrete valuation ring A with field of fractions K and residue field k. Let Δ be a central simple K-algebra of dimension n^2.

Definition 2.44 An A-subalgebra Λ of Δ will be called an A-*order* if it is a free A-module of rank n^2 with $\Lambda.K = \Delta$. An A-order is said to be maximal if it is not properly contained in any other order.

In order to study maximal orders in Δ (they will turn out to be all conjugated), we consider the completion $\hat{A}$ with respect to the m-adic filtration where $m = At$ with t a uniformizing parameter of A. $\hat{K}$ will denote the field of fractions of $\hat{A}$ and $\hat{\Delta} = \Delta \otimes_K \hat{K}$.

Because $\hat{\Delta}$ is a central simple $\hat{K}$-algebra of dimension n^2 it is of the form

$$\hat{\Delta} = M_t(D)$$

where D is a division algebra with center $\hat{K}$ of dimension s^2 and hence $n = s.t$. We call t the capacity of Δ at A.

In D we can construct a unique maximal $\hat{A}$-order Γ, namely the integral closure of $\hat{A}$ in D. We can view Γ as a discrete valuation ring extending the valuation v defined by A on K. If $v : \hat{K} \longrightarrow \mathbb{Z}$, then this extended valuation

$$w : D \longrightarrow n^{-2}\mathbb{Z} \text{ is defined as } w(a) = (\hat{K}(a) : \hat{K})^{-1} v(N_{\hat{K}(a)/\hat{K}}(a))$$

for every $a \in D$ where $\hat{K}(a)$ is the subfield generated by a and N is the norm map of fields.

The image of w is a subgroup of the form $e^{-1}\mathbb{Z} \hookrightarrow n^{-2}.\mathbb{Z}$. The number $e = e(D/\hat{K})$ is called the ramification index of D over $\hat{K}$. We can use it to normalize the valuation w to

$$v_D : D \longrightarrow \mathbb{Z} \text{ defined by } v_D(a) = \frac{e}{n^2} v(N_{D/\hat{K}}(a))$$

With these conventions we have that $v_D(t) = e$.

The maximal order Γ is then the subalgebra of all elements $a \in D$ with $v_D(a) \geq 0$. It has a unique maximal ideal generated by a prime element T and we have that $\overline{\Gamma} = \frac{\Gamma}{T\ \Gamma}$ is a division algebra finite dimensional over $\hat{A}/t\hat{A} = k$ (but not necessarily having k as its center).

The inertial degree of D over $\hat{K}$ is defined to be the number $f = f(D/\hat{K}) = (\overline{\Gamma} : k)$ and one shows that

$$s^2 = e.f \text{ and } e \mid s \text{ whence } s \mid f$$

After this detour, we can now take $\Lambda = M_t(\Gamma)$ as a maximal $\hat{A}$-order in $\hat{\Delta}$. One shows that all other maximal $\hat{A}$-orders are conjugated to Λ. Λ has a unique maximal ideal M with $\overline{\Lambda} = M_t(\overline{\Gamma})$.

Definition 2.45 With notations as above, we call the numbers $e = e(D/\hat{K})$, $f = f(D/\hat{K})$ and t resp. the ramification, inertia and capacity of the central simple algebra Δ at A. If $e = 1$ we call Λ an Azumaya algebra over A, or equivalently, if $\Lambda/t\Lambda$ is a central simple k-algebra of dimension n^2.

Now let us consider the case of a discrete valuation ring A in K such that the residue field k is $Tsen^1$. The center of the division algebra $\overline{\Gamma}$ is a finite dimensional field extension of k and hence is also $Tsen^1$ whence has trivial Brauer group and therefore must coincide with $\overline{\Gamma}$. Hence,

$$\overline{\Gamma} = k(\overline{a})$$

a commutative field, for some $a \in \Gamma$. But then, $f \leq s$ and we have $e = f = s$ and $k(\overline{a})$ is a cyclic degree s field extension of k.

Because $s \mid n$, the cyclic extension $k(\overline{a})$ determines an element of $H^1_{et}(k, \mathbb{Z}_n)$.

Definition 2.46 Let Z be a normal domain with field of fractions K and let Δ be a central simple K-algebra of dimension n^2. A Z-order B is a subalgebra which is a finitely generated Z-module. It is called maximal if it is not properly contained in any other order. One can show that B is a maximal Z-order if and only if $\Lambda = B_p$ is a maximal order over the discrete valuation ring $A = Z_p$ for every height one prime ideal p of Z.

Return to the situation of an irreducible smooth projective surface S. If Δ is a central simple $\mathbb{C}(S)$-algebra of dimension n^2, we define a maximal order as a sheaf $\mathcal{A}$ of $\mathcal{O}_S$-orders in Δ which for an open affine cover $U_i \hookrightarrow S$ is such that

$$A_i = \Gamma(U_i, \mathcal{A}) \text{ is a maximal } Z_i = \Gamma(U_i, \mathcal{O}_S) \text{ order in } \Delta$$

Any irreducible curve C on S defines a discrete valuation ring on $\mathbb{C}(S)$ with residue field $\mathbb{C}(C)$ which is $Tsen^1$. Hence, the above argument can be applied to obtain from $\mathcal{A}$ a cyclic extension of $\mathbb{C}(C)$, that is, an element of $\mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$.

Definition 2.47 We call the union of the curves C such that $\mathcal{A}$ determines a non-trivial cyclic extension of $\mathbb{C}(C)$ the *ramification divisor* of Δ (or of $\mathcal{A}$).

The map in the Artin-Mumford exact sequence

$$Br_n(\mathbb{C}(S)) \longrightarrow \bigoplus_C H^1_{et}(\mathbb{C}(C), \mu_n)$$

assigns to the class of Δ the cyclic extensions introduced above.

Definition 2.48 An S-Azumaya algebra (of index n) is a sheaf of maximal orders in a central simple $\mathbb{C}(S)$-algebra Δ of dimension n^2 such that it is Azumaya at each curve C, that is, such that $[\Delta]$ lies in the kernel of the above map.

Observe that this definition of Azumaya algebra coincides with the one given in the discussion of twisted forms of matrices. One can show that if $\mathcal{A}$ and $\mathcal{A}'$ are S-Azumaya algebras of index n resp. n', then $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}'$ is an Azumaya algebra of index $n.n'$. We call an Azumaya algebra trivial if it is of the form $End(\mathcal{P})$ where $\mathcal{P}$ is a vectorbundle over S. The equivalence classes of S-Azumaya algebras can be given a group-structure called the Brauer-group $Br(S)$ of the surface S.

2.9 Brauer-Severi schemes

Now that we have some control over the central simple algebras over functionfields, we will generalize the classical notion of Brauer-Severi variety of a central simple algebra to the setting of (maximal) orders.

Fix a projective *normal* variety X with function field $\mathbb{C}(X)$ and let Δ be a central simple $\mathbb{C}(X)$-algebra of dimension n^2. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras. We call $\mathcal{A}$ an $\mathcal{O}_X$-order in Δ if and only if for every affine open subset $U \hookrightarrow X$ we have that the sections $A(U) = \Gamma(U, \mathcal{A})$ is a finite module over the integrally closed domain $R(U) = \Gamma(U, \mathcal{O}_X)$ such that

$$A(U) \otimes_{R(U)} \mathbb{C}(X) \simeq \Delta$$

We will define the Brauer-Severi scheme $BS(\mathcal{A})$ of $\mathcal{A}$ locally so we fix an affine open set U and denote $A = A(U)$ and $R = R(U)$. Let $\mathbb{K}$ be the algebraic closure of $\mathbb{C}(X)$, then we have the natural inclusions

$$\begin{tikzcd} A \arrow[r, hook] & \Delta \arrow[r, hook] & \Delta \otimes \mathbb{K} \simeq M_n(\mathbb{K}) \\ R \arrow[r, hook] \arrow[u, hook] & \mathbb{C}(X) \arrow[r, hook] \arrow[u, hook] & \mathbb{K} \arrow[u, hook] \end{tikzcd}$$

By Galois descent we can define a linear trace map $\Delta \xrightarrow{t} \mathbb{C}(X)$ such that for all $\delta \in \Delta$

$$t(\delta) = tr(\delta \otimes 1)$$

with tr the usual trace map on $M_n(\mathbb{K})$. That is, $A \xrightarrow{t} A$ is a trace map on the order A satisfying the Cayley-Hamilton identities of $n \times n$ matrices such that the image of the trace map is the center R..

One of the major results we will prove in chapter 4 is that this allows to reconstruct both the order A and the center R from geometrical data. Consider the affine scheme of all n-dimensional trace preserving representations of A,

$$rep^t_n\ A = \{A \xrightarrow{\phi} M_n(\mathbb{C}) \ \mid \ tr \circ \phi = \phi \circ t\ \}$$

where tr is the ordinary trace map on $M_n(\mathbb{C})$. Conjugation by $GL_n(\mathbb{C})$ in the target space $M_n(\mathbb{C})$ induces a $GL_n(\mathbb{C})$-action on $rep^t_n\ A$. We will show in chapter 4 that

- A is the ring of $GL_n(\mathbb{C})$-equivariant maps from $rep_n^t\ A$ to $M_n(\mathbb{C})$, and
- R is the ring of polynomial $GL_n(\mathbb{C})$-invariants on $rep_n^t\ A$.

We use these representation spaces to define the Brauer-Severi scheme $BS(A)$ in a fashion very similar to the Hilbert scheme construction in the previous chapter. In fact, historically these varieties were introduced and studied by M. Nori [22] who called them noncommutative Hilbert schemes. We follow here the account of M. Van den Bergh in [31].

Consider the $GL_n(\mathbb{C})$ action on the product scheme $rep_n^t\ A \times \mathbb{C}^n$ given by

$$g.(\phi, v) = (g\ \phi\ g^{-1}, gv)$$

In this product we consider the set of *Brauer stable* points which are

$$Brauer^s(A) = \{(\phi, v)\ \mid\ \phi(A)v = \mathbb{C}^n\}$$

which is also the subset of points with trivial stabilizer subgroup. Hence, every $GL_n(\mathbb{C})$-orbit in $Brauer^s(A)$ is closed and we can form the orbit space which we call the *Brauer-Severi scheme* of the order A

$$BS(A) = Brauer^s(A)/GL_n(\mathbb{C}).$$

This is shown to be a projective space bundle over the quotient variety $rep_n^t\ A/GL_n(\mathbb{C})$ which by the above is the variety corresponding to R, that is, the chosen affine open subset of the projective normal variety X.

For arbitrary orders not much can be said about these Brauer-Severi schemes. We will now restrict to *smooth orders* A that is such that their representation space $rep_n^t\ A$ is a smooth $GL_n(\mathbb{C})$-variety. In chapter 5 we will prove that this geometric condition is equivalent to the algebraic characterization of A via the lifting property modulo nilpotent ideals in the category of algebras equipped with a trace map satisfying the Cayley-Hamilton identities of $n \times n$ matrices.

Lemma 2.49 *Let A be a smooth order, that is, $rep_n^t\ A$ is a smooth variety. Then, the Brauer-Severi scheme $BS(A)$ is a smooth variety.*

Proof. As the action of $GL_n(\mathbb{C})$ on $Brauer^s(A)$ is free, it suffices to prove that $Brauer^s(A)$ is a smooth variety. As $Brauer^s(A)$ is a Zariski open subset of the variety $rep_n^t\ A \times \mathbb{C}^n$ which is smooth by assumption, the result follows. □

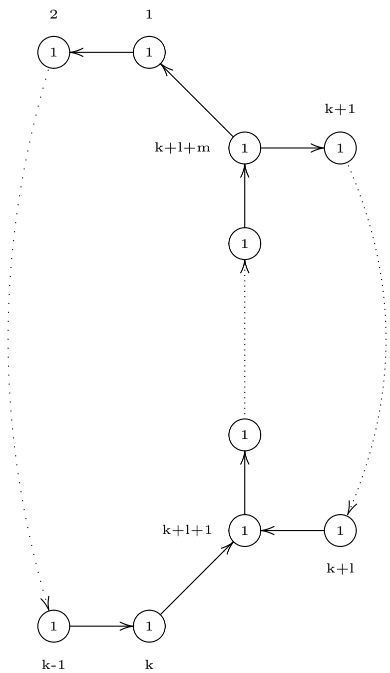
Remains to classify the smooth orders A. The strategy we will follow is : first compute the étale local structure of these orders, that is, if $\mathfrak{m} \triangleleft R$ is a maximal ideal of R we describe

$$\hat{A}_{\mathfrak{m}} = A \otimes_R \hat{R}_{\mathfrak{m}}$$

These structures will follow by combining the étale slice results in invariant theory with the geometric reconstruction of an order A from its representation space $rep_n^t\ A$. The local structures can be classified combinatorial by quiver-data.

In the special case of orders over smooth surfaces we will show that the relevant data is given a $\mathbb{Z}_n$-loop. That is,

- A two loop quiver A_{klm} with dimension vector $\alpha = (1, \ldots, 1)$



  and the vertices numbered as indicated. In this picture we make the natural changes whenever k or l is zero.

- An unordered partition of n in $k + l + m$ nonzero parts

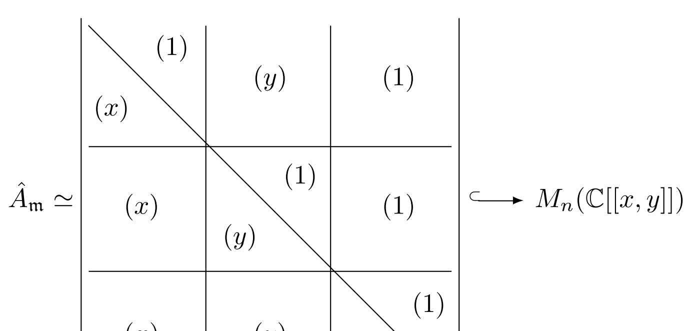
$$p = (p_1, \ldots, p_{k+l+m})$$

  associated to the vertices of the quiver

We will prove in chapter 6 that a $\mathbb{Z}_n$-loop encodes the following algebraic data. Let $\mathfrak{m}$ be a maximal ideal of R, then there is a closed $GL_n(\mathbb{C})$-orbit in $rep_n^t\ A$ corresponding to $\mathfrak{m}$. This closed orbit determines a semi-simple n-dimensional representation of A. the fact that all dimension components are equal to one asserts that all the simple components of this representation occur with multiplicity one and the components of the partition p give the dimensions of these simple components.

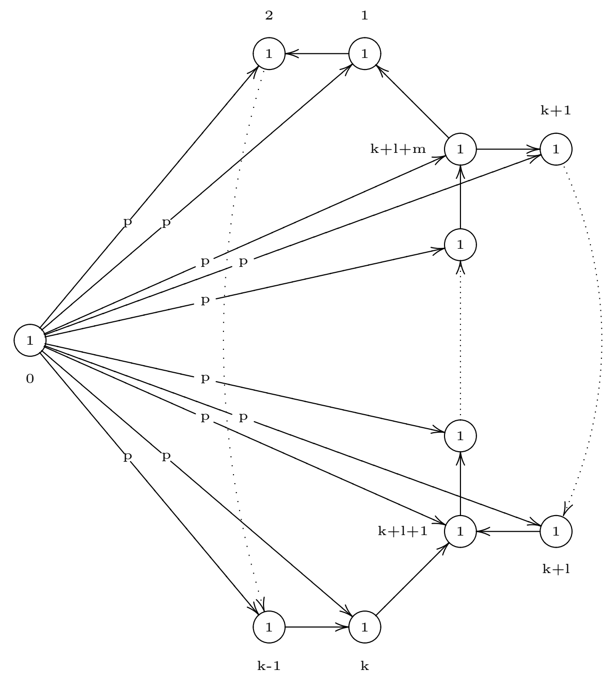
We will prove in chapter 6 a local characterization of smooth orders in arbitrary dimension. In the special case of surfaces we have the following result

Theorem 2.50 *Let Δ be a central simple algebra of dimension n^2 over $\mathbb{C}(S)$ where S is a smooth projective surface and let $\mathcal{A}$ be an $\mathcal{O}_S$-order in Δ. Then $\mathcal{A}$ is a sheaf of smooth orders if and only if for every affine open subset $U \hookrightarrow S$ and section algebras $A = \Gamma(U, \mathcal{A})$ and $R = \Gamma(U, \mathcal{O}_S)$ we have for every maximal ideal $\mathfrak{m}$ a $\mathbb{Z}_n$-loop (A_{klm}, p) such that in local coordinates x, y of S near the point corresponding to $\mathfrak{m}$*



where at spot (i,j) *with* $1 \leq i,j \leq k+l+m$ *there is a block of dimension* $p_i \times p_j$ *with entries the indicated ideal of* $\mathbb{C}[[x,y]]$.

Using such an explicit local description of the order, it is also possible to determine the étale local structure of the Brauer-Severi variety $BS(A)$ in a neighborhood of the closed fiber corresponding to $\mathfrak{m}$ as well as the structure of the fiber $\pi^{-1}(\mathfrak{m})$. Assume A is locally of type (A_{klm}, p) and construct the extended quiver



That is, we add on extra vertex labeled zero and connect it to vertex i by p_i directed arrows where p_i is the i-th component of the unordered partition p. In chapter 8 we will prove that the local structure of $BS(A)$ is determined by the *moduli space* of θ-stable representations of this extended quiver for a certain character θ.

The fiber of the structural morphism $BS(\mathcal{B}) \xrightarrow{\pi} S$ over the point corresponding to $\mathfrak{m}$ we will show in chapter 8 to be the moduli space of the θ-stable representations in the nullcone of the quiver.

Rather than introducing all these concepts here we will illustrate these results in the case of the smooth orders in quaternion algebras considered by M. Artin and D. Mumford in [3].

Example 2.51 Smooth quaternion orders over surfaces.

Let $\mathcal{A}$ be a maximal order in a quaternion division algebra over a smooth projective surface S such that the ramification divisor is a disjoint union of smooth curves. We restrict to affine sections on an affine open subset U and call them again A and R. If $\mathfrak{m} \triangleleft R$ is a maximal ideal corresponding to a point on S not contained in the ramification divisor of $\mathcal{A}$, then A_m is an Azumaya algebra and as the Brauer group of every Henselian local ring is trivial, it follows that in these points the étale local structure of A must be

$$\hat{A}_{\mathfrak{m}} \simeq M_2(\mathbb{C}[[x,y]])$$

for suitable local variables x and y. If however $\mathfrak{m}$ corresponds to a point on the ramification divisor we have seen before that a local description of $A_{\mathfrak{m}}$ is the free

$R_{\mathfrak{m}}$-module spanned by $\{1, i, j, ij\}$ such that

$$\begin{cases} i^2 &= a \\ j^2 &= bt \\ ji &= -ij \end{cases}$$

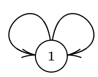
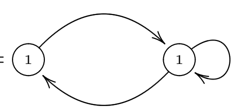
where a, b are units in $R_{\mathfrak{m}}$ and t is a local equation of the divisor in the point. That is, after splitting Δ with say the quadratic extension by adding $\sqrt{a}$ we can view the up-tensored $A_{\mathfrak{m}}$ to be the subalgebra over the center by the matrices

$$\begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix} \qquad \begin{bmatrix} 0 & t \\ -b & 0 \end{bmatrix}$$

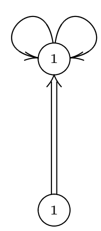
Hence, if we take our local variables to be such that $x = t$ we obtain for the étale local structure

$$\hat{A}_{\mathfrak{m}} \simeq \begin{bmatrix} \mathbb{C}[[x,y]] & (x) \\ \mathbb{C}[[x,y]] & \mathbb{C}[[x,y]] \end{bmatrix}.$$

In our quiver-approach there are just two possibilities for $\mathbb{Z}_2$-loops. They are

- type 1 : $A_{001} =$  and $p = \boxed{2}$,
- type 2 : $A_{101} =$  and $p = \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array}$.

We observe that type 1 is precisely the Azumaya case and type 2 corresponds to a point on the ramification divisor. To compute the fiber of $BS(\mathcal{A})$ over a type 1 point (an Azumaya point) we have to consider the quiver

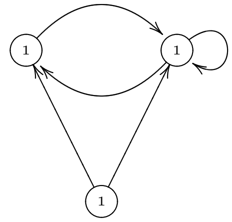


and we need to classify orbits of θ-stable representations where $\theta = (2, -2)$ in the nullcone containing the vertex space in v_0. Being in the nullcone means that all evaluations around oriented cycles in the quiver should be zero, so the two loop-matrices must be zero. Being θ-stable means that the representation has no proper subrepresentation, say with dimension vector $\beta = (1, n_1)$ such that $\langle \theta, \beta >= 2 - 2n_1 > 0$. In this case this means that either of the two matrices corresponding to the two extra arrows must be nonzero. That is, the relevant representation space is $\mathbb{C}^2$. Considering the $\mathbb{C}^* \times \mathbb{C}^*$-action on these representations

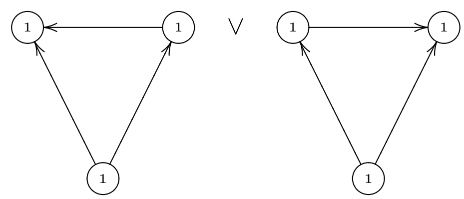
$$(\lambda, \mu).(a, b) = \tfrac{\mu}{\lambda}(a, b)$$

we see that the classifying space is $\mathbb{P}^1(\mathbb{C})$ as it should be over an Azumaya point.

The fiber over a type 2 point (a ramified point) is determined by the quiver



Again, we have to consider representations in the nullcone, meaning that the loop-matrix is zero and that at least one of the two arrows in the A_{101}-quiver must be zero. This time we have to consider θ-stable representations where $\theta = (2, -1, -1)$ which means that the representation is not allowed to have a proper subrepresentation of dimension vector $\beta = (1, n_1, n_2)$ such that $\langle \theta, \beta \rangle = 2 - n_1 - n_2 > 0$. If one of the arrows a in A_{101} is non-zero this means that the extra arrow ending in the source of a must be nonzero and if both arrows in A_{101} are zero the two extra arrows must be non-zero. That is, we have to classify the orbits of the quiver-representations



giving us the required $\mathbb{P}^1 \vee \mathbb{P}^1$ as classifying space.

In this book we will give combinatorial tools to extend these descriptions of Brauer-Severi schemes of smooth orders both the higher n and to higher dimensional base varieties.

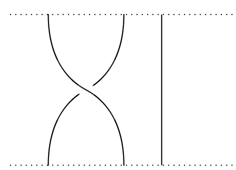
Classical Structures.

Among the plenitude of Quillen-smooth algebras there is a small subclass of well understood examples : the path algebras of quivers. The main result we will prove in this part asserts that for an arbitrary Quillen-smooth algebra A, the local study of the approximation at level n, $A@_n$, is controlled by a quiver setting. In this introduction, we will first give an example of this theory and then we will briefly indicate the relationship between this reduction result and the theory of A_∞-algebras.

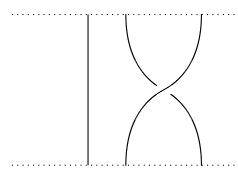
Consider Artin's braid group B_3 on three strings. B_3 has the presentation

$$B_3 \simeq \langle L, R \mid LR^{-1}L = R^{-1}LR^{-1} \rangle$$

where L and R are the fundamental 3-braids



L



R

If we let $S = LR^{-1}L$ and $T = R^{-1}L$, an algebraic manipulation shows that

$$B_3 = \langle S, T \mid T^3 = S^2 \rangle$$

is an equivalent presentation for B_3. The center of B_3 is the infinite cyclic group generated by the braid

$$Z = S^2 = (LR^{-1}L)^2 = (R^{-1}L)^3 = T^3$$

It follows from the second presentation of B_3 that the quotient group modulo the center is isomorphic to

$$\frac{B_3}{\langle Z \rangle} \simeq \langle s, t \mid s^2 = 1 = t^3 \rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_3$$

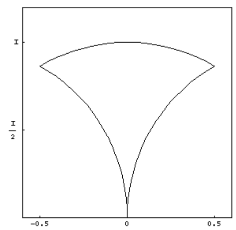
the free product of the cyclic group of order 2 (with generator s) and the cyclic group of order 3 (with generator t). This group is isomorphic to the modular group $PSL_2(\mathbb{Z})$ via

$$\overline{L} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \overline{R} \longrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

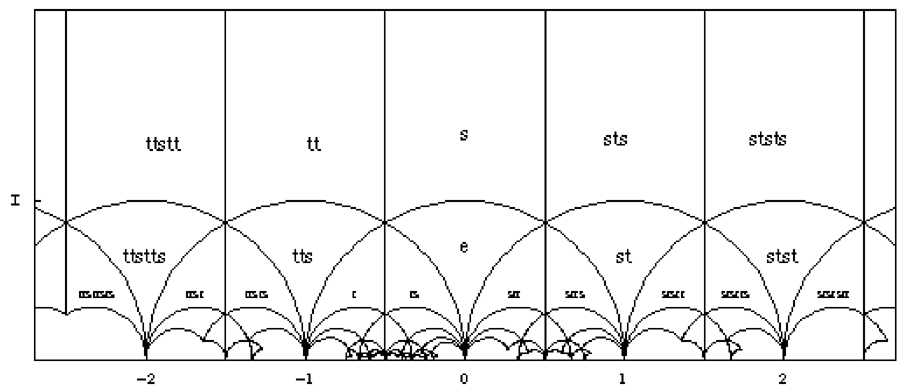
It is well known that the modular group $PSL_2(\mathbb{Z})$ acts on the upper half-plane H^2 by left multiplication in the usual way, that is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : H^2 \longrightarrow H^2 \quad \text{given by} \quad z \longrightarrow \frac{az+b}{cz+d}$$

The fundamental domain $H^2/PSL_2(\mathbb{Z})$ for this action is the hyperbolic triangle



and the action defines a quilt-tiling [?] on the hyperbolic plane, indexed by elements of $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$



By the above, the representation theory of the 3-string braid group B_3 is essentially reduced to that of the modular group $PSL_2(\mathbb{Z})$. The latter can be studied using noncommutative geometry as the group algebra $\mathbb{C}PSL_2(\mathbb{Z})$ is a Quillen-smooth algebra. Indeed,

$$\mathbb{C}PSL_2(\mathbb{Z}) \simeq \mathbb{C}\mathbb{Z}_2 * \mathbb{C}\mathbb{Z}_3 \simeq (\mathbb{C} \times \mathbb{C}) * (\mathbb{C} \times \mathbb{C} \times \mathbb{C})$$

and the free product of Quillen-smooth algebras is again Quillen-smooth as follows immediately from the universal property of free products. Phrased differently, the group algebra $\mathbb{C}PSL_2(\mathbb{Z})$ is the coordinate ring of the noncommutative product of two commutative points with three commutative points.

For a fixed integer n we want to determine the isomorphism classes of all n-dimensional representations of $PSL_2(\mathbb{Z})$, or equivalently, of the Quillen-smooth algebra $\mathbb{C}PSL_2(\mathbb{Z})$. We give a geometric reformulation of this problem. Let $rep_n\ \mathbb{C}PSL_2(\mathbb{Z})$ be the representation variety of $\mathbb{C}PSL_2(\mathbb{Z})$, that is, its geometric points are algebra morphisms

$$\mathbb{C}PSL_2(\mathbb{Z}) \xrightarrow{\phi} M_n(\mathbb{C})$$

From the presentation of $PSL_2(\mathbb{Z})$ we see that we can identify it with the closed subvariety of the affine space $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$

$$rep_n\ \mathbb{C}PSL_2(\mathbb{Z}) = \{(A, B) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \ \mid\ A^2 = \mathbb{1}_n = B^3\}$$

It follows from Quillen-smoothness that $rep_n\ \mathbb{C}PSL_2(\mathbb{Z})$ is a smooth affine variety (though not necessarily connected) for every n. On this representation space, the group of invertible matrices $GL_n(\mathbb{C})$ acts by simultaneous conjugation, that is

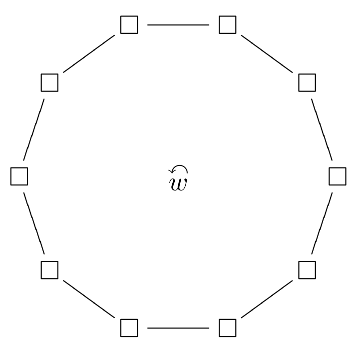
$$g.(A, B) = (gAg^{-1}, gBg^{-1})$$

and isomorphism classes of representations correspond to $GL_n(\mathbb{C})$-orbits. That is, we have to classify the orbits in $rep_n\ \mathbb{C}PSL_2(\mathbb{Z})$. For $n > 1$ there is no Hausdorff orbit-space due to the existence of non-closed orbits. For this reason, the classification of isomorphism classes of n-dimensional representations of $PSL_2(\mathbb{Z})$ is split into two sub-problems :

- The construction of an algebraic quotient map
$$rep_n\ \mathbb{C}PSL_2(\mathbb{Z}) \xrightarrow{\pi} iss_n\ \mathbb{C}PSL_2(\mathbb{Z})$$
classifying closed $GL_n(\mathbb{C})$-orbits, which we will show to be the same as isomorphism classes of semi-simple n-dimensional representations.

- For $x \in iss_n\ \mathbb{C}PSL_2(\mathbb{Z})$, the classification of all orbits in the fiber $\pi^{-1}(\xi)$. That is, if M_ξ is the corresponding semi-simple n-dimensional representation, we want to classify all representations having M_ξ as the direct sum of its Jordan-Hölder components.

To solve the first, we introduce the approximation at level n, denoted by $\mathbb{C}PSL_2(\mathbb{Z})@_n$. To define it we first adjoin formal traces to the groupalgebra. That is, we consider the commutative polynomial algebra in the variables t_w where w runs through all necklaces w of length $l \geq 0$



where each of beads is either $\boxed{s}$ or $\boxed{t}$ subject to the conditions that no two (resp. three) consecutive beads are labeled s (resp. t). With $\mathbb{C}PSL_2(\mathbb{Z})^t$ we denote the tensor product $\mathbb{C}[t_w \mid w \text{ necklace}] \otimes_{\mathbb{C}} \mathbb{C}PSL_2(\mathbb{Z})$ and there is a natural $\mathbb{C}$-linear trace map
$$\mathbb{C}PSL_2(\mathbb{Z})^t \xrightarrow{tr} \mathbb{C}[t_w \mid w \text{ necklace}]$$
defined by sending each monomial $m = s^{i_1} \ldots t^{i_l}$ in the noncommuting variables s and t to t_w where w is the cyclic word constructed from m. As we are interested in n-dimensional representations we would like to interpret t_w as the character $tr(m) = Tr(A^{i_1} \ldots B^{i_l})$. For this reason we consider the quotient
$$\mathbb{C}PSL_2(\mathbb{Z}) = \frac{\mathbb{C}PSL_2(\mathbb{Z})^t}{(t_1 - n, \chi_m^{(n)}(m))}$$
where for each monomial m we define the formal Cayley-Hamilton polynomial $\chi_m^{(n)}(t)$ of m of degree n to be the polynomial in $\mathbb{C}[t_w \mid w][t]$ obtained after expressing the coefficients of the polynomial $f(t) = \prod_{i=1}^n (t - \lambda_i)$ which are symmetric functions in the λ_i as polynomials in the Newton functions $\eta_i = \sum \lambda_j^i$ and replacing η_i by $tr(m^i)$. By construction it follows that there is a one-to-one correspondence between n-dimensional representations of the group algebra $\mathbb{C}PSL_2(\mathbb{Z})$ and n-dimensional trace preserving representations of $\mathbb{C}PSL_2(\mathbb{Z})@_n$. We will prove in

chapter 4 that $\mathbb{C}PSL_2(\mathbb{Z})@_n$ is a finitely generated module over the commutative central subalgebra $tr(\mathbb{C}PSL_2(\mathbb{Z})@_n)$ which is also

$$tr(\mathbb{C}PSL_2(\mathbb{Z})@_n) = \mathbb{C}[iss_n \ PSL_2(\mathbb{Z})]$$

the coordinate ring of the quotient variety as semi-simple representations are determined by their characters. The algebraic quotient map $rep_n \ PSL_2(\mathbb{Z}) \twoheadrightarrow iss_n \ PSL_2(\mathbb{Z})$ is given by sending an n-dimensional representation to its set of characters $tr(A^{i_1} \dots B^{i_l})$. Moreover, we will prove in chapter 5 that the approximation at level n can be geometrically reconstructed

$$\mathbb{C}PSL_2(\mathbb{Z})@_n = \{ \ rep_n \ PSL_2(\mathbb{Z}) \xrightarrow{equiv} M_n(\mathbb{C}) \ \}$$

as the algebra of all $GL_n(\mathbb{C})$-equivariant maps from the representation space to $n \times n$ matrices. Both results are valid for any affine $\mathbb{C}$-algebra A and follow from invariant theory and the generic case of m-tuples of $n \times n$ matrices under simultaneous conjugation, which we will prove in chapter 3.

Now, let $\xi \in iss_n \ PSL_2(\mathbb{Z})$ be the point corresponding to the semi-simple n-dimensional representation

$$M_\xi = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

where the S_i are distinct irreducible d_i-dimensional representations which occur in M_ξ with multiplicity e_i (hence, $n = \sum d_i e_i$). Using the theory of étale slices we will prove in chapter 5 that the étale local structure of the quotient variety $iss_n \ PSL_2(\mathbb{Z})$ in a neighborhood of ξ is fully determined by combinatorial data consisting of

- a quiver Q_ξ on k vertices corresponding to the distinct simple components of M_ξ, and
- a dimension vector $\alpha_\xi = (e_1, \dots, e_k)$ corresponding to the multiplicities of these simple components in M_ξ.

The local quiver Q_ξ is constructed as follows (a proof will be given in chapter 7). If S is a simple $PSL_2(\mathbb{Z})$-representation, we can decompose its restrictions to the cyclic subgroups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ into one-dimensional eigenspaces

$$\begin{cases} S \ \downarrow_{\mathbb{Z}_2} & \simeq E_1^{\oplus a_1} \oplus E_{-1}^{\oplus a_2} \\ S \ \downarrow_{\mathbb{Z}_3} & \simeq F_1^{\oplus b_1} \oplus F_\zeta^{\oplus b_2} \oplus F_{\zeta^2}^{\oplus b_3} \end{cases}$$

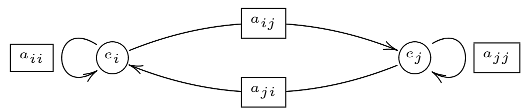
where E_λ resp. F_λ are the one-dimensional simple representations on which s resp. t acts via multiplication with λ. We will call this 5-tuple

$$\tau(S) = \begin{array}{|cc|} \hline & b_1 \\ a_1 & \\ & b_2 \\ a_2 & \\ & b_3 \\ \hline \end{array}$$

to be the type of S. If the dimension of S, $d(S) = d$, we will show in chapter 7 that these numbers must satisfy the relations (or see [**?**] for another proof)

$$\begin{cases} d & = a_1 + a_2 = b_1 + b_2 + b_3 \\ a_i & \geq b_j \text{ for all } i, j \end{cases}$$

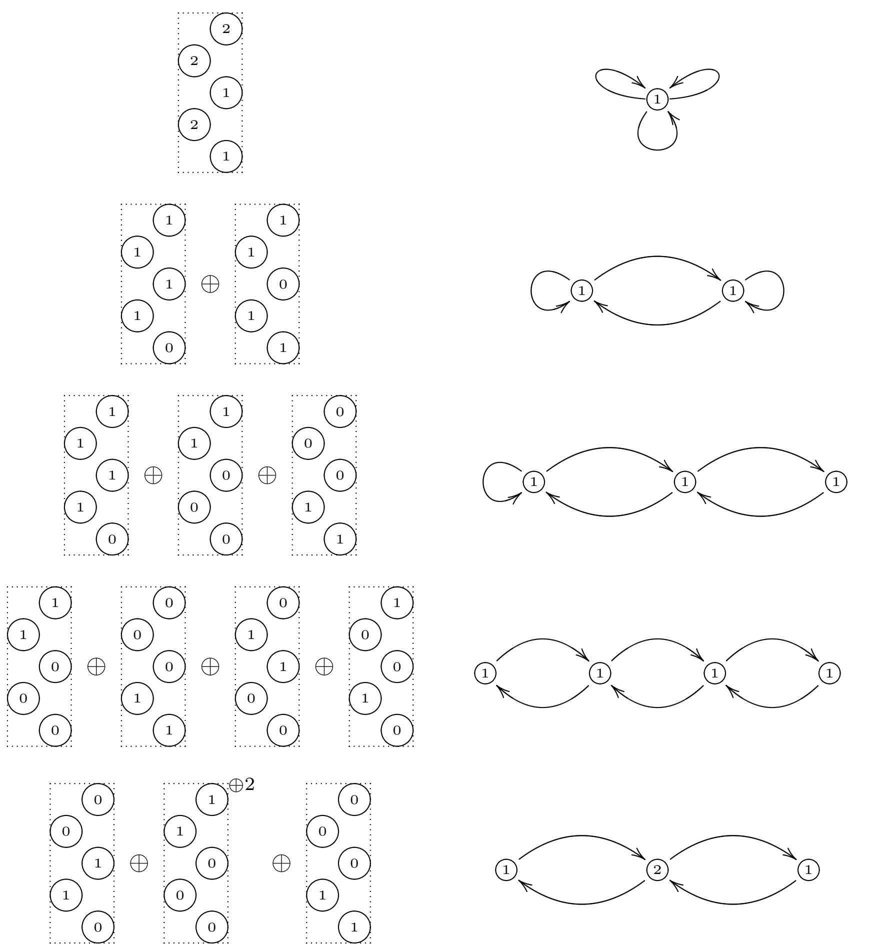
With these notations, the local quiver Q_ξ has the following local shape for every two of its vertices v_i and v_j



where the numbers of multiple arrows and loops are determined by the formulas

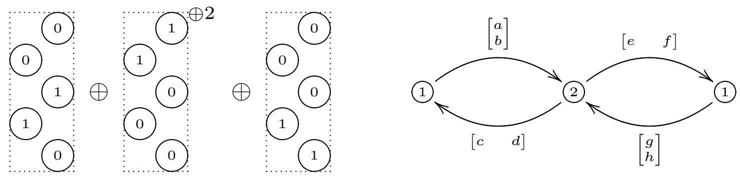
$$\begin{cases} a_{ij} & = d(S_i)d(S_j) - \langle \tau(S_i), \tau(S_j) \rangle \text{ when } i \neq j \\ a_{ii} & = 1 + d(S_i)^2 - \langle \tau(S_i), \tau(S_i) \rangle \end{cases}$$

where $\langle -, - \rangle$ is the usual inproduct on 5-tuples. For example, $iss_4\ PSL_2(\mathbb{Z})$ has several components of dimension 3 and 2. For one of the three 3-dimensional components, the different types of semi-simples M_ξ and corresponding local quivers Q_ξ of the 3-dimensional component are listed below.



In chapter 5 we prove that the étale local structure of $iss_n\ PSL_2(\mathbb{Z})$ near ξ is isomorphic to that of $iss_{\alpha_\xi}\ Q_\xi$ near the trivial representation. The local algebra of the latter is generated by traces along oriented cycles in Q_ξ. That is, for every arrow $e_j \xleftarrow{a} e_i$ we take an $e_j \times e_i$ matrix M_a of indeterminates. Multiplying these matrices along an oriented cycle in Q_ξ and taking the trace of the square matrix obtained gives an invariant function. Such invariants generate the local algebra of $iss_{\alpha_\xi}\ Q_\xi$ in the trivial semi-simple representation. Therefore, to verify whether

$iss_n\ PSL_2(\mathbb{Z})$ is smooth in ξ it suffices to prove that the traces along oriented cycle for the quiver-setting (Q_ξ, α_ξ) generate a polynomial algebra. For example, consider a point $\xi \in iss_4\ PSL_2(\mathbb{Z})$ of type

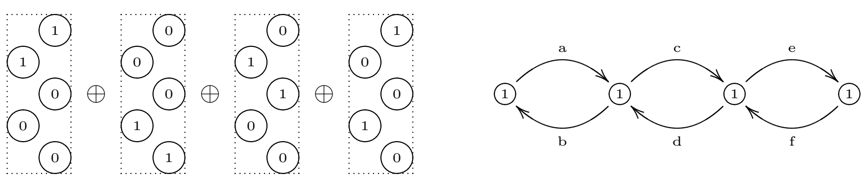


Then, the traces along oriented cycles in Q_ξ are generated by the following three algebraic independent polynomials

$$\begin{cases} x &= ac + bd \\ y &= eg + fh \\ z &= (cg + dh)(ea + fb) \end{cases}$$

and hence $iss_4\ PSL_2(\mathbb{Z})$ is smooth in ξ. The other cases being easier, we see that this component of $iss_4\ PSL_2(\mathbb{Z})$ is a smooth manifold.

Another application of this local quiver-setting (Q_ξ, α_ξ) is that one can construct families of irreducible representations of $PSL_2(\mathbb{Z})$ starting from known ones. For example consider the point ξ of type



Then, M_ξ is determined by the following matrices

$$(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix})$$

The quiver-setting (Q_ξ, α_ξ) implies that any nearby orbit is determined by a matrix-couple

$$(\begin{bmatrix} 1 & b_1 & 0 & 0 \\ a_1 & -1 & d_1 & 0 \\ 0 & c_1 & 1 & f_1 \\ 0 & 0 & e_1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & b_2 & 0 & 0 \\ a_2 & \zeta^2 & d_2 & 0 \\ 0 & c_2 & \zeta & f_2 \\ 0 & 0 & e_2 & 1 \end{bmatrix})$$

and as there is just one arrow in each direction these entries must satisfy

$$0 = a_1a_2 = b_1b_2 = c_1c_2 = d_1d_2 = e_1e_2 = f_1f_2$$

As the square of the first matrix must be the identity matrix $\mathbb{1}_4$, we have in addition that

$$0 = a_1b_1 = c_1d_1 = e_1f_1$$

Hence, we get several sheets of 3-dimensional families of representations (possibly, matrix-couples lying on different sheets give isomorphic $PSL_2(\mathbb{Z})$-representations, as the isomorphism holds in the étale topology and not necessarily in the Zariski topology). One of the sheets has representatives

$$(\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & -1 & d & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e & -1 \end{bmatrix}, \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & c & \zeta & f \\ 0 & 0 & 0 & 1 \end{bmatrix})$$

From the description of dimension vectors of semi-simple quiver representations we will give in chapter 6 it follows that such a representation is simple if and only if

$$ab \neq 0 \quad cd \neq 0 \quad \text{and} \quad ef \neq 0$$

Moreover, these simples are not-isomorphic unless their traces ab, cd and ef evaluate to the same numbers.

A final application of the local quiver-setting is that it solves the second sub-problem. That is, assume that $\xi \in iss_n\ PSL_2(\mathbb{Z})$ has local quiver-setting (Q_ξ, α_ξ), then the isomorphism classes of $PSL_2(\mathbb{Z})$-representations having as direct sum of its Jordan-Hölder components the semi-simple representation M_ξ are in one-to-one correspondence with the $GL(\alpha) = GL_{e_1}(\mathbb{C}) \times \ldots \times GL_{e_k}(\mathbb{C})$-orbits in the nullcone of the quiver representation space $rep_{\alpha_\xi}\ Q_\xi$. In chapter 8 we will see how we can stratify these nullcones to get a handle on this problem. In the above example, this nullcone problem is quite trivial. A representation has M_ξ as Jordan-Hölder sum if and only if all traces vanish, that is,

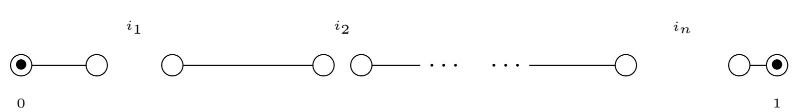
$$ab = cd = ef = 0$$

Under the action of the group $GL(\alpha_\xi) = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$, these orbits are easily seen to be classified by the arrays

a	c	e
b	d	f

filled with zeroes and ones subject to the rule that no column can have two 1's, giving $27 = 3^3$-orbits. In chapter 13 we will give more applications to the representation theory of B_3, $PSL_2(\mathbb{Z})$ and, more generally, knot groups.

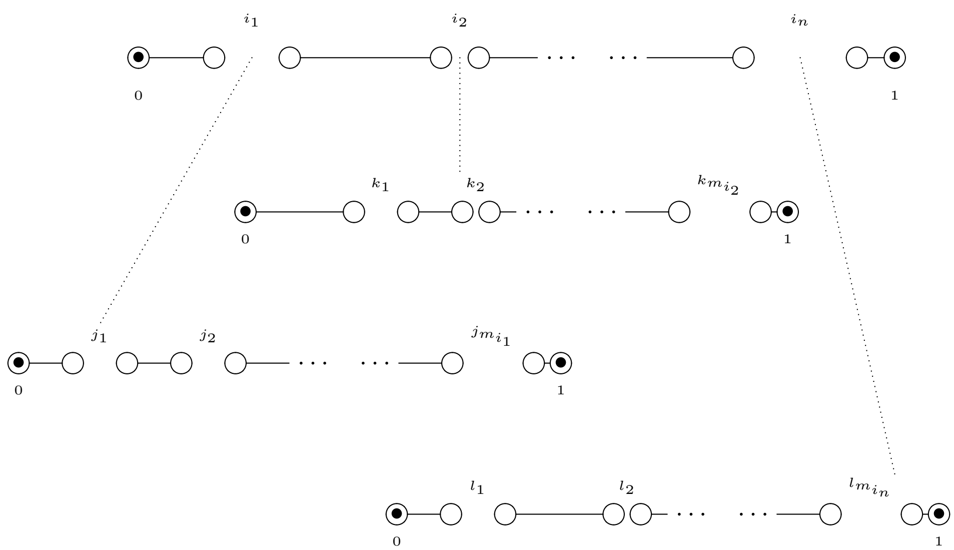
Although we will arrive at the local quiver-setting (Q_ξ, α_ξ) by invariant theory we will indicate an alternative approach using the theory of A_∞-algebras. More details can be found in the excellent notes of B. Keller [**?**]. In recent years some families of multi-linear objects satisfying certain extended associativity constraints have been studied which are naturally associated to topological operads. For our purposes, the relevant operad is the *tiny interval operad*. That is, let $D_1(n)$ be the collection of all configurations



consisting of the unit interval with n closed intervals cut out, each gap given a label i_j where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$. Clearly, $D_1(n)$ is a $2n$-dimensional $\mathcal{C}^\infty$-manifold having $n!$ connected components, each of which is a contractible space. the operadic structure comes from the collection of composition maps

$$D_1(n) \times (D_1(m_1) \times \ldots D_1(m_n)) \longrightarrow D_1(m_1 + \ldots + m_n)$$

defined by resizing the configuration in the $D_1(m_i)$-component such that it fits precisely in the i-th gap of the configuration of the $D_1(n)$-component. That is,



We then obtain a unit interval having $m_1 + \ldots + m_n$ gaps which are labeled in the natural way, that is the first m_1 labels are for the gaps in the $D_1(m_1)$-configuration fitted in gap 1, the next m_2 labels are for the gaps in the $D_1(m_2)$-configuration fitted in gap 2 and so on. The tiny interval operator consists of

- a collection of topological spaces $D_1(n)$ for $n \geq 0$,
- a continuous action of S_n on $D_1(n)$ by relabeling, for every n,
- an identity element $id \in D_1(1)$,
- continuous composition maps $m_{(n,m_1,\ldots,m_n)}$ satisfying a list of axioms.

For every topological operad, we can take its homology operad and define a class of algebras over it, see for example [**?**] or [**?**] for details. Rather than introducing all these concepts here we will list the set of axioms defining the algebra-objects associated to the tiny interval operad : the A_∞-algebras.

Definition 2.52 An A_∞-algebra is a $\mathbb{Z}$-graded complex vectorspace

$$B = \oplus_{p \in \mathbb{Z}} B_p$$

endowed with homogeneous $\mathbb{C}$-linear maps

$$m_n : B^{\otimes n} \longrightarrow B$$

of degree $2 - n$ for all $n \geq 1$, satisfying the following relations

- We have $m_1 \circ m_1 = 0$, that is (B, m_1) is a differential complex

$$\ldots \xrightarrow{m_1} B_{i-1} \xrightarrow{m_1} B_i \xrightarrow{m_1} B_{i+1} \xrightarrow{m_1} \ldots$$

- We have the equality of maps $B \otimes B \longrightarrow B$

$$m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1)$$

  where $\mathbb{1}$ is the identity map on the vectorspace B. That is, m_1 is a derivation with respect to the multiplication $B \otimes B \xrightarrow{m_2} B$.

- We have the equality of maps $B \otimes B \otimes B \longrightarrow B$

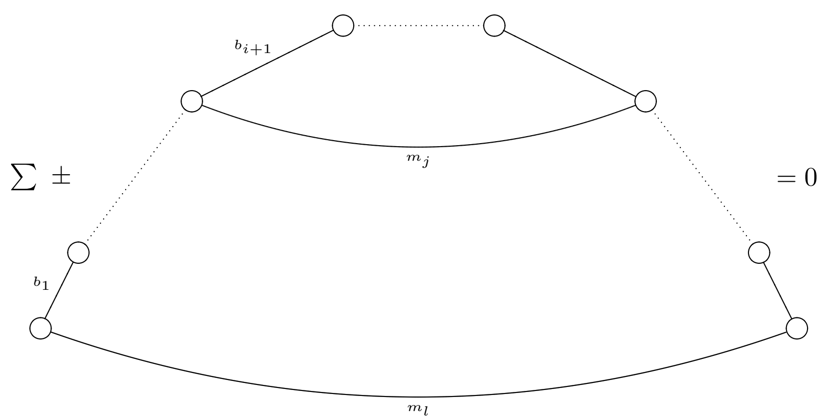
$$\begin{aligned} & m_2 \circ (\mathbb{1} \otimes m_2 - m_2 \otimes \mathbb{1}) \\ & = m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_1 \otimes 1 + \mathbb{1} \otimes \mathbb{1} \otimes m_1) \end{aligned}$$

  where the right second expression is the associator for the multiplication m_2 and the first is a boundary of m_3, implying that m_2 is associative up to homology.

- More generally, for $n \geq 1$ we have the relations

$$\sum (-1)^{i+j+k} m_l \circ (\mathbb{1}^{\otimes i} \otimes m_j \otimes \mathbb{1}^{\otimes k}) = 0$$

  where the sum runs over all decompositions $n = i+j+k$ and where $l = i+1+k$. These identities can be pictorially represented by



Observe that an A_∞-algebra B is in general not associative for the multiplication m_2, but its homology

$$H^* \; B = H^*(B, m_2)$$

is an associative graded algebra for the multiplication induced by m_2. Further, if $m_n = 0$ for all $n \geq 3$, then B is an associative differentially graded algebra and conversely every differentially graded algebra yields an A_∞-algebra with $m_n = 0$ for all $n \geq 3$.

Let A be an associative $\mathbb{C}$-algebra and M a left A-module. Choose an injective resolution of M

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

with the I^k injective left A-modules and denote by $I^\bullet$ the complex

$$I^\bullet \; : \; 0 \longrightarrow I^0 \xrightarrow{d} I^1 \xrightarrow{d} \dots$$

Let $B = HOM_A^\bullet(I^\bullet, I^\bullet)$ be the morphism complex. That is, its n-th component are the graded A-linear maps $I^\bullet \longrightarrow I^\bullet$ of degree n. This space can be equipped with a differential

$$d(f) = d \circ f - (-1)^n f \circ d \quad \text{for } f \text{ in the } n\text{-th part}$$

Then, B is a differentially graded algebra where the multiplication is the natural composition of graded maps. The homology algebra

$$H^* \; B = Ext_A^*(M, M)$$

is the extension algebra of M. This extension algebra has a canonical structure of A_∞-algebra with $m_1 = 0$ and m_2 he usual multiplication.

Now, let $M_1, \ldots, M_k$ be A-modules (for example, finite dimensional representations) and with $filt(M_1, \ldots, M_k)$ we denote the full subcategory of all A-modules whose objects admit finite filtrations with subquotients among the M_i. We have the following result, see for example [**?**, §6].

Theorem 2.53 *Let $M = M_1 \oplus \ldots \oplus M_k$. The canonical A_∞-structure on the extension algebra $Ext^*_A(M,M)$ contains enough information to reconstruct the category $filt(M_1, \ldots, M_k)$.*

Finally, let us elucidate the connection between this result and the local quiver-setting (Q_ξ, α_ξ) associated to a semi-simple n-dimensional representation

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

of a Quillen-smooth algebra A. In chapter 9 we will prove that for a Quillen-smooth algebra $Ext^i_A(M,M) = 0$ whenever $i \geq 2$. That is, the extension algebra

$$B_\xi = Ext^*_A(M_\xi, M_\xi)$$

contains only two terms

- $Ext^0_A(M_\xi, M_\xi) = Hom_A(M_\xi, M_\xi)$ and using the above decomposition this space is equal to
$$M_{e_1}(\mathbb{C}) \oplus \ldots \oplus M_{e_k}(\mathbb{C})$$
and hence determines the dimension vector $\alpha_\xi = (e_1, \ldots, e_k)$.
- $Ext^1_A(M_\xi, M_\xi)$ which by the decomposition is equal to
$$\oplus_{i,j=1}^k \, M_{e_j \times e_i}(Ext^1_A(S_i, S_j))$$
and we will prove that $dim_{\mathbb{C}} \; Ext^1_A(S_i, S_j)$ determines the number of arrows (or loops) in Q_ξ between the vertices v_i and v_j.

By the theorem above, this quiver-setting must contain enough information to describe $filt(S_1, \ldots, S_k)$ and hence in particular all n-dimensional representations having as their Jordan-Hölder components the simples S_i.

Chapter 3

Generic Matrices.

The results of this section are essential in the geometric study of noncommutative smooth algebras. If A is an affine $\mathbb{C}$-algebra, say with generators $\{a_1, \ldots, a_m\}$, we will study in chapter 4 the variety (actually, a scheme) of all n-dimensional representations $\underline{rep}_n\ A$ of A. The crucial result we will prove in chapter 4 is that the canonical Cayley-Hamilton algebra of degree n, $\int_n A$, associated to A can be recovered from the natural $GL_n(\mathbb{C})$-structure on $\underline{rep}_n\ A$ as the ring of all $GL_n(\mathbb{C})$-equivariant polynomial maps from $\underline{rep}_n\ A$ to $M_n(\mathbb{C})$.

In this chapter we will study the generic case, that is, when A is the free associative $\mathbb{C}$-algebra $\mathbb{C}\langle x_1, \ldots, x_m\rangle$ on m noncommuting generators. In this case,

$$\underline{rep}_n\ \mathbb{C}\langle x_1, \ldots, x_m\rangle = \underbrace{M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})}_{m}$$

as every $\mathbb{C}$-algebra map to $M_n(\mathbb{C})$ is determined by the images of the generators x_i. The $GL_n(\mathbb{C})$-action on the representation variety, determining isomorphisms of representations, is given by simultaneous conjugation, that is,

$$g.(A_1, \ldots, A_m) = (gA_1g^{-1}, \ldots, gA_mg^{-1}).$$

In the special case when $m = 1$, the Jordan-normal form of an $n \times n$ matrix provides us with a set-theoretical description of the $GL_n(\mathbb{C})$-orbits. However, as we will see in section 1, one cannot define a Hausdorff topology on this set or orbits due to the existence of non-closed orbits. Invariant theory provides us with the best continuous approximation to such an orbit-space. We will see that all functions on $M_n(\mathbb{C})$ which are constant on conjugacy classes are actually functions in the coefficients of the characteristic polynomial of the matrix. That is, we have an algebraic quotient map

$$M_n(\mathbb{C}) \twoheadrightarrow \mathbb{C}^n \qquad A \mapsto (\sigma_1(A), \ldots, \sigma_n(A))$$

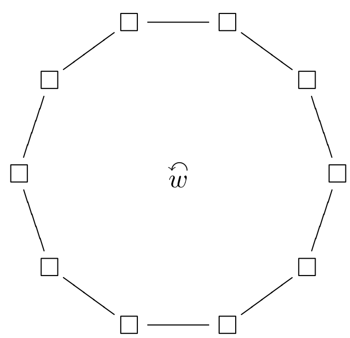
where $\sigma_i(A)$ is the i-th elementary symmetric function in the eigenvalues of A. A characteristic property of this quotient map is that every fiber contains a unique closed orbit.

In trying to extend this to arbitrary m we are faced with the problem that there are no known canonical forms for m-tuples of $n \times n$ matrices, except for small values of m and n such as $(m, n) = (2, 2)$, in which case a complete description of the orbits is given in section 2. The combinatorial tools which will be developed in part 3 will allow us later to extend such a complete classification (at least in principle) of all orbits for moderate values of n. In this chapter we will prove the important results, due to C. Procesi [24] on the invariant theory of m-tuples of $n \times n$ matrices under simultaneous conjugation.

In section 3 we will determine the ring of invariant polynomials for simultaneous conjugation. The approach is classical in invariant theory. First we determine the multilinear invariant polynomials and then we will use polarization and restitution to find all invariants. It turns out that all invariants are generated by traces of necklaces. Consider a noncommutative word in the variables x_i

$$w = x_{i_1} x_{i_2} \ldots x_{i_k}$$

determined only up to cyclic permutation of the terms, that is w should really be viewed as a necklace having k beads



Replace each of the beads x_i by an $n \times n$ matrix X_i having all its coefficients being indeterminates. That is X_i is a generic $n \times n$ matrix

$$\boxed{i} = X_i = \begin{bmatrix} x_{11}(i) & \ldots & x_{1n}(i) \\ \vdots & & \vdots \\ x_{n1}(i) & \ldots & x_{nn}(i) \end{bmatrix}$$

Then multiplying these generic matrices along the necklace and taking the trace of the $n \times n$ matrix obtained, we get an invariant polynomial. We will then use some results on the representation theory of the symmetric groups to bound the length of necklaces necessary to generate the whole algebra of invariants $\mathbb{N}_n^m = \mathbb{C}[rep_n\ \mathbb{C}\langle x_1, \ldots, x_m\rangle]^{GL_n(\mathbb{C})}$. The best bound on this length we will obtain is $n^2 + 1$.

Further, we will study in section 4 the $\mathbb{C}$-algebra of all $GL_n(\mathbb{C})$-equivariant polynomial maps

$$M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$$

which form the trace algebra $\mathbb{T}_n^m$. We will prove that $\mathbb{T}_n^m$ is generated by the ring of invariants $\mathbb{N}_n^m$ and the generic matrices X_i introduced above. In sections 6 and 7 we will then determine all relations holding among the necklace invariants and prove that they are all formal consequences of the Cayley-Hamilton equation holding for $n \times n$ matrices. In fact, we will show in the next chapter that the trace algebra $\mathbb{T}_n^m$ is the generic object in the category of all Cayley-Hamilton algebras of degree n.

We have tried to keep this chapter as self-contained as possible. More details on symmetric groups can be found for example in [**?**].

3.1 Conjugacy classes of matrices

From now on we will denote by M_n the space of all $n \times n$ matrices $M_n(\mathbb{C})$ and by GL_n the general linear group $GL_n(\mathbb{C})$. A matrix $A \in M_n$ determines by left multiplication a linear operator on the n-dimensional vectorspace $V_n = \mathbb{C}^n$ of *column vectors*. If $g \in GL_n$ is the matrix describing the *base change* from the canonical basis of V_n to a new basis, then the linear operator expressed in this new basis is represented by the matrix gAg^{-1}. For a given matrix A we want to find an suitable basis such that the *conjugated matrix* gAg^{-1} has a simple form.

That is, we consider the linear action of GL_n on the n^2-dimensional vectorspace M_n of $n \times n$ matrices determined by

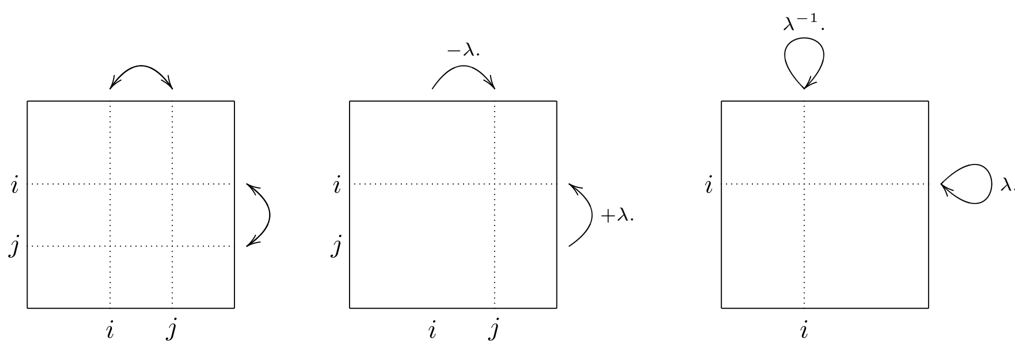
$$GL_n \times M_n \longrightarrow M_n \qquad (g, A) \mapsto g.A = gAg^{-1}.$$

The *orbit* $\mathcal{O}(A) = \{gAg^{-1} \mid g \in GL_n\}$ of A under this action is called the *conjugacy class* of A. We look for a particularly nice *representant* in a given conjugacy class. The answer to this problem is, of course, given by the *Jordan normal form* of the matrix.

With e_{ij} we denote the matrix whose unique non-zero entry is 1 at entry (i, j). Recall that the group GL_n is generated by the following three classes of matrices :

- the *permutation* matrices $p_{ij} = \mathbb{1}_n + e_{ij} + e_{ji} - e_{ii} - e_{jj}$ for all $i \neq j$,
- the *addition* matrices $a_{ij}(\lambda) = \mathbb{1}_n + \lambda e_{ij}$ for all $i \neq j$ and $0 \neq \lambda$, and
- the *multiplication* matrices $m_i(\lambda) = \mathbb{1}_n + (\lambda - 1)e_{ii}$ for all i and $0 \neq \lambda$.

Conjugation by these matrices determine the three types of *Jordan moves* on $n \times n$ matrices, where the altered rows and columns are dashed :



type p type a type m

Therefore, it suffices to consider sequences of these moves on a given $n \times n$ matrix $A \in M_n$. The *characteristic polynomial* of A is defined to be the polynomial of degree n in the variable t

$$\chi_A(t) = det(t\mathbb{1}_n - A) \in \mathbb{C}[t].$$

As $\mathbb{C}$ is algebraically closed, $\chi_A(t)$ decomposes as a product of linear terms

$$\prod_{i=1}^{e} (t - \lambda_i)^{d_i}$$

where the $\{\lambda_1, \ldots, \lambda_e\}$ are called the *eigenvalues* of the matrix A. Observe that λ_i is an eigenvalue of A if and only if there is a non-zero *eigenvector* $v \in V_n = \mathbb{C}^n$ with eigenvalue λ_i, that is, $A.v = \lambda_i v$. In particular, the *rank* r_i of the matrix $A_i = \lambda_i \mathbb{1}_n - A$ satisfies $n - d_i \leq r_i < n$. A nice inductive procedure using Jordan moves given in [**?**] gives a proof of the following *Jordan-Weierstrass theorem.*

Theorem 3.1 *Let $A \in M_n$ with characteristic polynomial $\chi_A(t) = \prod_{i=1}^{e}(t - \lambda_i)^{d_i}$. Then, A determines unique partitions*

$$p_i = (a_{i1}, a_{i2}, \ldots, a_{im_i}) \quad \textit{of} \quad d_i$$

associated to the eigenvalues λ_i of A such that A is conjugated to a unique (up to permutation of the blocks) block-diagonal matrix

$$J_{(p_1,\ldots,p_e)} = \begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_m} \end{bmatrix}$$

with $m = m_1 + \ldots + m_e$ and exactly one block B_l of the form $J_{a_{ij}}(\lambda_i)$ for all $1 \leq i \leq e$ and $1 \leq j \leq m_i$ where

$$J_{a_{ij}}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in M_{a_{ij}}(\mathbb{C})$$

For example, let us prove uniqueness of the partitions p_i of d_i corresponding to the eigenvalue λ_i of A. Assume A is conjugated to another Jordan block matrix $J_{(q_1,\ldots,q_e)}$, necessarily with partitions $q_i = (b_{i1}, \ldots, b_{im'_i})$ of d_i. To begin, observe that for a Jordan block of size k we have that

$$rk\ J_k(0)^l = k - l \quad \text{for all } l \leq k \text{ and if } \mu \neq 0 \text{ then} \quad rk\ J_k(\mu)^l = k$$

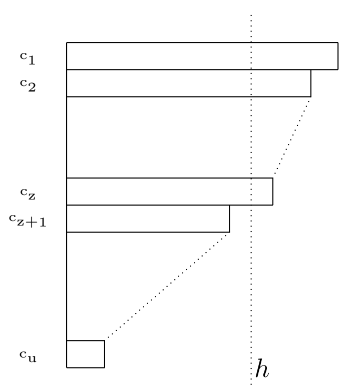
for all l. As $J_{(p_1,\ldots,p_e)}$ is conjugated to $J_{(q_1,\ldots,q_e)}$ we have for all $\lambda \in \mathbb{C}$ and all l

$$rk\ (\lambda \mathbb{1}_n - J_{(p_1,\ldots,p_e)})^l = rk\ (\lambda \mathbb{1}_n - J_{(q_1,\ldots,q_e)})^l$$

Now, take $\lambda = \lambda_i$ then only the Jordan blocks with eigenvalue λ_i are important in the calculation and one obtains for the ranks

$$n - \sum_{h=1}^{l} \#\{j \mid a_{ij} \geq h\} \quad \text{respectively} \quad n - \sum_{h=1}^{l} \#\{j \mid b_{ij} \geq h\}.$$

Now, for any partition $p = (c_1, \ldots, c_u)$ and any natural number h we see that the number $z = \#\{j \mid c_j \geq h\}$



is the number of blocks in the h-th row of the dual partition p^* which is defined to be the partition obtained by interchanging rows and columns in the *Young diagram* of p. Therefore, the above rank equality implies that $p_i^* = q_i^*$ and hence that $p_i = q_i$. As we can repeat this argument for the other eigenvalues we have the required uniqueness. Hence, the Jordan normal form shows that the classification of GL_n-orbits in M_n consists of two parts : a *discrete part* choosing

- a partition $p = (d_1, d_2, \ldots, d_e)$ of n, and for each d_i,
- a partition $p_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$ of d_i,

determining the sizes of the Jordan blocks and a *continuous* part choosing

- an e-tuple of distinct complex numbers $(\lambda_1, \lambda_2, \ldots, \lambda_e)$.

fixing the eigenvalues. Moreover, this e-tuple $(\lambda_1, \ldots, \lambda_e)$ is determined only up to permutations of the subgroup of all permutations π in the symmetric group S_e such that $p_i = p_{\pi(i)}$ for all $1 \leq i \leq e$. Whereas this gives a satisfactory set-theoretical description of the orbits we cannot put an Hausdorff topology on this set due to the existence of non-closed orbits in M_n. For example, if $n = 2$, consider the matrices

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

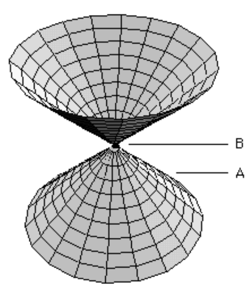
which are in different normal form so correspond to distinct orbits. For any $\epsilon \neq 0$ we have that

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & \epsilon \\ 0 & \lambda \end{bmatrix}$$

belongs to the orbit of A. Hence if $\epsilon \longrightarrow 0$, we see that B lies in the closure of $\mathcal{O}(A)$. As any matrix in $\mathcal{O}(A)$ has trace 2λ, the orbit is contained in the 3-dimensional subspace

$$\begin{bmatrix} \lambda + x & y \\ z & \lambda - x \end{bmatrix} \hookrightarrow M_2$$

In this space, the orbit-closure $\overline{\mathcal{O}(A)}$ is the set of points satisfying $x^2 + yz = 0$ (the determinant has to be λ^2), which is a cone having the origin as its top :



The orbit $\mathcal{O}(B)$ is the top of the cone and the orbit $\mathcal{O}(A)$ is the complement.

Still, for general n we can try to find the best separated topological quotient space for the action of GL_n on M_n. We will prove that this space coincide with the quotient variety determined by the invariant polynomial functions.

If two matrices are conjugated $A \sim B$, then A and B have the same *unordered* n-tuple of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ (occurring with multiplicities). Hence any symmetric function in the λ_i will have the same values in A as in B. In particular this is the case for the *elementary symmetric functions* σ_l

$$\sigma_l(\lambda_1, \ldots, \lambda_l) = \sum_{i_1 < i_2 < \ldots < i_l} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_l}.$$

Observe that for every $A \in M_n$ with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ we have

$$\prod_{j=1}^{n} (t - \lambda_j) = \chi_A(t) = det(t\mathbb{1}_n - A) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(A) t^{n-i}$$

Developing the determinant $det(t\mathbb{1}_n - A)$ we see that each of the coefficients $\sigma_i(A)$ is in fact a *polynomial function* in the entries of A. A fortiori, $\sigma_i(A)$ is a complex valued continuous function on M_n. The above equality also implies that the functions $\sigma_i : M_n \longrightarrow \mathbb{C}$ are constant along orbits. We now construct the continuous map

$$M_n \xrightarrow{\pi} \mathbb{C}^n$$

sending a matrix $A \in M_n$ to the point $(\sigma_1(A), \ldots, \sigma_n(A))$ in $\mathbb{C}^n$. Clearly, if $A \sim B$ then they map to the same point in $\mathbb{C}^n$. We claim that π is surjective. Take any point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ and consider the matrix $A \in M_n$

$$A = \begin{bmatrix} 0 & & & & a_n \\ -1 & 0 & & & a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & a_2 \\ & & & -1 & a_1 \end{bmatrix} \tag{3.1}$$

then we will show that $\pi(A) = (a_1, \ldots, a_n)$, that is,

$$det(t\mathbb{1}_n - A) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \ldots + (-1)^n a_n.$$

Indeed, developing the determinant of $t\mathbb{1}_n - A$ along the first column we obtain

$$\begin{vmatrix} t & 0 & 0 & \cdots & 0 & -a_n \\ 1 & t & 0 & & 0 & -a_{n-1} \\ 0 & 1 & t & & 0 & -a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & t & -a_2 \\ 0 & 0 & & & 1 & t-a_1 \end{vmatrix} - \begin{vmatrix} t & 0 & 0 & \cdots & 0 & -a_n \\ 1 & t & 0 & & 0 & -a_{n-1} \\ 0 & 1 & t & & 0 & -a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & t & -a_2 \\ 0 & 0 & & & 1 & t-a_1 \end{vmatrix}$$

Here, the second determinant is equal to $(-1)^{n-1}a_n$ and by induction on n the first determinant is equal to $t.(t^{n-1} - a_1 t^{n-2} + \ldots + (-1)^{n-1}a_{n-1})$, proving the claim.

Next, we will determine which $n \times n$ matrices can be conjugated to a matrix in the canonical form A as above. We call a matrix $B \in M_n$ *cyclic* if there is a (column) vector $v \in \mathbb{C}^n$ such that $\mathbb{C}^n$ is spanned by the vectors $\{v, B.v, B^2.v, \ldots, B^{n-1}.v\}$. Let $g \in GL_n$ be the basechange transforming the standard basis to the ordered basis

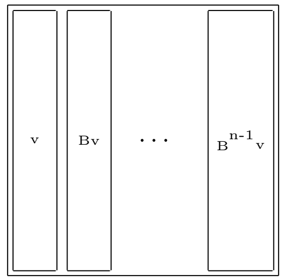
$$(v, -B.v, B^2.v, -B^3.v, \ldots, (-1)^{n-1}B^{n-1}.v).$$

In this new basis, the linear map determined by B (or equivalently, $g.B.g^{-1}$) is equal to the matrix in canonical form

$$\begin{bmatrix} 0 & & & & b_n \\ -1 & 0 & & & b_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & b_2 \\ & & & -1 & b_1 \end{bmatrix}$$

where $B^n.v$ has coordinates $(b_n, \ldots, b_2, b_1)$ in the new basis. Conversely, any matrix in this form is a cyclic matrix.

We claim that the set of all cyclic matrices in M_n is a *dense* open subset. To see this take $v = (x_1, \ldots, x_n)^\tau \in \mathbb{C}^n$ and compute the determinant of the $n \times n$ matrix



This gives a polynomial of total degree n in the x_i with all its coefficients polynomial functions c_j in the entries b_{kl} of B. Now, B is a cyclic matrix if and only if at least one of these coefficients is non-zero. That is, the set of non-cyclic matrices is exactly the intersection of the finitely many *hypersurfaces*

$$V_j = \{B = (b_{kl})_{k,l} \in M_n \mid c_j(b_{11}, b_{12}, \ldots, b_{nn}) = 0\}$$

in the vectorspace M_n.

Theorem 3.2 *The best continuous approximation to the orbit space is given by the surjection*

$$M_n \xrightarrow{\pi} \mathbb{C}^n$$

mapping a matrix $A \in M_n(\mathbb{C})$ to the n-tuple $(\sigma_1(A), \ldots, \sigma_n(A))$.

Let $f : M_n \longrightarrow \mathbb{C}$ be a continuous function which is constant along conjugacy classes. We will show that f factors through π, that is, f is really a continuous function in the $\sigma_i(A)$. Consider the diagram

$$\begin{tikzcd}
M_n \arrow[r, "f"] \arrow[d, shift left, "\pi"] & \mathbb{C} \\
\mathbb{C}^n \arrow[u, shift left, "s"] \arrow[ur, dotted, "f' = f \circ s"'] &
\end{tikzcd}$$

where s is the *section* of π (that is, $\pi \circ s = id_{\mathbb{C}^n}$) determined by sending a point $(a_1, \ldots, a_n)$ to the cyclic matrix in canonical form A as in equation (3.1). Clearly, s is continuous, hence so is $f' = f \circ s$. The approximation property follows if we prove that $f = f' \circ \pi$. By continuity, it suffices to check equality on the dense open set of cyclic matrices in M_n.

There it is a consequence of the following three facts we have proved before : (1) : any cyclic matrix lies in the same orbit as one in standard form, (2) : s is a section of π and (3) : f is constant along orbits.

Example 3.3 Orbits in M_2.

A 2×2 matrix A can be conjugated to an upper triangular matrix with diagonal entries the eigenvalues λ_1, λ_2 of A. As the trace and determinant of both matrices are equal we have

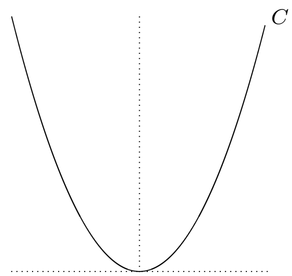
$$\sigma_1(A) = tr(A) \text{ and } \sigma_2(A) = det(A).$$

The best approximation to the orbitspace is therefore given by the surjective map

$$M_2 \xrightarrow{\pi} \mathbb{C}^2 \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a + d, ad - bc)$$

The matrix A has two equal eigenvalues if and only if the discriminant of the characteristic polynomial $t^2 - \sigma_1(A)t + \sigma_2(A)$ is zero, that is when $\sigma_1(A)^2 - 4\sigma_2(A) = 0$. This condition determines a closed curve C in $\mathbb{C}^2$ where

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 - 4y = 0\}.$$



Observe that C is a smooth 1-dimensional submanifold of $\mathbb{C}^2$. We will describe the *fibers* (that is, the inverse images of points) of the surjective map π .

If $p = (x, y) \in \mathbb{C}^2 - C$, then $\pi^{-1}(p)$ consists of precisely one orbit (which is then necessarily closed in M_2) namely that of the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{where} \quad \lambda_{1,2} = \frac{-x \pm \sqrt{x^2 - 4y}}{2}$$

If $p = (x, y) \in C$ then $\pi^{-1}(p)$ consists of two orbits,

$$\mathcal{O}_{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}} \quad \text{and} \quad \mathcal{O}_{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}$$

where $\lambda = \frac{1}{2}x$. We have seen that the second orbit lies in the closure of the first. Observe that the second orbit reduces to one point in M_2 and hence is closed. Hence, also $\pi^{-1}(p)$ contains a unique closed orbit.

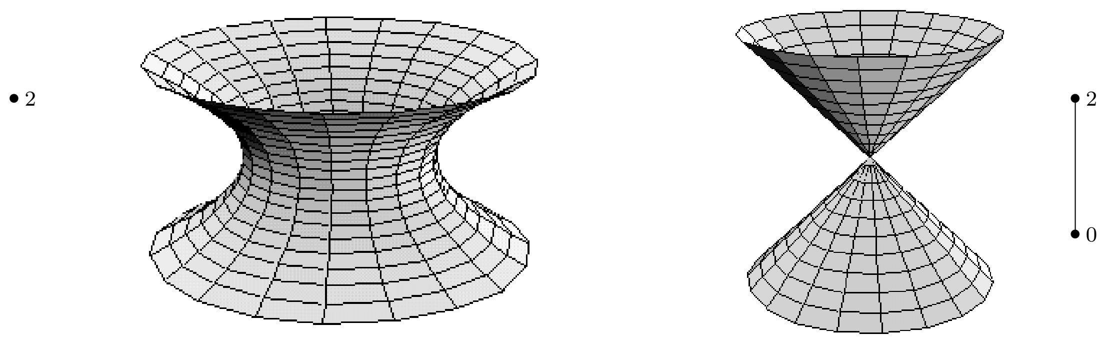
To describe the fibers of π as closed subsets of M_2 it is convenient to write any matrix A as a linear combination

$$A = u(A) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + v(A) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + w(A) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z(A) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Expressed in the coordinate functions u, v, w and z the fibers $\pi^{-1}(p)$ of a point $p = (x, y) \in \mathbb{C}^2$ are the common zeroes of

$$\begin{cases} u & = x \\ v^2 + 4wz & = x^2 - 4y \end{cases}$$

The first equation determines a three dimensional affine subspace of M_2 in which the second equation determines a *quadric*.



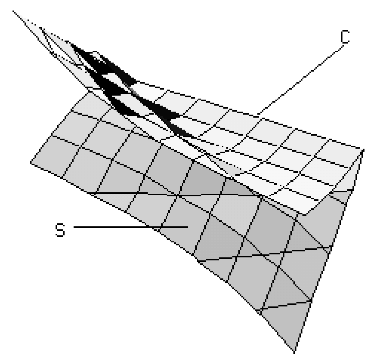
If $p \notin C$ this quadric is non-degenerate and thus $\pi^{-1}(p)$ is a smooth 2-dimensional submanifold of M_2. If $p \in C$, the quadric is a cone with top lying in the point $\frac{x}{2}\mathbb{1}_2$. Under the GL_2-action, the unique singular point of the cone must be clearly fixed giving us the closed orbit of dimension 0 corresponding to the diagonal matrix. The other orbit is the complement of the top and hence is a smooth 2-dimensional (non-closed) submanifold of M_2. The graphs represent the orbit-closures and the dimensions of the orbits.

Example 3.4 Orbits in M_3.

We will describe the fibers of the surjective map $M_3 \xrightarrow{\pi} \mathbb{C}^3$. If a 3×3 matrix has multiple eigenvalues then the *discriminant* $d = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2$ is zero. Clearly, d is a symmetric polynomial and hence can be expressed in terms of σ_1, σ_2 and σ_3. More precisely,

$$d = 4\sigma_1^3\sigma_3 + 4\sigma_2^3 + 27\sigma_3^2 - \sigma_1^2\sigma_2^2 - 18\sigma_1\sigma_2\sigma_3$$

The set of points in $\mathbb{C}^3$ where d vanishes is a surface S with *singularities*.



These singularities are the common zeroes of the $\frac{\partial d}{\partial \sigma_i}$ for $1 \leq i \leq 3$. One computes that these singularities form a *twisted cubic* curve C in $\mathbb{C}^3$, that is,

$$C = \{(3c, 3c^2, c^3) \mid c \in \mathbb{C}\}.$$

The description of the fibers $\pi^{-1}(p)$ for $p = (x, y, z) \in \mathbb{C}^3$ is as follows. When $p \notin S$, then $\pi^{-1}(p)$ consists of a unique orbit (which is therefore closed in M_3), the conjugacy class of a matrix with paired distinct eigenvalues. If $p \in S - C$, then $\pi^{-1}(p)$ consists of the orbits of

$$A_1 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

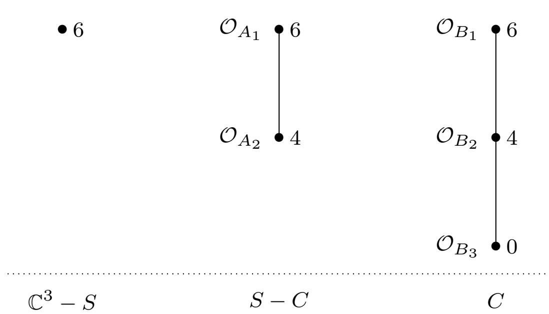
Finally, if $p \in C$, then the matrices in the fiber $\pi^{-1}(p)$ have a single eigenvalue $\lambda = \frac{1}{3}x$ and the fiber consists of the orbits of the matrices

$$B_1 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad B_2 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad B_3 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

We observe that the *strata* with distinct fiber behavior (that is, $\mathbb{C}^3 - S$, $S - C$ and C) are all submanifolds of $\mathbb{C}^3$.

The dimension of an orbit $\mathcal{O}(A)$ in M_n is computed as follows. Let C_A be the subspace of all matrices in M_n commuting with A. Then, the *stabilizer* subgroup of A is a dense open subset of C_A whence the dimension of $\mathcal{O}(A)$ is equal to $n^2 - \dim C_A$.

Performing these calculations for the matrices given above, we obtain the following graphs representing orbit-closures and the dimensions of orbits



Returning to M_n, the set of cyclic matrices is a Zariski open subset of M_n. For, consider the generic matrix of coordinate functions and generic column vector

$$X = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

and form the square matrix

$$\begin{bmatrix} v & X.v & X^2.v & \dots & X^{n-1}.v \end{bmatrix} \in M_n(\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, v_1, \dots, v_n])$$

Then its determinant can be written as $\sum_{l=1}^{z} p_l(x_{ij})q_l(v_k)$ where the q_l are polynomials in the v_k and the p_l polynomials in the x_{ij}. Let $A \in M_n$ be such that at least one of the $p_l(A) \neq 0$, then the polynomial $d = \sum_l p_l(A)q_l(v_k) \in \mathbb{C}[v_1, \dots, v_k]$ is non-zero. But then there is a $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ such that $d(c) \neq 0$ and hence c^τ is a cyclic vector for A. The converse implication is obvious.

Theorem 3.5 *Let $f : M_n \longrightarrow \mathbb{C}$ is a regular (that is, polynomial) function on M_n which is constant along conjugacy classes, then*

$$f \in \mathbb{C}[\sigma_1(X), \dots, \sigma_n(X)]$$

Proof. Consider again the diagram

$$\begin{tikzcd}
M_n \arrow[r, "f"] \arrow[d, shift left, "\pi"] & \mathbb{C} \\
\mathbb{C}^n \arrow[u, shift left, "s"] \arrow[ur, dotted, "f'=f\circ s"'] &
\end{tikzcd}$$

The function $f' = f \circ s$ is a regular function on $\mathbb{C}^n$ whence is a polynomial in the coordinate functions of $\mathbb{C}^m$ (which are the $\sigma_i(X)$), so

$$f' \in \mathbb{C}[\sigma_1(X), \dots, \sigma_n(X)] \hookrightarrow \mathbb{C}[M_n].$$

Moreover, f and f' are equal on a Zariski open (dense) subset of M_n whence they are equal as polynomials in $\mathbb{C}[M_n]$. $\square$

The ring of polynomial functions on M_n which are constant along conjugacy classes can also be viewed as a ring of invariants. The group GL_n acts as algebra automorphisms on the polynomial ring $\mathbb{C}[M_n]$. The automorphism ϕ_g determined by $g \in GL_n$ sends the variable x_{ij} to the (i, j)-entry of the matrix $g^{-1}.X.g$ which is a linear form in $\mathbb{C}[M_n]$. This action is determined by the property that for all $g \in GL_n$, $A \in A$ and $f \in \mathbb{C}[M_n]$ we have that

$$\phi_g(f)(A) = f(g.A.g^{-1})$$

The *ring of polynomial invariants* is the algebra of polynomials left invariant under this action

$$\mathbb{C}[M_n]^{GL_n} = \{f \in \mathbb{C}[M_n] \mid \phi_g(f) = f \text{ for all } \; g \in GL_n\}$$

and hence is the ring of polynomial functions on M_n which are constant along orbits. The foregoing theorem determines the ring of polynomials invariants

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[\sigma_1(X), \dots, \sigma_n(X)]$$

We will give an equivalent description of this ring below.

Consider the variables $\lambda_1, \dots, \lambda_n$ and consider the polynomial

$$f_n(t) = \prod_{i=1}^{n}(t - \lambda_i) = t^n + \sum_{i=1}^{n}(-1)^i \sigma_i t^{n-i}$$

then σ_i is the i-th elementary symmetric polynomial in the λ_j. We know that these polynomials are algebraically independent and generate the *ring of symmetric polynomials* in the λ_j, that is,

$$\mathbb{C}[\sigma_1, \ldots, \sigma_n] = \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{S_n}$$

where S_n is the *symmetric group* on n letters acting by automorphisms on the polynomial ring $\mathbb{C}[\lambda_1, \ldots, \lambda_n]$ via $\pi(\lambda_i) = \lambda_{\pi(i)}$ and the algebra of polynomials which are fixed under these automorphisms are precisely the symmetric polynomials in the λ_j.

Consider the symmetric Newton functions $s_i = \lambda_1^i + \ldots + \lambda_n^i$, then we claim that this is another generating set of symmetric polynomials, that is,

$$\mathbb{C}[\sigma_1, \ldots, \sigma_n] = \mathbb{C}[s_1, \ldots, s_n].$$

To prove this it suffices to express each σ_i as a polynomial in the s_j. More precisely, we claim that the following identities hold for all $1 \leq j \leq n$

$$s_j - \sigma_1 s_{j-1} + \sigma_2 s_{j-2} - \ldots + (-1)^{j-1}\sigma_{j-1}s_1 + (-1)^j \sigma_j.j = 0 \tag{3.2}$$

For $j = n$ this identity holds because we have

$$0 = \sum_{i=1}^{n} f_n(\lambda_i) = s_n + \sum_{i=1}^{n}(-1)^i \sigma_i s_{n-i}$$

if we take $s_0 = n$. Assume now $j < n$ then the left hand side of equation 3.2 is a symmetric function in the λ_i of degree $\leq j$ and is therefore a polynomial $p(\sigma_1, \ldots, \sigma_j)$ in the first j elementary symmetric polynomials. Let ϕ be the algebra epimorphism

$$\mathbb{C}[\lambda_1, \ldots, \lambda_n] \xrightarrow{\phi} \mathbb{C}[\lambda_1, \ldots, \lambda_j]$$

defined by mapping $\lambda_{j+1}, \ldots, \lambda_j$ to zero. Clearly, $\phi(\sigma_i)$ is the i-th elementary symmetric polynomial in $\{\lambda_1, \ldots, \lambda_j\}$ and $\phi(s_i) = \lambda_1^i + \ldots + \lambda_j^i$. Repeating the above $j = n$ argument (replacing n by j) we have

$$0 = \sum_{i=1}^{j} f_j(\lambda_i) = \phi(s_j) + \sum_{i=1}^{j}(-1)^i \phi(\sigma_i)\phi(s_{n-i})$$

(this time with $s_0 = j$). But then, $p(\phi(\sigma_1), \ldots, \phi(\sigma_j)) = 0$ and as the $\phi(\sigma_k)$ for $1 \leq k \leq j$ are algebraically independent we must have that p is the zero polynomial finishing the proof of the claimed identity.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix A, then A can be conjugated to an upper triangular matrix B with diagonal entries $(\lambda_1, \ldots, \lambda_1)$. Hence, the *trace* $tr(A) = tr(B) = \lambda_1 + \ldots + \lambda_n = s_1$. In general, A^i can be conjugated to B^i which is an upper triangular matrix with diagonal entries $(\lambda_1^i, \ldots, \lambda_n^i)$ and hence the traces of A^i and B^i are equal to $\lambda_1^i + \ldots + \lambda_n^i = s_i$. Concluding, we have

Theorem 3.6 *Consider the action of conjugation by GL_n on M_n. Let X be the generic matrix of coordinate functions on M_n*

$$X = \begin{bmatrix} x_{11} & \ldots & x_{nn} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix}$$

Then, the ring of polynomial invariants is generated by the traces of powers of X, that is,

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[tr(X), tr(X^2), \ldots, tr(X^n)]$$

Proof. The result follows from theorem 3.5 and the fact that

$$\mathbb{C}[\sigma_1(X),\ldots,\sigma_n(X)] = \mathbb{C}[tr(X),\ldots,tr(X^n)]$$

□

3.2 Simultaneous conjugacy classes.

For applications to noncommutative algebras it is crucial to extend what we have done for conjugacy classes of matrices to simultaneous conjugacy classes of *m-tuples of matrices*. Consider the mn^2-dimensional complex vectorspace

$$M_n^m = \underbrace{M_n \oplus \ldots \oplus M_n}_{m}$$

of m-tuples $(A_1,\ldots,A_m)$ of $n \times n$-matrices $A_i \in M_n$. On this space we let the group GL_n act by simultaneous conjugation, that is

$$g.(A_1,\ldots,A_m) = (g.A_1.g^{-1},\ldots,g.A_m.g^{-1})$$

for all $g \in GL_n$ and all m-tuples $(A_1,\ldots,A_m)$. Unfortunately, there is no substitute for the Jordan normalform result in this more general setting. Still, for small m and n one can work out the GL_n-orbits by ad hoc methods.

Example 3.7 Orbits in $M_2^2 = M_2 \oplus M_2$.
We can try to mimic the geometric approach to the conjugacy class problem, that is, we will try to approximate the orbitspace via polynomial functions on M_2^2 which are constant along orbits. For $(A,B) \in M_2^2 = M_2 \oplus M_2$ clearly the polynomial functions we have encountered before $tr(A), det(A)$ and $tr(B), det(B)$ are constant along orbits. However, there are more : for example $tr(AB)$. Later, we will show that these five functions generate all polynomials functions which are constant along orbits. Here, we will show that the map $M_2^2 = M_2 \oplus M_2 \xrightarrow{\pi} \mathbb{C}^5$ defined by

$$(A,B) \mapsto (tr(A), det(A), tr(B), det(B), tr(AB))$$

is surjective such that each fiber contains precisely one closed orbit. In the next chapter, we will see that this property characterizes the best polynomial approximation to the (non-existent) orbit space.

First, we will show surjectivity of π, that is, for every $(x_1,\ldots,x_5) \in \mathbb{C}^5$ we will construct a couple of 2×2 matrices (A,B) (or rather its orbit) such that $\pi(A,B) = (x_1,\ldots,x_5)$. Consider the open set where $x_1^2 \neq 4x_2$. We have seen that this property characterizes those $A \in M_2$ such that A has distinct eigenvalues and hence diagonalizable. Hence, we can take a representative of the orbit $\mathcal{O}(A,B)$ to be a couple

$$\left(\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} , \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right)$$

with $\lambda \neq \mu$. We need a solution to the set of equations

$$\begin{cases} x_3 = & c_1 + c_4 \\ x_4 = & c_1c_4 - c_2c_3 \\ x_5 = & \lambda c_1 + \mu c_4 \end{cases}$$

Because $\lambda \neq \mu$ the first and last equation uniquely determine c_1, c_4 and substitution in the second gives us c_2c_3. Analogously, points of $\mathbb{C}^5$ lying in the open set $x_3^2 \neq x_4$ lie in the image of π. Finally, for a point in the complement of these open sets, that is when $x_1^2 = x_2$ and $x_3^2 = 4x_4$ we can consider a couple (A,B)

$$\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} , \begin{bmatrix} \mu & 0 \\ c & \mu \end{bmatrix} \right)$$

where $\lambda = \frac{1}{2}x_1$ and $\mu = \frac{1}{2}x_3$. Observe that the remaining equation $x_5 = tr(AB) = 2\lambda\mu + c$ has a solution in c.

Now, we will describe the fibers of π. Assume (A,B) is such that A and B have a *common* eigenvector v. Simultaneous conjugation with a $g \in GL_n$ expressing a basechange from the standard basis to $\{v,w\}$ for some w shows that the orbit $\mathcal{O}(A,B)$ contains a couple of upper-triangular matrices. We want to describe the image of these matrices under π. Take an upper triangular representative in $\mathcal{O}(A,B)$

$$(\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} , \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}).$$

with π-image $(x_1,\ldots,x_5)$. The coordinates x_1,x_2 determine the eigenvalues a_1,a_3 of A only as an unordered set (similarly, x_3,x_4 only determine the set of eigenvalues $\{b_1,b_3\}$ of B). Hence, $tr(AB)$ is one of the following two expressions

$$a_1b_1 + a_3b_3 \quad \text{or} \quad a_1b_3 + a_3b_1$$

and therefore satisfies the equation

$$(tr(AB) - a_1b_1 - a_3b_3)(tr(AB) - a_1b_3 - a_3b_1) = 0.$$

Recall that $x_1 = a_1 + a_3, x_2 = a_1a_3$, $x_3 = b_1 + b_3, x_4 = b_1b_3$ and $x_5 = tr(AB)$ we can express this equation as

$$x_5^2 - x_1x_3x_5 + x_1^2x_4 + x_3^2x_2 - 4x_2x_4 = 0.$$

This determines an hypersurface $H \hookrightarrow \mathbb{C}^5$. If we view the left-hand side as a polynomial f in the coordinate functions of $\mathbb{C}^5$ we see that H is a four dimensional subset of $\mathbb{C}^5$ with singularities the common zeroes of the partial derivatives

$$\frac{\partial f}{\partial x_i} \text{ for } 1 \leq i \leq 5$$

These singularities for the 2-dimensional submanifold S of points of the form $(2a, a^2, 2b, b^2, 2ab)$. We now claim that the smooth submanifolds $\mathbb{C}^5 - H$, $H - S$ and S of $\mathbb{C}^5$ describe the different types of fiber behavior. In chapter 6 we will see that the subsets of points with different fiber behavior (actually, of different representation type) are manifolds for m-tuples of $n \times n$ matrices.

If $p \notin H$ we claim that $\pi^{-1}(p)$ is a unique orbit, which is therefore closed in M_2^2. Let $(A,B) \in \pi^{-1}$ and assume first that $x_1^2 \neq 4x_2$ then there is a representative in $\mathcal{O}(A,B)$ of the form

$$(\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} , \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix})$$

with $\lambda \neq \mu$. Moreover, $c_2c_3 \neq 0$ (for otherwise A and B would have a common eigenvector whence $p \in H$) hence we may assume that $c_2 = 1$ (eventually after simultaneous conjugation with a suitable diagonal matrix $diag(t, t^{-1})$). The value of λ, μ is determined by x_1, x_2. Moreover, c_1, c_3, c_4 are also completely determined by the system of equations

$$\begin{cases} x_3 &= c_1 + c_4 \\ x_4 &= c_1c_4 - c_3 \\ x_5 &= \lambda c_1 + \mu c_4 \end{cases}$$

and hence the point $p = (x_1,\ldots,x_5)$ completely determines the orbit $\mathcal{O}(A,B)$. Remains to consider the case when $x_1^2 = 4x_2$ (that is, when A has a single eigenvalue). Consider the couple $(uA+vB, B)$ for $u,v \in \mathbb{C}^*$. To begin, $uA+vB$ and B do not have a common eigenvalue. Moreover, $p = \pi(A,B)$ determines $\pi(uA+vB, B)$ as

$$\begin{cases} tr(uA+vB) &= utr(A) + vtr(B) \\ det(uA+vB) &= u^2det(A) + v^2det(B) + uv(tr(A)tr(B) - tr(AB)) \\ tr((uA+vB)B) &= utr(AB) + v(tr(B)^2 - 2det(B)) \end{cases}$$

Assume that for all $u,v \in \mathbb{C}^*$ we have the equality $tr(uA+vB)^2 = 4det(uA+vB)$ then comparing coefficients of this equation expressed as a polynomial in u and v we obtain the conditions $x_1^2 = 4x_2$, $x_3^2 = 4x_4$ and $2x_5 = x_1x_3$ whence $p \in S \hookrightarrow H$, a contradiction. So, fix u,v such that $uA+vB$ has distinct eigenvalues. By the above argument $\mathcal{O}(uA+vB, B)$ is the unique orbit lying over $\pi(uA+vB, B)$, but then $\mathcal{O}(A,B)$ must be the unique orbit lying over p.

Let $p \in H - S$ and $(A,B) \in \pi^{-1}(p)$, then A and B are simultaneous upper triangularizable, with eigenvalues a_1, a_2 respectively b_1, b_2. Either $a_1 \neq a_2$ or $b_1 \neq b_2$ for otherwise $p \in S$. Assume $a_1 \neq a_2$, then there is a representative in the orbit $\mathcal{O}(A,B)$ of the form

$$(\begin{bmatrix} a_i & 0 \\ 0 & a_j \end{bmatrix} , \begin{bmatrix} b_k & b \\ 0 & b_l \end{bmatrix})$$

for $\{i,j\} = \{1,2\} = \{k,l\}$. If $b \neq 0$ we can conjugate with a suitable diagonal matrix to get $b = 1$ hence we get at most 9 possible orbits. Checking all possibilities we see that only three of them are distinct, those corresponding to the couples

$$(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}) \quad (\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}) \quad (\begin{bmatrix} a_2 & 0 \\ 0 & a_1 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix})$$

Clearly, the first and last orbit have the middle one lying in its closure. Observe that the case assuming that $b_1 \neq b_2$ is handled similarly. Hence, if $p \in H - S$ then $\pi^{-1}(p)$ consists of three orbits, two of dimension three whose closures intersect in a (closed) orbit of dimension two.

Finally, consider the case when $p \in S$ and $(A, B) \in \pi^{-1}(p)$. Then, both A and B have a single eigenvalue and the orbit $\mathcal{O}(A, B)$ has a representative of the form

$$(\begin{bmatrix} a & x \\ 0 & a \end{bmatrix}, \begin{bmatrix} b & y \\ 0 & b \end{bmatrix})$$

for certain $x, y \in \mathbb{C}$. If either x or y are non-zero, then the subgroup of GL_2 fixing this matrix consists of the matrices of the form

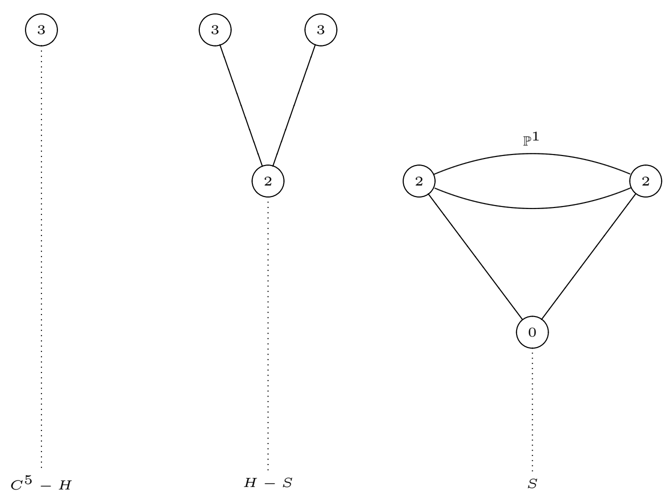
$$Stab \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} = \{ \begin{bmatrix} u & v \\ 0 & u \end{bmatrix} \mid u \in \mathbb{C}^*, v \in \mathbb{C} \}$$

but these matrices also fix the second component. Therefore, if either x or y is nonzero, the orbit is fully determined by $[x : y] \in \mathbb{P}^1$. That is, for $p \in S$, the fiber $\pi^{-1}(p)$ consists of an infinite family of orbits of dimension 2 parameterized by the points of the projective line $\mathbb{P}^1$ together with the orbit of

$$(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} , \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix})$$

which consists of one point (hence is closed in M_2^2) and lies in the closure of each of the 2-dimensional orbits.

Concluding, we see that each fiber $\pi^{-1}(p)$ contains a unique closed orbit (that of minimal dimension). The orbitclosure and dimension diagrams have the following shapes



3.3 Matrix invariants and necklaces

In this section we will determine the ring of all polynomial maps

$$M_n^m = \underbrace{M_n \oplus \ldots \oplus M_n}_{m} \xrightarrow{f} \mathbb{C}$$

which are constant along orbits under the action of GL_n on M_n^m by simultaneous conjugation. The strategy we will use is classical in invariant theory.

- First, we will determine the *multilinear* maps which are constant along orbits, equivalently, the *linear* maps

$$M_n^{\otimes m} = \underbrace{M_n \otimes \ldots \otimes M_n}_{m} \longrightarrow \mathbb{C}$$

which are constant along GL_n-orbits where GL_n acts by the diagonal action, that is,

$$g.(A_1 \otimes \ldots \otimes A_m) = gA_1g^{-1} \otimes \ldots \otimes gA_mg^{-1}.$$

- Afterwards, we will be able to obtain from them all polynomial invariant maps by using *polarization* and *restitution* operations.

First, we will translate our problem into one studied in classical invariant theory of GL_n.

Let $V_n \simeq \mathbb{C}^n$ be the n-dimensional vectorspace of column vectors on which GL_n acts naturally by left multiplication

$$V_n = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \\ \vdots \\ \mathbb{C} \end{bmatrix} \quad \text{with action} \quad g. \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix}$$

In order to define an action on the dual space $V_n^* = Hom(V_n, \mathbb{C}) \simeq \mathbb{C}^n$ of *covectors* (or, row vectors) we have to use the *contragradient* action

$$V_n^* = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \dots & \mathbb{C} \end{bmatrix} \quad \text{with action} \quad \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_n \end{bmatrix} .g^{-1}$$

Observe, that we have an *evaluation* map $V_n^* \times V_n \longrightarrow \mathbb{C}$ which is given by the scalar product $f(v)$ for all $f \in V_n^*$ and $v \in V_n$

$$\begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_n \end{bmatrix} . \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix} = \phi_1\nu_1 + \phi_2\nu_2 + \dots + \phi_n\nu_n$$

which is invariant under the diagonal action of GL_n on $V_n^* \times V_n$. Further, we have the natural identification

$$M_n = V_n \otimes V_n^* = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \\ \vdots \\ \mathbb{C} \end{bmatrix} \otimes \begin{bmatrix} \mathbb{C} & \mathbb{C} & \dots & \mathbb{C} \end{bmatrix}.$$

Under this identification, a *pure tensor* $v \otimes f$ corresponds to the rank one matrix or rank one endomorphism of V_n defined by

$$v \otimes f : V_n \longrightarrow V_n \quad \text{with} \quad w \mapsto f(w)v$$

and observe that the rank one matrices span M_n. The diagonal action of GL_n on $V_n \otimes V_n^*$ is then determined by its action on the pure tensors where it is equal to

$$g. \begin{bmatrix} \nu_1 \\ \nu_2 \\ \dots \\ \nu_n \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_n \end{bmatrix} .g^{-1}$$

and therefore coincides with the action of conjugation on M_n. Now, let us consider the identification

$$(V_n^{*\otimes m} \otimes V_n^{\otimes m})^* \simeq End(V_n^{\otimes m})$$

obtained from the nondegenerate pairing

$$End(V_n^{\otimes m}) \times (V_n^{*\otimes m} \otimes V_n^{\otimes m}) \longrightarrow \mathbb{C}$$

given by the formula

$$\langle \lambda, f_1 \otimes \dots \otimes f_m \otimes v_1 \otimes \dots \otimes v_m \rangle = f_1 \otimes \dots \otimes f_m(\lambda(v_1 \otimes \dots \otimes v_m))$$

GL_n acts diagonally on $V_n^{\otimes m}$ and hence again by conjugation on $End(V_n^{\otimes m})$ after embedding $GL_n \hookrightarrow GL(V_n^{\otimes m}) = GL_{mn}$. Thus, the above identifications are isomorphism as vectorspaces with GL_n-action. But then, the space of GL_n-invariant linear maps

$$V_n^{*\otimes m} \otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

can be identified with the space $End_{GL_n}(V_n^{\otimes m})$ of GL_n-linear endomorphisms of $V_n^{\otimes m}$. We will now give a different presentation of this vectorspace relating it to the symmetric group.

Apart from the diagonal action of GL_n on $V_n^{\otimes m}$ given by

$$g.(v_1 \otimes \ldots \otimes v_m) = g.v_1 \otimes \ldots \otimes g.v_m$$

we have an action of the symmetric group S_m on m letters on $V_n^{\otimes m}$ given by

$$\sigma.(v_1 \otimes \ldots \otimes v_m) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)}$$

These two actions commute with each other and give embeddings of GL_n and S_m in $End(V_n^{\otimes m})$.

$$\begin{tikzcd}
GL_n \arrow[dr] & & \\
& GL(V_n^{\otimes m}) \arrow[r, hook] & End(V_n^{\otimes m}) \\
S_m \arrow[ur] & &
\end{tikzcd}$$

The subspace of $V_n^{\otimes m}$ spanned by the image of GL_n will be denoted by $\langle GL_n \rangle$. Similarly, with $\langle S_m \rangle$ we denote the subspace spanned by the image of S_m.

Theorem 3.8 *With notations as above we have :*

1. $\langle GL_n \rangle = End_{S_m}(V_n^{\otimes m})$
2. $\langle S_m \rangle = End_{GL_n}(V_n^{\otimes m})$

Proof. (1) : Under the identification $End(V_n^{\otimes m}) = End(V_n)^{\otimes m}$ an element $g \in GL_n$ is mapped to the symmetric tensor $g \otimes \ldots \otimes g$. On the other hand, the image of $End_{S_m}(V_n^{\otimes m})$ in $End(V_n)^{\otimes m}$ is the subspace of all symmetric tensors in $End(V)^{\otimes m}$. We can give a basis of this subspace as follows. Let $\{e_1, \ldots, e_{n^2}\}$ be a basis of $End(V_n)$, then the vectors $e_{i_1} \otimes \ldots \otimes e_{i_m}$ form a basis of $End(V_n)^{\otimes m}$ which is stable under the S_m-action. Further, any S_m-orbit contains a unique representative of the form

$$e_1^{\otimes h_1} \otimes \ldots \otimes e_{n^2}^{\otimes h_{n^2}}$$

with $h_1 + \ldots + h_{n^2} = m$. If we denote by $r(h_1, \ldots, h_{n^2})$ the sum of all elements in the corresponding S_m-orbit then these vectors are a basis of the symmetric tensors in $End(V_n)^{\otimes m}$.

The claim follows if we can show that every linear map λ on the symmetric tensors which is zero on all $g \otimes \ldots \otimes g$ with $g \in GL_n$ is the zero map. Write $e = \sum x_i e_i$, then

$$\lambda(e \otimes \ldots \otimes e) = \sum x_1^{h_1} \ldots x_{n^2}^{h_{n^2}} \lambda(r(h_1, \ldots, h_{n^2}))$$

is a polynomial function on $End(V_n)$. As GL_n is a Zariski open subset of $End(V)$ on which by assumption this polynomial vanishes, it must be the zero polynomial. Therefore, $\lambda(r(h_1,\dots,h_{n^2}))=0$ for all $(h_1,\dots,h_{n^2})$ finishing the proof.

(2) : Recall that the groupalgebra $\mathbb{C}S_m$ of S_m is a *semisimple algebra*. Any epimorphic image of a semisimple algebra is semisimple. Therefore, $\langle S_m\rangle$ is a semisimple subalgebra of the matrixalgebra $End(V_n^{\otimes m}) \simeq M_{nm}$. By the *double centralizer theorem* (see for example [23]), it is therefore equal to the centralizer of $End_{S_m}(V_m^{\otimes m})$. By the first part, it is the centralizer of $\langle GL_n\rangle$ in $End(V_n^{\otimes m})$ and therefore equal to $End_{GL_n}(V_n^{\otimes m})$. □

Because $End_{GL_n}(V_n^{\otimes m})=\langle S_m\rangle$, every GL_n-endomorphism of $V_n^{\otimes m}$ can be written as a linear combination of the morphisms λ_σ describing the action of $\sigma\in S_m$ on $V_n^{\otimes m}$. Our next job is to trace back these morphisms λ_σ through the canonical identifications until we can express them in terms of matrices.

To start let us compute the linear invariant

$$\mu_\sigma : V_n^{*\otimes m}\otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

corresponding to λ_σ under the identification $(V_n^{*\otimes m}\otimes V_n^{\otimes m})^* \simeq End(V_n^{\otimes m})$. By the identification we know that $\mu_\sigma(f_1\otimes\dots f_m\otimes v_1\otimes\dots\otimes v_m)$ is equal to

$$\begin{aligned}\langle\lambda_\sigma, f_1\otimes\dots f_m\otimes v_1\otimes\dots\otimes v_m\rangle &= f_1\otimes\dots\otimes f_m(v_{\sigma(1)}\otimes\dots v_{\sigma(m)})\\ &= \textstyle\prod_i f_i(v_{\sigma(i)})\end{aligned}$$

That is, we have proved

Proposition 3.9 *Any multilinear GL_n-invariant map*

$$\gamma : V_n^{*\otimes m}\otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

is a linear combination of the invariants

$$\mu_\sigma(f_1\otimes\dots f_m\otimes v_1\otimes\dots\otimes v_m)=\prod_i f_i(v_{\sigma(i)})$$

for $\sigma\in S_m$.

Using the identification $M_n(\mathbb{C})=V_n\otimes V_n^{*\otimes}$ a multilinear GL_n-invariant map

$$(V_n^*\otimes V)_n^{\otimes m}=V_n^{*\otimes m}\otimes V_n^{\otimes m} \longrightarrow \mathbb{C}$$

corresponds to a multilinear GL_n-invariant map

$$M_n(\mathbb{C})\otimes\dots\otimes M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

We will now give a description of the generating maps μ_σ in terms of matrices. Under the identification, matrix multiplication is induced by composition on rank one endomorphisms and here the rule is given by

$$v\otimes f.v'\otimes f'=f(v')v\otimes f'$$

$$\begin{bmatrix}\nu_1\\ \vdots\\ \nu_n\end{bmatrix}\otimes\begin{bmatrix}\phi_1 & \dots & \phi_n\end{bmatrix}.\begin{bmatrix}\nu_1'\\ \vdots\\ \nu_n'\end{bmatrix}\otimes\begin{bmatrix}\phi_1' & \dots & \phi_n'\end{bmatrix}=\begin{bmatrix}\nu_1\\ \vdots\\ \nu_n\end{bmatrix}f(v')\otimes\begin{bmatrix}\phi_1' & \dots & \phi_n'\end{bmatrix}.$$

Moreover, the trace map on M_n is induced by that on rank one endomorphisms where it is given by the rule

$$tr(v\otimes f)=f(v)$$

$$tr(\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} \otimes \begin{bmatrix} \phi_1 & \dots & \phi_n \end{bmatrix}) = tr(\begin{bmatrix} \nu_1\phi_1 & \dots & \nu_1\phi_n \\ \vdots & \ddots & \vdots \\ \nu_n\phi_1 & \dots & \nu_n\phi_n \end{bmatrix}) = \sum_i \nu_i\phi_i = f(v)$$

With these rules we can now give a matrix-interpretation of the GL_n-invariant maps μ_σ.

Proposition 3.10 *Let $\sigma = (i_1 i_2 \dots i_\alpha)(j_1 j_2 \dots j_\beta)\dots(z_1 z_2 \dots z_\zeta)$ be a decomposition of $\sigma \in S_m$ into cycles (including those of length one). Then, under the above identification we have*

$$\mu_\sigma(A_1 \otimes \dots \otimes A_m) = tr(A_{i_1}A_{i_2}\dots A_{i_\alpha})tr(A_{j_1}A_{j_2}\dots A_{j_\beta})\dots tr(A_{z_1}A_{z_2}\dots A_{z_\zeta})$$

Proof. Both sides are multilinear hence it suffices to verify the equality for rank one matrices. Write $A_i = v_i \otimes f_i$, then we have that

$$\begin{aligned} \mu_\sigma(A_1 \otimes \dots \otimes A_m) = & \quad \mu_\sigma(v_1 \otimes \dots v_m \otimes f_1 \otimes \dots \otimes f_m) \\ = & \quad \textstyle\prod_i f_i(v_{\sigma(i)}) \end{aligned}$$

Consider the subproduct

$$f_{i_1}(v_{i_2})f_{i_2}(v_{i_3})\dots f_{i_{\alpha-1}}(v_{i_\alpha}) = S$$

Now, look at the matrixproduct

$$v_{i_1} \otimes f_{i_1}.v_{i_2} \otimes f_{i_2}. \ \dots \ .v_{i_\alpha} \otimes f_{i_\alpha}$$

which is by the product rule equal to

$$f_{i_1}(v_{i_2})f_{i_2}(v_{i_3})\dots f_{i_{\alpha-1}}(v_{i_\alpha})v_{i_1} \otimes f_{i_\alpha}$$

Hence, by the trace rule we have that

$$tr(A_{i_1}A_{i_2}\dots A_{i_\alpha}) = \prod_{j=1}^{\alpha} f_{i_j}(v_{\sigma(i_j)}) = S$$

$\square$

Having found a description of the multilinear invariant polynomial maps

$$M_n^m = \underbrace{M_n \oplus \dots \oplus M_n}_{m} \longrightarrow \mathbb{C}$$

we will now describe all polynomial maps which are constant along orbits by polarization. The coordinate algebra $\mathbb{C}[M_n^m]$ is the polynomial ring in mn^2 variables $x_{ij}(k)$ where $1 \leq k \leq m$ and $1 \leq i, j \leq n$. Consider the m generic $n \times n$ matrices

$$\boxed{k} = X_k = \begin{bmatrix} x_{11}(k) & \dots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \dots & x_{nn}(k) \end{bmatrix} \in M_n(\mathbb{C}[M_n^m]).$$

The action of GL_n on polynomial maps $f \in \mathbb{C}[M_n^m]$ is fully determined by the action on the coordinate functions $x_{ij}(k)$. As in the case of one $n \times n$ matrix we see that this action is given by

$$g.x_{ij}(k) = (g^{-1}.X_k.g)_{ij}.$$

We see that this action preserves the subspaces spanned by the entries of any of the generic matrices. Hence, we can define a gradation on $\mathbb{C}[M_n^m]$ by $deg(x_{ij}(k)) = (0,\ldots,0,1,0,\ldots,0)$ (with 1 at place k) and decompose

$$\mathbb{C}[M_n^m] = \underset{(d_1,\ldots,d_m)\in\mathbb{N}^m}{\oplus} \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)}$$

where $\mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)}$ is the subspace of all multihomogeneous forms f in the $x_{ij}(k)$ of degree $(d_1,\ldots,d_m)$, that is, in each monomial term of f there are exactly d_k factors coming from the entries of the generic matrix X_k for all $1 \leq k \leq m$. The action of GL_n stabilizes each of these subspaces, that is,

$$\text{if} \quad f \in \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)} \quad \text{then} \quad g.f \in \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)} \quad \text{for all } g \in GL_n.$$

In particular, if f determines a polynomial map on M_n^m which is constant along orbits, that is, if f belongs to the ring of invariants $\mathbb{C}[M_n^m]^{GL_n}$ then each of its multihomogeneous components is also an invariant and therefore it suffices to determine all multihomogeneous invariants.

Let $f \in \mathbb{C}[M_n^m]_{(d_1,\ldots,d_m)}$ and take for each $1 \leq k \leq m$ d_k new variables $t_1(k),\ldots,t_{d_k}(k)$. Expand

$$f(t_1(1)A_1(1) + \ldots + t_{d_1}A_{d_1}(1),\ldots,t_1(m)A_1(m) + \ldots + t_{d_m}(m)A_{d_m}(m))$$

as a polynomial in the variables $t_i(k)$, then we get an expression

$$\sum t_1(1)^{s_1(1)} \ldots t_{d_1}^{s_{d_1}(1)} \ldots t_1(m)^{s_1(m)} \ldots t_{d_m}(m)^{s_{d_m}(m)}.$$
$$f_{(s_1(1),\ldots,s_{d_1}(1),\ldots,s_1(m),\ldots,s_{d_m}(m))}(A_1(1),\ldots,A_{d_1}(1),\ldots,A_1(m),\ldots,A_{d_m}(m))$$

such that for all $1 \leq k \leq m$ we have $\sum_{i=1}^{d_k} s_i(k) = d_k$. Moreover, each of the $f_{(s_1(1),\ldots,s_{d_1}(1),\ldots,s_1(m),\ldots,s_{d_m}(m))}$ is a multi-homogeneous polynomial function on

$$\underbrace{M_n \oplus \ldots \oplus M_n}_{d_1} \oplus \underbrace{M_n \oplus \ldots \oplus M_n}_{d_2} \oplus \ldots \oplus \underbrace{M_n \oplus \ldots \oplus M_n}_{d_m}$$

of multi-degree $(s_1(1),\ldots,s_{d_1}(1),\ldots,s_1(m),\ldots,s_{d_m}(m))$. Observe that if f is an invariant polynomial function on M_n^m, then each of these multi homogeneous functions is an invariant polynomial function on M_n^D where $D = d_1 + \ldots + d_m$.

In particular, we consider the multi-*linear* function

$$f_{1,\ldots,1} : M_n^D = M_n^{d_1} \oplus \ldots \oplus M_n^{d_m} \longrightarrow \mathbb{C}$$

which we call the *polarization* of the polynomial f and denote with $Pol(f)$. Observe that $Pol(f)$ in symmetric in each of the entries belonging to a block $M_n^{d_k}$ for every $1 \leq k \leq m$. If f is invariant under GL_n, then so is the multilinear function $Pol(f)$ and we know the form of all such functions by the results given before (replacing M_n^m by M_n^D).

Finally, we want to recover f back from its polarization. We claim to have the equality

$$Pol(f)(\underbrace{A_1,\ldots,A_1}_{d_1},\ldots,\underbrace{A_m,\ldots,A_m}_{d_m}) = d_1!\ldots d_m! f(A_1,\ldots,A_m)$$

and hence we recover f. This process is called *restitution*. The claim follows from the observation that

$$\begin{aligned}
&f(t_1(1)A_1 + \ldots + t_{d_1}(1)A_1,\ldots,t_1(m)A_m + \ldots + t_{d_m}(m)A_m) = \\
&\qquad f((t_1(1) + \ldots + t_{d_1}(1))A_1,\ldots,(t_1(m) + \ldots + t_{d_m}(m))A_m) = \\
&\qquad\quad (t_1(1) + \ldots + t_{d_1}(1))^{d_1} \ldots (t_1(m) + \ldots + t_{d_m}(m))^{d_m} f(A_1,\ldots,A_m)
\end{aligned}$$

and the definition of $Pol(f)$. Hence we have proved that any multi-homogeneous invariant polynomial function f on M_n^m of multidegree $(d_1, \ldots, d_m)$ can be obtained by restitution of a multilinear invariant function

$$Pol(f) : M_n^D = M_n^{d_1} \oplus \ldots \oplus M_n^{d_m} \longrightarrow \mathbb{C}$$

If we combine this fact with our description of all multilinear invariant functions on $M_n \oplus \ldots \oplus M_n$ we finally obtain :

Theorem 3.11 *Any polynomial function $M_n^m \xrightarrow{f} \mathbb{C}$ which is constant along orbits under the action of GL_n by simultaneous conjugation is a polynomial in the invariants*

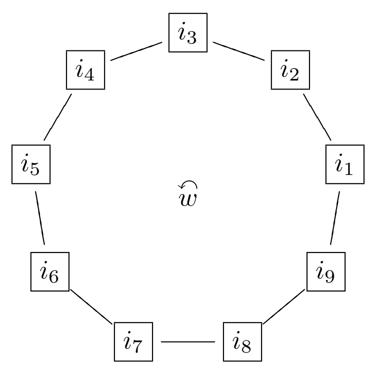
$$tr(X_{i_1} \ldots X_{i_l})$$

where $X_{i_1} \ldots X_{i_l}$ run over all possible noncommutative polynomials in the generic matrices $\{X_1, \ldots, X_m\}$.

We will call the algebra $\mathbb{C}[M_n^m]$ generated by these invariants the *necklace algebra* $\mathbb{N}_n^m = \mathbb{C}[M_n^m]^{GL_n}$. The terminology is justified by the observation that the generators

$$tr(X_{i_1} X_{i_2} \ldots X_{i_l})$$

are only determined up to cyclic permutation of the generic matrices X_j. That is, the generators are determined by *necklace words* w such as



where each bead corresponds to a generic matrix $\boxed{i} = X_i$. They are multiplied cyclicly to obtain an $n \times n$ matrix with coefficients in $M_n(\mathbb{C}[M_n^m])$. The trace of this matrix is called $tr(w)$ and the result asserts that these elements generate the ring of polynomial invariants.

3.4 The trace algebra.

In this section we will prove a bound on the length of the necklace words w necessary for the $tr(w)$ to generate $\mathbb{N}_n^m$. In the last section, after we have determined the relations between these necklaces $tr(w)$, we will be able to improve this bound.

First, we will characterize all GL_n-*equivariant maps* from M_n^m to M_n, that is all polynomial maps $M_n^m \xrightarrow{f} M_n$ such that for all $g \in GL_n$ the diagram below is commutative

$$\begin{tikzcd}
M_n^m \arrow[r, "f"] \arrow[d, "g.g^{-1}"'] & M_n \arrow[d, "g.g^{-1}"] \\
M_n^m \arrow[r, "f"] & M_n
\end{tikzcd}$$

With pointwise addition and multiplication in the target algebra M_n, these polynomial maps form a noncommutative algebra $\mathbb{T}_n^m$ called the *trace algebra*. Obviously, the trace algebra is a subalgebra of the algebra of *all* polynomial maps from M_n^m to M_n, that is,

$$\mathbb{T}_n^m \hookrightarrow M_n(\mathbb{C}[M_n^m])$$

Clearly, using the diagonal embedding of $\mathbb{C}$ in M_n any invariant polynomial on M_n^m determines a GL_n-equivariant map. Equivalently, using the diagonal embedding of $\mathbb{C}[M_n^m]$ in $M_n(\mathbb{C}[M_n^m])$ we can embed the necklace algebra

$$\mathbb{N}_n^m = \mathbb{C}[M_n^m]^{GL_n} \hookrightarrow \mathbb{T}_n^m$$

Another source of GL_n-equivariant maps are the *coordinate maps*

$$X_i : M_n^m = M_n \oplus \ldots \oplus M_n^m \longrightarrow M_n \qquad (A_1, \ldots, A_m) \mapsto A_i$$

Observe that the coordinate map X_i is represented by the generic matrix $\boxed{i} = X_i$ in $M_n(\mathbb{C}[M_n^m])$.

Proposition 3.12 *As an algebra over the necklace algebra $\mathbb{N}_n^m$, the trace algebra $\mathbb{T}_n^m$ is generated by the elements $\{X_1, \ldots, X_m\}$.*

Proof. Consider a GL_n-equivariant map $M_n^m \xrightarrow{f} M_n$ and associate to it the polynomial map

$$M_n^{m+1} = M_n^m \oplus M_n \xrightarrow{tr(fX_{m+1})} \mathbb{C}$$

defined by sending $(A_1, \ldots, A_m, A_{m+1})$ to $tr(f(A_1, \ldots, A_m).A_{m+1})$. For all $g \in GL_n$ we have that $f(g.A_1.g^{-1}, \ldots, g.A_m.g^{-1})$ is equal to $g.f(A_1, \ldots, A_m).g^{-1}$ and hence

$$\begin{aligned} tr(f(g.A_1.g^{-1}, \ldots, g.A_m.g^{-1}).g.A_{m+1}.g^{-1}) &= tr(g.f(A_1, \ldots, A_m).g^{-1}.g.A_{m+1}.g^{-1}) \\ &= tr(g.f(A_1, \ldots, A_m).A_{m+1}.g^{-1}) \\ &= tr(f(A_1, \ldots, A_m).A_{m+1}) \end{aligned}$$

so $tr(fX_{m+1})$ is an invariant polynomial function on M_n^{m+1} which is *linear* in X_{m+1}. By theorem 3.11 we can therefore write

$$tr(fX_{m+1}) = \sum \underbrace{g_{i_1 \ldots i_l}}_{\in \mathbb{N}_n^m} tr(X_{i_1} \ldots X_{i_l} X_{m+1})$$

Here, we used the necklace property allowing to permute cyclicly the trace terms in which X_{m+1} occurs such that X_{m+1} occurs as the last factor. But then, $tr(fX_{m+1}) = tr(gX_{m+1})$ where

$$g = \sum g_{i_1 \ldots i_l} X_{i_1} \ldots X_{i_l}.$$

Finally, using the *nondegeneracy* of the trace map on M_n (that is, if $A, B \in M_n$ such that $tr(AC) = tr(BC)$ for all $C \in M_n$, then $A = B$) it follows that $f = g$. $\square$

If we give each of the generic matrices X_i degree one, we see that the trace algebra $\mathbb{T}_n^m$ is a *connected positively graded algebra*

$$\mathbb{T}_n^m = T_0 \oplus T_1 \oplus T_2 \oplus \ldots \qquad \text{with} \quad T_0 = \mathbb{C}.$$

Our aim is to bound the length of the monomials in the X_i necessary to generate $\mathbb{T}_n^m$ as a module over the necklace algebra $\mathbb{N}_n^m$. Before we can do this we need to make a small detour in one of the more exotic realms of noncommutative algebra : the *Nagata-Higman problem*.

Theorem 3.13 (Nagata-Higman) *Let R be an associative algebra without a unit element. Assume there is a fixed natural number n such that $x^n = 0$ for all $x \in R$. Then, $R^{2^n-1} = 0$, that is*

$$x_1.x_2.\ldots.x_{2^n-1} = 0$$

for all $x_j \in R$.

Proof. We use induction on n, the case $n = 1$ being obvious. Consider for all $x, y \in R$

$$f(x,y) = yx^{n-1} + xyx^{n-2} + x^2yx^{n-3} + \ldots + x^{n-2}yx + x^{n-1}y.$$

Because for all $c \in \mathbb{C}$ we must have that

$$0 = (y + cx)^n = x^n c^n + f(x,y)c^{n-1} + \ldots + y^n$$

it follows that all the coefficients of the c^i with $1 \leq i < n$ must be zero, in particular $f(x,y) = 0$. But then we have for all $x, y, z \in R$ that

$$\begin{aligned} 0 &= f(x,z)y^{n-1} + f(x,zy)y^{n-2} + f(x,zy^2)y^{n-3} + \ldots + f(x,zy^{n-1}) \\ &= nx^{n-1}zy^{n-1} + zf(y,x^{n-1}) + xzf(y,x^{n-2}) + x^2zf(y,x^{n-3}) + \ldots + x^{n-2}zf(y,x) \end{aligned}$$

and therefore $x^{n-1}zy^{n-1} = 0$. Let $I \triangleleft R$ be the twosided ideal of R generated by all elements x^{n-1}, then we have that $I.R.I = 0$. In the quotient algebra $\overline{R} = R/I$ every element $\overline{x}$ satisfies $\overline{x}^{n-1} = 0$.

By induction we may assume that $\overline{R}^{2^{n-1}-1} = 0$, or equivalently that $R^{2^{n-1}-1}$ is contained in I. But then,

$$R^{2^n-1} = R^{2(2^{n-1}-1)+1} = R^{2^{n-1}-1}.R.R^{2^{n-1}-1} \hookrightarrow I.R.I = 0$$

finishing the proof. □

Proposition 3.14 *The trace algebra $\mathbb{T}_n^m$ is spanned as a module over the necklace algebra $\mathbb{N}_n^m$ by all monomials in the generic matrices*

$$X_{i_1}X_{i_2}\ldots X_{i_l}$$

of degree $l \leq 2^n - 1$.

Proof. By the diagonal embedding of $\mathbb{N}_n^m$ in $M_n(\mathbb{C}[M_n^m])$ it is clear that $\mathbb{N}_n^m$ commutes with any of the X_i. Let $\mathbb{T}_+$ and $\mathbb{N}_+$ be the strict positive degrees of $\mathbb{T}_n^m$ and $\mathbb{N}_n^m$ and form the graded associative algebra (without unit element)

$$R = \mathbb{T}_+/\mathbb{N}_+.\mathbb{T}_+$$

Observe that any element $t \in \mathbb{T}_+$ satisfies an equation of the form

$$t^n + c_1t^{n-1} + c_2t^{n-2} + \ldots + c_n = 0$$

with all of the $c_i \in \mathbb{N}_+$. Indeed we have seen that all the coefficients of the characteristic polynomial of a matrix can be expressed as polynomials in the traces of powers of the matrix. But then, for any $x \in R$ we have that $x^n = 0$.

By the Nagata-Higman theorem we know that $R^{2^n-1} = (R_1)^{2^n-1} = 0$. Let $\mathbb{T}'$ be the graded $\mathbb{N}_n^m$-submodule of $\mathbb{T}_n^m$ spanned by all monomials in the generic matrices X_i of degree at most $2^n - 1$, then the above can be reformulated as

$$\mathbb{T}_n^m = \mathbb{T}' + \mathbb{N}_+.\mathbb{T}_n^m.$$

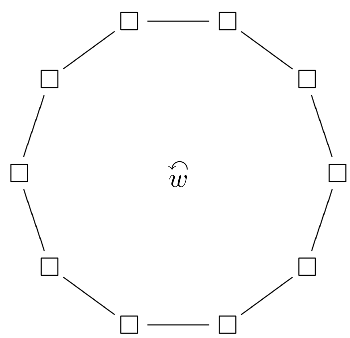
We claim that $\mathbb{T}_m^n = \mathbb{T}'$. Assume not, then there is a homogeneous $t \in \mathbb{T}_n^m$ *of minimal degree* d not contained in $\mathbb{T}'$ but still we have a description

$$t = t' + c_1.t_1 + \ldots + c_s.t_s$$

with t' and all c_i, t_i homogeneous elements. As $deg(t_i) < d$, $t_i \in \mathbb{T}'$ for all i but then is $t \in \mathbb{T}'$ a contradiction. $\square$

Finally we are in a position to bound the length of the necklaces generating $\mathbb{N}_n^m$ as an algebra.

Theorem 3.15 *The necklace algebra $\mathbb{N}_n^m$ is generated by all necklaces $tr(w)$ where w is a necklace word*



of length $l \leq 2^n$ where each of the beads is a generic matrix $\boxed{i} = X_i$.

Proof. Let $\mathbb{T}'$ be the $\mathbb{C}$-subalgebra of $\mathbb{T}_n^m$ generated by the generic matrices X_i. Then, $tr(\mathbb{T}'_+)$ generates the ideal $\mathbb{N}_+$. Let $\mathbb{S}$ be the set of all monomials in the X_i of degree at most $2^n - 1$. By the foregoing proposition we know that $\mathbb{T}' \hookrightarrow \mathbb{N}_n^m.\mathbb{S}$. The trace map

$$tr : \mathbb{T}_n^m \longrightarrow \mathbb{N}_n^m$$

is $\mathbb{N}_n^m$-linear and therefore, because $\mathbb{T}'_+ \subset \mathbb{T}'.(\mathbb{C}X_1 + \ldots + \mathbb{C}X_m)$ we have

$$tr(\mathbb{T}'_+) \subset tr(\mathbb{N}_n^m.\mathbb{S}.(\mathbb{C}X_1 + \ldots + \mathbb{C}X_m)) \subset \mathbb{N}_n^m.tr(\mathbb{S}')$$

where $\mathbb{S}'$ is the set of monomials in the X_i of degree at most 2^n. If $\mathbb{N}'$ is the $\mathbb{C}$-subalgebra of $\mathbb{N}_n^m$ generated by all $tr(S')$, then we have $tr(\mathbb{T}'_+) \subset \mathbb{N}_n^m.\mathbb{N}'_+$. But then, we have

$$\mathbb{N}_+ = \mathbb{N}_n^m tr(\mathbb{T}_+) \subset \mathbb{N}_n^m \mathbb{N}'_+ \quad \text{and thus} \quad \mathbb{N}_n^m = \mathbb{N}' + \mathbb{N}_n^m \mathbb{N}'_+$$

from which it follows that $\mathbb{N}_n^m = \mathbb{N}'$ by a similar argument as in the foregoing proof. $\square$

Example 3.16 The algebras $\mathbb{N}_2^2$ and $\mathbb{T}_2^2$.

When working with 2×2 matrices, the following identities are often helpful

$$0 = A^2 - tr(A)A + det(A)$$
$$A.B + B.A = tr(AB) - tr(A)tr(B) + tr(A)B + tr(B)A$$

for all $A, B \in M_2$. Let $\mathbb{N}'$ be the subalgebra of $\mathbb{N}_2^2$ generated by $tr(X_1), tr(X_2)$, $det(X_1), det(X_2)$ and $tr(X_1X_2)$. Using the two formulas above and $\mathbb{N}_2^2$-linearity of the trace on $\mathbb{T}_2^2$ we see that the trace of any monomial in X_1 and X_2 of degree $d \geq 3$ can be expressed in elements of $\mathbb{N}'$ and traces of monomials of degree $\leq d - 1$. Hence, we have

$$\mathbb{N}_2^2 = \mathbb{C}[tr(X_1), tr(X_2), det(X_1), det(X_2), tr(X_1X_2)].$$

Observe that there can be no algebraic relations between these generators as we have seen that the induced map $\pi : M_2^2 \longrightarrow \mathbb{C}^5$ is surjective. Another consequence of the above identities is that over $\mathbb{N}_2^2$ any monomial in the X_1, X_2 of degree $d \geq 3$ can be expressed as a linear combination of $1, X_1, X_2$ and $X_1 X_2$ and so these elements generate $\mathbb{T}_2^2$ as a $\mathbb{N}_2^2$-module. In fact, they are a basis of $\mathbb{T}_2^2$ over $\mathbb{N}_2^2$. Assume otherwise, there would be a relation say

$$X_1 X_2 = \alpha I_2 + \beta X_1 + \gamma X_2$$

with $\alpha, \beta, \gamma \in \mathbb{C}(tr(X_1), tr(X_2), det(X_1), det(X_2), tr(X_1 X_2))$. Then this relation has to hold for all matrix couples $(A, B) \in M_2^2$ and we obtain a contradiction if we take the couple

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{whence} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Concluding, we have the following description of $\mathbb{N}_2^2$ and $\mathbb{T}_2^2$ as a subalgebra of $\mathbb{C}[M_2^2]$ respectively $M_2(\mathbb{C}[M_2^2])$

$$\begin{cases} \mathbb{N}_2^2 = & \mathbb{C}[tr(X_1), tr(X_2), det(X_1), det(X_2), tr(X_1 X_2)] \\ \mathbb{T}_2^2 = & \mathbb{N}_2^2.I_2 \oplus \mathbb{N}_2^2.X_1 \oplus \mathbb{N}_2^2.X_2 \oplus \mathbb{N}_2^2.X_1 X_2 \end{cases}$$

Observe that we might have taken the generators $tr(X_i^2)$ rather than $det(X_i)$ because $det(X_i) = \frac{1}{2}(tr(X_i)^2 - tr(X_i)^2)$ as follows from taking the trace of characteristic polynomial of X_i.

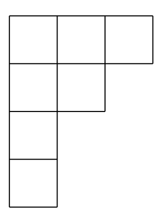
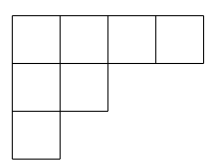
3.5 The symmetric group.

Let S_d be the symmetric group of all permutations on d letters. The group algebra $\mathbb{C}\ S_d$ is a semisimple algebra. In particular, any simple S_d-representation is isomorphic to a minimal left ideal of $\mathbb{C}\ S_d$ which is generated by an *idempotent*. We will now determine these idempotents.

To start, conjugacy classes in S_d correspond naturally to *partitions* $\lambda = (\lambda_1, \ldots, \lambda_k)$ of d, that is, decompositions in natural numbers

$$d = \lambda_1 + \ldots + \lambda_k \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$$

The correspondence associates to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ the conjugacy class of a permutation consisting of disjoint cycles of lengths $\lambda_1, \ldots, \lambda_k$. It is traditional to assign to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ a *Young diagram* with λ_i boxes in the i-th row, the rows of boxes lined up to the left. The *dual partition* $\lambda^* = (\lambda_1^*, \ldots, \lambda_r^*)$ to λ is defined by interchanging rows and columns in the Young diagram of λ. For example, to the partition $\lambda = (3, 2, 1, 1)$ of 7 we assign the Young diagram

$\lambda =$  $\lambda^* =$ 

with dual partition $\lambda^* = (4, 2, 1)$. A *Young tableau* is a numbering of the boxes of a Young diagram by the integers $\{1, 2, \ldots, d\}$. For example, two distinct Young tableaux of type λ are

1	2	3
4	5	
6		
7		

1	3	5
2	4	
6		
7		

Now, fix a Young tableau T of type λ and define subgroups of S_d by

$$P_\lambda = \{\sigma \in S_d \mid \sigma \text{ preserves each row } \}$$

$$Q_\lambda = \{\sigma \in S_d \mid \sigma \text{ preserves each column } \}$$

For example, for the second Young tableaux given above we have that

$$\begin{cases} P_\lambda & = S_{\{1,3,5\}} \times S_{\{2,4\}} \times \{(6)\} \times \{(7)\} \\ Q_\lambda & = S_{\{1,2,6,7\}} \times S_{\{3,4\}} \times \{(5)\} \end{cases}$$

Observe that different Young tableaux for the same λ define different subgroups and different elements to be defined below. Still, the simple representations we will construct from them turn out to be isomorphic.

Using these subgroups, we define the following elements in the groupalgebra $\mathbb{C}S_d$

$$a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma \quad , \quad b_\lambda = \sum_{\sigma \in Q_\lambda} sgn(\sigma) e_\sigma \quad \text{and} \quad c_\lambda = a_\lambda . b_\sigma$$

The element c_λ is called a *Young symmetrizer*. The next result gives an explicit one-to-one correspondence between the simple representations of $\mathbb{C}S_d$ and the conjugacy classes in S_d (or, equivalently, Young diagrams).

Theorem 3.17 *For every partition λ of d the left ideal $\mathbb{C}S_d.c_\lambda = V_\lambda$ is a simple S_d-representations and, conversely, any simple S_d-representation is isomorphic to V_λ for a unique partition λ.*

Proof. Observe that $P_\lambda \cap Q_\lambda = \{e\}$ (any permutation preserving rows as well as columns preserves all boxes) and so any element of S_d can be written in at most one way as a product $p.q$ with $p \in P_\lambda$ and $q \in Q_\lambda$. In particular, the Young symmetrizer can be written as $c_\lambda = \sum \pm e_\sigma$ with $\sigma = p.q$ for unique p and q and the coefficient $\pm 1 = sgn(q)$. From this it follows that for all $p \in P_\lambda$ and $q \in Q_\lambda$ we have

$$p.a_\lambda = a_\lambda.p = a_\lambda \quad , \quad sgn(q)q.b_\lambda = b_\lambda.sgn(q)q = b_\lambda \quad , \quad p.c_\lambda.sgn(q)q = c_\lambda$$

Moreover, we claim that c_λ is the unique element in $\mathbb{C}S_d$ (up to a scalar factor) satisfying the last property. This requires a few preparations.

Assume $\sigma \notin P_\lambda.Q_\lambda$ and consider the tableaux $T' = \sigma T$, that is, replacing the label i of each box in T by $\sigma(i)$. We claim that there are two distinct numbers which belong to the same row in T and to the same column in T'. If this were not the case, then all the distinct numbers in the first row of T appear in different columns of T'. But then we can find an element q_1' in the subgroup $\sigma.Q_\lambda.\sigma^{-1}$ preserving the columns of T' to take all these elements to the first row of T'. But then, there is an element $p_1 \in T_\lambda$ such that $p_1 T$ and $q_1' T'$ have the same first row. We can proceed to the second row and so on and obtain elements $p \in P_\lambda$ and $q' \in \sigma.Q_\lambda, \sigma^{-1}$ such that the tableaux pT and $q'T'$ are equal. Hence, $pT = q'\sigma T$ entailing that $p = q'\sigma$. Further, $q' = \sigma.q.\sigma^{-1}$ but then $p = q'\sigma = \sigma q$ whence $\sigma = p.q^{-1} \in P_\lambda.Q_\lambda$, a contradiction. Therefore, to $\sigma \notin P_\lambda.Q_\lambda$ we can assign a *transposition* $\tau = (ij)$ (replacing the two distinct numbers belonging to the same row in T and to the same column in T') for which $p = \tau \in P_\lambda$ and $q = \sigma^{-1}.\tau.\sigma \in Q_\lambda$.

After these preliminaries, assume that $c' = \sum a_\sigma e_\sigma$ is an element such that

$$p.c'.sgn(q)q = c' \quad \text{for all} \quad p \in P_\lambda, q \in Q_\lambda$$

We claim that $a_\sigma = 0$ whenever $\sigma \notin P_\lambda.Q_\lambda$. For take the transposition τ found above and $p = \tau$, $q = \sigma^{-1}.\tau.\sigma$, then $p.\sigma.q = \tau.\sigma.\sigma^{-1}.\tau.\sigma = \sigma$. However, the coefficient of σ in c' is a_σ and that of $p.c'.q$ is $-a_\sigma$ proving the claim. That is,

$$c' = \sum_{p,q} a_{pq} e_{p.q}$$

but then by the property of c' we must have that $a_{pq} = sgn(q)a_e$ whence $c' = a_e c_\lambda$ finishing the proof of the claimed uniqueness of the element c_λ.

As a consequence we have for all elements $x \in \mathbb{C}S_d$ that $c_\lambda.x, c_\lambda = \alpha_x c_\lambda$ for some scalar $\alpha_x \in \mathbb{C}$ and in particular that $c_\lambda^2 = n_\lambda c_\lambda$, for,

$$\begin{aligned} p.(c_\lambda.x.c_\lambda).sgn(q)q &= p.a_\lambda.b_\lambda.x.a_\lambda.b_\lambda.sgn(q)q \\ &= a_\lambda.b_\lambda.x.a_\lambda.b_\lambda = c_\lambda.x.c_\lambda \end{aligned}$$

and the statement follows from the uniqueness result for c_λ.

Define $V_\lambda = \mathbb{C}S_d.c_\lambda$ then we have $c_\lambda.V_\lambda \subset \mathbb{C}c_\lambda$. We claim that V_λ is a simple S_d-representation. Let $W \subset V_\lambda$ be a simple subrepresentation, then being a left ideal of $\mathbb{C}S_d$ we can write $W = \mathbb{C}S_d.x$ with $x^2 = x$ (note that W is a direct summand). Assume that $c_\lambda.W = 0$, then $W.W \subset \mathbb{C}S_d.c_\lambda.W = 0$ implying that $x = 0$ whence $W = 0$, a contradiction. Hence, $c_\lambda.W = \mathbb{C}c_\lambda \subset W$, but then

$$V_\lambda = \mathbb{C}S_d.c_\lambda \subset W \quad \text{whence} V_\lambda = W$$

is simple. Remains to show that for different partitions, the corresponding simple representations cannot be isomorphic.

We put a *lexicographic* ordering on the partitions by the rule that

$$\lambda > \mu \quad \text{if the first nonvanishing } \lambda_i - \mu_i \text{ is positive}$$

We claim that if $\lambda > \mu$ then $a_\lambda.\mathbb{C}S_d.b_\mu = 0$. It suffices to check that $a_\lambda.\sigma.b_\mu = 0$ for $\sigma \in S_d$. As $\sigma.b_\mu.\sigma^{-1}$ is the "b-element" constructed from the tableau $b.T'$ where T' is the tableaux fixed for μ, it is sufficient to check that $a_\lambda.b_\mu = 0$. As $\lambda > \mu$ there are distinct numbers i and j belonging to the same row in T and to the same column in T'. If not, the distinct numbers in any fixed row of T must belong to different columns of T', but this can only happen for all rows if $\mu \geq \lambda$. So consider $\tau = (ij)$ which belongs to P_λ and to Q_μ, whence $a_\lambda.\tau = a_\lambda$ and $\tau.b_\mu = -b_\mu$. But then,

$$a_\lambda.b_\mu = a_\lambda.\tau, \tau, b_\mu = -a_\lambda.b_\mu$$

proving the claim.

If $\lambda \neq \mu$ we claim that V_λ is not isomorphic to V_μ. Assume that $\lambda > \mu$ and ϕ a $\mathbb{C}S_d$-isomorphism with $\phi(V_\lambda) = V_\mu$, then

$$\phi(c_\lambda V_\lambda) = c_\lambda \phi(V_\lambda) = c_\lambda V_\mu = c_\lambda \mathbb{C}S_d c_\mu = 0$$

Hence, $c_\lambda V_\lambda = \mathbb{C}c_\lambda \neq 0$ lies in the kernel of an isomorphism which is clearly absurd.

Summarizing, we have constructed to distinct partitions of d, λ and μ nonisomorphic simple $\mathbb{C}S_d$-representations V_λ and V_μ. As we know that there are as many isomorphism classes of simples as there are conjugacy classes in S_d (or partitions), the V_λ form a complete set of isomorphism classes of simple S_d-representations, finishing the proof of the theorem. $\square$

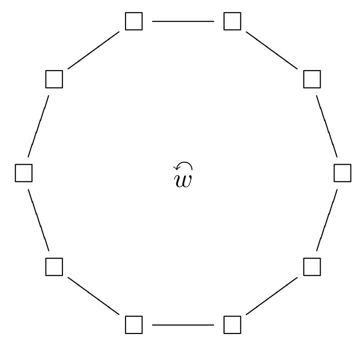
3.6 Necklace relations.

In this section we will prove that all the relations holding among the elements of the necklace algebra $\mathbb{N}_n^m$ are formal consequences of the Cayley-Hamilton equation. First, we will have to set up some notation to clarify what we mean by this.

For technical reasons it is sometimes convenient to have an infinite supply of noncommutative variables $\{x_1, x_2, \ldots, x_i, \ldots\}$. Two monomials of the same degree d in these variables

$$M = x_{i_1} x_{i_2} \ldots x_{i_d} \quad \text{and} \quad M' = x_{j_1} x_{j_2} \ldots x_{j_d}$$

are said to be *equivalent* if M' is obtained from M by a cyclic permutation, that is, there is a k such that $i_1 = j_k$ and all $i_a = j_b$ with $b = k + a - 1 \ mod \ d$. That is, if they determine the same necklace word



with each of the beads one of the noncommuting variables $\boxed{i} = x_i$. To each equivalence class we assign a formal variable that we denote by

$$t(x_{i_1} x_{i_2} \dots x_{i_d}).$$

The *formal necklace algebra* $\mathbb{N}^\infty$ is then the polynomial algebra on all these (infinitely many) letters. Similarly, we define the *formal trace algebra* $\mathbb{T}^\infty$ to be the algebra

$$\mathbb{T}^\infty = \mathbb{N}^\infty \otimes_{\mathbb{C}} \mathbb{C}\langle x_1, x_2, \dots, x_i, \dots \rangle$$

that is, the free associative algebra on the noncommuting variables x_i with coefficients in the polynomial algebra $\mathbb{N}^\infty$.

Crucial for our purposes is the existence of an $\mathbb{N}^\infty$-linear *formal trace map*

$$t : \mathbb{T}^\infty \twoheadrightarrow \mathbb{N}^\infty$$

defined by the formula

$$t(\sum a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k}) = \sum a_{i_1 \dots i_k} t(x_{i_1} \dots x_{i_k})$$

where $a_{i_1 \dots i_k} \in \mathbb{N}^\infty$.

In an analogous manner we will define infinite versions of the necklace and trace algebras. Let M_n^∞ be the space of all ordered sequences $(A_1, A_2, \dots, A_i, \dots)$ with $A_i \in M_n$ and all but finitely many of the A_i are the zero matrix. Again, GL_n acts on M_n^∞ by simultaneous conjugation and we denote the *infinite necklace algebra* $\mathbb{N}_n^\infty$ to be the algebra of polynomial functions f

$$M_n^\infty \xrightarrow{f} \mathbb{C}$$

which are constant along orbits. Clearly, $\mathbb{N}_n^\infty$ is generated as $\mathbb{C}$-algebra by the invariants $tr(M)$ where M runs over all monomials in the coordinate generic matrices $X_k = (x_{ij}(k))_{i,j}$ belonging to the k-th factor of M_n^∞. Similarly, the *infinite trace algebra* $\mathbb{T}_n^\infty$ is the algebra of GL_n-equivariant polynomial maps

$$M_n^\infty \longrightarrow M_n.$$

Clearly, $\mathbb{T}_n^\infty$ is the $\mathbb{C}$-algebra generated by $\mathbb{N}_n^\infty$ and the generic matrices X_k for $1 \leq k < \infty$. Observe that $\mathbb{T}_n^\infty$ is a subalgebra of the matrixring

$$\mathbb{T}_n^\infty \hookrightarrow M_n(\mathbb{C}[M_n^\infty])$$

and as such has a trace map tr defined on it and from our knowledge of the generators of $\mathbb{N}_n^\infty$ we know that $tr(\mathbb{T}_n^\infty) = \mathbb{N}_n^\infty$.

Now, there are natural algebra epimorphisms

$$\mathbb{T}^\infty \xrightarrow{\tau} \mathbb{T}_n^\infty \quad \text{and} \quad \mathbb{N}^\infty \xrightarrow{\nu} \mathbb{N}_n^\infty$$

defined by $\tau(t(x_{i_1} \ldots x_{i_k})) = \nu(t(x_{i_1} \ldots x_{i_k})) = tr(X_{i_1} \ldots X_{i_k})$ and $\tau(x_i) = X_i$. That is, ν and τ are compatible with the trace maps

$$\begin{tikzcd} \mathbb{T}^\infty \arrow[r, two heads, "\tau"] \arrow[d, "t"] & \mathbb{T}_n^\infty \arrow[d, "tr"] \\ \mathbb{N}^\infty \arrow[r, two heads, "\nu"] & \mathbb{N}_n^\infty \end{tikzcd}$$

We are interested in describing the *necklace relations*, that is, the kernel of ν. In the next section we will describe the *trace relations* which is the kernel of τ. Note that we obtain the relations holding among the necklaces in $\mathbb{N}_n^m$ by setting all $x_i = 0$ with $i > m$ and all $t(x_{i_1} \ldots x_{i_k}) = 0$ containing a variable with $i_j > m$.

In the description a map $T : \mathbb{C}S_d \longrightarrow \mathbb{N}^\infty$ will be important. Let S_d be the symmetric group of permutations on $\{1, \ldots, d\}$ and let

$$\sigma = (i_1 i_1 \ldots i_\alpha)(j_1 j_2 \ldots j_\beta) \ldots (z_1 z_2 \ldots z_\zeta)$$

be a decomposition of $\sigma \in S_d$ into cycles including those of length one. The map T assigns to σ a formal necklace $T_\sigma(x_1, \ldots, x_d)$ defined by

$$T_\sigma(x_1, \ldots, x_d) = t(x_{i_1} x_{i_2} \ldots x_{i_\alpha}) t(x_{j_1} x_{j_2} \ldots x_{j_\beta}) \ldots t(x_{z_1} x_{z_2} \ldots x_{z_\zeta})$$

Let $V = V_n$ be again the n-dimensional vectorspace of column vectors, then S_d acts naturally on $V^{\otimes d}$ via

$$\sigma.(v_1 \otimes \ldots \otimes v_d) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$$

hence determines a linear map $\lambda_\sigma \in End(V^{\otimes d})$. Recall from section 3 that under the natural identifications

$$(M_n^{\otimes d})^* \simeq (V^{*\otimes d} \otimes V^{\otimes d})^* \simeq End(V^{\otimes d})$$

the map λ_σ defines the multilinear map

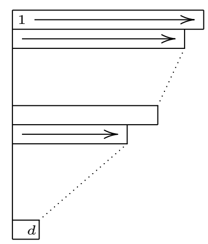
$$\mu_\sigma : \underbrace{M_n \otimes \ldots \otimes M_n}_{d} \longrightarrow \mathbb{C}$$

defined by (using the cycle decomposition of σ as before)

$$\mu_\sigma(A_1 \otimes \ldots \otimes A_d) = tr(A_{i_1} A_{i_2} \ldots A_{i_\alpha}) tr(A_{j_1} A_{j_2} \ldots A_{j_\beta}) \ldots tr(A_{z_1} A_{z_2} \ldots A_{z_\zeta}) \quad .$$

Therefore, a linear combination $\sum a_\sigma T_\sigma(x_1, \ldots, x_d)$ is a necklace relation (that is, belongs to $Ker\ \nu$) if and only if the multilinear map $\sum a_\sigma \mu_\sigma : M_n^{\otimes d} \longrightarrow \mathbb{C}$ is zero. This, in turn, is equivalent to the endomorphism $\sum a_\sigma \lambda_\sigma \in End(V^{\otimes m})$, induced by the action of the element $\sum a_\sigma e_\sigma \in \mathbb{C}S_d$ on $V^{\otimes d}$, being zero. In order to answer the latter problem we have to understand the action of a Young symmetrizer $c_\lambda \in \mathbb{C}S_d$ on $V^{\otimes d}$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of d and equip the corresponding Young diagram with the standard tableau (that is, order first the boxes in the first row from left to right, then the second row from left to right and so on).



The subgroup P_λ of S_d which preserves each row then becomes

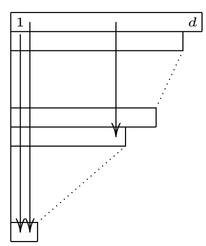
$$P_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_k} \hookrightarrow S_d.$$

As $a_\lambda = \sum_{p \in P_\lambda} e_p$ we see that the image of the action of a_λ on $V^{\otimes d}$ is the subspace

$$Im(a_\lambda) = Sym^{\lambda_1}\ V \otimes Sym^{\lambda_2}\ V \otimes \ldots \otimes Sym^{\lambda_k}\ V \hookrightarrow V^{\otimes d} \quad .$$

Here, $Sym^i\ V$ denotes the subspace of symmetric tensors in $V^{\otimes i}$.

Similarly, equip the Young diagram of λ with the tableau by ordering first the boxes in the first column from top to bottom, then those of the second column from top to bottom and so on.



Equivalently, give the Young diagram corresponding to the dual partition of λ

$$\lambda^* = (\mu_1, \mu_2, \ldots, \mu_l)$$

the standard tableau. Then, the subgroup Q_λ of S_d which preserves each row of λ (or equivalently, each column of λ^*) is

$$Q_\lambda = S_{\mu_1} \times S_{\mu_2} \times \ldots \times S_{\mu_l} \hookrightarrow S_d$$

As $b_\lambda = \sum_{q \in Q_\lambda} sgn(q) e_q$ we see that the image of b_λ on $V^{\otimes d}$ is the subspace

$$Im(b_\lambda) = \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \ldots \otimes \bigwedge^{\mu_l} V \hookrightarrow V^{\otimes d} \quad .$$

Here, $\bigwedge^i\ V$ is the subspace of all anti-symmetric tensors in $V^{\otimes i}$. Note that $\bigwedge^i\ V = 0$ whenever i is greater than the dimension $dim\ V = n$. That is, the image of the action of b_λ on $V^{\otimes d}$ is zero whenever the dual partition λ^* contains a row of length $\geq n+1$, or equivalently, whenever λ has $\geq n+1$ rows. Because the Young symmetrizer $c_\lambda = a_\lambda . b_\lambda \in \mathbb{C}\ S_d$ we have proved the first result on necklace relations.

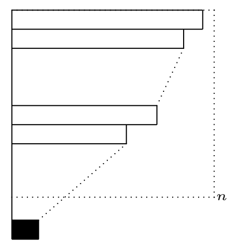
Proposition 3.18 *A formal necklace*

$$\sum_{\sigma \in S_d} a_\sigma T_\sigma(x_1, \ldots, x_d)$$

is a necklace relation (for $n \times n$ matrices) if and only if the element

$$\sum a_\sigma e_\sigma \in \mathbb{C}S_d$$

belongs to the ideal of $\mathbb{C}S_d$ spanned by the Young symmetrizers c_λ relative to partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$



with a least $n+1$ rows, that is, $k \geq n+1$.

Example 3.19 (Fundamental necklace and trace relation.)
Consider the partition $\lambda = (1, 1, \ldots, 1)$ of $n+1$, with corresponding Young tableau

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n+1 \\ \hline \end{array}$$

Then, $P_\lambda = \{e\}$, $Q_\lambda = S_{n+1}$ and we have the Young symmetrizer

$$a_\lambda = 1 \qquad b_\lambda = c_\lambda = \sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma.$$

The corresponding element is called the *fundamental necklace relation*

$$\mathfrak{F}(x_1, \ldots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} sgn(\sigma) T_\sigma(x_1, \ldots, x_{n+1}).$$

Clearly, $\mathfrak{F}(x_1, \ldots, x_{n+1})$ is multilinear of degree $n+1$ in the variables $\{x_1, \ldots, x_{n+1}\}$. Conversely, any multilinear necklace relation of degree $n+1$ must be a scalar multiple of $\mathfrak{F}(x_1, \ldots, x_{n+1})$. This follows from the proposition as the ideal described there is for $d = n+1$ just the scalar multiples of $\sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma$.

Because $\mathfrak{F}(x_1, \ldots, x_{n+1})$ is multilinear in the variables x_i we can use the cyclic permutation property of the formal trace t to write it in the form

$$\mathfrak{F}(x_1, \ldots, x_{n+1}) = t(\mathfrak{CH}(x_1, \ldots, x_n) x_{n+1}) \quad \text{with} \quad \mathfrak{CH}(x_1, \ldots, x_n) \in \mathbb{T}^\infty$$

Observe that $\mathfrak{CH}(x_1, \ldots, x_n)$ is multilinear in the variables x_i. Moreover, by the nondegeneracy of the trace map tr and the fact that $\mathfrak{F}(x_1, \ldots, x_{n+1})$ is a necklace relation, it follows that $\mathfrak{CH}(x_1, \ldots, x_n)$ is a trace relation. Again, any multilinear trace relation of degree n in the variables $\{x_1, \ldots, x_n\}$ is a scalar multiple of $\mathfrak{CH}(x_1, \ldots, x_n)$. This follows from the corresponding uniqueness result for $\mathfrak{F}(x_1, \ldots, x_{n+1})$.

We can give an explicit expression of this *fundamental trace relation*

$$\mathfrak{CH}(x_1, \ldots, x_n) = \sum_{k=0}^{n} (-1)^k \sum_{i_1 \neq i_2 \neq \ldots \neq i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \sum_{\sigma \in S_J} sgn(\sigma) T_\sigma(x_{j_1}, \ldots, x_{j_{n-k}})$$

where $J = \{1, \ldots, n\} - \{i_1, \ldots, i_k\}$. In a moment we will see that $\mathfrak{CH}(x_1, \ldots, x_n)$ and hence also $\mathfrak{F}(x_1, \ldots, x_{n+1})$ is obtained by polarization of the Cayley-Hamilton identity for $n \times n$ matrices.

We will explain what we mean by the Cayley-Hamilton polynomial for an element of $\mathbb{T}^\infty$. Recall that when $X \in M_n(A)$ is a matrix with coefficients in a commutative $\mathbb{C}$-algebra A its *characteristic polynomial* is defined to be

$$\chi_X(t) = det(t\mathbb{1}_n - X) \in A[t]$$

and by the *Cayley-Hamilton theorem* we have the basic relation that $\chi_X(X) = 0$. We have seen that the coefficients of the characteristic polynomial can be expressed as polynomial functions in the $tr(X^i)$ for $1 \leq i \leq n$.

For example if $n = 2$, then the characteristic polynomial can we written as

$$\chi_X(t) = t^2 - tr(X)t + \frac{1}{2}(tr(X)^2 - tr(X^2)).$$

For general n the method for finding these polynomial functions is based on the formal recursive algorithm expressing elementary symmetric functions in term of *Newton functions*, usually expressed by the formulae

$$\begin{aligned} f(t) &= \prod_{i=1}^{n}(t - \lambda_i), \\ \frac{f'(t)}{f(t)} &= \frac{d\ log\ f(t)}{dt} = \sum_{i=1}^{n} \frac{1}{t - \lambda_i} = \sum_{k=0}^{\infty} \frac{1}{t^{k+1}}(\sum_{i=1}^{n} \lambda_i^k) \end{aligned}$$

Note, if λ_i are the eigenvalues of $X \in M_n$, then $f(t) = \chi_X(t)$ and $\sum_{i=1}^n \lambda_i^k = tr(X^k)$. Therefore, one can use the formulae to express $f(t)$ in terms of the elements $\sum_{i=1}^n \lambda_i^k$. To get the required expression for the characteristic polynomial of X one only has to substitute $\sum_{i=1}^n \lambda_i^k$ with $tr(X^k)$.

This allows us to construct a *formal Cayley-Hamilton polynomial* $\chi_x(x) \in \mathbb{T}^\infty$ of an element $x \in \mathbb{T}^\infty$ by replacing in the above characteristic polynomial the term $tr(X^k)$ with $t(x^k)$ and t^l with x^l. If x is one of the variables x_i then $\chi_x(x)$ is an element of $\mathbb{T}^\infty$ homogeneous of degree n. Moreover, by the Cayley-Hamilton theorem it follows immediately that $\chi_x(x)$ is a trace relation. Hence, if we fully polarize $\chi_x(x)$ (say, using the variables $\{x_1, \ldots, x_n\}$) we obtain a multilinear trace relation of degree n. By the argument given in the example above we know that this element must be a scalar multiple of $\mathfrak{CH}(x_1, \ldots, x_n)$. In fact, one can see that this scale factor must be $(-1)^n$ as the leading term of the multilinearization is $\sum_{\sigma \in S_n} x_{\sigma(1)} \ldots x_{\sigma(n)}$ and compare this with the explicit form of $\mathfrak{CH}(x_1, \ldots, x_n)$.

Example 3.20 Consider the case $n = 2$. The formal Cayley-Hamilton polynomial of an element $x \in \mathbb{T}^\infty$ is

$$\chi_x(x) = x^2 - t(x)x + \frac{1}{2}(t(x)^2 - t(x^2)) \quad .$$

Polarization with respect to the variables x_1 and x_2 gives the expression

$$x_1x_2 + x_2x_1 - t(x_1)x_2 - t(x_2)x_1 + t(x_1)t(x_2) - t(x_1x_2)$$

which is $\mathfrak{CH}(x_1, x_2)$. Indeed, multiplying it on the right with x_3 and applying the formal trace t to it we obtain

$$\begin{aligned} &t(x_1x_2x_3) + t(x_2x_1x_3) - t(x_1)t(x_2x_3) - t(x_2)t(x_1x_3) + t(x_1)t(x_2)t(x_3) - t(x_1x_2)t(x_3) \\ &\quad = T_{(123)}(x_1, x_2, x_3) + T_{(213)}(x_1, x_2, x_3) - T_{(1)(23)}(x_1, x_2, x_3) - T_{(2)(13)}(x_1, x_2, x_3) \\ &\quad\quad + T_{(1)(2)(3)}(x_1, x_2, x_3) - T_{(12)(3)}(x_1, x_2, x_3) \\ &\quad = \sum_{\sigma \in S_3} T_\sigma(x_1, x_2, x_3) = \mathfrak{F}(x_1, x_2, x_3) \end{aligned}$$

as required.

Theorem 3.21 *The necklace relations $Ker\ \nu$ is the ideal of $\mathbb{N}^\infty$ generated by all the elements*

$$\mathfrak{F}(m_1, \ldots, m_{n+1})$$

where the m_i run over all monomials in the variables $\{x_1, x_2, \ldots, x_i, \ldots\}$.

Proof. Take a homogeneous necklace relation $f \in Ker\ \nu$ of degree d and polarize it to get a multilinear element $f' \in \mathbb{N}^\infty$. Clearly, f' is also a necklace relation and if we can show that f' belongs to the described ideal, then so does f as the process of restitution maps this ideal into itself.

Therefore, we may assume that f is multilinear of degree d. A priori f may depend on more than d variables x_k, but we can separate f as a sum of multilinear polynomials f_i each depending on precisely d variables such that for $i \neq j$ f_i and f_j do not depend on the same variables. Setting some of the variables equal to zero, we see that each of the f_i is again a necklace relation.

Thus, we may assume that f is a multilinear necklace identity of degree d depending on the variables $\{x_1, \ldots, x_d\}$. But then we know from proposition 3.18 that we can write

$$f = \sum_{\tau \in S_d} a_\tau T_\tau(x_1, \ldots, x_d)$$

where $\sum a_\tau e_\tau \in \mathbb{C}S_d$ belongs to the ideal spanned by the Young symmetrizers of Young diagrams λ having at least $n+1$ rows.

We claim that this ideal is generated by the Young symmetrizer of the partition $(1, \ldots, 1)$ of $n+1$ under the natural embedding of S_{n+1} into S_d. Let λ be a Young diagram having $k \geq n+1$ boxes and let c_λ be a Young symmetrizer with respect to a tableau where the boxes in the first column are labeled by the numbers $I = \{i_1, \ldots, i_k\}$ and let S_I be the obvious subgroup of S_d. As $Q_\lambda = S_I \times Q'$ we see that $b_\lambda = (\sum_{\sigma \in S_I} sgn(\sigma) e_\sigma).b'$ with $b' \in \mathbb{C}Q'$. Hence, c_λ belongs to the twosided ideal generated by $c_I = \sum_{\sigma \in S_I} sgn(\sigma) e_\sigma$ but this is also the twosided ideal generated by $c_k = \sum_{\sigma \in S_k} sgn(\sigma) e_\sigma$ as one verifies by conjugation with a partition sending I to $\{1, \ldots, k\}$. Moreover, by induction one shows that the twosided ideal generated by c_k belongs to the twosided ideal generated by $c_d = \sum_{\sigma \in S_d} sgn(\sigma) e_\sigma$, finishing the proof of the claim.

From this claim, we can write

$$\sum_{\tau \in S_d} a_\tau e_\tau = \sum_{\tau_i, \tau_j \in S_d} a_{ij} e_{\tau_i}.(\sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma).e_{\tau_j}$$

and therefore it suffices to analyze the form of the necklace identity associated to an element of the form

$$e_\tau.(\sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma).e_{\tau'} \quad \text{with} \quad \tau, \tau' \in S_d$$

Now, if a groupelement $\sum_{\mu \in S_d} b_\mu e_\mu$ corresponds to the formal necklace polynomial $\mathfrak{G}(x_1, \ldots, x_d)$, then the element $e_\tau.(\sum_{\mu \in S_d} b_\mu e_\mu).e_{\tau^{-1}}$ corresponds to the formal necklace polynomial $\mathfrak{G}(x_{\tau(1)}, \ldots, x_{\tau(d)})$.

Therefore, we may replace the element $e_\tau.(\sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma).e_{\tau'}$ by the element

$$(\sum_{\sigma \in S_{n+1}} sgn(\sigma) e_\sigma).e_\eta \quad \text{with} \quad \eta = \tau'.\tau \in S_d$$

We claim that we can write $\eta = \sigma'.\theta$ with $\sigma' \in S_{n+1}$ and $\theta \in S_d$ such that each cycle of θ contains at most one of the elements from $\{1, 2, \ldots, n+1\}$. Indeed assume that η contains a cycle containing more than one element from $\{1, \ldots, n+1\}$, say 1 and 2, that is

$$\eta = (1 i_1 i_2 \ldots i_r 2 j_1 j_2 \ldots j_s)(k_1 \ldots k_\alpha) \ldots (z_1 \ldots z_\zeta)$$

then we can express the product $(12).\eta$ in cycles as

$$(1 i_1 i_2 \ldots i_r)(2 j_1 j_2 \ldots j_s)(k_1 \ldots k_\alpha) \ldots (z_1 \ldots z_\zeta)$$

Continuing in this manner we reduce the number of elements from $\{1. \ldots, n+1\}$ in every cycle to at most one.

But then as $\sigma' \in S_{n+1}$ we have seen that $(\sum sgn(\sigma)e_\sigma).e_{\sigma'} = sgn(\sigma')(\sum sgn(\sigma)e_\sigma)$ and consequently

$$(\sum_{\sigma\in S_{n+1}} sgn(\sigma)e_\sigma).e_\eta = \pm(\sum_{\sigma\in S_{n+1}} sgn(\sigma)e_\sigma).e_\theta$$

where each cycle of θ contains at most one of $\{1,\ldots,n+1\}$. Let us write

$$\theta = (1i_1\ldots i_\alpha)(2j_1\ldots j_\beta)\ldots(n+1s_1\ldots s_\kappa)(t_1\ldots t_\lambda)\ldots(z_1\ldots z_\zeta)$$

Now, let $\sigma \in S_{n+1}$ then the cycle decomposition of $\sigma.\theta$ is obtained as follows : substitute in each cycle of σ the element 1 formally by the string $1i_1\ldots i_\alpha$, the element 2 by the string $2j_1\ldots j_\beta$, and so on until the element $n+1$ by the string $n+1s_1\ldots s_\kappa$ and finally adjoin the cycles of θ in which no elements from $\{1,\ldots,n+1\}$ appear.

Finally, we can write out the formal necklace element corresponding to the element $(\sum_{\sigma\in S_{n+1}} sgn(\sigma)e_\sigma).e_\theta$ as

$$\mathfrak{F}(x_1x_{i_1}\ldots x_{i_\alpha}, x_2x_{j_1}\ldots x_{j_\beta},\ldots,x_{n+1}x_{s_1}\ldots x_{s_\kappa})t(x_{t_1}\ldots x_{t_\lambda})\ldots t(x_{z_1}\ldots x_{z_\zeta})$$

finishing the proof of the theorem. □

3.7 Trace relations.

We will again use the non-degeneracy of the trace map to deduce the trace relations, that is, $Ker\ \tau$ from the description of the necklace relations.

Theorem 3.22 *The trace relations $Ker\ \tau$ is the twosided ideal of the formal trace algebra $\mathbb{T}^\infty$ generated by all elements*

$$\mathfrak{F}(m_1,\ldots,m_{n+1}) \quad \textit{and} \quad \mathfrak{CH}(m_1,\ldots,m_n)$$

where the m_i run over all monomials in the variables $\{x_1,x_2,\ldots,x_i,\ldots\}$.

Proof. Consider a trace relation $\mathfrak{H}(x_1,\ldots,x_d) \in Ker\ \tau$. Then, we have a necklace relation of the form

$$t(\mathfrak{H}(x_1,\ldots,x_d)x_{d+1}) \in Ker\ \nu$$

By theorem 3.21 we know that this element must be of the form

$$\sum n_{i_1\ldots i_{n+1}}\mathfrak{F}(m_{i_1},\ldots,m_{i_{n+1}})$$

where the m_i are monomials, the $n_{i_1\ldots i_{n+1}} \in \mathbb{N}^\infty$ and the expression must be linear in the variable x_{d+1}. That is, x_{d+1} appears linearly in each of the terms

$$n\mathfrak{F}(m_1,\ldots,m_{n+1})$$

so appears linearly in n or in precisely one of the monomials m_i. If x_{d+1} appears linearly in n we can write

$$n = t(n'.x_{d+1}) \quad \text{where} \quad n' \in \mathbb{T}^\infty.$$

If x_{d+1} appears linearly in one of the monomials m_i we may assume that it does so in m_{n+1}, permuting the monomials if necessary. That is, we may assume $m_{n+1} = m'_{n+1}.x_{d+1}.m"_{n+1}$ with m, m' monomials. But then, we can write

$$\begin{aligned} n\mathfrak{F}(m_1,\ldots,m_{n+1}) = &\quad nt(\mathfrak{CH}(m_1,\ldots,m_n).m'_{n+1}.x_{d+1}.m"_{n+1}) \\ = &\quad t(n.m"_{n+1}.\mathfrak{CH}(m_1,\ldots,m_n).m'_{n+1}.x_{d+1}) \end{aligned}$$

using $\mathbb{N}^\infty$-linearity and the cyclic permutation property of the formal trace t. But then, separating the two cases, one can write the total expression

$$t(\mathfrak{H}(x_1,\ldots,x_d)x_{d+1}) = t([\sum_{\underline{i}} n'_{i_1\ldots i_{n+1}}\mathfrak{F}(m_{i_1},\ldots,m_{i_{n+1}}) \\ + \sum_{\underline{j}} n_{j_1\ldots j_{n+1}}.m"_{j_{n+1}}.\mathfrak{CH}(m_{j_1},\ldots,m_{j_n}).m'_{j_{n+1}}] \quad x_{d+1})$$

Finally, observe that two formal trace elements $\mathfrak{H}(x_1,\ldots,x_d)$ and $\mathfrak{K}(x_1,\ldots,x_d)$ are equal if the formal necklaces

$$t(\mathfrak{H}(x_1,\ldots,x_d)x_{d+1}) = t(\mathfrak{K}(x_1,\ldots,x_d)x_{d+1})$$

are equal, finishing the proof. □

We will give another description of the necklace relations $Ker\ \tau$ which is better suited for the categorical interpretation of $\mathbb{T}^\infty_n$ to be given in the next chapter. Consider formal trace elements $m_1, m_2, \ldots, m_i, \ldots$ with $m_j \in \mathbb{T}^\infty$. The *formal substitution*

$$f \mapsto f(m_1, m_2, \ldots, m_i, \ldots)$$

is the uniquely determined algebra endomorphism of $\mathbb{T}^\infty$ which maps the variable x_i to m_i and is compatible with the formal trace t. That is, the substitution sends a monomial $x_{i_1}x_{i_1}\ldots x_{i_k}$ to the element $g_{i_1}g_{i_2}\ldots g_{i_k}$ and an element $t(x_{i_1}x_{i_2}\ldots x_{i_k})$ to the element $t(g_{i_1}g_{i_2}\ldots g_{i_k})$. A *substitution invariant ideal* of $\mathbb{T}^\infty$ is a twosided ideal of $\mathbb{T}^\infty$ that is closed under all possible substitutions as well as under the formal trace t. For any subset of elements $E \subset \mathbb{T}^\infty$ there is a minimal substitution invariant ideal containing E. This is the ideal generated by all elements obtained from E by making all possible substitutions and taking all their formal traces. We will refer to this ideal as the *substitution invariant ideal generated by E*.

Recall the definition of the formal Cayley-Hamilton polynomial $\chi_x(x)$ of an element $x \in \mathbb{T}^\infty$ given in the previous section.

Theorem 3.23 *The trace relations $Ker\ \tau$ is the substitution invariant ideal of $\mathbb{T}^\infty$ generated by the formal Cayley-Hamilton polynomials*

$$\chi_x(x) \quad \textit{for all} \quad x \in \mathbb{T}^\infty$$

Proof. The result follows from theorem 3.22 and the definition of a substitution invariant ideal once we can show that the full polarization of $\chi_x(x)$, which we have seen is $\mathfrak{CH}(x_1,\ldots,x_n)$, lies in the substitution invariant ideal generated by the $\chi_x(x)$.

This is true since we may replace the process of polarization with the process of multilinearization, whose first step is to replace, for instance

$$\chi_x(x) \quad \text{by} \quad \chi_{x+y}(x+y) - \chi_x(x) - \chi_y(y).$$

The final result of multilinearization is the same as of full polarization and the claim follows as multilinearizing a polynomial in a substitution invariant ideal, we remain in the same ideal. □

We will use our knowledge on the necklace and trace relations to improve the bound of $2^n - 1$ in the *Nagata-Higman problem* to n^2. Recall that this problem asks for a number $N(n)$ with the property that if R is an associative $\mathbb{C}$-algebra without unit such that $r^n = 0$ for all $r \in R$, then we must have for all $r_i \in R$ the identity

$$r_1 r_2 \dots r_{N(n)} = 0 \quad \text{in} \quad R.$$

We start by reformulating the problem. Consider the positive part $\mathbb{F}_+$ of the free $\mathbb{C}$-algebra generated by the variables $\{x_1, x_2, \dots, x_i, \dots\}$

$$\mathbb{F}_+ = \mathbb{C}\langle x_1, x_2, \dots, x_i, \dots \rangle_+$$

which is an associative $\mathbb{C}$-algebra without unit. Let $T(n)$ be the twosided ideal of $\mathbb{F}_+$ generated by all n-powers f^n with $f \in \mathbb{F}_+$. Note that the ideal $T(n)$ is invariant under all substitutions of $\mathbb{F}_+$. The Nagata-Higman problem then asks for a number $N(n)$ such that the product

$$x_1 x_2 \dots x_{N(n)} \in T(n).$$

We will now give an alternative description of the quotient algebra $\mathbb{F}_+/T(n)$. Let $\mathbb{N}_+$ be the positive part of the infinite necklace algebra $\mathbb{N}_n^\infty$ and $\mathbb{T}_+$ the positive part of the infinite trace algebra $\mathbb{T}_n^\infty$. Consider the quotient associative $\mathbb{C}$-algebra without unit

$$\overline{\mathbb{T}_+} = \mathbb{T}_+/(\mathbb{N}_+\mathbb{T}_n^\infty).$$

Observe the following facts about $\overline{\mathbb{T}_+}$: as a $\mathbb{C}$-algebra it is generated by the variables $X_1, X_2, \dots$ as all the other algebra generators of the form $t(x_{i_1} \dots x_{i_r})$ of $\mathbb{T}^\infty$ are mapped to zero in $\overline{\mathbb{T}_+}$. Further, from the form of the Cayley-Hamilton polynomial it follows that every $t \in \overline{\mathbb{T}_+}$ satisfies $t^n = 0$. That is, we have an algebra epimorphism

$$\mathbb{F}_+/T(n) \twoheadrightarrow \overline{\mathbb{T}_+}$$

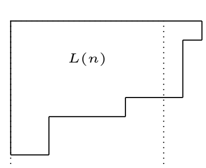
and we claim that it is also injective. To see this, observe that the quotient $\mathbb{T}^\infty/\mathbb{N}_+^\infty\mathbb{T}^\infty$ is just the free $\mathbb{C}$-algebra on the variables $\{x_1, x_2, \dots\}$. To obtain $\overline{\mathbb{T}_+}$ we have to factor out the ideal of trace relations. Now, a formal Cayley-Hamilton polynomial $\chi_x(x)$ is mapped to x^n in $\mathbb{T}^\infty/\mathbb{N}_+^\infty\mathbb{T}^\infty$. That is, to obtain $\overline{\mathbb{T}_+}$ we factor out the substitution invariant ideal (observe that t is zero here) generated by the elements x^n, but this is just the definition of $\mathbb{F}_+/T(n)$.

Therefore, a reformulation of the Nagata-Higman problem is to find a number $N = N(n)$ such that the product of the first N generic matrices

$$X_1 X_2 \dots X_N \in \mathbb{N}_+^\infty \mathbb{T}_n^\infty \qquad \text{or, equivalently that} \quad tr(X_1 X_2 \dots X_N X_{N+1})$$

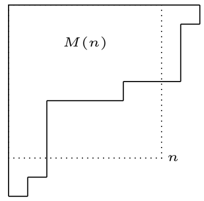
can be expressed as a linear combination of products of traces of lower degree. Using the description of the necklace relations given in proposition 3.18 we can reformulate this conditions in terms of the group algebra $\mathbb{C}S_{N+1}$. Let us introduce the following subspaces of the groupalgebra :

- A will be the subspace spanned by all $N+1$ cycles in S_{N+1},
- B will be the subspace spanned by all elements except $N+1$ cycles,
- $L(n)$ will be the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions



with $\leq n$ rows, and

- $M(n)$ will be the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions



  having more than n rows.

With these notations, we can reformulate the above condition as

$$(12\ldots NN+1) \in B + M(n) \quad \text{and consequently} \quad \mathbb{C}S_{N+1} = B + M(n)$$

Define an inner product on the groupalgebra $\mathbb{C}S_{N+1}$ such that the groupelements form an orthonormal basis, then A and B are orthogonal complements and also $L(n)$ and $M(n)$ are orthogonal complements. But then, taking orthogonal complements the condition can be rephrased as

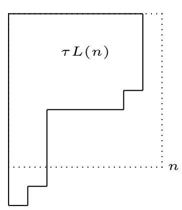
$$(B + M(n))^{\perp} = A \cap L(n) = 0.$$

Finally, let us define an automorphism τ on $\mathbb{C}S_{N+1}$ induced by sending e_σ to $sgn(\sigma)e_\sigma$. Clearly, τ is just multiplication by $(-1)^N$ on A and therefore the above condition is equivalent to

$$A \cap L(n) \cap \tau L(n) = 0.$$

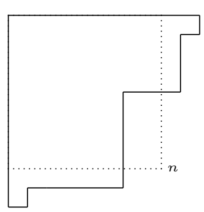
Moreover, for any Young tableau λ we have that $\tau(a_\lambda) = b_{\lambda^*}$ and $\tau(b_\lambda) = a_{\lambda^*}$. Hence, the automorphism τ sends the Young symmetrizer associated to a partition to the Young symmetrizer of the dual partition. This gives the following characterization

- $\tau L(n)$ is the ideal of $\mathbb{C}S_{N+1}$ spanned by the Young symmetrizers associated to partitions



  with $\leq n$ columns.

Now, specialize to the case $N = n^2$. Clearly, any Young diagram having $n^2 + 1$ boxes must have either more than n columns or more than n rows



and consequently we indeed have for $N = n^2$ that

$$A \cap L(n) \cap \tau L(n) = 0$$

finishing the proof of the promised refinement of the Nagata-Higman bound

Theorem 3.24 *Let R be an associative $\mathbb{C}$-algebra without unit element. Assume that $r^n = 0$ for all $r \in R$. Then, for all $r_i \in R$ we have*

$$r_1 r_2 \ldots r_{n^2} = 0$$

Theorem 3.25 *The necklace algebra $\mathbb{N}_n^m$ is generated as a $\mathbb{C}$-algebra by all elements of the form*

$$tr(X_{i_1} X_{i_2} \ldots X_{i_l})$$

with $l \leq n^2 + 1$. The trace algebra $\mathbb{T}_n^m$ is spanned as a module over the necklace algebra $\mathbb{N}_n^m$ by all monomials in the generic matrices

$$X_{i_1} X_{i_2} \ldots X_{i_l}$$

of degree $l \leq n^2$.

Chapter 4

Reconstructing Algebras.

Let A be an affine $\mathbb{C}$-algebra, generated by $\{a_1, \ldots, a_m\}$. A running theme of this book is to study A by investigating its level n approximations $A@_n$ for all natural numbers n. These algebras are defined in two stages. First, we define the category alg^t of $\mathbb{C}$-algebras equipped with a linear trace map $A \xrightarrow{t} A$. This map has to satisfy for all $a, b \in A$

$$t(a)b = bt(a) \quad t(ab) = t(ba) \quad \text{and} \quad t(t(a)b) = t(a)t(b)$$

Morphisms in alg^t are $\mathbb{C}$-algebra morphisms compatible with the trace structure. The forgetful functor $alg^t \longrightarrow alg$ has a left adjoint

$$alg \xrightarrow{(.)^t} alg^t \qquad A \mapsto A^t$$

where A^t is constructed (as in the special case of $\mathbb{T}^\infty$ in the foregoing chapter) by adding formal traces to necklaces with beads running through a $\mathbb{C}$-basis of A. The algebra A^t is trace-generated by m elements, that is, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}\langle x_1, \ldots, x_m \rangle \arrow[r, two heads] \arrow[d] & A \arrow[d] \\
\mathbb{T}^m \arrow[r, two heads] & A^t
\end{tikzcd}$$

where $\mathbb{T}^m$ is the subalgebra with trace of $\mathbb{T}^\infty$ generated by $\{x_1, \ldots, x_m\}$. The vertical maps are the natural ones and the lower map is trace preserving. For any $a \in A$ and natural number n we can define a formal Cayley-Hamilton polynomial $\chi_a^{(n)}(t)$ of degree n by expressing

$$f(t) = \prod_{i=1}^{n} (t - \lambda_i) \qquad \text{with the } \lambda_i \text{ indeterminates}$$

as a polynomial in t with coefficients which are polynomials in the Newton functions $\sum_{i=1}^n \lambda_i^k$ (as they are symmetric functions in the λ_i). Replacing each occurrence of $\sum_{i=1}^n \lambda_i^k$ by $t(a^k)$ gives $\chi_a^{(n)}(t) \in A[t]$. The approximation of A at level n is then defined to be

$$A@_n = \frac{A^t}{(t(1) - n, \chi_a^{(n)}(a) \ \forall a \in A)}$$

with the induced trace map, that is we can complete the above diagram

$$\begin{tikzcd} \mathbb{T}^m \arrow[r, two heads] \arrow[d, two heads] & A^t \arrow[d, two heads] \\ \mathbb{T}^m_n \arrow[r, two heads] & A@_n \end{tikzcd}$$

The main result of this chapter gives a geometric interpretation of the algebra $A@_n$. The representation variety $\underline{rep}_n\ A$ has as its geometric points the n-dimensional representations of A, that is, $\mathbb{C}$-algebra morphisms

$$A \xrightarrow{\rho} M_n(\mathbb{C})$$

It will be shown that this variety comes equipped with a GL_n-structure, the orbits of which correspond to isomorphism classes of n-dimensional representations. We will prove that $A@_n$ is the algebra of GL_n-equivariant polynomial maps

$$\underline{rep}_n\ A \longrightarrow M_n$$

with the algebra structure coming from those of the target space $M_n(\mathbb{C})$. Further, we will prove that the the commutative central subalgebra $t(A@_n)$ of $A@_n$ classifies the isomorphism classes of n-dimensional semi-simple representations of A. The geometric interpretation of $t(A@_n)$ proves it to be the coordinate ring of the quotient variety $\underline{rep}_n\ A/GL_n$ classifying the closed orbits in $\underline{rep}_n\ A$ which correspond by the Artin-Voigt result to semi-simple representations.

These two main results follow from the description of necklace and trace algebras given in the foregoing chapter and invariant theory, the basics of which we will recall in this chapter. At an intermediate stage, we will introduce also trace preserving representation varieties $\underline{rep}^{tr}_n\ A$ when the algebra A is equipped with a trace map. The above results then follow from the natural GL_n-isomorphisms

$$\underline{rep}_n\ A \simeq \underline{rep}^{tr}_n\ A^t \simeq \underline{rep}^{tr}_n\ A@_n$$

coming from the left adjointness. The level n approximation $A@_n$ is a special case of a Cayley-Hamilton algebra of degree n, other natural examples are orders over normal affine varieties in central simple algebras of dimension n^2 over the functionfield. The results in this chapter prove that there is a functor from the category $CH(n)$ of Cayley-Hamilton algebras of degree n to the category of affine GL_n-varieties (actually schemes) which assigns to an algebra A with trace map t the trace preserving representation variety $\underline{rep}^{tr}_n\ A$ and that this functor has a left inverse assigning to an affine GL_n-variety the algebra of GL_n-equivariant maps from the variety to M_n. This left inverse is not an equivalence of categories, however, and the characterization of those affine GL_n-varieties which are trace preserving representation varieties is a difficult problem. In chapter 11 we will show that the formal structure defined on them may be a first step in solving this riddle.

4.1 Representation varieties.

When A is a noncommutative affine algebra with generating set $\{a_1, \ldots, a_m\}$, there is an epimorphism

$$\mathbb{C}\langle x_1, \ldots, x_m \rangle \xrightarrow{\phi} A$$

defined by $\phi(x_i) = a_i$. Hence, we have a presentation of A as

$$A \simeq \mathbb{C}\langle x_1, \ldots, x_m \rangle / I_A$$

where I_A is the *twosided ideal of relations* holding among the a_i. For example, if $A = \mathbb{C}[x_1, \ldots, x_m]$, then I_A is the twosided ideal of $\mathbb{C}\langle x_1, \ldots, x_m \rangle$ generated by the elements $x_i x_j - x_j x_i$ for all $1 \leq i, j \leq m$. Observe that there is no analog of the Hilbert basis theorem for $\mathbb{C}\langle x_1, \ldots, x_m \rangle$, that is, I_A is not necessarily finitely generated as a twosided ideal. If it is, we say that A is *finitely presented*.

An *n-dimensional representation* of A is an algebra morphism

$$A \xrightarrow{\psi} M_n$$

from A to $n \times n$ matrices over $\mathbb{C}$. If A is generated by $\{a_1, \ldots, a_m\}$, then ψ is fully determined by the point

$$(\psi(a_1), \psi(a_2), \ldots, \psi(a_m)) \in M_n^m = \underbrace{M_n \oplus \ldots \oplus M_n}_{m}.$$

We claim that $mod_n(A)$, the set of all n-dimensional representations of A, forms a Zariski closed subset of M_n^m. To begin, observe that

$$rep_n(\mathbb{C}\langle x_1, \ldots, x_m \rangle) = M_n^m$$

as any m-tuple of $n \times n$ matrices $(A_1, \ldots, A_m) \in M_n^m$ determines an algebra morphism $\mathbb{C}\langle x_1, \ldots, x_m \rangle \xrightarrow{\psi} M_n$ by taking $\psi(x_i) = A_i$.

Now, given a presentation $A = \mathbb{C}\langle x_1, \ldots, x_m \rangle / I_A$ an m-tuple $(A_1, \ldots, A_m) \in M_n^m$ determines an n-dimensional representation of A if (and only if) for every noncommutative polynomial $r(x_1, \ldots, x_m) \in I_A \triangleleft \mathbb{C}\langle x_1, \ldots, x_m \rangle$ we have that

$$r(A_1, \ldots, A_m) = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in M_n.$$

Hence, consider the ideal $I_A(n)$ of $\mathbb{C}[M_n^m] = \mathbb{C}[x_{ij}(k) \mid 1 \leq i, j \leq n, 1 \leq k \leq m]$ generated by all the entries of the matrices in $M_n(\mathbb{C}[M_n^m])$ of the form

$$r(X_1, \ldots, X_m) \quad \text{for all} \quad r(x_1, \ldots, x_m) \in I_A.$$

By the above observation we see that the *reduced representation variety $rep_n\ A$ is the set of simultaneous zeroes of the ideal $I_A(n)$, that is,*

$$rep_n\ A = \mathbb{V}(I_A(n)) \hookrightarrow M_n^m$$

proving the claim, where $\mathbb{V}$ denotes the closed set in the Zariski topology determined by an ideal, the complement of which we will denotye with $\mathbb{X}$). Observe that even when A is not finitely presented, the ideal $I_A(n)$ is finitely generated as an ideal of the commutative polynomial algebra $\mathbb{C}[M_n^m]$.

Often, the ideal $I_A(n)$ contains more information than the closed subset $rep_n(A) = \mathbb{V}(I_A(n))$ which only determines the radical ideal of $I_A(n)$. This forces us to consider also the representation variety *(or module scheme) $\underline{rep}_n\ A$ which we will introduce in a moment.*

Example 4.1 It may happen that $rep_n\ A = \emptyset$. For example, consider the *Weyl algebra*

$$A_1(\mathbb{C}) = \mathbb{C}\langle x, y \rangle / (xy - yx - 1)$$

If a couple of $n \times n$-matrices $(A, B) \in rep_n\ A_1(\mathbb{C})$ then we must have

$$A.B - B.A = \mathbb{1}_n \in M_n$$

However, taking traces on both sides gives a contradiction as $tr(AB) = tr(BA)$ and $tr(\mathbb{1}_n) = n \neq 0$.

In the foregoing chapter we studied the action of GL_n by simultaneous conjugation on M_n^m. We claim that $rep_n\ A \hookrightarrow M_n^m$ is stable under this action, that is, if $(A_1,\ldots,A_m) \in rep_n\ A$, then also $(gA_1g^{-1},\ldots,gA_mg^{-1}) \in rep_n\ A$. This is clear by composing the n-dimensional representation ψ of A determined by $(A_1,\ldots,A_m)$ with the algebra automorphism of M_n given by conjugation with $g \in GL_n$,

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[dr, dotted, "g.\psi"'] & M_n \arrow[d, "g.g^{-1}"] \\ & M_n \end{tikzcd}$$

That is, $rep_n\ A$ is a GL_n-variety. We will give an interpretation of the orbits under this action.

Recall that a left A-module M is a vectorspace on which elements of A act on the left as linear operators satisfying the conditions

$$1.m = m \quad \text{and} \quad a.(b.m) = (ab).m$$

for all $a,b \in A$ and all $m \in M$. An A-module morphism $M \xrightarrow{f} N$ between two left A-modules is a linear map such that $f(a.m) = a.f(m)$ for all $a \in A$ and all $m \in M$. An A-module automorphism is an A-module morphism $M \xrightarrow{f} N$ such that there is an A-module morphism $N \xrightarrow{g} M$ such that $f \circ g = id_M$ and $g \circ f = id_N$.

Assume the A-module M has complex dimension n, then after fixing a basis we can identify M with $\mathbb{C}^n$ (column vectors). For any $a \in A$ we can represent the linear action of a on M by an $n \times n$ matrix $\psi(a) \in M_n$. The condition that $a.(b.m) = (ab).m$ for all $m \in M$ asserts that $\psi(ab) = \psi(a)\psi(b)$ for all $a,b \in A$, that is, ψ is an algebra morphism $A \xrightarrow{\psi} M_n$ and hence M determines an n-dimensional representation of A. Conversely, an n-dimensional representation $A \xrightarrow{\psi} M_n$ determines an A-module structure on $\mathbb{C}^n$ by the rule

$$a.v = \psi(a)v \quad \text{for all} \quad v \in \mathbb{C}^n.$$

Hence, there is a one-to-one correspondence between the n-dimensional representations of A and the A-module structures on $\mathbb{C}^n$. For this reason we call the reduced variety $rep_n\ A$ the reduced representation variety of A. If two n-dimensional A-module structures M and N on $\mathbb{C}^n$ are isomorphic (determined by a linear invertible map $g \in GL_n$) then for all $a \in A$ we have the commutative diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "a."'] & N \arrow[d, "a."] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

Hence, if the action of a on M is represented by the matrix A, then the action of a on M is represented by the matrix $g.A.g^{-1}$. Therefore, two A-module structures on $\mathbb{C}^n$ are isomorphic if and only if the points of $rep_n\ A$ corresponding to them lie in the same GL_n-orbit. Concluding, studying n-dimensional A-modules up to isomorphism is the same as studying the GL_n-orbits in the reduced representation variety $rep_n\ A$.

If the defining ideal $I_A(n)$ is a radical ideal (as we will see, this is the case when A is a Quillen-smooth algebra) the above suffices. In general, the scheme *structure* *of the representation variety* $\underline{rep}_n\ A$ *will be important. By definition, the module scheme* $\underline{rep}_n\ A$ *is the functor assigning to any (affine) commutative* $\mathbb{C}$*-algebra* R*, the set*

$$\underline{rep}_n\ A(R) = Alg_{\mathbb{C}}(\mathbb{C}[M_n^m]/I_A(n), R)$$

of $\mathbb{C}$*-algebra morphisms* $\frac{\mathbb{C}[M_n^m]}{I_A(n)} \xrightarrow{\psi} R$*. Such a map* ψ *is determined by the image* $\psi(x_{ij}(k)) = r_{ij}(k) \in R$*. That is,* $\psi \in \underline{rep}_n\ A(R)$ *determines an* m*-tuple of* $n \times n$ *matrices with coefficients in* R

$$(r_1, \ldots, r_m) \in \underbrace{M_n(R) \oplus \ldots \oplus M_n(R)}_{m} \quad where \quad r_k = \begin{bmatrix} r_{11}(k) & \ldots & r_{1n}(k) \\ \vdots & & \vdots \\ r_{n1}(k) & \ldots & r_{nn}(k) \end{bmatrix}.$$

Clearly, for any $r(x_1, \ldots, x_m) \in I_A$ *we must have that* $r(r_1, \ldots, r_m)$ *is the zero matrix in* $M_n(R)$*. That is,* ψ *determines uniquely an* R*-algebra morphism*

$$\psi : R \otimes_{\mathbb{C}} A \longrightarrow M_n(R) \quad by\ mapping \quad x_k \mapsto r_k.$$

Alternatively, we can identify the set $\underline{rep}_n(R)$ *with the set of left* $R \otimes A$*-module structures on the free* R*-module* $R^{\oplus n}$ *of rank* n*. In section 8, we will introduce the representation variety* $\underline{rep}^t_n\ A$ *and teh reduced representation variety* $rep^t_n\ A$ *of trace preserving* n*-dimensional representations.*

4.2 Some algebraic geometry.

Throughout this book we assume that the reader has some familiarity with algebraic geometry, such as the first two chapters of the textbook [9]. In this section we restrict to the dimension formulas and the relation between Zariski and analytic closure, illustrating them with examples from module varieties. We will work only with reduced varieties in this section.

A morphism $X \xrightarrow{\phi} Y$ *between two affine irreducible varieties is said to be dominant if the image* $\phi(X)$ *is Zariski dense in* Y*. On the level of the coordinate algebras dominance is equivalent to* $\phi^* : \mathbb{C}[Y] \longrightarrow \mathbb{C}[X]$ *being injective and hence inducing a fieldextension* $\phi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ *between the functionfields. Indeed, for* $f \in \mathbb{C}[Y]$ *the function* $\phi^*(f)$ *is by definition the composition*

$$X \xrightarrow{\phi} Y \xrightarrow{f} \mathbb{C}$$

and therefore $\phi^*(f) = 0$ *iff* $f(\phi(X)) = 0$ *iff* $f(\overline{\phi(X)}) = 0$*.*

A morphism $X \xrightarrow{\phi} Y$ *between two affine varieties is said to be finite if under the algebra morphism* ϕ^* *the coordinate algebra* $\mathbb{C}[X]$ *is a finite* $\mathbb{C}[Y]$*-module. An important property of finite morphisms is that they are closed, that is the image of a closed subset is closed. Indeed, we can replace without loss of generality* Y *by the closed subset* $\overline{\phi(X)} = \mathbb{V}_Y(Ker\ \phi^*)$ *and hence assume that* ϕ^* *is an inclusion* $\mathbb{C}[Y] \hookrightarrow \mathbb{C}[X]$*. The claim then follows from the fact that in a finite extension there exists for any maximal ideal* $N \triangleleft \mathbb{C}[Y]$ *a maximal ideal* $M \triangleleft \mathbb{C}[X]$ *such that* $M \cap \mathbb{C}[Y] = \mathbb{C}[X]$*.*

Example 4.2 Let X be an irreducible affine variety of dimension d. By the Noether normalization result $\mathbb{C}[X]$ is a finite module over a polynomial subalgebra $\mathbb{C}[f_1, \ldots, f_d]$. But then, the finite inclusion $\mathbb{C}[f_1, \ldots, f_d] \hookrightarrow \mathbb{C}[X]$ determines a finite surjective morphism

$$X \overset{\phi}{\twoheadrightarrow} \mathbb{C}^d$$

An important source of finite morphisms is given by integral extensions. Recall that, if $R \hookrightarrow S$ is an inclusion of domains we call S integral over R if every $s \in S$ satisfies an equation

$$s^n = \sum_{i=0}^{n-1} r_i s^i \quad \textit{with} \quad r_i \in R.$$

A normal domain R has the property that any element of its field of fractions which is integral over R belongs already to R. If $X \xrightarrow{\phi} Y$ is a dominant morphism between two irreducible affine varieties, then ϕ is finite if and only if $\mathbb{C}[X]$ in integral over $\mathbb{C}[Y]$ for the embedding coming from ϕ^*.

Proposition 4.3 *Let $X \xrightarrow{\phi} Y$ be a dominant morphism between irreducible affine varieties. Then, for any $x \in X$ and any irreducible component C of the fiber $\phi^{-1}(\phi(z))$ we have*

$$dim\ C \geq dim\ X - dim\ Y.$$

Moreover, there is a nonempty open subset U of Y contained in the image $\phi(X)$ such that for all $u \in U$ we have

$$dim\ \phi^{-1}(u) = dim\ X - dim\ Y.$$

Proof. *Let $d = dim\ X - dim\ Y$ and apply the Noether normalization result to the affine $\mathbb{C}(Y)$-algebra $\mathbb{C}(Y)\mathbb{C}[X]$. Then, we can find a function $g \in \mathbb{C}[Y]$ and algebraic independent functions $f_1, \ldots, f_d \in \mathbb{C}[X]_g$ (g clears away any denominators that occur after applying the normalization result) such that $\mathbb{C}[X]_g$ is integral over $\mathbb{C}[Y]_g[f_1, \ldots, f_d]$. That is, we have the commutative diagram*

$$\begin{tikzcd}
\mathbb{X}_X(g) \arrow[rr, two heads, "\rho"] \arrow[d, hook] & & \mathbb{X}_Y(g) \times \mathbb{C}^d \arrow[d, "pr_1"] \\
X \arrow[r, "\phi"] & Y & \mathbb{X}_Y(g) \arrow[l, hook']
\end{tikzcd}$$

where we know that ρ is finite and surjective. But then we have that the open subset $\mathbb{X}_Y(g)$ lies in the image of ϕ and in $\mathbb{X}_Y(g)$ all fibers of ϕ have dimension d. For the first part of the statement we have to recall the statement of Krull's Hauptideal result : if X is an irreducible affine variety and $g_1, \ldots, g_r \in \mathbb{C}[X]$ with $(g_1, \ldots, g_r) \neq \mathbb{C}[X]$, then any component C of $\mathbb{V}_X(g_1, \ldots, g_r)$ satisfies the inequality

$$dim\ C \geq dim\ X - r.$$

If dim $Y = r$ apply this result to the g_i determining the morphism

$$X \xrightarrow{\phi} Y \twoheadrightarrow \mathbb{C}^r$$

where the latter morphism is the one from example 4.2. □

In fact, a stronger result holds. Chevalley's theorem asserts the following.

Theorem 4.4 *Let $X \xrightarrow{\phi} Y$ be a morphism between affine varieties, the function*

$$X \longrightarrow \mathbb{N} \quad \textit{defined by} \quad x \mapsto dim_x\ \phi^{-1}(\phi(x))$$

is upper-semicontinuous. That is, for all $n \in \mathbb{N}$, the set

$$\{x \in X \mid dim_x\ \phi^{-1}(\phi(x)) \leq n\}$$

is Zariski open in X.

Proof. *Let $Z(\phi, n)$ be the set $\{x \in X \mid dim_x\ \phi^{-1}(\phi(x)) \geq n\}$. We will prove that $Z(\phi, n)$ is closed by induction on the dimension of X. We first make some reductions. We may assume that X is irreducible. For, let $X = \cup_i X_i$ be the decomposition of X into irreducible components, then $Z(\phi, n) = \cup Z(\phi \mid X_i, n)$. Next, we may assume that $Y = \overline{\phi(X)}$ whence Y is also irreducible and ϕ is a dominant map. Now, we are in the setting of proposition 4.3. Therefore, if $n \leq dim\ X - dim\ Y$ we have $Z(\phi, n) = X$ by that proposition, so it is closed. If $n > dim\ X - dim\ Y$ consider the open set U in Y of proposition 4.3. Then, $Z(\phi, n) = Z(\phi \mid (X - \phi^{-1}(U)), n)$. the dimension of the closed subvariety $X - \phi^{-1}(U)$ is strictly smaller that dim X hence by induction we may assume that $Z(\phi \mid (X - \phi^{-1}(U)), n)$ is closed in $X - \phi^{-1}(U)$ whence closed in X.* $\square$

An immediate consequence of the foregoing proposition is that for any morphism $X \xrightarrow{\phi} Y$ between affine varieties, the image $\phi(X)$ contains an open dense subset of $\overline{\phi(Z)}$ (reduce to irreducible components and apply the proposition).

Example 4.5 Let A be an affine $\mathbb{C}$-algebra and $M \in rep_n\ A$. We claim that the orbit

$$\mathcal{O}(M) = GL_n.M \quad \text{is Zariski open in its closure} \quad \overline{\mathcal{O}(M)}.$$

Consider the 'orbit-map' $GL_n \xrightarrow{\phi} rep_n\ A$ defined by $g \mapsto g.M$. then, by the above remark $\overline{\mathcal{O}(M)} = \overline{\phi(GL_n)}$ contains a Zariski open subset U of $\overline{\mathcal{O}(M)}$ contained in the image of ϕ which is $\mathcal{O}(M)$. But then,

$$\mathcal{O}(M) = GL_n.M = \cup_{g \in GL_n} g.U$$

is also open in $\overline{\mathcal{O}(M)}$. Next, we claim that $\overline{\mathcal{O}(M)}$ contains a closed orbit. Indeed, assume $\mathcal{O}(M)$ is not closed, then the complement $C_M = \overline{\mathcal{O}(M)} - \mathcal{O}(M)$ is a proper Zariski closed subset whence $dim\ C < dim\ \overline{\mathcal{O}(M)}$. But, C is the union of GL_n-orbits $\mathcal{O}(M_i)$ with $dim\ \overline{\mathcal{O}(M_i)} < dim\ \overline{\mathcal{O}(M)}$. Repeating the argument with the M_i and induction on the dimension we will obtain a closed orbit in $\overline{\mathcal{O}(M)}$.

Next, we want to relate the Zariski closure with the $\mathbb{C}$-closure. Whereas they are usually not equal (for example, the unit circle in $\mathbb{C}^1$), we will show that they coincide for the important class of constructible subsets. A subset Z of an affine variety X is said to be locally closed if Z is open in its Zariski closure $\overline{Z}$. A subset Z is said to be constructible if Z is the union of finitely many locally closed subsets. Clearly, finite unions, finite intersections and complements of constructible subsets are again constructible. The importance of constructible sets for algebraic geometry is clear from the following result.

Proposition 4.6 *Let $X \xrightarrow{\phi} Y$ be a morphism between affine varieties. If Z is a constructible subset of X, then $\phi(Z)$ is a constructible subset of Y.*

Proof. *Because every open subset of X is a finite union of special open sets which are themselves affine varieties, it suffices to show that $\phi(X)$ is constructible. We will use induction on dim $\overline{\phi(X)}$. There exists an open subset $U \subset \overline{\phi(X)}$ which is contained in $\phi(X)$. Consider the closed complement $W = \overline{\phi(X)} - U$ and its inverse image $X' = \phi^{-1}(W)$. Then, X' is an affine variety and by induction we may assume that $\phi(X')$ is constructible. But then, $\phi(X) = U \cup \phi(X')$ is also constructible.* $\square$

Example 4.7 Let A be an affine $\mathbb{C}$-algebra. The subset $ind_n\ A \hookrightarrow rep_n\ A$ of the *indecomposable* n-dimensional A-modules is constructible. Indeed, define for any pair k, l such that $k + l = n$ the morphism

$$GL_n \times rep_k\ A \times rep_l\ A \longrightarrow rep_n\ A$$

by sending a triple (g, M, N) to $g.(M \oplus N)$. By the foregoing result the image of this map is constructible. The decomposable n-dimensional A-modules belong to one of these finitely many sets whence are constructible, but then so is its complement which in $ind_n\ A$.

Apart from being closed, finite morphisms often satisfy the going-down property. That is, consider a finite and surjective morphism

$$X \xrightarrow{\phi} Y$$

where X is irreducible and Y is normal (that is, $\mathbb{C}[Y]$ is a normal domain). Let $Y' \hookrightarrow Y$ an irreducible Zariski closed subvariety and $x \in X$ with image $\phi(x) = y' \in Y'$. Then, the going-down property asserts the existence of an irreducible Zariski closed subvariety $X' \hookrightarrow X$ such that $x \in X'$ and $\phi(X') = Y'$. In particular, the morphism $X' \xrightarrow{\phi} Y'$ is again finite and surjective and in particular $\dim X' = \dim Y'$.

An important application of this property is that any two points of an irreducible affine variety can be connected through an irreducible curve.

Lemma 4.8 *Let $x \in X$ an irreducible affine variety and U a Zariski open subset. Then, there is an irreducible curve $C \hookrightarrow X$ through x and intersecting U.*

Proof. *If $d = \dim X$ consider the finite surjective morphism $X \xrightarrow{\phi} \mathbb{C}^d$ of example 4.2. Let $y \in \mathbb{C}^d - \phi(X - U)$ and consider the line L through y and $\phi(x)$. By the going-down property there is an irreducible curve $C \hookrightarrow X$ containing x such that $\phi(C) = L$ and by construction $C \cap U \neq \emptyset$.* □

Proposition 4.9 *Let $X \xrightarrow{\phi} Y$ be a dominant morphism between irreducible affine varieties any $y \in Y$. Then, there is an irreducible curve $C \hookrightarrow X$ such that $y \in \overline{\phi(C)}$.*

Proof. *Consider an open dense subset $U \hookrightarrow Y$ contained in the image $\phi(X)$. By the lemma there is a curve $C' \hookrightarrow Y$ containing y and such that $C' \cap U \neq \emptyset$. Then, again applying the lemma to an irreducible component of $\phi^{-1}(C')$ not contained in a fiber, we obtain an irreducible curve $C \hookrightarrow X$ with $\overline{\phi(C)} = \overline{C'}$.* □

Any affine variety $X \hookrightarrow \mathbb{C}^k$ can also be equipped with the induced $\mathbb{C}$-topology from $\mathbb{C}^k$ which is much finer than the Zariski topology. Usually there is no relation between the closure $\overline{Z}^{\mathbb{C}}$ of a subset $Z \hookrightarrow X$ in the $\mathbb{C}$-topology and the Zariski closure $\overline{Z}$.

Lemma 4.10 *Let $U \subset \mathbb{C}^k$ containing a subset V which is Zariski open and dense in $\overline{U}$. Then,*

$$\overline{U}^{\mathbb{C}} = \overline{U}$$

Proof. *By reducing to irreducible components, we may assume that $\overline{U}$ is irreducible. Assume first that $\dim \overline{U} = 1$, that is, $\overline{U}$ is an irreducible curve in $\mathbb{C}^k$. Let U_s be the subset of points where $\overline{U}$ is a complex manifold, then $\overline{U} - U_s$ is finite and by the implicit function theorem in analysis every $u \in U_s$ has a $\mathbb{C}$-open neighborhood which is $\mathbb{C}$-homeomorphic to the complex line $\mathbb{C}^1$, whence the result holds in this case.*

If $\overline{U}$ is general and $x \in \overline{U}$ we can take by the lemma above an irreducible curve $C \hookrightarrow \overline{U}$ containing z and such that $C \cap V \neq \emptyset$. Then, $C \cap V$ is Zariski open and dense in C and by the curve argument above $x \in \overline{(C \cap V)}^{\mathbb{C}} \subset \overline{U}^{\mathbb{C}}$. We can do this for any $x \in \overline{U}$ finishing the proof. □

Consider the embedding of an affine variety $X \hookrightarrow \mathbb{C}^k$, proposition 4.6 and the fact that any constructible set Z contains a subset U which is open and dense in $\overline{Z}$ we deduce from the lemma at one the next result.

Proposition 4.11 *If Z is a constructible subset of an affine variety X, then*

$$\overline{Z}^{\mathbb{C}} = \overline{Z}$$

Example 4.12 Let A be an affine $\mathbb{C}$-algebra and $M \in rep_n\ A$. We have proved in example 4.5 that the orbit $\mathcal{O}(M) = GL_n.M$ is Zariski open in its closure $\overline{\mathcal{O}(M)}$. Therefore, the orbit $\mathcal{O}(M)$ is a constructible subset of $rep_n\ A$. By the proposition above, the Zariski closure $\overline{\mathcal{O}(M)}$ of the orbit coincides with the closure of $\mathcal{O}(M)$ in the $\mathbb{C}$-topology.

4.3 The Gerstenhaber-Hesselink theorem.

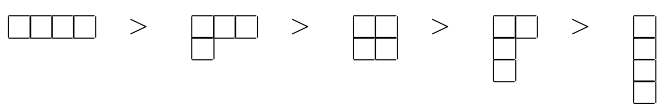
In the next sections we will study orbit-closure and closed orbits in $rep_n\ A$. In this section we give one of the rare instances (but which is very important in applications) where everything can be fully determined : the orbits in $rep_n\ \mathbb{C}[x]$ or, equivalent, conjugacy classes of $n \times n$ matrices.

It is sometimes convenient to relax our definition of partitions to include zeroes at its tail. That is, a partition p of n is an integral n-tuple $(a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ with $\sum_{i=1}^n a_i = n$. As before, we represent a partition by a Young diagram by omitting rows corresponding to zeroes.

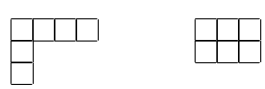
If $q = (b_1, \ldots, b_n)$ is another partition of n we say that p dominates q and write

$$p > q \quad \text{if and only if} \quad \sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i \quad \text{for all } 1 \leq r \leq n.$$

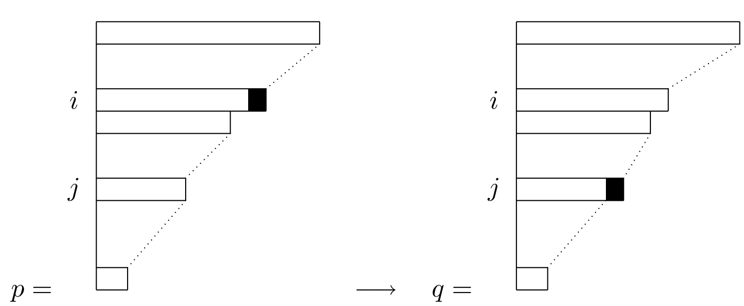
For example, the partitions of 4 are ordered as indicated below



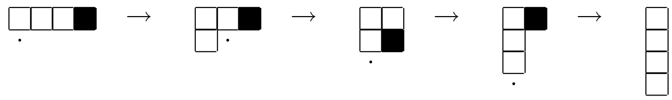
Note however that the dominance relation is not a total ordering whenever $n \geq 6$. For example, the following two partition of 6



are not comparable. The dominance order is induced by the Young move of throwing a row-ending box down the diagram. Indeed, let p and q be partitions of n such that $p > q$ and assume there is no partition r such that $p > r$ and $r > q$. Let i be the minimal number such that $a_i > b_i$. Then by the assumption $a_i = b_i + 1$. Let $j > i$ be minimal such that $a_j \neq b_j$, then we have $b_j = a_j + 1$ because p dominates q. But then, the remaining rows of p and q must be equal. That is, a Young move can be depicted as



For example, the Young moves between the partitions of 4 given above are as indicated



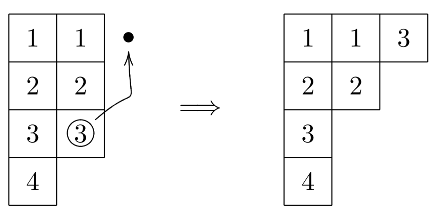
A Young p-tableau is the Young diagram of p with the boxes labeled by integers from $\{1,2,\ldots,s\}$ for some s such that each label appears at least ones. A Young p-tableau is said to be of type q for some partition $q=(b_1,\ldots,b_n)$ of n if the following conditions are met :

- *the labels are non-decreasing along rows,*
- *the labels are strictly increasing along columns, and*
- *the label i appears exactly b_i times.*

For example, if $p=(3,2,1,1)$ and $q=(2,2,2,1)$ then the p-tableau below

1	1	3
2	2	
3		
4		

is of type q (observe that $p>q$ and even $p\rightarrow q$). In general, let $p=(a_1,\ldots,a_n)$ and $q=(b_1,\ldots,b_n)$ be partitions of n and assume that $p\rightarrow q$. Then, there is a Young p-tableau of type q. For, fill the Young diagram of q by putting 1's in the first row, 2's in the second and so on. Then, upgrade the fallen box together with its label to get a Young p-tableau of type q. In the example above



Conversely, assume there is a Young p-tableau of type q. The number of boxes labeled with a number $\leq i$ is equal to $b_1+\ldots+b_i$. Further, any box with label $\leq i$ must lie in the first i rows (because the labels strictly increase along a column). There are $a_1+\ldots+a_i$ boxes available in the first i rows whence

$$\sum_{j=1}^{i} b_i \leq \sum_{j=1}^{i} a_i \quad \text{for all} \quad 1\leq i\leq n$$

and therefore $p>q$. After these preliminaries on partitions, let us return to nilpotent matrices.

Let A be a nilpotent matrix of type $p=(a_1,\ldots,a_n)$, that is, conjugated to a matrix with Jordan blocks (all with eigenvalue zero) of sizes a_i. We have seen before that the subspace V_l of column vectors $v\in\mathbb{C}^n$ such that $A^l.v=0$ has dimension

$$\sum_{h=1}^{l} \#\{j \mid a_j \geq h\} = \sum_{h=1}^{l} a_h^*$$

where $p^* = (a_1^*, \ldots, a_n^*)$ *is the dual partition of* p*. Choose a basis* $\{v_1, \ldots, v_n\}$ *of* $\mathbb{C}^n$ *such that for all* l *the first* $a_1^* + \ldots + a_l^*$ *base vectors span the subspace* V_l*. For example, if* A *is in Jordan normal form of type* $p = (3, 2, 1, 1)$

$$\begin{bmatrix} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & & & & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

then $p^* = (4, 2, 1)$ *and we can choose the standard base vectors ordered as follows*

$$\{\underbrace{e_1, e_4, e_6, e_7}_{4}, \underbrace{e_2, e_5}_{2}, \underbrace{e_3}_{1}\}.$$

Take a partition $q = (b_1, \ldots, b_n)$ *with* $p \to q$ *(in particular,* $p > q$*), then for the dual partitions we have* $q^* \to p^*$ *(and thus* $q^* > p^*$*). But then there is a Young* q^**-tableau of type* p^**. In the example with* $q = (2, 2, 2, 1)$ *we have* $q^* = (4, 3)$ *and* $p^* = (4, 2, 1)$ *and we can take the Young* q^**-tableau of type* p^*

1	1	1	1
2	2	3	

Now label the boxes of this tableau by the base vectors $\{v_1, \ldots, v_n\}$ *such that the boxes labeled* i *in the Young* q^**-tableau of type* p^* *are filled with the base vectors from* $V_i - V_{i-1}$*. Call this tableau* T*. In the example, we can take*

$T =$

e_1	e_4	e_6	e_7
e_2	e_5	e_3	

Define a linear operator F *on* $\mathbb{C}^n$ *by the rule that* $F(v_i) = v_j$ *if* v_j *is the label of the box in* T *just above the box labeled* v_i*. In case* v_i *is a label of a box in the first row of* T *we take* $F(v_i) = 0$*. Obviously,* F *is a nilpotent* $n \times n$ *matrix and by construction we have that*

$$rk\ F^l = n - (b_1^* + \ldots + b_l^*)$$

That is, F *is nilpotent of type* $q = (b_1, \ldots, b_n)$*. Moreover,* F *satisfies* $F(V_i) \subset V_{i-1}$ *for all* i *by the way we have labeled the tableau* T *and defined* F*.*

In the example above, we have $F(e_2) = e_1$*,* $F(e_5) = e_4$*,* $F(e_3) = e_6$ *and all other* $F(e_i) = 0$*. That is,* F *is the matrix*

$$\begin{bmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & & & 0 & \\ & & & 0 & 1 & & \\ & & & 0 & 0 & & \\ & & 1 & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

which is seen to be of type $(2, 2, 2, 1)$ *after performing a few Jordan moves.*

Returning to the general case, consider for all $\epsilon \in \mathbb{C}$ the $n \times n$ matrix :

$$F_\epsilon = (1-\epsilon)F + \epsilon A.$$

We claim that for all but finitely many values of ϵ we have $F_\epsilon \in \mathcal{O}(A)$. Indeed, we have seen that $F(V_i) \subset V_{i-1}$ where V_i is defined as the subspace such that $A^i(V_i) = 0$. Hence, $F(V_1) = 0$ and therefore

$$F_\epsilon(V_1) = (1-\epsilon)F + \epsilon A(V_1) = 0.$$

Assume by induction that $F_\epsilon^i(V_i) = 0$ holds for all $i < l$, then we have that

$$\begin{array}{rcl} F_\epsilon^l(V_l) & = & F_\epsilon^{l-1}((1-\epsilon)F + \epsilon A)(V_l) \\ & \subset & F_\epsilon^{l-1}(V_{l-1}) = 0 \end{array}$$

because $A(V_l) \subset V_{l-1}$ and $F(V_l) \subset V_{l-1}$. But then we have for all l that

$$rk\ F_\epsilon^l \leq dim\ V_l = n - (a_1^* + \ldots + a_l^*) = rk\ A^l \overset{def}{=} r_l.$$

Then for at least one $r_l \times r_l$ submatrix of F_ϵ^l its determinant considered it as a polynomial of degree r_l in ϵ is not identically zero (as it is nonzero for $\epsilon = 1$). But then this determinant is non-zero for all but finitely many ϵ. Hence, $rk\ F_\epsilon^l = rk\ A^l$ for all l for all but finitely many ϵ. As these numbers determine the dual partition p^* of the type of A, F_ϵ is a nilpotent $n \times n$ matrix of type p for all but finitely many values of ϵ, proving the claim. But then, $F_0 = F$ which we have proved to be a nilpotent matrix of type q belongs to the closure of the orbit $\mathcal{O}(A)$. That is, we have proved the difficult part of the Gerstenhaber-Hesselink theorem.

Theorem 4.13 *Let A be a nilpotent $n \times n$ matrix of type $p = (a_1, \ldots, a_n)$ and B nilpotent of type $q = (b_1, \ldots, b_n)$. Then, B belongs to the closure of the orbit $\mathcal{O}(A)$, that is,*

$$B \in \overline{\mathcal{O}(A)} \quad \text{if and only if} \quad p > q$$

in the domination order on partitions of n.

To prove the theorem we only have to observe that if B is contained in the closure of A, then B^l is contained in the closure of A^l and hence $rk\ A^l \geq rk\ B^l$ (because $rk\ A^l < k$ is equivalent to vanishing of all determinants of $k \times k$ minors which is a closed condition). But then,

$$n - \sum_{i=1}^{l} a_i^* \geq n - \sum_{i=1}^{l} b_i^*$$

for all l, that is, $q^* > p^*$ and hence $p > q$. The other implication was proved above if we remember that the domination order was induced by the Young moves and clearly we have that if $B \in \overline{\mathcal{O}(C)}$ and $C \in \overline{\mathcal{O}(A)}$ then also $B \in \overline{\mathcal{O}(A)}$.

Example 4.14 Nilpotent matrices for small n.

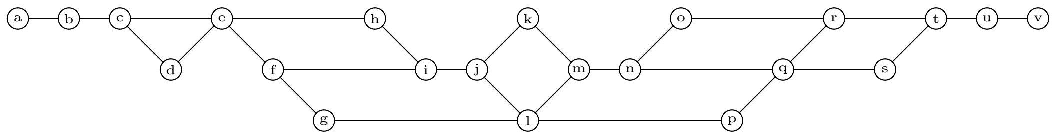
We will apply theorem 4.13 to describe the orbit-closures of nilpotent matrices of 8×8 matrices. The following table lists all partitions (and their dual in the other column)

The partitions of 8.

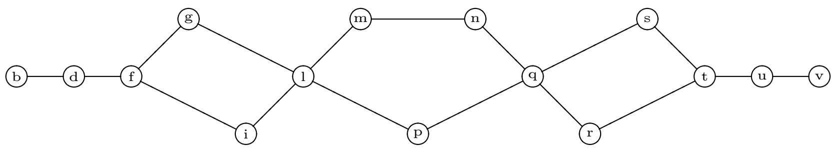
a	(8)	v	(1,1,1,1,1,1,1,1)
b	(7,1)	u	(2,1,1,1,1,1,1)
c	(6,2)	t	(2,2,1,1,1,1)
d	(6,1,1)	s	(3,1,1,1,1,1)
e	(5,3)	r	(2,2,2,1,1)

f	(5,2,1)	q	(3,2,1,1,1)
g	(5,1,1,1)	p	(4,1,1,1,1)
h	(4,4)	o	(2,2,2,2)
i	(4,3,1)	n	(3,2,2,1)
j	(4,2,2)	m	(3,3,1,1)
k	(3,3,2)	k	(3,3,2)
l	(4,2,1,1)	l	(4,2,1,1)

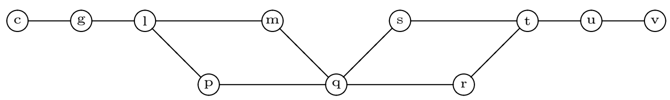
The domination order between these partitions can be depicted as follows where all the Young moves are from left to right



Of course, from this graph we can read off the dominance order graphs for partitions of $n \leq 8$. The trick is to identify a partition of n with that of 8 by throwing in a tail of ones and to look at the relative position of both partitions in the above picture. Using these conventions we get the following graph for partitions of 7



and for partitions of 6 the dominance order is depicted as follows



The dominance order on partitions of $n \leq 5$ is a total ordering.

The Gerstenhaber-Hesselink theorem can be applied to describe the module varieties of the algebras $\frac{\mathbb{C}[x]}{(x^r)}$.

Example 4.15 The representation variety $rep_n \, \frac{\mathbb{C}[x]}{(x^r)}$.

Any algebra morphism $\mathbb{C}[x] \longrightarrow M_n$ is determined by the image of x, whence $rep_n(\mathbb{C}[x]) = M_n$. We have seen that conjugacy classes in M_n are classified by the Jordan normalform. Let A is conjugated to a matrix in normalform

$$\begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_s} \end{bmatrix}$$

where J_i is a Jordan block of size d_i, hence $n = d_1 + d_2 + \ldots + d_s$. Then, the n-dimensional $\mathbb{C}[x]$-module M determined by A can be decomposed uniquely as

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_s$$

where M_i is a $\mathbb{C}[x]$-module of dimension d_i which is *indecomposable*, that is, cannot be decomposed as a direct sum of proper submodules.

Now, consider the quotient algebra $R = \mathbb{C}[x]/(x^r)$, then the ideal $I_R(n)$ of $\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$ is generated by the n^2 entries of the matrix

$$\begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix}^r.$$

For example if $r = m = 2$, then the ideal is generated by the entries of the matrix

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^2 = \begin{bmatrix} x_1^2 + x_2x_3 & x_2(x_1 + x_4) \\ x_3(x_1 + x_4) & x_4^2 + x_2x_3 \end{bmatrix}$$

That is, the ideal with generators

$$I_R = (x_1^2 + x_2x_3, x_2(x_1 + x_4), x_3(x_1 + x_4), (x_1 - x_4)(x_1 + x_4))$$

The variety $\mathbb{V}(I_R) \hookrightarrow M_2$ consists of all matrices A such that $A^2 = 0$. Conjugating A to an upper triangular form we see that the eigenvalues of A must be zero, hence

$$rep_2\ \mathbb{C}[x]/(x^2) = \mathcal{O}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \cup \mathcal{O}(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$$

and we have seen that this variety is a cone with top the zero matrix and defining equations

$$\mathbb{V}(x_1 + x_4, x_1^2 + x_2x_3)$$

and we see that I_R is properly contained in this ideal. Still, we have that

$$rad(I_R) = (x_1 + x_4, x_1^2 + x_3x_4)$$

for an easy computation shows that $\overline{x_1 + x_4}^3 = 0 \in \mathbb{C}[x_1, x_2, x_3, x_4]/I_R$. Therefore, even in the easiest of examples, the representation variety does not have to be reduced.

For the general case, observe that when J is a Jordan block of size d with eigenvalue zero an easy calculation shows that

$$J^{d-1} = \begin{bmatrix} 0 & \ldots & 0 & d-1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix} \quad \text{and} \quad J^d = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}$$

Therefore, we see that the representation variety $rep_n\ \mathbb{C}[x]/(x^r)$ is the union of all conjugacy classes of matrices having 0 as only eigenvalue and all of which Jordan blocks have size $\leq r$. Expressed in module theoretic terms, any n-dimensional $R = \mathbb{C}[x]/(x^r)$-module M is isomorphic to a direct sum of indecomposables

$$M = I_1^{\oplus e_1} \oplus I_2^{\oplus e_2} \oplus \ldots \oplus I_r^{\oplus e_r}$$

where I_j is the unique indecomposable j-dimensional R-module (corresponding to the Jordan block of size j). Of course, the multiplicities e_i of the factors must satisfy the equation

$$e_1 + 2e_2 + 3e_3 + \ldots + re_r = n$$

In M we can consider the subspaces for all $1 \leq i \leq r-1$

$$M_i = \{m \in M \mid x^i.m = 0\}$$

the dimension of which can be computed knowing the powers of Jordan blocks (observe that the dimension of M_i is equal to $n - \text{rank}(A^i)$)

$$t_i = dim_{\mathbb{C}}\ M_i = e_1 + 2e_2 + \ldots (i-1)e_i + i(e_i + e_{i+1} + \ldots + e_r)$$

Observe that giving n and the $r-1$-tuple $(t_1, t_2, \ldots, t_{n-1})$ is the same as giving the multiplicities e_i because

$$\begin{cases} 2t_1 & = t_2 + e_1 \\ 2t_2 & = t_3 + t_1 + e_2 \\ 2t_3 & = t_4 + t_2 + e_3 \\ & \vdots \\ 2t_{n-2} & = t_{n-1} + t_{n-3} + e_{n-2} \\ 2t_{n-1} & = n + t_{n-2} + e_{n-1} \\ n & = t_{n-1} + e_n \end{cases}$$

Let n-dimensional $\mathbb{C}[x]/(x^r)$-modules M and M' (or associated matrices A and A') be determined by the $r-1$-tuples $(t_1, \ldots, t_{r-1})$ respectively $(t'_1, \ldots, t'_{r-1})$ then we have that

$$\mathcal{O}(A') \hookrightarrow \overline{\mathcal{O}(A)} \quad \text{if and only if} \quad t_1 \leq t'_1, t_2 \leq t'_2, \ldots, t_{r-1} \leq t'_{r-1}$$

Therefore, we have an inverse order isomorphism between the orbits in $rep_n(\mathbb{C}[x]/(x^r))$ and the $r-1$-tuples of natural numbers $(t_1, \ldots, t_{r-1})$ satisfying the following linear inequalities (which follow from the above system)

$$2t_1 \geq t_2, 2t_2 \geq t_3 + t_1, 2t_3 \geq t_4 + t_2, \ldots, 2t_{n-1} \geq n + t_{n-2}, n \geq t_{n-2}.$$

Let us apply this general result in a few easy cases. First, consider $r = 2$, then the orbits in $rep_n\ \mathbb{C}[x]/(x^2)$ are parameterized by a natural number t_1 satisfying the inequalities $n \geq t_1$ and $2t_1 \geq n$, the multiplicities are given by $e_1 = 2t_1 - n$ and $e_2 = n - t_1$. Moreover, the orbit of the module $M(t'_1)$ lies in the closure of the orbit of $M(t_1)$ whenever $t_1 \leq t'_1$.

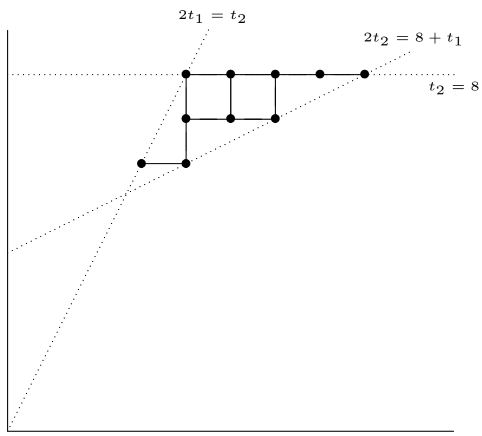
That is, if $n = 2k + \delta$ with $\delta = 0$ or 1, then $rep_n\ \mathbb{C}[x]/(x^2)$ is the union of $k+1$ orbits and the orbitclosures form a linear order as follows (from big to small)

$$I_1^{\delta} \oplus I_2^{\oplus k} \;\text{——}\; I_1^{\oplus \delta+2} \oplus I_2^{\oplus k-1} \;\text{——}\; \dots \;\text{——}\; I_1^{\oplus n}$$

If $r = 3$, orbits in $rep_n\ \mathbb{C}[x]/(x^3)$ are determined by couples of natural numbers (t_1, t_2) satisfying the following three linear inequalities

$$\begin{cases} 2t_1 & \geq t_2 \\ 2t_2 & \geq n + t_1 \\ n & \geq t_2 \end{cases}$$

For example, for $n = 8$ we obtain the following situation



Therefore, $rep_8\ \mathbb{C}[x]/(x^3)$ consists of 10 orbits with orbitclosure diagram as below (the nodes represent the multiplicities $[e_1 e_2 e_3]$).

$$\begin{tikzcd}
 & & {[012]} \arrow[d, dash] & \\
 & & {[202]} \arrow[d, dash] & \\
 & & {[121]} \arrow[dl, dash] \arrow[dr, dash] & \\
 & {[040]} \arrow[dl, dash] \arrow[dr, dash] & & {[311]} \arrow[dl, dash] \\
{[501]} \arrow[dr, dash] & & {[230]} \arrow[dl, dash] & \\
 & {[420]} \arrow[d, dash] & & \\
 & {[610]} \arrow[d, dash] & & \\
 & {[800]} & &
\end{tikzcd}$$

Here we used the equalities $e_1 = 2t_1 - t_2$, $e_2 = 2t_2 - n - t_1$ and $e_3 = n - t_2$. For general n and r this result shows that $rep_n\ \mathbb{C}[x]/(x^r)$ is the closure of the orbit of the module with decomposition

$$M_{gen} = I_r^{\oplus e} \oplus I_s \quad \text{if} \quad n = er + s$$

4.4 The Hilbert criterium.

A one parameter subgroup of a linear algebraic group G is a morphism

$$\lambda : \mathbb{C}^* \longrightarrow G$$

of affine algebraic groups. That is, λ is both a groupmorphism and a morphism of affine varieties. The set of all one parameter subgroup of G will be denoted by $Y(G)$.

If G is commutative algebraic group, then $Y(G)$ is an Abelian group with additive notation

$$\lambda_1 + \lambda_2 : \mathbb{C}^* \longrightarrow G \quad \text{with } (\lambda_1+\lambda_2)(t) = \lambda_1(t).\lambda_2(t)$$

Recall that an n-dimensional torus is an affine algebraic group isomorphic to

$$\underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{n} = T_n$$

the closed subgroup of invertible diagonal matrices in GL_n.

Lemma 4.16 *$Y(T_n) \simeq \mathbb{Z}^n$. The correspondence is given by assigning to $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ the one-parameter subgroup*

$$\lambda : \mathbb{C}^* \longrightarrow T_n \quad \text{given by } t \mapsto (t^{r_1}, \ldots, t^{r_n})$$

Proof. *For any two affine algebraic groups G and H there is a canonical bijection $Y(G \times H) = Y(G) \times Y(H)$ so it suffices to verify that $Y(\mathbb{C}^*) \simeq \mathbb{Z}$ with any $\lambda : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ given by $t \mapsto t^r$ for some $r \in \mathbb{Z}$. This is obvious as λ induces the algebra morphism*

$$\mathbb{C}[\mathbb{C}^*] = \mathbb{C}[x, x^{-1}] \xrightarrow{\lambda^*} \mathbb{C}[x, x^{-1}] = \mathbb{C}[\mathbb{C}^*]$$

which is fully determined by the image of x which must be an invertible element. Now, any invertible element in $\mathbb{C}[x, x^{-1}]$ is homogeneous of the form cx^r for some $r \in \mathbb{Z}$ and $c \in \mathbb{C}^*$. The corresponding morphism maps t to ct^r which is only a groupmorphism if it maps the identity element 1 to 1 so $c = 1$, finishing the proof. □

Proposition 4.17 *Any one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ is of the form*

$$t \mapsto g^{-1}. \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} .g$$

for some $g \in GL_n$ and some n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$.

Proof. *Let H be the image under λ of the subgroup μ of roots of unity in $\mathbb{C}^*$. We claim that there is a basechange matrix $g \in GL_n$ such that*

$$g.H.g^{-1} \hookrightarrow \begin{bmatrix} \mathbb{C}^* & & 0 \\ & \ddots & \\ 0 & & \mathbb{C}^* \end{bmatrix}$$

Assume $h \in H$ not a scalar matrix, then h has a proper eigenspace decomposition $V \oplus W = \mathbb{C}^n$. We use that $h^l = \mathbb{1}_n$ and hence all its Jordan blocks must have size one as for any $\lambda \neq 0$ we have

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}^l = \begin{bmatrix} \lambda^l & l\lambda^{l-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & l\lambda^{l-1} \\ & & & \lambda^l \end{bmatrix} \neq \mathbb{1}_n$$

Because H is commutative, both V and W are stable under H. By induction on n we may assume that the images of H in $GL(V)$ and $GL(W)$ are diagonalizable, but then the same holds in GL_n.

As μ is infinite, it is Zariski dense in $\mathbb{C}^*$ and because the diagonal matrices are Zariski closed in GL_n we have

$$g.\lambda(\mathbb{C}^*).g^{-1} = g.\overline{H}.g^{-1} \hookrightarrow T_n$$

and the result follows from the lemma above $\square$

Let V be a general GL_n-representation considered as an affine space with GL_n-action, let X be a GL_n-stable closed subvariety and consider a point $x \in X$. A one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ determines a morphism

$$\mathbb{C}^* \xrightarrow{\lambda_x} X \quad \text{defined by} \quad t \mapsto \lambda(t).x$$

Observe that the image $\lambda_x(\mathbb{C}^*)$ lies in the orbit $GL_n.x$ of x. Assume there is a continuous extension of this map to the whole of $\mathbb{C}$. We claim that this extension must then be a morphism. If not, the induced algebra morphism

$$\mathbb{C}[X] \xrightarrow{\lambda_x^*} \mathbb{C}[t, t^{-1}]$$

does not have its image in $\mathbb{C}[t]$, so for some $f \in \mathbb{C}[Z]$ we have that

$$\lambda_x^*(f) = \frac{a_0 + a_1 t + \ldots + a_z t^z}{t^s} \quad \text{with} \quad a_0 \neq 0 \text{ and } s > 0$$

But then $\lambda_x^*(f)(t) \longrightarrow \pm\infty$ when t goes to zero, that is, λ_x^* cannot have a continuous extension, a contradiction.

So, if a continuous extension exists there is morphism $\lambda_x : \mathbb{C} \longrightarrow X$. Then, $\lambda_x(0) = y$ and we denote this by

$$\lim_{t \to 0} \lambda(t).x = y$$

Clearly, the point $y \in X$ must belong to the orbitclosure $\overline{GL_n.x}$ in the Zariski topology (or in the $\mathbb{C}$-topology as orbits are constructible). Conversely, one might ask whether if $y \in \overline{GL_n.x}$ we can always approach y via a one-parameter subgroup. The Hilbert criterium gives situations when this is indeed possible.

The only ideals of the formal power series $\mathbb{C}[[t]]$ are principal and generated by t^r for some $r \in \mathbb{N}_+$. With $\mathbb{C}((t))$ we will denote the field of fractions of the domain $\mathbb{C}((t))$.

Lemma 4.18 *Let V be a GL_n-representation, $v \in V$ and a point $w \in V$ lying in the orbitclosure $\overline{GL_n.v}$. Then, there exists a matrix g with coefficients in the field $\mathbb{C}((t))$ and $det(g) \neq 0$ such that*

$$(g.v)_{t=0} \quad \text{is well defined and is equal to} \quad w$$

Proof. *Note that $g.v$ is a vector with coordinates in the field $\mathbb{C}((t))$. If all coordinates belong to $\mathbb{C}[[t]]$ we can set $t = 0$ in this vector and obtain a vector in V. It is this vector that we denote with $(g.v)_{t=0}$.*

Consider the orbit map $\mu : GL_n \longrightarrow V$ defined by $g' \mapsto g'.v$. As $w \in \overline{GL_n.v}$ we have seen that there is an irreducible curve $C \hookrightarrow GL_n$ such that $w \in \overline{\mu(C)}$.

We obtain a diagram of $\mathbb{C}$*-algebras*

$$\begin{tikzcd}
\mathbb{C}[GL_n] \arrow[r] & \mathbb{C}[C] \arrow[r, hook] & \mathbb{C}(C) \\
\mathbb{C}[V] \arrow[u, "\mu^*"] \arrow[r] & \mathbb{C}[\overline{\mu(C)}] \arrow[u, hook, "\mu^*"] \arrow[r, hook] & \mathbb{C}[C'] \arrow[u, hook]
\end{tikzcd}$$

Here, $\mathbb{C}[C]$ *is defined to be the integral closure of* $\mathbb{C}[\overline{\mu(C)}]$ *in the functionfield* $\mathbb{C}(C)$ *of* C*. Two things are important to note here :* $C' \longrightarrow \overline{\mu(C)}$ *is finite, so surjective and take* $c \in C'$ *be a point lying over* $w \in \overline{\mu(C)}$*. Further,* C' *having an integrally closed coordinate ring is a complex manifold. Hence, by the implicit function theorem polynomial functions on* C *can be expressed in a neighborhood of* c *as power series in one variable, giving an embedding* $\mathbb{C}[C'] \hookrightarrow \mathbb{C}[[t]]$ *with* $(t) \cap \mathbb{C}[C'] = M_c$*. This inclusion extends to one on the level of their fields of fractions. That is, we have a diagram of* $\mathbb{C}$*-algebra morphisms*

$$\begin{tikzcd}
\mathbb{C}[GL_n] \arrow[r] & \mathbb{C}(C) \arrow[r, phantom, "="] & \mathbb{C}(C') \arrow[r, hook] & \mathbb{C}((t)) \\
\mathbb{C}[V] \arrow[u, "\mu^*"] \arrow[r] & \mathbb{C}[\overline{\mu(C)}] \arrow[u, hook] \arrow[r, hook] & \mathbb{C}[C'] \arrow[u, hook] \arrow[r, hook] & \mathbb{C}[[t]] \arrow[u, hook]
\end{tikzcd}$$

The upper composition defines an invertible matrix $g(t)$ *with coefficients in* $\mathbb{C}((t))$*, its* (i,j)*-entry being the image of the coordinate function* $x_{ij} \in \mathbb{C}[GL_n]$*. Moreover, the inverse image of the maximal ideal* $(t) \triangleleft \mathbb{C}[[t]]$ *under the lower composition gives the maximal ideal* $M_w \triangleleft \mathbb{C}[V]$*. This proves the claim.* $\square$

Lemma 4.19 *Let* g *be an* $n \times n$ *matrix with coefficients in* $\mathbb{C}((t))$ *and* $\det g \neq 0$*. Then there exist* $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ *such that*

$$g = u_1 . \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} . u_2$$

with $r_i \in \mathbb{Z}$ *and* $r_1 \leq r_2 \leq \ldots \leq r_n$*.*

Proof. *By multiplying* g *with a suitable power of* t *we may assume that* $g = (g_{ij}(t))_{i,j} \in M_n(\mathbb{C}[[t]])$*. If* $f(t) = \sum_{i=0}^{\infty} f_i t^i \in \mathbb{C}[[t]]$ *define* $v(f(t))$ *to be the minimal* i *such that* $a_i \neq 0$*. Let* (i_0, j_0) *be an entry where* $v(g_{ij}(t))$ *attains a minimum, say* r_1*. That is, for all* (i,j) *we have* $g_{ij}(t) = t^{r_1} t^r f(t)$ *with* $r \geq 0$ *and* $f(t)$ *an invertible element of* $\mathbb{C}[[t]]$*.*

By suitable row and column interchanges we can take the entry (i_0, j_0) *to the* $(1,1)$*-position. Then, multiplying with a unit we can replace it by* t^{r_1} *and by elementary row and column operations all the remaining entries in the first row and column can be made zero. That is, we have invertible matrices* $a_1, a_2 \in GL_n(\mathbb{C}[[t]])$ *such that*

$$g = a_1 . \begin{bmatrix} t^{r_1} & \underline{0}^{\tau} \\ \underline{0} & \boxed{g_1} \end{bmatrix} . a_2$$

Repeating the same idea on the submatrix g_1 *and continuing gives the result.* $\square$

We can now state and prove the Hilbert criterium which allows us to study orbit-closures by one parameter subgroups.

Theorem 4.20 *Let V be a GL_n-representation and $X \hookrightarrow V$ a closed GL_n-stable subvariety. Let $\mathcal{O}(x) = GL_n.x$ be the orbit of a point $x \in X$. Let $Y \hookrightarrow \overline{\mathcal{O}(x)}$ be a closed GL_n-stable subset. Then, there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ such that*

$$\underset{t \to 0}{lim} \lambda(t).x \in Y$$

Proof. *It suffices to prove the result for $X = V$. By lemma 4.18 there is an invertible matrix $g \in M_n(\mathbb{C}((t)))$ such that*

$$(g.x)_{t=0} = y \in Y$$

By lemma 4.19 we can find $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ such that

$$g = u_1.\lambda'(t).u_2 \quad \text{with} \quad \lambda'(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}$$

a one-parameter subgroup. There exist $x_i \in V$ such that $u_2.x = \sum_{i=0}^{\infty} z_i t^i$ in particular $u_2(0).x = x_0$. But then,

$$\begin{aligned}(\lambda'(t).u_2.x)_{t=0} &= \sum_{i=0}^{\infty} (\lambda'(t).x_i t^i)_{t=0} \\ &= (\lambda'(t).x_0)_{t=0} + (\lambda'(t).x_1 t)_{t=0} + \dots\end{aligned}$$

But one easily verifies (using a basis of eigenvectors of $\lambda'(t)$) that

$$\underset{s \to 0}{lim} \lambda^{'-1}(s).(\lambda'(t) x_i t^i)_{t=0} = \begin{cases} (\lambda'(t).x_0)_{t=0} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases}$$

As $(\lambda'(t).u_2.x)_{t=0} \in Y$ and Y is a closed GL_n-stable subset, we also have that

$$\underset{s \longrightarrow 0}{lim} \lambda^{'-1}(s).(\lambda'(t).u_2.x)_{t=0} \in Y \qquad \text{that is,} \qquad (\lambda'(t).x_0)_{t=0} \in Y$$

But then, we have for the one-parameter subgroup $\lambda(t) = u_2(0)^{-1}.\lambda'(t).u_2(0)$ that

$$\underset{t \to 0}{lim} \lambda(t).x \in Y$$

finishing the proof. □

An important special case occurs when $x \in V$ belongs to the nullcone, that is, when the orbit closure $\overline{\mathcal{O}(x)}$ contains the fixed point $0 \in V$. The original Hilbert criterium asserts the following.

Proposition 4.21 *Let V be a GL_n-representation and $x \in V$ in the nullcone. Then, there is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that*

$$\underset{t \to 0}{lim} \ \lambda(t).x = 0$$

In the statement of theorem 4.20 it is important that Y is closed. In particular, it does not follow that any orbit $\mathcal{O}(y) \hookrightarrow \overline{\mathcal{O}(x)}$ can be reached via one-parameter subgroups. In the next section we will give an example of such a situation.

4.5 Semisimple modules

In this section we will characterize the closed GL_n-orbits in the module variety $rep_n\ A$ for an affine $\mathbb{C}$-algebra A. We have seen that any point $\psi \in rep_n\ A$, that is any n-dimensional representation $A \xrightarrow{\psi} M_n$ determines an n-dimensional A-module which we will denote with M_ψ.

A finite filtration F on an n-dimensional module M is a sequence of A-submodules

$$F \quad : \quad 0 = M_{t+1} \subset M_t \subset \ldots \subset M_1 \subset M_0 = M.$$

The associated graded A-module is the n-dimensional module

$$gr_F\ M = \oplus_{i=0}^t M_i/M_{i+1}.$$

We have the following ringtheoretical interpretation of the action of one-parameter subgroups of GL_n on the representation variety $rep_n\ A$.

Lemma 4.22 *Let $\psi, \rho \in rep_n\ A$. Equivalent are,*

1. *There is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that*

$$\lim_{t \rightarrow 0} \lambda(t).\psi = \rho$$

2. *There is a finite filtration F on the A-module M_ψ such that*

$$gr_F\ M_\psi \simeq M_\rho$$

as A-modules.

Proof. $(1) \Rightarrow (2)$ *: If V is any GL_n-representation and $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ a one-parameter subgroup, we have an induced weight space decomposition of V*

$$V = \oplus_i V_{\lambda,i} \quad \textit{where} \quad V_{\lambda,i} = \{ v \in V \mid \lambda(t).v = t^i v, \forall t \in \mathbb{C}^* \}.$$

In particular, we apply this to the underlying vectorspace of M_ψ which is $V = \mathbb{C}^n$ (column vectors) on which GL_n acts by left multiplication. We define

$$M_j = \oplus_{i > j} V_{\lambda,i}$$

and claim that this defines a finite filtration on M_ψ with associated graded A-module M_ρ. For any $a \in A$ (it suffices to vary a over the generators of A) we can consider the linear maps

$$\phi_{ij}(a) : V_{\lambda,i} \hookrightarrow V = M_\psi \xrightarrow{a.} M_\psi = V \twoheadrightarrow V_{\lambda,j}$$

(that is, we express the action of a in a blockmatrix Φ_a with respect to the decomposition of V). Then, the action of a on the module corresponding to $\lambda(t).\psi$ is given by the matrix $\Phi'_a = \lambda(t).\Phi_a.\lambda(t)^{-1}$ with corresponding blocks

$$\begin{tikzcd}
V_{\lambda,i} \arrow[r, "\phi_{ij}(a)"] & V_{\lambda,j} \arrow[d, "\lambda(t)"] \\
V_{\lambda,i} \arrow[u, "\lambda(t)^{-1}"] \arrow[r, "\phi'_{ij}(a)"'] & V_{\lambda,j}
\end{tikzcd}$$

that is $\phi'_{ij}(a) = t^{j-i}\phi_{ij}(a)$. *Therefore, if* $lim_{t\to 0}\lambda(t).\psi$ *exists we must have that*

$$\phi_{ij}(a) = 0 \quad \textit{for all} \quad j < i.$$

But then, the action by a *sends any* $M_k = \oplus_{i>k} V_{\lambda,i}$ *to itself, that is, the* M_k *are* A*-submodules of* M_ψ. *Moreover, for* $j > i$ *we have*

$$\lim_{t\to 0} \phi'_{ij}(a) = \lim_{t\to 0} t^{j-i}\phi_{ij}(a) = 0$$

Consequently, the action of a *on* ρ *is given by the diagonal blockmatrix with blocks* $\phi_{ii}(a)$, *but this is precisely the action of* a *on* $V_i = M_{i-1}/M_i$, *that is,* ρ *corresponds to the associated graded module.*

(2) $\Rightarrow$ (1) : *Given a finite filtration on* M_ψ

$$F \quad : \quad 0 = M_{t+1} \subset \ldots \subset M_1 \subset M_0 = M_\psi$$

we have to find a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ *which induces the filtration* F *as in the first part of the proof. Clearly, there exist subspaces* V_i *for* $0 \leq i \leq t$ *such that*

$$V = \oplus_{i=0}^{t} V_i \quad \textit{and} \quad M_j = \oplus_{j=i}^{t} V_i.$$

Then we take λ *to be defined by* $\lambda(t) = t^i Id_{V_i}$ *for all* i *and verifies the claims.* □

Example 4.23 Let M_ψ we the 2-dimensional $\mathbb{C}[x]$-module determined by the Jordan block and consider the canonical basevectors

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then, $\mathbb{C}e_1$ is a $\mathbb{C}[x]$-submodule of M_ψ and we have a filtration

$$0 = M_2 \subset \mathbb{C}e_1 = M_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 = M_0 = M_\psi$$

Using the conventions of the second part of the above proof we then have

$$V_1 = \mathbb{C}e_1 \qquad V_2 = \mathbb{C}e_2 \quad \text{hence} \quad \lambda(t) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

Indeed, we then obtain that

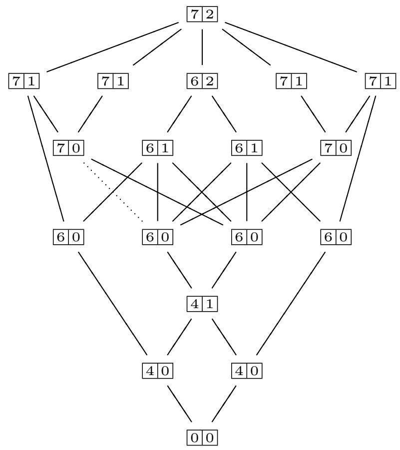
$$\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix}$$

and the limit $t \longrightarrow 0$ exists and is the associated graded module $gr_F\ M_\psi = M_\rho$ determined by the diagonal matrix.

Consider two modules $M_\psi, M_\psi \in rep_n\ A$. *Assume that* $\mathcal{O}(M_\rho) \hookrightarrow \overline{\mathcal{O}(M_\psi)}$ *and that we can reach the orbit of* M_ρ *via a one-parameter subgroup. Then, lemma 4.22 asserts that* M_ρ *must be decomposable as it is the associated graded of a nontrivial filtration on* M_ψ. *This gives us a criterium to construct examples showing that the closedness assumption in the formulation of Hilbert's criterium is essential.*

Example 4.24 (Nullcone of $M_3^2 = M_3 \oplus M_3$)

In chapter 8 we will describe a method to work-out the nullcones of m-tuples of $n \times n$ matrices. The special case of 2 3×3 matrices has been worked out by H.P. Kraft in [14, p.202]. We depict the orbits here and refer to chapter 8 for more details.



In this picture, each node corresponds to a torus. The right hand number is the dimension of the torus and the left hand number is the dimension of the orbit represented by a point in the torus. The solid or dashed lines indicate orbitclosures. For example, the dashed line corresponds to the following two points in $M_3^2 = M_3 \oplus M_3$

$$\psi = (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) \qquad \rho = (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

We claim that M_ρ is an indecomposable 3-dimensional module of $\mathbb{C}\langle x, y\rangle$. Indeed, the only subspace of the column vectors $\mathbb{C}^3$ left invariant under both x and y is equal to

$$\begin{bmatrix} \mathbb{C} \\ 0 \\ 0 \end{bmatrix}$$

hence M_ρ cannot have a direct sum decomposition of two or more modules. Next, we claim that $\mathcal{O}(M_\rho) \hookrightarrow \overline{\mathcal{O}(M_\psi)}$. Indeed, simultaneous conjugating ψ with the invertible matrix

$$\begin{bmatrix} 1 & \epsilon - 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon^{-1} \end{bmatrix} \quad \text{we obtain the couple} \quad (\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \epsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

and letting $\epsilon \longrightarrow 0$ we see that the limiting point is ρ.

The Jordan-Hölder theorem, see for example [23, 2.6] asserts that any finite dimensional A-module M has a composition series, that is, M has a finite filtration

$$F \quad : \quad 0 = M_{t+1} \subset M_t \subset \ldots \subset M_1 \subset M_0 = M$$

such that the successive quotients $S_i = M_i/M_{i+1}$ are all simple A-modules for $0 \leq i \leq t$. Moreover, these composition factors S and their multiplicities are independent of the chosen composition series, that is, the set $\{S_0, \ldots, S_t\}$ is the same for every composition series. In particular, the associated graded module for a composition series is determined only up to isomorphism and is the semisimple n-dimensional module

$$gr\ M = \oplus_{i=0}^t S_i$$

Theorem 4.25 *Let A be an affine $\mathbb{C}$-algebra and $M \in rep_n\ A$.*

1. *The orbit $\mathcal{O}(M)$ is closed in $rep_n\ A$ if and only if M is an n-dimensional semisimple A-module.*

2. *The orbitclosure $\overline{\mathcal{O}(M)}$ contains exactly one closed orbit, corresponding to the direct sum of the composition factors of M.*

3. *The points of the quotient variety of $rep_n\ A$ under GL_n parameterize the isomorphism classes of n-dimensional semisimple A-modules. We will denote the quotient variety by $iss_n\ A$.*

Proof. (1) : *Assume that the orbit $\mathcal{O}(M)$ is Zariski closed. Let $gr\ M$ be the associated graded module for a composition series of M. From lemma 4.22 we know that $\mathcal{O}(gr\ M)$ is contained in $\overline{\mathcal{O}(M)} = \mathcal{O}(M)$. But then $gr\ M \simeq M$ whence M is semisimple.*

Conversely, assume M is semisimple. We know that the orbitclosure $\overline{\mathcal{O}(M)}$ contains a closed orbit, say $\mathcal{O}(N)$. By the Hilbert criterium we have a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that

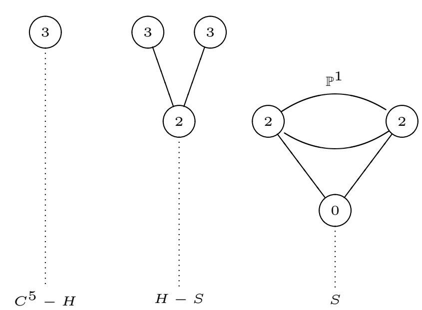
$$\lim_{t \to 0} \lambda(t).M = N' \simeq N.$$

By lemma 4.22 this means that there is a finite filtration F on M with associated graded module $gr_F\ M \simeq N$. For the semisimple module M the only possible finite filtrations are such that each of the submodules is a direct sum of simple components, so $gr_F\ M \simeq M$, whence $M \simeq N$ and hence the orbit $\mathcal{O}(M)$ is closed.

(2) : *Remains only to prove uniqueness of the closed orbit in $\overline{\mathcal{O}(M)}$. This either follows from the Jordan-Hölder theorem or, alternatively, from the separation property of the quotient map to be proved in the next section.*

(3) : *We will prove in the next section that the points of a quotient variety parameterize the closed orbits.* □

Example 4.26 Recall the description of the orbits in $M_2^2 = M_2 \oplus M_2$ given in the previous chapter.



and each fiber contains a unique closed orbit. The one over a point in $H - S$ corresponding to the matrix couple

$$(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix})$$

which is indeed a semi-simple module of $\mathbb{C}\langle x, y\rangle$ (the direct sum of teh two 1-dimensional simple representations determined by $x \mapsto a_i$ and $y \mapsto b_i$. In case $a_1 = a_2$ and $b_1 = b_2$ then these two simples coincide and the semi-simple module having this factor with multiplicity two is the unique closed orbit in the fiber of a point in S.

Example 4.27 Assume A is a finite dimensional $\mathbb{C}$-algebra. Then, there are only a finite number, say k, of nonisomorphic n-dimensional semisimple A-modules. Hence $iss_n\ A$ is a finite number of k points, whence $rep_n\ A$ is the disjoint union of k connected components, each consisting of all n-dimensional A-modules with the same composition factors. Connectivity follows from the fact that the orbit of the sum of the composition factors lies in the closure of each orbit.

Example 4.28 Let A be an affine commutative algebra with presentation $A = \mathbb{C}[x_1, \ldots, x_m]/I_A$ and let X be the affine variety $\mathbb{V}(I_A)$. Observe that any simple A-module is one-dimensional hence corresponds to a point in X. (Indeed, for any algebra A a simple k-dimensional module determines an epimorphism $A \twoheadrightarrow M_k$ and M_k is only commutative if $k = 1$). Applying the Jordan-Hölder theorem we see that

$$iss_n\ A \simeq X^{(n)} = \underbrace{X \times \ldots \times X}_{n} / S_n$$

the n-th symmetric product of X.

4.6 Some invariant theory.

The results in this section hold for any reductive algebraic group. As we will need them primarily for GL_n (or products of GL_{n_i}) we will prove them only in that case. Our first aim is to prove that GL_n is a reductive group, that is, all GL_n-representations are completely reducible. Consider the unitary group

$$U_n = \{A \in GL_n \mid A.A^* = \mathbb{1}_n\}$$

where A^* is the Hermitian transpose of A. Clearly, U_n is a compact Lie group. Any compact Lie group has a so called Haar measure which allows one to integrate continuous complex valued functions over the group in an invariant way. That is, there is a linear function assigning to every continuous function $f : U_n \longrightarrow \mathbb{C}$ its integral

$$f \mapsto \int_{U_n} f(g)dg \in \mathbb{C}$$

which is normalized such that $\int_{U_n} dg = 1$ and is left and right invariant, which means that for all $u \in U_n$ we have the equalities

$$\int_{U_n} f(gu)dg = \int_{U_n} f(g)dg = \int_{U_n} f(ug)dg.$$

This integral replaces the classical idea in representation theory of averaging functions over a finite group.

Proposition 4.29 *Every U_n-representation is completely reducible.*

Proof. *Take a finite dimensional complex vectorspace V with a U_n-action and assume that W is a subspace of V left invariant under this action. Extending a basis of W to V we get a linear map $V \overset{\phi}{\twoheadrightarrow} W$ which is the identity on W. For any $v \in V$ we have a continuous map*

$$U_n \longrightarrow W \qquad g \mapsto g.\phi(g^{-1}.v)$$

(use that W is left invariant) and hence we can integrate it over U_n (integrate the coordinate functions). Hence we can define a map $\phi_0 : V \longrightarrow W$ by

$$\phi_0(v) = \int_{U_n} g.\phi(g^{-1}.v)dg.$$

Clearly, ϕ_0 is linear and is the identity on W. Moreover,

$$\begin{aligned}\phi_0(u.v) &= \int_{U_n} g.\phi(g^{-1}u.v)dg = u.\int_{U_n} u^{-1}g.\phi(g^{-1}u.v)dg \\ &\overset{*}{=} u.\int_{U_n} g\phi(g^{-1}.v)dg = u.\phi_0(v)\end{aligned}$$

where the starred equality uses the invariance of the Haar measure. Hence, $V = W \oplus Ker\ \phi_0$ is a decomposition as U_n-representations. Continuing whenever one of the components has a nontrivial subrepresentation we arrive at a decomposition of V into simple U_n-representations. □

We claim that for any n, U_n is Zariski dense in GL_n. Let D_n be the group of all diagonal matrices in GL_n. The Cartan decomposition for GL_n asserts that

$$GL_n = U_n.D_n.U_n$$

For, take $g \in GL_n$ then $g.g^*$ is an Hermitian matrix and hence diagonalizable by unitary matrices. So, there is a $u \in U_n$ such that

$$u^{-1}.g.g^*.u = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix} = \underbrace{s^{-1}.g.s}_{p} . \underbrace{s^{-1}.g^*.s}_{p^*}$$

Then, each $\alpha_i > 0 \in \mathbb{R}$ as $\alpha_i = \sum_{j=1}^n \parallel p_{ij} \parallel^2$. Let $\beta_i = \sqrt{\alpha_i}$ and let d the diagonal matrix $diag(\beta_1, \ldots, \beta_n)$. Clearly,

$$g = u.d.(d^{-1}.u^{-1}.g) \quad \text{and we claim} \quad v = d^{-1}.u^{-1}.g \in U_n.$$

Indeed, we have

$$\begin{aligned} v.v^* =&(d^{-1}.u^{-1}.g).(g^*.u.d^{-1}) = d^{-1}.(u^{-1}.g.g^*.u).d^{-1} \\ =&d^{-1}.d^2.d^{-1} = \mathbb{1}_n \end{aligned}$$

proving the Cartan decomposition. Now, $D_n = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ and $D_n \cap U_n = U_1 \times \ldots \times U_1$ and because $U_1 = \mu$ is Zariski dense (being infinite) in $D_1 = \mathbb{C}^*$, we have that D_n is contained in the Zariski closure of U_n. By the Cartan decomposition we then have that the Zariski closure of U_n is GL_n.

Theorem 4.30 *GL_n is a reductive group. That is, all GL_n-representations are completely reducible.*

Proof. *Let V be a GL_n-representation having a subrepresentation W. In particular, V and W are U_n-representations, so by the foregoing proposition we have a decomposition $V = W \oplus W'$ as U_n-representations. Consider the subgroup*

$$N = N_{GL_n}(W') = \{g \in GL_n \mid g.W' \subset W'\}$$

then N is a Zariski closed subgroup of GL_n containing U_n. As the Zariski closure of U_n is GL_n we have $N = GL_n$ and hence that W' is a representation of GL_n. Continuing gives a decomposition of V in simple GL_n-representations. □

Let $S = S_{GL_n}$ be the set of isomorphism classes of simple GL_n-representations. If W is a simple GL_n-representation belonging to the isomorphism class $s \in S$, we say that W is of type s and denote this by $W \in s$. Let X be a complex vectorspace (not necessarily finite dimensional) with a linear action of GL_n. We say that the action is locally finite on X if, for any finite dimensional subspace Y of X, there exists a finite dimensional subspace $Y \subset Y' \subset X$ which is a GL_n-representation. The isotypical component of X of type $s \in S$ is defined to be the subspace

$$X_{(s)} = \sum \{W \mid W \subset X, W \in s\}.$$

If V is a GL_n-representation, we have seen that V is completely reducible. Then, $V = \oplus V_{(s)}$ and every isotypical component $V_{(s)} \simeq W^{\oplus e_s}$ for $W \in s$ and some number e_s. Clearly, $e_s \neq 0$ for only finitely many classes $s \in S$. We call the decomposition $V = \oplus_{s \in S} V_{(s)}$ the isotypical decomposition of V and we say that the simple representation $W \in s$ occurs with multiplicity e_s in V.

If V' is another GL_n-representation and if $V \xrightarrow{\phi} V'$ is a morphism of GL_n-representations (that is, a linear map commuting with the action), then for any $s \in S$ we have that $\phi(V_{(s)}) \subset V'_{(s)}$. If the action of GL_n on X is locally finite, we can reduce to finite dimensional GL_n-subrepresentation and obtain a decomposition

$$X = \oplus_{s \in S} X_{(s)},$$

which is again called the isotypical decomposition of X.

Let V be a GL_n-representation of some dimension m. Then, we can view V as an affine space $\mathbb{C}^m$ and we have an induced action of GL_n on the polynomial functions $f \in \mathbb{C}[V]$ by the rule

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g."'] & \mathbb{C} \\ V \arrow[ur, dotted, "g.f"'] & \end{tikzcd}$$

that is $(g.f)(v) = f(g^{-1}.v)$ for all $g \in GL_n$ and all $v \in V$. If $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_m]$ is graded by giving all the x_i degree one, then each of the homogeneous components of $\mathbb{C}[V]$ is a finite dimensional GL_n-representation. Hence, the action of GL_n on $\mathbb{C}[V]$ is locally finite. Indeed, let $\{y_1, \ldots, y_l\}$ be a basis of a finite dimensional subspace $Y \subset \mathbb{C}[V]$ and let d be the maximum of the $deg(y_i)$. Then $Y' = \oplus_{i=0}^{d} \mathbb{C}[V]_i$ is a GL_n-representation containing Y.

Therefore, we have an isotypical decomposition $\mathbb{C}[V] = \oplus_{s \in S} \mathbb{C}[V]_{(s)}$. In particular, if $0 \in S$ denotes the isomorphism class of the trivial GL_n-representation ($\mathbb{C}_{triv} = \mathbb{C}x$ with $g.x = x$ for every $g \in GL_n$) then we have

$$\mathbb{C}[V]_{(0)} = \{f \in \mathbb{C}[V] \mid g.f = f, \forall g \in GL_n\} = \mathbb{C}[V]^{GL_n}$$

the ring of polynomial invariants, that is, of polynomial functions which are constant along orbits in V.

Lemma 4.31 *Let V be a GL_n-representation.*

1. *Let $I \triangleleft \mathbb{C}[V]$ be a GL_n-stable ideal, that is, $g.I \subset I$ for all $g \in GL_n$, then*

$$(\mathbb{C}[V]/I)^{GL_n} \simeq \mathbb{C}[V]^{GL_n}/(I \cap \mathbb{C}[V]^{GL_n}).$$

2. *Let $J \triangleleft \mathbb{C}[V]^{GL_n}$ be an ideal, then we have a lying-over property*

$$J = J\mathbb{C}[V] \cap \mathbb{C}[V]^{GL_n}.$$

   Hence, $\mathbb{C}[V]^{GL_n}$ is Noetherian, that is, every increasing chain of ideals stabilizes.

3. *Let I_j be a family of GL_n-stable ideals of $\mathbb{C}[V]$, then*

$$(\sum_j I_j) \cap \mathbb{C}[V]^{GL_n} = \sum_j (I_j \cap \mathbb{C}[V]^{GL_n}).$$

Proof. *(1) : As I has the induced GL_n-action which is locally finite we have the isotypical decomposition $I = \oplus I_{(s)}$ and clearly $I_{(s)} = \mathbb{C}[V]_{(s)} \cap I$. But then also, taking quotients we have*

$$\oplus_s (\mathbb{C}[V]/I)_{(s)} = \mathbb{C}[V]/I = \oplus_s \mathbb{C}[V]_{(s)}/I_{(s)}.$$

Therefore, $(\mathbb{C}[V]/I)_{(s)} = \mathbb{C}[V]_{(s)}/I_{(s)}$ *and taking the special case* $s = 0$ *is the statement.*

(2) : For any $f \in \mathbb{C}[V]^{GL_n}$ *left-multiplication by* f *in* $\mathbb{C}[V]$ *commutes with the* GL_n*-action, whence* $f.\mathbb{C}[V]_{(s)} \subset \mathbb{C}[V]_{(s)}$*. That is,* $\mathbb{C}[V]_{(s)}$ *is a* $\mathbb{C}[V]^{GL_n}$*-module. But then, as* $J \subset \mathbb{C}[V]^{GL_n}$ *we have*

$$\oplus_s (J\mathbb{C}[V])_{(s)} = J\mathbb{C}[V] = \oplus_s J\mathbb{C}[V]_{(s)}.$$

Therefore, $(J\mathbb{C}[V])_{(s)} = J\mathbb{C}[V]_{(s)}$ *and again taking the special value* $s = 0$ *we obtain* $J\mathbb{C}[V] \cap \mathbb{C}[V]^{GL_n} = (J\mathbb{C}[V])_{(0)} = J$*. The Noetherian statement follows from the fact that* $\mathbb{C}[V]$ *is Noetherian (the Hilbert basis theorem).*

(3) : For any j *we have the decomposition* $I_j = \oplus_s (I_j)_{(s)}$*. But then, we have*

$$\oplus_s (\sum_j I_j)_{(s)} = \sum_j I_j = \sum_j \oplus_s (I_j)_{(s)} = \oplus_s \sum_j (I_j)_{(s)}.$$

Therefore, $(\sum_j I_j)_{(s)} = \sum_j (I_j)_{(s)}$ *and taking* $s = 0$ *gives the required statement.* □

Theorem 4.32 *Let* V *be a* GL_n*-representation. Then, the ring of polynomial invariants* $\mathbb{C}[V]^{GL_n}$ *is an affine* $\mathbb{C}$*-algebra.*

Proof. *Because the action of* GL_n *on* $\mathbb{C}[V]$ *preserves the gradation, the ring of invariants is also graded*

$$\mathbb{C}[V]^{GL_n} = R = \mathbb{C} \oplus R_1 \oplus R_2 \oplus \ldots \quad .$$

From lemma 4.31(2) we know that $\mathbb{C}[V]^{GL_n}$ *is Noetherian and hence the ideal* $R_+ = R_1 \oplus R_2 \oplus \ldots$ *is finitely generated* $R_+ = Rf_1 + \ldots + Rf_l$ *by homogeneous elements* $f_1, \ldots, f_l$*. We claim that as a* $\mathbb{C}$*-algebra* $\mathbb{C}[V]^{GL_n}$ *is generated by the* f_i*. Indeed, we have* $R_+ = \sum_{i=1}^l \mathbb{C}f_i + R_+^2$ *and then also*

$$R_+^2 = \sum_{i,j=1}^{l} \mathbb{C}f_i f_j + R_+^3$$

and iterating this procedure we obtain for all powers m *that*

$$R_+^m = \sum_{\sum m_i = m} \mathbb{C}f_1^{m_1} \ldots f_l^{m_l} + R_+^{m+1}.$$

Now, consider the subalgebra $\mathbb{C}[f_1, \ldots, f_l]$ *of* $R = \mathbb{C}[V]^{GL_n}$*, then we obtain for any power* $d > 0$ *that*

$$\mathbb{C}[V]^{GL_n} = \mathbb{C}[f_1, \ldots, f_l] + R_+^d.$$

For any i *we then have for the homogeneous components of degree* i

$$R_i = \mathbb{C}[f_1, \ldots, f_l]_i + (R_+^d)_i.$$

Now, if $d > i$ *we have that* $(R_+^d)_i = 0$ *and hence that* $R_i = \mathbb{C}[f_1, \ldots, f_l]_i$*. As this holds for all* i *we proved the claim.* □

Choose generating invariants $f_1, \ldots, f_l$ *of* $\mathbb{C}[V]^{GL_n}$*, consider the morphism*

$$V \xrightarrow{\phi} \mathbb{C}^l \qquad \text{defined by} \qquad v \mapsto (f_1(v), \ldots, f_l(v))$$

and define W to be the Zariski closure $\overline{\phi(V)}$ in $\mathbb{C}^l$. Then, we have a diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \mathbb{C}^l \\ & W \arrow[u, hook] \end{tikzcd}$$

and an isomorphism $\mathbb{C}[W] \xrightarrow{\pi^*} \mathbb{C}[V]^{GL_n}$. More general, let X be a closed GL_n-stable subvariety of V, then $X = \mathbb{V}_V(I)$ for some GL_n-stable ideal I of $\mathbb{C}[V]$. From lemma 4.31(1) we obtain

$$\mathbb{C}[X]^{GL_n} = (\mathbb{C}[V]/I)^{GL_n} = \mathbb{C}[V]^{GL_n}/(I \cap \mathbb{C}[V]^{GL_n})$$

whence $\mathbb{C}[X]^{GL_n}$ is also an affine algebra (and generated by the images of the f_i). Define Y to be the Zariski closure of $\phi(X)$ in $\mathbb{C}^l$, then we have a diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \mathbb{C}^l \\ & Y \arrow[u, hook] \end{tikzcd}$$

and an isomorphism $\mathbb{C}[Y] \xrightarrow{\pi} \mathbb{C}[X]^{GL_n}$. We call the morphism $X \xrightarrow{\pi} Y$ an algebraic quotient of X under GL_n. We will now prove some important properties of this quotient.

Proposition 4.33 (universal property) *If $X \xrightarrow{\mu} Z$ is a morphism which is constant along GL_n-orbits in X, then there exists a unique factoring morphism $\overline{\mu}$*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\mu"'] & & Y \arrow[dl, dotted, "\overline{\mu}"] \\ & Z & \end{tikzcd}$$

Proof. *As μ is constant along GL_n-orbits in X, we have an inclusion $\mu^*(\mathbb{C}[Z]) \subset \mathbb{C}[X]^{GL_n}$. We have the commutative diagram*

$$\begin{tikzcd} & \mathbb{C}[X] & \\ & \mathbb{C}[X]^{GL_n} \arrow[u, hook] & \\ \mathbb{C}[Z] \arrow[uur, "\mu^*"] \arrow[ur] \arrow[rr, dotted, "\overline{\mu}^*"'] & & \mathbb{C}[Y] \arrow[uul, "\pi_*"'] \arrow[ul, "\cong"'] \end{tikzcd}$$

from which the existence and uniqueness of $\overline{\mu}$ follows. $\square$

As a consequence, an algebraic quotient is uniquely determined up to isomorphism (that is, we might have started from other generating invariants and still obtain the same quotient variety up to isomorphism).

Proposition 4.34 (onto property) *The algebraic quotient* $X \xrightarrow{\pi} Y$ *is surjective. Moreover, if* $Z \hookrightarrow X$ *is a closed* GL_n*-stable subset, then* $\pi(Z)$ *is closed in* Y *and the morphism*

$$\pi_X \mid Z : Z \longrightarrow \pi(Z)$$

is an algebraic quotient, that is, $\mathbb{C}[\pi(Z)] \simeq \mathbb{C}[Z]^{GL_n}$.

Proof. *Let* $y \in Y$ *with maximal ideal* $M_y \triangleleft \mathbb{C}[Y]$. *By lemma 4.31(2) we have* $M_y\mathbb{C}[X] \neq \mathbb{C}[X]$ *and hence there is a maximal ideal* M_x *of* $\mathbb{C}[X]$ *containing* $M_y\mathbb{C}[X]$, *but then* $\pi(x) = y$. *Let* $Z = \mathbb{V}_X(I)$ *for a* G*-stable ideal* I *of* $\mathbb{C}[X]$, *then* $\overline{\pi(Z)} = \mathbb{V}_Y(I \cap \mathbb{C}[Y])$. *That is,* $\mathbb{C}[\overline{\pi(Z)}] = \mathbb{C}[Y]/(I \cap \mathbb{C}[Y])$. *However, we have from lemma 4.31(1) that*

$$\mathbb{C}[Y]/(\mathbb{C}[Y] \cap I) \simeq (\mathbb{C}[X]/I)^{GL_n} = \mathbb{C}[Z]^{GL_n}$$

and hence $\mathbb{C}[\overline{\pi(Z)}] = \mathbb{C}[Z]^{GL_n}$. *Finally, surjectivity of* $\pi \mid Z$ *is proved as above.* □

An immediate consequence is that the Zariski topology on Y *is the quotient topology of that on* X. *For, take* $U \subset Y$ *with* $\pi^{-1}(U)$ *Zariski open in* X. *Then,* $X - \pi^{-1}(U)$ *is a* GL_n*-stable closed subset of* X. *Then,* $\pi(X - \pi^{-1}(U)) = Y - U$ *is Zariski closed in* Y.

Proposition 4.35 (separation property) *The quotient* $X \xrightarrow{\pi} Y$ *separates disjoint closed* GL_n*-stable subvarieties of* X.

Proof. *Let* Z_j *be closed* GL_n*-stable subvarieties of* X *with defining ideals* $Z_j = \mathbb{V}_X(I_j)$. *Then,* $\cap_j Z_j = \mathbb{V}_X(\sum_j I_j)$. *Applying lemma 4.31(3) we obtain*

$$\begin{aligned}\overline{\pi(\cap_j Z_j)} &= \mathbb{V}_Y((\sum_j I_j) \cap \mathbb{C}[Y]) = \mathbb{V}_Y(\sum_j (I_j \cap \mathbb{C}[Y])) \\ &= \cap_j \mathbb{V}_Y(I_j \cap \mathbb{C}[Y]) = \cap_j \overline{\pi(Z_j)}\end{aligned}$$

The onto property implies that $\overline{\pi(Z_j)} = \pi(Z_j)$ *from which the statement follows.* □

It follows from the universal property that the quotient variety Y *determined by the ring of polynomial invariants* $\mathbb{C}[Y]^{GL_n}$ *is the best algebraic approximation to the orbit space problem. From the separation property a stronger fact follows.*

Proposition 4.36 *The algebraic quotient* $X \xrightarrow{\pi} Y$ *is the best continuous approximation to the orbit space. That is, points of* Y *parameterize the closed* GL_n*-orbits in* X. *In fact, every fiber* $\pi^{-1}(y)$ *contains exactly one closed orbit* C *and we have*

$$\pi^{-1}(y) = \{x \in X \mid C \subset \overline{GL_n.x}\}$$

Proof. *The fiber* $F = \pi^{-1}(y)$ *is a* GL_n*-stable closed subvariety of* X. *Take any orbit* $GL_n.x \subset F$ *then either it is closed or contains in its closure an orbit of strictly smaller dimension. Induction on the dimension then shows that* $\overline{G.x}$ *contains a closed orbit* C. *On the other hand, assume that* F *contains two closed orbits, then they have to be disjoint contradicting the separation property.* □

4.7 Cayley-Hamilton algebras.

A trace map on an (affine) $\mathbb{C}$-algebra A is a $\mathbb{C}$-linear map

$$tr : A \longrightarrow A$$

satisfying the following three properties for all $a, b \in A$:

1. $tr(a)b = btr(a)$,
2. $tr(ab) = tr(ba)$ *and*
3. $tr(tr(a)b) = tr(a)tr(b)$.

Note that it follows from the first property that the image $tr(A)$ of the trace map is contained in the center of A. Consider two algebras A and B equipped with a trace map which we will denote by tr_A respectively tr_B. A trace morphism $\phi : A \longrightarrow B$ will be a $\mathbb{C}$-algebra morphism which is compatible with the trace maps, that is, the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "tr_A"'] & B \arrow[d, "tr_B"] \\ A \arrow[r, "\phi"] & B \end{tikzcd}$$

This definition turns algebras with a trace map into a category. We will say that an algebra A with trace map tr is trace generated by a subset of elements $I \subset A$ if the $\mathbb{C}$-algebra generated by B and $tr(B)$ is equal to A where B is the $\mathbb{C}$-subalgebra generated by the elements of I. Note that A does not have to be generated as a $\mathbb{C}$-algebra by the elements from I.

Observe that for $\mathbb{T}^\infty$ the formal trace $t : \mathbb{T}^\infty \twoheadrightarrow \mathbb{N}^\infty \hookrightarrow \mathbb{T}^\infty$ is a trace map. Property (1) follows because $\mathbb{N}^\infty$ commutes with all elements of $\mathbb{T}^\infty$, property (2) is the cyclic permutation property for t and property (3) is the fact that t is a $\mathbb{N}^\infty$-linear map. The formal trace algebra $\mathbb{T}^\infty$ is trace generated by the variables $\{x_1, x_2, \ldots, x_i, \ldots\}$ but not as a $\mathbb{C}$-algebra.

Actually, $\mathbb{T}^\infty$ is the free algebra in the generators $\{x_1, x_2, \ldots, x_i, \ldots\}$ in the category of algebras with a trace map. That is, if A is an algebra with trace tr which is trace generated by $\{a_1, a_2, \ldots\}$, then there is a trace preserving algebra epimorphism

$$\mathbb{T}^\infty \overset{\pi}{\twoheadrightarrow} A \quad .$$

For example, define $\pi(x_i) = a_i$ and $\pi(t(x_{i_1} \ldots x_{i_l})) = tr(\pi(x_{i_1}) \ldots \pi(x_{i_l}))$. Also, the formal trace algebra $\mathbb{T}^m$, that is the subalgebra of $\mathbb{T}^\infty$ trace generated by $\{x_1, \ldots, x_m\}$, is the free algebra in the category of algebras with trace that are trace generated by at most m elements.

Given a trace map tr on A, we can define for any $a \in A$ a formal Cayley-Hamilton polynomial of degree n. Indeed, express

$$f(t) = \prod_{i=1}^{n} (t - \lambda_i)$$

as a polynomial in t with coefficients polynomial functions in the Newton functions $\sum_{i=1}^n \lambda_i^k$. Replacing the Newton function $\sum \lambda_i^k$ by $tr(a^k)$ we obtain the Cayley-Hamilton polynomial of degree n

$$\chi_a^{(n)}(t) \in A[t] \quad .$$

Definition 4.37 An (affine) $\mathbb{C}$-algebra A with trace map $tr : A \longrightarrow A$ is said to be a *Cayley-Hamilton algebra of degree n* if the following two properties are satisfied :

1. $tr(1) = n$, and
2. For all $a \in A$ we have $\chi_a^{(n)}(a) = 0$ in A.

Observe that if R is a commutative $\mathbb{C}$-algebra, then $M_n(R)$ is a Cayley-Hamilton algebra of degree n. The corresponding trace map is the composition of the usual trace with the inclusion of $R \longrightarrow\!\!\!\!\!\rightarrow M_n(R)$ via scalar matrices. As a consequence, the infinite trace algebra $\mathbb{T}_n^\infty$ has a trace map induced by the natural inclusion

$$\begin{tikzcd}
\mathbb{T}_n^\infty \arrow[r, hook] \arrow[d, dotted, "tr"'] & M_n(\mathbb{C}[M_n^\infty]) \arrow[d, "tr"] \\
\mathbb{N}_n^\infty \arrow[r, hook] & \mathbb{C}[M_n^\infty]
\end{tikzcd}$$

which has image $tr(\mathbb{T}_n^\infty)$ the infinite necklace algebra $\mathbb{N}_n^\infty$. Clearly, being a trace preserving inclusion, $\mathbb{T}_n^\infty$ is a Cayley-Hamilton algebra of degree n. With this definition, we have the following categorical description of the trace algebra $\mathbb{T}_n^\infty$.

Theorem 4.38 *The trace algebra $\mathbb{T}_n^\infty$ is the free algebra in the generic matrix generators $\{X_1, X_2, \ldots, X_i, \ldots\}$ in the category of Cayley-Hamilton algebras of degree n.*

For any m, the trace algebra $\mathbb{T}_n^m$ is the free algebra in the generic matrix generators $\{X_1, \ldots, X_m\}$ in the category of Cayley-Hamilton algebras of degree n which are trace generated by at most m elements.

Proof. *Let F_n be the free algebra in the generators $\{y_1, y_2, \ldots\}$ in the category of Cayley-Hamilton algebras of degree n, then by freeness of $\mathbb{T}^\infty$ there is a trace preserving algebra epimorphism*

$$\mathbb{T}^\infty \xrightarrow{\pi} F_n \quad with \quad \pi(x_i) = y_i.$$

By the universal property of F_n, the ideal $Ker\ \pi$ is the minimal ideal I of $\mathbb{T}^\infty$ such that $\mathbb{T}^\infty/I$ is Cayley-Hamilton of degree n.

We claim that $Ker\ \pi$ is substitution invariant. Consider a substitution endomorphism ϕ of $\mathbb{T}^\infty$ and consider the diagram

$$\begin{tikzcd}
\mathbb{T}^\infty \arrow[r, "\phi"] \arrow[d, dotted] \arrow[dr, dotted, "\chi"] & \mathbb{T}^\infty \arrow[d, two heads, "\pi"] \\
\mathbb{T}^\infty/Ker\ \chi \arrow[r, hook] & F_n
\end{tikzcd}$$

then $Ker\ \chi$ is an ideal closed under traces such that $\mathbb{T}^\infty/Ker\ \chi$ is a Cayley-Hamilton algebra of degree n (being a subalgebra of F_n). But then $Ker\ \pi \subset Ker\ \chi$ (by minimality of $Ker\ \pi$) and therefore χ factors over F_n, that is, the substitution endomorphism ϕ descends to an endomorphism $\overline{\phi} : F_n \longrightarrow F_n$ meaning that $Ker\ \pi$ is left invariant under ϕ, proving the claim. Further, any formal Cayley-Hamilton polynomial $\chi_x^{(n)}(x)$ of degree n of $x \in \mathbb{T}^\infty$ maps to zero under π. By substitution invariance it follows that the ideal of trace relations $Ker\ \tau \subset Ker\ \pi$. We have *seen that $\mathbb{T}^\infty / Ker\ \tau = \mathbb{T}^\infty_n$ is the infinite trace algebra which is a Cayley-Hamilton algebra of degree n. Thus, by minimality of $Ker\ \pi$ we must have $Ker\ \tau = Ker\ \pi$ and hence $F_n \simeq \mathbb{T}^\infty_n$. The second assertion follows immediately.* $\square$

Let A be a Cayley-Hamilton algebra of degree n which is trace generated by the elements $\{a_1, \ldots, a_m\}$. We have a trace preserving algebra epimorphism p_A defined by $p(X_i) = a_i$

$$\begin{tikzcd}
\mathbb{T}^m_n \arrow[r, two heads, "p_a"] \arrow[d, "tr"'] & A \arrow[d, "tr_A"] \\
\mathbb{T}^m_n \arrow[r, two heads, "p_a"] & A
\end{tikzcd}$$

and hence a presentation $A \simeq \mathbb{T}^m_n / T_A$ where $T_A \triangleleft \mathbb{T}^m_n$ is the ideal of trace relations holding among the generators a_i. We recall that $\mathbb{T}^m_n$ is the ring of GL_n-equivariant polynomial maps $M^m_n \xrightarrow{f} M_n$, that is,

$$M_n(\mathbb{C}[M^m_n])^{GL_n} = \mathbb{T}^m_n$$

where the action of GL_n is the diagonal action on $M_n(\mathbb{C}[M^m_n]) = M_n \otimes \mathbb{C}[M^m_n]$.

Observe that if R is a commutative algebra, then any twosided ideal $I \triangleleft M_n(R)$ is of the form $M_n(J)$ for an ideal $J \triangleleft R$. Indeed, the subsets J_{ij} of (i,j) entries of elements of I is an ideal of R as can be seen by multiplication with scalar matrices. Moreover, by multiplying on both sides with permutation matrices one verifies that $J_{ij} = J_{kl}$ for all i,j,k,l proving the claim.

Applying this to the induced ideal $M_n(\mathbb{C}[M^m_n])\ T_A\ M_n(\mathbb{C}[M^m_n]) \triangleleft M_n(\mathbb{C}[M^m_n])$ we find an ideal $N_A \triangleleft \mathbb{C}[M^m_n]$ such that

$$M_n(\mathbb{C}[M^m_n])\ T_A\ M_n(\mathbb{C}[M^m_n]) = M_n(N_A)$$

Observe that both the induced ideal and N_A are stable under the respective GL_n-actions.

Assume that V and W are two (not necessarily finite dimensional) $\mathbb{C}$-vectorspaces with a locally finite GL_n-action and that $V \xrightarrow{f} W$ is a linear map commuting with the GL_n-action. Decomposing V and W in their isotypical components and recalling that $V_{(0)} = V^{GL_n}$ respectively $W_{(0)} = W^{GL_n}$ we obtain a commutative diagram

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "R"'] & W \arrow[d, "R"] \\
V^{GL_n} \arrow[r, "f_0"] & W^G
\end{tikzcd}$$

where R is the Reynolds operator, that is, the canonical projection to the isotypical component of the trivial representation. Clearly, the Reynolds operator commutes with the GL_n-action. Moreover, using complete decomposability we see that f_0 is surjective (resp. injective) if f is surjective (resp. injective).

Because N_A is a GL_n-stable ideal of $\mathbb{C}[M_n^m]$ we can apply the above in the situation

$$\begin{tikzcd}
M_n(\mathbb{C}[M_n^m]) \arrow[r, two heads, "\pi"] \arrow[d, "R"'] & M_n(\mathbb{C}[M_n^m]/N_A) \arrow[d, "R"] \\
\mathbb{T}_n^m \arrow[r, two heads, "\pi_0"] & M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n}
\end{tikzcd}$$

and the bottom map factorizes through $A = \mathbb{T}_n^m/T_A$ giving a surjection

$$A \twoheadrightarrow M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n}.$$

In order to verify that this map is injective (and hence an isomorphism) it suffices to check that

$$M_n(\mathbb{C}[M_n^m])\ T_A\ M_n(\mathbb{C}[M_n^m]) \cap \mathbb{T}_n^m = T_A.$$

Using the functoriality of the Reynolds operator with respect to multiplication in $M_n(\mathbb{C}[M_n^\infty])$ with an element $x \in \mathbb{T}_n^m$ or with respect to the trace map (both commuting with the GL_n-action) we deduce the following relations :

- *For all $x \in \mathbb{T}_n^m$ and all $z \in M_n(\mathbb{C}[M_n^\infty])$ we have $R(xz) = xR(z)$ and $R(zx) = R(z)x$.*
- *For all $z \in M_n(\mathbb{C}[M_n^\infty])$ we have $R(tr(z)) = tr(R(z))$.*

Assume that $z = \sum_i t_i x_i n_i \in M_n(\mathbb{C}[M_n^m])\ T_A\ M_n(\mathbb{C}[M_n^m]) \cap \mathbb{T}_n^m$ with $m_i, n_i \in M_n(\mathbb{C}[M_n^m])$ and $t_i \in T_A$. Now, consider $X_{m+1} \in \mathbb{T}_n^\infty$. Using the cyclic property of traces we have

$$tr(zX_{m+1}) = \sum_i tr(m_i t_i n_i X_{m+1}) = \sum_i tr(n_i X_{m+1} m_i t_i)$$

and if we apply the Reynolds operator to it we obtain the equality

$$tr(zX_{m+1}) = tr(\sum_i R(n_i X_{m+1} m_i) t_i)$$

For any i, the term $R(n_i X_{m+1} m_i)$ is invariant so belongs to $\mathbb{T}_n^{m+1}$ and is linear in X_{m+1}. Knowing the generating elements of $\mathbb{T}_n^{m+1}$ we can write

$$R(n_i X_{m+1} m_i) = \sum_j s_{ij} X_{m+1} t_{ij} + \sum_k tr(u_{ik} X_{m+1}) v_{ik}$$

with all of the elements s_{ij}, t_{ij}, u_{ik} and v_{ik} in $\mathbb{T}_n^m$. Substituting this information and again using the cyclic property of traces we obtain

$$tr(zX_{m+1}) = tr((\sum_{i,j,k} s_{ij} t_{ij} t_i + tr(v_{ik} t_i)) X_{m+1})$$

and by the nondegeneracy of the trace map we again deduce from this the equality

$$z = \sum_{i,j,k} s_{ij} t_{ij} t_i + tr(v_{ik} t_i)$$

Because $t_i \in T_A$ and T_A is stable under taking traces we deduce from this that $z \in T_A$ as required.

Because $A = M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n}$ we can apply functoriality of the Reynolds operator to the setting

$$\begin{tikzcd} M_n(\mathbb{C}[M_n^m]/N_A) \arrow[r, shift left, two heads, "tr"] \arrow[d, "R"'] & \mathbb{C}[M_n]/N_A \arrow[l, shift left, hook'] \arrow[d, "R"] \\ A \arrow[r, shift left, two heads, "tr_A"] & (\mathbb{C}[M_n]/N_A)^{GL_n} \arrow[l, shift left, hook'] \end{tikzcd}$$

Concluding we also have the equality

$$tr_A(A) = (\mathbb{C}[M_n^m]/J_A)^{GL_n}.$$

Summarizing, we have proved the following invariant theoretic reconstruction result for Cayley-Hamilton algebras.

Theorem 4.39 *Let A be a Cayley-Hamilton algebra of degree n, with trace map tr_A, which is trace generated by at most m elements. Then , there is a canonical ideal $N_A \triangleleft \mathbb{C}[M_n^m]$ from which we can reconstruct the algebras A and $tr_A(A)$ as invariant algebras*

$$A = M_n(\mathbb{C}[M_n^m]/N_A)^{GL_n} \quad \textit{and} \quad tr_A(A) = (\mathbb{C}[M_n^m]/N_A)^{GL_n}$$

A direct consequence of the above proof is the universal property of the embedding

$$A \overset{i_A}{\hookrightarrow} M_n(\mathbb{C}[M_n^m]/N_A).$$

Let R be a commutative $\mathbb{C}$-algebra, then $M_n(R)$ with the usual trace is a Cayley-Hamilton algebra of degree n. If $f : A \longrightarrow M_n(R)$ is a trace preserving morphism, we claim that there exists a natural algebra morphism $\overline{f} : \mathbb{C}[M_n^m]/N_A \longrightarrow R$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i_A"'] & M_n(R) \\ M_n(\mathbb{C}[M_n^m]/N_A) \arrow[ur, dotted, "M_n(\overline{f})"'] & \end{tikzcd}$$

where $M_n(\overline{f})$ is the algebra morphism defined entrywise. To see this, consider the composed trace preserving morphism $\phi : \mathbb{T}_n^m \twoheadrightarrow A \xrightarrow{f} M_n(R)$. Its image is fully determined by the images of the trace generators X_k of $\mathbb{T}_n^m$ which are say $m_k = (m_{ij}(k))_{i,j}$. But then we have an algebra morphism $\mathbb{C}[M_n^m] \xrightarrow{g} R$ defined by sending the variable $x_{ij}(k)$ to $m_{ij}(k)$. Clearly, $T_A \subset Ker\ \phi$ and after inducing to $M_n(\mathbb{C}[M_n^m])$ it follows that $N_A \subset Ker\ g$ proving that g factors through $\mathbb{C}[M_n^m]/J_A \longrightarrow R$. This morphism has the required universal property.

4.8 Geometric reconstruction

In this section we will give a geometric interpretation of the reconstruction result. Again, let A be a Cayley-Hamilton algebra of degree n, with trace map tr_A, which is generated by at most m elements $a_1, \ldots, a_m$. We will give a functorial interpretation to the affine scheme determined by the canonical ideal $N_A \triangleleft \mathbb{C}[M_n^m]$. First, let us *identify the reduced affine variety $\mathbb{V}(N_A)$. A point $m = (m_1, \ldots, m_m) \in \mathbb{V}(N_A)$ determines an algebra map $f_m : \mathbb{C}[M_n^m]/N_A \longrightarrow \mathbb{C}$ and hence an algebra map ϕ_m*

$$\begin{tikzcd} A \arrow[r, dotted, "\phi_m"] \arrow[d, hook] & M_n(\mathbb{C}) \\ M_n(\mathbb{C}[M_n^m]/N_A) \arrow[ur, "M_n(f_m)"'] & \end{tikzcd}$$

which is trace preserving. Conversely, from the universal property it follows that any trace preserving algebra morphism $A \longrightarrow M_n(\mathbb{C})$ is of this form by considering the images of the trace generators $a_1, \ldots, a_m$ of A. Alternatively, the points of $\mathbb{V}(N_A)$ parameterize n-dimensional trace preserving representations of A. That is, n-dimensional representations for which the morphism $A \longrightarrow M_n(\mathbb{C})$ describing the action is trace preserving. For this reason we will denote the variety $\mathbb{V}(N_A)$ by $rep_n^{tr}\ A$ and call it the trace preserving reduced representation variety of A.

Assume that A is generated as a $\mathbb{C}$-algebra by $a_1, \ldots, a_m$ (observe that this is no restriction as trace affine algebras are affine) then clearly $I_A(n) \subset N_A$. That is,

Lemma 4.40 *For A a Cayley-Hamilton algebra of degree n generated by $\{a_1, \ldots, a_m\}$, the reduced trace preserving representation variety*

$$rep_n^{tr}\ A \hookrightarrow rep_n\ A$$

is a closed subvariety of the reduced representation variety.

It is easy to determine the additional defining equations. For, write any trace monomial out in the generators

$$tr_A(a_{i_1} \ldots a_{i_k}) = \sum \alpha_{j_1 \ldots j_l} a_{j_1} \ldots a_{j_l}$$

then for a point $m = (m_1, \ldots, m_m) \in rep_n\ A$ to belong to $rep_n^{tr}\ A$, it must satisfy all the relations of the form

$$tr(m_{i_1} \ldots m_{i_k}) = \sum \alpha_{j_1 \ldots j_l} m_{j_1} \ldots m_{j_l}$$

with tr the usual trace on $M_n(\mathbb{C})$. These relations define the closed subvariety $rep_n^{tr}(A)$. Usually, this is a proper subvariety.

Example 4.41 Let A be a finite dimensional semi-simple algebra $A = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C})$, then A has precisely k distinct simple modules $\{M_1, \ldots, M_k\}$ of dimensions $\{d_1, \ldots, d_k\}$. Here, M_i can be viewed as column vectors of size d_i on which the component $M_{d_i}(\mathbb{C})$ acts by left multiplication and the other factors act as zero. Because A is semi-simple every n-dimensional A-representation M is isomorphic to

$$M = M_1^{\oplus e_1} \oplus \ldots \oplus M_k^{\oplus e_k}$$

for certain multiplicities e_i satisfying the numerical condition

$$n = e_1 d_1 + \ldots + e_k d_k$$

That is, $rep_n\ A$ is the disjoint union of a finite number of (closed) orbits each determined by an integral vector $(e_1, \ldots, e_k)$ satisfying the condition called the *dimension vector* of M.

$$rep_n\ A \simeq \bigsqcup_{(e_1,\ldots,e_k)} GL_n/(GL_{e_1} \times \ldots GL_{e_k})$$

Let $f_i \geq 1$ be natural numbers such that $n = f_1 d_1 + \ldots f_k d_k$ and consider the embedding of A into $M_n(\mathbb{C})$ defined by

$$(a_1, \ldots, a_k) \in A \longrightarrow \begin{bmatrix} \underbrace{\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_1 \end{bmatrix}}_{f_1} & & \\ & \ddots & \\ & & \overbrace{\begin{bmatrix} a_k & & 0 \\ & \ddots & \\ 0 & & a_k \end{bmatrix}}^{f_k} \end{bmatrix} \in M_n(\mathbb{C})$$

Via this embedding, A becomes a Cayley-Hamilton algebra of degree n when equipped with the induced trace tr from $M_n(\mathbb{C})$.

Let M be the n-dimensional A-representation with dimension vector $(e_1, \ldots, e_k)$ and choose a basis compatible with this decomposition. Let E_i be the idempotent of A corresponding to the identity matrix I_{d_i} of the i-th factor. Then, the trace of the matrix defining the action of E_i on M is clearly $e_i d_i . I_n$. On the other hand, $tr(E_i) = f_i d_i . I_n$, hence the only trace preserving n-dimensional A-representation is that of dimension vector $(f_1, \ldots, f_k)$. Therefore, $rep_n^{tr}\ A$ consists of the single closed orbit determined by the integral vector $(f_1, \ldots, f_k)$.

$$rep_n^{tr}\ A \simeq GL_n/(GL_{f_1} \times \ldots \times GL_{f_k})$$

Consider the scheme structure of the trace preserving representation variety $\underline{rep}_n^{tr}\ A$. The corresponding functor assigns to a commutative affine $\mathbb{C}$-algebra R

$$\underline{rep}_n^{tr}(R) = Alg_{\mathbb{C}}(\mathbb{C}[M_n^m]/N_A, R).$$

An algebra morphism $\psi : \mathbb{C}[M_n^m]/N_A \longrightarrow R$ determines uniquely an m-tuple of $n \times n$ matrices with coefficients in R by

$$r_k = \begin{bmatrix} \psi(x_{11}(k)) & \ldots & \psi(x_{1n}(k)) \\ \vdots & & \vdots \\ \psi(x_{n1}(k)) & \ldots & \psi(x_{nn}(k)) \end{bmatrix}$$

Composing with the canonical embedding

$$\begin{tikzcd} A \arrow[r, dotted, "\phi"] \arrow[d, hook] & M_n(R) \\ M_n(\mathbb{C}[M_n^m]/N_A) \arrow[ur, "M_n(\psi)"'] & \end{tikzcd}$$

determines the trace preserving algebra morphism $\phi : A \longrightarrow M_n(R)$ where the trace map on $M_n(R)$ is the usual trace. By the universal property any trace preserving map $A \longrightarrow M_n(R)$ is also of this form.

Lemma 4.42 *Let A be a Cayley-Hamilton algebra of degree n which is generated by $\{a_1, \ldots, a_m\}$. The trace preserving representation variety $\underline{rep}_n^{tr}\ A$ represents the functor*

$$\underline{rep}_n^{tr}\ A(R) = \{A \xrightarrow{\phi} M_n(R) \mid \phi \text{ is trace preserving } \}$$

Moreover, $\underline{rep}_n^{tr}\ A$ is a closed subscheme of $\underline{rep}_n\ A$.

Recall that there is an action of GL_n on $\mathbb{C}[M_n^m]$ and from the definition of the ideals $I_A(n)$ and N_A it is clear that they are stable under the GL_n-action. That is, there is an action by automorphisms on the quotient algebras $\mathbb{C}[M_n^m]/I_A(n)$ and $\mathbb{C}[M_n^m]/N_A$. But then, their algebras of invariants are equal to

$$\begin{cases} \mathbb{C}[\underline{rep}_n\ A]^{GL_n} & = (\mathbb{C}[M_n^m]/I_A(n))^{GL_n} = \frac{\mathbb{N}_n^m}{(I_A(n)\cap\mathbb{N}_n^m)} \\ \mathbb{C}[\underline{rep}_n^{tr}\ A]^{GL_n} & = (\mathbb{C}[M_n^m]/N_A)^{GL_n} = \frac{\mathbb{N}_n^m}{(N_A\cap\mathbb{N}_n^m)} \end{cases}$$

That is, these rings of invariants define closed subschemes of the affine (reduced) variety associated to the necklace algebra $\mathbb{N}_n^m$. We will call these schemes the quotient schemes for the action of GL_n and denote them respectively by

$$\underline{iss}_n\ A = \underline{rep}_n\ A/GL_n \quad \text{and} \quad \underline{iss}_n^{tr}\ A = \underline{rep}_n^{tr}\ A/GL_n.$$

We have seen that the geometric points of the reduced variety $iss_n\ A$ of the affine quotient scheme $\underline{iss}_n\ A$ parameterize the isomorphism classes of n-dimensional semisimple A-representations. Similarly, the geometric points of the reduced variety $iss_n^{tr}\ A$ of the quotient scheme $\underline{iss}_n^{tr}\ A$ parameterize isomorphism classes of trace preserving n-dimensional semisimple A-representations.

Proposition 4.43 *Let A be a Cayley-Hamilton algebra of degree n with trace map tr_A. Then, we have that*

$$tr_A(A) = \mathbb{C}[\underline{iss}_n^{tr}\ A],$$

the coordinate ring of the quotient scheme $\underline{iss}_n^{tr}\ A$. In particular, maximal ideals of $tr_A(A)$ parameterize the isomorphism classes of trace preserving n-dimensional semi-simple A-representations.

By definition, a GL_n-equivariant map between the affine GL_n-schemes

$$\underline{rep}_n^{tr}\ A \xrightarrow{f} M_n = \underline{M}_n$$

means that for any commutative affine $\mathbb{C}$-algebra R the corresponding map

$$\underline{rep}_n^{tr}\ A(R) \xrightarrow{f(R)} M_n(R)$$

commutes with the action of $GL_n(R)$. Alternatively, the ring of all morphisms $\underline{rep}_n^{tr}\ A \longrightarrow M_n$ is the matrixalgebra $M_n(\mathbb{C}[M_n^m]/N_A)$ and those that commute with the GL_n action are precisely the invariants. That is, we have the following description of A.

Proposition 4.44 *Let A be a Cayley-Hamilton algebra of degree n with trace map tr_A. Then, we can recover A as the ring of GL_n-equivariant maps*

$$A = \{ f : \underline{rep}_n^{tr}\ A \longrightarrow M_n \quad \textit{equivariant} \ \}$$

of affine GL_n-schemes.

Summarizing the results of this and the previous section we have

Theorem 4.45 *The functor which assigns to a Cayley-Hamilton algebra A of degree n the GL_n-affine scheme $\underline{rep}_n^{tr}\ A$ of trace preserving n-dimensional representations has a left inverse.*

This left inverse functor assigns to a GL_n-affine scheme $\underline{X}$ its witness algebra $M_n(\mathbb{C}[\underline{X}])^{GL_n}$ which is a Cayley-Hamilton algebra of degree n.

Note however that this functor is not an equivalence of categories. For, there are many affine GL_n-schemes having the same witness algebra.

Example 4.46 Consider the action of GL_n on M_n by conjugation and take a nilpotent matrix A. All eigenvalues of A are zero, so the conjugacy class of A is fully determined by the sizes of its Jordan blocks. These sizes determine a partition $\lambda(A) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. Moreover, we have given an algorithm to determine whether an orbit $\mathcal{O}(B)$ of another nilpotent matrix B is contained in the orbit closure $\overline{\mathcal{O}(A)}$, the criterium being that

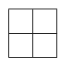
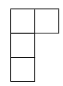
$$\mathcal{O}(B) \subset \overline{\mathcal{O}(A)} \Longleftrightarrow \lambda(B)^* \geq \lambda(A)^*.$$

where λ^* denotes the dual partition. We see that the witness algebra of $\overline{\mathcal{O}(A)}$ is equal to

$$M_n(\mathbb{C}[\overline{\mathcal{O}(A)}])^{GL_n} = \mathbb{C}[X]/(X^k)$$

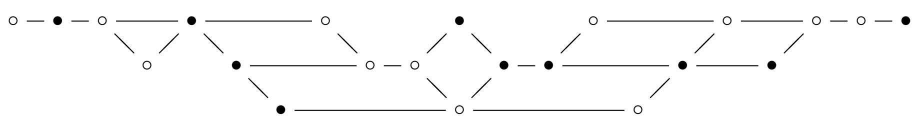
where k is the number of columns of the Young diagram $\lambda(A)$.

Hence, the orbit closures of nilpotent matrices such that their associated Young diagrams have equal number of columns have the same witness algebras. For example, if $n = 4$ then the closures of the orbits corresponding to

 and 

have the same witness algebra, although the closure of the second is a proper closed subscheme of the closure of the first.

Recall the orbitclosure diagram of conjugacy classes of nilpotent 8×8 matrices given by the Gerstenhaber-Hesselink theorem. In the picture below, the closures of orbits corresponding to connected nodes of the same colour have the same witness algebra.



Chapter 5

Etale Slices.

Let A be an affine $\mathbb{C}$-algebra. In the foregoing chapter we have found a geometric reconstruction of the approximation at level n of A

$$\begin{cases} A@_n & \simeq M_n(\mathbb{C}[\underline{rep}_n\ A])^{GL_n} \\ t(A@_n) & \simeq \mathbb{C}[\underline{rep}_n\ A]^{GL_n} = \mathbb{C}[\underline{iss}_n\ A] \end{cases}$$

In this chapter we will use the GL_n-geometry to determine the étale local structure of the Cayley-Hamilton algebra $A@_n$. By this we mean the following. Let $\mathfrak{m}$ be a maximal ideal of the central subalgebra $t(A@_n)$, then we want to determine the $\mathfrak{m}$-adic completion

$$\widehat{(A@_n)}_{\mathfrak{m}}$$

of $A@_n$. We know that $\mathfrak{m}$ determines a point ξ in the quotient variety $\underline{iss}_n\ A$ and so there is an n-dimensional semi-simple representation M_ξ of A with decomposition

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i are distinct simple A-representations of dimension d_i and occurring in M_ξ with multiplicity e_i, in particular $n = \sum_{i=1}^k d_i e_i$.

To determine the local structure of $A@_n$ in $\mathfrak{m}$ we determine the GL_n-local structure of $\underline{rep}_n\ A$ in a neighborhood of the closed orbit $\mathcal{O}(M_\xi)$. This can be done by the theory of Luna's étale slices, or rather by the elegant extension of it to not necessarily reduced varieties, due to F. Knop [?], whose proof we will outline in section 4. When $\underline{rep}_n\ A$ is smooth in M_ξ (which is always the case when A is Quillen-smooth) then this local structure is determined by the normal space to the orbit, considered as a module over the stabilizer subgroup.

In the case of representation varieties, this normal space can be identified with the vectorspace of self-extensions $Ext^1_A(M_\xi, M_\xi)$ and the stabilizer subgroup with the centralizer of V_ξ. The main result we will prove in this chapter is that this local data is encoded in a quiver-setting, or rather a marked quiver-setting where we allow some loops in the quiver to acquire a mark. We will prove that this marked quiver has k vertices (corresponding to the distinct simple components of M_ξ) and we have to consider α-dimensional representations of this quiver where $\alpha = (e_1, \ldots, e_k)$, the multiplicities with which these simples occur in M_ξ. In the next chapter we will show that the arrows and loops in the quiver are determined by (trace preserving) self-extensions of M_ξ.

The étale local structure of $A@_n$ in $\mathfrak{m}$ is then given by

$$\widehat{(A@_n)}_{\mathfrak{m}} \simeq \widehat{\mathbb{T}_\alpha\ Q_\xi}$$

where Q_ξ is the local quiver determined by M_ξ, $\mathbb{T}_\alpha\ Q_\xi$ is the ring of $GL(\alpha)$-equivariant maps from $rep_\alpha\ Q_\xi$ to $M_n(\mathbb{C})$ and we take its completion at the maximal graded ideal of the corresponding ring of invariants.

5.1 $\mathcal{C}^\infty$ slices.

Let A be an affine $\mathbb{C}$-algebra and $\xi \in \underline{iss}_n\ A$ a point in the quotient space corresponding to an n-dimensional semi-simple representation M_ξ of A. In this chapter we will present a method to study the étale local structure of $\underline{iss}_n\ A$ near ξ and the étale local GL_n-structure of the representation variety $\underline{rep}_n\ A$ near the closed orbit $\mathcal{O}(M_\xi) = GL_n.M_\xi$. In this section we will outline the basic idea in the setting of differential geometry.

Let M be a compact $\mathcal{C}^\infty$-manifold on which a compact Lie group G acts differentially. By a usual averaging process we can define a G-invariant Riemannian metric on M. For a point $m \in M$ we define

- *The G-orbit $\mathcal{O}(m) = G.m$ of m in M,*
- *the stabilizer subgroup $H = Stab_G(m) = \{g \in G \mid g.m = m\}$ and*
- *the normal space N_m defined to be the orthogonal complement to the tangent space in m to the orbit in the tangent space to M. That is, we have a decomposition of H-vectorspaces*

$$T_m\ M = T_m\ \mathcal{O}(m) \oplus N_m$$

The normal spaces N_x when x varies over the points of the orbit $\mathcal{O}(m)$ define a vectorbundle $\mathcal{N} \overset{p}{\twoheadrightarrow} \mathcal{O}(m)$ over the orbit. We can identify the bundle with the associated fiber bundle

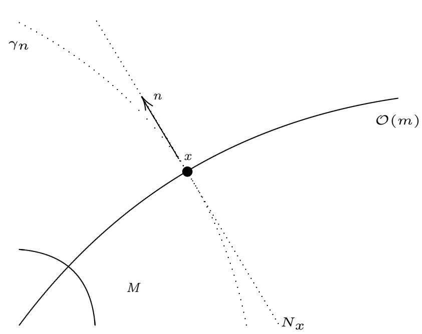
$$\mathcal{N} \simeq G \times^H N_m$$

Any point $n \in \mathcal{N}$ in the normal bundle determines a geodesic

$$\gamma_n : \mathbb{R} \longrightarrow M \quad \textit{defined by} \quad \begin{cases} \gamma_n(0) & = p(n) \\ \frac{d\gamma_n}{dt}(0) & = n \end{cases}$$

Using this geodesic we can define a G-equivariant exponential map from the normal bundle $\mathcal{N}$ to the manifold M via

$$\mathcal{N} \xrightarrow{exp} M \qquad \textit{where} \qquad exp(n) = \gamma_n(1)$$



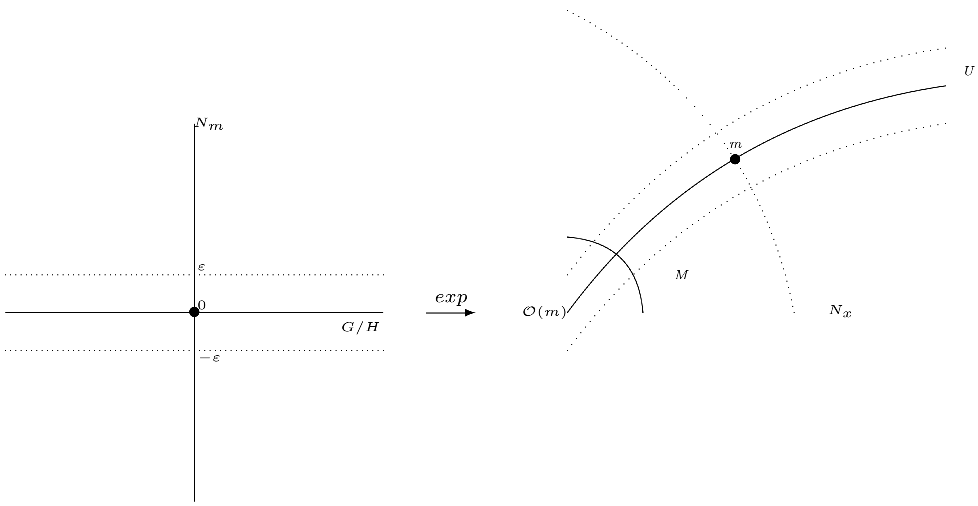
Now, take $\varepsilon > 0$ and define the $\mathcal{C}^\infty$ slice S_ε to be

$$S_\varepsilon = \{n \in N_m \mid \ \parallel n \parallel < \varepsilon\ \}$$

then $G \times^H S_\varepsilon$ *is a neighborhood of the zero section in the normal bundle* $\mathcal{N} = G \times^H N_m$. *But then we have a G-equivariant exponential*

$$G \times^H S_\varepsilon \xrightarrow{exp} M$$

which for small enough ε *gives a diffeomorphism with a G-stable tubular neighborhood* U *of the orbit* $\mathcal{O}(m)$ *in* M.



If we assume moreover that the action of G *on* M *and the action of* H *on* N_m *are such that the orbit-spaces are manifolds* M/G *and* N_m/H, *then we have the situation*

$$\begin{tikzcd}
G \times^H S_\varepsilon \arrow[r, "exp", "\simeq"'] \arrow[d] & U \arrow[r, hook] \arrow[d] & M \arrow[d] \\
S_\varepsilon/H \arrow[r, "\simeq"'] & U/G \arrow[r, hook] & M/G
\end{tikzcd}$$

giving a local diffeomorphism between a neighborhood of $\overline{0}$ *in* N_m/H *and a neighborhood of the point* $\overline{m}$ *in* M/G *corresponding to the orbit* $\mathcal{O}(m)$.

Returning to the setting of the orbit $\mathcal{O}(M_\xi)$ *in* $\underline{rep}_n\ A$ *we would equally like to define a* GL_n*-equivariant morphism from an associated fiber bundle*

$$GL_n \times^{GL(\alpha)} N_\xi \xrightarrow{e} \underline{rep}_n\ A$$

where $GL(\xi)$ *is the stabilizer subgroup of* M_ξ *and* N_ξ *is a normal space to the orbit* $\mathcal{O}(M_\xi)$. *Because we do not have an exponential-map in the setting of algebraic geometry, the map e will have to be an étale map. Before we come to the description of these étale slices we will first study the tangent spaces to* $\underline{rep}_n\ A$ *and give a ringtheoretical interpretation of the normal space* N_ξ.

5.2 Tangent spaces.

Let $\underline{X}$ *be a not necessarily reduced affine variety with coordinate ring* $\mathbb{C}[\underline{X}] = \mathbb{C}[x_1, \ldots, x_n]/I$. *If the origin* $o = (0, \ldots, 0) \in \mathbb{V}(I)$, *elements of* I *have no constant terms and we can write any* $p \in I$ *as*

$$p = \sum_{i=1}^{\infty} p^{(i)} \qquad \text{with } p^{(i)} \text{ homogeneous of degree } i.$$

The order $ord(p)$ is the least integer $r \geq 1$ such that $p^{(r)} \neq 0$. Define the following two ideals in $\mathbb{C}[x_1, \ldots, x_n]$

$$I_l = \{p^{(1)} \mid p \in I\} \quad \text{and } I_m = \{p^{(r)} \mid p \in I \quad \text{and } ord(p) = r\}.$$

The subscripts l (respectively m) stand for linear terms (respectively, terms of minimal degree).

The tangent space to $\underline{X}$ in o, $T_o(\underline{X})$ is by definition the subscheme of $\mathbb{C}^n$ determined by I_l. Observe that

$$I_l = (a_{11}x_1 + \ldots + a_{1n}x_n, \ldots, a_{l1}x_1 + \ldots + a_{ln}x_n)$$

for some $l \times n$ matrix $A = (a_{ij})_{i,j}$ of rank l. That is, we can express all x_k as linear combinations of some $\{x_{i_1}, \ldots, x_{i_{n-l}}\}$, but then clearly

$$\mathbb{C}[T_o(\underline{X})] = \mathbb{C}[x_1, \ldots, x_n]/I_l = \mathbb{C}[x_{i_1}, \ldots, x_{i_{n-l}}]$$

In particular, $T_o(\underline{X})$ is reduced and is a linear subspace of dimension $n - l$ in $\mathbb{C}^n$ through the point o.

Next, consider an arbitrary geometric point x of $\underline{X}$ with coordinates $(a_1, \ldots, a_n)$. We can translate x to the origin o and the translate of $\underline{X}$ is then the scheme defined by the ideal

$$(f_1(x_1 + a_1, \ldots, x_n + a_n), \ldots, f_k(x_1 + a_1, \ldots, x_n + a_n))$$

Now, the linear term of the translated polynomial $f_i(x_1 + a_1, \ldots, x_n + a_n)$ is equal to

$$\frac{\partial f_i}{\partial x_1}(a_1, \ldots, a_n)x_1 + \ldots + \frac{\partial f_i}{\partial x_n}(a_1, \ldots, a_n)x_n$$

and hence the tangent space to $\underline{X}$ in x, $T_x(\underline{X})$ is the linear subspace of $\mathbb{C}^n$ defined by the set of zeroes of the linear terms

$$T_x(\underline{X}) = \mathbb{V}(\sum_{j=1}^n \frac{\partial f_1}{\partial x_j}(x)x_j, \ldots, \sum_{j=1}^n \frac{\partial f_k}{\partial x_j}(x)x_j) \hookrightarrow \mathbb{C}^n.$$

In particular, the dimension of this linear subspace can be computed from the Jacobian matrix in x associated with the polynomials $(f_1, \ldots, f_k)$

$$dim\ T_x(\underline{X}) = n - rk \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(x) & \ldots & \frac{\partial f_k}{\partial x_n}(x) \end{bmatrix}.$$

We now give an alternative description of the tangent spaces using the associated functor of $\underline{X}$. Let $\mathbb{C}[\varepsilon]$ be the algebra of dual numbers, that is, $\mathbb{C}[\varepsilon] \simeq \mathbb{C}[y]/(y^2)$. Consider a $\mathbb{C}$-algebra morphism

$$\mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\phi} \mathbb{C}[\varepsilon] \quad \text{defined by } x_i \mapsto a_i + c_i\varepsilon.$$

Because $\varepsilon^2 = 0$ it is easy to verify that the image of a polynomial $f(x_1, \ldots, x_n)$ under ϕ is of the form

$$\phi(f(x_1, \ldots, x_n)) = f(a_1, \ldots, a_n) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a_1, \ldots, a_n)c_j\varepsilon$$

Therefore, ϕ factors through I, that is $\phi(f_i) = 0$ for all $1 \leq i \leq k$, if and only if $(c_1, \ldots, c_n) \in T_x(\underline{X})$. Hence, we can also identify the tangent space to $\underline{X}$ in x

with the algebra morphisms $\mathbb{C}[\underline{X}] \xrightarrow{\phi} \mathbb{C}[\varepsilon]$ *whose composition with the projection* $\pi : \mathbb{C}[\varepsilon] \twoheadrightarrow \mathbb{C}$ *(sending* ε *to zero) is the evaluation in* $x = (a_1, \ldots, a_n)$*. That is, let* $ev_x \in \underline{X}(\mathbb{C})$ *be the point corresponding to evaluation in* x*, then*

$$T_x(\underline{X}) = \{\phi \in \underline{X}(\mathbb{C}[\varepsilon]) \mid \underline{X}(\pi)(\phi) = ev_x\}.$$

Example 5.1 $\underline{GL}_n(\mathbb{C}[\varepsilon])$ is the group of invertible $n \times n$ matrices with coefficients in $\mathbb{C}[\varepsilon]$. By the above we have for any $g \in GL_n$ that

$$T_g(\underline{GL}_n) = \{m \in M_n(\mathbb{C}) \mid g + m\varepsilon \text{ is invertible in } M_n(\mathbb{C}[\varepsilon])\ \} = M_n(\mathbb{C})$$

because $(g + m\varepsilon)^{-1} = g^{-1} - g^{-1}.m.g^{-1}\varepsilon$ for any $m \in M_n(\mathbb{C})$. This computation is consistent with the observation that GL_n is an open subset of M_n. For any affine algebraic group scheme $\underline{G}$ one defines the *Lie algebra* $\mathfrak{g}$ of $\underline{G}$ to be the tangentspace $T_e(\underline{G})$ at $\underline{G}$ in the neutral element e. In particular, the Lie algebra $\mathfrak{gl}_n$ of $\underline{GL}_n$ is the vectorspace $M_n(\mathbb{C})$.

The following two examples compute the tangent spaces to the (trace preserving) representation varieties.

Example 5.2 Let A be an affine $\mathbb{C}$-algebra generated by $\{a_1, \ldots a_m\}$ and $\rho : A \longrightarrow M_n(\mathbb{C})$ an algebra morphism, that is, $\rho \in rep_n\ A$. We call a linear map $A \xrightarrow{D} M_n(\mathbb{C})$ a *ρ-derivation* if and only if for all $a, a' \in A$ we have that

$$D(aa') = D(a).\rho(a') + \rho(a).D(a').$$

We denote the vectorspace of all ρ-derivations of A by $Der_\rho(A)$. Observe that any ρ-derivation is determined by its image on the generators a_i, hence $Der_\rho(A) \subset M_n^m$. We claim that

$$T_\rho(\underline{rep}_n\ A) = Der_\rho(A).$$

Indeed, we know that $\underline{rep}_n\ A(\mathbb{C}[\varepsilon])$ is the set of algebra morphisms

$$A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$$

By the functorial characterization of tangentspaces we have that $T_\rho(\underline{rep}_n\ A)$ is equal to

$$\{D : A \longrightarrow M_n(\mathbb{C}) \text{ linear} \mid \rho + D\varepsilon : A \longrightarrow M_n(\mathbb{C}[\varepsilon]) \text{ is an algebra map}\}.$$

Because ρ is an algebra morphism, the algebra map condition

$$\rho(aa') + D(aa')\varepsilon = (\rho(a) + D(a)\varepsilon).(\rho(a') + D(a')\varepsilon)$$

is equivalent to D being a ρ-derivation.

Example 5.3 Let A be a Cayley-Hamilton algebra of degree n with trace map tr_A and trace generated by $\{a_1, \ldots, a_m\}$. Let $\rho \in rep_n^{tr}\ A$, that is, $\rho : A \longrightarrow M_n(\mathbb{C})$ is a *trace preserving* algebra morphism. Because $\underline{rep}_n^{tr}\ A(\mathbb{C}[\varepsilon])$ is the set of all trace preserving algebra morphisms $A \longrightarrow M_n(\mathbb{C}[\varepsilon])$ (with the usual trace map tr on $M_n(\mathbb{C}[\varepsilon])$) and the previous example one verifies that

$$T_\rho(\underline{rep}_n^{tr}\ A) = Der_\rho^{tr}(A) \subset Der_\rho(A)$$

the subset of *trace preserving ρ-derivations* D, that is, those satisfying

$$D \circ tr_A = tr \circ D \qquad \begin{array}{ccc} A & \xrightarrow{D} & M_n(\mathbb{C}) \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr} \\ A & \xrightarrow{D} & M_n(\mathbb{C}) \end{array}$$

Again, using this property and the fact that A is *trace* generated by $\{a_1, \ldots, a_m\}$ a trace preserving ρ-derivation is determined by its image on the a_i so is a subspace of M_n^m.

The tangent cone to $\underline{X}$ *in* o*,* $TC_o(\underline{X})$*, is by definition the subscheme of* $\mathbb{C}^n$ *determined by* I_m*, that is,*

$$\mathbb{C}[TC_o(\underline{X})] = \mathbb{C}[x_1, \ldots, x_n]/I_m.$$

It is called a cone because if c is a point of the underlying variety of $TC_o(\underline{X})$, then the line $l = \overrightarrow{oc}$ is contained in this variety because I_m is a graded ideal. Further, observe that as $I_l \subset I_m$, the tangent cone is a closed subscheme of the tangent space at $\underline{X}$ in o. Again, if x is an arbitrary geometric point of $\underline{X}$ we define the tangent cone to $\underline{X}$ in x, $TC_x(\underline{X})$ as the tangent cone $TC_o(\underline{X}')$ where $\underline{X}'$ is the translated scheme of $\underline{X}$ under the translation taking x to o.

Both the tangent space and tangent cone contain local information of the scheme $\underline{X}$ in a neighborhood of x. We will now present a ringtheoretical description of both using only the local algebra $\mathcal{O}_x(\underline{X})$ of $\underline{X}$ in x. These descriptions have the additional advantage of providing a description of tangent space and tangent cone independent of the embedding of $\underline{X}$.

Let m_x be the maximal ideal of $\mathbb{C}[\underline{X}]$ corresponding to x (that is, the ideal of polynomial functions vanishing in x). Then, its complement $\mathcal{S}_x = \mathbb{C}[\underline{X}] - m_x$ is a multiplicatively closed subset the local algebra $\mathcal{O}_x(\underline{X})$ is the corresponding localization $\mathbb{C}[\underline{X}]_{\mathcal{S}_x}$. It has a unique maximal ideal $\mathfrak{m}_x$ with residue field $\mathcal{O}_x(\underline{X})/\mathfrak{m}_x = \mathbb{C}$. We equip the local algebra $\mathcal{O}_x = \mathcal{O}_x(\underline{X})$ with the $\mathfrak{m}_x$-adic filtration that is the $\mathbb{Z}$-filtration

$$\mathcal{F}_x : \qquad ... \subset \mathfrak{m}^i \subset \mathfrak{m}^{i-1} \subset \ldots \subset \mathfrak{m} \subset \mathcal{O}_x = \mathcal{O}_x = \ldots = \mathcal{O}_x = \ldots$$

with associated graded algebra

$$gr(\mathcal{O}_x) = \qquad \ldots \oplus \frac{\mathfrak{m}_x^i}{\mathfrak{m}_x^{i+1}} \oplus \frac{\mathfrak{m}_x^{i-1}}{\mathfrak{m}_x^i} \oplus \ldots \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \oplus \mathbb{C} \oplus 0 \oplus \ldots \oplus 0 \oplus \ldots$$

Proposition 5.4 *If x is a geometric point of the affine scheme $\underline{X}$, then*

1. *$\mathbb{C}[T_x(\underline{X})]$ is isomorphic to the polynomial algebra $\mathbb{C}[\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}]$.*
2. *$\mathbb{C}[TC_x(\underline{X})]$ is isomorphic to the associated graded algebra $gr(\mathcal{O}_x(\underline{X}))$.*

Proof. *After translating we may assume that $x = o$ lies in $\mathbb{V}(I) \hookrightarrow \mathbb{C}^n$. That is,*

$$\mathbb{C}[\underline{X}] = \mathbb{C}[x_1, \ldots, x_n]/I \quad \textit{and} \quad m_x = (x_1, \ldots, x_n)/I.$$

(1) *: Under these identifications we have*

$$\begin{aligned} \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} &\simeq \frac{m_x}{m_x^2} \\ &\simeq \frac{(x_1,\ldots,x_n)}{(x_1,\ldots,x_n)^2+I} \\ &\simeq \frac{(x_1,\ldots,x_n)}{(x_1,\ldots,x_n)^2+I_l} \end{aligned}$$

and as I_l is generated by linear terms it follows that the polynomial algebra on $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ is isomorphic to the quotient algebra $\mathbb{C}[x_1, \ldots, x_n]/I_l$ which is by definition the coordinate ring of the tangent space.
(2) *: Again using the above identifications we have*

$$\begin{aligned} gr(\mathcal{O}_x) &\simeq \oplus_{i=0}^{\infty} \frac{\mathfrak{m}_x^i}{\mathfrak{m}_x^{i+1}} \\ &\simeq \oplus_{i=0}^{\infty} \frac{m_x^i}{m_x^{i+1}} \\ &\simeq \oplus_{i=0}^{\infty} \frac{(x_1,\ldots,x_n)^i}{(x_1,\ldots,x_n)^{i+1}+(I\cap(x_1,\ldots,x_n)^i)} \\ &\simeq \oplus_{i=0}^{\infty} \frac{(x_1,\ldots,x_n)^i}{(x_1,\ldots,x_n)^{i+1}+I_m(i)} \end{aligned}$$

where $I_m(i)$ is the homogeneous part of I_m of degree i. On the other hand, the i-th homogeneous part of $\mathbb{C}[x_1,\ldots,x_n]/I_m$ is equal to

$$\frac{(x_1,\ldots,x_n)^i}{(x_1,\ldots,x_n)^{i+1}+I_m(i)}$$

we obtain the required isomorphism. □

This gives a third interpretation of the tangent space as

$$T_x(\underline{X}) = Hom_{\mathbb{C}}(\frac{m_x}{m_x^2},\mathbb{C}) = Hom_{\mathbb{C}}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2},\mathbb{C}).$$

Hence, we can also view the tangent space $T_x(\underline{X})$ as the space of point derivations $Der_x(\mathcal{O}_x)$ on $\mathcal{O}_x(\underline{X})$ (or of the point derivations $Der_x(\mathbb{C}[\underline{X}])$ on $\mathbb{C}[\underline{X}]$). That is, $\mathbb{C}$-linear maps $D : \mathcal{O}_x \longrightarrow \mathbb{C}$ (or $D : \mathbb{C}[\underline{X}] \longrightarrow \mathbb{C}$) such that for all functions f,g we have

$$D(fg) = D(f)g(x) + f(x)D(g).$$

If we define the local dimension of an affine scheme $\underline{X}$ in a geometric point x, $dim_x\ \underline{X}$ to be the maximal dimension of irreducible components of the reduced variety X passing through x, then

$$dim_x\ \underline{X} = dim_o\ TC_x(\underline{X}).$$

We say that $\underline{X}$ is nonsingular at x (or equivalently, that x is a nonsingular point of $\underline{X}$) if the tangent cone to $\underline{X}$ in x coincides with the tangent space to $\underline{X}$ in x. An immediate consequence is

Proposition 5.5 *If $\underline{X}$ is nonsingular at x, then $\mathcal{O}_x(\underline{X})$ is a domain. That is, in a Zariski neighborhood of x , $\underline{X}$ is an irreducible variety.*

Proof. *If $\underline{X}$ is nonsingular at x, then*

$$gr(\mathcal{O}_x) \simeq \mathbb{C}[TC_x(\underline{X})] = \mathbb{C}[T_x(\underline{X})]$$

the latter one being a polynomial algebra whence a domain. Now, let $0 \neq a,b \in \mathcal{O}_x$ then there exist k,l such that $a \in \mathfrak{m}^k - \mathfrak{m}^{k+1}$ and $b \in \mathfrak{m}^l - \mathfrak{m}^{l+1}$, that is $\overline{a}$ is a nonzero homogeneous element of $gr(\mathcal{O}_x)$ of degree $-k$ and $\overline{b}$ one of degree $-l$. But then, $\overline{a}.\overline{b} \in \mathfrak{m}^{k+l} - \mathfrak{m}^{k+l-1}$ hence certainly $a.b \neq 0$ in $\mathcal{O}_x$.

Now, consider the natural map $\phi : \mathbb{C}[\underline{X}] \longrightarrow \mathcal{O}_x$. Let $\{P_1,\ldots,P_l\}$ be the minimal prime ideals of $\mathbb{C}[\underline{X}]$. All but one of them, say $P_1 = \phi^{-1}(0)$, extend to the whole ring $\mathcal{O}_x$. Taking the product of f functions $f_i \in P_i$ nonvanishing in x for $2 \leq i \leq l$ gives the Zariski open set $\mathbb{X}(f)$ containing x and whose coordinate ring is a domain, whence $\mathbb{X}(f)$ is an affine irreducible variety. □

When restricting to nonsingular points we reduce to irreducible affine varieties. From the Jacobian condition it follows that nonsingularity is a Zariski open condition on $\underline{X}$ and by the implicit function theorem $\underline{X}$ is a complex manifold in a neighborhood of a nonsingular point.

5.3 Normal spaces.

Let $\underline{X} \xrightarrow{\phi} \underline{Y}$ be a morphism of affine varieties corresponding to the algebra morphism $\mathbb{C}[\underline{Y}] \xrightarrow{\phi^*} \mathbb{C}[\underline{X}]$. Let x be a geometric point of $\underline{X}$ and $y = \phi(x)$. As

$\phi^*(m_y) \subset m_x$, ϕ induces a linear map $\frac{m_y}{m_y^2} \longrightarrow \frac{m_x}{m_x^2}$ and taking the dual map gives the differential of ϕ in x which is a linear map

$$d\phi_x : T_x(\underline{X}) \longrightarrow T_{\phi(x)}(\underline{Y}).$$

Assume $\underline{X}$ a closed subscheme of $\mathbb{C}^n$ and $\underline{Y}$ a closed subscheme of $\mathbb{C}^m$ and let ϕ be determined by the m polynomials $\{f_1, \ldots, f_m\}$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then, the Jacobian matrix in x

$$J_x(\phi) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_m}{\partial x_1}(x) \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

defines a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ and the differential $d\phi_x$ is the induced linear map from $T_x(\underline{X}) \subset \mathbb{C}^n$ to $T_{\phi(x)}(\underline{Y}) \subset \mathbb{C}^m$. Let $D \in T_x(\underline{X}) = Der_x(\mathbb{C}[\underline{X}])$ and x_D the corresponding element of $\underline{X}(\mathbb{C}[\varepsilon])$ defined by $x_D(f) = f(x) + D(f)\varepsilon$, then $x_D \circ \phi^* \in \underline{Y}(\mathbb{C}[\varepsilon])$ is defined by

$$x_D \circ \phi^*(g) = g(\phi(x)) + (D \circ \phi^*)\varepsilon = g(\phi(x)) + d\phi_x(D)\varepsilon$$

giving us the ε-interpretation of the differential

$$\phi(x + v\varepsilon) = \phi(x) + d\phi_x(v)\varepsilon$$

for all $v \in T_x(\underline{X})$.

Proposition 5.6 *Let $X \xrightarrow{\phi} Y$ be a dominant morphism between irreducible affine varieties. There is a Zariski open dense subset $U \hookrightarrow X$ such that $d\phi_x$ is surjective for all $x \in U$.*

Proof. *We may assume that ϕ factorizes into*

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, "\phi"'] & Y \times \mathbb{C}^d \arrow[d, "pr_Y"] \\ & Y \end{tikzcd}$$

with ϕ a finite and surjective morphism. Because the tangent space of a product is the sum of the tangent spaces of the components we have that $d(pr_W)_z$ is surjective for all $z \in Y \times \mathbb{C}^d$, hence it suffices to verify the claim for a finite morphism ϕ. That is, we may assume that $S = \mathbb{C}[Y]$ is a finite module over $R = \mathbb{C}[X]$ and let L/K be the corresponding extension of the function fields. By the principal element theorem we know that $L = K[s]$ for an element $s \in L$ which is integral over R with minimal polynomial

$$F = t^n + g_{n-1}t^{n-1} + \ldots + g_1 t + g_0 \qquad \text{with } g_i \in R$$

Consider the ring $S' = R[t]/(F)$ then there is an element $r \in R$ such that the localizations S'_r and S_r are isomorphic. By restricting we may assume that $X = \mathbb{V}(F) \hookrightarrow Y \times \mathbb{C}$ and that

$$\begin{tikzcd} X = \mathbb{V}(F) \arrow[r, hook] \arrow[dr, "\phi"'] & Y \times \mathbb{C} \arrow[d, "pr_Y"] \\ & Y \end{tikzcd}$$

Let $x = (y, c) \in X$ then we have (again using the identification of the tangent space of a product with the sum of the tangent spaces of the components) that

$$T_x(X) = \{(v, a) \in T_y(Y) \oplus \mathbb{C} \mid c\frac{\partial F}{\partial t}(x) + vg_{n-1}c^{n-1} + \ldots + vg_1c + vg_0 = 0\}.$$

But then, $d\phi_x$ i surjective whenever $\frac{\partial F}{\partial t}(x) \neq 0$. This condition determines a non-empty open subset of X as otherwise $\frac{\partial F}{\partial t}$ would belong to the defining ideal of X in $\mathbb{C}[Y \times \mathbb{C}]$ (which is the principal ideal generated by F) which is impossible by a degree argument $\square$

Example 5.7 Let $\underline{X}$ be a closed GL_n-stable subscheme of a GL_n-representation V and x a geometric point of $\underline{X}$. Consider the orbitclosure $\overline{\mathcal{O}(x)}$ of x in V. Because the orbit map

$$\mu : GL_n \twoheadrightarrow GL_n.x \hookrightarrow \overline{\mathcal{O}(x)}$$

is dominant we have that $\mathbb{C}[\overline{\mathcal{O}(x)}] \hookrightarrow \mathbb{C}[GL_n]$ and therefore a domain, so $\overline{\mathcal{O}(x)}$ is an irreducible affine variety. We define the *stabilizer subgroup* $Stab(x)$ to be the fiber $\mu^{-1}(x)$, then $Stab(x)$ is a closed subgroup of GL_n. We claim that the differential of the orbit map in the identity matrix $e = \mathbb{1}_n$

$$d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\underline{X})$$

satisfies the following properties

$$Ker\ d\mu_e = \mathfrak{stab}(x) \quad \text{and} \quad Im\ d\mu_e = T_x(\overline{\mathcal{O}(x)}).$$

By the proposition we know that there is a dense open subset U of GL_n such that $d\mu_g$ is surjective for all $g \in U$. By GL_n-equivariance of μ it follows that $d\mu_g$ is surjective for all $g \in GL_n$, in particular $d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\overline{\mathcal{O}(x)})$ is surjective. Further, all fibers of μ over $\mathcal{O}(x)$ have the same dimension. But then it follows from the *dimension formula* of proposition that

$$dim\ GL_n = dim\ Stab(x) + dim\ \overline{\mathcal{O}(x)}$$

(which, incidentally gives us an algorithm to compute the dimensions of orbitclosures). Combining this with the above surjectivity, a dimension count proves that $Ker\ d\mu_e = \mathfrak{stab}(x)$, the Lie algebra of $Stab(x)$.

Let M and N two A-representations of dimensions say m and n. An A-representation P of dimension $m+n$ is said to be an extension of N by M if there exists a short exact sequence of left A-modules

$$e : \quad 0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

We define an equivalence relation on extensions (P, e) of N by M : $(P, e) \cong (P', e')$ if and only if there is an isomorphism $P \xrightarrow{\phi} P'$ of left A-modules such that the diagram below is commutative

$$\begin{array}{lccccccccc}
e: & 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle id_M} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id_N} & & \\
e': & 0 & \longrightarrow & M & \longrightarrow & P' & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

The set of equivalence classes of extensions of N by M will be denoted by $Ext^1_A(N, M)$.

An alternative description of $Ext^1_A(N, M)$ is as follows. Let $\rho : A \longrightarrow M_m$ and $\sigma : A \longrightarrow M_n$ be the representations defining M and N. For an extension (P, e) we can identify the $\mathbb{C}$-vectorspace with $M \oplus N$ and the A-module structure on P gives a algebra map $\mu : A \longrightarrow M_{m+n}$ and we can represent the action of a on P by left multiplication of the block-matrix

$$\mu(a) = \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix},$$

where $\lambda(a)$ is an $m \times n$ matrix and hence defines a linear map

$$\lambda : A \longrightarrow Hom_{\mathbb{C}}(N, M).$$

The condition that μ is an algebra morphism is equivalent to the condition

$$\lambda(aa') = \rho(a)\lambda(a') + \lambda(a)\sigma(a')$$

and we denote the set of all liner maps $\lambda : A \longrightarrow Hom_{\mathbb{C}}(N, M)$ by $Z(N, M)$ and call it the space of cycles. (Observe already that if $M = N$ and $m = n$ then $Z(M, M)$ is the vectorspace of ρ-derivations $Der_\rho(A)$ from $A \longrightarrow M_n$.)

The extensions of N by M corresponding to two cycles λ and λ' from $Z(N, M)$ are equivalent if and only if we have an A-module isomorphism in block form

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{with } \beta \in Hom_{\mathbb{C}}(N, M)$$

between them. A-linearity of this map translates into the matrix relation

$$\begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} . \begin{bmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{bmatrix} = \begin{bmatrix} \rho(a) & \lambda'(a) \\ 0 & \sigma(a) \end{bmatrix} . \begin{bmatrix} id_M & \beta \\ 0 & id_N \end{bmatrix} \qquad \text{for all } a \in A$$

or equivalently, that $\lambda(a) - \lambda'(a) = \rho(a)\beta - \beta\sigma(a)$ for all $a \in A$. We will now define the subspace of $Z(N, M)$ of boundaries $B(N, M)$

$$\{\delta \in Hom_{\mathbb{C}}(N, M) \mid \exists \beta \in Hom_{\mathbb{C}}(N, M) : \forall a \in A : \delta(a) = \rho(a)\beta - \beta\sigma(a)\}.$$

We then have the description $Ext^1_A(N, M) = \frac{Z(N,M)}{B(N,M)}$.

Example 5.8 Let A be an affine $\mathbb{C}$-algebra generated by $\{a_1, \ldots, a_m\}$ and $\rho : A \longrightarrow M_n(\mathbb{C})$ an algebra morphism, that is, $\rho \in rep_n\ A$ determines an n-dimensional A-representation M. We claim to have the following description of the *normal space* to the orbitclosure $C_\rho = \overline{\mathcal{O}(\rho)}$ of ρ

$$N_\rho(\underline{rep}_n\ A) \overset{def}{=} \frac{T_\rho(\underline{rep}_n\ A)}{T_\rho(C_\rho)} = Ext^1_A(M, M).$$

We have already seen that the space of cycles $Z(M, M)$ is the space of ρ-derivations of A in $M_n(\mathbb{C})$, $Der_\rho(A)$, which we know to be the tangent space $T_\rho(\underline{rep}_n\ A)$. Moreover, we know that the differential $d\mu_e$ of the orbit map $GL_n \xrightarrow{\mu} C_\rho \hookrightarrow M_n^m$

$$d\mu_e : \quad \mathfrak{gl}_n = M_n \longrightarrow T_\rho(C_\rho)$$

is surjective. Now, $\rho = (\rho(a_1), \ldots, \rho(a_m)) \in M_n^m$ and the action of action of GL_n is given by simultaneous conjugation. But then we have for any $A \in \mathfrak{gl}_n = M_n$ that

$$(I_n + A\varepsilon).\rho(a_i).(I_n - A\varepsilon) = \rho(a_i) + (A\rho(a_i) - \rho(a_i)A)\varepsilon.$$

Therefore, by definition of the differential we have that

$$d\mu_e(A)(a) = A\rho(a) - \rho(a)A \quad \text{ for all } a \in A.$$

that is, $d\mu_e(A) \in B(M, M)$ and as the differential map is surjective we have $T_\rho(C_\rho) = B(M, M)$ from which the claim follows.

Example 5.9 Let A be a Cayley-Hamilton algebra with trace map tr_A and trace generated by $\{a_1, \ldots, a_m\}$. Let $\rho \in rep_n^{tr}\ A$, that is, $\rho : A \longrightarrow M_n(\mathbb{C})$ is a trace preserving algebra morphism. Any cycle $\lambda : A \longrightarrow M_n(\mathbb{C})$ in $Z(M, M) = Der_\rho(A)$ determines an algebra morphism

$$\rho + \lambda\varepsilon : A \longrightarrow M_n(\mathbb{C}[\varepsilon])$$

We know that the tangent space $T_\rho(\underline{rep}_n^{tr}\ A)$ is the subspace $Der_\rho^{tr}(A)$ of trace preserving ρ-derivations, that is, those satisfying

$$\lambda(tr_A(a)) = tr(\lambda(a)) \quad \text{ for all } a \in A$$

Observe that for all boundaries $\delta \in B(M,M)$, that is, such that there is an $m \in M_n(\mathbb{C})$ with $\delta(a) = \rho(a).m - m.\rho(a)$ are trace preserving as

$$\begin{aligned}\delta(tr_A(a)) = & \quad \rho(tr_A(a)).m - m.\rho(tr_A(a)) = tr(\rho(a)).m - m.tr(\rho(a)) \\ = & \quad 0 = tr(m.\rho(a) - \rho(a).m) = tr(\delta(a))\end{aligned}$$

Hence, we can define the space of *trace preserving self-extensions*

$$Ext^{tr}_A(M,M) = \frac{Der^{tr}_\rho(A)}{B(M,M)}$$

and obtain as before that the *normal space* to the orbit closure $C_\rho = \overline{\mathcal{O}(\rho)}$ is equal to

$$N_\rho(\underline{rep}^{tr}_n\ A) \overset{def}{=} \frac{T_\rho(\underline{rep}^{tr}_n\ A)}{T_\rho(C_\rho)} = Ext^{tr}_A(M,M)$$

5.4 Luna's étale slices.

The results of this section hold for any reductive algebraic group G. As we will use them only in the case $G = GL_n$ or $GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k}$ we will restrict to the case of GL_n. Also all affine GL_n-varieties we will consider are representation varieties or associated fiber bundles. We fix the setting : $\underline{X}$ and $\underline{Y}$ are not necessarily reduced affine GL_n-varieties, ψ is a GL_n-equivariant map

$$\begin{array}{ccccccc} x = \psi(y) & \vdash & \underline{X} & \xleftarrow{\psi} & \underline{Y} & \dashv & y \\ & & \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} & & \\ & & \underline{X}/GL_n & & \underline{Y}/GL_n & & \end{array}$$

and we assume the following restrictions :

- *ψ is étale in y,*
- *the GL_n-orbits $\mathcal{O}(y)$ in $\underline{Y}$ and $\mathcal{O}(x)$ in $\underline{X}$ are closed. That is, in representation varieties we restrict to semi-simple representations,*
- *the stabilizer subgroups are equal $Stab(x) = Stab(y)$. In the case of representation varieties, for a semi-simple n-dimensional representation with decomposition*

$$M = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

into distinct simple components, this stabilizer subgroup is

$$GL(\alpha) = \begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{f_1}) & & \\ & \ddots & \\ & & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{f_k}) \end{bmatrix} \hookrightarrow GL_n$$

where $f_i = dim\ S_i$. In particular, the stabilizer subgroup is again reductive.

In algebraic terms : consider the coordinate rings $R = \mathbb{C}[\underline{X}]$ and $S = \mathbb{C}[\underline{Y}]$ and the dual morphism $R \xrightarrow{\psi^*} S$. Let $I \triangleleft R$ be the ideal describing $\mathcal{O}(x)$ and $J \triangleleft S$ the ideal describing $\mathcal{O}(y)$. With $\widehat{R}$ we will denote the I-adic completion $\underset{\leftarrow}{lim}\ \frac{R}{I^n}$ of R and with $\widehat{S}$ the J-adic completion of S.

Lemma 5.10 *The morphism ψ^* induces for all n an isomorphism*

$$\frac{R}{I^n} \xrightarrow{\psi^*} \frac{S}{J^n}$$

In particular, $\widehat{R} \simeq \widehat{S}$.

Proof. *Let $\underline{Z}$ be the closed GL_n-stable subvariety of $\underline{Y}$ where ψ is not étale. By the separation property, there is an invariant function $f \in S^{GL_n}$ vanishing on $\underline{Z}$ such that $f(y) = 1$ because the two closed GL_n-subschemes $\underline{Z}$ and $\mathcal{O}(y)$ are disjoint. Replacing S by S_f we may assume that ψ^* is an étale morphism. Because $\mathcal{O}(x)$ is smooth, $\psi^{-1}\ \mathcal{O}(x)$ is the disjoint union of its irreducible components and restricting $\underline{Y}$ if necessary we may assume that $\psi^{-1}\ \mathcal{O}(x) = \mathcal{O}(y)$. But then $J = \psi^*(I)S$ and as $\mathcal{O}(y) \xrightarrow{\simeq} \mathcal{O}(x)$ we have $\frac{R}{I} \simeq \frac{S}{J}$ so the result holds for $n = 1$.*

Because étale maps are flat, we have $\psi^*(I^n)S = I^n \otimes_R S = J^n$ and an exact sequence

$$0 \longrightarrow I^{n+1} \otimes_R S \longrightarrow I^n \otimes_R S \longrightarrow \frac{I^n}{I^{n+1}} \otimes_R S \longrightarrow 0$$

But then we have

$$\frac{I^n}{I^{n+1}} = \frac{I^n}{I^{n+1}} \otimes_{R/I} \frac{S}{J} = \frac{I^n}{I^{n+1}} \otimes_R S \simeq \frac{J^n}{J^{n+1}}$$

and the result follows from induction on n and the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{I^n}{I^{n+1}} \arrow[r] \arrow[d, "\simeq"'] & \frac{R}{I^{n+1}} \arrow[r] \arrow[d, dotted] & \frac{R}{I^n} \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \frac{J^n}{J^{n+1}} \arrow[r] & \frac{S}{J^{n+1}} \arrow[r] & \frac{S}{J^n} \arrow[r] & 0
\end{tikzcd}$$

□

As in the previous chapter we will denote for any irreducible GL_n-representation s and any locally finite GL_n-module X its s-isotypical component by $X_{(s)}$.

Lemma 5.11 *Let s be an irreducible GL_n-representation. There are natural numbers $m \geq 1$ (independent of s) and $n \geq 0$ such that for all $k \in \mathbb{N}$ we have*

$$I^{mk+n} \cap R_{(s)} \hookrightarrow (I^{GL_n})^k R_{(s)} \hookrightarrow I^k \cap R_{(s)}$$

Proof. *Consider $A = \oplus_{i=0}^{\infty} I^n t^n \hookrightarrow R[t]$, then A^{GL_n} is affine so certainly finitely generated as R^{GL_n}-algebra say by*

$$\{r_1 t^{m_1}, \ldots, r_z t^{m_z}\} \quad \text{with } r_i \in R \text{ and } m_i \geq 1.$$

Further, $A_{(s)}$ is a finitely generated A^{GL_n}-module, say generated by

$$\{s_1 t^{n_1}, \ldots, s_y t^{n_y}\} \quad \text{with } s_i \in R_{(s)} \text{ and } n_i \geq 0.$$

Take $m = max\ m_i$ and $n = max\ n_i$ and $r \in I^{mk+n} \cap R_{(s)}$, then $rt^{mk+n} \in A_{(s)}$ and

$$rt^{mk+n} = \sum_j p_j(r_1 t^{m_1}, \ldots, r_z t^{m_z}) s_j t^{n_j}$$

with p_j a homogeneous polynomial of t-degree $mk + n - n_j \geq mk$. But then each monomial in p_j occurs at least with ordinary degree $\frac{mk}{m} = k$ and therefore is contained in $(I^{GL_n})^k R_{(s)} t^{mk+n}$. □

Let $\widehat{R^{GL_n}}$ be the I^{GL_n}-adic completion of the invariant ring R^{GL_n} and let $\widehat{S^{GL_n}}$ be the J^{GL_n}-adic completion of S^{GL_n}.

Lemma 5.12 *The morphism ψ^* induces an isomorphism*

$$R \otimes_{R^{GL_n}} \widehat{R^{GL_n}} \xrightarrow{\simeq} S \otimes S^{GL_n} \widehat{S^{GL_n}}$$

Proof. *Let s be an irreducible GL_n-module, then the I^{GL_n}-adic completion of $R_{(s)}$ is equal to $\widehat{R_{(s)}} = R_{(s)} \otimes_{R^{GL_n}} \widehat{R^{GL_n}}$. Moreover,*

$$\widehat{R}_{(s)} = \underset{\leftarrow}{lim}(\frac{R}{I^k})_{(s)} = \underset{\leftarrow}{lim}\frac{R_{(s)}}{(I^k \cap R_{(s)})}$$

which is the I-adic completion of $R_{(s)}$. By the foregoing lemma both topologies coincide on $R_{(s)}$ and therefore

$$\widehat{R_{(s)}} = \widehat{R}_{(s)} \quad \text{and similarly} \quad \widehat{S_{(s)}} = \widehat{S}_{(s)}$$

Because $\widehat{R} \simeq \widehat{S}$ it follows that $\widehat{R}_{(s)} \simeq \widehat{S}_{(s)}$ from which the result follows as the foregoing holds for all s. □

Theorem 5.13 (Luna's fundamental lemma) *Consider a GL_n-equivariant map $\underline{Y} \xrightarrow{\psi} \underline{X}$, $y \in \underline{Y}$, $x = \psi(y)$ and ψ étale in y. Assume that the orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are closed and that ψ is injective on $\mathcal{O}(y)$. Then, there is an affine open subset $U \hookrightarrow \underline{Y}$ containing y such that*

1. *$U = \pi_Y^{-1}(\pi_Y(U))$ and $\pi_Y(U) = U/GL_n$.*
2. *ψ is étale on U with affine image.*
3. *The induced morphism $U/GL_n \xrightarrow{\overline{\psi}} \underline{X}/GL_n$ is étale.*
4. *The diagram below is commutative*

$$\begin{tikzcd}
U \arrow[r, "\psi"] \arrow[d, "\pi_U"'] & \underline{X} \arrow[d, "\pi_X"] \\
U/GL_n \arrow[r, "\overline{\psi}"] & \underline{X}/GL_n
\end{tikzcd}$$

Proof. *By the foregoing lemma we have $\widehat{R^{GL_n}} \simeq \widehat{S^{GL_n}}$ which means that $\overline{\psi}$ is étale in $\pi_Y(y)$. As étaleness is an open condition, there is an open affine neighborhood V of $\pi_Y(y)$ on which $\overline{\psi}$ is étale. If $\overline{R} = R \otimes_{R^{GL_n}} S^{GL_n}$ then the above lemma implies that*

$$\overline{R} \otimes_{S^{GL_n}} \widehat{S^{GL_n}} \simeq S \otimes_{S^{GL_n}} \widehat{S^{GL_n}}$$

Let $S^{GL_n}_{loc}$ be the local ring of S^{GL_n} in J^{GL_n}, then as the morphism $S^{GL_n}_{loc} \longrightarrow \widehat{S^{GL_n}}$ is faithfully flat we deduce that

$$\overline{R} \otimes_{S^{GL_n}} S^{GL_n}_{loc} \simeq S \otimes_{S^{GL_n}} S^{GL_n}_{loc}$$

but then there is an $f \in S^{GL_n} - J^{GL_n}$ such that $\overline{R}_f \simeq S_f$. Now, intersect V with the open affine subset where $f \neq 0$ and let U' be the inverse image under π_Y of this set. Remains to prove that the image of ψ is affine. As $U' \xrightarrow{\psi} X$ is étale, its image is open and GL_n-stable. By the separation property we can find an invariant $h \in R^{GL_n}$ such that h is zero on the complement of the image and $h(x) = 1$. But then we take U to be the subset of U' of points u such that $h(u) \neq 0$. □

Theorem 5.14 (Luna's slice theorem) *Let $\underline{X}$ be an affine GL_n-variety with quotient map $\underline{X} \overset{\pi}{\twoheadrightarrow} \underline{X}/GL_n$. Let $x \in X$ be such that its orbit $\mathcal{O}(x)$ is closed and its stabilizer subgroup $Stab(x) = H$ is reductive. Then, there is a locally closed affine subscheme $\underline{S} \hookrightarrow \underline{X}$ containing x with the following properties*

1. *$\underline{S}$ is an affine H-variety,*
2. *the action map $GL_n \times \underline{S} \longrightarrow \underline{X}$ induces an étale GL_n-equivariant morphism*
$$GL_n \times^H \underline{S} \xrightarrow{\psi} \underline{X}$$
with affine image,
3. *the induced quotient map ψ/GL_n is étale*
$$(GL_n \times^H \underline{S})/GL_n \simeq \underline{S}/H \xrightarrow{\psi/GL_n} \underline{X}/GL_n$$
4. *the diagram below is commutative*
$$\begin{tikzcd} GL_n \times^H \underline{S} \arrow[r, "\psi"] \arrow[d] & \underline{X} \arrow[d, "\pi"] \\ \underline{S}/H \arrow[r, "\psi/GL_n"] & \underline{X}/GL_n \end{tikzcd}$$

If we assume moreover that $\underline{X}$ is smooth in x, then we can choose the slice $\underline{S}$ such that also the following properties are satisfied

1. *$\underline{S}$ is smooth,*
2. *there is an H-equivariant morphism*
$$\underline{S} \xrightarrow{\phi} T_x\ \underline{S} = N_x$$
with $\phi(x) = 0$ having an affine image,
3. *the induced morphism is étale*
$$\underline{S}/H \xrightarrow{\phi/H} N_x/H$$
4. *the diagram below is commutative*
$$\begin{tikzcd} GL_n \times^H \underline{S} \arrow[r, "GL_n \times^H \phi"] \arrow[d] & GL_n \times^H N_x \arrow[d] \\ \underline{S}/H \arrow[r, "\phi/H"] & N_x/H \end{tikzcd}$$

Proof. *Choose a finite dimensional GL_n-subrepresentation V of $\mathbb{C}[\underline{X}]$ that generates the coordinate ring as algebra. This gives a GL_n-equivariant embedding*
$$\underline{X} \overset{i}{\hookrightarrow} W = V^*$$

Choose in the vectorspace W an H-stable complement S_0 of $\mathfrak{gl}_n.i(x) = T_{i(x)}\ \mathcal{O}(x)$ and denote $S_1 = i(x)+S_0$ and $\underline{S}_2 = i^{-1}(S_1)$. Then, the diagram below is commutative

$$\begin{tikzcd} GL_n \times^H \underline{S}_2 \arrow[r, hook] \arrow[d, "\psi"'] & GL_n \times^H S_1 \arrow[d, "\psi_0"] \\ \underline{X} \arrow[r, hook, "i"] & W \end{tikzcd}$$

By construction we have that ψ_0 induces an isomorphism between the tangent spaces in $\overline{(1, i(x))} \in GL_n \times^H S_0$ and $i(x) \in W$ which means that ψ_0 is étale in $i(x)$, whence ψ is étale in $\overline{(1,x)} \in GL_n \times^H \underline{S}_2$. By the fundamental lemma we ge an affine neighborhood U which must be of the form $U = GL_n \times^H \underline{S}$ giving a slice $\underline{S}$ with the required properties.

Assume that $\underline{X}$ is smooth in x, then S_1 is transversal to $\underline{X}$ in $i(x)$ as

$$T_{i(x)}\ i(\underline{X}) + S_0 = W$$

Therefore, $\underline{S}$ is smooth in x. Again using the separation property we can find an invariant $f \in \mathbb{C}[\underline{S}]^H$ such that f is zero on the singularities of $\underline{S}$ (which is a H-stable closed subscheme) and $f(x) = 1$. Then replace $\underline{S}$ with its affine reduced subvariety of points s such that $f(s) \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}[\underline{S}]$ in x, then we have an exact sequence of H-modules

$$0 \longrightarrow \mathfrak{m}^2 \longrightarrow \mathfrak{m} \xrightarrow{\alpha} N_x^* \longrightarrow 0$$

Choose a H-equivariant section $\phi^* : N_x^* \longrightarrow \mathfrak{m} \hookrightarrow \mathbb{C}[\underline{S}]$ of α then this gives an H-equivariant morphism $\underline{S} \xrightarrow{\phi} N_x$ which is étale in x. Applying again the fundamental lemma to this setting finishes the proof. □

5.5 Grothendieck smoothness.

In this section we prove that an affine variety $\underline{X}$ is smooth if and only if its coordinate ring $\mathbb{C}[\underline{X}]$ satisfies a certain lifting property in the category of all commutative $\mathbb{C}$-algebras. This allows us to define formally smooth algebras in other categories such as the category of Cayley-Hamilton algebras of degree n or the category of all $\mathbb{C}$-algebras.

Let $\underline{X}$ be a possibly non-reduced affine variety and x a geometric point of $\underline{X}$. As we are interested in local properties of $\underline{X}$ near x, we may assume (after translation) that $x = o$ in $\mathbb{C}^n$ and that we have a presentation

$$\mathbb{C}[\underline{X}] = \mathbb{C}[x_1, \ldots, x_n]/I \quad \textit{with } I = (f_1, \ldots, f_m) \textit{ and } m_x = (x_1, \ldots, x_n)/I.$$

Denote the polynomial algebra $P = \mathbb{C}[x_1, \ldots, x_n]$ and consider the map

$$d \quad : I \longrightarrow (Pdx_1 \oplus \ldots \oplus Pdx_n) \otimes_P \mathbb{C}[\underline{X}] = \mathbb{C}[\underline{X}]dx_1 \oplus \ldots \oplus \mathbb{C}[\underline{X}]dx_n$$

where the dx_i are a formal basis of the free module of rank n and the map is defined by

$$d(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \textit{ mod } I.$$

This gives a $\mathbb{C}[\underline{X}]$-linear mapping

$$\frac{I}{I^2} \xrightarrow{d} \mathbb{C}[\underline{X}]dx_1 \oplus \ldots \oplus \mathbb{C}[\underline{X}]dx_n.$$

Extending to the local algebra $\mathcal{O}_x$ at x and then quotient out the maximal ideal $\mathfrak{m}_x$ we get a $\mathbb{C} = \mathcal{O}_x/\mathfrak{m}_x$- linear map

$$\frac{\mathfrak{I}}{\mathfrak{I}^2} \xrightarrow{d(x)} \mathbb{C}dx_1 \oplus \ldots \oplus \mathbb{C}dx_n$$

Clearly, x is a nonsingular point of $\underline{X}$ if and only if the $\mathbb{C}$-linear map $d(x)$ is injective. This is equivalent to the existence of a $\mathbb{C}$-section and by the Nakayama lemma also to the existence of a $\mathcal{O}_x$-linear splitting s_x of the induced $\mathcal{O}_x$-linear map d_x

$$\begin{tikzcd} \frac{\mathfrak{I}}{\mathfrak{I}^2} \arrow[r, hook, shift left, "d_x"] & \mathcal{O}_x dx_1 \oplus \ldots \oplus \mathcal{O}_x dx_n \arrow[l, two heads, shift left, "s_x"] \end{tikzcd}$$

satisfying $s_x \circ d_x = id_{\frac{\mathfrak{I}}{\mathfrak{I}^2}}$

A $\mathbb{C}$-algebra epimorphism (between commutative algebras) $R \overset{\pi}{\twoheadrightarrow} S$ with square zero kernel is called an infinitesimal extension of S. It is called a trivial infinitesimal extension if π has an algebra section $\sigma : S \hookrightarrow R$ satisfying $\pi \circ \sigma = id_S$. An infinitesimal extension $R \overset{\pi}{\twoheadrightarrow} S$ of S is said to be versal if for any other infinitesimal extension $R' \overset{\pi'}{\twoheadrightarrow} S$ of S there is a $\mathbb{C}$-algebra morphism

$$\begin{tikzcd} R \arrow[rr, two heads, "\pi"] \arrow[dr, dotted, "\exists\, g"'] & & S \\ & R' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

making the diagram commute. From this universal property it is clear that versal infinitesimal extensions are uniquely determined up to isomorphism. Moreover, if a versal infinitesimal extension is trivial, then so is any infinitesimal extension.

Definition 5.15 A commutative $\mathbb{C}$-algebra S is said to be *Grothendieck smooth* if and only if it has the following universal property. Let T be a commutative $\mathbb{C}$-algebra and I a nilpotent ideal of T. Then, any $\mathbb{C}$-algebra morphism $\kappa : S \longrightarrow T/I$

$$\begin{tikzcd} & S \arrow[dl, dotted, "\exists\, \lambda"'] \arrow[d, "\kappa"] \\ T \arrow[r, two heads] & T/I \end{tikzcd}$$

can be lifted to a $\mathbb{C}$-algebra morphism $\lambda : S \longrightarrow T$ making the diagram commutative.

Clearly, by iterating, S is Grothendieck smooth if and only if it has the lifting property with respect to nilpotent ideals I with square zero. Therefore, assume we have a test object (T, I) with $I^2 = 0$, then we have a commuting diagram

$$\begin{tikzcd} S \times_{T/I} T \arrow[r, two heads, "pr_1"] \arrow[d, two heads, "pr_2"'] & S \arrow[d, "\kappa"] \\ T \arrow[r, two heads, "p"'] & T/I \end{tikzcd}$$

where we define the pull-back algebra

$$S \times_{T/I} T = \{(s,t) \in S \times T \mid \kappa(s) = p(t)\}.$$

Observe that $pr_1 : S \times_{T/I} T \twoheadrightarrow S$ *is a* $\mathbb{C}$*-algebra epimorphism with kernel* $0 \times_{T/I} I$ *having square zero, that is, it is an infinitesimal extension of* S. *Moreover, the existence of a lifting* λ *of* κ *is equivalent to the existence of a* $\mathbb{C}$*-algebra section*

$$\sigma : S \longrightarrow S \times_{T/I} T \quad \text{defined by } s \mapsto (s, \lambda(s)).$$

Hence, S *is Grothendieck smooth if and only if a versal infinitesimal extension of* S *is trivial.*

Returning to the situation of interest to us, we claim that the algebra epimorphism

$$\mathcal{O}_x(\mathbb{C}^n)/I_x^2 \xrightarrow{c_x} \mathcal{O}_x$$

is a versal infinitesimal extension of $\mathcal{O}_x$. *Indeed, consider any other infinitesimal extension* $R \xrightarrow{\pi} \mathcal{O}_x$ *then we define a* $\mathbb{C}$*-algebra morphism* $\mathcal{O}_x(\mathbb{C}^n)/I_x^2 \longrightarrow R$ *as follows : let* $r_i \in R$ *such that* $\pi(r_i) = c_x(x_i)$ *and define an algebra morphism* $\mathbb{C}[x_1, \ldots, x_n] \longrightarrow R$ *by sending the variable* x_i *to* r_i. *As the image of any polynomial non-vanishing in* x *is a unit in* R, *this algebra map extends to one from the local algebra* $\mathcal{O}_x(\mathbb{C}^n)$ *and it factors over* $\mathcal{O}_x(\mathbb{C}^n)/I_x^2$ *as the image of* I_x *lies in the kernel of* π *which has square zero, proving the claim. Hence,* $\mathcal{O}_x$ *is Grothendieck smooth if and only if there is a* $\mathbb{C}$*-algebra section*

$$\begin{tikzcd} \mathcal{O}_x(\mathbb{C}^n)/I_x^2 \arrow[r, two heads, shift left, "c_x"] & \mathcal{O}_x \arrow[l, hook, shift left, "r_x"] \end{tikzcd}$$

satisfying $c_x \circ r_x = id_{\mathcal{O}_x}$.

Proposition 5.16 *The affine scheme* $\underline{X}$ *is non-singular at the geometric point* x *if and only if the local algebra* $\mathcal{O}_x(\underline{X})$ *is Grothendieck smooth.*

Proof. *The result will follow once we prove that there is a natural one-to-one correspondence between* $\mathcal{O}_x$*-module splittings* s_x *of* d_x *and* $\mathbb{C}$*-algebra sections* r_x *of* c_x. *This correspondence is given by assigning to an algebra section* r_x *the map* s_x *defined by*

$$s_x(dx_i) = (x_i - r_x \circ c_x(x_i)) \ mod \ I_x^2$$

□

If $\underline{X}$ *is an affine scheme which is smooth in all of its geometric points, then we have seen before that* $\underline{X} = X$ *must be reduced, that is, an affine variety. Restricting to its disjoint irreducible components we may assume that*

$$\mathbb{C}[\underline{X}] = \cap_{x \in X} \mathcal{O}_x.$$

Clearly, if $\mathbb{C}[\underline{X}]$ *is Grothendieck smooth, so is any of the local algebras* $\mathcal{O}_x$. *Conversely, if all* $\mathcal{O}_x$ *are Grothendieck smooth and* $\mathbb{C}[\underline{X}] = \mathbb{C}[x_1, \ldots, x_n]/I$ *one knows that the algebra epimorphism*

$$\mathbb{C}[x_1, \ldots, x_n]/I^2 \xrightarrow{c} \mathbb{C}[\underline{X}]$$

has local sections in every x, *but then there is an algebra section. Because c is clearly a versal infinitesimal deformation of* $\mathbb{C}[\underline{X}]$, *it follows that* $\mathbb{C}[\underline{X}]$ *is Grothendieck smooth.*

Proposition 5.17 *Let* $\underline{X}$ *be an affine scheme. Then,* $\mathbb{C}[\underline{X}]$ *is Grothendieck smooth if and only if* $\underline{X}$ *is non-singular in all of its geometric points. In this case,* $\underline{X}$ *is a reduced affine variety.*

5.6 Cayley smoothness.

Observe that the commutative $\mathbb{C}$-algebras are precisely the Cayley-Hamilton algebras of degree one, so we recover the notion of Grothendieck smoothness for commutative algebras from the following one.

Definition 5.18 A Cayley-Hamilton algebra A of degree n with trace map tr_A is said to be *Cayley smooth* if it satisfies the following lifting property. Let T be a Cayley-Hamilton algebra of degree n with trace map tr_T and I a twosided nilpotent ideal of T such that $tr_T(I) \subset I$. Assume there is a trace preserving $\mathbb{C}$-algebra morphism $\kappa : A \longrightarrow T/I$, *then* there is a trace preserving $\mathbb{C}$-algebra lift $\lambda : A \longrightarrow T$

$$\begin{tikzcd} & A \arrow[dl, dotted, "\exists \lambda"'] \arrow[d, "\kappa"] \\ T \arrow[r, two heads] & T/I \end{tikzcd}$$

making the diagram commutative.

Let B be a Cayley-Hamilton algebra of degree n with trace map tr_B and trace generated by m elements say $\{b_1, \ldots, b_m\}$. Then, we can write

$$B = \mathbb{T}_n^m / T_B \qquad \textit{with } T_B \textit{ closed under traces.}$$

Now, consider the extended ideal

$$E_B = M_n(\mathbb{C}[M_n^m]).T_B.M_n(\mathbb{C}[M_n^m]) = M_n(N_B)$$

and we have seen that $\mathbb{C}[\underline{rep}_n^{tr}\ B] = \mathbb{C}[M_n^m]/N_B$. We need the following technical result.

Lemma 5.19 *With notations as above, we have for all k that*

$$E_B^{kn^2} \cap \mathbb{T}_n^m \subset T_B^k.$$

Proof. *Let $\mathbb{T}_n^m$ be the trace algebra on the generic $n \times n$ matrices $\{X_1, \ldots, X_m\}$ and $\mathbb{T}_n^{l+m}$ the trace algebra on the generic matrices $\{Y_1, \ldots, Y_l, X_1, \ldots, X_m\}$. Let $\{U_1, \ldots, U_l\}$ be elements of $\mathbb{T}_n^m$ and consider the trace preserving map $\mathbb{T}_n^{l+m} \xrightarrow{u} \mathbb{T}_n^m$ induced by the map defined by sending Y_i to U_i. Then, by the universal property we have a commutative diagram of Reynold operators*

$$\begin{tikzcd} M_n(\mathbb{C}[M_n^{l+m}]) \arrow[r, "\tilde{u}"] \arrow[d, "R"] & M_n(\mathbb{C}[M_n^m]) \arrow[d, "R"] \\ \mathbb{T}_n^{l+m} \arrow[r, "u"] & \mathbb{T}_n^m \end{tikzcd} .$$

Now, let $A_1, \ldots, A_{l+1}$ be elements from $M_n(\mathbb{C}[M_n^m])$, then we can calculate $R(A_1 U_1 A_2 U_2 A_3 \ldots A_l U_l A_{l+1})$ by first computing

$$r = R(A_1 Y_1 A_2 Y_2 A_3 \ldots A_l Y_l A_{l+1})$$

and then substituting the Y_i with U_i. The Reynolds operator preserves the degree in each of the generic matrices, therefore r will be linear in each of the Y_i and is a *sum of trace algebra elements. By our knowledge of the generators of necklaces and the trace algebra we can write each term of the sum as an expression*

$$tr(M_1)tr(M_2)\dots tr(M_z)M_{z+1}$$

where each of the M_i *is a monomial of degree* $\leq n^2$ *in the generic matrices* $\{Y_1,\dots,Y_l,X_1,\dots,X_m\}$. *Now, look at how the generic matrices* Y_i *are distributed among the monomials* M_j. *Each* M_j *contains at most* n^2 *of the* Y_i*'s, hence the monomial* M_{z+1} *contains at least* $l-vn^2$ *of the* Y_i *where* $v \leq z$ *is the number of* M_i *with* $i \leq z$ *containing at least one* Y_j.

Now, assume all the U_i *are taken from the ideal* $T_B \triangleleft \mathbb{T}^m_n$ *which is closed under taking traces, then it follows that*

$$R(A_1U_1A_2U_2A_3\dots A_lU_lA_{l+1}) \in T_B^{v+(l-vn^2)} \subset T_B^k$$

if we take $l = kn^2$ *as* $v + (k-v)n^2 \geq k$. *But this finishes the proof of the required inclusion.* $\square$

Let B *be a Cayley-Hamilton algebra of degree* n *with trace map* tr_B *and* I *a twosided ideal of* B *which is closed under taking traces. We will denote by* $E(I)$ *the extended ideal with respect to the universal embedding, that is,*

$$E(I) = M_n(\mathbb{C}[\underline{rep}_n^{tr}\ B])IM_n(\mathbb{C}[\underline{rep}_n^{tr}\ B]).$$

Then, for all powers k *we have the inclusion* $E(I)^{kn^2} \cap B \subset I^k$.

Theorem 5.20 *Let* A *be a Cayley-Hamilton algebra of degree* n *with trace map* tr_A. *Then,* A *is Cayley smooth if and only if the trace preserving representation variety* $\underline{rep}_n^{tr}\ A$ *is non-singular in all points (in particular,* $\underline{rep}_n^{tr}\ A$ *is reduced).*

Proof. *Let* A *be Cayley smooth, then we have to show that* $\mathbb{C}[\underline{rep}_n^{tr}\ A]$ *is Grothendieck smooth. Take a commutative test-object* (T,I) *with* I *nilpotent and an algebra map* $\kappa : \mathbb{C}[\underline{rep}_n^{tr}\ A] \longrightarrow T/I$. *Composing with the universal embedding* i_A *we obtain a trace preserving morphism* μ_0

$$\begin{tikzcd}
A \arrow[rr, dotted, "\mu_1"] \arrow[d, hook, "i_A"'] \arrow[drr, "\mu_0"] & & M_n(T) \arrow[d] \\
M_n(\mathbb{C}[\underline{rep}_n^{tr}\ A]) \arrow[rr, "M_n(\kappa)"'] & & M_n(T/I)
\end{tikzcd}$$

Because $M_n(T)$ *with the usual trace is a Cayley-Hamilton algebra of degree* n *and* $M_n(I)$ *a trace stable ideal and* A *is Cayley smooth there is a trace preserving algebra map* μ_1. *But then, by the universal property of the embedding* i_A *there exists a* $\mathbb{C}$*-algebra morphism*

$$\lambda : \mathbb{C}[\underline{rep}_n^{tr}\ A] \longrightarrow T$$

such that $M_n(\lambda)$ *completes the diagram. The morphism* λ *is the required lift.*

Conversely, assume that $\mathbb{C}[\underline{rep}_n^{tr}\ A]$ *is Grothendieck smooth. Assume we have a Cayley-Hamilton algebra of degree* n *with trace map* tr_T *and a trace-stable nilpotent ideal* I *of* T *and a trace preserving* $\mathbb{C}$*-algebra map* $\kappa : A \longrightarrow T/I$. *If we combine* *this test-data with the universal embeddings we obtain a diagram*

$$\begin{tikzcd}
 & T \arrow[r, hook, "i_T"] \arrow[d] & M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]) \arrow[d] \\
A \arrow[ur, dotted, "?\exists\lambda"] \arrow[r, "\kappa"] \arrow[d, hook, "i_A"'] & T/I \arrow[r, hook, "i_{T/I}"] & M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T/I]) = M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]/J) \\
M_n(\mathbb{C}[\underline{rep}_n^{tr}\ A]) \arrow[urr, dotted, "M_n(\alpha)"'] & &
\end{tikzcd}$$

Here, $J = M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]) I M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T])$ *and we know already that* $J \cap T = I$. *By the universal property of the embedding* i_A *we obtain a* $\mathbb{C}$*-algebra map*

$$\mathbb{C}[\underline{rep}_n^{tr}\ A] \xrightarrow{\alpha} \mathbb{C}[\underline{rep}_n^{tr}\ T]/J$$

which we would like to lift to $\mathbb{C}[\underline{rep}_n^{tr}\ T]$. *This does not follow from Grothendieck smoothness of* $\mathbb{C}[\underline{rep}_n^{tr}\ A]$ *as* J *is usually not nilpotent. However, as* I *is a nilpotent ideal of* T *there is some* h *such that* $I^h = 0$. *As* I *is closed under taking traces we know by the remark preceding the theorem that*

$$E(I)^{hn^2} \cap T \subset I^h = 0.$$

Now, by definition $E(I) = M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]) I M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T])$ *which is equal to* $M_n(J)$. *That is, the inclusion can be rephrased as* $M_n(J)^{hn^2} \cap T = 0$, *whence there is a trace preserving embedding* $T \hookrightarrow M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]/J^{hn^2})$. *Now, we have the following situation*

$$\begin{tikzcd}
 & T \arrow[r, hook] \arrow[d] & M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]/J^{kn^2}) \arrow[d] \\
A \arrow[ur, dotted, "\lambda"] \arrow[r, "\kappa"] \arrow[d, hook, "i_A"'] & T/I \arrow[r, hook, "i_{T/I}"] & M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]/J) \\
M_n(\mathbb{C}[\underline{rep}_n^{tr}\ A]) \arrow[uurr, dotted] \arrow[urr, "M_n(\alpha)"'] & &
\end{tikzcd}$$

This time we can lift α *to a* $\mathbb{C}$*-algebra morphism*

$$\mathbb{C}[\underline{rep}_n^{tr}\ A] \longrightarrow \mathbb{C}[\underline{rep}_n^{tr}\ T]/J^{hn^2}.$$

This in turn gives us a trace preserving morphism

$$A \xrightarrow{\lambda} M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]/J^{hn^2})$$

the image of which is contained in the algebra of GL_n*-invariants. Because* $T \hookrightarrow M_n(\mathbb{C}[\underline{rep}_n^{tr}\ T]/J^{hn^2})$ *and by surjectivity of invariants under surjective maps, the* GL_n*-equivariants are equal to* T, *giving the required lift* λ. $\square$

5.7 Quillen smoothness.

In this section we introduce Quillen smooth algebras which are the basic building blocks to construct noncommutative manifolds.

Definition 5.21 A $\mathbb{C}$-algebra A is said to be *Quillen smooth* if it satisfies the following lifting property. Let T be a $\mathbb{C}$-algebra and $I \lhd T$ a nilpotent ideal. If there is a $\mathbb{C}$-algebra morphism $A \xrightarrow{\kappa} T/I$ then there exists a $\mathbb{C}$-algebra lift $A \xrightarrow{\lambda} T$

$$\begin{tikzcd} & A \arrow[dl, dotted, "\exists \lambda"'] \arrow[d, "\kappa"] \\ T \arrow[r, two heads] & T/I \end{tikzcd}$$

making the diagram commutative.

This definition is rather restrictive. In particular, a commutative (Grothendieck) smooth algebra does not have to satisfy the lifting property in the category of all $\mathbb{C}$-algebras.

Example 5.22 consider the polynomial algebra $\mathbb{C}[x_1, \ldots, x_d]$ and the 4-dimensional noncommutative local algebra

$$T = \frac{\mathbb{C}\langle x, y\rangle}{(x^2, y^2, xy + yx)} = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy$$

Consider the one-dimensional nilpotent ideal $I = \mathbb{C}(xy - yx)$ of T, then the 3-dimensional quotient $\frac{T}{I}$ is commutative and we have a morphism $\mathbb{C}[x_1, \ldots, x_d] \xrightarrow{\phi} \frac{T}{I}$ by $x_1 \mapsto x, x_2 \mapsto y$ and $x_i \mapsto 0$ for $i \geq 2$. This morphism admits no lift to T as for any potential lift the commutator

$$[\tilde{\phi}(x), \tilde{\phi}(y)] \neq 0 \quad \text{in } T.$$

Therefore, $\mathbb{C}[x_1, \ldots, x_d]$ can only be Quillen smooth if $d = 1$. In fact, we will see in chapter 9 that the only commutative affine Quillen smooth algebras are the coordinate rings of a disjoint union of points and affine smooth curves.

Still, the world of Quillen smooth algebras is rather exotic containing many algebras determined by universal constructions. There is a fairly innocent class of Quillen smooth algebras determined by combinatorial data : path algebras of quivers. *Consider the commutative $\mathbb{C}$-algebra*

$$C_k = \mathbb{C}[e_1, \ldots, e_k]/(e_i^2 - e_i, e_i e_j, \sum_{i=1}^{k} e_i - 1).$$

C_k is the universal $\mathbb{C}$-algebra in which 1 is decomposed into k orthogonal idempotents, that is, if R is any $\mathbb{C}$-algebra such that $1 = r_1 + \ldots + r_k$ with $r_i \in R$ idempotents satisfying $r_i r_j - 0$, then there is an embedding $C_k \hookrightarrow R$ sending e_i to r_i. Observe that as a $\mathbb{C}$-algebra

$$C_k \simeq \underbrace{\mathbb{C} \oplus \ldots \oplus \mathbb{C}}_{k}$$

or equivalently, the coordinate ring of k distinct points.

Proposition 5.23 *C_k is Quillen smooth. That is, if I be a nilpotent ideal of a $\mathbb{C}$-algebra T and if $1 = \overline{e}_1 + \ldots + \overline{e}_k$ is a decomposition of 1 into orthogonal idempotents $\overline{e}_i \in T/I$. Then, we can lift this decomposition to $1 = e_1 + \ldots + e_k$ for orthogonal idempotents $e_i \in T$ such that $\pi(e_i) = \overline{e}_i$ where $T \xrightarrow{\pi} T/I$ is the canonical projection.*

Proof. *Assume that* $I^l = 0$*, clearly any element* $1 - i$ *with* $i \in I$ *is invertible in* T *as*

$$(1-i)(1+i+i^2+\ldots+i^{l-1}) = 1 - i^l = 1.$$

If $\overline{e}$ *is an idempotent of* T/I *and* $x \in T$ *such that* $\pi(x) = \overline{e}$*. Then,* $x - x^2 \in I$ *whence*

$$0 = (x-x^2)^l = x^l - lx^{l+1} + \binom{l}{2} x^{l+2} - \ldots + (-1)^l x^{2l}$$

and therefore $x^l = ax^{l+1}$ *where* $a = l - \binom{l}{2} x + \ldots + (-1)^{l-1} x^{l-1}$ *and so* $ax = xa$*. If we take* $e = (ax)^l$*, then* e *is an idempotent in* T *as*

$$e^2 = (ax)^{2l} = a^l(a^l x^{2l}) = a^l x^l = e$$

the next to last equality follows from $x^l = ax^{l+1} = a^2x^{l+2} = \ldots = a^l x^{2l}$*. Moreover,*

$$\pi(e) = \pi(a)^l \pi(x)^l = \pi(a)^l \pi(x)^{2l} = \pi(a^l x^{2l}) = \pi(x)^l = \overline{e}.$$

If $\overline{f}$ *is another idempotent in* T/I *such that* $\overline{e}\overline{f} = 0 = \overline{f}\overline{e}$ *then as above we can lift* $\overline{f}$ *to an idempotent* f' *of* T*. As* $f'e \in I$ *we can form the element*

$$f = (1-e)(1-f'e)^{-1} f'(1-f'e).$$

Because $f'(1-f'e) = f'(1-e)$ *one verifies that* f *is idempotent,* $\pi(f) = \overline{f}$ *and* $e.f = 0 = f.e$*. Assume by induction that we have already lifted the pairwise orthogonal idempotents* $\overline{e}_1, \ldots, \overline{e}_{k-1}$ *to pairwise orthogonal idempotents* $e_1, \ldots, e_{k-1}$ *of* R*, then* $e = e_1 + \ldots + e_{k-1}$ *is an idempotent of* T *such that* $\overline{e}\overline{e}_k = 0 = \overline{e}_k \overline{e}$*. Hence, we can lift* $\overline{e_k}$ *to an idempotent* $e_k \in T$ *such that* $ee_k = 0 = e_k e$*. But then also*

$$e_i e_k = (e_i e) e_k = 0 = e_k(ee_i) = e_k e_i.$$

Finally, as $e_1 + \ldots + e_k - 1 = i \in I$ *we have that*

$$e_1 + \ldots + e_k - 1 = (e_1 + \ldots + e_k - 1)^l = i^l = 0$$

finishing the proof. $\square$

Let Q *be a quiver, that is, a directed graph determined by*

- *a finite set* $Q_v = \{v_1, \ldots, v_k\}$ *of vertices, and*
- *a finite set* $Q_a = \{a_1, \ldots, a_l\}$ *of arrows where we allow multiple arrows between vertices and loops in vertices.*

For now, we will depict vertex v_i *by* $\textcircled{i}$ *and an arrow* a *from vertex* v_i *to* v_j *by* $\textcircled{j} \xleftarrow{a} \textcircled{i}$*. Note however than once we come to dimension vectors, we will encircle the vector components and will indicate the ordering of the vertices by subscripts when necessary.*

The path algebra $\mathbb{C}Q$ *has as underlying* $\mathbb{C}$*-vectorspace basis the set of all oriented paths in* Q*, including those of length zero corresponding to the vertices* v_i*. Multiplication in* $\mathbb{C}Q$ *is induced by (left) concatenation of paths. More precisely,* $1 = v_1 + \ldots + v_k$ *is a decomposition of* 1 *into mutually orthogonal idempotents and further we define*

- $v_j.a$ *is always zero unless* $\textcircled{j} \xleftarrow{a} \bigcirc$ *in which case it is the path* a*,*
- $a.v_i$ *is always zero unless* $\bigcirc \xleftarrow{a} \textcircled{i}$ *in which case it is the path* a*,*

- $a_i.a_j$ *is always zero unless* $\circ \xleftarrow{a_i} \circ \xleftarrow{a_j} \circ$ *in which case it is the path* $a_i a_j$.

Proposition 5.24 *For any quiver Q, the path algebra $\mathbb{C}Q$ is Quillen smooth.*

Proof. *Take an algebra T with a nilpotent twosided ideal $I \lhd T$ and consider*

$$\begin{tikzcd} T \arrow[r, two heads] & \frac{T}{I} \\ & \mathbb{C}Q \arrow[u, "\phi"'] \arrow[ul, dotted, "\exists \tilde{\phi}"] \end{tikzcd}$$

The decomposition $1 = \phi(v_1) + \ldots + \phi(v_k)$ into mutually orthogonal idempotents in $\frac{T}{I}$ can be lifted up the nilpotent ideal I to a decomposition $1 = \tilde{\phi}(v_1) + \ldots + \tilde{\phi}(v_k)$ into mutually orthogonal idempotents in T. But then, taking for every arrow a

$$\textcircled{j} \xleftarrow{a} \textcircled{i} \quad \text{an arbitrary element} \quad \tilde{\phi}(a) \in \tilde{\phi}(v_j)(\phi(a) + I)\tilde{\phi}(v_i)$$

gives a required lifted algebra morphism $\mathbb{C}Q \xrightarrow{\tilde{\phi}} T$. $\square$

A Quillen smooth algebra A determines for every integer n a Cayley smooth algebra $A@_n$. Let alg^{tr} be the category of all $\mathbb{C}$-algebras equipped with a trace map and with trace preserving morphisms. The forgetful functor $alg^{tr} \longrightarrow alg$ has a left adjoint

$$alg \xrightarrow{\tau} alg^{tr}$$

that is, given an algebra A we construct an algebra A^τ by formally adjoining traces (as in the case of $\mathbb{T}^\infty$ given before). If $tr : A^\tau \longrightarrow A^\tau$ is the trace map on A^τ we define for given n a Cayley-Hamilton algebra $A@_n$ to be the quotient

$$A@_n = \frac{A^\tau}{(tr(1) - n, \chi_a^{(n)}(a)\ \forall a \in A)}$$

In general it may happen that $A@_n = 0$ for example if A has no n-dimensional representations. The characteristic feature of $A@_n$ is that any $\mathbb{C}$-algebra map $A \longrightarrow B$ with B a Cayley-Hamilton algebra of degree n factors through $A@_n$

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "can"'] & & B \\ & A@_n \arrow[ur, dotted, "\exists \phi_n"'] & \end{tikzcd}$$

with ϕ_n a trace preserving algebra morphism. From this universal property the next result is immediate

Proposition 5.25 *If A is Quillen smooth, then for every integer n, the Cayley-Hamilton algebra of degree n, $A@_n$, is Cayley smooth. Moreover,*

$$\underline{rep}_n\ A \simeq \underline{rep}_n^{tr}\ A@_n$$

is a smooth affine variety.

This result allows us to study Quillen smooth algebras. We know that the algebra $A@_n$ is given by the GL_n-equivariant maps from $\underline{rep}_n^{tr}\ A@_n$ to $M_n(\mathbb{C})$. Because this representation variety is smooth we will apply the Luna étale slices to determine the local structure of the GL_n-variety $\underline{rep}_n^{tr}\ A@_n$ and hence of $A@_n$. The major result we will prove in a moment is that this local structure is fully determined by a quiver situation. That is, the local study of arbitrary Quillen smooth algebras can be reduced to that of the better understood subclass of path algebras of quivers.

5.8 Local structure.

Let A be an affine $\mathbb{C}$-algebra generated by m elements $\{a_1, \ldots, a_m\}$. The Cayley-Hamilton algebra of $A@_n$ of degree n is then trace generated by m elements, that is, there is a trace preserving epimorphism $\mathbb{T}_n^m \overset{\psi^*}{\twoheadrightarrow} A@_n$. That is, we have a GL_n-equivariant closed embedding of affine schemes

$$\underline{rep}_n\ A = \underline{rep}_n^{tr}\ A@_n \overset{\psi}{\hookrightarrow} \underline{rep}_n^{tr}\ \mathbb{T}_n^m = M_n^m$$

Take a point ξ of the quotient scheme $\underline{iss}_n\ A = \underline{rep}_n^{tr}\ A@_n/GL_n$. We know that ξ determines the isomorphism class of a semi-simple n-dimensional representation of A, say

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i are distinct simple A-representations, say of dimension d_i and occurring in M_ξ with multiplicity e_i. These numbers determine the representation type *$\tau(\xi)$ of ξ (or of the semi-simple representation M_ξ), that is*

$$\tau(\xi) = (e_1, d_1; e_2, d_2; \ldots; e_k, d_k)$$

Choosing a basis of M_ξ adapted to this decomposition we find a point $x = (X_1, \ldots, X_m)$ in the orbit $\mathcal{O}(M_\xi) \hookrightarrow M_n^m$ such that each $n \times n$ matrix X_i is of the form

$$X_i = \begin{bmatrix} m_1^{(i)} \otimes \mathbb{1}_{e_1} & 0 & \ldots & 0 \\ 0 & m_2^{(i)} \otimes \mathbb{1}_{e_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_k^{(i)} \otimes \mathbb{1}_{e_k} \end{bmatrix}$$

where each $m_j^{(i)} \in M_{d_j}(\mathbb{C})$. Using this description we can compute the stabilizer subgroup $Stab(x)$ of GL_n consisting of those invertible matrices $g \in GL_n$ commuting with every X_i. That is, $Stab(x)$ is the multiplicative group of units of the centralizer of the algebra generated by the X_i, that is

$$\begin{bmatrix} M_{d_1}(\mathbb{C}) \otimes \mathbb{1}_{e_1} & 0 & \ldots & 0 \\ 0 & M_{d_2}(\mathbb{C}) \otimes \mathbb{1}_{e_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{d_k}(\mathbb{C}) \otimes \mathbb{1}_{e_k} \end{bmatrix}$$

It is easy to verify that this group is isomorphic to

$$Stab(x) \simeq GL_{e_1} \times GL_{e_2} \times \ldots \times GL_{e_k}$$

with the embedding $Stab(x) \hookrightarrow GL_n$ given by

$$\begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{d_1}) & 0 & \ldots & 0 \\ 0 & GL_{e_2}(\mathbb{C} \otimes \mathbb{1}_{d_2}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{d_k}) \end{bmatrix}$$

Clearly, a different choice of point in the orbit $\mathcal{O}(M_\xi)$ gives a subgroup of GL_n conjugated to $Stab(x)$. Consider the vector $\alpha = (e_1, e_2, \ldots, e_k)$, then we see that

$$Stab(x) \simeq GL(\alpha) = GL_{e_1} \times GL_{e_2} \times \ldots \times GL_{e_k}$$

We will compute the normal space N_x^{big} to the orbit $\mathcal{O}(M_{xi})$ in $M_n^m = \underline{rep}_n^{tr}\ \mathbb{T}_n^m$. This is an elaborate book-keeping operation involving $GL(\alpha)$-representations. As $x = (X_1, \ldots, X_m)$ the tangent space $T_x\ \mathcal{O}(M_{xi})$ in M_n^m to the orbit is equal to the image of the linear map

$$\begin{array}{ccc} \mathfrak{gl}_n = M_n & \longrightarrow & M_n \oplus \ldots \oplus M_n = T_x\ M_n^m \\ A & \mapsto & ([A, X_1], \ldots, [A, X_m]) \end{array}$$

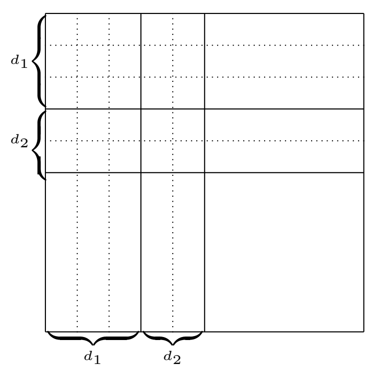
Observe that the kernel of this map is the centralizer of the subalgebra generated by the X_i, so we have an exact sequence of $Stab(x) = GL(\alpha)$-modules

$$0 \longrightarrow \mathfrak{gl}(\alpha) = Lie\ GL(\alpha) \longrightarrow \mathfrak{gl}_n = M_n \longrightarrow T_x\ \mathcal{O}(x) \longrightarrow 0$$

As $GL(\alpha)$ is a reductive group every $GL(\alpha)$-module is completely reducible and so the sequence splits. But then, the normal space in $M_n^m = T_x\ M_n^m$ to the orbit is isomorphic as $GL(\alpha)$-module to

$$N_x^{big} = \underbrace{M_n \oplus \ldots \oplus M_n}_{m-1} \oplus \mathfrak{gl}(\alpha)$$

with the action of $GL(\alpha)$ (embedded as above in GL_n) is given by simultaneous conjugation. If we consider the $GL(\alpha)$-action on M_n



we see that it decomposes into a direct sum of subrepresentations

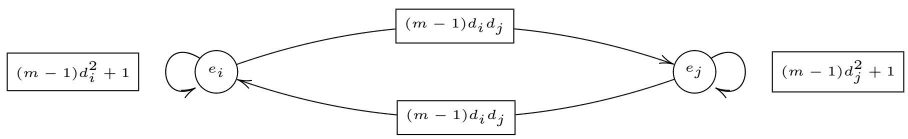
- *for each $1 \leq i \leq k$ we have d_i^2 copies of the $GL(\alpha)$-module M_{e_i} on which GL_{e_i} acts by conjugation and the other factors of $GL(\alpha)$ act trivially,*
- *for all $1 \leq i, j \leq k$ we have $d_i d_j$ copies of the $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ acts via $g.m = g_i m g_j^{-1}$ and the other factors of $GL(\alpha)$ act trivially.*

These $GL(\alpha)$ components are precisely the modules appearing in representation spaces of quivers, which are defined in chapter 1 or more precisely in chapter 6.

Theorem 5.26 *Let ξ be of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ and let $\alpha = (e_1, \ldots, e_k)$. Then, the $GL(\alpha)$-module structure of the normal space N_x^{big} in $\underline{rep}_n^{tr}\ \mathbb{T}_n^m = M_n^m$ to the orbit of the semi-simple n-dimensional representation $\mathcal{O}(M_\xi)$ is isomorphic to*

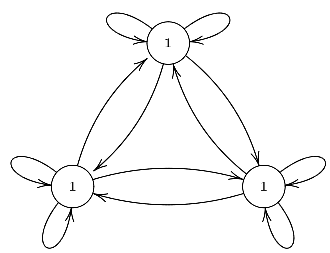
$$rep_\alpha\ Q_\xi$$

where the quiver Q_ξ has k vertices and the subquiver on any two vertices v_i, v_j for $1 \leq i, j \leq k$ has the following shape



That is, in each vertex v_i there are $(m-1)d_i^2+1$-loops and there are $(m-1)d_id_j$ arrows from vertex v_i to vertex v_j for all $1 \leq i, j \leq k$.

Example 5.27 If $m = 2$ and $n = 3$ and the representation type is $\tau = (1, 1; 1, 1; 1, 1)$ (that is, M_ξ is the direct sum of three distinct one-dimensional simple representations) then the quiver Q_ξ is



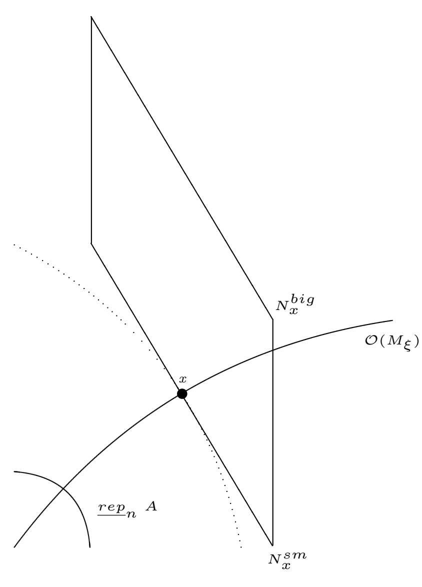
We say that A is smooth at $\xi \in \underline{iss}_n\ A$ if the representation variety $\underline{rep}_n^{tr}\ A@_n$ is smooth in M_ξ. Before we can apply the Luna slice theorem we have to control the normal space N_x^{sm} to the orbit $\mathcal{O}(M_\xi)$ in $\underline{rep}_n^{tr}\ A@_n$. We have GL_n-equivariant embeddings

$$\mathcal{O}(M_\xi) \hookrightarrow \underline{rep}_n^{tr}\ A@_n \hookrightarrow \underline{rep}_n^{tr}\ \mathbb{T}_n^m = M_n^m$$

and corresponding embeddings of the tangent spaces in x

$$T_x\ \mathcal{O}(M_\xi) \hookrightarrow T_x\ \underline{rep}_n^{tr}\ A@_n \hookrightarrow T_x\ M_n^m$$

Because $GL(\alpha)$ is reductive we then obtain for the normal spaces to the orbit



$$N_x^{sm} = \frac{T_x\ \underline{rep}_n^{tr}\ A@_n}{T_x\ \mathcal{O}(M_\xi)} \lhd N_x^{big} = \frac{T_x\ M_n^m}{T_x\ \mathcal{O}(M_\xi)}$$

a direct summand as $GL(\alpha)$-modules. As we know the isotypical decomposition of N_x^{big} as the $GL(\alpha)$-module $rep_\alpha\ Q_\xi$ this allows us to control N_x^{sm}. We only have to observe that arrows in Q_ξ correspond to simple $GL(\alpha)$-modules, whereas a loop at vertex v_i decomposes as $GL(\alpha)$-module into the simples

$$M_{e_i} = M_{e_i}^0 \oplus \mathbb{C}_{triv}$$

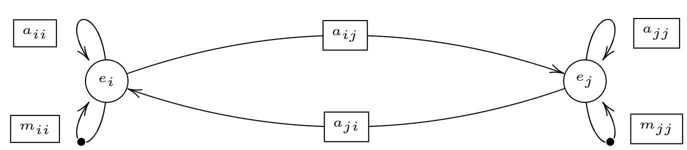
where $\mathbb{C}_{triv}$ is the one-dimensional simple with trivial $GL(\alpha)$-action and $M_{e_i}^0$ is the space of trace zero matrices in M_{e_i}. Again, we can represent the $GL(\alpha)$-module structure of N_x^{sm} graphically, this time by a marked quiver *using the dictionary*

- *a loop at vertex v_i corresponds to the $GL(\alpha)$-module M_{e_i} on which GL_{e_i} acts by conjugation and the other factors act trivially,*
- *a marked loop at vertex v_i corresponds to the simple $GL(\alpha)$-module $M_{e_i}^0$ on which GL_{e_i} acts by conjugation and the other factors act trivially,*
- *an arrow from vertex v_i to vertex v_j corresponds to the simple $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ acts via $g.m = g_i m g_j^{-1}$ and the other factors act trivially,*

Theorem 5.28 *Let ξ be such that M_ξ is a smooth point on $\underline{rep}_n\ A$ of representation type $\tau = (e_1, d_1; \ldots ; e_k, d_k)$ and let $\alpha = (e_1, \ldots, e_k)$. Then, the $GL(\alpha)$-module structure of the normal space N_x to the orbit is isomorphic to the $GL(\alpha)$-module of representations of the marked quiver*

$$rep_\alpha\ Q_\xi^\bullet$$

on k vertices $\{v_1, \ldots, v_k\}$ such that the marked subquiver on any two vertices v_i, v_j with $1 \leq i, j \leq k$ has the form



where these numbers satisfy $a_{ij} \leq (m-1)d_i d_j$ and $a_{ii} + m_{ii} \leq (m-1)d_i^2 + 1$.

Under the assumptions of the theorem, the étale slice result enables us with a slice $S_x \xrightarrow{\phi} N_x^{sm}$ and a commutative diagram

$$\begin{tikzcd}
& GL_n \times^{GL(\alpha)} S_x \arrow[dl, "GL_n \times^{GL(\alpha)} \phi"'] \arrow[dr, "\psi"] \arrow[dd] & \\
GL_n \times^{GL(\alpha)} N_x^{sm} \arrow[dd] & & \underline{rep}_n\ A \arrow[dd] \\
& S_x/GL(\alpha) \arrow[dl, "\phi/GL(\alpha)"] \arrow[dr, "\psi/GL_n"] & \\
N_x^{sm}/GL(\alpha) & & \underline{iss}_n\ A
\end{tikzcd}$$

where the vertical maps are the quotient maps, all diagonal maps are étale and the upper ones are GL_n-equivariant. Hence, the GL_n-local structure of the representation variety $\underline{rep}_n\ A = \underline{rep}_n^{tr}\ A@_n$ in a neighborhood of the orbit of x is the same as that of the associated fiber bundle $GL_n \times^{GL(\alpha)} N_x^{sm}$ in a neighborhood of the orbit of $\overline{(\mathbb{1}_n, 0)}$. Further, the local structure of the quotient scheme $\underline{iss}_n\ A$ in a neighborhood of ξ is the same as that of the quotient variety of the marked quiver representations $N_x^{sm}/GL(\alpha)$ in a neighborhood of the trivial representation $\overline{0}$.

Let $\mathfrak{m} \triangleleft \mathbb{C}[\underline{iss}_n\ A]$ be the maximal ideal corresponding to ξ. As the ring of polynomial invariants $\mathbb{C}[\underline{iss}_n\ A]$ is via the diagonal embedding a central subalgebra of the ring of GL_n-equivariant maps from $\underline{rep}_n\ A$ to $M_n(\mathbb{C})$ we can localize this ring of equivariant maps $A@_n$ at $\mathfrak{m}$ and also take its $\mathfrak{m}$-adic completion which we denote by $\widehat{(A@_n)}_{\mathfrak{m}}$.

Let $\mathfrak{m}_0$ be the maximal ideal of the ring of $GL(\alpha)$-polynomial invariants of the marked quiver representation space $\mathbb{C}[rep_\alpha\ Q_\xi^\bullet]^{GL(\alpha)} = \mathbb{C}[N_x^{sm}/GL(\alpha)]$. Let $\mathbb{T}_\alpha\ Q_\xi^\bullet$ denote the ring of $GL(\alpha)$-equivariant maps from $rep_\alpha\ Q_\xi^\bullet$ to $M_n(\mathbb{C})$ and denote the $\mathfrak{m}_0$-adic filtration of it with $\widehat{\mathbb{T}_\alpha\ Q^\bullet_\xi}_{\mathfrak{m}_0}$. The above diagram then implies

Theorem 5.29 *Let ξ correspond to a semi-simple n-dimensional representation of A such that the representation variety is smooth along this closed orbit. Then, with notations as before we have an isomorphism of complete local algebras*

$$\widehat{(A@_n)}_{\mathfrak{m}} \simeq \widehat{\mathbb{T}_\alpha\ Q^\bullet_\xi}_{\mathfrak{m}_0}$$

In the following sections we will determine the algebra structure of $\mathbb{T}_\alpha\ Q_\xi^\bullet$.

5.9 Finite dimensional algebras.

Let A be a Cayley-Hamilton algebra of degree n wit trace map tr, then we can define a norm-map on A by defining

$$N(a) = \sigma_n(a) \quad \text{for all } a \in A.$$

Recall that the elementary symmetric function σ_n is a polynomial function $f(t_1, t_2, \ldots, t_n)$ in the Newton functions $t_i = \sum_{j=1}^n x_j^i$. Then, $\sigma(a) = f(tr(a), tr(a^2), \ldots, tr(a^n))$. Because, we have a trace preserving embedding $A \hookrightarrow M_n(\mathbb{C}[\underline{rep}_n^{tr}\ A])$ and the norm map N coincides with the determinant in this matrix-algebra, we have that

$$N(1) = 1 \quad \text{and} \quad \forall a, b \in A \quad N(ab) = N(a)N(b).$$

Furthermore, the norm-map extends to a polynomial map on $A[t]$ and we have that $\chi_a^{(n)}(t) = N(t-a)$, in particular we can obtain the trace by polarization of the norm map. Consider a finite dimensional semi-simple $\mathbb{C}$-algebra

$$A = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C}),$$

then all the Cayley-Hamilton structures of degree n on A with trace values in $\mathbb{C}$ are given by the following result.

Lemma 5.30 *Let A be a semi-simple algebra as above and tr a trace map on A making it into a Cayley-Hamilton algebra of degree n. Then, there exist a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}_+^k$ such that $n = \sum_{i=1}^k m_i d_i$ and for any $a = (A_1, \ldots, A_k) \in A$ with $A_i \in M_{d_i}(\mathbb{C})$ we have that*

$$tr(a) = m_1 Tr(A_1) + \ldots + m_k Tr(A_k)$$

where Tr are the usual trace maps on matrices.

Proof. *The norm-map N on A defined by the trace map tr induces a group morphism on the invertible elements of A*

$$N : A^* = GL_{d_1}(\mathbb{C}) \times \ldots \times GL_{d_k}(\mathbb{C}) \longrightarrow \mathbb{C}^*$$

that is, a character. Now, any character is of the following form, let $A_i \in GL_{d_i}(\mathbb{C})$, then for $a = (A_1, \ldots, A_k)$ we must have

$$N(a) = det(A_1)^{m_1} det(A_2)^{m_2} \ldots det(A_k)^{m_k}$$

for certain integers $m_i \in \mathbb{Z}$. Since N extends to a polynomial map on the whole of A we must have that all $m_i \geq 0$. By polarization it then follows that

$$tr(a) = m_1 Tr(A_1) + \ldots m_k Tr(A_k)$$

and it remains to show that no $m_i = 0$. Indeed, if $m_i = 0$ then tr would be the zero map on $M_{d_i}(\mathbb{C})$, but then we would have for any $a = (0, \ldots, 0, A, 0, \ldots, 0)$ with $A \in M_{d_i}(\mathbb{C})$ that

$$\chi_a^{(n)}(t) = t^n$$

whence $\chi_a^{(n)}(a) \neq 0$ whenever A is not nilpotent. This contradiction finishes the proof. □

We can extend this to all finite dimensional $\mathbb{C}$-algebras. Let A be a finite dimensional algebra with radical J and assume there is a trace map tr on A making A into a Cayley-Hamilton algebra of degree n and such that $tr(A) = \mathbb{C}$. We claim that the norm map $N : A \longrightarrow \mathbb{C}$ is zero on J. Indeed, any $j \in J$ satisfies $j^l = 0$ for some l whence $N(j)^l = 0$. But then, polarization gives that $tr(J) = 0$ and we have that the semisimple algebra

$$A^{ss} = A/J = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C})$$

is a semisimple Cayley-hamilton algebra of degree n on which we can apply the foregoing lemma. Finally, note that $A \simeq A^{ss} \oplus J$ as $\mathbb{C}$-vectorspaces. This concludes the proof of

Proposition 5.31 *Let A be a finite dimensional $\mathbb{C}$-algebra with radical J and semisimple part*

$$A^{ss} = A/J = M_{d_1}(\mathbb{C}) \oplus \ldots \oplus M_{d_k}(\mathbb{C}).$$

If $tr : A \longrightarrow \mathbb{C} \hookrightarrow A$ is a trace map such that A is a Cayley-Hamilton algebra of degree n, there exists a dimension vector $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}_+^k$ such that for all $a = (A_1, \ldots, A_k, j)$ with $A_i \in M_{d_i}(\mathbb{C})$ and $j \in J$ we have

$$tr(a) = m_1 Tr(A_1) + \ldots m_k Tr(A_k)$$

with Tr the usual traces on $M_{d_i}(\mathbb{C})$ and $\sum_i m_i d_i = n$.

However, there can be other trace maps on A making A into a Cayley-Hamilton algebra of degree n. For example let C be a finite dimensional commutative $\mathbb{C}$-algebra with radical N, then $A = M_n(C)$ is finite dimensional with radical $J = M_n(N)$ and the usual trace map $tr : M_n(C) \longrightarrow C$ makes A into a Cayley-Hamilton algebra of degree n such that $tr(J) = N \neq 0$. Still, if A is semi-simple, the center $Z(A) = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$ (as many terms as there are matrix components in A) and any subring of $Z(A)$ is of the form $\mathbb{C} \oplus \ldots \oplus \mathbb{C}$. In particular, $tr(A)$ has this form and composing the trace map with projection on the j-th component we have a trace map tr_j on which we can apply the lemma.

With $C_k@_n$ we will denote the universal algebra in the category of Cayley-Hamilton algebras of degree n such that 1 has a decomposition into k orthogonal idempotents $\{e_1, \ldots, e_k\}$. That is, take generic $n \times n$ matrices

$$X_l = \begin{bmatrix} x_{11}(l) & \ldots & x_{1n}(l) \\ \vdots & & \vdots \\ x_{n1}(l) & \ldots & x_{nn}(l) \end{bmatrix}$$

for the idempotents e_l for $1 \leq l \leq k$. As $e_i e_j = 0$ if $i \neq j$ and $e_i^2 = e_i$, the only nonvanishing traces of monomials in the e_i (up to cyclic permutation) form the polynomial algebra

$$P = \mathbb{C}[t_1, \ldots, t_k]/(t_1 + \ldots + t_k - n)$$

where $t_i = tr(X_i) = \sum_{j=1}^n x_{jj}(i)$. Now, consider the quotient R of the polynomial algebra $\mathbb{C}[x_{ij}(l) \mid 1 \leq i,j \leq n, 1 \leq l \leq k]$ by the ideal of all entries coming from the matrix identities

$$\begin{cases} X_i^2 = X_i \\ X_iX_j = 0 \text{ for } i \neq j \\ X_1 + \ldots + X_k = I_n \\ \chi_{X_i}^{(n)}(X_i) = 0 \end{cases}$$

and observe that all the coefficients of the Cayley-Hamilton polynomial of X_i are polynomials in t_i. Then, $C_k@_n$ is the subalgebra of $M_n(R)$ generated by the images of the X_l and t_l.

Theorem 5.32 *With notations as above, we have :*

1. *$C_k@_n$ is a smooth Cayley-Hamilton algebra of degree n.*
2. *$\underline{rep}_n^{tr}\ C_k@_n$ is the disjoint union of the homogeneous varieties*

   $$GL_n/(GL_{m_1} \times \ldots \times GL_{m_k})$$

   where $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}_+^k$ is a dimension vector such that $m_1 + \ldots + m_k = n$.
3. *The Cayley-Hamilton algebra corresponding to the component determined by $\alpha = (m_1, \ldots, m_k)$ is the semi-simple algebra $C_k(\alpha)$ which is the subalgebra of $M_n(\mathbb{C})$ generated by the images*

   $$e_i \mapsto \sum_{j=\sum_{k=1}^{i-1} m_k + 1}^{\sum_{k=1}^{i} m_i} e_{jj} \in M_n(\mathbb{C})$$

Proof. (1) *: Let T be a Cayley-Hamilton algebra of degree n with trace map tr and I a twosided nilpotent ideal of T such that $tr(I) \subset I$. Assume there is a trace preserving algebra map*

$$C_k@_n \xrightarrow{\phi} T/I$$

which is determined by the $\phi(X_l) = \overline{f_l}$ which are idempotents in T/I such that $1 = \overline{f_1} + \ldots + \overline{f_k}$. By the lemma above we can lift this decomposition to $1 = f_1 + \ldots + f_k$ where f_i are orthogonal idempotents of T. Clearly, there is a $\mathbb{C}$-algebra morphism $\psi : C_k@_n \longrightarrow T$ lifting ϕ by sending X_l to f_l and t_l to $tr(f_l)$. Observe that this is possible as the only relation holding among the t_l is $t_1 + \ldots + t_k = n$ and because T is of degree n we have that $tr(f_1) + \ldots + tr(f_k) = n$.

(2) : *By the previous part we know that the trace preserving representation variety* $\underline{rep}_n^{tr}\ C_k@_n$ *is smooth and hence reduced. Therefore, it suffices to describe the points. Take a trace preserving* n*-dimensional representation* M *determined by the trace preserving algebra morphism*

$$C_k@_n \xrightarrow{\chi} M_n(\mathbb{C})$$

The image $\chi(C_k@_n)$ *is a finite dimensional semi-simple commutative Cayley-hamilton algebra with trace image* $\mathbb{C}$ *and hence one of the algebras* $C_k(\alpha)$ *for some dimension vector* $\alpha = (m_1, \ldots, m_k)$ *with* $\sum m_i = n$*. Therefore,* M *is a semi-simple representation and the multiplicities of the simple components* S_l *corresponding to the* l*-th factor of* $C_k(\alpha)$ *is equal to the dimension of* $e_l.M$ *which is the number* m_l*. Hence, there is a unique trace preserving representation factoring via* $C_k(\alpha)$*. Therefore,* $\underline{rep}_n^{tr}\ C_k@_n$ *is the disjoint union of finitely many closed orbits (since they are semi-simple) one for each dimension vector* $\alpha = (m_1, \ldots, m_k)$ *with* $\sum m_i = n$*. The stabilizer group of* M *is the group of module automorphisms and hence equal to* $GL_{m_1} \times \ldots \times GL_{m_k}$ *proving the assertion.*

(3) : *We have already seen that there is a unique isomorphism class of trace preserving representation for* $C_k(\alpha)$ *so the reduced variety of* $\underline{rep}_n^{tr}\ C_k(\alpha)$ *is the orbit* $GL_n/(GL_{m_1} \times \ldots \times GL_{m_k})$*. Moreover,*

$$C_k(\alpha) \simeq C_k@_n/(t_1 - m_1, \ldots, t_k - m_k) = C_k@_n(\alpha)$$

We claim that $\mathbb{C}[\underline{rep}_n^{tr}\ C_k@_n(\alpha)]$ *is formally smooth. Let* C *be a commutative algebra and* $I \triangleleft C$ *a nilpotent ideal. Any algebra morphism* $\mathbb{C}[\underline{rep}_n^{tr}\ C_k@_n(\alpha)] \xrightarrow{\phi} C/I$ *determines a decomposition* $1 = \overline{c}_1 + \ldots + \overline{c}_k$ *into orthogonal idempotents* $\overline{c}_l \in M_n(C/I)$ *with* $tr(\overline{c}_l) = m_l$*. This decomposition can be lifted to* $1 = c_1 + \ldots + c_k$ *where* c_l *are orthogonal idempotents in* $M_n(C)$*. The entries of the* v_l *determine an algebra morphism* $\mathbb{C}[\underline{rep}_n^{tr}\ C_k@_n(\alpha)] \longrightarrow C$ *lifting* ϕ *proving formal smoothness. Therefore* $\underline{rep}_n^{tr}\ C_k(\alpha)$ *is smooth and reduced hence is the orbit.* □

5.10 Invariant and equivariant maps.

Let A *be a Cayley-Hamilton algebra of degree* n *with trace map* tr*. Assume we have a decomposition*

$$1 = a_1 + \ldots + a_k$$

of 1 *as a sum of orthogonal idempotents* a_l *in* A*. Then, we have a trace preserving embedding*

$$C_k@_n \overset{\phi}{\hookrightarrow} A \quad \text{defined by} \quad X_i \mapsto a_i, \quad t_i \mapsto tr(a_i).$$

On the level of representation schemes this embedding gives rise to a morphism between the representation varieties

$$\underline{rep}_n^{tr}\ A \xrightarrow{\pi} \underline{rep}_n^{tr}\ C_k@_n$$

defined by composition. By the foregoing result we have a decomposition of $\underline{rep}_n^{tr}\ C_k@_n$ *as a disjoint union of finitely many orbits* $\mathcal{O}(\alpha)$ *determined by a dimension vector* $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}_+^k$ *such that* $\sum m_i = n$*. Therefore, we can similarly decompose*

$$\underline{rep}_n^{tr}\ A = \cup_\alpha \pi^{-1}\ \mathcal{O}(\alpha)$$

into a disjoint union of finitely many closed and open subschemes. We will denote the component $\pi^{-1}\ \mathcal{O}(\alpha)$ *by* $\underline{rep}_\alpha\ A$*. Observe that of course some of these* *components may be empty. On the level of coordinate algebras this decomposition translates itself into*

$$\mathbb{C}[\underline{rep}_n^{tr}\ A] = \oplus_\alpha \mathbb{C}[\underline{rep}_\alpha\ A] \quad \text{and} \quad M_n(\mathbb{C}[\underline{rep}_n^{tr}\ A]) = \oplus_\alpha M_n(\mathbb{C}[\underline{rep}_\alpha\ A])$$

Since each of the components $\underline{rep}_\alpha\ A$ is stable under the GL_n-action we have that

$$M_n(\mathbb{C}[\underline{rep}_n^{tr}\ A])^{GL_n} = \oplus_\alpha M_n(\mathbb{C}[\underline{rep}_\alpha\ A])^{GL_n}$$

as the left term equals A this finishes the proof of the following result.

Proposition 5.33 *Let A be a Cayley-Hamilton algebra of degree n having a decomposition $1 = a_1 + \ldots + a_k$ into orthogonal idempotents, then*

1. *$A = \oplus_\alpha A_\alpha$ the sum ranging over all dimension vectors $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}_+^k$ satisfying $\sum m_i = n$.*
2. *The projection of a_i in the component A_α has trace m_i where $\alpha = (m_1, \ldots, m_k)$.*

Again, observe that usually most components in the above direct sum decomposition are zero. We will now concentrate on one of the components A_α, that is we assume that A is a Cayley-Hamilton algebra of degree n with decomposition $1 = a_1 + \ldots + a_n$ into orthogonal idempotents such that $tr(a_i) = m_i \in \mathbb{N}_+$ and $\sum m_i = n$. Then, we have a trace preserving embedding $C_k(\alpha) \xhookrightarrow{i} A$ making A into a $C_k(\alpha) = \times_{i=1}^k \mathbb{C}$-algebra. We have constructed a trace preserving embedding $C_k(\alpha) \xhookrightarrow{i'} M_n(\mathbb{C})$ by sending the idempotent e_i to the diagonal idempotent $d_i \in M_n(\mathbb{C})$ with ones from position $\sum_{j=1}^{i-1} m_j - 1$ to $\sum_{j=1}^{i} m_i$. This calls for the introduction of a restricted representation scheme of all trace preserving algebra morphisms χ such that the diagram below is commutative

$$\begin{tikzcd}
A \arrow[r, "\chi"] & M_n(\mathbb{C}) \\
C_k(\alpha) \arrow[u, hook, "i"] \arrow[ur, hook, "i'"'] &
\end{tikzcd}$$

that is, such that $\chi(a_i) = d_i$. This again determines an affine scheme $\underline{rep}_\alpha^{res}\ A$ which is in fact a closed subscheme of $\underline{rep}_n^{tr}\ A$. The functorial description of the restricted module scheme is as follows. Let C be any commutative $\mathbb{C}$-algebra, then $M_n(C)$ is a $C_k(\alpha)$-algebra and the idempotents d_i allow for a block decomposition

$$M_n(C) = \oplus_{i,j} d_i M_n(C) d_j = \begin{bmatrix} d_1 M_n(C) d_1 & \ldots & d_1 M_n(C) d_k \\ \vdots & & \vdots \\ d_k M_n(C) d_1 & \ldots & d_k M_n(C) d_k \end{bmatrix}.$$

The scheme $\underline{rep}_\alpha^{res}\ A$ assigns to the algebra C the set of all trace preserving algebra maps

$$A \xrightarrow{\phi} M_n(B) \qquad \text{such that } \phi(a_i) = d_i.$$

Equivalently, the idempotents a_i decompose A into block form $A = \oplus_{i,j} a_i A a_j$ and then $\underline{rep}_\alpha^{res}\ A(C)$ are the trace preserving algebra morphisms $A \longrightarrow M_n(B)$ compatible with the block decompositions.

The embedding $C_k(\alpha) \overset{i'}{\hookrightarrow} M_n(\mathbb{C})$ sending e_i to d_i gives a special point p of the homogeneous variety

$$\underline{rep}_n^{tr}\ C_k(\alpha) = GL_n/(GL_{m_1} \times \ldots \times GL_{m_k}).$$

Still another description of the restricted representation scheme is therefore that $\underline{rep}_\alpha^{res}\ A$ is the scheme theoretic fiber $\pi^{-1}(p)$ of the point p under the GL_n-equivariant morphism

$$\underline{rep}_n^{tr}\ A \xrightarrow{\pi} \underline{rep}_n^{tr}\ C_k(\alpha).$$

Hence, the stabilizer subgroup of p acts on $\underline{rep}_\alpha^{res}\ A$. This stabilizer is the subgroup $GL(\alpha) = GL_{m_1} \times \ldots \times GL_{m_k}$ embedded in GL_n along the diagonal

$$GL(\alpha) = \begin{bmatrix} GL_{m_1} & & \\ & \ddots & \\ & & GL_{m_k} \end{bmatrix} \hookrightarrow GL_n.$$

Clearly, $GL(\alpha)$ acts via this embedding by conjugation on $M_n(\mathbb{C})$. The main implication of the existence of a decomposition of 1 into orthogonal idempotents is the following reduction result both of the affine scheme and of the acting group.

Theorem 5.34 *Let A be a Cayley-Hamilton algebra of degree n such that $1 = a_1 + \ldots + a_k$ is a decomposition into orthogonal idempotents with $tr(a_i) = m_i \in \mathbb{N}_+$. Then, A is isomorphic to the ring of $GL(\alpha)$-equivariant maps*

$$\underline{rep}_\alpha^{res}\ A \longrightarrow M_n.$$

Proof. *We know that A is the ring of GL_n-equivariant maps $\underline{rep}_n^{tr}\ A \longrightarrow M_n$. Further, we have a GL_n-equivariant map*

$$\underline{rep}_n^{tr}\ A \xrightarrow{\pi} \underline{rep}_n tr\ C_k(\alpha) = GL_n.p \simeq GL_n/GL(\alpha)$$

Thus, the GL_n-equivariant maps from $\underline{rep}_n^{tr}\ A$ to M_n coincide with the $Stab(p) = GL(\alpha)$-equivariant maps from the fiber $\pi^{-1}(p) = \underline{rep}_\alpha^{res}\ A$ to M_n. □

That is, we have a block matrix decomposition for A. Indeed, we have

$$A \simeq (\mathbb{C}[\underline{rep}_\alpha^{res}\ A] \otimes M_n(\mathbb{C}))^{GL(\alpha)}$$

and this isomorphism is clearly compatible with the block decomposition and thus we have for all i, j that

$$a_i A a_j \simeq (\mathbb{C}[\underline{rep}_\alpha^{res}\ A] \otimes M_{m_i \times m_j}(\mathbb{C}))^{GL(\alpha)}$$

where $M_{m_i \times m_j}(\mathbb{C})$ is the space of rectangular $m_i \times m_j$ matrices M with coefficients in $\mathbb{C}$ on which $GL(\alpha)$ acts via

$$g.M = g_i M g_j^{-1} \quad \text{where } g = (g_1, \ldots, g_k) \in GL(\alpha).$$

Another consequence of a idempotent decomposition is.

Theorem 5.35 *Let A be a Cayley-Hamilton algebra of degree n such that $1 = a_1 + \ldots + a_k$ is a decomposition into orthogonal idempotents with $tr(a_i) = m_i \in \mathbb{N}_+$. If the restricted representation scheme is a smooth $GL(\alpha)$-variety, then A is a smooth Cayley-Hamilton algebra.*

Proof. *Consider again the projection* $\underline{rep}_n^{tr}\ A \xrightarrow{\pi} \underline{rep}_n^{tr}\ C_k(\alpha)$. *As the base is a homogeneous variety it is smooth. Moreover all the fibers are isomorphic to* $\underline{rep}_\alpha^{res}\ A$ *whence smooth by assumption. Then, the total space* $\underline{rep}_n^{tr}\ A$ *is also smooth whence is* A *a smooth algebra.* □

If Q *be a quiver on* k *vertices, then the vertex idempotents* e_i *give a decomposition* $1 = e_1 + \ldots + e_k$ *of one into orthogonal idempotents and make the path algebra* $\mathbb{C}Q$ *into a* C_k*-algebra. Fix a dimension vector* $\alpha = (d_1, \ldots, d_k) \in \mathbb{N}^k$ *and let* $n = \sum d_i$. *Observe that we may assume that* $\alpha \in \mathbb{N}_+^k$ *(and hence that the map* $C_k \xrightarrow{\phi} \mathbb{C}Q$ *is an embedding. If not, we can restrict to the full subquiver of* Q *on the vertices* v_i *such that* $d_i \neq 0$.

The algebra embedding $C_k \xrightarrow{\phi} \mathbb{C}Q$ *determines a morphism*

$$\underline{rep}_n\ \mathbb{C}Q \xrightarrow{\pi} \underline{rep}_n\ C_k = \cup_\beta \mathcal{O}(\beta)$$

where the disjoint union is taken over all the dimension vectors $\beta = (b_1, \ldots, b_k)$ *such that* $n = \sum b_i$. *Again, consider the point* $p \in \mathcal{O}(\alpha)$ *determined by sending the idempotents* e_i *to the canonical diagonal idempotents of* $M_n(\mathbb{C})$. *As* $rep_\alpha\ Q$ *can be identified with the variety of* n*-dimensional representations of* $\mathbb{C}Q$ *in block form determined by these idempotents we see that* $rep_\alpha\ Q = \pi^{-1}(p)$.

We construct the algebra $\mathbb{T}_\alpha\ Q$ *as follows : first adjoin formally all traces to the path algebra* $\mathbb{C}Q$*, that is, consider the path algebra of* Q *over the polynomial algebra* R *in the variables* t_p *where* p *is a word in the arrows* $a_j \in Q_a$ *and is determined only up to cyclic permutation. As a consequence we only retain the variables* t_p *where* p *is an oriented cycle in* Q *(as all the others have a cyclic permutation which is the zero element in* $\mathbb{C}Q$*). This way we put a formal trace map on* $R \otimes \mathbb{C}Q$ *by defining* $tr(p) = t_p$ *is* p *is an oriented cycle in* Q *and* $tr(p) = 0$ *otherwise.*

The algebra $\mathbb{T}_\alpha\ Q$ *is obtained from this formal trace algebra* $R \otimes \mathbb{C}Q$ *by dividing out the substitution invariant twosided ideal generated by all the evaluations of the formal Cayley-Hamilton algebras of degree* n, $\chi_a^{(n)}(a)$ *for* $a \in R \otimes \mathbb{C}Q$ *together with the additional relations that* $tr(e_i) = d_i$. $\mathbb{T}_\alpha\ Q$ *is a Cayley-Hamilton algebra of degree* n *with a decomposition* $1 = e_1 + \ldots + e_k$ *into orthogonal idempotents such that* $tr(e_i) = d_i$. *Consequently, the restricted representation scheme*

$$\underline{rep}_\alpha^{res}\ \mathbb{T}_\alpha\ Q \simeq rep_\alpha\ Q$$

as $GL(\alpha)$*-varieties. Summarizing the results proved before in this special we obtain*

Theorem 5.36 *With notations as before,*

1. *The algebra* $\mathbb{T}_\alpha\ Q$ *is a smooth Cayley-Hamilton algebra.*

2. $\mathbb{T}_\alpha\ Q$ *is the algebra of* $GL(\alpha)$*-equivariant maps from* $rep_\alpha\ Q$ *to* M_n*, that is,*

$$\mathbb{T}_\alpha\ Q = M_n(\mathbb{C}[rep_\alpha\ Q])^{GL(\alpha)}$$

3. *The ring of* $GL(\alpha)$*-polynomial invariants of* $rep_\alpha\ Q$,

$$\mathbb{N}_\alpha\ Q = \mathbb{C}[rep_\alpha\ Q]^{GL(\alpha)}$$

is generated by traces along oriented cycles in the quiver Q *of length bounded by* $n^2 + 1$.

A realization of these algebras is as follows. To an arrow $j \xleftarrow{a} i$ *corresponds a* $d_j \times d_i$ *matrix of variables from* $\mathbb{C}[rep_\alpha\ Q]$

$$\boxed{M_a} = \begin{bmatrix} x_{11}(a) & \ldots & \ldots & x_{1d_i}(a) \\ \vdots & & & \vdots \\ x_{d_j1}(a) & \ldots & \ldots & x_{d_jd_i}(a) \end{bmatrix}$$

where $x_{ij}(a)$ *are the coordinate functions of the entries of* V_a *of a representation* $V \in rep_\alpha\ Q$*. Let* $p = a_1a_2\ldots a_r$ *be an oriented cycle in* Q*, then we can compute the following matrix*

$$M_p = M_{a_r} \ldots M_{a_2}M_{a_1}$$

over $\mathbb{C}[rep_\alpha\ Q]$*. As we have that* $s(a_r) = t(a_1) = v_i$*, this is a square* $d_i \times d_i$ *matrix with coefficients in* $\mathbb{C}[rep_\alpha\ Q]$ *and we can take its ordinary trace*

$$Tr(M_p) \in \mathbb{C}[rep_\alpha\ Q].$$

Then, $\mathbb{N}_\alpha\ Q$ *is the* $\mathbb{C}$*-subalgebra of* $\mathbb{C}[rep_\alpha\ Q]$ *generated by these elements.*

Consider the block structure of $M_n(\mathbb{C}[rep_\alpha\ Q])$ *with respect to the idempotents* e_i

$$\begin{bmatrix} M_{d_1}(S) & \ldots & \ldots & M_{d_1\times d_k}(S) \\ \vdots & & & \vdots \\ \vdots & M_{d_j\times d_i}(S) & & \vdots \\ M_{d_k\times d_i}(S) & \ldots & \ldots & M_{d_k}(S) \end{bmatrix}$$

where $S = \mathbb{C}[rep_\alpha\ Q]$*. Then, we can also view the matrix* M_a *for an arrow* $j \xleftarrow{a} i$ *as a block matrix in* $M_n(\mathbb{C}[rep_\alpha\ Q])$

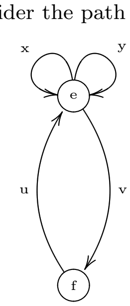
$$\begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & & & \vdots \\ \vdots & \boxed{M_a} & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}$$

Then, $\mathbb{T}_\alpha\ Q$ *is the* $C_k(\alpha)$*-subalgebra of* $M_n(\mathbb{C}[rep_\alpha\ Q])$ *generated by* $\mathbb{N}_\alpha\ Q$ *and these block matrices for all arrows* $a \in Q_a$*.* $\mathbb{T}_\alpha\ Q$ *itself has a block decomposition*

$$\mathbb{T}_\alpha\ Q = \begin{bmatrix} P_{11} & \ldots & \ldots & P_{1k} \\ \vdots & & & \vdots \\ \vdots & P_{ij} & & \vdots \\ P_{k1} & \ldots & \ldots & P_{kk} \end{bmatrix}$$

where P_{ij} *is the* $\mathbb{N}_\alpha\ Q$*-module spanned by all matrices* M_p *where* p *is a path from* v_i *to* v_j *of length bounded by* n^2*.*

Example 5.37 This result proves the claims we made in chapter 1 on the algebras related to the study of Calogero particles. For consider the path algebra $\mathbb{M}$ of the quiver

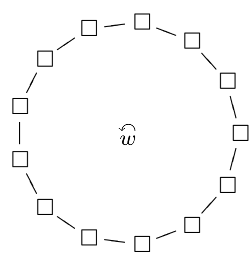


and take as dimension vector $\alpha = (n, 1)$. The total dimension is in this case $\overline{n} = n+1$ and we fix the embedding $C_2 = \mathbb{C} \times \mathbb{C} \hookrightarrow \mathbb{M}$ given by the decomposition $1 = e + f$. Then, the above realization of $\mathbb{T}_\alpha\ \mathbb{M}$ consists in taking the following $\overline{n} \times \overline{n}$ matrices

$$e_n = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad f_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad x_n = \begin{bmatrix} x_{11} & \dots & x_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \dots & x_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$y_n = \begin{bmatrix} y_{11} & \dots & y_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ y_{n1} & \dots & y_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad u_n = \begin{bmatrix} 0 & \dots & 0 & u_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & u_n \\ 0 & \dots & 0 & 0 \end{bmatrix} \quad v_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ v_1 & \dots & v_n & 0 \end{bmatrix}$$

In order to determine the ring of $GL(\alpha)$-polynomial invariants of $rep_\alpha\ \mathbb{M}$ we have to consider the traces along oriented cycles in the quiver. Any nontrivial such cycle must pass through the vertex e and then we can decompose the cycle into factors x, y and uv (observe that if we wanted to describe circuits based at the vertex f they are of the form $c = vc'u$ with c' a circuit based at e and we can use the cyclic property of traces to bring it into the claimed form). That is, all relevant oriented cycles in the quiver can be represented by a necklace word w



where each bead is one of the elements

$$\boxed{\bullet} = x \qquad \boxed{\circ} = y \quad \text{and} \quad \boxed{\blacktriangledown} = uv$$

In calculating the trace, we first have to replace each occurrence of x, y, u or v by the relevant $\overline{n} \times \overline{n}$-matrix above. This results in replacing each of the beads in the necklace by one of the following $n \times n$ matrices

$$\boxed{\bullet} = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \quad \boxed{\circ} = \begin{bmatrix} y_{11} & \dots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \dots & y_{nn} \end{bmatrix} \quad \boxed{\blacktriangledown} = \begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ \vdots & & \vdots \\ u_nv_1 & \dots & u_nv_n \end{bmatrix}$$

and taking the trace of the $n \times n$ matrix obtained after multiplying these bead-matrices cyclicly in the indicated orientation. This concludes the description of the invariant ring $\mathbb{N}_\alpha\ \mathbb{Q}$. The algebra $\mathbb{T}_\alpha\ \mathbb{M}$ of $GL(\alpha)$-equivariant maps from $rep_\alpha\ \mathbb{M}$ to $M_{\overline{n}}$, that is, $\mathbb{T}_\alpha\ \mathbb{M} = \mathbb{M}(n)$ defined in chapter 1, is then the subalgebra of $M_{\overline{n}}(\mathbb{C}[rep_\alpha\ \mathbb{M}])$ generated as $C_2(\alpha)$-algebra (using the idempotent $\overline{n} \times \overline{n}$ matrices corresponding to e and f) by $\mathbb{N}_\alpha\ \mathbb{M}$ and the $\overline{n} \times \overline{n}$-matrices corresponding to x, y, u and v.

We will have to extend these results to a marked quiver $Q^\bullet$. Let $\{l_1, \dots, l_m\}$ be the marked loops in $Q^\bullet$, then we define

$$\begin{cases} \mathbb{N}_\alpha\ Q^\bullet = \frac{\mathbb{N}_\alpha\ Q}{(tr(l_1),\dots,tr(l_m))} \\ \mathbb{T}_\alpha\ Q^\bullet = \frac{\mathbb{T}_\alpha\ Q}{(tr(l_1),\dots,tr(l_m))} \end{cases}$$

Clearly, $\mathbb{T}_\alpha\ Q^\bullet$ is a Cayley-Hamilton algebra of degree n having a decomposition $1 = e_1 + \dots + e_k$ into orthogonal idempotents and such that

$$\underline{rep}_\alpha^{res}\ \mathbb{T}(Q^\bullet, \alpha) = rep_\alpha\ Q^\bullet$$

Theorem 5.38 *1. The algebra $\mathbb{T}_\alpha\ Q^\bullet$ is a smooth Cayley-Hamilton algebra.*

2. *$\mathbb{T}_\alpha\ Q^\bullet$ is the algebra of $GL(\alpha)$-equivariant maps from $rep_\alpha\ Q^\bullet$ to $M_n(\mathbb{C})$, that is,*

$$\mathbb{T}_\alpha\ Q^\bullet = M_n(\mathbb{C}[rep_\alpha\ Q^\bullet])^{GL(\alpha)}$$

3. *The algebra $\mathbb{N}_\alpha\ Q^\bullet$ is the ring of $GL(\alpha)$-polynomial invariants of $rep_\alpha\ Q^\bullet$,*

$$\mathbb{N}_\alpha\ Q^\bullet = \mathbb{C}[rep_\alpha\ Q^\bullet]^{GL(\alpha)}$$

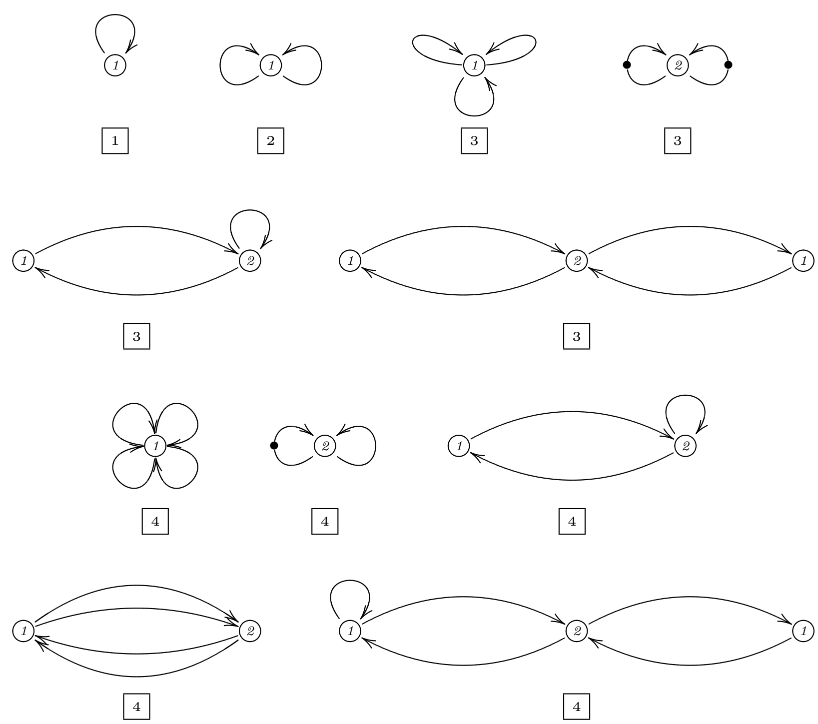
and is generated by traces along oriented cycles in the quiver Q of length bounded by n^2+1.

Combinatorics.

Chapter 6

Local Classification.

Every commutative smooth variety of dimension d is locally in the étale topology isomorphic to affine d-space $\mathbb{A}^d$. In this chapter we study the corresponding problem for Cayley-smooth orders A of degree n. In the foregoing chapter we have described the étale local structure of A near a point $\xi \in iss_n\ A$ by a marked quiver setting $(Q^\bullet_\xi, \alpha_\xi)$. In this chapter we will classify those quiver settings which can occur for given n and given dimension $d = dim\ iss_n\ A$ of the central variety. We will show that for fixed d and n only a finite number of such settings do arise, that is, Cayley-smooth algebras have a finite number of possible étale local behaviour. We prove this by simplifying a quiver-setting by shrinking (identifying arrow-connected vertices with the same vertex-dimension) to one of a finite list of settings in reduced form. For dimension $d \leq 4$, the complete list is



where the boxed value is the dimension d. To arrive at this result we need to classify the dimension vectors of simple representations of (marked) quivers and be able to compute the dimension of the corresponding quotient varieties. Further, we show that the local quiver setting $(Q^\bullet_\xi, \alpha_\xi)$ contains enough information to determine the *quiver-settings in nearby points and even to give the dimension of the strata of points of equal type. We then apply these results to the local characterization of Cayley-smooth orders over curves and surfaces. Smooth curve orders are proved to coincide with hereditary orders and smooth surface orders must have a smooth surface as their center, a ramification divisor having as worst singularities normal crossings and must be étale splittable. If we combine these results with the Artin-Mumford sequence, describing the Brauer group of the functionfield of a smooth projective surface X, we are able to prove that any central simple algebra Δ of degree n over $\mathbb{C}(X)$ contains a Cayley-Hamilton order $\mathcal{A}$ having at worst a finite number of noncommutative singularities, all of which are étale isomorphic to those appearing in the origin of a quantum plane. Finally, we classify all central simple algebras over $\mathbb{C}(X)$ admitting a Cayley-smooth model.*

Whereas we restrict attention mainly to orders, it is clear that the strategy can be extended to Cayley-Hamilton algebras of degree n which are finite modules over a central subring C, provided we have some control on the commutative extension $C \hookrightarrow Z(A)$. For an application of this to the theory of quantum groups, the reader may consult [19].

6.1 Marked quivers.

In this section we recall the basics on representations of (marked) quivers. Recall that a quiver Q is a directed graph determined by

- *a finite set $Q_v = \{v_1, \ldots, v_k\}$ of vertices, and*
- *a finite set $Q_a = \{a_1, \ldots, a_l\}$ of arrows where we allow multiple arrows between vertices and loops in vertices.*

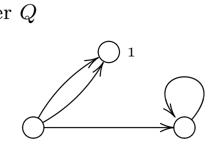
Every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ has a starting vertex $s(a) = i$ and a terminating vertex $t(a) = j$. Multiplication in the path algebra $\mathbb{C}Q$ is induced by (left) concatenation of paths. More precisely, $1 = v_1 + \ldots + v_k$ is a decomposition of 1 into mutually orthogonal idempotents and further we define

- *$v_j.a$ is always zero unless $\textcircled{j} \xleftarrow{a} \bigcirc$ in which case it is the path a,*
- *$a.v_i$ is always zero unless $\bigcirc \xleftarrow{a} \textcircled{i}$ in which case it is the path a,*
- *$a_i.a_j$ is always zero unless $\bigcirc \xleftarrow{a_i} \bigcirc \xleftarrow{a_j} \bigcirc$ in which case it is the path $a_i a_j$.*

The description of the quiver Q can be encoded in an integral $k \times k$ matrix

$$\chi_Q = \begin{bmatrix} \chi_{11} & \cdots & \chi_{1k} \\ \vdots & & \vdots \\ \chi_{k1} & \cdots & \chi_{kk} \end{bmatrix} \quad \text{with} \quad \chi_{ij} = \delta_{ij} - \#\ \{\ \textcircled{j} \longleftarrow \textcircled{i}\ \}$$

Example 6.1 Consider the quiver Q



Then, with the indicated ordering of the vertices we have that the integral matrix is

$$\chi_Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the path algebra of Q is isomorphic to the block-matrix algebra

$$\mathbb{C}Q' \simeq \begin{bmatrix} \mathbb{C} & \mathbb{C} \oplus \mathbb{C} & 0 \\ 0 & \mathbb{C} & 0 \\ 0 & \mathbb{C}[x] & \mathbb{C}[x] \end{bmatrix}$$

where x is the loop in vertex v_3.

The subspace $\mathbb{C}Qv_i$ *has as basis the paths starting in vertex* v_i *and because* $\mathbb{C}Q = \oplus_i \mathbb{C}Qv_i$, $\mathbb{C}Qv_i$ *is a projective left ideal of* $\mathbb{C}Q$. *Similarly,* $v_i\mathbb{C}Q$ *has as basis the paths ending at* v_i *and is a projective right ideal of* $\mathbb{C}Q$. *The subspace* $v_i\mathbb{C}Qv_j$ *has as basis the paths starting at* v_j *and ending at* v_i *and* $\mathbb{C}Qv_i\mathbb{C}Q$ *is the twosided ideal of* $\mathbb{C}Q$ *having as basis all paths passing through* v_i. *If* $0 \neq f \in \mathbb{C}Qv_i$ *and* $0 \neq g \in v_i\mathbb{C}Q$, *then* $f.g \neq 0$ *for let* p *be a longest path occurring in* f *and* q *a longest path in* g, *then the coefficient of* $p.q$ *in* $f.g$ *cannot be zero. As a consequence we have*

Lemma 6.2 *The projective left ideals* $\mathbb{C}Qv_i$ *are indecomposable and paired non-isomorphic.*

Proof. *If* $\mathbb{C}Qv_i$ *is not indecomposable, then there exists a projection idempotent* $f \in Hom_{\mathbb{C}Q}(\mathbb{C}Qv_i, \mathbb{C}Qv_i) \simeq v_i\mathbb{C}Qv_i$. *But then,* $f^2 = f = f.v_i$ *whence* $f.(f - v_i) = 0$, *contradicting the remark above. Further, for any left* $\mathbb{C}Q$*-module* M *we have that* $Hom_{\mathbb{C}Q}(\mathbb{C}Qv_i, M) \simeq v_iM$. *So, if* $\mathbb{C}Qv_i \simeq \mathbb{C}Qv_j$ *then the isomorphism gives elements* $f \in v_i\mathbb{C}Qv_j$ *and* $g \in v_j\mathbb{C}Qv_i$ *such that* $f.g = v_i$ *and* $g.f = v_j$. *But then,* $v_i \in \mathbb{C}Qv_j\mathbb{C}Q$, *a contradiction unless* $i = j$ *as this space has basis all paths passing through* v_j. $\square$

Example 6.3 Let Q be a quiver, then the following properties hold :

1. $\mathbb{C}Q$ is finite dimensional if and only if Q has no oriented cycles.
2. $\mathbb{C}Q$ is prime (that is, $I.J \neq 0$ for all twosided ideals $I, J \neq 0$) if and only if Q is *strongly connected*, that is, for all vertices v_i and v_j there is a path from v_i to v_j.
3. $\mathbb{C}Q$ is Noetherian (that is, satisfies the ascending chain condition on left (or right) ideals) if and only if for every vertex v_i belonging to an oriented cycle there is only one arrow starting at v_i and only one arrow terminating at v_i.
4. The radical of $\mathbb{C}Q$ has as basis all paths from v_i to v_j for which there is no path from v_j to v_i.
5. The center of $\mathbb{C}Q$ is of the form $\mathbb{C} \times \ldots \times \mathbb{C} \times \mathbb{C}[x] \times \ldots \times \mathbb{C}[x]$ with one factor for each connected component C of Q (that is, connected component for the underlying graph forgetting the orientation) and this factor is isomorphic to $\mathbb{C}[x]$ if and only if C is one oriented cycle.

Recall that a representation V *of the quiver* Q *is given by*

- *a finite dimensional* $\mathbb{C}$*-vector space* V_i *for each vertex* $v_i \in Q_v$, *and*
- *a linear map* $V_j \xleftarrow{V_a} V_i$ *for every arrow* $(j) \xleftarrow{a} (i)$ *in* Q_a.

If $\dim V_i = d_i$ *we call the integral vector* $\alpha = (d_1, \ldots, d_k) \in \mathbb{N}^k$ *the dimension vector of* V *and denote it with* $\dim V$. *A morphism* $V \xrightarrow{\phi} W$ *between two representations* V *and* W *of* Q *is determined by a set of linear maps*

$$V_i \xrightarrow{\phi_i} W_i \qquad \text{for all vertices } v_i \in Q_v$$

satisfying the following compatibility conditions for every arrow $(j) \xleftarrow{a} (i)$ *in* Q_a

$$\begin{tikzcd} V_i \arrow[r, "V_a"] \arrow[d, "\phi_i"] & V_j \arrow[d, "\phi_j"] \\ W_i \arrow[r, "W_a"] & W_j \end{tikzcd}$$

Clearly, composition of morphisms $V \xrightarrow{\phi} W \xrightarrow{\psi} X$ *is given by the rule that* $(\psi \circ \phi)_i = \psi_i \circ \psi_i$ *and one readily verifies that this is again a morphism of representations of* Q. *In this way we form a category rep* Q *of all finite dimensional representations of the quiver* Q.

Proposition 6.4 *The category rep* Q *is equivalent to the category of finite dimensional* $\mathbb{C}Q$*-representations (left modules).*

Proof. *Let* M *be an* n*-dimensional* $\mathbb{C}Q$*-representation. Then, we construct a representation* V *of* Q *by taking*

- $V_i = v_i M$, *and for any arrow* $\textcircled{j} \xleftarrow{a} \textcircled{i}$ *in* Q_a *define*
- $V_a : V_i \longrightarrow V_j$ *by* $V_a(x) = v_j a x$.

Observe that the dimension vector $dim(V) = (d_1, \ldots, d_k)$ *satisfies* $\sum d_i = n$. *If* $\phi : M \longrightarrow N$ *is* $\mathbb{C}Q$*-linear, then we have a linear map* $V_i = v_i M \xrightarrow{\phi_i} W_i = v_i N$ *which clearly satisfies the compatibility condition.*

Conversely, let V *be a representation of* Q *with dimension vector* $dim(V) = (d_1, \ldots, d_k)$. *Then, consider the* $n = \sum d_i$*-dimensional space* $M = \oplus_i V_i$ *which we turn into a* $\mathbb{C}Q$*-representation as follows. Consider the canonical injection and projection maps* $V_j \xhookrightarrow{i_j} M \xrightarrow{\pi_j} V_j$. *Then, define the action of* $\mathbb{C}Q$ *by fixing the action of the algebra generators* v_j *and* a_l *to be*

$$\begin{cases} v_j m & = i_j(\pi_j(m)) \\ a_l m & = i_j(V_a(\pi_i(m))) \end{cases}$$

for all arrows $\textcircled{j} \xleftarrow{a_l} \textcircled{i}$. *A computation verifies that these two operations are inverse to each other and induce an equivalence of categories.* $\square$

The Euler form *of the quiver* Q *is the bilinear form on* $\mathbb{Z}^k$

$$\chi_Q(.,.) : \mathbb{Z}^k \times \mathbb{Z}^k \longrightarrow \mathbb{Z} \quad \text{defined by} \quad \chi_Q(\alpha, \beta) = \alpha.\chi_Q.\beta^\tau$$

for all row vectors $\alpha, \beta \in \mathbb{Z}^k$.

Theorem 6.5 *Let* V *and* W *be two representations of* Q, *then*

$$dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, W) - dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V, W) = \chi_Q(dim(V), dim(W))$$

Proof. *We claim that there exists an exact sequence of* $\mathbb{C}$*-vectorspaces*

$$0 \longrightarrow Hom_{\mathbb{C}Q}(V, W) \xrightarrow{\gamma} \oplus_{v_i \in Q_v} Hom_{\mathbb{C}}(V_i, W_i) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} \oplus_{a \in Q_a} Hom_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) \xrightarrow{\epsilon} Ext^1_{\mathbb{C}Q}(V, W) \longrightarrow 0$$

Here, $\gamma(\phi) = (\phi_1, \ldots, \phi_k)$ *and* δ *maps a family of linear maps* $(f_1, \ldots, f_k)$ *to the linear maps* $\mu_a = f_{t(a)} V_a - W_a f_{s(a)}$ *for any arrow* a *in* Q, *that is, to the obstruction of the following diagram to be commutative*

$$\begin{tikzcd} V_{s(a)} \arrow[r, "V_a"] \arrow[d, "f_{s(a)}"] \arrow[dr, dotted, "\mu_a"] & V_{t(a)} \arrow[d, "f_{t(a)}"] \\ W_{s(a)} \arrow[r, "W_a"] & W_{t(a)} \end{tikzcd}$$

By the definition of morphisms between representations of Q it is clear that the kernel of δ coincides with $Hom_{\mathbb{C}Q}(V,W)$.

Further, the map ϵ is defined by sending a family of maps $(g_1,\ldots,g_s) = (g_a)_{a\in Q_a}$ to the equivalence class of the exact sequence

$$0 \longrightarrow W \xrightarrow{i} E \xrightarrow{p} V \longrightarrow 0$$

where for all $v_i \in Q_v$ we have $E_i = W_i \oplus V_i$ and the inclusion i and projection map p are the obvious ones and for each generator $a \in Q_a$ the action of a on E is defined by the matrix

$$E_a = \begin{bmatrix} W_a & g_a \\ 0 & V_a \end{bmatrix} : E_{s(a)} = W_{s(a)} \oplus V_{s(a)} \longrightarrow W_{t(a)} \oplus V_{t(a)} = E_{t(a)}$$

Clearly, this makes E into a $\mathbb{C}Q$-module and one verifies that the above short exact sequence is one of $\mathbb{C}Q$-modules. Remains to prove that the cokernel of δ can be identified with $Ext^1_{\mathbb{C}Q}(V,W)$. For this, we need to look back at the description of Ext^1 in terms of cycles and boundaries.

A set of algebra generators of $\mathbb{C}Q$ is given by $\{v_1,\ldots,v_k,a_1,\ldots,a_l\}$. A cycle is given by a linear map $\lambda : \mathbb{C}Q \longrightarrow Hom_{\mathbb{C}}(V,W)$ such that for all $f,f' \in \mathbb{C}Q$ we have the condition

$$\lambda(ff') = \rho(f)\lambda(f') + \lambda(f)\sigma(f')$$

where ρ determines the action on W and σ that on V. First, consider v_i then the condition says $\lambda(v_i^2) = \lambda(v_i) = p_i^W\lambda(v_i) + \lambda(v_i)p_i^V$ whence $\lambda(v_i) : V_i \longrightarrow W_i$ but then applying again the condition we see that $\lambda(v_i) = 2\lambda(v_i)$ so $\lambda(v_i) = 0$. Similarly, using the condition on $a = v_{t(a)}a = av_{s(a)}$ we deduce that $\lambda(a) : V_{s(a)} \longrightarrow W_{t(a)}$. That is, we can identify $\oplus_{a\in Q_a} Hom_{\mathbb{C}}(V_{s(a)},W_{t(a)})$ with $Z(V,W)$ under the map ϵ. Moreover, the image of δ gives under δ rise to a family of morphisms $\lambda(a) = f_{t(a)}V_a - W_af_{s(a)}$ for a linear map $f = (f_i) : V \longrightarrow W$ so this image coincides precisely to the subspace of boundaries $B(V,W)$ proving that indeed the cokernel of δ is $Ext^1_{\mathbb{C}Q}(V,W)$ finishing the proof of exactness of the long sequence of vectorspaces. But then, if $dim(V) = (r_1,\ldots,r_k)$ and $dim(W) = (s_1,\ldots,s_k)$, we have that $dim\ Hom(V,W) - dim\ Ext^1(V,W)$ is equal to

$$\begin{aligned}
&\sum_{v_i\in Q_v} dim\ Hom_{\mathbb{C}}(V_i,W_i) - \sum_{a\in Q_a} dim\ Hom_{\mathbb{C}}(V_{s(a)},W_{t(a)}) \\
&= \sum_{v_i\in Q_v} r_is_i - \sum_{a\in Q_a} r_{s(a)}s_{t(a)} \\
&= (r_1,\ldots,r_k)M_Q(s_1,\ldots,s_k)^\tau = \chi_Q(dim(V),dim(W))
\end{aligned}$$

finishing the proof. □

Fix a dimension vector $\alpha = (d_1,\ldots,d_k) \in \mathbb{N}^k$ and consider the set $rep_\alpha\ Q$ of all representations V of Q such that $dim(V) = \alpha$. Because V is completely determined by the linear maps

$$V_a : V_{s(a)} = \mathbb{C}^{d_{s(a)}} \longrightarrow \mathbb{C}^{d_{t(a)}} = V_{t(a)}$$

we see that $rep_\alpha\ Q$ is the affine space

$$rep_\alpha\ Q = \bigoplus_{\textcircled{j} \xleftarrow{a} \textcircled{i}} M_{d_j\times d_i}(\mathbb{C}) \simeq \mathbb{C}^r$$

where $r = \sum_{a \in Q_a} d_{s(a)} d_{t(a)}$. *On this affine space we have an action of the algebraic group* $GL(\alpha) = GL_{d_1} \times \ldots \times GL_{d_k}$ *by conjugation. That is, if* $g = (g_1, \ldots, g_k) \in GL(\alpha)$ *and if* $V = (V_a)_{a \in Q_a}$ *then* $g.V$ *is determined by the matrices*

$$(g.V)_a = g_{t(a)} V_a g_{s(a)}^{-1}.$$

If V *and* W *in* $rep_\alpha\ Q$ *are isomorphic as representations of* Q*, such an isomorphism is determined by invertible matrices* $g_i : V_i \longrightarrow W_i \in GL_{d_i}$ *such that for every arrow* $\textcircled{j} \xleftarrow{a} \textcircled{i}$ *we have a commutative diagram*

$$\begin{tikzcd} V_i \arrow[r, "V_a"] \arrow[d, "g_i"] & V_j \arrow[d, "g_j"] \\ W_i \arrow[r, "W_a"] & W_j \end{tikzcd}$$

or equivalently, $g_j V_a = W_a g_i$. *That is, two representations are isomorphic if and only if they belong to the same orbit under* $GL(\alpha)$. *In particular, we see that*

$$Stab_{GL(\alpha)}\ V \simeq Aut_{\mathbb{C}Q}\ V$$

and the latter is an open subvariety of the affine space $End_{\mathbb{C}Q}(V) = Hom_{\mathbb{C}Q}(V, V)$ *whence they have the same dimension. The dimension of the orbit* $\mathcal{O}(V)$ *of* V *in* $rep_\alpha\ Q$ *is equal to*

$$dim\ \mathcal{O}(V) = dim\ GL(\alpha) - dim\ Stab_{GL(\alpha)}\ V.$$

But then we have a geometric reformulation of the above theorem.

Lemma 6.6 *Let* $V \in rep_\alpha\ Q$*, then*

$$dim\ rep_\alpha\ Q - dim\ \mathcal{O}(V) = dim\ End_{\mathbb{C}Q}(V) - \chi_Q(\alpha, \alpha) = dim\ Ext^1_{\mathbb{C}Q}(V, V)$$

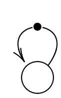
Proof. *We have seen that* $dim\ rep_\alpha\ Q - dim\ \mathcal{O}(V)$ *is equal to*

$$\sum_a d_{s(a)} d_{t(a)} - (\sum_i d_i^2 - dim\ End_{\mathbb{C}Q}(V)) = dim\ End_{\mathbb{C}Q}(V) - \chi_Q(\alpha, \alpha)$$

and the foregoing theorem asserts that the latter term is equal to dim $Ext^1_{\mathbb{C}Q}(V, V)$. $\square$

In particular it follows that the orbit $\mathcal{O}(V)$ *is open in* $rep_\alpha\ Q$ *if and only if* V *has no self-extensions. Moreover, as* $rep_\alpha\ Q$ *is irreducible there can be at most one isomorphism class of a representation without self-extensions.*

For our purposes we have to generalize the setting slightly. A marked quiver $Q^\bullet$ *has an underlying quiver* Q *such that some of its loops can acquire a marking. Such a marked loop will be depicted by*



A representation V *of a marked quiver* $Q^\bullet$ *is a representation of the underlying quiver* Q *such that the matrices corresponding to marked loops have trace equal to zero. If we fix a dimension vector* α,

$$rep_\alpha\ Q^\bullet = \{ V \in rep_\alpha\ Q \ \mid\ tr(V_a) = 0 \textit{ if } a \textit{ is a marked loop in } Q^\bullet \}$$

$rep_\alpha\ Q^\bullet$ is an affine subspace of $rep_\alpha\ Q$ of codimension equal to the number of marked loops in $Q^\bullet$. This subspace is stable under the action of $GL(\alpha)$ on $rep_\alpha\ Q$ and $GL(\alpha)$-orbits in $rep_\alpha\ Q^\bullet$ correspond to isomorphism classes of representations of $Q^\bullet$. The Euler form *of the marked quiver Q^* is the Euler form of the underlying quiver Q. We denote*

$$\chi_{ii} = 1 - u_i - m_i$$

where u_i is the number of unmarked loops at v_i and m_i is the number of marked loops at v_i.

6.2 Simple dimension vectors.

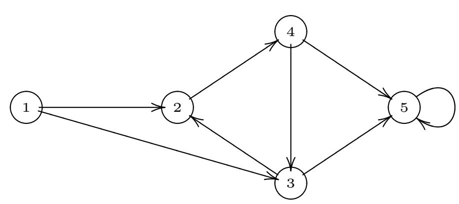
In this section we characterize the dimension vectors α such that the marked quiver $Q^\bullet$ has a simple representation V (that is, contains no proper subrepresentations) with $dim(V) = \alpha$.

Consider the underlying quiver Q with vertex set $Q_v = \{v_1, \ldots, v_k\}$. To a subset $S \hookrightarrow Q_v$ we associate the full subquiver Q_S of Q, that is, Q_S has as set of vertices the subset S and as set of arrows all arrows $(j) \xleftarrow{a} (i)$ in Q_a such that v_i and v_j belong to S. A full subquiver Q_S is said to be strongly connected *if and only if for all $v_i, v_j \in V$ there is an oriented cycle in Q_S passing through v_i and v_j. We can partition*

$$Q_v = S_1 \sqcup \ldots \sqcup S_s$$

such that the Q_{S_i} are maximal strongly connected components of Q. Clearly, the direction of arrows in Q between vertices in S_i and S_j is the same by the maximality assumption and can be used to define an orientation between S_i and S_j. The strongly connected component quiver *$SC(Q)$ is then the quiver on s vertices $\{w_1, \ldots, w_s\}$ with w_i corresponding to S_i and there is one arrow from w_i to w_j if and only if there is an arrow in Q from a vertex in S_i to a vertex in S_j. Observe that when the underlying graph of Q is connected, then so is the underlying graph of $SC(Q)$ and $SC(Q)$ is a quiver without oriented cycles.*

Example 6.7 Consider the connected quiver Q



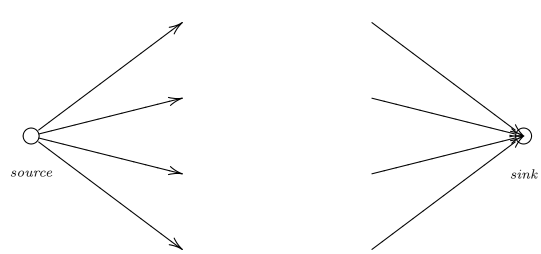
then the partitioning of Q_v into maximal strongly connected components is

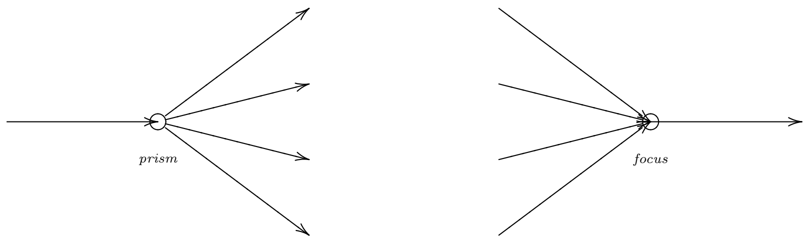
$$Q_v = \{v_1\} \sqcup \{v_2, v_3, v_4\} \sqcup \{v_5\}$$

and the strictly connected component quiver $SC(Q)$ of Q has the following form

$$(1) \longrightarrow (2) \longrightarrow (3)$$

We will give names to vertices with very specific in- and out-going arrows





For example, for the quiver Q of the above example, v_1 is a source, there are no sinks, v_2 and v_5 are focuses and v_4 is a prism (observe that a loop gives one incoming and one outgoing arrow). If $\alpha = (d_1, \ldots, d_k)$ is a dimension vector, then $supp(\alpha) = \{v_i \in Q_v \mid d_i \neq 0\}$.

Lemma 6.8 *If α is the dimension vector of a simple representation of Q, then $Q_{supp(\alpha)}$ is a strongly connected subquiver.*

Proof. *If not, we consider the strongly connected component quiver $SC(Q_{supp(\alpha)})$ and by assumption there must be a sink in it corresponding to a proper subset $S \overset{\neq}{\hookrightarrow} Q_v$. If $V \in rep_\alpha\ Q$ we can then construct a representation W by*

- $W_i = V_i$ *for* $v_i \in S$ *and* $W_i = 0$ *if* $v_i \notin S$,
- $W_a = V_a$ *for an arrow* a *in* Q_S *and* $W_a = 0$ *otherwise.*

One verifies that W is a proper subrepresentation of V, so V cannot be simple, a contradiction. $\square$

The second necessary condition involves the Euler form of Q. With ϵ_i be denote the dimension vector of the simple representation having a one-dimensional space at vertex v_i and zero elsewhere and all arrows zero matrices.

Lemma 6.9 *If α is the dimension vector of a simple representation of Q, then*

$$\begin{cases} \chi_Q(\alpha, \epsilon_i) & \leq 0 \\ \chi_Q(\epsilon_i, \alpha) & \leq 0 \end{cases}$$

for all $v_i \in supp(\alpha)$.

Proof. *Let V be a simple representation of Q with dimension vector $\alpha = (d_1, \ldots, d_k)$. One verifies that*

$$\chi_Q(\epsilon_i, \alpha) = d_i - \sum_{(j) \xleftarrow{a} (i)} d_j$$

Assume that $\chi_Q(\epsilon_i, \alpha) > 0$, then the natural linear map

$$\bigoplus_{(j) \xleftarrow{a} (i)} V_a \ : \ V_i \longrightarrow \bigoplus_{(j) \xleftarrow{a} (i)} V_j$$

has a nontrivial kernel, say K. But then we consider the representation W of Q determined by

- $W_i = K$ *and* $W_j = 0$ *for all* $j \neq i$,
- $W_a = 0$ *for all* $a \in Q_a$.

It is clear that W is a proper subrepresentation of V, a contradiction.

Similarly, assume that $\chi_Q(\alpha, \epsilon_i) = d_i - \sum_{(i) \xleftarrow{a} (j)} d_j > 0$, *then the linear map*

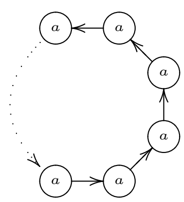
$$\bigoplus_{(i) \xleftarrow{a} (j)} V_a \; : \; \bigoplus_{(i) \xleftarrow{a} (j)} V_j \longrightarrow V_i$$

has an image I which is a proper subspace of V_i. The representation W of Q determined by

- $W_i = I$ *and* $W_j = V_j$ *for* $j \neq i$,
- $W_a = V_a$ *for all* $a \in Q_a$.

is a proper subrepresentation of V, a contradiction finishing the proof. □

Example 6.10 The necessary conditions of the foregoing two lemmas are not sufficient. Consider the extended Dynkin quiver of type $\tilde{A}_k$ with cyclic orientation.



and dimension vector $\alpha = (a, \ldots, a)$. For a simple representation all arrow matrices must be invertible but then, under the action of $GL(\alpha)$, they can be diagonalized. Hence, the only simple representations (which are not the trivial simples concentrated in a vertex) have dimension vector $(1, \ldots, 1)$.

Nevertheless, we will show that these are the only exceptions. A vertex v_i is said to be large *with respect to a dimension vector $\alpha = (d_1, \ldots, d_k)$ whenever d_i is maximal among the d_j. The vertex v_i is said to be* good *if v_i is large and has no direct successor which is a large prism nor a direct predecessor which is a large focus.*

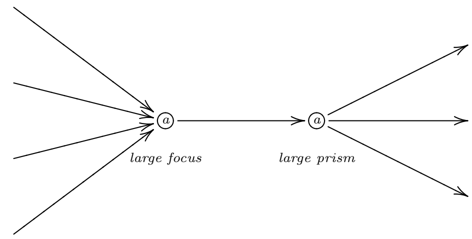
Lemma 6.11 *Let Q be a strongly connected quiver, not of type $\tilde{A}_k$, then one of the following hold*

1. *Q has a good vertex, or,*
2. *Q has a large prism having no direct large prism successors, or*
3. *Q has a large focus having no direct large focus predecessors.*

Proof. *If neither of the cases hold, we would have an oriented cycle in Q consisting of prisms (or consisting of focusses). Assume $(v_{i_1}, \ldots, v_{i_l})$ is a cycle of prisms, then the unique incoming arrow of v_{i_j} belongs to the cycle. As $Q \neq \tilde{A}_k$ there is at least one extra vertex v_a not belonging to the cycle. But then, there can be no oriented path from v_a to any of the v_{i_j}, contradicting the assumption that Q is strongly connected.* □

If we are in one of the two last cases, let a be the maximum among the components of the dimension vector α and assume that α satisfies $\chi_Q(\alpha, \epsilon_i) \leq 0$ and

$\chi_Q(\epsilon_i, \alpha) \leq 0$ for all $1 \leq i \leq k$, then we have the following subquiver in Q

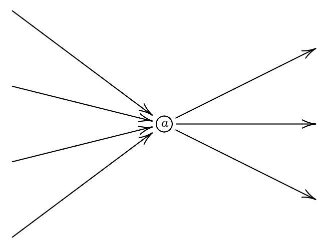


We can reduce to a quiver situation with strictly less vertices.

Lemma 6.12 *Assume Q is strongly connected and we have a vertex v_i which is a prism with unique predecessor the vertex v_j which is a focus. Consider the dimension vector $\alpha = (d_1, \ldots, d_k)$ with $d_i = d_j = a \neq 0$. Then, α is the dimension of a simple representation of Q if and only if*

$$\alpha' = (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_k) \in \mathbb{N}^{k-1}$$

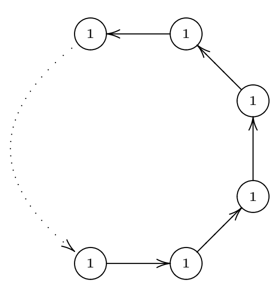
is the dimension vector of a simple representation of the quiver Q' on $k-1$ vertices, obtained from Q by identifying the vertices v_i and v_j, that is, the above subquiver in Q is simplified to the one below in Q'



Proof. *If b is the unique arrow from v_j to v_i and if $V \in rep_\alpha\ Q$ is a simple representation then V_b is an isomorphism, so we can identify V_i with V_j and obtain a simple representation of Q'. Conversely, if $V' \in rep_{\alpha'}\ Q'$ is a simple representation, define $V \in rep_\alpha\ Q$ by $V_i = V'_j$ and $V_z = V'_z$ for $z \neq i$, $V_{b'} = V'_{b'}$ for all arrows $b' \neq b$ and $V_b = \mathbb{1}_a$. Clearly, V is a simple representation of Q.* $\square$

Theorem 6.13 *$\alpha = (d_1, \ldots, d_k)$ is the dimension vector of a simple representation of Q if and only if one of the following two cases holds*

1. *$supp(\alpha) = \tilde{A}_k$, the extended Dynkin quiver on k vertices with cyclic orientation and $d_i = 1$ for all $1 \leq i \leq k$*



2. *$supp(\alpha) \neq \tilde{A}_k$. Then, $supp(\alpha)$ is strongly connected and for all $1 \leq i \leq k$ we have*

$$\begin{cases} \chi_Q(\alpha, \epsilon_i) & \leq 0 \\ \chi_Q(\epsilon_i, \alpha) & \leq 0 \end{cases}$$

Proof. *We will use induction, both on the number of vertices k in $supp(\alpha)$ and on the total dimension $n = \sum_i d_i$ of the representation. If $supp(\alpha)$ does not possess a good vertex, then the above lemma finishes the proof by induction on k. Observe that the Euler-form conditions are preserved in passing from Q to Q' as $d_i = d_j$.*

Hence, assume v_i is a good vertex in $supp(\alpha)$. If $d_i = 1$ then all $d_j = 1$ for $v_j \in supp(\alpha)$ and we can construct a simple representation by taking $V_b = 1$ for all arrows b in $supp(\alpha)$. Simplicity follows from the fact that $supp(\alpha)$ is strongly connected.

If $d_i > 1$, consider the dimension vector $\alpha' = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_k)$. Clearly, $supp(\alpha') = supp(\alpha)$ is strongly connected and we claim that the Euler-form conditions still hold for α'. the only vertices v_l where things might go wrong are direct predecessors or direct successors of v_i. Assume for one of them $\chi_Q(\epsilon_l, \alpha) > 0$ holds, then

$$d_l = d'_l > \sum_{\textcircled{m} \xleftarrow{a} \textcircled{l}} d'_m \geq d'_i = d_i - 1$$

But then, $d_l = d_i$ whence v_l is a large vertex of α and has to be also a focus with end vertex v_i (if not, $d_l > d_i$), contradicting goodness of v_i.

Hence, by induction on n we may assume that there is a simple representation $W \in rep_{\alpha'}\ Q$. Consider the space rep_W of representations $V \in rep_\alpha\ Q$ such that $V \mid \alpha' = W$. That is, for every arrow

$$\textcircled{j} \xleftarrow{\quad a \quad} \textcircled{i} \qquad V_a = \left[\begin{array}{c} W_a \\ \hline v_1 \quad \ldots \quad v_{d_j} \end{array}\right]$$

$$\textcircled{i} \xleftarrow{\quad a \quad} \textcircled{j} \qquad V_a = \left[\begin{array}{c|c} W_a & \begin{matrix} v_1 \\ \vdots \\ v_{d_j} \end{matrix} \end{array}\right]$$

Hence, rep_W is an affine space consisting of all representations degenerating to $W \oplus S_i$ where S_i is the simple one-dimensional representation concentrated in v_i. As $\chi_Q(\alpha', \epsilon_i) < 0$ and $\chi_Q(\epsilon_i, \alpha') < 0$ we have that $Ext^1(W, S_i) \neq 0 \neq Ext^1(S_i, W)$ so there is an open subset of representations which are not isomorphic to $W \oplus S_i$.

As there are simple representations of Q having a one-dimensional component at each vertex in $supp(\alpha)$ and as the subset of simple representations in $rep_{\alpha'}\ Q$ is open, we can choose W such that rep_W contains representations V such that a trace of an oriented cycle differs from that of $W \oplus S_i$. Hence, by the description of the invariant ring $\mathbb{C}[rep_\alpha\ Q]^{GL(\alpha)}$ as being generated by traces along oriented cycles and by the identification of points in the quotient variety as isomorphism classes of semi-simple representations, it follows that the Jordan-Hölder factors of V are different from W and S_i. In view of the definition of rep_W, this can only happen if V is a simple representation, finishing the proof of the theorem. □

From this result we can easily deduce the characterization of dimension vectors of simple representations of a marked quiver $Q^\bullet$. For, if l is a marked loop in a vertex v_i with $d_i > 1$, then we may replace the matrix V_l of a simple representation $V \in rep_\alpha\ Q$ by $V'_l = V_l - \frac{1}{d_i} \mathbb{1}_{d_i}$ and retain the property that V' is a simple representation. Things are different, however, for a marked loop in a vertex v_i with $d_i = 1$ as this 1×1-matrix factor is removed from the representation space. That is, we have the following characterization result.

Theorem 6.14 *$\alpha = (d_1, \ldots, d_k)$ is the dimension vector of a simple representation of a marked quiver $Q^\bullet$ if and only if $\alpha = (d_1, \ldots, d_k)$ is the dimension vector of a simple representation of the quiver Q' obtained from the underlying quiver Q of $Q^\bullet$ after removing the loops in Q which are marked in $Q^\bullet$ in all vertices v_i such that $d_i = 1$.*

6.3 The local quiver.

Consider the underlying quiver Q and a fixed dimension vector α. Closed $GL(\alpha)$-orbits is $rep_\alpha\ Q$ correspond to isomorphism classes of semi-simple representations of Q of dimension vector α. We have a quotient map

$$rep_\alpha\ Q \xrightarrow{\pi} rep_\alpha\ Q/GL(\alpha) = iss_\alpha\ Q$$

and we have seen that the coordinate ring $\mathbb{C}[iss_\alpha\ Q]$ is generated by traces along oriented cycles in the quiver Q. Consider a point $\xi \in iss_\alpha\ Q$ and assume that the corresponding semi-simple representation V_ξ has a decomposition

$$V_\xi = V_1^{\oplus e_1} \oplus \ldots \oplus V_z^{\oplus e_z}$$

into distinct simple representations V_i of dimension vector say α_i and occurring in V_ξ with multiplicity e_i. We then say that ξ is a point of representation-type

$$\tau = t(\xi) = (e_1, \alpha_1; \ldots, e_z, \alpha_z) \quad with \quad \alpha = \sum_{i=1}^{z} e_i \alpha_i$$

We want to apply the Luna slice theorem to obtain the étale $GL(\alpha)$-local structure of the representation space $rep_\alpha\ Q$ in a neighborhood of V_ξ and the étale local structure of the quotient variety $iss_\alpha\ Q$ in a neighborhood of ξ. That is, we have to calculate the normal space N_ξ to the orbit $\mathcal{O}(V_\xi)$ as a representation over the stabilizer subgroup $GL(\alpha)_\xi = Stab_{GL(\alpha)}(V_\xi)$.

Denote $a_i = \sum_{j=1}^k a_{ij}$ where $\alpha_i = (a_{i1}, \ldots, a_{ik})$, that is, $a_i = dim\ V_i$. We will choose a basis of the underlying vectorspace

$$\oplus_{v_i \in Q_v} \mathbb{C}^{\oplus d_i} \qquad of \qquad V_\xi = V_1^{\oplus e_1} \oplus \ldots \oplus V_z^{\oplus e_z}$$

as follows : the first $e_1 a_1$ vectors give a basis of the vertex spaces of all simple components of type V_1, the next $e_2 a_2$ vectors give a basis of the vertex spaces of all simple components of type V_2, and so on. If $n = \sum_{i=1}^k e_i d_i$ is the total dimension of V_ξ, then with respect to this basis, the subalgebra of $M_n(\mathbb{C})$ generated by the representation V_ξ has the following block-decomposition

$$\begin{bmatrix} M_{a_1}(\mathbb{C}) \otimes \mathbb{1}_{e_1} & 0 & \ldots & 0 \\ 0 & M_{a_2}(\mathbb{C}) \otimes \mathbb{1}_{e_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & M_{a_z}(\mathbb{C}) \otimes \mathbb{1}_{e_z} \end{bmatrix}$$

But then, the stabilizer subgroup

$$Stab_{GL(\alpha)}(V_\xi) \simeq GL_{e_1} \times \ldots \times GL_{e_z}$$

embedded in $GL(\alpha)$ with respect to this particular basis as

$$\begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{a_1}) & 0 & \ldots & 0 \\ 0 & GL_{e_2}(\mathbb{C} \otimes \mathbb{1}_{a_2}) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & GL_{e_z}(\mathbb{C} \otimes \mathbb{1}_{a_z}) \end{bmatrix}$$

The tangentspace to the $GL(\alpha)$-orbit in V_ξ is equal to the image of the natural linear map

$$Lie\ GL(\alpha) \longrightarrow rep_\alpha\ Q$$

sending a matrix $m \in Lie\ GL(\alpha) \simeq M_{d_1} \oplus \ldots \oplus M_{d_k}$ to the representation determined by the commutator $[m, V_\xi] = mV_\xi - V_\xi m$. By this we mean that the matrix $[m, V_\xi]_a$ corresponding to an arrow a is obtained as the commutator in $M_n(\mathbb{C})$ using the canonical embedding with respect to the above choice of basis. The kernel of this linear map is the centralizer subalgebra. That is, we have an exact sequence of $GL(\alpha)_\xi$-modules

$$0 \longrightarrow C_{M_n(\mathbb{C})}(V_\xi) \longrightarrow Lie\ GL(\alpha) \longrightarrow T_{V_\xi}\ \mathcal{O}(V_\xi) \longrightarrow 0$$

where

$$C_{M_n(\mathbb{C})}(V_\xi) = \begin{bmatrix} M_{e_1}(\mathbb{C} \otimes \mathbb{1}_{a_1}) & 0 & \ldots & 0 \\ 0 & M_{e_2}(\mathbb{C} \otimes \mathbb{1}_{a_2}) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & M_{e_z}(\mathbb{C} \otimes \mathbb{1}_{a_z}) \end{bmatrix}$$

where the action of $GL(\alpha)_{V_\xi}$ is given by conjugation on $M_n(\mathbb{C})$ via the above embedding. We will now engage in a book-keeping operation counting the factors of the relevant $GL(\alpha)_\xi$-spaces. We identify the factors by the action of the GL_{e_i}-components of $GL(\alpha)_\xi$

1. *The centralizer $C_{M_n(\mathbb{C})}(V_\xi)$ decomposes as a $GL(\alpha)_\xi$-module into*
   - *one factor M_{e_i} on which GL_{e_1} acts via conjugation and the other factors act trivially,*

     $\vdots$
   - *one factor M_{e_z} on which GL_{e_z} acts via conjugation and the other factors act trivially.*
2. *Recall the notation $\alpha_i = (a_{i1}, \ldots, a_{ik})$,then the Lie algebra $Lie\ GL(\alpha)$ decomposes as a $GL(\alpha)_\xi$-module into*
   - *$\sum_{j=1}^k a_{1j}^2$ factors M_{e_1} on which GL_{e_1} acts via conjugation and the other factors act trivially,*

     $\vdots$
   - *$\sum_{j=1}^k a_{zj}^2$ factors M_{e_z} on which GL_{e_z} acts via conjugation and the other factors act trivially,*
   - *$\sum_{j=1}^k a_{1j}a_{2j}$ factors $M_{e_1 \times e_2}$ on which $GL_{e_1} \times GL_{e_2}$ acts via $\gamma_1.m.\gamma_2^{-1}$ and the other factors act trivially,*

     $\vdots$
   - *$\sum_{j=1}^k a_{zj}a_{z-1\ j}$ factors $M_{e_z \times e_{z-1}}$ on which $GL_{e_z} \times GL_{e_{z-1}}$ acts via $\gamma_z.m.\gamma_{z-1}^{-1}$ and the other factors act trivially.*
3. *The representation space $rep_\alpha\ Q$ decomposes as a $GL(\alpha)_\xi$-modulo into the following factors, for every arrow $\textcircled{j} \xleftarrow{a} \textcircled{i}$ in Q (or every loop in v_i by setting $i = j$ in the expressions below) we have*

- *$a_{1i}a_{1j}$ factors M_{e_1} on which GL_{e_1} acts via conjugation and the other factors act trivially,*
- *$a_{1i}a_{2j}$ factors $M_{e_1 \times e_2}$ on which $GL_{e_1} \times GL_{e_2}$ acts via $\gamma_1.m.\gamma_2^{-1}$ and the other factors act trivially,*

$$\vdots$$

- *$a_{zi}a_{z-1\ j}$ factors $M_{e_z \times e_{z-1}}$ on which $GL_{e_z} \times GL_{e_{z-1}}$ act via $\gamma_z.m.\gamma_{z-1}^{-1}$ and the other factors act trivially,*
- *$a_{zi}a_{zj}$ factors M_{e_z} on which GL_{e_z} acts via conjugation and the other factors act trivially.*

Removing the factors of 1. from those of 2. we obtain a description of the tangentspace to the orbit $T_{V_\xi}\ \mathcal{O}(V_\xi)$. But then, removing these factors from those of 3. we obtain the description of the normal space N_{V_ξ} as a $GL(\alpha)_\xi$-module as there is an exact sequence of $GL(\alpha)_\xi$-modules

$$0 \longrightarrow T_{V_\xi}\ \mathcal{O}(V_\xi) \longrightarrow rep_\alpha\ Q \longrightarrow N_{V_\xi} \longrightarrow 0$$

The upshot of this book-keeping is that we have proved that the normal space to the orbit in V_ξ depends only on the representation type $\tau = t(\xi)$ of the point ξ and can be identified with the representation space of a specific local quiver Q_τ.

Theorem 6.15 *Let $\xi \in iss_\alpha\ Q$ be a point of representation type*

$$\tau = t(\xi) = (e_1, \alpha_1; \ldots, e_z, \alpha_z)$$

Then, the normal space N_{V_ξ} to the orbit, as a module over the stabilizer subgroup, is identical to the representation space of a local quiver situation

$$N_{V_\xi} \simeq rep_{\alpha_\tau}\ Q_\tau$$

where Q_τ is the quiver on z vertices (the number of distinct simple components of V_ξ) say $\{w_1, \ldots, w_z\}$ such that in Q_τ

$$\#\ \textcircled{j} \xleftarrow{a} \textcircled{i} \quad = \quad -\chi_Q(\alpha_i, \alpha_j) \quad \textit{for } i \neq j, \textit{ and}$$

$$\#\ \circlearrowright \textcircled{i} \quad = \quad 1 - \chi_Q(\alpha_i, \alpha_i)$$

and such that the dimension vector $\alpha_\tau = (e_1, \ldots, e_z)$ (the multiplicities of the simple components in V_ξ).

We can repeat this argument in the case of a marked quiver $Q^\bullet$. the only difference lies in the description of the factors of $rep_\alpha\ Q^\bullet$ where we need to replace the factors M_{e_j} in the description of a loop in v_i by $M_{e_i}^0$ (trace zero matrices) in case the loop gets a mark in $Q^\bullet$. Recall the notation of u_i as the number of unmarked loops in v_i and m_i the number of marked loops in v_i. We define

$$\chi^1_{Q^\bullet} = \begin{bmatrix} 1-u_1 & \chi_{12} & \ldots & \chi_{1k} \\ \chi_{21} & 1-u_2 & \ldots & \chi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{k1} & \chi_{k2} & \ldots & 1-u_k \end{bmatrix} \qquad \textit{and} \qquad \chi^2_{Q^\bullet} = \begin{bmatrix} -m_1 & & & \\ & -m_2 & & \\ & & \ddots & \\ & & & -m_k \end{bmatrix}$$

such that $\chi_Q = \chi^1_{Q^\bullet} + \chi^2_{Q^\bullet}$ where Q is the underlying quiver of $Q^\bullet$.

Theorem 6.16 *Let $\xi \in iss_\alpha \; Q^\bullet$ be a point of representation type*

$$\tau = t(\xi) = (e_1, \alpha_1; \ldots, e_z, \alpha_z)$$

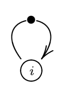
Then, the normal space N_{V_ξ} to the orbit, as a module over the stabilizer subgroup, is identical to the representation space of a local marked quiver situation

$$N_{V_\xi} \simeq rep_{\alpha_\tau} \; Q^\bullet_\tau$$

where $Q^\bullet_\tau$ is the quiver on z vertices (the number of distinct simple components of V_ξ) say $\{w_1, \ldots, w_z\}$ such that in $Q^\bullet_\tau$

$$\# \; (j) \xleftarrow{\;a\;} (i) \quad = \quad -\chi_Q(\alpha_i, \alpha_j) \quad \textit{for } i \neq j, \textit{ and}$$

$\#$  $= \; 1 - \chi^1_{Q^\bullet}(\alpha_i, \alpha_i)$

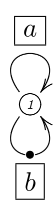
$\#$  $= \; -\chi^2_{Q^\bullet}(\alpha_i, \alpha_i)$

and such that the dimension vector $\alpha_\tau = (e_1, \ldots, e_z)$ (the multiplicities of the simple components in V_ξ).

Proposition 6.17 *If $\alpha = (d_1, \ldots, d_k)$ is the dimension vector of a simple representation of $Q^\bullet$, then the dimension of the quotient variety $iss_\alpha \; Q^\bullet$ is equal to*

$$1 - \chi^1_{Q^\bullet}(\alpha, \alpha)$$

Proof. *There is a Zariski open subset of $iss_\alpha \; Q^\bullet$ consisting of points ξ such that the corresponding semi-simple module V_ξ is simple, that is, ξ has representation type $\tau = (1, \alpha)$. But then the local quiver setting (Q_τ, α_τ) is*



where $a = 1 - \chi^1_{Q^\bullet}(\alpha, \alpha)$ and $b = -\chi^2_{Q^\bullet}(\alpha, \alpha)$. The corresponding representation space has coordinate ring

$$\mathbb{C}[rep_{\alpha_\tau} \; Q^\bullet_\tau] = \mathbb{C}[x_1, \ldots, x_a]$$

on which $GL(\alpha_\tau) = \mathbb{C}^*$ acts trivially. That is, the quotient variety is

$$rep_{\alpha_\tau} \; Q^\bullet_\tau / GL(\alpha_\tau) = rep_{\alpha_\tau} \; Q^\bullet_\tau \simeq \mathbb{C}^a$$

As $iss_\alpha \; Q^\bullet$ has the same local structure near ξ as this quotient space near the origin, by the Luna slice result, the result follows. □

6.4 The stratification.

In this section we will draw some consequences from the description of the local quiver. usually, the quotient varieties $iss_\alpha\ Q^\bullet$ *have lots of singularities. Still, we can decompose these quotient varieties in smooth pieces according to the representation types of its points.*

Proposition 6.18 *Let* $iss_\alpha\ Q^\bullet(\tau)$ *be the set of points* $\xi \in iss_\alpha\ Q^\bullet$ *of representation type*

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$$

Then, $iss_\alpha\ Q^\bullet(\tau)$ *is a locally closed smooth subvariety of* $iss_\alpha\ Q^\bullet$ *and*

$$iss_\alpha\ Q^\bullet = \bigsqcup_\tau iss_\alpha\ Q^\bullet(\tau)$$

is a finite smooth stratification of the quotient variety.

Proof. *Let* $Q^\bullet_\tau$ *be the local marked quiver in* ξ. *Consider a nearby point* ξ'. *If some trace of an oriented cycles of length* > 1 *in* $Q^\bullet_\tau$ *is non-zero in* ξ', *then* ξ' *cannot be of representation type* τ *as it contains a simple factor composed of vertices of that cycle. That is, locally in* ξ *the subvariety* $iss_\alpha\ Q^\bullet(\tau)$ *is determined by the traces of unmarked loops in vertices of the local quiver* $Q^\bullet_\tau$ *and hence is locally in the étale topology an affine space whence smooth. All other statements are direct.* □

Given a stratification of a topological space, one always wants to determine which strata make up the boundary of a given stratum. In the stratification of $iss_\alpha\ Q^\bullet$ *given by the above result, we have a combinatorial solution to this problem. Two representation types*

$$\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z) \quad \textit{and} \quad \tau' = (e'_1, \alpha'_1; \ldots; e'_{z'}, \alpha'_{z'})$$

are said to be direct successors $\tau < \tau'$ *if and only if one of the following two cases occurs*

- *(splitting of one simple) :* $z' = z + 1$ *and for all but one* $1 \leq i \leq z$ *we have that* $(e_i, \alpha_i) = (e'_j, \alpha'_j)$ *for a uniquely determined* j *and for the remaining* i_0 *we have that the remaining couples of* τ' *are*

$$(e_i, \alpha'_u; e_i, \alpha'_v) \quad \textit{with} \quad \alpha_i = \alpha'_u + \alpha'_v$$

- *(combining two simple types) :* $z' = z - 1$ *and for all but one* $1 \leq i \leq z'$ *we have that* $(e'_i, \alpha'_i) = (e_j, \alpha_j)$ *for a uniquely determined* j *and for the remaining* i *we have that the remaining couples of* τ *are*

$$(e_u, \alpha'_i; e_v, \alpha'_i) \quad \textit{with} \quad e_u + e_v = e'_i$$

This direct successor relation $<$ *induces an ordering which we will denote with* $<<$. *Observe that* $\tau << \tau'$ *if and only if the stabilizer subgroup* $GL(\alpha)_\tau$ *is conjugated to a subgroup of* $GL(\alpha)_{\tau'}$. *The following result either follows from general theory, see for example [29], or from the description of the local marked quivers.*

Proposition 6.19 *The stratum* $iss_\alpha\ Q^\bullet(\tau')$ *lies in the closure of the stratum* $iss_\alpha\ Q^\bullet$ *if and only if* $\tau << \tau'$.

Using the dimension of the quotient variety $iss_\alpha\ Q^\bullet$ *given in the precious section when* α *is the dimension vector of a simple representation can be used to determine the dimensions of the different strata* $iss_\alpha\ Q^\bullet(\tau)$ *in general.*

Proposition 6.20 *Let $\tau = (e_1, \alpha_1; \ldots; e_z, \alpha_z)$ a representation type of α. Then,*

$$dim\ iss_\alpha\ Q^\bullet = \sum_{j=1}^{z} (1 - \chi^1_{Q^\bullet}(\alpha_j, \alpha_j)$$

Because $rep_\alpha\ Q^\bullet$ and hence $iss_\alpha\ Q^\bullet$ is irreducible, there is a unique representation type τ_{gen} such that $iss_\alpha\ Q^\bullet(\tau_{gen})$ is Zariski open. We call τ_{gen} the generic representation type for $rep_\alpha\ Q$. The generic representation type can be determined as follows.

- *Let Q' be the full marked subquiver of $Q^\bullet$ on the support of α and consider its strongly connected component quiver $SC(Q')$.*
- *Let $V \in rep_\alpha\ Q^\bullet$ be in general position, then a simple subrepresentation $S \subset V$ must have its support in a strongly connected component G of Q which is a sink in $SC(Q')$. Restrict attention to this subquiver G say on l vertices.*
- *As $(1, \ldots, 1) \in \mathbb{N}^l$ is the dimension vector of a simple representation of G, there exists a dimension vector β with support equal to G satisfying the following properties*
  1. *β is the dimension vector of a simple representation of G.*
  2. *If α' is the restriction of the dimension vector α to G, then a representation of $rep_{\alpha'}\ G$ in general position has a subrepresentation of dimension vector β.*
  3. *β is minimal among all dimension vectors $(1, \ldots, 1) \leq \beta \leq \alpha'$ satisfying 1. and 2.*

  A representation in $rep_\alpha\ Q^\bullet$ will then contain a simple subrepresentation of dimension vector β.
- *Continue the process with starting dimension vector $\alpha - \beta$ until this difference is the zero vector. We will ten have found the generic decomposition $\alpha = \beta_1 + \ldots + \beta_z$ into dimension vectors of simple representations.*
- *Calculate $1 - \chi^1_{Q^\bullet}(\beta_i, \beta_i)$. If it is zero, then β_i occurs with multiplicity e_i in the generic representation type τ_{gen} if e_i is the number of components β_j in the generic decomposition which are equal to β_i. This determines the generic representation type.*

The difficult part in the procedure is determining when a representation in general position has a subrepresentation of given dimension vector. In the next chapter we will prove a combinatorial procedure to verify this, due to A. Schofield [27].

6.5 The Cayley-smooth locus.

Let A be a Cayley-Hamilton algebra of degree n equipped with a trace map $A \xrightarrow{tr} A$ and consider the quotient map

$$\underline{rep}^t_n\ A \xrightarrow{\pi} \underline{iss}^t_n\ A$$

Let ξ be a geometric point of he quotient scheme $\underline{iss}^t_n\ A$ with corresponding n-dimensional trace preserving semi-simple representation V_ξ with decomposition

$$V_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i are distinct simple representations of A of dimension d_i such that $n = \sum_{i=1}^k d_i e_i$.

Definition 6.21 The Cayley-smooth locus of A is the subset of $\underline{iss}_n^t\ A$

$$Sm_n\ A = \{\xi \in \underline{iss}_n^t\ A \mid \underline{iss}_n^t\ A \text{ is smooth along } \pi^{-1}(\xi)\ \}$$

As the singular locus of $\underline{iss}_n^t\ A$ is a GL_n-stable closed subscheme of $\underline{iss}_n^t\ A$ this is equivalent to

$$Sm_n\ A = \{\xi \in \underline{iss}_n^t\ A \mid \underline{iss}_n^t\ A \text{ is smooth in } V_\xi\ \}$$

We will give some conditions on ξ to be in the smooth locus $Sm_n\ A$. To begin, $\underline{rep}_n^t\ A$ is sooth in V_ξ if and only if the dimension of the tangent space in V_ξ is equal to the local dimension of $\underline{rep}_n^t\ A$ in V_ξ. In the previous chapter we have calculated the tangent space to be the set of trace preserving derivations $A \xrightarrow{D} M_n(\mathbb{C})$ satisfying

$$D(aa') = D(a)\rho(a') + \rho(a)D(a')$$

where $A \xrightarrow{\rho} M_n(\mathbb{C})$ is the $\mathbb{C}$-algebra morphism determined by the action of A on V_ξ and such that D is compatible with the traces, that is, the diagram

$$\begin{tikzcd}
A \arrow[r, "D"] \arrow[d, "tr_A"'] & M_n(\mathbb{C}) \arrow[d, "tr"] \\
A \arrow[r, "D"] & M_n(\mathbb{C})
\end{tikzcd}$$

is commutative. The $\mathbb{C}$-vectorspace of such derivations is denoted by $Der_\rho^t\ A$. Therefore,

$$\xi \in Sm_n\ A \Longleftrightarrow dim_{\mathbb{C}}\ Der_\rho^t\ A = dim_{V_\xi}\ \underline{rep}_n^t\ A$$

Further, if $\xi \in Sm_n\ A$, then we know from the Luna slice theorem that the local GL_n-structure of $\underline{rep}_n^t\ A$ near V_ξ is determined by a marked quiver setting $(Q_\xi^\bullet, \alpha_\xi)$ where $Q_\xi^\bullet$ is a marked quiver on k vertices (the number of distinct simple components of V_ξ) and $\alpha_\xi = (e_1, \ldots, e_k)$ (the multiplicities with which these simples occur in V_ξ). Recall that $GL(\alpha_\xi)$ is the stabilizer subgroup in GL_n of V_ξ and can be embedded in GL_n after a suitable choice of basis via

$$GL(\alpha_\xi) = \begin{bmatrix} GL_{e_1}(\mathbb{C} \otimes \mathbb{1}_{d_1}) & 0 & \ldots & 0 \\ 0 & GL_{e_2}(\mathbb{C} \otimes \mathbb{1}_{d_2}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & GL_{e_k}(\mathbb{C} \otimes \mathbb{1}_{d_k}) \end{bmatrix} \hookrightarrow GL_n$$

and we have an isomorphism of $GL(\alpha_\xi)$-modules

$$rep_{\alpha_\xi}\ Q_\xi^\bullet \simeq N_{V_\xi}\ \mathcal{O}(V_\xi) \simeq Ext_A^t(V_\xi, V_\xi),$$

the last isomorphism was proved in the previous chapter. This fact also allows us to determine the quiver Q_ξ. Indeed,

$$Ext_A^1(V_\xi, V_\xi) = \oplus_{i,j=1}^k Ext_A^1(S_i, S_j)^{\oplus e_i e_j}$$

That is, as a $GL(\alpha_\xi)$-module $Ext_A^1(V_\xi, V_\xi)$ is isomorphic to a quiver setting $rep_{\alpha_\xi}\ Q_\xi^{big}$ where the arrows and loops in the quiver Q_ξ^{big} are given by

$$\#\ (j) \xleftarrow{a} (i) \quad = \quad dim_{\mathbb{C}}\ Ext_A^1(S_i, S_j) \qquad \textit{if } i \neq j, \textit{ and}$$

$$\#\ \circlearrowleft (i) \quad = \quad dim_{\mathbb{C}}\ Ext_A^1(S_i, S_i)$$

Proposition 6.22 *With notations as before, $Q^\bullet_\xi$ is a marked sub-quiver of the extension-quiver Q^{big}_ξ determined by the $GL(\alpha_\xi)$-submodule $Ext^t_A(V_\xi, V_\xi) \hookrightarrow Ext^1_A(V_\xi, V_\xi)$.*

It follows from the definition of trace preserving extensions $Ext^t_A(V_\xi, V_\xi)$ that $Q^\bullet_\xi$ and Q^{big}_ξ have the same number of arrows $\textcircled{j} \xleftarrow{a} \textcircled{i}$ when $i \neq j$, but some of the loops in Q^{big}_ξ may vanish or get a marking in $Q^\bullet_\xi$. Observe that we can define this quiver $Q^\bullet_\xi$ to any point $\xi \in \underline{rep}^t_n\ A$ whether $\xi \in Sm_n\ A$ or not. However, if $\xi \in Sm_n\ A$ then by the Luna slice theorem, we have local étale isomorphisms between the varieties

$$GL_n \times^{GL(\alpha_\xi)} rep_{\alpha_\xi}\ Q^\bullet_\xi \xleftrightarrow{et} \underline{rep}^t_n\ A \quad \text{and} \quad rep_{\alpha_\xi}\ Q^\bullet_\xi / GL(\alpha_\xi) \xleftrightarrow{et} \underline{iss}^t_n\ A$$

Which gives us the following numerical restrictions on $\xi \in Sm_n\ A$:

Proposition 6.23 *$\xi \in Sm_n\ A$ if and only if the following two equalities hold*

$$\begin{cases} dim_{V_\xi}\ \underline{rep}^t_n\ A & = n^2 - (e_1^2 + \ldots + e_k^2) + dim_{\mathbb{C}}\ Ext^t_A(V_\xi, V_\xi) \\ dim_\xi\ \underline{iss}^t_n\ A & = dim_{\overline{0}}\ rep_{\alpha_\xi}\ Q^\bullet_\xi / GL(\alpha_\xi) \end{cases}$$

Moreover, if $\xi \in Sm_n\ A$, then $\underline{rep}^t_n\ A$ is a normal variety in a neighborhood of ξ

Proof. *The last statement follows from the fact that $\mathbb{C}[rep_{\alpha_\xi}\ Q^\bullet_\xi]^{GL(\alpha_\xi)}$ is integrally closed and this property is preserved under the étale map.* $\square$

In general, the difference between these numbers gives a measure for the non-commutative singularity of A in ξ. In the next section we will refine these conditions under the extra assumption that A is an order in a central simple algebra.

Example 6.24 Consider the affine $\mathbb{C}$-algebra $A = \frac{\mathbb{C}\langle x,y\rangle}{(xy+yx)}$ then $u = x^2$ and $v = y^2$ are central elements of A and A is a free module of rank 4 over $\mathbb{C}[u,v]$. In fact, A is a $\mathbb{C}[u,v]$-order in the quaternion division algebra

$$\Delta = \begin{pmatrix} u & & v \\ & \mathbb{C}(u,v) & \end{pmatrix}$$

and the reduced trace map on Δ makes A into a Cayley-Hamilton algebra of degree 2. More precisely, tr is the linear map on A such that

$$\begin{cases} tr(x^i y^j) = 0 & \text{if either } i \text{ or } j \text{ are odd, and} \\ tr(x^i y^j) = 2x^i y^j & \text{if } i \text{ and } j \text{ are even.} \end{cases}$$

In particular, a trace preserving 2-dimensional representation is determined by a couple of 2×2 matrices

$$\rho = (\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} , \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix}) \quad \text{with} \quad tr(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} . \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix}) = 0$$

That is, $\underline{rep}^t_2\ A$ is the hypersurface in $\mathbb{C}^6$ determined by the equation

$$\underline{rep}^t_2\ A = \mathbb{V}(2x_1x_4 + x_2x_6 + x_3x_5) \hookrightarrow \mathbb{C}^6$$

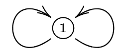
and is therefore irreducible of dimension 5 with an isolated singularity at $p = (0, \ldots, 0)$. The image of the trace map is equal to the center of A which is $\mathbb{C}[u,v]$ and the quotient map

$$\underline{rep}^t_2\ A \xrightarrow{\pi} \underline{iss}^t_2\ A = \mathbb{C}^2 \qquad \pi(x_1, \ldots, x_6) = (x_1^2 + x_2x_3, x_4^2 + x_5x_6)$$

There are three different representation types to consider. Let $\xi = (a,b) \in \mathbb{C}^2 = \underline{iss}^t_2\ A$ with $ab \neq 0$, then $\pi^{-1}(\xi)$ is a closed GL_2-orbit and a corresponding simple A-module is given by the matrixcouple

$$(\begin{bmatrix} i\sqrt{a} & 0 \\ 0 & -i\sqrt{a} \end{bmatrix} , \begin{bmatrix} 0 & \sqrt{b} \\ -\sqrt{b} & 0 \end{bmatrix})$$

That is, ξ is of type $(1,2)$ and the stabilizer subgroup are the scalar matrixes $\mathbb{C}^*\mathbb{1}_2 \hookrightarrow GL_2$. So, the action on both the tangentspace to $\underline{rep}_2^t\ A$ and the tangent space to the orbit are trivial. As they have respectively dimension 5 and 3, the normalspace corresponds to the quiver setting

$N_\xi =$ 

which is compatible with the numerical restrictions. Next, consider a point $\xi = (0,b)$ (or similarly, $(a,0)$), then ξ is of type $(1,1;1,1)$ and the corresponding semi-simple representation is given by the matrices

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} i\sqrt{b} & 0 \\ 0 & -i\sqrt{b} \end{bmatrix} \right)$$

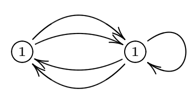
The stabilizer subgroup is in this case the maximal torus of diagonal matrices $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow GL_2$. The tangent space in this point to $\underline{rep}_2^t\ A$ are the 6-tuples $(a_1, \ldots, a_6)$ such that

$$tr\ \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{bmatrix} \right).\left(\begin{bmatrix} i\sqrt{b} & 0 \\ 0 & -i\sqrt{b} \end{bmatrix} + \epsilon \begin{bmatrix} b_4 & b_5 \\ b_6 & -b_4 \end{bmatrix} \right) = 0 \quad \text{where } \epsilon^2 = 0$$

This leads to the condition $a_1 = 0$, so the tangentspace are the matrix couples

$$\left(\begin{bmatrix} 0 & a_2 \\ a_3 & 0 \end{bmatrix} , \begin{bmatrix} a_4 & a_5 \\ a_6 & -a_4 \end{bmatrix} \right) \quad \text{on which the stabilizer} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

acts via conjugation. That is, the tangentspace corresponds to the quiver setting



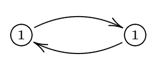
Moreover, the tangentspace to the orbit is the image of the linear map

$$\left(\mathbb{1}_2 + \epsilon \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}\right).\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{bmatrix} \right).\left(\mathbb{1}_2 - \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}\right)$$

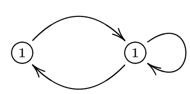
which is equal to

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{bmatrix} + \epsilon \begin{bmatrix} 0 & -2m_2\sqrt{b} \\ 2m_3\sqrt{b} & 0 \end{bmatrix} \right)$$

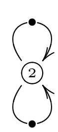
on which the stabilizer acts again via conjugation giving the quiver setting



Therefore, the normal space to the orbit corresponds to the quiver setting



which is again compatible with the numerical restrictions. Finally, consider $\xi = (0,0)$ which is of type $(2,1)$ and whose semi-simple representation corresponds to the zero matrix-couple. The action fixes this point, so the stabilizer is GL_2 and the tangent space to the orbit is the trivial space. Hence, the tangent space to $\underline{rep}_2^t\ A$ coincides with the normalspace to the orbit and both spaces are acted on by GL_2 via simultaneous conjugation leading to the quiver setting

$N_\xi =$ 

This time, the data is not compatible with the numerical restriction as

$$5 = dim\ \underline{rep}_2^t\ A \neq n^2 - e^2 + dim\ rep_\alpha\ Q^\bullet = 4 - 4 + 6$$

consistent with the fact that the zero matrix-couple is a (in fact, the only) singularity on $\underline{rep}_2^t\ A$.

6.6 Cayley-smooth orders.

Let X be a normal affine variety with coordinate ring $\mathbb{C}[X]$ and functionfield $\mathbb{C}(X)$. Let Δ be a central simple $\mathbb{C}(X)$-algebra of dimension n^2 which is a Cayley-Hamilton algebra of degree n using the reduced trace map tr. Let A be a $\mathbb{C}[X]$-order in Δ, *that is, the center of A is $\mathbb{C}[X]$ and $A \otimes_{\mathbb{C}[X]} \mathbb{C}(X) \simeq \Delta$. Because $\mathbb{C}[X]$ is integrally closed, the restriction of the reduced trace tr to A has its image in $\mathbb{C}[X]$, that is, A is a Cayley-Hamilton algebra of degree n and*

$$tr(A) = \mathbb{C}[X]$$

Consider the quotient morphism for the representation variety

$$\underline{rep}_n^t\ A \xrightarrow{\pi} \underline{iss}_n^t\ A$$

then the above argument shows that $X \simeq \underline{iss}_n^t\ A$ and in particular the quotient scheme is reduced.

Proposition 6.25 *Let A be a Cayley-Hamilton order of degree n over $\mathbb{C}[X]$. Then, its smooth locus $Sm_n\ A$ is a nonempty Zariski open subset of X. In particular, the set az_A of Azumaya points, that is, of points $x \in X = \underline{iss}_n\ A$ of representation type $(1,n)$ is a non-empty Zariski open subset of X and its intersection with the Zariski open subset X_{reg} of smooth points of X satisfies*

$$X_{az} \cap X_{reg} \hookrightarrow Sm_n\ A$$

Proof. *Because $A\mathbb{C}(X) = \Delta$, there is an $f \in \mathbb{C}[X]$ such that $A_f = A \otimes_{\mathbb{C}[X]} \mathbb{C}[X]_f$ is a free $\mathbb{C}[X]_f$-module of rank n^2 say with basis $\{a_1, \ldots, a_{n^2}\}$. Consider the $n^2 \times n^2$ matrix with entries in $\mathbb{C}[X]_f$*

$$R = \begin{bmatrix} tr(a_1 a_1) & \ldots & tr(a_1 a_{n^2}) \\ \vdots & & \vdots \\ tr(a_{n^2} a_1) & \ldots & tr(a_{n^2} a_{n^2}) \end{bmatrix}$$

The determinant $d = det\ R$ is nonzero in $\mathbb{C}[X]_f$. For, let $\mathbb{K}$ be the algebraic closure of $\mathbb{C}(X)$ then $A_f \otimes_{\mathbb{C}[X]_f} \mathbb{K} \simeq M_n(\mathbb{K})$ and for any $\mathbb{K}$-basis of $M_n(\mathbb{K})$ the corresponding matrix is invertible (for example, verify this on the matrixes e_{ij}). As $\{a_1, \ldots, a_{n^2}\}$ is such a basis, $d \neq 0$. Next, consider the Zariski open subset $U = \mathbb{X}(f) \cap \mathbb{X}(d) \hookrightarrow X$. For any $x \in X$ with maximal ideal $\mathfrak{m}_x \triangleleft \mathbb{C}[X]$ we claim that

$$\frac{A}{A\mathfrak{m}_x A} \simeq M_n(\mathbb{C})$$

Indeed, the images of the a_i give a $\mathbb{C}$-basis in the quotient such that the $n^2 \times n^2$-matrix of their product-traces is invertible. This property is equivalent to the quotient being $M_n(\mathbb{C})$. Such points x are called Azumaya points of A. The corresponding semi-simple representation of A is simple, proving that az_A is a non-empty Zariski open subset of X. But then, over U the restriction of the quotient map

$$\underline{rep}_n^t\ A \mid \pi^{-1}(U) \twoheadrightarrow U$$

is a principal PGL_n-fibration. In fact, this restricted quotient map determines an element in $H^1_{et}(U, PGL_n)$ determining the class of the central simple $\mathbb{C}(X)$-algebra Δ in $H^1_{et}(\mathbb{C}(X), PGL_n)$. Restrict this quotient map further to $U \cap X_{reg}$, then the PGL_n-fibration

$$\underline{rep}_n^t\ A \mid \pi^{-1}(U \cap X_{reg}) \twoheadrightarrow U \cap X_{reg}$$

has a smooth base and therefore also the total space is smooth. But then, $U \cap X_{reg}$ is a non-empty Zariski open subset of $Sm_n\ A$. $\square$

We will now determine the étale local structure of A in points $\xi \in Sm_n\ A$. Observe that the normality assumption on X is no restriction as the quotient scheme is locally normal in a point of $Sm_n\ A$. Our next result drastically limits the local dimension vectors α_ξ.

Proposition 6.26 *Let A be a Cayley-Hamilton order and $\xi \in Sm_n\ A$ such that the normal space to the orbit of the corresponding semi-simple n-dimensional representation is*

$$N_\xi = rep_{\alpha_\xi}\ Q_\xi^\bullet$$

Then, α_ξ is the dimension vector of a simple representation of $Q_\xi^\bullet$.

Proof. *Let V_ξ be the semi-simple representation of A determined by ξ. Let S_ξ be the slice variety in V_ξ then we have by the Luna slice theorem the following diagram of étale GL_n-equivariant maps*

$$\begin{tikzcd} & GL_n \times^{GL(\alpha_\xi)} S_\xi \arrow[dl, "et"'] \arrow[dr, "et"] & \\ GL_n \times^{GL(\alpha_\xi)} rep_{\alpha_\xi}\ Q_\xi^\bullet & & \underline{rep}_n^t\ A \end{tikzcd}$$

linking a neighborhood of V_ξ with one of $\overline{(\mathbb{1}_n, 0)}$. Because A is an order, every Zariski neighborhood of V_ξ in $\underline{rep}_n^t\ A$ contains simple n-dimensional representations, that is, closed GL_n-orbits with stabilizer subgroup isomorphic to $\mathbb{C}^*$. Transporting this property via the GL_n-equivariant étale maps, every Zariski neighborhood of $\overline{(\mathbb{1}_n, 0)}$ contains closed GL_n-orbits with stabilizer $\mathbb{C}^*$. By the correspondence of orbits is associated fiber bundles, every Zariski neighborhood of the trivial representation $0 \in rep_{\alpha_\xi}\ Q_\xi^\bullet$ contains closed $GL(\alpha_\xi)$-orbits with stabilizer subgroup $\mathbb{C}^*$. We have seen that closed $GL(\alpha_\xi)$-orbits correspond to semi-simple representations of $Q_\xi^\bullet$. However, if the stabilizer subgroup of a semi-simple representation is $\mathbb{C}^*$ this representation must be simple. □

These two results allow us to refine the numerical characterization of smooth points given in the previous section.

Theorem 6.27 *Let A be a Cayley-Hamilton order of degree n with center $\mathbb{C}[X]$ where X is a normal variety of dimension d. For $\xi \in X = \underline{iss}_n^t\ A$ with corresponding semi-simple representation*

$$V_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

and normal space to the orbit $\mathcal{O}(V_\xi)$ isomorphic to $rep_{\alpha_\xi}\ Q_\xi^\bullet$ as $GL(\alpha_\xi)$-modules where $\alpha_\xi = (e_1, \ldots, e_k)$. Then, $\xi \in Sm_n\ A$ if and only if the following two conditions are met

$$\begin{cases} \alpha_\xi & \text{is the dimension vector of a simple representation of } Q^\bullet, \text{ and} \\ d & = 1 - \chi_Q(\alpha_\xi, \alpha_\xi) - \sum_{i=1}^k m_i \end{cases}$$

where Q is the underlying quiver of $Q_\xi^\bullet$ and m_i is the number of marked loops in $Q_\xi^\bullet$ in vertex v_i.

Proof. *By the Luna slice theorem we have étale maps*

$$rep_{\alpha_\xi}\ Q^\bullet_\xi/GL(\alpha_\xi) \xleftarrow{et} S_\xi/GL(\alpha_\xi) \xrightarrow{et} \underline{iso}^t_n\ A = X$$

connecting a neighborhood of $\xi \in X$ with one of the trivial semi-simple representation $\overline{0}$. By definition of the Euler-form of Q we have that

$$\chi_Q(\alpha_\xi, \alpha_\xi) = -\sum_{i \neq j} e_i e_j \chi_{ij} + \sum_i e_i^2 (1 - u_i - m_i)$$

On the other hand we have the following dimensions

$$dim\ rep_\alpha\ Q^\bullet_{\alpha_\xi} = \sum_{i \neq j} e_i e_j \chi_{ij} + \sum_i e_i^2 (u_i + m_i) - \sum_i m_i$$

$$dim\ GL(\alpha_\xi) = \sum_i e_i^2$$

As any Zariski open neighborhood of ξ contains an open set where the quotient map is a $PGL(\alpha_\xi) = \frac{GL(\alpha_\xi)}{\mathbb{C}^*}$-fibration we see that the quotient variety $rep_{\alpha_\xi}\ Q^\bullet_\xi$ has dimension equal to

$$dim\ rep_{\alpha_\xi}\ Q^\bullet_\xi - dim\ GL(\alpha_\xi) + 1$$

and plugging in the above information we see that this is equal to $1 - \chi_Q(\alpha_\xi, \alpha_\xi) - \sum_i m_i$. □

Example 6.28 The quantum plane.

We will generalize the discussion of example 6.24 to the algebra

$$A = \frac{\mathbb{C}\langle x, y\rangle}{(yx - qxy)}$$

where q is a primitive n-th root of unity. Let $u = x^n$ and $v = y^n$ then it is easy to see that A is a free module of rank n^2 over its center $\mathbb{C}[u, v]$ and is a Cayley-Hamilton algebra of degree n with the trace determined on the basis

$$tr(x^i y^j) = \begin{cases} 0 & \text{when either } i \text{ or } j \text{ is not a multiple of } n, \\ n x^i y^j & \text{when } i \text{ and } j \text{ are multiples of } n, \end{cases}$$

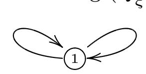
Let $\xi \in \underline{iss}_n\ A = \mathbb{C}^2$ be a point (a^n, b) with $a.b \neq 0$, then ξ is of representation type $(1, n)$ as the corresponding (semi)simple representation V_ξ is determined by (if m is odd, for even n we replace a by ia and b by $-b$)

$$\rho(x) = \begin{bmatrix} a & & & \\ & qa & & \\ & & \ddots & \\ & & & q^{n-1}a \end{bmatrix} \quad \text{and} \quad \rho(y) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ b & 0 & 0 & \dots & 0 \end{bmatrix}$$

One computes that $Ext^1_A(V_\xi, V_\xi) = \mathbb{C}^2$ where the algebra map $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ corresponding to (α, β) is given by

$$\begin{cases} \phi(x) & = \rho(x) + \varepsilon\ \alpha \mathbb{1}_n \\ \phi(y) & = \rho(y) + \varepsilon\ \beta \mathbb{1}_n \end{cases}$$

and all these algebra maps are trace preserving. That is, $Ext^1_A(V_\xi, V_\xi) = Ext^t_A(V_\xi, V_\xi)$ and as the stabilizer subgroup is $\mathbb{C}^*$ the marked quiver-setting $(Q^\bullet_\xi, \alpha_\xi)$ is



and $d = 1 - \chi_Q(\alpha, \alpha) - \sum_i m_i$ as $2 = 1 - (-1) + 0$, compatible with the fact that over these points the quotient map is a principal PGL_n-fibration.

Next, let $\xi = (a^n, 0)$ with $a \neq 0$ (or, by a similar argument $(0, b^n)$ with $b \neq 0$). Then, the representation type of ξ is $(1, 1; \dots; 1, 1)$ because

$$V_\xi = S_1 \oplus \dots \oplus S_n$$

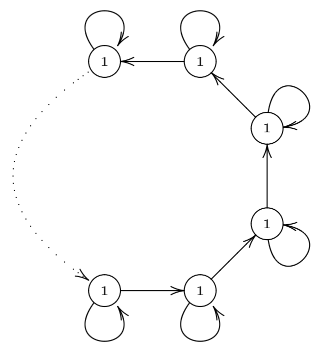
where the simple one-dimensional representation S_i is given by

$$\begin{cases} \rho(x) &= q^i a \\ \rho(y) &= 0 \end{cases}$$

One verifies that

$$Ext^1_A(S_i, S_i) = \mathbb{C} \quad \text{and} \quad Ext^1_A(S_i, S_j) = \delta_{i+1,j}\ \mathbb{C}$$

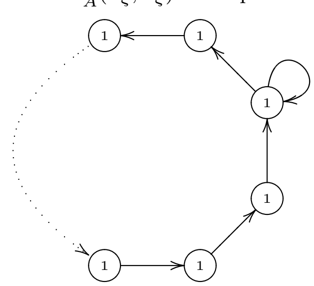
and as the stabilizer subgroup is $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$, the Ext-quiver setting is



The algebra map $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ corresponding to the extension $(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) \in Ext^1_A(V_\xi, V_\xi)$ is given by

$$\begin{cases} \phi(x) &= \begin{bmatrix} a + \varepsilon\ \alpha_1 & & & \\ & qa + \varepsilon\ \alpha_2 & & \\ & & \ddots & \\ & & & q^{n-1}a + \varepsilon\ \alpha_n \end{bmatrix} \\ \phi(y) &= \varepsilon \begin{bmatrix} 0 & \beta_1 & 0 & \ldots & 0 \\ 0 & 0 & \beta_2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & \beta_{n-1} \\ \beta_n & 0 & 0 & \ldots & 0 \end{bmatrix} \end{cases}$$

The conditions $tr(x^j) = 0$ for $1 \leq i < n$ impose $n-1$ linear conditions among the α_j, whence the space of trace preserving extensions $Ext^t_A(V_\xi, V_\xi)$ corresponds to the quiver setting



The Euler-form of this quiver $Q^\bullet$ is given by the $n \times n$ matrix

$$\begin{bmatrix} 0 & -1 & 0 & \ldots & 0 \\ & 1 & -1 & & 0 \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{bmatrix}$$

giving the numerical restriction as $\alpha_\xi = (1, \ldots, 1)$

$$1 - \chi_Q(\alpha, \alpha) - \sum_i m_i = 1 - (-1) - 0 = 2 = dim\ \underline{iss}^t_n\ A$$

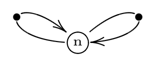
so $\xi \in Sm_n\ A$. Finally, the only remaining point is $\xi = (0, 0)$. This has representation type $(n, 1)$ as the corresponding semi-simple representation V_ξ is the trivial one. The stabilizer subgroup is GL_n and the (trace preserving) extensions are given by

$$Ext^1_A(V_\xi, V_\xi) = M_n \oplus M_n \quad \text{and} \quad Ext^t_A(V_\xi, V_\xi) = M^0_n \oplus M^0_n$$

determined by the algebra maps $A \xrightarrow{\phi} M_n(\mathbb{C}[\varepsilon])$ given by

$$\begin{cases} \phi(x) & = \varepsilon\; m_1 \\ \phi(y) & = \varepsilon\; m_2 \end{cases}$$

That is, the relevant quiver setting $(Q^\bullet_\xi, \alpha_\xi)$ is in this point



This time, $\xi \notin Sm_n\; A$ as the numerical condition fails

$$1 - \chi_Q(\alpha,\alpha) - \sum_i m_i = 1 - (-n^2) - 0 \neq 2 = dim\; \underline{iss}^t_n\; A$$

unless $n = 1$. That is, $Sm_n\; A = \mathbb{C}^2 - \{(0,0)\}$.

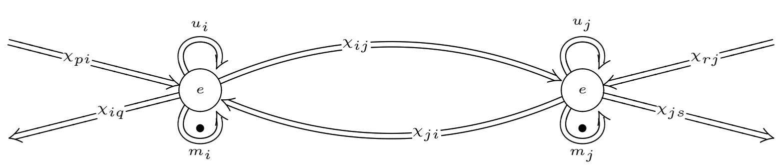
6.7 Smooth local types.

If we want to study the local structure of Cayley-Hamilton orders A of degree n over a central normal variety X of dimension d, we have to compile a list of admissible marked quiver settings, that is couples $(Q^\bullet, \alpha)$ satisfying the two properties

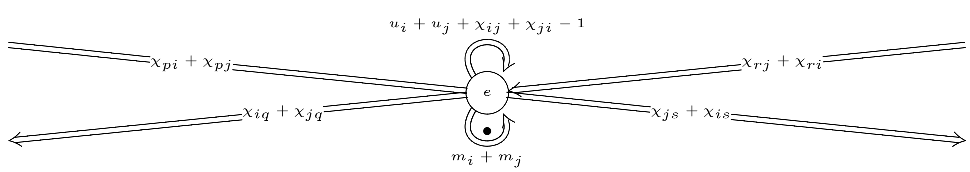
$$\begin{cases} \alpha & \text{is the dimension vector of a simple representation of } Q^\bullet\text{, and} \\ d & = 1 - \chi_Q(\alpha,\alpha) - \sum_i m_i \end{cases}$$

In this section, we will give the first steps in such a classification project

The basic idea that we use is to shrink a marked quiver-setting to its simplest form and classify these simplest forms for given d. By shrinking *we mean the following process. Assume $\alpha = (e_1, \ldots, e_k)$ is the dimension vector of a simple representation of $Q^\bullet$ and let v_i and v_j be two vertices connected with an arrow such that $e_i = e_j = e$. That is, locally we have the following situation*



We will use one of the arrows connecting v_i with v_j to identify the two vertices. That is, we form the shrinked marked quiver-setting $(Q^\bullet_s, \alpha_s)$ where $Q^\bullet_s$ is the marked quiver on $k-1$ vertices $\{v_1, \ldots, \hat{v}_i, \ldots, v_k\}$ and α_s is the dimension vector with e_i removed. That is, $Q^\bullet_s$ has the following form in a neighborhood of the contracted vertex



That is, in $Q^\bullet_s$ we have for all $k, l \neq i$ that $\chi^s_{kl} = \chi_{kl}$. Moreover, the number of arrows and (marked) loops connected to v_j are determined as follows

- $\chi^s_{jk} = \chi_{ik} + \chi_{jk}$
- $\chi^s_{kj} = \chi_{ki} + \chi_{kj}$
- $u^s_j = u_i + u_j + \chi_{ij} + \chi_{ji} - 1$

- $m_j^s = m_i + m_j$

Lemma 6.29 *α is the dimension vector of a simple representation of $Q^\bullet$ if and only if α_s is the dimension vector of a simple representation of $Q_s^\bullet$. Moreover,*

$$dim\ rep_\alpha\ Q^\bullet/GL(\alpha) = dim\ rep_{\alpha_s}\ Q_s^\bullet/GL(\alpha_s)$$

Proof. *Fix an arrow $(j) \xleftarrow{a} (i)$. As $e_i = e_j = e$ there is a Zariski open subset $U \hookrightarrow rep_\alpha\ Q^\bullet$ of points V such that V_a is invertible. By basechange in either v_i or v_j we can find a point W in its orbit such that $W_a = \mathbb{1}_e$. If we think of W_a as identifying $\mathbb{C}^{e_i}$ with $\mathbb{C}^{e_j}$ we can view the remaining maps of W as a representation in $rep_{\alpha_s}\ Q_s^\bullet$ and denote it by W^s. The map $U \longrightarrow rep_{\alpha_s}\ Q_s^\bullet$ is well-defined and maps $GL(\alpha)$-orbits to $GL(\alpha_s)$-orbits. Conversely, given a representation $W' \in rep_{\alpha_s}\ Q_s^\bullet$ we can uniquely determine a representation $W \in U$ mapping to W'. Both claims follow immediately from this observation.* □

It is clear that any marked quiver-setting can uniquely be reduced to its simplest form, which has the characteristic property that no arrow-connected vertices can have the same dimension. The shrinking process has a converse operation which we will call **splitting of a vertex**. *However, this splitting operation is usually not uniquely determined. Before we can compile lists of marked-quiver settings in simplified form for a specific base-dimension d, we need to bound the components of the occurring dimension vectors α. We will do this in the case of quivers and leave the extension to marked quivers an an exercise.*

Proposition 6.30 *Let $\alpha = (e_1, \ldots, e_k)$ be the dimension vector of a simple representation of Q and let $1 - \chi_Q(\alpha, \alpha) = d = dim\ rep_\alpha\ Q^\bullet/GL(\alpha)$. Then, if $e = max\ e_i$, we have that $d \geq e + 1$.*

Proof. *By the above lemma we may assume that (Q, α) is brought in its simplest form, that is, no two arrow-connected vertices have the same dimension. Let χ_{ii} denote the number of loops in a vertex v_i, then*

$$-\chi_Q(\alpha,\alpha) = \begin{cases} \sum_i\ e_i\ (\sum_j\ \chi_{ij}e_j - e_i) \\ \sum_i\ e_i\ (\sum_j\ \chi_{ji}e_j - e_i) \end{cases}$$

and observe that the bracketed terms are positive by the requirement that α is the dimension vector of a simple representation. We call them the incoming in_i, respectively outgoing out_i, contribution of the vertex v_i to d. Let v_m be a vertex with maximal vertex-dimension e.

$$in_m = e(\sum_{j \neq m} \chi_{jm}e_j + (\chi_{ii} - 1)e) \quad and \quad out_m = e(\sum_{j \neq m} \chi_{ij}e_j + (\chi_{ii} - 1)e)$$

If there are loops in v_m, then $in_m \geq 2$ or $out_m \geq 2$ unless the local structure of Q is

$$\begin{tikzcd} (1) \arrow[r] & (e) \arrow[loop above] \arrow[r] & (1) \end{tikzcd}$$

in which case $in_m = e = out_m$. Let v_i be the unique incoming vertex of v_m, then we have $out_i \geq e - 1$. But then,

$$d = 1 - \chi_Q(\alpha,\alpha) = 1 + \sum_j out_j \geq 2e$$

If v_m has no loops, consider the incoming vertices $\{v_{i_1}, \ldots, v_{i_s}\}$, then

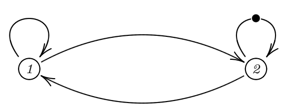
$$in_m = e(\sum_{j=1}^{s} \chi_{i_j m} e_{i_j} - e)$$

which is $\geq e$ unless $\sum \chi_{i_j m} e_{i_j} = e$, but in that case we have

$$\sum_{j=1}^{s} out_{i_j} \geq e^2 - \sum_{j=1}^{s} e_{i_j}^2 \geq e$$

the last inequality because all $e_{i_j} < e$. In either case we have that $d = 1 - \chi_Q(\alpha, \alpha) = 1 + \sum_i out_i = 1 + \sum_i in_i \geq e + 1$. $\square$

This result allows us to compile a list of all possible marked quiver-settings in simplest form for small values of d. In such a list we are only interested in $rep_\alpha\, Q^\bullet$ as $GL(\alpha)$-module and we call two setting equivalent if they determine the same $GL(\alpha)$-module. For example, the marked quiver-settings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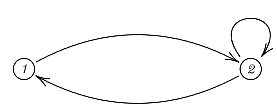
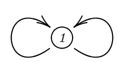
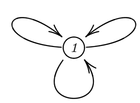


determine the same $\mathbb{C}^* \times GL_2$-module, hence are equivalent.

Theorem 6.31 *Let A be a Cayley-Hamilton order of degree n over a central normal variety X of degree d. Then, the local quiver of A in a point $\xi \in X = \underline{iss}_n^t\, A$ belonging to the smooth locus $Sm_n\, A$ can be shrinked to one of a finite list of equivalence classes of marked quiver-settings. For $d \leq 4$, the complete lists are given below where the boxed values are the dimension d of X.*


$\boxed{1}$


$\boxed{2}$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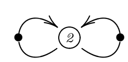
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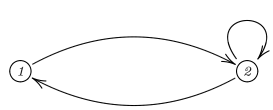
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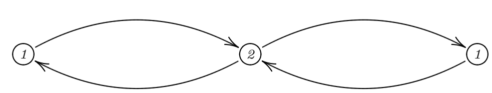
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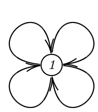
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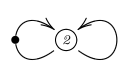
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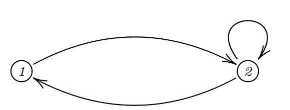
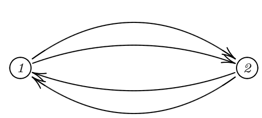
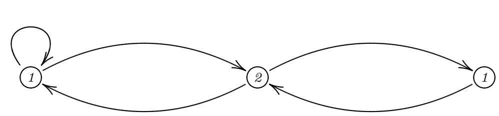

$\boxed{4}$


$\boxed{4}$


$\boxed{4}$

An immediate consequence is the following analog of the fact that commutative smooth varieties have only one type of analytic (or étale) local behavior.

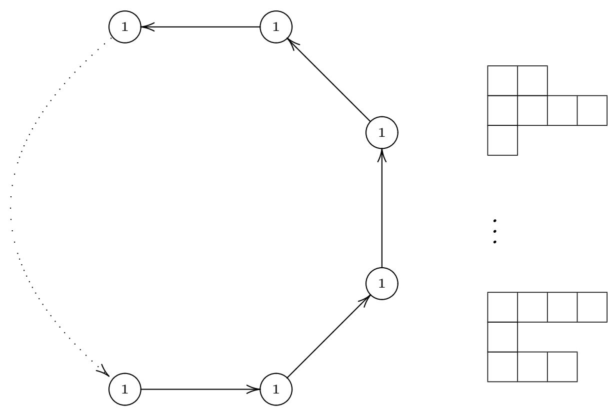
Theorem 6.32 *There are only finitely many types of étale local behaviour of smooth Cayley-Hamilton orders of degree n over a central variety of dimension d.*

Proof. *The foregoing reduction shows that for fixed d there are only a finite number of marked quiver-settings shrinked to their simplest form. As $\sum e_i \leq n$, we can only apply the splitting operations on vertices a finite number of times.* $\square$

6.8 Curve orders.

W. Schelter has proved in [26] that in dimension one, smooth orders are hereditary. In this section we will give an alternative proof of this result using the étale local classification. The result below can also be proved by the splitting operation and the above classification. We give this direct proof as an illustration of the stratification result of § 4.

Theorem 6.33 *Let A be a Cayley-Hamilton order of degree n over an affine curve $X = \underline{iss}_n^t\ A$. If $\xi \in Sm_n\ A$, then the étale local structure of A in ξ is determined by a marked quiver-setting which is an oriented cycle on k vertices with $k \leq n$*



and an unordered partition $p = (d_1, \ldots, d_k)$ having precisely k parts such that $\sum_i d_i = n$ determining the dimensions of the simple components of V_ξ.

Proof. *Let $(Q^\bullet, \alpha)$ be the corresponding local marked quiver-setting. Because $Q^\bullet$ is strongly connected, there exist oriented cycles in $Q^\bullet$. Fix one such cycle of length $s \leq k$ and renumber the vertices of $Q^\bullet$ such that the first s vertices make up the cycle. If $\alpha = (e_1, \ldots, e_k)$, then there exist semi-simple representations in $rep_\alpha\ Q^\bullet$ with composition*

$$\alpha_1 = (\underbrace{1, \ldots, 1}_{s}, \underbrace{0, \ldots, 0}_{k-s}) \oplus \epsilon_1^{\oplus e_1 - 1} \oplus \ldots \oplus \epsilon_s^{\oplus e_s - 1} \oplus \epsilon_{s+1}^{\oplus e_{s+1}} \oplus \ldots \oplus \epsilon_k^{\oplus e_k}$$

where ϵ_i stands for the simple one-dimensional representation concentrated in vertex v_i. There is a one-dimensional family of simple representations of dimension vector α_1, hence the stratum of semi-simple representations in $iss_\alpha\ Q^\bullet$ of representation type $\tau = (1, \alpha_1; e_1 - 1, \epsilon_1; \ldots ; e_s - 1, \epsilon_s; e_{s+1}, \epsilon_{s+1}; e_k, \epsilon_k)$ is at least one-dimensional. However, as dim $iss_\alpha\ Q^\bullet = 1$ this can only happen if this semi-simple representation is actually simple. That is, when $\alpha = \alpha_1$ and $k = s$. $\square$

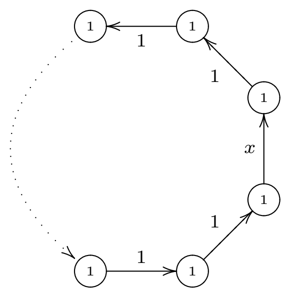
Hence, if V_ξ is the semi-simple n-dimensional representation of A corresponding to ξ, then

$$V_\xi = S_1 \oplus \ldots \oplus S_k \quad \text{with} \quad \dim\, S_i = d_i$$

That is, the stabilizer subgroup is $GL(\alpha) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ embedded in GL_n via the diagonal embedding

$$(\lambda_1, \ldots, \lambda_k) \longrightarrow diag(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{d_k})$$

Further, using basechange in $rep_\alpha\ Q^\bullet$ we can bring every simple α-dimensional representation of Q^α in standard form



where $x \in \mathbb{C}^*$ is the arrow from v_k to v_1. That is, $\mathbb{C}[rep_\alpha\ Q^\bullet]^{GL(\alpha)} \simeq \mathbb{C}[x]$ proving that the quotient (or central) variety X must be smooth in ξ by the Luna slice result. Moreover, as $\widehat{A_\xi} \simeq \widehat{\mathbb{T}_\alpha\ Q^\bullet}$ we have, using the numbering conventions of the vertices) the following block decomposition

$$\widehat{A_\xi} \simeq \begin{bmatrix} M_{d_1}(\mathbb{C}[[x]]) & M_{d_1 \times d_2}(\mathbb{C}[[x]]) & \ldots & M_{d_1 \times d_k}(\mathbb{C}[[x]]) \\ M_{d_2 \times d_1}(x\mathbb{C}[[x]]) & M_{d_2}(\mathbb{C}[[x]]) & \ldots & M_{d_2 \times d_k}(\mathbb{C}[[x]]) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_k \times d_1}(x\mathbb{C}[[x]]) & M_{d_k \times d_2}(x\mathbb{C}[[x]]) & \ldots & M_{d_k}(\mathbb{C}[[x]]) \end{bmatrix}$$

and from the local description of hereditary orders given in [25, Thm. 39.14] we deduce that A_ξ is an hereditary order. That is, we have the following characterization of the smooth locus

Proposition 6.34 *Let A be a Cayley-Hamilton order of degree n over a central affine curve X. Then, $Sm_n\ A$ is the locus of points $\xi \in X$ such that A_ξ is an hereditary order (in particular, ξ must be a smooth point of X).*

Globalizing this result, we obtain the following characterization of noncommutative smooth models in dimension one.

Theorem 6.35 *Let $\mathcal{A}$ be a Cayley-Hamilton central $\mathcal{O}_X$-order of degree n where X is a projective curve. Equivalent are*

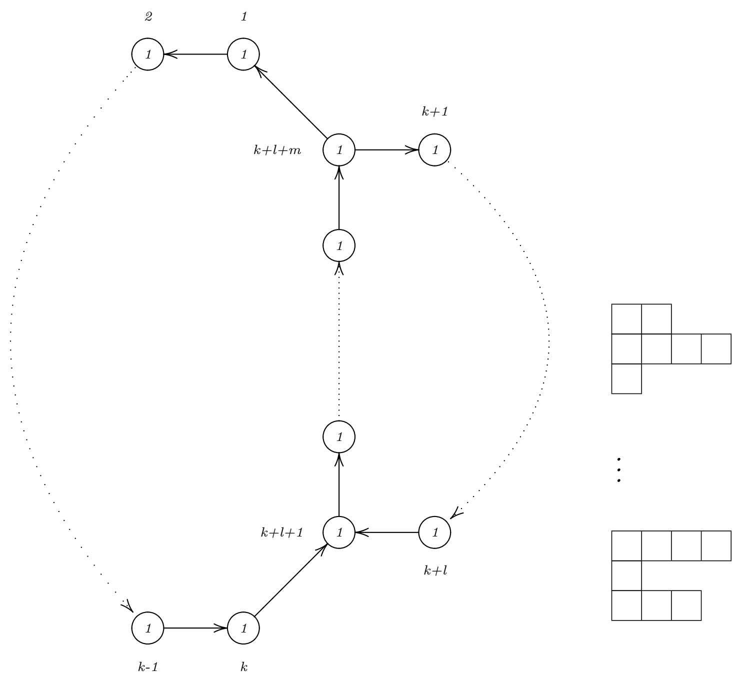
1. *$\mathcal{A}$ is a sheaf of Cayley-smooth orders*

2. *X is smooth and $\mathcal{A}$ is a sheaf of hereditary $\mathcal{O}_X$-orders*

6.9 Surface orders.

The result below can equally be proved using the splitting operation and the classification result.

Theorem 6.36 *Let A be a Cayley-Hamilton order of degree n over an affine surface $X = \underline{iss}_n^t\ A$. If $\xi \in Sm_n\ A$, then the étale local structure of A in ξ is determined by a marked local quiver-setting A_{klm} on $k+l+m \leq n$ vertices*



and an unordered partition $p = (d_1, \ldots, d_{k+l+m})$ of n with $k+l+m$ non-zero parts determined by the dimensions of the simple components of V_ξ.

Proof. *Let $(Q^\bullet, \alpha)$ be the marked quiver-setting on r vertices with $\alpha = (e_1, \ldots, e_r)$ corresponding to ξ. As $Q^\bullet$ is strongly connected and the quotient variety is two-dimensional, $Q^\bullet$ must contain more than one oriented cycle, hence it contains a sub-quiver of type A_{klm}, possibly degenerated with k or l equal to zero. Order the first $k+l+m$ vertices of $Q^\bullet$ as indicated. One verifies that A_{klm} has simple representations of dimension vector $(1, \ldots, 1)$. Assume that A_{klm} is a proper subquiver and denote $s = k+l+m+1$ then $Q^\bullet$ has semi-simple representations in $rep_\alpha\ Q^\bullet$ with dimension-vector decomposition*

$$\alpha_1 = (\underbrace{1, \ldots, 1}_{k+l+m}, 0, \ldots, 0) \oplus \epsilon_1^{\oplus e_1 - 1} \oplus \ldots \oplus \epsilon_{k+l+m}^{\oplus e_{k+l+m}-1} \oplus \epsilon_s^{\oplus e_s} \oplus \ldots \oplus \epsilon_r^{\oplus e_r}$$

Applying the formula for the dimension of the quotient variety shows that $iss_{(1,\ldots,1)}\ A_{klm}$ has dimension 2 so there is a two-dimensional family of such semi-simple representation in the two-dimensional quotient variety $iss_\alpha\ Q^\bullet$. This is only possible if this semi-simple representation is actually simple, whence $r = k+l+m$, $Q^\bullet = A_{klm}$ and $\alpha = (1, \ldots, 1)$. $\square$

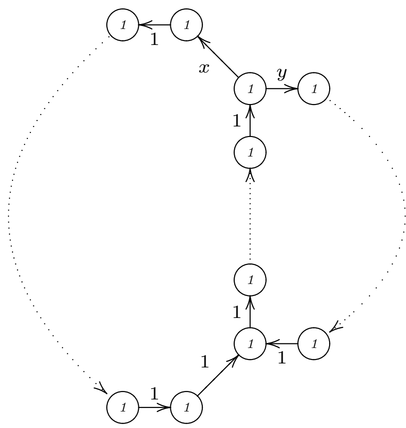
If V_ξ is the semi-simple n-dimensional representation of A corresponding to ξ, then

$$V_\xi = S_1 \oplus \ldots \oplus S_r \quad \textit{with} \quad \dim\ S_i = d_i$$

and the stabilizer subgroup $GL(\alpha) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ embedded diagonally in GL_n

$$(\lambda_1, \ldots, \lambda_r) \mapsto diag(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{d_r})$$

By basechange in $rep_\alpha\ A_{klm}$ we can bring every simple α-dimensional representation in the following standard form



with $x, y \in \mathbb{C}^*$ and as $\mathbb{C}[iss_\alpha\ A_{klm}] = \mathbb{C}[rep_\alpha\ A_{klm}]^{GL(\alpha)}$ is the ring generated by traces along oriented cycles in A_{klm}, it is isomorphic to $\mathbb{C}[x,y]$. From the Luna slice results one deduces that ξ must be a smooth point of X and because $\widehat{A_\xi} \simeq \mathbb{T}_\alpha\ \widehat{A_{klm}}$ we deduce it must have the following block-decomposition

$$\hat{A}_\xi \simeq \left|\begin{array}{c|c|c} \begin{smallmatrix} & (1) \\ (x) & \end{smallmatrix} & (y) & (1) \\ \hline (x) & \begin{smallmatrix} & (1) \\ (y) & \end{smallmatrix} & (1) \\ \hline (x) & (y) & \begin{smallmatrix} & (1) \\ (x,y) & \end{smallmatrix} \\ \underbrace{\qquad}_{k} & \underbrace{\qquad}_{l} & \underbrace{\qquad}_{m} \end{array}\right| \hookrightarrow M_n(\mathbb{C}[[x,y]])$$

where at spot (i,j) with $1 \leq i, j \leq k+l+m$ there is a block of dimension $d_i \times d_j$ with entries the indicated ideal of $\mathbb{C}[[x,y]]$.

Definition 6.37 Let A be a Cayley-Hamilton central $\mathbb{C}[X]$-order of degree n in a central simple $\mathbb{C}(X)$- algebra Δ of dimension n^2.

1. A is said to be étale locally split in ξ if and only if $\widehat{A_\xi}$ is a central $\hat{\mathcal{O}}_{X,x}$-order in $M_n(\hat{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}(X))$.

2. The ramification locus ram_A of A is the locus of points $\xi \in X$ such that

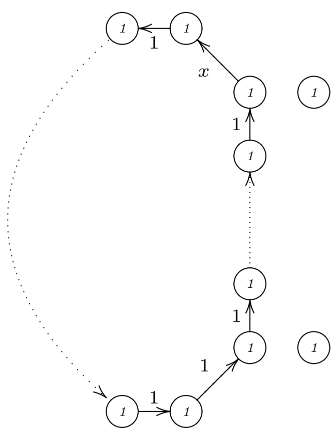
$$\frac{A}{\mathfrak{m}_\xi A \mathfrak{m}_\xi} \not\simeq M_n(\mathbb{C})$$

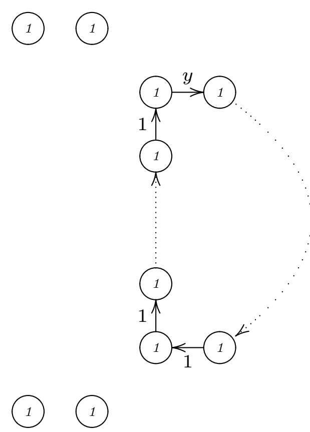
   The complement $X - ram_A$ is called the Azumaya locus az_A of A.

Theorem 6.38 *Let $\mathcal{A}$ be a Cayley-smooth central $\mathcal{O}_X$-order of degree n over a projective surface X. Then,*

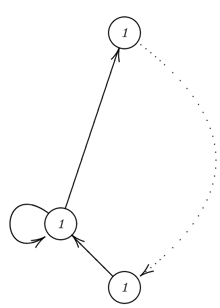
1. *X is smooth.*
2. *$\mathcal{A}$ is étale locally split in all points of X.*
3. *The ramification divisor $ram_{\mathcal{A}} \hookrightarrow X$ is either empty or consists of a finite number of isolated (possibly embedded) points and a reduced divisor having as its worst singularities normal crossings.*

Proof. (1) *and* (2) *follow from the above local description of $\mathcal{A}$. As for* (3) *we have to compute the local quiver-settings in proper semi-simple representations of $rep_\alpha\ A_{klm}$. As simples have a strongly connected support, the decomposition types of these proper semi-simple can be depicted by one of the following two situations*

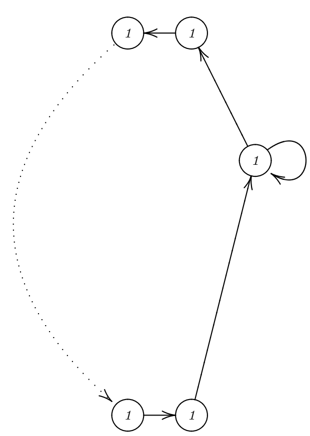




with $x, y \in \mathbb{C}^*$. By the description of local quivers given in section 3 we see that they are respectively of the following form

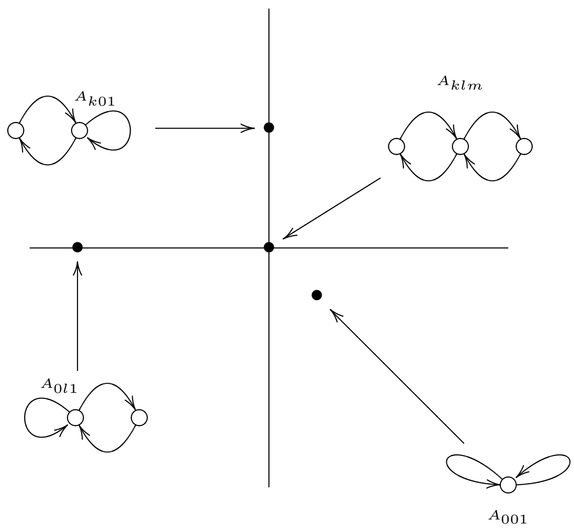


A_{0l1}

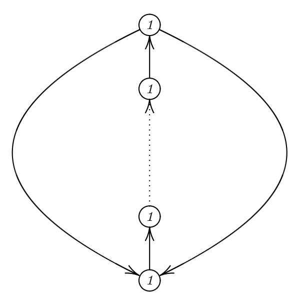


A_{k01}

and the associated unordered partitions are defined in the obvious way, that is, to the looped vertex one assigns the sum of the d_i belonging to the loop-contracted circuit and the other components of the partition are preserved. Using the étale local isomorphism between X in a neighborhood of ξ and of $iss_\alpha\ A_{klm}$ in a neighborhood of the trivial representation, we see that the local picture of quiver-settings of $\mathcal{A}$ in *a neighborhood of ξ can be represented by*



The Azumaya points are the points in which the quiver-setting is A_{001} (the two-loop quiver). From this local description the result follows if we take care of possibly degenerated cases. For example, an isolated point in ξ can occur if the quiver-setting in ξ is of type A_{00m} with $m \geq 2$, that is,



□

In the next section we will characterize those central simple $\mathbb{C}(X)$-algebras Δ allowing a Cayley-smooth model. We first need to perform a local calculation. Consider the ring of algebraic functions in two variables $\mathbb{C}\{x,y\}$ and let $X_{loc} = Spec\ \mathbb{C}\{x,y\}$. There is only one codimension two subvariety : $m = (x,y)$. Let us compute the coniveau spectral sequence for X_{loc}. If K is the field of fractions of $\mathbb{C}\{x,y\}$ and if we denote with k_p the field of fractions of $\mathbb{C}\{x,y\}/p$ where p is a *height one prime, we have as its first term*

0	0	0	0	$\ldots$
$H^2(K,\mu_n)$	$\oplus_p\ H^1(k_p,\mathbb{Z}_n)$	μ_n^{-1}	0	$\ldots$
$H^1(K,\mu_n)$	$\oplus_p \mathbb{Z}_n$	0	0	$\ldots$
μ_n	0	0	0	$\ldots$

Because $\mathbb{C}\{x,y\}$ is a unique factorization domain, we see that the map

$$H^1_{et}(K,\mu_n) = K^*/(K^*)^n \xrightarrow{\gamma} \oplus_p \mathbb{Z}_n$$

is surjective. Moreover, all fields k_p are isomorphic to the field of fractions of $\mathbb{C}\{z\}$ whose only cyclic extensions are given by adjoining a root of z and hence they are all ramified in m. Therefore, the component maps

$$\mathbb{Z}_n = H^1_{et}(k_p,\mathbb{Z}_n) \xrightarrow{\beta_L} \mu^{-1}$$

are isomorphisms. But then, the second (and limiting) term of the spectral sequence has the form

0	0	0	0	$\ldots$
$Ker\ \alpha$	$Ker\ \beta/Im\ \alpha$	0	0	$\ldots$
$Ker\ \gamma$	0	0	0	$\ldots$
μ_n	0	0	0	$\ldots$

Finally, we use the fact that $\mathbb{C}\{x,y\}$ is strict Henselian whence has no proper étale extensions. But then,

$$H^i_{et}(X_{loc},\mu_n) = 0 \text{ for } i \geq 1$$

and substituting this information in the spectral sequence we obtain that the top sequence of the coniveau spectral sequence

$$0 \longrightarrow Br_n\ K \xrightarrow{\alpha} \oplus_p \mathbb{Z}_n \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

is exact. From this sequence we immediately obtain the following

Lemma 6.39 *With notations as before, we have*

1. *Let* $U = X_{loc} - V(x)$*, then* $Br_n\ U = 0$

2. *Let* $U = X_{loc} - V(xy)$*, then* $Br_n\ U = \mathbb{Z}_n$ *with generator the quantum-plane algebra*
$$\mathbb{C}_\zeta[u,v] = \frac{\mathbb{C}\langle u,v\rangle}{(vu - \zeta uv)}$$
where ζ *is a primitive* n*-th root of one*

6.10 Noncommutative smooth surfaces.

Let Δ *be a central simple algebra of dimension* n^2 *over a field of transcendence degree* 2 *say* L*. We want to determine when* Δ *admits a Cayley-smooth order* $\mathcal{A}$*, that is, a sheaf of Cayley-smooth* $\mathcal{O}_X$*-algebras where* X *is a projective surface with functionfield* $\mathbb{C}(X) = L$*. In the previous section we have seen that if such a model exists, then* X *has to be a smooth projective surface. So we may assume that* X *is a commutative smooth model for* L*. But then we know from the Artin-Mumford exact sequence, proved in chapter 2, that the class of* Δ *in* $Br_n\ \mathbb{C}(X)$ *is determined by the following geo-combinatorial data*

- *A finite collection* $\mathcal{C} = \{C_1,\ldots,C_k\}$ *of* irreducible curves *in* X*.*

- *A finite collection* $\mathcal{P} = \{P_1,\ldots,P_l\}$ *of* points *of* X *where each* P_i *is either an intersection point of two or more* C_i *or a singular point of some* C_i*.*

- *For each* $P \in \mathcal{P}$ *the* branch-data $b_P = (b_1,\ldots,b_{i_P})$ *with* $b_i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ *and* $\{1,\ldots,i_P\}$ *the different branches of* $\mathcal{C}$ *in* P*. These numbers must satisfy the admissibility condition*
$$\sum_i b_i = 0 \in \mathbb{Z}_n$$
for every $P \in \mathcal{P}$

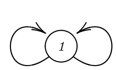
- *for each* $C \in \mathcal{C}$ *we fix a cyclic* $\mathbb{Z}_n$*-cover of smooth curves*
$$D \twoheadrightarrow \tilde{C}$$
of the desingularization $\tilde{C}$ *of* C *which is compatible with the branch-data.*

We have seen in chapter 2 that if $\mathcal{A}$ *is a maximal* $\mathcal{O}_X$*-order in* Δ*, then the ramification locus* $ram_{\mathcal{A}}$ *coincides with the collection of curves* $\mathcal{C}$*. We fix such a maximal* $\mathcal{O}_X$*-order* $\mathcal{A}$ *and investigate its smooth locus.*

Proposition 6.40 *Let* $\mathcal{A}$ *be a maximal* $\mathcal{O}_X$*-order in* Δ *with* X *a projective smooth surface and with geo-combinatorial data* $(\mathcal{C},\mathcal{P},b,\mathcal{D})$ *determining the class of* Δ *in* $Br_n\ \mathbb{C}(X)$*.*

If $\xi \in X$ *lies in* $X - \mathcal{C}$ *or if* ξ *is a non-singular point of* $\mathcal{C}$*, then* $\mathcal{A}$ *is smooth in* ξ*.*

Proof. *If* $\xi \notin \mathcal{C}$*, then* $\mathcal{A}_\xi$ *is an Azumaya algebra over* $\mathcal{O}_{X,x}$*. As* X *is smooth in* ξ*,* $\mathcal{A}$ *is Cayley-smooth in* ξ*. Alternatively, we know that Azumaya algebras are split by étale extensions, whence* $\hat{\mathcal{A}}_\xi \simeq M_n(\mathbb{C}[[x,y]])$ *which shows that the behaviour of* $\mathcal{A}$ *near* ξ *is controlled by the local data*



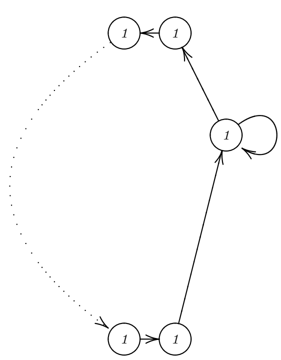
$$\underbrace{\square\ldots\square}_{n}$$

and hence $\xi \in Sm_n \mathcal{A}$. *Next, assume that* ξ *is a nonsingular point of the ramification divisor* $\mathcal{C}$. *Consider the pointed spectrum* $X_\xi = Spec\ \mathcal{O}_{X,\xi} - \{\mathfrak{m}_\xi\}$. *The only prime ideals are of height one, corresponding to the curves on* X *passing through* ξ *and hence this pointed spectrum is a Dedekind scheme. Further,* $\mathcal{A}$ *determines a maximal order over* X_ξ. *But then, tensoring* $\mathcal{A}$ *with the strict henselization* $\mathcal{O}^{sh}_{X,\xi} \simeq \mathbb{C}\{x,y\}$ *determines a sheaf of hereditary orders on the pointed spectrum* $\hat{X}_\xi = Spec\ \mathbb{C}\{x,y\} - \{(x,y)\}$ *and we may choose the local variable* x *such that* x *is a local parameter of the ramification divisor* $\mathcal{C}$ *near* ξ.

Using the characterization result for hereditary orders over discrete valuation rings, given in [25, Thm. 39.14] we know the structure of this extended sheaf if hereditary orders over every height one prime of $\hat{X}_\xi$. *Because* $\mathcal{A}_\xi$ *is a reflexive (even a projective)* $\mathcal{O}_{X,\xi}$*-module this height one information determines* $\mathcal{A}^{sh}_\xi$ *or* $\widehat{\mathcal{A}}_\xi$. *This proves that* $\mathcal{A}^{sh}_\xi$ *must be isomorphic to the following blockdecomposition*

$$\left[\begin{array}{c|c|c|c}
M_{d_1}(\mathbb{C}\{x,y\}) & M_{d_1 \times d_2}(\mathbb{C}\{x,y\}) & \ldots & M_{d_1 \times d_k}(\mathbb{C}\{x,y\}) \\
\hline
M_{d_2 \times d_1}(x\mathbb{C}\{x,y\}) & M_{d_2}(\mathbb{C}\{x,y\}) & \ldots & M_{d_2 \times d_k}(\mathbb{C}\{x,y\}) \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
M_{d_k \times d_1}(x\mathbb{C}\{x,y\}) & M_{d_k \times d_2}(x\mathbb{C}\{x,y\}) & \ldots & M_{d_k}(\mathbb{C}\{x,y\})
\end{array}\right]$$

for a certain partition $p = (d_1, \ldots, d_k)$ *of* n *having* k *parts. In fact, as we started out with a maximal order* $\mathcal{A}$ *one can even show that all these integers* d_i *must be equal. Anyway, this local form corresponds to the following quiver-setting*



$p = (d_1, \ldots, d_k)$

A_{k01}

whence $\xi \in Sm_n\ A$ *as this is one of the allowed surface settings.* □

Concluding, a maximal $\mathcal{O}_X$*-order in* Δ *can have at worst noncommutative singularities in the singular points of the ramification divisor* $\mathcal{C}$. *We have seen that a Cayley-smooth order over a surface has as ramification-singularities at worst normal crossings. We are always able to reduce to normal crossings by the following classical result on commutative surfaces, see for example [9, V.3.8].*

Theorem 6.41 (Embedded resolution of curves in surfaces) *Let* $\mathcal{C}$ *be any curve on the surface* X. *Then, there exists a finite sequence of blow-ups*

$$X' = X_s \longrightarrow X_{s-1} \longrightarrow \ldots \longrightarrow X_0 = X$$

and, if $f : X' \twoheadrightarrow X$ is their composition, then the total inverse image $f^{-1}(\mathcal{C})$ is a divisor with normal crossings.

Fix now a series of blow-ups $X' \xrightarrow{f} X$ such that the inverse image $f^{-1}(\mathcal{C})$ is a divisor on X' having as worst singularities normal crossings. We will now replace the Cayley-Hamilton $\mathcal{O}_X$-order $\mathcal{A}$ by a Cayley-Hamilton $\mathcal{O}_{X'}$-order $\mathcal{A}'$ where $\mathcal{A}'$ is a sheaf of $\mathcal{O}_{X'}$-maximal orders in Δ. In order to determine the ramification divisor of $\mathcal{A}'$ we need to be able to keep track how the ramification divisor $\mathcal{C}$ of Δ changes if we blow up a singular point $p \in \mathcal{P}$.

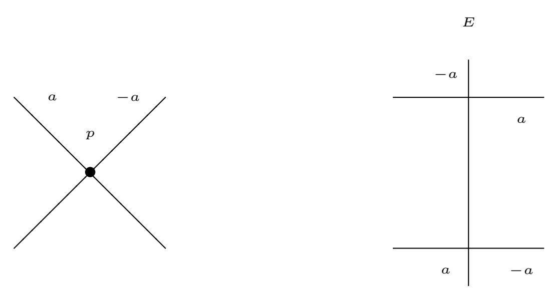
Lemma 6.42 *Let $\tilde{X} \twoheadrightarrow X$ be the blow-up of X at a singular point p of $\mathcal{C}$, the ramification divisor of Δ on X. Let $\tilde{\mathcal{C}}$ be the strict transform of $\mathcal{C}$ and E the exceptional line on $\tilde{X}$. Let $\mathcal{C}'$ be the ramification divisor of Δ on the smooth model $\tilde{X}$. Then,*

1. *Assume the local branch data at p distribute in an admissible way on $\tilde{\mathcal{C}}$, that is,*
$$\sum_{i \text{ at } q} b_{i,p} = 0 \text{ for all } q \in E \cap \tilde{\mathcal{C}}$$
where the sum is taken only over the branches at q. Then, $\mathcal{C}' = \tilde{\mathcal{C}}$.

2. *Assume the local branch data at p do not distribute in an admissible way, then $\mathcal{C}' = \tilde{\mathcal{C}} \cup E$.*

Proof. *Clearly, $\tilde{\mathcal{C}} \hookrightarrow \mathcal{C}' \hookrightarrow \tilde{\mathcal{C}} \cup E$. By the Artin-Mumford sequence applied to X' we know that the branch data of $\mathcal{C}'$ must add up to zero at all points q of $\tilde{\mathcal{C}} \cap E$. We investigate the two cases*

1. *: Assume $E \subset \mathcal{C}'$. Then, the E-branch number at q must be zero for all $q \in \tilde{\mathcal{C}} \cap E$. But there are no non-trivial étale covers of $\mathbb{P}^1 = E$ so $ram(\Delta)$ gives the trivial element in $H^1_{et}(\mathbb{C}(E), \mu_n)$, a contradiction. Hence $\mathcal{C}' = \tilde{\mathcal{C}}$.*



2. *: If at some $q \in \tilde{\mathcal{C}} \cap E$ the branch numbers do not add up to zero, the only remedy is to include E in the ramification divisor and let the E-branch number be such that the total sum is zero in $\mathbb{Z}_n$.* □

Theorem 6.43 *Let Δ be a central simple algebra of dimension n^2 over a field L of transcendence degree two. Then, there exists a smooth projective surface S with functionfield $\mathbb{C}(S) = L$ such that any maximal $\mathcal{O}_S$-order $\mathcal{A}_S$ in Δ has at worst a finite number of isolated noncommutative singularities. Each of these singularities is locally étale of quantum-plane type.*

Proof. *We take any projective smooth surface X with functionfield $\mathbb{C}(X) = L$. By the Artin-Mumford exact sequence, the class of Δ determines a geo-combinatorial set of data*
$$(\mathcal{C}, \mathcal{P}, b, \mathcal{D})$$

as before. In particular, $\mathcal{C}$ is the ramification divisor $ram(\Delta)$ and $\mathcal{P}$ is the set of singular points of $\mathcal{C}$. We can separate $\mathcal{P}$ in two subsets

- $\mathcal{P}_{unr} = \{P \in \mathcal{P}$ *where all the branch-data* $b_P = (b_1, \ldots, b_{i_P})$ *are trivial, that is, all* $b_i = 0$ *in* $\mathbb{Z}_n\}$
- $\mathcal{P}_{ram} = \{P \in \mathcal{P}$ *where some of the branch-data* $b_P = (b_1, \ldots, b_{i_P})$ *are non-trivial, that is, some* $b_i \neq 0$ *in* $\mathbb{Z}_n\}$

After a finite number of blow-ups we get a birational morphism $S_1 \xrightarrow{\pi} X$ *such that* $\pi^{-1}(\mathcal{C})$ *has as its worst singularities normal crossings and all branches in points of* $\mathcal{P}$ *are separated in* S. *Let* $\mathcal{C}_1$ *be the ramification divisor of* Δ *in* S_1. *By the foregoing argument we have*

- *If* $P \in \mathcal{P}_{unr}$, *then we have that* $\mathcal{C}' \cap \pi^{-1}(P)$ *consists of smooth points of* $\mathcal{C}_1$,
- *If* $P \in \mathcal{P}_{ram}$, *then* $\pi^{-1}(P)$ *contains at least one singular points* Q *of* $\mathcal{C}_1$ *with branch data* $b_Q = (a, -a)$ *for some* $a \neq 0$ *in* $\mathbb{Z}_n$.

In fact, after blowing-up singular points Q' *in* $\pi^{-1}(P)$ *with trivial branch-data we obtain a smooth surface* $S \twoheadrightarrow S_1 \twoheadrightarrow X$ *such that the only singular points of the ramification divisor* $\mathcal{C}'$ *of* Δ *have non-trivial branch-data* $(a, -a)$ *for some* $a \in \mathbb{Z}_n$. *Then, take a maximal* $\mathcal{O}_S$*-order* $\mathcal{A}$ *in* Δ. *By the local calculation of* $Br_n\ \mathbb{C}\{x, y\}$ *performed in the last section we know that locally étale* $\mathcal{A}$ *is of quantum-plane type in these remaining singularities. As the quantum-plane is not étale locally split,* $\mathcal{A}$ *is not Cayley-smooth in these finite number of singularities.* □

In fact, the above proof gives also a complete classification of those central simple algebras admitting a Cayley-smooth model.

Theorem 6.44 *Let* Δ *be a central simple* $\mathbb{C}(X)$*-algebra of dimension* n^2 *determined by the geo-combinatorial data* $(\mathcal{C}, \mathcal{P}, b, \mathcal{D})$ *given by the Artin-Mumford sequence. Then,* Δ *admits a Cayley-smooth model if and only if all branch-data are trivial.*

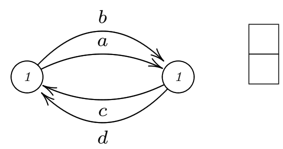
Proof. *If all branch-data are trivial, the foregoing proof constructs a Cayley-smooth model of* Δ. *Conversely, if* $\mathcal{A}$ *is a Cayley-smooth* $\mathcal{O}_S$*-order in* Δ *with* S *a smooth projective model of* $\mathbb{C}(X)$, *then* $\mathcal{A}$ *is locally étale split in every point* $s \in S$. *But then, so is any maximal* $\mathcal{O}_S$*-order* $\mathcal{A}_{max}$ *containing* $\mathcal{A}$. *By the foregoing arguments this can only happen if all branch-data are trivial.* □

6.11 Higher dimensional orders.

The strategy we used to characterize the central simple algebras over a surface admitting a Cayley-smooth model can also be applied (at least in principle) to higher dimensional varieties. First, one uses the classification result of marked quiver-settings to compile a list of allowed étale local behaviour of Cayley-smooth orders and of their ramification. Next, if a subclass of central simple algebras is determined by ramification data, the obtained local behaviour puts restrictions on those admitting a smooth model. We have seen that in the case of curves and surfaces, the central variety X *of a Cayley-smooth model* $\mathcal{A}$ *had to be smooth and that* $\mathcal{A}$ *is étale locally split in every point* $\xi \in X$. *Both of these properties are no longer valid in higher dimensions.*

Lemma 6.45 *For dimension* $d \geq 3$, *the center* Z *of a Cayley-smooth order of degree* n *can have singularities.*

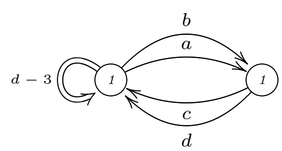
Proof. *Consider the following marked quiver-setting*



which is allowed for dimension $d = 3$ and degree $n = 2$. The quiver-invariants are generated by the traces along oriented cycles, that is, are generated by ac, ad, bc and bd. That is,

$$\mathbb{C}[iss_\alpha\ Q] \simeq \frac{\mathbb{C}[x, y, z, v]}{(xv - yz)}$$

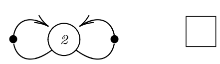
which has a singularity in the origin. This example can be extended to dimensions $d \geq 3$ by adding loops in one of the vertices.



$\square$

Lemma 6.46 *For dimension $d \geq 3$, a Cayley-smooth algebra does not have to be locally étale split in every point of its central variety.*

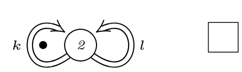
Proof. *Consider the following allowable quiver-setting for $d = 3$ and $n = 2$*



The corresponding Cayley-smooth algebra A is generated by two generic 2×2 trace zero matrices, say A and B. Using our knowledge of $\mathbb{T}_2^2$ we see that its center is generated by $A^2 = x$, $B^2 = z$ and $AB + BA = z$. Alternatively, we can identify A with the Clifford-algebra over $\mathbb{C}[x, y, z]$ of the non-degenerate quadratic form

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

This is a noncommutative domain and remains to be so over the formal power series $\mathbb{C}[[x, y, z]]$. That is, A cannot be split by an étale extension in the origin. More generally, whenever the local marked quiver contains vertices with dimension ≥ 2, the corresponding Cayley-smooth algebra cannot be split by an étale extension as the local quiver-setting does not change and for a split algebra all vertex-dimensions have to be equal to 1. In particular, the Cayley-smooth algebra of degree 2 corresponding to the quiver-setting



cannot be split by an étale extension in the origin. Its corresponding dimension is

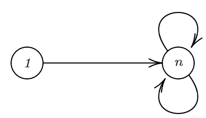
$$d = 3k + 4l - 3$$

whenever $k + l \geq 2$ and so all dimensions $d \geq 3$ are reached. $\square$

Chapter 7

Moduli Spaces.

In the study of the Hilbert scheme $Hilb_n$ of n points in $\mathbb{C}^2$ we ran into the quiver setting (Q, α)



In fact, we proved that $Hilb_n$ is the orbit space of the $GL(\alpha)$-orbits on triples $(A, B, u) \in rep_\alpha\ Q = M_n \oplus M_n \oplus \mathbb{C}^n$ such that $[A, B] = 0$ and u is a cyclic vector for (A, B). None of these triples (A, B, u) determines a closed $GL(\alpha)$-orbit in $rep_\alpha\ Q$ because

$$\lim_{t \to 0} (1, t\mathbb{1}_n).(A, B, v) = (A, B, \underline{0})$$

Still, a cyclic triple does determine a closed $GL(\alpha)$-orbit in some Zariski open subset rep (σ) determined by a Hilbert stair σ, as the dimension of all $GL(\alpha)$-orbits in rep (σ) is equal to n^2. Such situations, where a shortage of closed orbits is compensated when restricted to suitable open subsets, often occur such as in the study of linear dynamical systems as we will see in the first sections.

For a general quiver setting (Q, α) and a character $\chi_\theta : GL(\alpha) \longrightarrow \mathbb{C}^*$ we will study a moduli space $M_\alpha^{ss}(Q, \theta)$ classifying closed orbits in the Zariski open subset of so called θ-semistable representations of $rep_\alpha\ Q$. These moduli spaces were introduced and studied by A. King in [12]. The intuition we have formed on algebraic quotient varieties is helpful in studying these moduli spaces provided we use the following dictionary

$iss_\alpha\ Q$	$M_\alpha^{ss}(Q, \theta)$
closed orbits in $rep_\alpha\ Q$	*closed orbits in* $rep_\alpha^{ss}(Q, \theta)$
simple representation	θ*-stable representation*
semi-simple representation	*direct sum of* θ*-stable representations*
polynomial invariants	*semi-invariants of weight* θ

A first important problem is to determine which of these moduli spaces are non-empty, that is for which triples (Q, α, θ) do there exist θ-(semi)stable representations in $rep_\alpha\ Q$. A beautiful inductive combinatorial answer to this problem was discovered by A. Schofield [27]. His characterization of the dimension vectors α allowing θ-stables is as fundamental to the study of the moduli spaces $M_\alpha^{ss}(Q, \theta)$ as the description of the dimension vectors of simple representations is to the study of the quotient varieties $iss_\alpha\ Q$. These moduli spaces are defined to be the projective varieties of certain graded algebras of semi-invariant functions. Hence, we need to find generators of semi-invariants precisely as we needed to control the polynomial *invariants to study the quotient varieties. Of the many independent description of these semi-invariants we follow here the approach due to A. schofield and M. Van den Bergh in [28].*

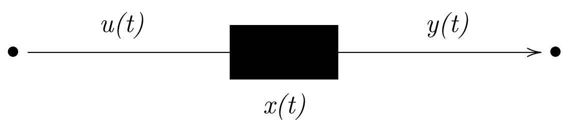
In the next chapter we will see that the investigation of these moduli spaces is crucial in the study of fibers of the representation spaces $rep_n\ A \twoheadrightarrow iss_n\ A$ *and of the Brauer-Severi fibration* $BS_n\ A \twoheadrightarrow iss_n\ A$ *for Cayley-smooth algebras* A *of degree* n.

7.1 Dynamical systems.

A linear time invariant dynamical system Σ *is determined by the following system of differential equations*

$$\begin{cases} \frac{dx}{dt} & = Bx + Au \\ y & = Cx. \end{cases} \tag{7.1}$$

Here, $u(t) \in \mathbb{C}^m$ *is the input or control of the system at tome* t, $x(t) \in \mathbb{C}^n$ *the state of the system and* $y(t) \in \mathbb{C}^p$ *the output of the system* Σ. *Time invariance of* Σ *means that the matrices* $A \in M_{n\times m}(\mathbb{C})$, $B \in M_n(\mathbb{C})$ *and* $C \in M_{p\times n}(\mathbb{C})$ *are constant. The system* Σ *can be represented as a black box*



which is in a certain state $x(t)$ *that we can try to change by using the input controls* $u(t)$. *By reading the output signals* $y(t)$ *we can try to determine the state of the system.*

Recall that the matrix exponential e^B *of any* $n \times n$ *matrix* B *is defined by the infinite series*

$$e^B = \mathbb{1}_n + B + \frac{B^2}{2!} + \ldots + \frac{B^m}{m!} + \ldots$$

The importance of this construction is clear from the fact that e^{Bt} *is the fundamental matrix for the homogeneous differential equation* $\frac{dx}{dt} = Bx$. *That is, the columns of* e^{Bt} *are a basis for the* n*-dimensional space of solutions of the equation* $\frac{dx}{dt} = Bx$.

Motivated by this, we look for a solution to equation (7.1) as the form $x(t) = e^{Bt}g(t)$ *for some function* $g(t)$. *Substitution gives the condition*

$$\frac{dg}{dt} = e^{-Bt}Au \quad \textit{whence} \quad g(\tau) = g(\tau_0) + \int_{\tau_0}^{\tau} e^{-Bt}Au(t)dt.$$

Observe that $x(\tau_0) = e^{B\tau_0}g(\tau_0)$ *and we obtain the solution of the linear dynamical system* $\Sigma = (A, B, C)$:

$$\begin{cases} x(\tau) & = e^{(\tau-\tau_0)B}x(\tau_0) + \int_{\tau_0}^{\tau} e^{(\tau-t)B}Au(t)dt \\ y(\tau) & = Ce^{B(\tau-\tau_0)}x(\tau_0) + \int_{\tau_0}^{\tau} Ce^{(\tau-t)B}Au(t)dt. \end{cases}$$

Differentiating we see that this is indeed a solution and it is the unique one having a prescribed starting state $x(\tau_0)$. *Indeed, given another solution* $x_1(\tau)$ *we have that* $x_1(\tau) - x(\tau)$ *is a solution to the homogeneous system* $\frac{dx}{dt} = Bt$, *but then*

$$x_1(\tau) = x(\tau) + e^{\tau B}e^{-\tau_0 B}(x_1(\tau_0) - x(\tau_0)).$$

We call the system Σ completely controllable if we can steer any starting state $x(\tau_0)$ to the zero state by some control function $u(t)$ in a finite time span $[\tau_0, \tau]$. That is, the equation

$$0 = x(\tau_0) + \int_{\tau_0}^{\tau} e^{(\tau_0 - t)B} A u(t) dt$$

has a solution in τ and $u(t)$. As the system is time-invariant we may always assume that $\tau_0 = 0$ and have to satisfy the equation

$$0 = x_0 + \int_0^{\tau} e^{tB} A u(t) dt \quad \textit{for every} \quad x_0 \in \mathbb{C}^n \tag{7.2}$$

Consider the control matrix $c(\Sigma)$ which is the $n \times mn$ matrix

$$c(\Sigma) = \boxed{\boxed{A} \; \boxed{BA} \; \boxed{B^2 A} \quad \cdots \quad \boxed{B^{n-1} A}}$$

Assume that $rk\ c(\Sigma) < n$ then there is a non-zero state $s \in \mathbb{C}^n$ such that $s^{\tau} c(\Sigma) = 0$, where s^{τ} denotes the transpose (row column) of s. Because B satisfies the characteristic polynomial $\chi_B(t)$, B^n and all higher powers B^m are linear combinations of $\{\mathbb{1}_n, B, B^2, \ldots, B^{n-1}\}$. Hence, $s^{\tau} B^m A = 0$ for all m. Writing out the power series expansion of e^{tB} in equation (7.2) this leads to the contradiction that $0 = s^{\tau} x_0$ for all $x_0 \in \mathbb{C}^n$. Hence, if $rk\ c(\Sigma) < n$, then Σ is not completely controllable.

Conversely, let $rk\ c(\Sigma) = n$ and assume that Σ is not completely controllable. That is, the space of all states

$$s(\tau, u) = \int_0^{\tau} e^{-tB} A u(t) dt$$

is a proper subspace of $\mathbb{C}^n$. But then, there is a non-zero state $s \in \mathbb{C}^n$ such that $s^{tr} s(\tau, u) = 0$ for all τ and all functions $u(t)$. Differentiating this with respect to τ we obtain

$$s^{tr} e^{-\tau B} A u(\tau) = 0 \quad \textit{whence} \quad s^{tr} e^{-\tau B} A = 0 \tag{7.3}$$

for any τ as $u(\tau)$ can take on any vector. For $\tau = 0$ this gives $s^{tr} A = 0$. If we differentiate (7.3) with respect to τ we get $s^{tr} B e^{-\tau B} A = 0$ for all τ and for $\tau = 0$ this gives $s^{tr} BA = 0$. Iterating this process we show that $s^{tr} B^m A = 0$ for any m, whence

$$s^{tr} \begin{bmatrix} A & BA & B^2 A & \ldots & B^{n-1} A \end{bmatrix} = 0$$

contradicting the assumption that $rk\ c(\Sigma) = n$. That is, we have proved :

Proposition 7.1 *A linear time-invariant dynamical system Σ determined by the matrices (A, B, C) is completely controllable if and only if $rk\ c(\Sigma)$ is maximal.*

We say that a state $x(\tau)$ at time τ is unobservable if $Ce^{(\tau - t)B} x(\tau) = 0$ for all t. Intuitively this means that the state $x(\tau)$ cannot be detected uniquely from the output of the system Σ. Again, if we differentiate this condition a number of times and evaluate at $t = \tau$ we obtain the conditions

$$Cx(\tau) = CBx(\tau) = \ldots = CB^{n-1} x(\tau) = 0.$$

We say that Σ is completely observable if the zero state is the only unobservable state at any time τ. Consider the observation matrix $o(\Sigma)$ of the system Σ which is the $pn \times n$ matrix

$$o(\Sigma) = \begin{bmatrix} C^{tr} & (CB)^{tr} & \ldots & (CB^{n-1})^{tr} \end{bmatrix}^{tr}$$

An analogous argument as in the proof of proposition 7.1 gives us that a linear time-invariant dynamical system Σ determined by the matrices (A,B,C) is completely observable if and only if rk $o(\Sigma)$ is maximal. For reasons which will become clear in a moment, we call linear time-invariant dynamical systems which are both completely controllable and completely observable Schurian systems. *An important problem in system theory is to classify the Schurian systems with the same input/output behavior. We reduce this problem to the study of GL_n-orbits in an open subset of a vectorspace. Assume we have two systems Σ and Σ', determined by matrix triples from $Sys = M_{n\times m}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{p\times n}(\mathbb{C})$ producing the same output $y(t)$ when given the same input $u(t)$, for all possible input functions $u(t)$. We recall that the output function y for a system $\Sigma = (A,B,C)$ is determined by*

$$y(\tau) = Ce^{B(\tau-\tau_0)}x(\tau_0) + \int_{\tau_0}^{\tau} Ce^{(\tau-t)B}Au(t)dt.$$

Differentiating this a number of times and evaluating at $\tau = \tau_0$ as in the proof of proposition 7.1 equality of input/output for Σ and Σ' gives the conditions

$$CB^iA = C'B'^iA' \quad \text{for all} \quad i.$$

As a consequence the systems Σ and Σ' have the same Hankel matrix which by definition is the product of the observation matrix with the control matrix of the system :

$$H(\Sigma) = \begin{bmatrix} C \\ CB \\ \vdots \\ CB^{n-1} \end{bmatrix} \begin{bmatrix} A & BA & \dots & B^{n-1}A \end{bmatrix} = \begin{bmatrix} \boxed{CA} & \boxed{CBA} & \cdots & \\ \boxed{CBA} & \boxed{CB^2A} & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & \boxed{CB^{2n-2}A} \end{bmatrix}$$

But then, we have for any $v \in \mathbb{C}^{mn}$ that $c(\Sigma)(v) = 0 \Leftrightarrow c(\Sigma')(v) = 0$ and we can decompose $\mathbb{C}^{pn} = V \oplus W$ such that the restriction of $c(\Sigma)$ and $c(\Sigma')$ to V are the zero map and the restrictions to W give isomorphisms with $\mathbb{C}^n$. Hence, there is an invertible matrix $g \in GL_n$ such that $c(\Sigma') = gc(\Sigma)$ and from the commutative diagram

$$\begin{tikzcd} \mathbb{C}^{mn} \arrow[r, two heads, "c(\Sigma)"] \arrow[d, equal] & \mathbb{C}^n \arrow[r, hook, "o(\Sigma)"] \arrow[d, "g"] & \mathbb{C}^{pn} \arrow[d, equal] \\ \mathbb{C}^{mn} \arrow[r, two heads, "c(\Sigma')"] & \mathbb{C}^n \arrow[r, hook, "o(\Sigma')"] & \mathbb{C}^{pn} \end{tikzcd}$$

we obtain that also $o(\Sigma') = o(\Sigma)g^{-1}$. Consider the system $\Sigma_1 = (A_1, B_1, C_1)$ equivalent with Σ under the base-change matrix g. That is, $\Sigma_1 = g.\Sigma = (gA, gBg^{-1}, Cg^{-1})$. Then,

$$\begin{bmatrix} A_1, B_1A_1, \dots, B_1^{n-1}A_1 \end{bmatrix} = gc(\Sigma) = c(\Sigma') = \begin{bmatrix} A', B'A', \dots, B'^{n-1}A' \end{bmatrix}$$

and so $A_1 = A'$. Further, as $B_1^{i+1}A_1 = B'^{i+1}A'$ we have by induction on i that the restriction of B_1 on the subspace of $B'^iIm(A')$ is equal to the restriction of B' on this space. Moreover, as $\sum_{i=0}^{n-1} B'^iIm(A') = \mathbb{C}^n$ it follows that $B_1 = B'$. Because $o(\Sigma') = o(\Sigma)g^{-1}$ we also have $C_1 = C'$. This finishes the proof of :

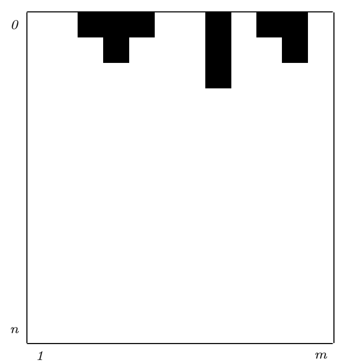
Proposition 7.2 *Let $\Sigma = (A, B, C)$ and $\Sigma' = (A', B', C')$ be two Schurian dynamical systems. The following are equivalent*

1. *The input/output behavior of Σ and Σ' are equal.*

2. *The systems Σ and Σ' are equivalent, that is, there exists an invertible matrix $g \in GL_n$ such that*

$$A' = gA, \quad B' = gBg^{-1} \quad \text{and} \quad C' = Cg^{-1}.$$

By definition, a dynamical system $\Sigma = (A, B, C)$ is Schurian if (and only if) the determinant of at least one $n \times n$ minor of $c(\Sigma)$ and $o(\Sigma)$ is non-zero. That is, the subset Sys^s of Schurian dynamical systems is open in Sys and is stable under the GL_n-action. Our next job is to classify the orbits under this action. We introduce a combinatorial gadget : the Kalman code. It is an array consisting of $(n+1) \times m$ boxes each having a position label (i, j) where $0 \leq i \leq n$ and $1 \leq j \leq m$. These boxes are ordered lexicographically that is $(i', j') < (i, j)$ if and only if either $i' < i$ or $i' = i$ and $j' < j$. Exactly n of these boxes are painted black subject to the rule that if box (i, j) is black, then so is box (i', j) for all $i' < i$. That is, a Kalman code looks like



We assign to a completely controllable system $\Sigma = (A, B, C)$ its Kalman code $K(\Sigma)$ as follows : let $A = \begin{bmatrix} A_1 & A_2 & \ldots & A_m \end{bmatrix}$, that is A_i is the i-th column of A. Paint the box (i, j) black if and only if the column vector $B^i A_j$ is linearly independent of the column vectors $B^k A_l$ for all $(k, l) < (i, j)$. The painted array $K(\Sigma)$ is indeed a Kalman code. Assume that box (i, j) is black but box (i', j) white for $i' < i$, then

$$B^{i'} A_j = \sum_{(k,l)<(i',j)} \alpha_{kl} B^k A_l \quad \text{but then,} \quad B^i A_j = \sum_{(k,l)<(i',j)} \alpha_{kl} B^{k+i-i'} A_l$$

and all $(k + i - i', l) < (i, l)$, a contradiction. Moreover, $K(\Sigma)$ has exactly n black boxes as there are n linearly independent columns of the control matrix $c(\Sigma)$ when Σ is completely controllable. The Kalman code is a discrete invariant of the orbit $\mathcal{O}_(\Sigma)$ under the action of GL_n. This follows from the fact that $B^i A_j$ is linearly independent of the $B^k A_l$ for all $(k, l) < (i, j)$ if and only if $gB^i A_j$ is linearly independent of the $gB^k A_l$ for any $g \in GL_n$ and the observation that $gB^k A_l = (gBg^{-1})^k (gA)_l$.

With V_c we will denote the open subset of all completely controllable pairs (A, B) that is, those for which the rank of the $n \times nm$ matrix $\begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A \end{bmatrix}$ is maximal. We consider the map

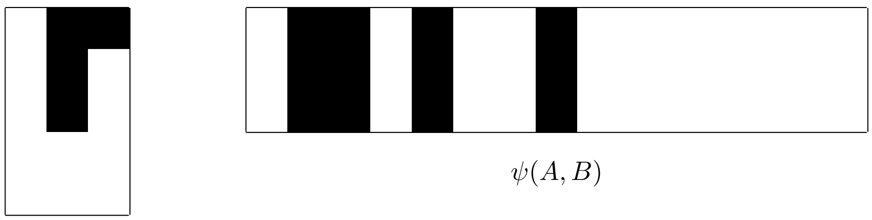
$$V = M_{n \times m}(\mathbb{C}) \oplus M_n(\mathbb{C}) \xrightarrow{\psi} M_{n \times (n+1)m}(\mathbb{C})$$

$$(A, B) \mapsto \begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A & B^nA \end{bmatrix}$$

The matrix $\psi(A,B)$ determines a linear map $\psi_{(A,B)} : \mathbb{C}^{(n+1)m} \longrightarrow \mathbb{C}^n$ and (A,B) is a completely controllable pair if and only if the corresponding linear map $\psi_{(A,B)}$ is surjective. Moreover, there is a linear action of GL_n on $M_{n\times(n+1)m}(\mathbb{C})$ by left multiplication and the map ψ is GL_n-equivariant.

7.2 Grassmannians.

The Kalman code induces a barcode on $\psi(A,B)$, that is the $n\times n$ minor of $\psi(A,B)$ determined by the columns corresponding to black boxes in the Kalman code.



$\psi(A,B)$

By construction this minor is an invertible matrix $g^{-1} \in GL_n$. We can choose a canonical point in the orbit $\mathcal{O}\ (A,B)\ :\ g.(A,B)$. It does have the characteristic property that the $n\times n$ minor of its image under ψ, determined by the Kalman code is the identity matrix $\mathbb{1}_n$. The matrix $\psi(g.(A,B))$ will be denoted by $b(A,B)$ and is called barcode of the pair (A,B). We claim that the barcode determines the orbit uniquely.

The map ψ is injective on the open set V_c of completely controllable pairs. Indeed, if

$$\begin{bmatrix} A & BA & \dots & B^nA \end{bmatrix} = \begin{bmatrix} A' & B'A' & \dots & {B'}^nA' \end{bmatrix}$$

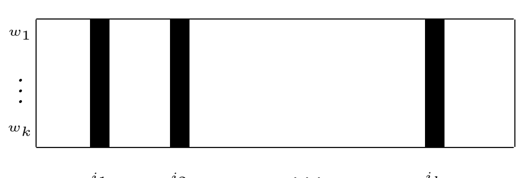
then $A = A'$, $B \mid Im(A) = B' \mid Im(A)$ and hence by induction also

$$B \mid B^iIm(A) = B' \mid {B'}^iIm(A') \quad \text{for all } i \leq n-1.$$

But then, $B = B'$ as both pairs (A,B) and (A',B') are completely controllable, that is, $\sum_{i=0}^{n-1} B^iIm(A) = \mathbb{C}^n = \sum_{i=0}^{n-1} {B'}^iIm(A')$. Hence, the barcode $b(A,B)$ determines the orbit $\mathcal{O}\ (A,B)$ and is a point in the Grassmannian *$Grass_n(m(n+1))$.*

We briefly recall the definition of these Grassmannians. Let $k \leq l$ be integers, then the points of the Grassmannian $Grass_k(l)$ are in one-to-one correspondence with k-dimensional subspaces of $\mathbb{C}^l$. For example, if $k = 1$ then $Grass_1(l) = \mathbb{P}^{l-1}$. We know that projective space can be covered by affine spaces defining a manifold structure on it. Also Grassmannians admit a cover by affine spaces.

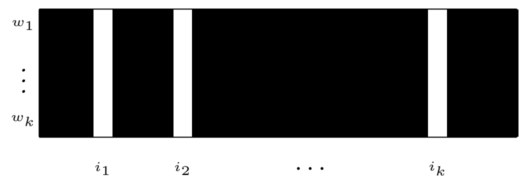
Let W be a k-dimensional subspace of $\mathbb{C}^l$ then fixing a basis $\{w_1,\dots,w_k\}$ of W determines an $k\times l$ matrix M having as i-th row the coordinates of w_i with respect to the standard basis of $\mathbb{C}^l$. Linear independence of the vectors w_i means that there is a barcode design I on M



where $I = 1 \leq i_1 < i_2 < \dots < i_k \leq l$ such that the corresponding $k\times k$ minor M_I of M is invertible. Observe that M can have several such designs.

Conversely, given a $k\times l$ matrix M of rank k determines a k-dimensional subspace of l spanned by the transposed rows. Two $k\times l$ M and M' matrices of rank

k determine the same subspace provided there is a basechange matrix $g \in GL_k$ such that $gM = M'$. That is, we can identify $Grass_k(l)$ with the orbit space of the linear action of GL_k by left multiplication on the open set $M^{max}_{k\times l}(\mathbb{C})$ of $M_{k\times l}(\mathbb{C})$ of matrices of maximal rank. Let I be a barcode design and consider the subset of $Grass_k(l)(I)$ of subspaces having a matrix representation M having I as barcode design. Multiplying on the left with M_I^{-1} the GL_k-orbit $\mathcal{O}_M$ has a unique representant N with $N_I = \mathbb{1}_k$. Conversely, any matrix N with $N_I = \mathbb{1}_k$ determines a point in $Grass_k(l)(I)$. Thus, $Grass_k(l)(I)$ depends on $k(l-k)$ free parameters (the entries of the negative of the barcode)



and we have an identification $Grass_k(l)(I) \xrightarrow{\pi_I} \mathbb{C}^{k(l-k)}$. For a different barcode design I' the image $\pi_I(Grass_k(l)(I) \cap Grass_k(l)(I'))$ is an open subset of $\mathbb{C}^{k(l-k)}$ (one extra nonsingular minor condition) and $\pi_{I'} \circ \pi_I^{-1}$ is a diffeomorphism on this set. That is, the maps π_I provide us with an atlas and determine a manifold structure on $Grass_k(l)$.

Returning to dynamical systems, the barcode $b(A,B)$ determined by the Kalman code determines a unique point in $Grass_n(m(n+1))$. We have

$$\begin{tikzcd} V_c \arrow[r, hook, "\psi"] \arrow[dr, "b(.)"'] & M^{max}_{n\times m(n+1)}(\mathbb{C}) \arrow[d, "\chi"] \\ & Grass_n(m(n+1)) \end{tikzcd}$$

where ψ is a GL_n-equivariant embedding and χ the orbit map. Observe that the barcode matrix $b(A,B)$ shows that the stabilizer of (A,B) is trivial. Indeed, the minor of $g.b(A,B)$ determined by the Kalman code is equal to g. Moreover, continuity of b implies that the orbit $\mathcal{O}\ (A,B)$ is closed in V_c. We claim that ψ is a diffeomorphism to a locally closed submanifold of $M_{n\times m(n+1)}(\mathbb{C})$. To prove this we have to consider the differential of ψ. For all $(A,B) \in W$ and $(X,Y) \in T_{(A,B)}(V) \simeq V$ we have

$$(B+\epsilon Y)^j(A+\epsilon X) = B^nA + \epsilon\ (B^nX + \sum_{i=0}^{j-1} B^iYB^{n-1-i}A).$$

Therefore the differential of ψ in $(A,B) \in V$, $d\psi_{(A,B)}(X,Y)$ is equal to

$$\begin{bmatrix} X & BX+YA & B^2X+BYA+YBA & \ldots & B^nX+\sum_{i=0}^{n-1}B^iYB^{n-1-i}A \end{bmatrix}.$$

Assume $d\psi_{(A,B)}(X,Y)$ is the zero matrix, then $X=0$ and substituting in the next term also $YA=0$. Substituting in the third gives $YBA=0$, then in the fourth $YB^2A=0$ and so on until $YB^{n-1}A=0$. But then,

$$Y\begin{bmatrix} A & BA & B^2A & \ldots & B^{n-1}A \end{bmatrix} = 0.$$

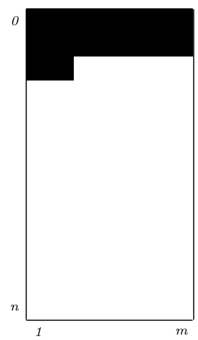
If (A,B) is a completely controllable pair, this implies that $Y=0$ and hence shows that $d\psi_{(A,B)}$ is injective for all $(A,B) \in V_c$. By the implicit function theorem, ψ induces a GL_n-equivariant diffeomorphism between the open subset V_c of completely controllable pairs and a locally closed submanifold of $M_{n\times(n+1)m}(\mathbb{C})^{max}$. The image of this submanifold under the orbit map χ is again a manifold as all fibers are equal to GL_n. This concludes the difficult part of the Kalman theorem *:*

Theorem 7.3 *The orbit space $O_c = V_c/GL_n$ of equivalence classes of completely controllable pairs is a locally closed submanifold of dimension $m.n$ of the Grassmannian $Grass_n(m(n+1))$. In fact $V_c \xrightarrow{b} O_c$ is a principal GL_n-bundle.*

To prove the dimension statement, consider $V_c(K)$ the set of completely controllable pairs (A,B) having Kalman code K and let $O_c(K)$ be the image under the orbit map. After identifying $V_c(K)$ with its image under ψ, the barcode matrix $b(A,B)$ gives a section $O_c(K) \overset{s}{\hookrightarrow} V_c(K)$. In fact,

$$GL_n \times O_c(K) \longrightarrow V_c(K) \qquad (g,x) \mapsto g.s(x)$$

is a GL_n-equivariant diffeomorphism because the $n \times n$ minor determined by K of $g.b(A,B)$ is g. Apply the local product decomposition to the generic Kalman code K^g



obtained by painting the top boxes black from left to right until one has n black boxes. Clearly $V_c(K^g)$ is open in V_c and one deduces

$$dim\ O_c = dim\ O_c(K^g) = dim\ V_c(K^g) - dim\ GL_n = mn + n^2 - n^2 = mn.$$

The Kalman theorem implies the existence of an orbit space for completely controllable and Schurian systems. Indeed, let $\Sigma = (A,B,C)$ completely controllable and let $g = g_{(A,B)} \in GL_n$ be the uniquely determined basechange such that $g.(A,B) = b(A,B)$, then we have a canonical representant (gA, gBg^{-1}, Cg^{-1}) in the orbit $\mathcal{O}(\Sigma)$. As the stabilizer $Stab(A,B)$ is trivial the orbits of (A,B,C) and (A,B,C') are distinct if $C = C'$. That is the natural projection pr_3

$$\begin{tikzcd}
Sys_c \arrow[r, two heads, "pr_3"] \arrow[d, dotted] & V_c \arrow[d, two heads, "b"] \\
Sys_c/GL_n \arrow[r, two heads] & O_c
\end{tikzcd}$$

descends to define an orbit space which is an $M_{p\times n}(\mathbb{C})$- bundle over O_c and hence is a manifold. The Schurian systems Sys_s form a GL_n-stable open subset of Sys_c and hence their orbit space is an open submanifold of Sys_c/GL_n.

Theorem 7.4 *Let Sys_c (resp. Sys_s) the the open subset of*

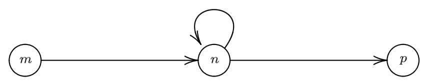
$$Sys = M_{n\times m}(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_{p\times n}(\mathbb{C})$$

determined by the completely controllable (resp. Schurian) linear dynamical systems.

1. *The orbit space for the GL_n action on Sys_c exists and is a vectorbundle of rank pn over O_c.*
2. *The orbit space for the GL_n-action on Sys_s exists and is a manifold of dimension mn^2p.*

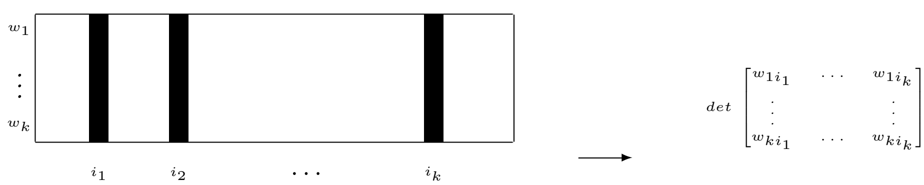
7.3 Mixed semi-invariants.

The space Sys of all linear systems determined by the numbers (m,n,p) can be identified with the representation space of the quiver situation $rep_\alpha\ Q$ with $\alpha = (m,n,p)$ and Q the quiver



Instead of considering $GL(\alpha)$-orbits we only consider the GL_n-action. Up till now we have only considered $GL(\alpha)$-invariant functions to classify (closed) orbits. Observe that a completely controllable system $\Sigma = (A,B)$ does not determine a closed GL_n-orbit in $V = M_{n\times m} \oplus M_n$ as the action of the scalar matrix $g = \epsilon\mathbb{1}_n$ gives the system $(\epsilon A, B)$ and hence $(0_{n\times m}, B)$ is a (not completely controllable) system belonging to the orbit closure $\overline{\mathcal{O}(\Sigma)}$. Still, we were able to construct a nice orbit space for such systems because the orbit $\mathcal{O}(\Sigma)$ is closed in the open subvariety V^s. We will give an interpretation of the orbit map in invariant-theoretic language.

There is a natural embedding $Grass_k(l) \hookrightarrow \mathbb{P}^N$ where $N = \binom{l}{k} + 1$ given by sending a point to the N-tuple of all determinants of the $k \times k$ minors determined by the different bar-code designs I



Composing the orbit map b with this embedding, a system $\Sigma = (A,B)$ is send to the N-tuple of determinants $det\ b_I(A,B)$. For a different point $g.(A,B)$ in the orbit $\mathcal{O}(\Sigma)$ we have that

$$det\ b_I(g.(A,B)) = det(g) det\ b_I(A,B)$$

That is, these functions are semi-invariants for GL_n. In general, if V is a GL_n-module, a polynomial function f on V is said to be a semi-invariant if for all $v \in V$ we have

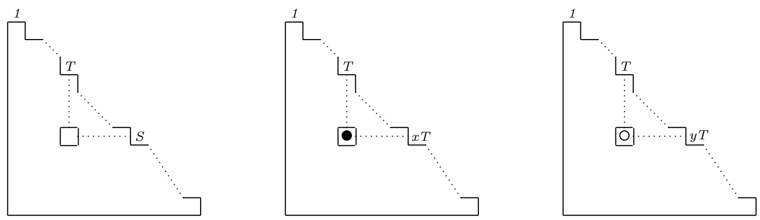
$$f(g.v) = \chi(g)f(v) \quad \text{for some character } GL_n \xrightarrow{\chi} \mathbb{C}^*$$

and we recall that every character of GL_n is of the form det^k for some $k \in \mathbb{Z}$. Equivalently, f is an invariant polynomial for the restricted action of the special linear group $SL_n = \{\ g \in GL_n \ \mid\ det(g) = 1\ \}$ on V.

In chapter 1, we ran into semi-invariants in the description of the orbit space for the GL_n-action on $rep_\alpha\ \mathbb{M} = M_n \oplus M_n \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$ using Hilbert stairs. Recall that a Hilbert stair σ, that is, the lower triangular part of a square of $n \times n$ array of boxes filled with go-stones subject to the rules

- *each row contains exactly one stone, and*
- *each column contains at most one stone of each color.*

determines a sequence $W(\sigma) = \{1, w_2, \ldots, w_n\}$ of monomials in the noncommuting variables x and y, placing 1 at the top of the stairs and descending the chair following the rule that every go-stone has a top word T which we may assume we have *constructed before and a side word S and they are related as indicated below*



For a quadruple $(X,Y,u,v) \in rep_\alpha \mathbb{M}$ we replace every occurrence of x in the word $w_i(x,y)$ by X and every occurrence of y by Y to obtain an $n \times n$ matrix $w_i = w_i(X,Y) \in M_n(\mathbb{C})$ and by left multiplication on u a column vector $w_i.v$. The evaluation of σ on (X,Y,u,v) is the determinant of the $n \times n$ matrix

$$\sigma(X,Y,u,v) = det \begin{bmatrix} u & w_2.u & w_3.u & \dots & w_n.u \end{bmatrix}$$

These functions were used to separate the orbits of cyclic quadruples. As for every monomial $w(x,y)$ and every $g \in GL_n$ we have that

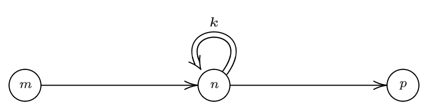
$$w(gX^{-1}, gYg^{-1})gu = gw(X,Y)u$$

we see that the functions $\sigma(X,Y,u,v)$ are again semi-invariants for the action of GL_n, or equivalently, SL_n-invariants on $rep_\alpha \mathbb{M}$.

In this section we will determine all such mixed semi-invariants for GL_n acting on the vectorspace

$$W = \underbrace{M_n \oplus \ldots \oplus M_n}_{k} \oplus \underbrace{V_n \oplus \ldots \oplus V_n}_{m} \oplus \underbrace{V_n^* \oplus \ldots \oplus V_n^*}_{p}$$

made up of k matrix-components M_n on which GL_n act by simultaneous conjugation, m vector-components V_n on which GL_n-acts by left-multiplication and p covector-components V_n^* on which GL_n act via the contragradient action. That is, W is the representation space of the quiver situation



where we restrict the usual $GL(\alpha)$-action to the GL_n-component. We will determine the generating semi-invariant polynomials, that is, the SL_n-invariant functions on W. In chapter 3 we worked out a similar problem in great detail, here we merely sketch the main steps. In section 6 we will generalize these calculations to determine the $GL(\alpha)$-semi-invariants on an arbitrary quiver situation $rep_\alpha Q$.

As always, we first determine the multilinear SL_n-invariants, that is the SL_n-invariant linear maps

$$\underbrace{M_n \otimes \ldots \otimes M_n}_{i} \otimes \underbrace{V_n \otimes \ldots \otimes V_n}_{j} \otimes \underbrace{V_n^* \otimes \ldots \otimes V_n^*}_{z} \xrightarrow{f} \mathbb{C}$$

By the identification $M_n = V_n \otimes V_n^*$ we have to determine the SL_n-invariant linear maps

$$V_n^{\otimes i+j} \otimes V_n^{*\otimes i+z} \xrightarrow{f} \mathbb{C}$$

The description of such invariants is given by classical invariant theory, see for example [32, II.5,Thm. 2.5.A].

Theorem 7.5 *The multilinear SL_n-invariants f of the situation above are linear combinations of invariants of one of the following two types*

1. *For $(i_1,\ldots,i_n,h_1,\ldots,h_n,\ldots,t_1,\ldots,t_n,s_1,\ldots,s_r)$ a permutation of the $i+j$ vector indices and $(u_1,\ldots,u_r)$ a permutation of the $i+z$ covector indices, consider the SL_n-invariant*

$$[v_{i_1},\ldots,v_{i_n}]\;[v_{h_1},\ldots,v_{h_n}]\;\ldots\;[v_{t_1},\ldots,v_{t_n}]\;\phi_{u_1}(v_{s_1})\ldots\phi_{u_r}(v_{s_r})$$

where the brackets are the determinantal invariants

$$[v_{a_1},\ldots,v_{a_n}] = det\;\begin{bmatrix} v_{a_1} & v_{a_2} & \ldots & v_{a_n}\end{bmatrix}$$

2. *For $(i_1,\ldots,i_n,h_1,\ldots,h_n,\ldots,t_1,\ldots,t_n,s_1,\ldots,s_r)$ a permutation of the $i+z$ covector indices and $(u_1,\ldots,u_r)$ a permutation of the $i+j$ vector indices, consider the SL_n-invariant*

$$[\phi_{i_1},\ldots,\phi_{i_n}]^*\;[\phi_{h_1},\ldots,\phi_{h_n}]^*\;\ldots\;[\phi_{t_1},\ldots,\phi_{t_n}]^*\;\phi_{u_1}(v_{s_1})\ldots\phi_{u_r}(v_{s_r})$$

where the brackets are the determinantal invariants

$$[\phi_{a_1},\ldots,\phi_{a_n}]^* = det\;\begin{bmatrix}\phi_{a_1}\\ \vdots \\ \phi_{a_n}\end{bmatrix}$$

Observe that we do not have at the same time brackets of vectors and of covectors, due to the relation

$$[v_1,\ldots,v_n]\;[\phi_1,\ldots,\phi_n] = det\;\begin{bmatrix}\phi_1(v_1) & \ldots & \phi_1(v_n)\\ \vdots & & \vdots\\ \phi_n(v_1) & \ldots & \phi_n(v_n)\end{bmatrix}$$

Our next job is to give a matrix-interpretation of these basic invariants. Let us consider the case of a bracket of vectors (the case of covectors is similar)

$$[v_{i_1},\ldots,v_{i_n}]$$

If all the indices $\{i_1,\ldots,i_n\}$ are original vector-indices (and so do not come from the matrix-terms) we save this term and go to the next factor. Otherwise, if say i_1 is one of the matrix indices, $A_{i_1} = \phi_{i_1}\otimes v_{i_1}$, then the covector ϕ_{i_1} must be paired up in a scalar product $\phi_{i_1}(v_{u_1})$ with a vector v_{u_1}. Again, two cases can occur. If u_1 is a vector index, we have that

$$\phi_{i_1}(v_{u_1})[v_{i_1},\ldots,v_{i_n}] = [A_{i_1}v_{u_1},v_{i_2},\ldots,v_{i_n}] = [v'_{i_1},v_{i_2},\ldots,v_{i_n}]$$

Otherwise, we can keep on matching the matrix indices and get an expression

$$\phi_{i_1}(v_{u_1})\;\phi_{u_1}(v_{u_2})\;\phi_{u_2}(v_{u_3})\;\ldots$$

until we finally hit again a vector index, say u_l, but then we have the expression

$$\phi_{i_1}(v_{u_1})\;\phi_{u_1}(v_{z_1})\;\ldots\;\phi_{u_{l-1}}(v_{u_l})\;[v_{i_1},\ldots,v_{i_n}] = [Mv_{u_l},v_{i_2},\ldots,v_{i_n}]$$

where $M = A_{i_1}A_{u_1}\ldots A_{u_{l-1}}$. One repeats the same argument for all vectors in the brackets. As for the remaining scalar product terms, we have a similar procedure of *matching up the matrix indices and one verifies that in doing so one obtains factors of the type*

$$\phi(Mv) \quad and \quad tr(M)$$

where M is a monomial in the matrices. As we mentioned, the case of covector-brackets is similar except that in matching the matrix indices with a covector ϕ, one obtains a monomial in the transposed matrices.

Having found these interpretations of the basic SL_n-invariant linear terms, we can proceed by polarization and restitution processes as explained in chapter 3, to finish the proof of the next result, due to C. Procesi [24, Thm 12.1].

Theorem 7.6 *The SL_n-invariants of $W = rep_\alpha\ Q$ where Q is the quiver*

$$\begin{tikzcd} m \arrow[r] & n \arrow[loop above, "k"] \arrow[r] & p \end{tikzcd}$$

are generated by the following four types of functions, where we write a typical element in W as

$$(\underbrace{A_1, \ldots, A_k}_{k}, \underbrace{v_1, \ldots, v_m}_{m}, \underbrace{\phi_1, \ldots, \phi_p}_{p})$$

with the A_i the matrices corresponding to the loops, the v_j making up the rows of the $n \times m$ matrix and the ϕ_j the columns of the $p \times n$ matrix.

- *$tr(M)$ where M is a monomial in the matrices A_i,*
- *scalar products $\phi_j(Mv_i)$ where M is a monomial in the matrices A_i,*
- *vector-brackets $[M_1 v_{i_1}, M_2 v_{i_2}, \ldots, M_n v_{i_n}]$ where the M_j are monomials in the matrices A_i,*
- *covector-brackets $[M_1 \phi^\tau_{i_1}, \ldots, M_n \phi^\tau_{i_n}]$ where the M_j are monomials in the matrices A_i,*

7.4 General subrepresentations.

Throughout this section we fix a quiver Q on k vertices $\{v_1, \ldots, v_k\}$ and dimension vectors $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$. We want to describe morphisms between representations $V \in rep_\alpha\ Q$ and $W \in rep_\beta\ Q$. That is, we consider the closed subvariety

$$Hom_Q(\alpha, \beta) \hookrightarrow M_{a_1 \times b_1} \oplus \ldots \oplus M_{a_k \times b_k} \oplus rep_\alpha\ Q \oplus rep_\beta\ Q$$

consisting of the triples (ϕ, V, W) where $\phi = (\phi_1, \ldots, \phi_k)$ is a morphism of quiver-representations $V \longrightarrow W$. Projecting to the two last components we have an onto morphism between affine varieties

$$Hom_Q(\alpha, \beta) \xrightarrow{h} \!\!\!\!\rightarrow rep_\alpha\ Q \oplus rep_\beta\ Q$$

In chapter 4.2 we have proved that the dimension of fibers is an upper-semicontinuous function. That is, for every natural number d, the set

$$\{\Phi \in Hom_Q(\alpha, \beta)\ \mid\ dim_\Phi\ h^{-1}(h(\Phi)) \leq d\}$$

is a Zariski open subset of $Hom_Q(\alpha, \beta)$. As the target space $rep_\alpha\ Q \oplus rep_\beta\ Q$ is irreducible, it contains a non-empty open subset hom_{min} where the dimension of *the fibers attains a minimal value. This minimal fiber dimension will be denoted by* $hom(\alpha, \beta)$.

Similarly, we could have defined an affine variety $Ext_Q(\alpha, \beta)$ *where the fiber over a point* $(V, W) \in rep_\alpha\ Q \oplus rep_\beta\ Q$ *is given by the extensions* $Ext^1_{\mathbb{C}Q}(V, W)$. *If* χ_Q *is the Euler-form of* Q *we recall that for all* $V \in rep_\alpha\ Q$ *and* $W \in rep_\beta\ Q$ *we have*

$$dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, W) - dim_{\mathbb{C}}\ Ext^1_Q(V, W) = \chi_Q(\alpha, \beta)$$

Hence, there is also an open set ext_{min} *of* $rep_\alpha\ Q \oplus rep_\beta\ Q$ *where the dimension of* $Ext^1(V, W)$ *attains a minimum. This minimal value we denote by* $ext(\alpha, \beta)$. *As* $hom_{min} \cap ext_{min}$ *is a non-empty open subset we have the numerical equality*

$$hom(\alpha, \beta) - ext(\alpha, \beta) = \chi_Q(\alpha, \beta).$$

In particular, if $hom(\alpha, \alpha+\beta) > 0$, *there will be an open subset where the morphism* $V \xrightarrow{\phi} W$ *is a monomorphism. Hence, there will be an open subset of* $rep_{\alpha+\beta}\ Q$ *consisting of representations containing a subrepresentation of dimension vector* α. *We say that* α *is a general subrepresentation of* $\alpha + \beta$ *and denote this with* $\alpha \hookrightarrow \alpha + \beta$. *We want to characterize this property. To do this, we introduce the quiver-Grassmannians*

$$Grass_\alpha(\alpha + \beta) = \prod_{i=1}^{k} Grass_{a_i}(a_i + b_i)$$

which is a projective manifold. Consider the following diagram of morphisms of reduced varieties

$$\begin{tikzcd}
& rep_{\alpha+\beta}\ Q \\
rep_\alpha^{\alpha+\beta}\ Q \arrow[ur, "s"] \arrow[r, hook] \arrow[dr, "p"'] & rep_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta) \arrow[u, "pr_1"'] \arrow[d, "pr_2"] \\
& Grass_\alpha(\alpha+\beta)
\end{tikzcd}$$

with the following properties

- $rep_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta)$ *is the trivial vectorbundle with fiber* $rep_{\alpha+\beta}\ Q$ *over the projective smooth variety* $Grass_\alpha(\alpha + \beta)$ *with structural morphism* pr_2.
- $rep_\alpha^{\alpha+\beta}\ Q$ *is the subvariety of* $rep_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta)$ *consisting of couples* (W, V) *where* V *is a subrepresentation of* W *(observe that this is for fixed* W *a linear condition). Because* $GL(\alpha+\beta)$ *acts transitively on the Grassmannian* $Grass_\alpha(\alpha+\beta)$ *(by multiplication on the right) we see that* $rep_\alpha^{\alpha+\beta}\ Q$ *is a sub-vectorbundle over* $Grass_\alpha(\alpha + \beta)$ *with structural morphism* p. *In particular,* $rep_\alpha^{\alpha+\beta}\ Q$ *is a reduced variety.*
- *The morphism* s *is a projective morphism, that is, can be factored via the natural projection*

$$\begin{tikzcd}
& rep_{\alpha+\beta}\ Q \times \mathbb{P}^N \arrow[d, "\pi_2"] \\
rep_\alpha^{\alpha+\beta}\ Q \arrow[ur, "f"] \arrow[r, "s"] & rep_{\alpha+\beta}\ Q
\end{tikzcd}$$

where f is the composition of the inclusion $rep_\alpha^{\alpha+\beta}\ Q \hookrightarrow rep_{\alpha+\beta}\ Q \times Grass_\alpha(\alpha+\beta)$ with the natural inclusion of Grassmannians in projective spaces recalled in the previous section $Grass_\alpha(\alpha+\beta) \hookrightarrow \prod_{i=1}^k \mathbb{P}^{n_i}$ with the Segre embedding $\prod_{i=1}^k \mathbb{P}^{n_i} \hookrightarrow \mathbb{P}^N$. In particular, s is proper by [9, Thm. II.4.9], that is, maps closed subsets to closed subsets.

We are interested in the scheme-theoretic fibers of s. If $W \in rep_{\alpha+\beta}\ Q$ lies in the image of s, we denote the fiber $s^{-1}(W)$ by $\underline{Grass}_\alpha(W)$. Its geometric points are couples (W,V) where V is an α-dimensional subrepresentation of W. Whereas $\underline{Grass}_\alpha(W)$ is a projective scheme, it is in general neither smooth, nor irreducible nor even reduced. Therefore, in order to compute the tangent space in a point (W,V) of $\underline{Grass}_\alpha(W)$ we have to clarify the functor it represents on the category commalg of commutative $\mathbb{C}$-algebras.

Let C be a commutative $\mathbb{C}$-algebra, a representation $\mathcal{R}$ of the quiver Q over C consists of a collection $\mathcal{R}_i = P_i$ of projective C-modules of finite rank and a collection of C-module morphisms for every arrow a in Q

$$(j) \xleftarrow{\ a\ } (i) \qquad\qquad \mathcal{R}_j = P_j \xleftarrow{\mathcal{R}_a} P_i = \mathcal{R}_i$$

The dimension vector of the representation $\mathcal{R}$ is given by the k-tuple $(rk_C\ \mathcal{R}_1, \ldots, rk_C\ \mathcal{R}_k)$. A subrepresentation $\mathcal{S}$ of $\mathcal{R}$ is determined by a collection of projective sub-summands (and not merely sub-modules) $\mathcal{S}_i \triangleleft \mathcal{R}_i$. In particular, for $W \in rep_{\alpha+\beta}\ Q$ we define the representation $\mathcal{W}_C$ of Q over the commutative ring C by

$$\begin{cases} (\mathcal{W}_C)_i &= C \otimes_{\mathbb{C}} W_i \\ (\mathcal{W}_C)_a &= id_C \otimes_{\mathbb{C}} W_a \end{cases}$$

With these definitions, we can now define the functor represented by $\underline{Grass}_\alpha(W)$ as the functor assigning to a commutative $\mathbb{C}$-algebra C the set of all subrepresentations of dimension vector α of the representation $\mathcal{W}_C$.

Lemma 7.7 *Let $x = (W,V)$ be a geometric point of $\underline{Grass}_\alpha(W)$, then*

$$T_x\ \underline{Grass}_\alpha(W) = Hom_{\mathbb{C}Q}(V, \frac{W}{V})$$

Proof. *The tangent space in $x = (W,V)$ are the $\mathbb{C}[\epsilon]$-points of $\underline{Grass}_\alpha(W)$ lying over (W,V). To start, let $V \xrightarrow{\psi} \frac{W}{V}$ be a homomorphism of representations of Q and consider a $\mathbb{C}$-linear lift of this map $\tilde{\psi} : V \longrightarrow W$. Consider the $\mathbb{C}$-linear subspace of $\mathcal{W}_{\mathbb{C}[\epsilon]} = \mathbb{C}[\epsilon] \otimes W$ spanned by the sets*

$$\{v + \epsilon \otimes \tilde{\psi}(v) \ \mid\ v \in V\} \quad \textit{and} \quad \epsilon \otimes V$$

This determines a $\mathbb{C}[\epsilon]$-subrepresentation of dimension vector α of $\mathcal{W}_{\mathbb{C}[\epsilon]}$ lying over (W,V) and is independent of the chosen linear lift $\tilde{\psi}$.

Conversely, if $\mathcal{S}$ is a $\mathbb{C}[\epsilon]$-subrepresentation of $\mathcal{W}_{\mathbb{C}[\epsilon]}$ lying over (W,V), then $\frac{\mathcal{S}}{\epsilon\mathcal{S}} = V \hookrightarrow W$. But then, a $\mathbb{C}$-linear complement of $\epsilon\mathcal{S}$ is spanned by elements of the form $v + \epsilon\psi(v)$ where $\psi(v) \in W$ and $\epsilon \otimes \psi$ is determined modulo an element of $\epsilon \otimes V$. But then, we have a $\mathbb{C}$-linear map $\tilde{\psi} : V \longrightarrow \frac{W}{V}$ and as $\mathcal{S}$ is a $\mathbb{C}[\epsilon]$-subrepresentation, $\tilde{\psi}$ must be a homomorphism of representations of Q. $\square$

We can now give a characterization for general α-dimensional subrepresentations, proved by A. Schofield in citeSchofield.

Theorem 7.8 *The following are equivalent*

1. $\alpha \hookrightarrow \alpha + \beta$.
2. *Every representation* $W \in rep_{\alpha+\beta}\ Q$ *has a subrepresentation* V *of dimension* α.
3. $ext(\alpha, \beta) = 0$.

Proof. *Assume 1. , then the image of the proper map* $s : rep_\alpha^{\alpha+\beta}\ Q \longrightarrow rep_{\alpha+\beta}\ Q$ *contains a Zariski open subset. As properness implies that the image of* s *must also be a closed subset of* $rep_{\alpha+\beta}\ Q$ *it follows that* $Im\ s = rep_{\alpha+\beta}\ Q$*, that is 2. holds. Conversely, 2. clearly implies 1. so they are equivalent.*

We compute the dimension of the vectorbundle $rep_\alpha^{\alpha+\beta}\ Q$ *over* $Grass_\alpha(\alpha+\beta)$*. Using that the dimension of a Grassmannians* $Grass_k(l)$ *is* $k(l-k)$ *we know that the base has dimension* $\sum_{i=1}^k a_i b_i$*. Now, fix a point* $V \hookrightarrow W$ *in* $Grass_\alpha(\alpha+\beta)$*, then the fiber over it determines all possible ways in which this inclusion is a subrepresentation of quivers. That is, for every arrow in* Q *of the form* $\textcircled{j} \xleftarrow{a} \textcircled{i}$ *we need to have a commuting diagram*

$$\begin{tikzcd} V_i \arrow[r] \arrow[d, hook] & V_j \arrow[d, hook] \\ W_i \arrow[r] & W_j \end{tikzcd}$$

Here, the vertical maps are fixed. If we turn $V \in rep_\alpha\ Q$*, this gives us the* $a_i a_j$ *entries of the upper horizontal map as degrees of freedom, leaving only freedom for the lower horizontal map determined by a linear map* $\frac{W_i}{V_i} \longrightarrow W_j$*, that is, having* $b_i(a_j + b_j)$ *degrees of freedom. Hence, the dimension of the vectorspace-fibers is*

$$\sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i a_j + b_i(a_j + b_j))$$

giving the total dimension of the reduced variety $rep_\alpha^{\alpha+\beta}\ Q$*. But then,*

$$\begin{aligned} dim\ rep_\alpha^{\alpha+\beta}\ Q - dim\ rep_{\alpha+\beta}\ Q &= \sum_{i=1}^k a_i b_i + \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i a_j + b_i(a_j + b_j)) \\ &\qquad - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} (a_i + b_i)(a_j + b_j) \\ &= \sum_{i=1}^k a_i b_i - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i b_j = \chi_Q(\alpha, \beta) \end{aligned}$$

Assume that 2. holds, then the proper map $rep_\alpha^{\alpha+\beta} \overset{s}{\twoheadrightarrow} rep_{\alpha+\beta}\ Q$ *is onto and as both varieties are reduced, the general fiber is a reduced variety of dimension* $\chi_Q(\alpha, \beta)$*, whence the general fiber contains points such that their tangentspaces have dimension* $\chi_Q(\alpha, \beta)$*. By the foregoing lemma we can compute the dimension of this tangentspace as dim* $Hom_{\mathbb{C}Q}(V, \frac{W}{V})$*. But then, as*

$$\chi_Q(\alpha, \beta) = dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, \frac{W}{V}) - dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V, \frac{W}{V})$$

it follows that $Ext^1(V, \frac{W}{V}) = 0$ *for some representation* V *of dimension vector* α *and* $\frac{W}{V}$ *of dimension vector* β*. But then,* $ext(\alpha, \beta) = 0$*, that is, 3. holds.*

Conversely, assume that $ext(\alpha, \beta) = 0$*. Then, for a general point* $W \in rep_{\alpha+\beta}\ Q$ *in the image of* s *and for a general point in its fiber* $(W, V) \in rep_{\alpha}^{\alpha+\beta}\ Q$ *we have* $dim_{\mathbb{C}}\ Ext^1_{\mathbb{C}Q}(V, \frac{W}{V}) = 0$ *whence* $dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, \frac{W}{V}) = \chi_Q(\alpha, \beta)$*. But then, the general fiber of* s *has dimension* $\chi_Q(\alpha, \beta)$ *and as this is the difference in dimension between the two irreducible varieties, the map is generically onto. Finally, properness of* s *then implies that it is onto, giving 2. and finishing the proof.* □

7.5 Schofield's criterium.

In all moduli space problems we will encounter, it will be crucial to determine the dimension vectors of general subrepresentations, or by the foregoing section, to compute $ext(\alpha, \beta)$*. An inductive algorithm to do this was discovered by A. Schofield [27].*

Recall that $\alpha \hookrightarrow \beta$ *iff a general representation* $W \in rep_\beta\ Q$ *contains a subrepresentation* $S \hookrightarrow W$ *of dimension vector* α*. Similarly, we denote* $\beta \twoheadrightarrow \gamma$ *if and only if a general representation* $W \in rep_\beta\ Q$ *has a quotient-representation* $W \twoheadrightarrow T$ *of dimension vector* γ*. As before,* Q *will be a quiver on* k*-vertices* $\{v_1, \ldots, v_k\}$ *and we denote dimension vectors* $\alpha = (a_1, \ldots, a_k)$*,* $\beta = (b_1, \ldots, b_k)$ *and* $\gamma = (c_1, \ldots, c_k)$*. We will first determine the* rank of a general homomorphism $V \longrightarrow W$ *between representations* $V \in rep_\alpha\ Q$ *and* $W \in rep_\beta\ Q$*. We denote*

$$Hom(\alpha, \beta) = \oplus_{i=1}^k M_{b_i \times a_i} \quad \text{and} \quad Hom(V, \beta) = Hom(\alpha, \beta) = Hom(\alpha, W)$$

for any representations V *and* W *as above. With these conventions we have*

Lemma 7.9 *There is an open subset* $Hom_m(\alpha, \beta) \hookrightarrow rep_\alpha\ Q \times rep_\beta\ Q$ *and a dimension vector* $\gamma \overset{def}{=} rk\ hom(\alpha, \beta)$ *such that for all* $(V, W) \in Hom_{min}(\alpha, \beta)$

- $dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, W)$ *is minimal, and*
- $\{\phi \in Hom_{\mathbb{C}Q}(V, W) \mid rk\ \phi = \gamma\}$ *is a non-empty Zariski open subset of* $Hom_{\mathbb{C}Q}(V, W)$*.*

Proof. *Consider the subvariety* $Hom_Q(\alpha, \beta)$ *of the trivial vectorbundle*

$$\begin{tikzcd} Hom_Q(\alpha,\beta) \arrow[r, hook] \arrow[dr, "\Phi"'] & Hom(\alpha,\beta)\times rep_\alpha\ Q \times rep_\beta\ Q \arrow[d, "pr"] \\ & rep_\alpha\ Q \times rep_\beta\ Q \end{tikzcd}$$

of triples (ϕ, V, W) *such that* $V \xrightarrow{\phi} W$ *is a morphism of representations of* Q*. The fiber* $\Phi^{-1}(V, W) = Hom_{\mathbb{C}Q}(V, W)$*. As the fiber dimension is upper semi-continuous, there is an open subset* $Hom_{min}(\alpha, \beta)$ *of* $rep_\alpha\ Q \times rep_\beta\ Q$ *consisting of points* (V, W) *where* $dim_{\mathbb{C}}\ Hom_{\mathbb{C}Q}(V, W)$ *is minimal. For given dimension vector* $\delta = (d_1, \ldots, d_k)$ *we consider the subset*

$$Hom_Q(\alpha, \beta, \delta) = \{(\phi, V, W) \in Hom_Q(\alpha, \beta) \mid rk\ \phi = \delta\} \hookrightarrow Hom_Q(\alpha, \beta)$$

This is a constructible subset of $Hom_Q(\alpha, \beta)$ *and hence there is a dimension vector* γ *such that* $Hom_Q(\alpha, \beta, \gamma) \cap \Phi^{-1}(Hom_{min}(\alpha, \beta))$ *is constructible and dense in* $\Phi^{-1}(Hom_{min}(\alpha, \beta))$*. But then,*

$$\Phi(Hom_Q(\alpha, \beta, \gamma) \cap \Phi^{-1}(Hom_{min}(\alpha, \beta)))$$

is constructible and dense in $Hom_{min}(V,W)$. Therefore it contains an open subset $Hom_m(V,W)$ satisfying the requirements of the lemma. □

Lemma 7.10 *Assume we have short exact sequences of representations of Q*

$$\begin{cases} \blacksquare_1 = & 0 \longrightarrow S \longrightarrow V \longrightarrow X \longrightarrow 0 \\ \blacksquare_2 = & 0 \longrightarrow Y \longrightarrow W \longrightarrow T \longrightarrow 0 \end{cases}$$

then there is a natural onto map

$$Ext^1_{\mathbb{C}Q}(V,W) \twoheadrightarrow Ext^1_{\mathbb{C}Q}(S,T)$$

Proof. *We will see in chapter 9 that gldim $\mathbb{C}Q \leq 1$, whence applying derived functors to the given sequences we obtain the following part of the natural long-exact sequences*

$$\begin{tikzcd}
 & \mathbb{R}Hom(W,\blacksquare_1) & \mathbb{R}Hom(T,\blacksquare_1) & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathbb{R}Hom(V,\blacksquare_2)\ldots \arrow[r] & Ext(V,W) \arrow[r] \arrow[d] & Ext(V,T) \arrow[r] \arrow[d] & 0 \\
\mathbb{R}Hom(S,\blacksquare_2)\ldots \arrow[r] & Ext(S,W) \arrow[r] \arrow[d] & Ext(S,T) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

from which the statement follows. □

Theorem 7.11 *Let $\gamma = rk\ hom(\alpha,\beta)$ (with notations as in lemma 7.9), then*

1. $\alpha - \gamma \hookrightarrow \alpha \twoheadrightarrow \gamma \hookrightarrow \beta \twoheadrightarrow \beta - \gamma$
2. $ext(\alpha,\beta) = -\chi_Q(\alpha-\gamma,\beta-\gamma) = ext(\alpha-\gamma,\beta-\gamma)$

Proof. *The first statement is obvious from the definitions, for if $\gamma = rk\ hom(\alpha,\beta)$, then a general representation of dimension α will have a quotient-representation of dimension γ (and hence a subrepresentation of dimension $\alpha-\gamma$) and a general representation of dimension β will have a subrepresentation of dimension γ (and hence a quotient-representation of dimension $\beta-\gamma$.*

The strategy of the proof of the second statement is to compute the dimension of the subvariety of $Hom(\alpha,\beta) \times rep_\alpha \times rep_\beta \times rep_\gamma$ defined by

$$H^{factor} = \{(\phi,V,W,X) \mid \begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr] & & W \\ & X = Im\ \phi \arrow[ur, hook] & \end{tikzcd} \text{ factors as representations} \}$$

in two different ways. Consider the intersection of the open set $Hom_m(\alpha,\beta)$ determined by lemma 7.9 with the open set of couples (V,W) such that $dim\ Ext(V,W) = ext(\alpha,\beta)$ and let (V,W) lie in this intersection. In the previous section we have proved that

$$dim\ \underline{Grass}_\gamma(W) = \chi_Q(\gamma,\beta-\gamma)$$

Let H be the subbundle of the trivial vectorbundle over $\underline{Grass}_\gamma(W)$

$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr] & Hom(\alpha,W)\times\underline{Grass}_\gamma(W) \arrow[d] \\ & \underline{Grass}_\gamma(W) \end{tikzcd}$$

consisting of triples (ϕ,W,U) with $\phi : \oplus_i\mathbb{C}^{\oplus a_i} \longrightarrow W$ a linear map such that $Im(\phi)$ is contained in the subrepresentation $U \hookrightarrow W$ of dimension γ. That is, the fiber over (W,U) is $Hom(\alpha,U)$ and therefore has dimension $\sum_{i=1}^k a_ic_i$. With H^{full} we consider the open subvariety of H of triples (ϕ,W,U) such that $Im\ \phi = U$. We have

$$dim\ H^{full} = \sum_{i=1}^k a_ic_i + \chi_Q(\gamma,\beta-\gamma)$$

But then, H^{factor} is the subbundle of the trivial vectorbundle over H^{full}

$$\begin{tikzcd} H^{factor} \arrow[r, hook] \arrow[dr, "\pi"] & rep_\alpha\ Q\times H^{full} \arrow[d] \\ & H^{full} \end{tikzcd}$$

consisting of quadruples (V,ϕ,W,X) such that $V \xrightarrow{\phi} W$ is a morphism of representations, with image the subrepresentation X of dimension γ. The fiber of π over a triple (ϕ,W,X) is determined by the property that for each arrow $j \xleftarrow{a} i$ the following diagram must be commutative, where we decompose the vertex spaces $V_i = X_i\oplus K_i$ for $K = Ker\ \phi$

$$\begin{tikzcd} X_i\oplus K_i \arrow[r, "\begin{bmatrix} A & B \\ C & D\end{bmatrix}"] \arrow[d, "\begin{bmatrix} \mathbb{1}_{c_i} & 0\end{bmatrix}"'] & X_j\oplus K_j \arrow[d, "\begin{bmatrix} \mathbb{1}_{c_j} & 0\end{bmatrix}"] \\ X_i \arrow[r, "A"'] & X_j \end{tikzcd}$$

where A is fixed, giving the condition $B = 0$ and hence the fiber has dimension equal to

$$\sum_{j \xleftarrow{a} i} (a_i-c_i)(a_j-c_j) + \sum_{j \xleftarrow{a} i} c_i(a_j-c_j) = \sum_{j \xleftarrow{a} i} a_i(a_j-c_j)$$

This gives our first formula for the dimension of H^{factor}

$$H^{factor} = \sum_{i=1}^k a_ic_i + \chi_Q(\gamma,\beta-\gamma) + \sum_{j \xleftarrow{a} i} a_i(a_j-c_j)$$

On the other hand, we can consider the natural map $H^{factor} \xrightarrow{\Phi} rep_\alpha\ Q$ *defined by sending a quadruple* (V, ϕ, W, X) *to* V*. the fiber in* V *is given by all quadruples* (V, ϕ, W, X) *such that* $V \xrightarrow{\phi} W$ *is a morphism of representations with* $Im\ \phi = X$ *a representation of dimension vector* γ*, or equivalently*

$$\Phi^{-1}(V) = \{V \xrightarrow{\phi} W \ \mid \ rk\ \phi = \gamma\}$$

Now, recall our restriction on the couple (V, W) *giving at the beginning of the proof. There is an open subset* max *of* $rep_\alpha\ Q$ *of such* V *and by construction* $max \hookrightarrow Im\ \Phi$, $\Phi^{-1}(max)$ *is open and dense in* H^{factor} *and the fiber* $\Phi^{-1}(V)$ *is open and dense in* $Hom_{\mathbb{C}Q}(V, W)$*. This provides us with the second formula for the dimension of* H^{factor}

$$dim\ H^{factor} = dim\ rep_\alpha\ Q + hom(\alpha, W) = \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i a_j + hom(\alpha, \beta)$$

Equating both formulas we obtain the equality

$$\chi_Q(\gamma, \beta - \gamma) + \sum_{i=1}^{k} a_i c_i - \sum_{\textcircled{j} \xleftarrow{a} \textcircled{i}} a_i c_j = hom(\alpha, \beta)$$

which is equivalent to

$$\chi_Q(\gamma, \beta - \gamma) + \chi_Q(\alpha, \gamma) - \chi_Q(\alpha, \beta) = ext(\alpha, \beta)$$

Now, for our (V, W) *we have that* $Ext(V, W) = ext(\alpha, \beta)$ *and we have exact sequences of representations*

$$0 \longrightarrow S \longrightarrow V \longrightarrow X \longrightarrow 0 \qquad 0 \longrightarrow X \longrightarrow W \longrightarrow T \longrightarrow 0$$

and using lemma 7.10 this gives a surjection $Ext(V, W) \twoheadrightarrow Ext(S, T)$*. On the other hand we always have from the homological interpretation of the Euler form the first inequality*

$$\begin{aligned} dim_{\mathbb{C}}\ Ext(S, T) &\geq -\chi_Q(\alpha - \gamma, \beta - \gamma) = \chi_Q(\gamma, \beta - \gamma) - \chi_Q(\alpha, \beta) + \chi_Q(\alpha, \gamma) \\ &= ext(\alpha, \beta) \end{aligned}$$

As the last term is $dim_{\mathbb{C}}\ Ext(V, W)$*, this implies that the above surjection must be an isomorphism and that*

$$dim_{\mathbb{C}}\ Ext(S, T) = -\chi_Q(\alpha - \gamma, \beta - \gamma) \quad whence \quad dim_{\mathbb{C}}\ Hom(S, T) = 0$$

But this implies that $hom(\alpha - \gamma, \beta - \gamma) = 0$ *and therefore* $ext(\alpha - \gamma, \beta - \gamma) = -\chi_Q(\alpha - \gamma, \beta - \gamma)$*. Finally,*

$$ext(\alpha - \gamma, \beta - \gamma) = dim\ Ext(S, T) = dim\ Ext(V, W) = ext(\alpha, \beta)$$

finishing the proof. □

Theorem 7.12 *For all dimension vectors* α *and* β *we have*

$$\begin{aligned} ext(\alpha, \beta) &= \max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} - \chi_Q(\alpha', \beta') \\ &= \max_{\beta \twoheadrightarrow \beta"} - \chi_Q(\alpha, \beta") \\ &= \max_{\alpha" \hookrightarrow \alpha} - \chi_Q(\alpha", \beta) \end{aligned}$$

Proof. *Let V and W be representation of dimension vector α and β such that dim $Ext(V,W) = ext(\alpha,\beta)$. Let $S \hookrightarrow V$ be a subrepresentation of dimension α' and $W \twoheadrightarrow T$ a quotient representation of dimension vector β'. Then, we have*

$$ext(\alpha,\beta) = dim_{\mathbb{C}}\ Ext(V,W) \geq dim_{\mathbb{C}}\ Ext(S,T) \geq -\chi_Q(\alpha',\beta')$$

where the first inequality is lemma 7.10 and the second follows from the interpretation of the Euler form. Therefore, $ext(\alpha,\beta)$ is greater or equal than all the terms in the statement of the theorem. The foregoing theorem asserts the first equality, as for rk $hom(\alpha,\beta) = \gamma$ we do have that $ext(\alpha,\beta) = -\chi_Q(\alpha-\gamma,\beta-\gamma)$.

In the proof of the above theorem, we have found for sufficiently general V and W an exact sequence of representations

$$0 \longrightarrow S \longrightarrow V \longrightarrow W \longrightarrow T \longrightarrow 0$$

where S is of dimension $\alpha-\gamma$ and T of dimension $\beta-\gamma$. Moreover, we have a commuting diagram of surjections

$$\begin{tikzcd} Ext(V,W) \arrow[r, two heads] \arrow[d] \arrow[dr, dotted] & Ext(V,T) \arrow[d] \\ Ext(S,W) \arrow[r, two heads] & Ext(S,T) \end{tikzcd}$$

and the dashed map is an isomorphism, hence so are all the epimorphisms. Therefore, we have

$$\begin{cases} ext(\alpha,\beta-\gamma) & \leq dim\ Ext(V,T) = dim\ Ext(V,W) = ext(\alpha,\beta) \\ ext(\alpha-\gamma,\beta) & \leq dim\ Ext(S,W) = dim\ Ext(V,W) = ext(\alpha,\beta) \end{cases}$$

Further, let T' be a sufficiently general representation of dimension $\beta-\gamma$, then it follows from $Ext(V,T') \twoheadrightarrow Ext(S,T)$ that

$$ext(\alpha-\gamma,\beta-\gamma) \leq dim\ Ext(S,T') \leq dim\ Ext(V,T') = ext(\alpha,\beta-\gamma)$$

but the left term is equal to $ext(\alpha,\beta)$ by the above theorem. But then, we have $ext(\alpha,\beta) = ext(\alpha,\beta-\gamma)$. Now, we may assume by induction that the theorem holds for $\beta-\gamma$. That is, there exists $\beta-\gamma \twoheadrightarrow \beta"$ such that $ext(\alpha,\beta-\gamma) = -\chi_Q(\alpha,\beta")$. Whence, $\beta \twoheadrightarrow \beta"$ and $ext(\alpha,\beta) = -\chi_Q(\alpha,\beta")$ and the middle equality of the theorem holds. By a dual argument so does the last. □

This gives us the following inductive procedure to find all the dimension vectors of general subrepresentations. Take a dimension vector α and assume by induction we know for all $\beta < \alpha$ the set of general subrepresentations $\beta' \hookrightarrow \beta$. Then, $\beta \hookrightarrow \alpha$ if and only if

$$0 = ext(\beta,\alpha-\beta) = \max_{\beta' \hookrightarrow \beta} -\chi_Q(\beta',\alpha-\beta)$$

where the first equality is the main result of the foregoing section and the last is the result above.

7.6 θ-semistable representations.

Let Q be a quiver on k vertices $\{v_1,\ldots,v_k\}$ and fix a dimension vector α. So far, we have considered the algebraic quotient map

$$rep_\alpha\ Q \twoheadrightarrow iss_\alpha\ Q$$

classifying closed $GL(\alpha)$-orbits in $rep_\alpha\ Q$, that is, isomorphism classes of semi-simple representations of dimension α. We have seen that the invariant polynomial maps are generated by traces along oriented cycles in the quiver. Hence, if Q has no oriented cycles, the quotient variety $iss_\alpha\ Q$ is reduced to one point corresponding to the semi-simple

$$S_1^{\oplus a_1} \oplus \ldots \oplus S_k^{\oplus a_k}$$

where S_i is the trivial one-dimensional simple concentrated in vertex v_i. Still, in these cases one can often classify nice families of representations. For example, consider the quiver situation

$$\begin{tikzcd} \circled{1} \arrow[r, bend left, "x"] \arrow[r, "y"] \arrow[r, bend right, "z"'] & \circled{1} \end{tikzcd}$$

Then, $rep_\alpha\ Q = \mathbb{C}^3$ and the action of $GL(\alpha) = \mathbb{C}^* \times \mathbb{C}^*$ is given by $(\lambda,\mu).(x,y,z) = (\frac{\lambda}{\mu}x, \frac{\lambda}{\mu}y, \frac{\lambda}{\mu}z)$. The only closed $GL(\alpha)$-orbit in $\mathbb{C}^3$ is $(0,0,0)$ as the one-parameter subgroup $\lambda(t) = (t,1)$ has the property

$$\lim_{t \rightarrow 0} \lambda(t).(x,y,z) = (0,0,0)$$

so $(0,0,0) \in \overline{\mathcal{O}(x,y,z)}$ for any representation (x,y,z). Still, if we trow away the zero-representation, then we have a nice quotient map

$$\mathbb{C}^3 - \{(0,0,0)\} \overset{\pi}{\twoheadrightarrow} \mathbb{P}^2 \qquad (x,y,z) \mapsto [x:y:z]$$

and as $\mathcal{O}(x,y,z) = \mathbb{C}^*(x,y,z)$ we see that every $GL(\alpha)$-orbit is closed in this complement $\mathbb{C}^3 - \{(0,0,0)\}$. We will generalize such settings to arbitrary quivers.

A character of $GL(\alpha)$ is an algebraic group morphism $\chi : GL(\alpha) \longrightarrow \mathbb{C}^*$. They are fully determined by an integral k-tuple $\theta = (t_1, \ldots, t_k) \in \mathbb{Z}^k$ where

$$GL(\alpha) \xrightarrow{\chi_\theta} \mathbb{C}^* \qquad (g_1, \ldots, g_k) \mapsto det(g_1)^{t_1}. \ \ldots \ .det(g_k)^{t_k}$$

For a fixed θ we can extend the $GL(\alpha)$-action to the space $rep_\alpha \oplus \mathbb{C}$ by

$$GL(\alpha) \times rep_\alpha\ Q \oplus \mathbb{C} \longrightarrow rep_\alpha\ Q \oplus \mathbb{C} \qquad g.(V,c) = (g.V, \chi_\theta^{-1}(g)c)$$

The coordinate ring $\mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}] = \mathbb{C}[rep_\alpha][t]$ can be given a $\mathbb{Z}$-gradation by defining $deg(t) = 1$ and $deg(f) = 0$ for all $f \in \mathbb{C}[rep_\alpha\ Q]$. The induced action of $GL(\alpha)$ on $\mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}]$ preserves this gradation. Therefore, the ring of invariant polynomial maps

$$\mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = \mathbb{C}[rep_\alpha\ Q][t]^{GL(\alpha)}$$

is also graded with homogeneous part of degree zero the ring of invariants $\mathbb{C}[rep_\alpha]^{GL(\alpha)}$. An invariant of degree n, say ft^n with $f \in \mathbb{C}[rep_\alpha\ Q]$ has the characteristic property that

$$f(g.V) = \chi_\theta^n(g) f(V)$$

that is, f is a semi-invariant of weight χ_θ^n. That is, the graded decomposition of the invariant ring is

$$\mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = R_0 \oplus R_1 \oplus \ldots \quad with \quad R_i = \mathbb{C}[rep_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$$

With these notations, the moduli space of semi-stable quiver representations of dimension *α was introduced by A. King in [12] to be the variety*

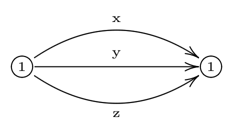
$$M_\alpha^{ss}(Q,\theta) = Proj\ \mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = Proj\ \oplus_{n=0}^{\infty} \mathbb{C}[rep_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$$

Recall that for a positively graded affine commutative $\mathbb{C}$*-algebra* $R = \oplus_{i=0}^{\infty} R_i$*, the geometric points of* $Proj\ R$ *correspond to graded-maximal ideals* $\mathfrak{m}$ *not containing the positive part* $R_+ = \oplus_{i=1}^{\infty} R_i$*. Intersecting* $\mathfrak{m}$ *with the part of degree zero* R_0 *determines a point of* $Spec\ R_0$*, the affine variety with coordinate ring* R_0 *and gives rise to a structural morphism* $Proj\ R \longrightarrow Spec\ R_0$*. The Zariski closed subsets of* $Proj\ R$ *are of the form*

$$\mathbb{V}(I) = \{\mathfrak{m} \in Proj\ R \mid I \subset \mathfrak{m}\}$$

for a homogeneous ideal $I \triangleleft R$*. Also recall that* $Proj\ R$ *can be covered by affine varieties of the form* $\mathbb{X}(f)$ *with* f *a homogeneous element in* R_+*. The coordinate ring of this affine variety is the part of degree zero of the graded localization* R_f^g*. We refer to [9, II.2] for more details.*

Example 7.13 Consider again the quiver-situation



and character $\theta = (-1, 1)$, then the three coordinate functions x, y and z of $\mathbb{C}[rep_\alpha\ Q]$ are semi-invariants of weight χ_θ. It is then clear that the invariant ring is equal to

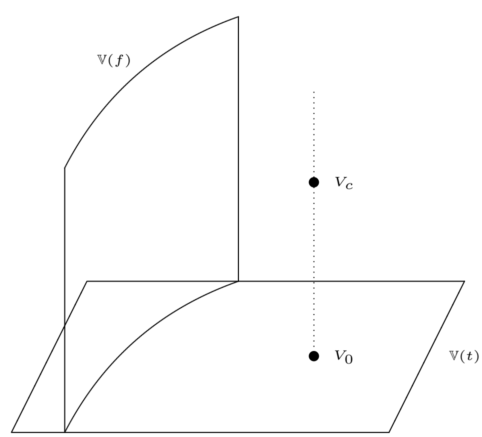
$$\mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)} = \mathbb{C}[xt, yt, zt]$$

where the three generators all have degree one. That is,

$$M_\alpha^{ss}(Q, \theta) = Proj\ \mathbb{C}[xt, yt, zt] = \mathbb{P}^2$$

as desired,

We will now investigate which orbits in $rep_\alpha\ Q$ *are parameterized by the moduli space* $M_\alpha^{ss}(Q, \theta)$*. We say that a representation* $V \in rep_\alpha\ Q$ *is* χ_θ*-semistable if and only if there is a semi-invariant* $f \in \mathbb{C}[rep_\alpha\ Q]^{GL(\alpha), \chi^n \theta}$ *for some* $n \geq 1$ *such that* $f(V) \neq 0$*. The subset of* $rep_\alpha\ Q$ *consisting of all* χ_θ*-semistable representations will be denoted by* $rep_\alpha^{ss}(Q, \theta)$*. Observe that* $rep_\alpha^{ss}(Q, \theta)$ *is Zariski open (but it may be empty for certain* (α, θ)*). We can lift a representation* $V \in rep_\alpha\ Q$ *to points* $V_c = (V, c) \in rep_\alpha\ Q \oplus \mathbb{C}$ *and use* $GL(\alpha)$*-invariant theory on this larger* $GL(\alpha)$*-module*



Let $c \neq 0$ *and assume that the orbit closure* $\overline{\mathcal{O}(V_c)}$ *does not intersect* $\mathbb{V}(t) = rep_\alpha\ Q \times \{0\}$*. As both are* $GL(\alpha)$*-stable closed subsets of* $rep_\alpha\ Q \oplus \mathbb{C}$ *we know from the separation property of invariant theory, see §4.6, that this is equivalent to the existence of a* $GL(\alpha)$*-invariant function* $g \in \mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)}$ *such that* $g(\overline{\mathcal{O}(V_c)}) \neq 0$ *but* $g(\mathbb{V}(t)) = 0$*. We have seen that the invariant ring is graded,* *hence we may assume g to be homogeneous, that is, of the form $g = ft^n$ for some n. But then, f is a semi-invariant on $rep_\alpha\ Q$ of weight χ_θ^n and we see that V must be χ_θ-semistable. Moreover, we must have that $\theta(\alpha) = \sum_{i=1}^k t_i a_i = 0$, for the one-dimensional central torus of $GL(\alpha)$*

$$\mu(t) = (t\mathbb{1}_{a_1}, \ldots, t\mathbb{1}_{a_k}) \hookrightarrow GL(\alpha)$$

acts trivially on $rep_\alpha\ Q$ but acts on $\mathbb{C}$ via multiplication with $\prod_{i=1}^k t^{-a_i t_i}$ hence if $\theta(\alpha) \neq 0$ then $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) \neq \emptyset$. More generally, we have from the strong form of the Hilbert criterium proved in §4.4 that $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) = \emptyset$ if and only if for every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ we must have that $\lim_{t \to 0} \lambda(t).V_c \notin \mathbb{V}(t)$. We can also formulate this in terms of the $GL(\alpha)$-action on $rep_\alpha\ Q$. The composition of a one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ with the character

$$\mathbb{C}^* \xrightarrow{\lambda(t)} GL(\alpha) \xrightarrow{\chi\theta} \mathbb{C}^*$$

is an algebraic group morphism and is therefore of the form $t \longrightarrow t^m$ for some $m \in \mathbb{Z}$ and we denote this integer by $\theta(\lambda) = m$. Assume that $\lambda(t)$ is a one-parameter subgroup such that $\lim_{t \to 0} \lambda(t).V = V'$ exists in $rep_\alpha\ Q$, then as

$$\lambda(t).(V, c) = (\lambda(t).V, t^{-m}c)$$

we must have that $\theta(\lambda) \geq 0$ for the orbitclosure $\overline{\mathcal{O}(V_c)}$ not to intersect $\mathbb{V}(t)$. That is, we have the following characterization of χ_θ-semistable representations.

Proposition 7.14 *The following are equivalent*

1. *$V \in rep_\alpha\ Q$ is χ_θ-semistable.*
2. *For $c \neq 0$, we have $\overline{\mathcal{O}(V_c)} \cap \mathbb{V}(t) = \emptyset$.*
3. *For every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ we have $\lim_{t \to 0} \lambda(t).V_c \notin \mathbb{V}(t) = rep_\alpha\ Q \times \{0\}$.*
4. *For every one-parameter subgroup $\lambda(t)$ of $GL(\alpha)$ such that $\lim_{t \to 0} \lambda(t).V$ exists in $rep_\alpha\ Q$ we have $\theta(\lambda) \geq 0$.*

Moreover, these cases can only occur if $\theta(\alpha) = 0$.

Assume that $g = ft^n$ is a homogeneous invariant function for the $GL(\alpha)$-action on $rep_\alpha\ Q \oplus \mathbb{C}$ and consider the affine open $GL(\alpha)$-stable subset $\mathbb{X}(g)$. The construction of the algebraic quotient in §4.6 and the fact that invariant rings here are graded asserts that the closed $GL(\alpha)$-orbits in $\mathbb{X}(g)$ are classified by the points of the graded localization at g which is of the form

$$(\mathbb{C}[rep_\alpha\ Q \oplus \mathbb{C}]^{GL(\alpha)})_g = R_f[h, h^{-1}]$$

for some homogeneous invariant h and where R_f is the coordinate ring of the affine open subset $\mathbb{X}(f)$ in $M_\alpha^{ss}(Q, \theta)$ determined by the semi-invariant f of weight χ_θ^n. As the moduli space is covered by such open subsets we have

Proposition 7.15 *The moduli space of θ-semistable representations of $rep_\alpha\ Q$*

$$M_\alpha^{ss}(Q, \theta)$$

classifies closed $GL(\alpha)$-orbits in the open subset $rep_\alpha^{ss}(Q, \theta)$ of all χ_θ-semistable representations of Q of dimension vector α.

Example 7.16 In the foregoing example $rep_\alpha^{ss}(Q,\theta) = \mathbb{C}^3 - \{(0,0,0)\}$ as for all these points one of the semi-invariant coordinate functions is non-zero. For $\theta = (-1,1)$ the lifted $GL(\alpha) = \mathbb{C}^* \times \mathbb{C}^*$-action to $rep_\alpha\ Q \oplus \mathbb{C} = \mathbb{C}^4$ is given by

$$(\lambda,\mu).(x,y,z,t) = (\frac{\mu}{\lambda}x, \frac{\mu}{\lambda}y, \frac{\mu}{\lambda}z, \frac{\lambda}{\mu}t)$$

We have seen that the ring of invariants is $\mathbb{C}[xt,yt,zt]$. Consider the affine open set $\mathbb{X}(xt)$ of $\mathbb{C}^4$, then the closed orbits in $\mathbb{X}(xt)$ are classified by

$$\mathbb{C}[xt,yt,zt]^g_{xt} = \mathbb{C}[\frac{y}{x},\frac{z}{x}][xt,\frac{1}{xt}]$$

and the part of degree zero $\mathbb{C}[\frac{y}{x},\frac{z}{x}]$ is the coordinate ring of the open set $\mathbb{X}(x)$ in $\mathbb{P}^2$.

In §4.5 we were able to classify closed GL_n-orbits in $rep_n\ A$ with semi-simple representations. We will now give a representation theoretic interpretation of closed $GL(\alpha)$-orbits in $rep_\alpha^{ss}(Q,\theta)$. Again, the starting point is that one-parameter subgroups $\lambda(t)$ of $GL(\alpha)$ correspond to filtrations of representations. Let us go through the motions one more time. For $\lambda : \mathbb{C}^* \longrightarrow GL(\alpha)$ a one-parameter subgroup and $V \in rep_\alpha\ Q$ we can decompose for every vertex v_i the vertex-space in weight spaces

$$V_i = \oplus_{n\in\mathbb{Z}} V_i^{(n)}$$

where $\lambda(t)$ acts on the weight space $V_i^{(n)}$ as multiplication by t^n. This decomposition allows us to define a filtration

$$V_i^{(\geq n)} = \oplus_{m\geq n} V_i^{(m)}$$

For every arrow $ⓙ \xleftarrow{a} ⓘ$, *$\lambda(t)$ acts on the components of the arrow maps*

$$V_i^{(n)} \xrightarrow{V_a^{m,n}} V_j^{(m)}$$

by multiplication with t^{m-n}. That is, a limit $\underset{t\to 0}{lim}\ V_a$ exists if and only if $V_a^{m,n} = 0$ for all $m < n$, that is, if V_a induces linear maps

$$V_i^{(\geq n)} \xrightarrow{V_a} V_j^{(\geq n)}$$

Hence, a limiting representation exists if and only if the vertex-filtration spaces $V_i^{(\geq n)}$ determine a subrepresentation $V_n \hookrightarrow V$ for all n. That is, a one-parameter subgroup λ such that $\underset{t\to}{lim}\ \lambda(t).V$ exists determines a decreasing filtration of V by subrepresentations

$$\dots \hookleftarrow V_n \hookleftarrow V_{n+1} \hookleftarrow \dots$$

Further, the limiting representation is then the associated graded representation

$$\underset{t\to 0}{lim}\ \lambda(t).V = \oplus_{n\in\mathbb{Z}} \frac{V_n}{V_{n+1}}$$

where of course only finitely many of these quotients can be nonzero. For the given character $\theta = (t_1,\dots,t_k)$ and a representation $W \in rep_\beta\ Q$ we denote

$$\theta(W) = t_1b_1 + \dots + t_kb_k \quad \textit{where} \quad \beta = (b_1,\dots,b_k)$$

Assume that $\theta(V) = 0$, then with the above notations, we have an interpretation of $\theta(\lambda)$ as

$$\theta(\lambda) = \sum_{i=1}^k t_i \sum_{n\in Z} n dim_{\mathbb{C}}\ V_i^{(n)} = \sum_{n\in\mathbb{Z}} n\theta(\frac{V_n}{V_{n+1}}) = \sum_{n\in\mathbb{Z}} \theta(V_n)$$

Definition 7.17 A representation $V \in rep_\alpha\ Q$ is said to be

- θ-semistable if $\theta(V) = 0$ and for all subrepresentations $W \hookrightarrow V$ we have $\theta(W) \geq 0$.
- θ-stable if V is θ-semistable and if the only subrepresentations $W \hookrightarrow V$ such that $\theta(W) = 0$ are V and 0.

Proposition 7.18 *For $V \in rep_\alpha\ Q$ the following are equivalent*

1. *V is χ_θ-semistable.*
2. *V is θ-semistable.*

Proof. $1. \Rightarrow 2.$ *: Let W be a subrepresentation of V and let λ be the one-parameter subgroup associated to the filtration $V \hookleftarrow W \hookleftarrow 0$, then $\lim_{t\to 0} \lambda(t).V$ exists whence by proposition 7.14.4 we have $\theta(\lambda) \geq 0$, but we have*

$$\theta(\lambda) = \theta(V) + \theta(W) = \theta(W)$$

$2. \Rightarrow 1.$ *: Let λ be a one-parameter subgroup of $GL(\alpha)$ such that $\lim_{t\to 0} \lambda(t).V$ exists and consider the induced filtration by subrepresentations V_n defined above. By assumption all $\theta(V_n) \geq 0$, whence*

$$\theta(\lambda) = \sum_{n \in Z} \theta(V_n) \geq 0$$

and again proposition 7.14.4 finishes the proof. □

Lemma 7.19 *Let $V \in rep_\alpha\ Q$ and $W \in rep_\beta\ Q$ be both θ-semistable and*

$$V \xrightarrow{f} W$$

a morphism of representations. Then, $Ker\ f$, $Im\ f$ and $Coker\ f$ are θ-semistable representations.

Proof. *Consider the two short exact sequences of representations of Q*

$$\begin{cases} 0 \longrightarrow Ker\ f \longrightarrow V \longrightarrow Im\ f \longrightarrow 0 \\ 0 \longrightarrow Im\ f \longrightarrow W \longrightarrow Coker\ f \longrightarrow 0 \end{cases}$$

As $\theta(-)$ is additive, we have $0 = \theta(V) = \theta(Ker\ f) + \theta(Im\ f)$ and as both are subrepresentations of θ-semistable representations V resp. W, the right-hand terms are ≥ 0 whence are zero. But then, from the second sequence also $\theta(Coker\ f) = 0$. Being submodules of θ-semistable representations, $Ker\ f$ and $Im\ f$ also satisfy $\theta(S) \geq 0$ for all their subrepresentations U. Finally, a subrepresentation $T \hookrightarrow Coker\ f$ can be lifted to a subrepresentation $T' \hookrightarrow W$ and $\theta(T) \geq 0$ follows from the short exact sequence $0 \longrightarrow Im\ f \longrightarrow T' \longrightarrow T \longrightarrow 0$. □

That is, the full subcategory $rep^{ss}(Q,\theta)$ of $rep\ Q$ consisting of all θ-semistable representations is an Abelian subcategory and clearly the simple objects in $rep^{ss}(Q,\theta)$ are precisely the θ-stable representations. As this Abelian subcategory has the necessary finiteness conditions, one can prove a version of the Jordan-Hölder theorem. That is, every θ-semistable representation V has a finite filtration

$$V = V_0 \hookleftarrow V_1 \hookleftarrow \ldots \hookleftarrow V_z = 0$$

of subrepresentation such that every factor $\frac{V_i}{V_{i+1}}$ is θ-stable. Moreover, the unordered set of these θ-stable factors are uniquely determined by V.

Theorem 7.20 *For a θ-semistable representation $V \in rep_\alpha\ Q$ the following are equivalent*

1. *The orbit $\mathcal{O}(V)$ is closed in $rep^{ss}_\alpha(Q,\alpha)$.*
2. *$V \simeq W_1^{\oplus e_1} \oplus \ldots \oplus W_l^{\oplus e_l}$ with every W_i a θ-stable representation.*

That is, the geometric points of the moduli space $M^{ss}_\alpha(Q,\theta)$ are in natural one-to-one correspondence with isomorphism classes of α-dimensional representations which are direct sums of θ-stable subrepresentations. The quotient map

$$rep^{ss}_\alpha(Q,\theta) \twoheadrightarrow M^{ss}_\alpha(Q,\theta)$$

maps a θ-semistable representation V to the direct sum of its Jordan-Hölder factors in the Abelian category $rep^{ss}(Q,\theta)$.

Proof. *Assume that $\mathcal{O}(V)$ is closed in $rep^{ss}_\alpha(Q,\theta)$ and consider the θ-semistable representation $W = gr_{ss}\ V$, the direct sum of the Jordan-Hölder factors in $rep^{ss}(Q,\theta)$. As W is the associated graded representation of a filtration on V, there is a one-parameter subgroup λ of $GL(\alpha)$ such that $\underset{t\to 0}{lim}\ \lambda(t).V \simeq W$, that is $\mathcal{O}(W) \subset \overline{\mathcal{O}(V)} = \mathcal{O}(V)$, whence $W \simeq V$ and 2. holds.*

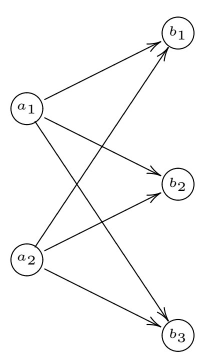
Conversely, assume that V is as in 2. and let $\mathcal{O}(W)$ be a closed orbit contained in $\overline{\mathcal{O}(V)}$ (one of minimal dimension). By the Hilbert criterium there is a one-parameter subgroup λ in $GL(\alpha)$ such that $\underset{t\to 0}{lim}\ \lambda(t).V \simeq W$. Hence, there is a finite filtration of V with associated graded θ-semistable representation W. As none of the θ-stable components of V admits a proper quotient which is θ-semistable (being a direct summand of W), this shows that $V \simeq W$ and so $\mathcal{O}(V) = \mathcal{O}(W)$ is closed. The other statements are clear from this. □

The striking similarities between θ-stable representations and simple representations will become more transparent in chapter 13 when we discuss universal localizations. It will turn out that θ-stable representations become simple representations of a certain universal localization of the path algebra $\mathbb{C}Q$.

Example 7.21 Consider the modular group $PSL_2(\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$, the free product of the cyclic groups of order two and three with generators σ resp. τ. Let S be an n-dimensional simple representation of $PSL_2(\mathbb{Z})$. Let ξ be a 3-rd root of unity, then restricting S to these finite Abelian subgroups we have

$$\begin{cases} S \downarrow_{\mathbb{Z}_2} & \simeq S_1^{\oplus a_1} \oplus S_{-1}^{\oplus a_2} \\ S \downarrow_{\mathbb{Z}_3} & \simeq T_1^{\oplus b_1} \oplus T_\xi^{\oplus b_2} \oplus T_{\xi^2}^{\oplus b_3} \end{cases}$$

where S_x resp. T_x are the one-dimensional representations on which σ resp. τ acts via multiplication with x. Observe that $a_1 + a_2 = b_1 + b_2 + b_3 = n$ and we associate to S a representation V of the quiver situation



with $V_{1i} = S_i^{\oplus a_i}$ and $V_{2j} = T_j^{\oplus b_j}$ and where the linear map corresponding to an arrow $b_j \xrightarrow{a_{ij}} a_i$ is the composition of

$$V_{a_{ij}} \quad : \quad S_i^{\oplus a_i} \hookrightarrow S \downarrow_{\mathbb{Z}_2} = V \downarrow_{\mathbb{Z}_3} \twoheadrightarrow T_j^{\oplus b_j}$$

of the canonical injections and projections. If $\alpha = (a_1, a_2, b_1, b_2, b_3)$ then we take as $\theta = (-1, -1, +1, +1, +1)$. Observe that $\oplus_{i,j} V_{a_{ij}} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a linear isomorphism. If $W \hookrightarrow V$ is a subrepresentation, then $\theta(W) \geq 0$. Indeed, if the dimension vector of W is $\beta = (c_1, c_2, d_1, d_2, d_3)$ and assume that $\theta(W) < 0$, then $k = c_1 + c_2 > l = d_1 + d_2 + d_3$, but then the restriction of $\oplus V_{a_{ij}}$ to W gives a linear map $\mathbb{C}^k \twoheadrightarrow \mathbb{C}^l$ having a kernel which is impossible. Hence, V is a θ-semistable representation of the quiver. In fact, V is even θ-stable, for consider a subrepresentation $W \hookrightarrow V$ with dimension vector β as before and $\theta(W) = 0$, that is, $c_1 + c_2 = d_1 + d_2 + d_3 = m$, then the isomorphism $\oplus_{i,j} V_{a_{ij}} \mid W$ and the decomposition into eigenspaces of $\mathbb{C}^m$ with respect to the $\mathbb{Z}_2$ and $\mathbb{Z}_3$-action, makes $\mathbb{C}^m$ into an m-dimensional representation of $PSL_2(\mathbb{Z})$ which is a subrepresentation of S. S being simple then implies that $W = V$ or $W = 0$, whence V is θ-stable. The underlying reason is that the group algebra $\mathbb{C}PSL_2(\mathbb{Z})$ is a universal localization of the path algebra $\mathbb{C}Q$ of the above quiver.

Remains to determine the situations (α, θ) such that the corresponding moduli space $M^{ss}_\alpha(Q, \theta)$ is non-empty, or equivalently, such that the Zariski open subset $rep^{ss}_\alpha(Q, \theta) \hookrightarrow rep_\alpha\ Q$ is non-empty.

Theorem 7.22 *Let α be a dimension vector such that $\theta(\alpha) = 0$. Then,*

1. *$rep^{ss}_\alpha(Q, \alpha)$ is a non-empty Zariski open subset of $rep_\alpha\ Q$ if and only if for every $\beta \hookrightarrow \alpha$ we have that $\theta(\beta) \geq 0$.*

2. *The θ-stable representations $rep^s_\alpha(Q, \alpha)$ are a non-empty Zariski open subset of $rep_\alpha\ Q$ if and only if for every $0 \neq \beta \hookrightarrow \alpha$ we have that $\theta(\beta) > 0$*

Observe that the Schofield criterium gives an inductive procedure to calculate these conditions. Sometimes we can bypass the troublesome inductive step using our description of dimension vectors of simple representations.

Example 7.23 It is possible to determine the weight space decomposition vectors $\alpha = (a_1, a_2, b_1, b_2, b_3)$ of simple $n = a_1 + a_2 = b_1 + b_2 + b_3$-dimensional representations of the modular group $PSL_2(\mathbb{Z})$ by first computing the dimension vectors $\beta = (c_1, c_2, d_1, d_2, d_3)$ of general subrepresentations of α and then to check whether for all of these $c_1 + c_2 < d_1 + d_2 + d_3$.

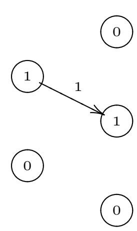
An alternative method is to compute local quiver settings and use the description of semi-simple dimension vectors. With S_{ij} we denote the simple 1-dimensional representation of $PSL_2(\mathbb{Z})$ determined by

$$S_{ij} \downarrow_{\mathbb{Z}_2} = S_i \quad \text{and} \quad S_{ij} \downarrow_{\mathbb{Z}_3}$$

Let $n = x_1 + \ldots + x_6$ and we aim to study the local structure of $rep_n\ \mathbb{C}PSL_2(\mathbb{Z})$ in a neighborhood of the semi-simple n-dimensional representation

$$V_\xi = S_{11}^{\oplus x_1} \oplus S_{12}^{\oplus x_2} \oplus S_{13}^{\oplus x_3} \oplus S_{21}^{\oplus x_4} \oplus S_{22}^{\oplus x_5} \oplus S_{23}^{\oplus x_6}$$

To determine the structure of $Q^\bullet_\xi$ we have to compute $dim\ Ext^1(S_{ij}, S_{kl})$. To do this we view the S_{ij} as representations of the quiver Q in the example above. For example S_{12} is the representation



of dimension vector $(1, 0; 0, 1, 0)$.For representations of Q, the dimensions of Hom and Ext-groups are determined by the bilinear form

$$\chi_Q = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

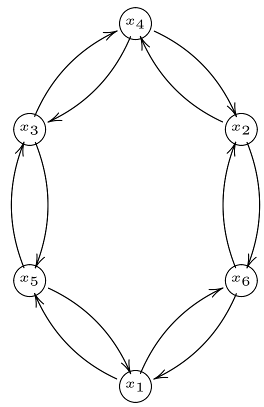
If $V \in rep_\alpha\ Q$ and $W \in rep_\beta\ Q$ where $\alpha = (a_1, a_2; b_1, b_2, b_3)$ with $a_1 + a_2 = b_1 + b_2 + b_3 = k$ and $\beta = (c_1, c_2; d_1, d_2, d_3)$ with $c_1 + c_2 = d_1 + d_2 + d_3 = l$ we have

$$dim\ Hom(V, W) - dim\ Ext^1(V, W) = \chi_Q(\alpha, \beta) = kl - (a_1c_1 + a_2c_2 + b_1d_1 + b_2d_2 + b_3d_3)$$

Because the translation from $PSL_2(\mathbb{Z})$-representations to Q-representations is full and faithful and as $Hom(S_{ij}, S_{kl}) = \mathbb{C}^{\oplus \delta_{ik}\delta_{jl}}$ we have that

$$dim\ Ext^1(S_{ij}, S_{kl}) = \begin{cases} 1 & \text{if } i \neq k \text{ and } j \neq l \\ 0 & \text{otherwise} \end{cases}$$

But then, the local quiver setting $(Q^\bullet_\xi, \alpha_\xi)$ is



We want to determine whether the irreducible component of $rep_n\ \mathbb{C}PSL_2(\mathbb{Z})$ containing V_ξ contains simple $PSL_2(\mathbb{Z})$-representations, or equivalently, whether α_ξ is the dimension vector of a simple representation of $Q^\bullet_\xi$, that is,

$$\chi_{Q^\bullet_\xi}(\alpha_\xi, \epsilon_j) \leq 0 \quad \text{and} \quad \chi_{Q^\bullet_\xi}(\epsilon_j, \alpha_\xi) \quad \text{for all } 1 \leq j \leq 6$$

The Euler-form of $Q^\bullet_\xi$ is determined by the matrix where we number the vertices cyclicly

$$\chi_{Q^\bullet_\xi} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

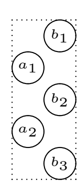
leading to the following set of inequalities

$$\begin{cases} x_1 & \leq x_5 + x_6 \\ x_2 & \leq x_4 + x_6 \\ x_3 & \leq x_4 + x_5 \end{cases} \qquad \begin{cases} x_4 & \leq x_2 + x_3 \\ x_5 & \leq x_1 + x_3 \\ x_6 & \leq x_1 + x_2 \end{cases}$$

Finally, observe that V_ξ corresponds to a Q-representation of dimension vector $(x_1 + x_2 + x_3, x_4 + x_5 + x_6; x_1 + x_4, x_2 + x_5, x_3 + x_6)$. If we write this dimension vector as $(a_1, a_2; b_1, b_2, b_3)$ then the inequalities are equivalent to the conditions

$$a_i \geq b_j \quad \text{for all } 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 3$$

which gives us the desired restriction on the quintuples



at least when $a_i \geq 3$ and $b_j \geq 2$. The remaining cases are handled similarly. Observe that we can use a similar strategy to determine the restrictions on simple representations of any free product $\mathbb{Z}_p * \mathbb{Z}_q$ of two cyclic groups.

The graded algebra $\mathbb{C}[rep_\alpha \oplus \mathbb{C}]^{GL(\alpha)}$ *of all semi-invariants on* $rep_\alpha\ Q$ *of weight* χ_θ^n *for some* $n \geq 0$ *has as degree zero part the ring of polynomial invariants* $\mathbb{C}[rep_\alpha\ Q]^{GL(\alpha)}$*. This embedding determines a proper morphism*

$$M_\alpha^{ss}(Q,\theta) \xrightarrow{\pi} iss_\alpha\ Q$$

which is onto whenever $rep_\alpha^{ss}(Q,\alpha)$ *is non-empty. In particular, if* Q *is a quiver without oriented cycles, then the moduli space of* θ*-semistable representations of dimension vector* α*,* $M_\alpha^{ss}(Q,\theta)$*, is a projective variety.*

7.7 Semi-invariants of quivers.

Because the moduli space $M_\alpha^{ss}(Q,\theta)$ *is defined to be the projective scheme of the graded algebra of semi-invariants of weight* χ_θ^n *for some* n

$$M_\alpha^{ss}(Q,\alpha) = Proj\ \oplus_{n=0}^{\infty} \mathbb{C}[rep_\alpha\ Q]^{GL(\alpha),\chi^n\theta}$$

we need some control on the semi-invariants of quivers. A generating set of semi-invariants was described by A. Schofield and M. Van den Bergh in [28]. The strategy of proof should be clear by now. First, we describe a large set of semi-invariants, apart from the invariant polynomials which we know to be generated by traces of oriented cycles in the quiver we expect determinantal semi-invariants as in the case of mixed GL_n*-semi-invariants of section 3. Then we use classical invariant theory to describe all multilinear semi-invariants of* $GL(\alpha)$*, or equivalently, all multilinear invariants of* $SL(\alpha) = SL_{a_1} \times \ldots \times SL_{a_k}$ *and describe them in terms of these determinantal semi-invariants. Finally, we show by polarization and restitution that these semi-invariants do indeed generate all semi-invariants.*

Let Q *be a quiver on* k *vertices* $\{v_1,\ldots,v_k\}$*. We introduce the additive* $\mathbb{C}$*-category* add Q *generated by the quiver. For every vertex* v_i *we introduce an indecomposable object which we denote by* (i)*. An arbitrary object in* add Q *is then a sum of those*

$$(1)^{\oplus e_1} \oplus \ldots \oplus (k)^{\oplus e_k}$$

Morphisms in the category add Q *are defined by the rules*

$$\begin{cases} Hom(\ (i)\ ,\ (j)\ ) = (j) \longleftarrow\cdots\ (i) \\ Hom(\ (i)\ ,\ (i)\ ) = (i) \end{cases}$$

where the right hand sides are the $\mathbb{C}$*-vectorspaces spanned by all oriented paths from* v_i *to* v_j *in the quiver* Q*, including the idempotent (trivial) path* e_i *when* $i = j$*. Clearly, for any* k*-tuples of positive integers* $\alpha = (u_1,\ldots,u_k)$ *and* $\beta = (v_1,\ldots,v_k)$

$$Hom(\ (1)^{\oplus u_1} \oplus \ldots \oplus (k)^{\oplus u_k}\ ,\ (1)^{\oplus v_1} \oplus \ldots \oplus (k)^{\oplus v_k}\ )$$

is defined in the usual way in the additive category add Q*, that is by the matrices* *where composition arises via matrix multiplication*

$$\begin{bmatrix} M_{v_1 \times u_1}(\; 1 \dashleftarrow 1 \;) & \dots & M_{v_1 \times u_k}(\; 1 \dashleftarrow k \;) \\ \vdots & \ddots & \vdots \\ M_{v_k \times u_1}(\; k \dashleftarrow 1 \;) & \dots & M_{v_k \times u_k}(\; k \dashleftarrow k \;) \end{bmatrix}$$

Now, fix a dimension vector $\alpha = (a_1, \dots, a_k)$ *and a morphism in add* Q

$$1^{\oplus u_1} \oplus \dots \oplus k^{\oplus u_k} \xrightarrow{\phi} 1^{\oplus v_1} \oplus \dots \oplus k^{\oplus v_k}$$

For any representation $V \in rep_\alpha\ Q$ *we can replace each occurrence of an arrow* $j \xleftarrow{a} i$ *of* Q *in* ϕ *by the* $a_j \times a_i$*-matrix* V_a*. This way we obtain a rectangular matrix*

$$V(\phi) \in M_{\sum_{i=1}^k a_i v_i \times \sum_{i=1}^k a_i u_i}(\mathbb{C})$$

If we are in a situation such that $\sum a_i v_i = \sum a_i u_i$*, then we can define a semi-invariant polynomial function on* $rep_\alpha\ Q$ *by*

$$P_{\alpha,\phi}(V) = det\ V(\phi)$$

We call such semi-invariants determinantal semi-invariants*. One verifies that* $P_{\phi,\alpha}$ *is a semi-invariant of weight* χ_θ *where* $\theta = (u_1 - v_1, \dots, u_k - v_k)$*. We will show that such determinantal semi-invariant together with traces along oriented cycles generate all semi-invariants.*

Because semi-invariants for the $GL(\alpha)$*-action on* $rep_\alpha\ Q$ *are the same as invariants for the restricted action of* $SL(\alpha) = SL_{a_1} \times \dots \times SL_{a_k}$*, we will describe the multilinear* $SL(\alpha)$*-invariants from classical invariant theory. Because*

$$\begin{aligned} rep_\alpha\ Q &= \bigoplus_{j \xleftarrow{a} i} M_{a_j \times a_i}(\mathbb{C}) \\ &= \bigoplus_{j \xleftarrow{a} i} \mathbb{C}^{a_i} \otimes \mathbb{C}^{*a_j} \end{aligned}$$

we have to consider multilinear $SL(\alpha)$*-invariants of*

$$\bigotimes_{j \leftarrow i} \mathbb{C}^{a_i} \otimes \mathbb{C}^{*a_j} = \bigotimes_{i} \Big[\bigotimes_{\circ \leftarrow i} \mathbb{C}^{a_i} \otimes \bigotimes_{i \leftarrow \circ} \mathbb{C}^{*a_i} \Big]$$

Hence, any multilinear $SL(\alpha)$*-invariant can be written as* $f = \prod_{i=1}^k f_i$ *where* f_i *is a* SL_{a_i}*-invariant of*

$$\bigotimes_{\circ \leftarrow i} \mathbb{C}^{a_i} \otimes \bigotimes_{i \leftarrow \circ} \mathbb{C}^{*a_i}$$

In section 3 we have recalled Weyl's result describing all multilinear SL_m*-invariants on* $\otimes_B \mathbb{C}^m \otimes \otimes_C \mathbb{C}^{*m}$*. By polarization and restitution it follows from this that the linear* SL_m*-invariants are determined by the following three sets*

- *traces, that is, for each pair* (b,c) *we have* $\mathbb{C}^m \otimes \mathbb{C}^{*m} = M_m(\mathbb{C}) \xrightarrow{Tr} \mathbb{C}$.
- *brackets, that is, for each* m*-tuple* $(b_1,\ldots,b_m)$ *we have an invariant* $\otimes_{b_j}\mathbb{C}^m \longrightarrow \mathbb{C}$ *defined by*

$$v_{b_1} \otimes \ldots \otimes v_{b_m} \mapsto det \begin{bmatrix} v_{b_1} & \ldots & v_{b_m} \end{bmatrix}$$

- *cobrackets, that is, for each* m*-tuple* $(c_1,\ldots,c_m)$ *we have an invariant* $\otimes_{c_i}\mathbb{C}^{*m} \longrightarrow \mathbb{C}$ *defined by*

$$\phi_{c_1} \otimes \ldots \otimes \phi_{c_m} \mapsto det \begin{bmatrix} \phi_{c_1} \\ \vdots \\ \phi_{c_m} \end{bmatrix}$$

Multilinear SL_m*-invariants of* $\otimes_B \mathbb{C}^m \otimes \otimes_C \mathbb{C}^{*m}$ *are then spanned by invariants constructed from the following data. Take three disjoint index-sets* I, J *and* K *and consider surjective maps*

$$\begin{cases} B & \overset{\mu}{\twoheadrightarrow} I \sqcup K \\ C & \overset{\nu}{\twoheadrightarrow} J \sqcup K \end{cases}$$

subject to the following conditions

$$\begin{cases} \#\ \mu^{-1}(k) = 1 = \#\ \nu^{-1}(k) & \text{for all } k \in K. \\ \#\ \mu^{-1}(i) = m = \#\ \nu^{-1}(j) & \text{for all } i \in I \text{ and } j \in J. \end{cases}$$

To this data $\gamma = (\mu,\nu,I,J,K)$ *we can associate a multilinear* SL_m*-invariant* $f_\gamma(\otimes_B v_b \otimes \otimes_C \phi_c)$ *defined by*

$$\prod_{k \in K} \phi_{\nu^{-1}(k)}(v_{\mu^{-1}(k)}) \prod_{i \in I} det \begin{bmatrix} v_{b_1} & \ldots & v_{b_m} \end{bmatrix} \prod_{j \in J} det \begin{bmatrix} \phi_{c_1} \\ \vdots \\ \phi_{c_m} \end{bmatrix}$$

where $\mu^{-1}(i) = \{b_1,\ldots,b_m\}$ *and* $\nu^{-1}(j) = \{c_1,\ldots,c_m\}$. *Observe that* f_γ *is determined only up to a sign by the data* γ. *But then, we also have a spanning set for the multilinear* $SL(\alpha)$*-invariants on*

$$rep_\alpha\ Q = \bigotimes_{v} [\ \bigotimes_{\circ \leftarrow v} \mathbb{C}^{a_v} \otimes \bigotimes_{v \leftarrow \circ} \mathbb{C}^{*a_v}\ ]$$

determined by quintuples $\Gamma = (\mu,\nu,I,J,K)$ *where we have disjoint index-sets partitioned over the vertices* $v \in \{v_1,\ldots,v_k\}$ *of* Q

$$\begin{cases} I & = \bigsqcup_v\ I_v \\ J & = \bigsqcup_v\ J_v \\ K & = \bigsqcup_v\ K_v \end{cases}$$

together with surjective maps from the set of all arrows A *of* Q

$$\begin{cases} A & \overset{\mu}{\twoheadrightarrow} I \sqcup K \\ A & \overset{\nu}{\twoheadrightarrow} J \sqcup K \end{cases}$$

where we have for every arrow $w \xleftarrow{a} v$ *that*

$$\begin{cases} \mu(a) & \in I_v \sqcup K_v \\ \nu(a) & \in J_w \sqcup K_w \end{cases}$$

and these maps μ and ν are subject to the numerical restrictions

$$\begin{cases} \#\ \mu^{-1}(k) = 1 = \#\ \nu^{-1}(k) & \text{for all } k \in K. \\ \#\ \mu^{-1}(i) = a_v = \#\ \nu^{-1}(j) & \text{for all } i \in I_v \text{ and all } j \in J_v. \end{cases}$$

Such a quintuple $\Gamma = (\mu, \nu, I, J, K)$ determines for every vertex v a quintuple

$$\gamma_v = (\mu_v = \mu \mid \{ \bigcirc \xleftarrow{a} \textcircled{v} \},\ \nu_v = \nu \mid \{ \textcircled{v} \xleftarrow{a} \bigcirc \}, I_v, J_v, K_v)$$

satisfying the necessary numerical restrictions to define the SL_{a_v}-invariant f_{γ_v} described before. Then, the multilinear $SL(\alpha)$-invariant on $rep_\alpha\ Q$ determined by Γ is defined to be

$$f_\gamma = \prod_v f_{\gamma_v}$$

and we have to show that these semi-invariants lie in the linear span of the determinantal semi-invariants.

First, consider the case where the index set K is empty. If we denote the total number of arrows in Q by n, then the numerical restrictions imposed give us two expressions for n

$$\sum_v a_v.\#\ I_v = n = \sum_v a_v.\#\ J_v$$

Every arrow $\textcircled{w} \xleftarrow{a} \textcircled{v}$ determines a pair of indices $\mu(a) \in I_v$ and $\nu(a) \in J_w$. To the quintuple Γ we assign a map Φ_Γ in add Q

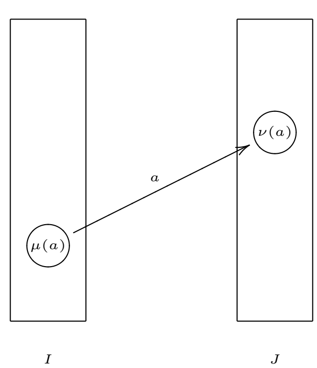
$$\textcircled{1}^{\oplus I_1} \oplus \ldots \oplus \textcircled{k}^{\oplus I_k} \xrightarrow{\Phi_\Gamma} \textcircled{1}^{\oplus J_1} \oplus \ldots \oplus \textcircled{k}^{\oplus J_k}$$

which decomposes as a block-matrix in blocks $M_{v,w} \in Hom(\ \textcircled{v}^{\oplus I_v}, \textcircled{w}^{\oplus J_w}\ )$ of which the (i,j) entry is given by the sum of arrows

$$\sum_{\substack{\mu(a)=i \\ \nu(a)=j}} \textcircled{w} \xleftarrow{a} \textcircled{v}$$

For a representation $V \in rep_\alpha\ Q$, $V(\Phi_\Gamma)$ is an $n \times n$ matrix and the determinant defines the determinantal semi-invariant $P_{\Phi_{\alpha,\Gamma}}$ which we claim to be equal to the basic invariant f_Γ possibly up to a sign.

We introduce a new quiver situation. Let Q' be the quiver with vertices the elements of $I \sqcup J$ and with arrows the set A of arrows of Q, but this time w take the starting point of the arrow $\bigcirc \xleftarrow{a} \bigcirc$ in Q to be $\mu(a) \in I$ and the terminating vertex to be $\nu(a) \in J$. That is, Q' is a bipartite quiver



On Q' we have the quintuple $\Gamma' = (\mu', \nu', I', J', K')$ where $K' = \emptyset$,

$$I' = \bigsqcup_{i \in I} I'_i = \bigsqcup_{i \in I} \{i\} \qquad J' = \bigsqcup_{j \in J} J'_j = \bigsqcup_{j \in J} \{j\}$$

and $\mu' = \mu$, $\nu' = \nu$. We define an additive functor add $Q' \xrightarrow{s}$ add Q by

$$\circledcirc_i \xrightarrow{s} \circledcirc_v \qquad \circledcirc_j \xrightarrow{s} \circledcirc_w \qquad \circ \xleftarrow{a} \circ \xrightarrow{s} \circ \xleftarrow{a} \circ$$

for all $i \in I_v$ and all $j \in J_w$. The functor s induces a functor rep $Q \xrightarrow{s}$ rep Q' defined by $V \xrightarrow{s} V \circ s$. If $V \in rep_\alpha\ Q$ then $s(V) \in rep_{\alpha'}\ Q'$ where

$$\alpha' = (\underbrace{c_1, \ldots, c_p}_{\#\ I}, \underbrace{d_1, \ldots, d_q}_{\#\ J}) \quad \textit{with} \quad \begin{cases} c_i = a_v & \textit{if } i \in I_v \\ d_j = a_w & \textit{if } j \in J_w \end{cases}$$

That is, the characteristic feature of Q' is that every vertex $i \in I$ is the source of exactly c_i arrows (follows from the numerical condition on μ) and that every vertex $j \in J$ is the sink of exactly d_j arrows in Q'. That is, locally Q' has the following form

$$\textcircled{c} \overset{c}{\Longrightarrow} \qquad \textit{or} \qquad \overset{d}{\Longrightarrow} \textcircled{d}$$

There are induced maps

$$rep_\alpha\ Q \xrightarrow{s} rep_{\alpha'}\ Q' \qquad GL(\alpha) \xrightarrow{s} GL(\alpha')$$

where the latter follows from functoriality by considering $GL(\alpha)$ as the automorphism group of the trivial representation in $rep_\alpha\ Q$. These maps are compatible with the actions as one checks that $s(g.V) = s(g).s(V)$. Also s induces a map on the coordinate rings $\mathbb{C}[rep_\alpha\ Q] \xrightarrow{s} \mathbb{C}[rep_{\alpha'}\ Q']$ by $s(f) = f \circ s$. In particular, for the determinantal semi-invariants we have

$$s(P_{\alpha',\phi'}) = P_{\alpha,s(\phi')}$$

and from the compatibility of the action it follows that when f is a semi-invariant the $GL(\alpha')$ action on $rep_{\alpha'}\ Q'$ with character χ', then $s(f)$ is a semi-invariant for the $GL(\alpha)$-action on $rep_\alpha\ Q$ with character $s(\chi) = \chi' \circ s$. In particular we have that

$$s(P_{\alpha',\Phi_{\Gamma'}}) = P_{\alpha,s(\Phi_{\Gamma'})} = P_{\alpha,\Phi_\Gamma} \quad \textit{and} \quad s(f_{\Gamma'}) = f_\Gamma$$

Hence in order to prove our claim, we may replace the triple (Q, α, Γ) by the triple (Q', α', Γ'). We will do this and forget the dashes from here on.

In order to verify that $f_\Gamma = \pm P_{\alpha,\Phi_\Gamma}$ it suffices to check this equality on the image of

$$W = \bigoplus_{\textcircled{i} \xrightarrow{a} \textcircled{j}} \mathbb{C}^{c_i} \oplus \mathbb{C}^{*d_j} \quad \textit{in} \quad \bigotimes_{\textcircled{i} \xrightarrow{a} \textcircled{j}} \mathbb{C}^{c_i} \otimes \mathbb{C}^{*d_j}$$

One verifies that both f_Γ and P_{α,Φ_Γ} are $GL(\alpha)$-semi-invariants on W of weight χ_θ where

$$\theta = (\underbrace{1, \ldots, 1}_{\#\ I}, \underbrace{-1, \ldots, -1}_{\#\ J})$$

Using the characteristic local form of $Q = Q'$, we see that W is isomorphic to the $GL(\alpha)$- module

$$W \simeq \bigoplus_{i \in I} (\underbrace{\mathbb{C}^{c_i} \oplus \ldots \oplus \mathbb{C}^{c_i}}_{c_i}) \oplus \bigoplus_{j \in J} (\underbrace{\mathbb{C}^{*d_j} \oplus \ldots \oplus \mathbb{C}^{*d_j}}_{d_j}) \simeq \bigoplus_{i \in I} M_{c_i}(\mathbb{C}) \oplus \bigoplus_{j \in J} M_{d_j}(\mathbb{C})$$

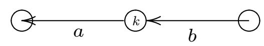
and the i factors of $GL(\alpha)$ act by inverse right-multiplication on the component M_{c_i} (and trivially on all others) and the j factors act by left-multiplication on the component M_{d_j} (and trivially on the others). That is, $GL(\alpha)$ acts on W with an open orbit, say that of the element

$$w = (\mathbb{1}_{c_1}, \ldots, \mathbb{1}_{c_p}, \mathbb{1}_{d_1}, \ldots, \mathbb{1}_{d_q}) \in W$$

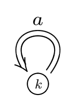
One verifies immediately from the definitions that that both f_Γ and P_{α,Φ_Γ} evaluate to ± 1 in w. Hence, indeed, f_Γ can be expressed as a determinantal semi-invariant.

Remains to consider the case when K is non-empty. For $k \in K$ two situations can occur

- *$\mu^{-1}(k) = a$ and $\nu^{-1}(k) = b$ are distinct, then k corresponds to replacing the arrows a and b by their concatenation*

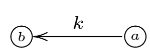


- *$\mu^{-1}(k) = a = \nu^{-1}(k)$ then a is a loop in Q and k corresponds*



to taking the trace of a.

This time we construct a new quiver Q" with vertices $\{w_1, \ldots, w_n\}$ corresponding to the set A of arrows in Q. The arrows in Q" will correspond to elements of K, that is if $k \in K$ we have the arrow (or loop) in Q" with notations as before

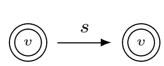


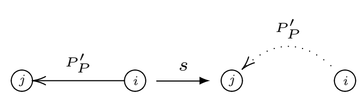
or


We consider the connected components of Q". They are of the following three types

- *(oriented cycle) : To an oriented cycle C in Q" corresponds an oriented cycle C'_C in the original quiver Q. We associate to it the trace $tr(C'_C)$ of this cycle.*
- *(open paths) : An open path P in Q" corresponds to an oriented path P'_P in Q which may be a cycle. To P we associate the corresponding path P'_P in Q.*
- *(isolated points) : They correspond to arrows in Q.*

We will now construct a new quiver Q' having the same vertex set $\{v_1, \ldots, v_k\}$ as Q but with arrows corresponding to the set of paths P'_P described above. The starting and ending vertex of the arrow corresponding to P'_P are of course the starting and ending vertex of the path P_P in Q. Again, we define an additive functor add $Q' \xrightarrow{s}$ add Q by the rules



and


If the path P'_P is the concatenation of the arrows $a_d \circ \ldots \circ a_1$ in Q, we define the maps

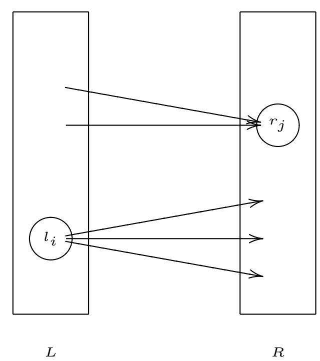
$$\begin{cases} \mu'(P'_P) & = \mu(a_1) \\ \nu'(P'_P) & = \nu(a_d) \end{cases} \quad \text{whence} \quad \begin{cases} \{P'_P\} & \xrightarrow{\mu} I' \\ \{P'_P\} & \xrightarrow{\nu} J' \end{cases}$$

that is, a quintuple $\Gamma' = (\mu', \nu', I', J', K' = \emptyset)$ for the quiver Q'. One then verifies that

$$f_\Gamma = s(f_{\Gamma'}) \prod_C tr(C'_C) = s(P_{\alpha,\Phi_{\Gamma'}}) \prod_C tr(C'_C)$$
$$= P_{\alpha,s(\Phi_{\Gamma'})} \prod_C tr(C'_C)$$

finishing the proof of the fact that multilinear semi-invariants lie in the linear span of determinantal semi-invariants (and traces of oriented cycles).

The arguments above can be reformulated in a more combinatorial form which is often useful in constructing semi-invariants of a specific weight, as is necessary in the study of the moduli spaces $M^{ss}_\alpha(Q,\theta)$. Let Q be a quiver on the vertices $\{v_1, \ldots, v_k\}$, fix a dimension vector $\alpha = (a_1, \ldots, a_k)$ and a character χ_θ where $\theta = (t_1, \ldots, t_k)$ such that $\theta(\alpha) = 0$. We will call a bipartite quiver Q'



on left vertex-set $L = \{l_1, \ldots, l_p\}$ and right vertex-set $R = \{r_1, \ldots, r_q\}$ and a dimension vector $\beta = (c_1, \ldots, c_p; d_1, \ldots, d_q)$ to be of type (Q, α, θ) if the following conditions are met

- *All left and right vertices correspond to vertices of Q, that is, there are maps*

  $$\begin{cases} L \xrightarrow{\ l\ } \{v_1, \ldots, v_k\} \\ R \xrightarrow{\ r\ } \{v_1, \ldots, v_k\} \end{cases}$$

  possibly occurring with multiplicities, that is there is a map

  $$L \cup R \xrightarrow{\ m\ } \mathbb{N}_+$$

  such that $c_i = m(l_i)a_z$ if $l(l_i) = v_z$ and $d_j = m(r_j)a_z$ if $r(r_j) = v_z$.

- *There can only be an arrow $l_i \longrightarrow r_j$ if for $v_k = l(l_i)$ and $v_l = r(r_i)$ there is an oriented path*

  $$v_k \dashrightarrow v_l$$

  in Q allowing the trivial path and loops if $v_k = v_l$.

- *Every left vertex l_i is the source of exactly c_i arrows in Q' and every right-vertex r_j is the sink of precisely d_j arrows in Q'.*

- *Consider the $u \times u$ matrix where $u = \sum_i c_i = \sum_j d_j$ (both numbers are equal to the total number of arrows in Q') where the i-th row contains the entries of the i-th arrow in Q' with respect to the obvious left and right bases. Observe* *that this is a $GL(\beta)$ semi-invariant on $rep_\beta\ Q'$ with weight determined by the integral $k+l$-tuple $(-1,\ldots,-1;1,\ldots,1)$. If we fix for every arrow a from l_i to r_j in Q' an $m(r_j)\times m(l_i)$ matrix p_a of linear combinations of paths in Q from $l(l_i)$ to $r(r_j)$, we obtain a morphism*

$$rep_\alpha\ Q \longrightarrow rep_\beta\ Q'$$

sending a representation $V\in rep_\alpha\ Q$ to the representation W of Q' defined by $W_a = p_a(V)$. Composing this map with the above semi-invariant we obtain a $GL(\alpha)$ semi-invariant of $rep_\alpha\ Q$ with weight determined by the k-tuple $\theta = (t_1,\ldots,t_k)$ where

$$t_i = \sum_{j\in r^{-1}(v_i)} m(r_j) - \sum_{j\in l^{-1}(v_i)} m(l_j)$$

.

We call such semi-invariants standard determinantal*. Summarizing the arguments of this section we have proved after applying polarization and restitution processes*

Theorem 7.24 *The semi-invariants of the $GL(\alpha)$-action on $rep_\alpha\ Q$ are generated by traces of oriented cycles and by standard determinantal semi-invariants.*

7.8 Brauer-Severi varieties.

In this section we will reconsider the Brauer-Severi scheme $BS_n(A)$ of an algebra, introduced in chapter 2. In the generic case, that is when A is the free algebra $\mathbb{C}\langle x_1,\ldots,x_m\rangle$, we show that it is a moduli space of a certain quiver situation. This then allows us to give the étale local description of $BS_n(A)$ whenever A is a Cayley-smooth algebra. Again, this local description will be a moduli space.

The generic Brauer-Severi scheme of degree n for m-generators, $BS_n^m(gen)$ is defined as follows. Consider the free algebra on m generators $\mathbb{C}\langle x_1,\ldots,x_m\rangle$ and consider the GL_n-action on $rep_n\ \mathbb{C}\langle x_1,\ldots,x_m\rangle\times\mathbb{C}^n = M_n^m\oplus\mathbb{C}^n$ given by

$$g.(A_1,\ldots,A_m,v) = (gA_1g^{-1},\ldots,gA_mg^{-1},gv)$$

and consider the open subset $Brauer^s(gen)$ consisting of those points $(A_1,\ldots,A_m,v)$ where v is a cyclic vector, that is, there is no proper subspace of $\mathbb{C}^n$ containing v and invariant under left multiplication by the matrices A_i. The GL_n-stabilizer is trivial in every point of $Brauer^s(gen)$ whence we can define the orbit space

$$BS_n^m(gen) = Brauer^s(gen)/GL_n$$

Consider the following quiver situation

on two vertices $\{v_1,v_2\}$ such that there are m loops in v_2 and consider the dimension vector $\alpha=(1,n)$. Then, clearly

$$rep_\alpha\ Q = \mathbb{C}^n\oplus M_n^m \simeq rep_n\ \mathbb{C}\langle x_1,\ldots,x_m\rangle\oplus\mathbb{C}^n$$

where the isomorphism is as GL_n-module. On $rep_\alpha\ Q$ we consider the action of the larger group $GL(\alpha)=\mathbb{C}^*\times GL_n$ acting as

$$(\lambda,g).(v,A_1,\ldots,A_m) = (gv\lambda^{-1},gA_1g^{-1},\ldots,gA_mg^{-1})$$

Consider the character χ_θ where $\theta=(-n,1)$, then $\theta(\alpha)=0$ and consider the open subset of θ-semistable representations in $rep_\alpha\ Q$.

Lemma 7.25 *The following are equivalent for $V = (v, A_1, \ldots, A_m) \in rep_\alpha\ Q$*

1. *V is θ-semistable.*
2. *V is θ-stable.*
3. *$V \in Brauer^s(gen)$.*

Consequently,

$$M^{ss}_\alpha(Q, \alpha) \simeq BS^m_n(gen)$$

Proof. 1. $\Rightarrow$ 2. *: If V is θ-semistable it must contain a largest θ-stable subrepresentation W (the first term in the Jordan-Hölder filtration for θ-semistables). In particular, if the dimension vector of W is $\beta = (a, b) < (1, n)$, then $\theta(\beta) = 0$ which is impossible unless $\beta = \alpha$ whence $W = V$ is θ-stable.*

2. $\Rightarrow$ 3. *: Observe that $v \neq 0$, for otherwise V would contain a subrepresentation of dimension vector $\beta = (1, 0)$ but $\theta(\beta) = -n$ is impossible. Assume that v is noncyclic and let $U \hookrightarrow \mathbb{C}^n$ be a proper subspace say of dimension $l < n$ containing v and stable under left multiplication by the A_i, then V has a subrepresentation of dimension vector $\beta' = (1, l)$ and again $\theta(\beta') = l - n < 0$ is impossible.*

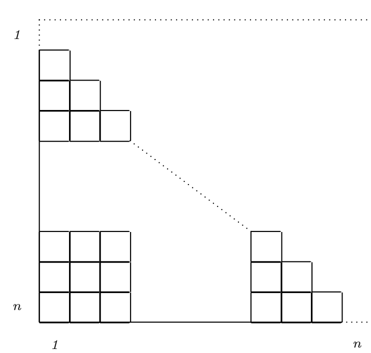
3. $\Rightarrow$ 1. *: By cyclicity of v, the only proper subrepresentations of V have dimension vector $\beta = (0, l)$ for some $0 < l \leq n$, but they satisfy $\theta(\beta) > 0$, whence V is θ-(semi)stable.*

As for the last statement, recall that geometric points of $M^{ss}_\alpha(Q, \alpha)$ classify isomorphism classes of direct sums of θ-stable representations. As there are no proper θ-stable subrepresentations, $M^{ss}_\alpha(Q, \alpha)$ classifies the $GL(\alpha)$-orbits in $Brauer^s(gen)$. Finally, as in chapter 1, there is a one-to-one orbits between the GL_n-orbits as described in the definition of the Brauer-Severi variety and the $GL(\alpha)$-orbits on $rep_\alpha\ Q$. $\square$

By definition, $M^{ss}_\alpha(Q, \theta) = Proj\ \oplus_{n=0}^\infty\ \mathbb{C}[rep_\alpha\ Q]^{GL(\alpha), \chi^n\theta}$ and we can either use the results of section 3 or the previous section to show that these semi-invariants f are generated by brackets, that is,

$$f(V) = det\ \begin{bmatrix} w_1(A_1, \ldots, A_m)v & \ldots & w_n(A_1, \ldots, A_m)v \end{bmatrix}$$

where the w_i are words in the noncommuting variables $x_1, \ldots, x_m$. As in section I.3 we can restrict these n-tuples of words $\{w_1, \ldots, w_n\}$ to sequences arising from multicolored Hilbert n-stairs. That is, the lower triangular part of a square $n \times n$ array

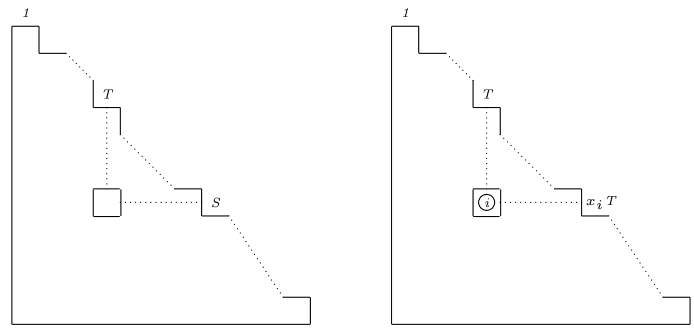


this time filled with colored stones ⓘ where $1 \leq i \leq m$ subject to the two coloring rules

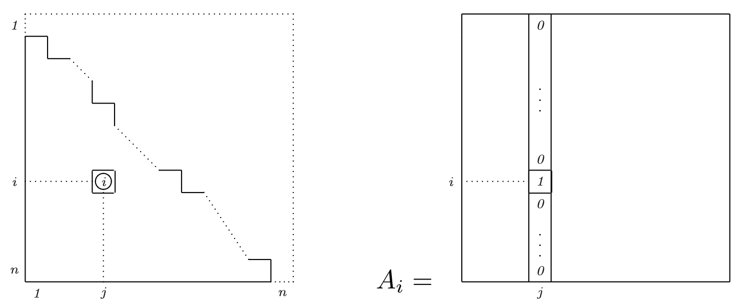
- *each row contains exactly one stone*
- *each column contains at most one stone of each color*

The relevant sequences $W(\sigma) = \{1, w_2, \ldots, w_n\}$ of words are then constructed by placing the identity element 1 at the top of the stair, and descend according to the rule

- *Every go-stone has a top word T which we may assume we have constructed before and a side word S and they are related as indicated below*



In a similar way to the argument in chapter 1 we can cover $M^{ss}_{\alpha}(Q, \alpha) = BS^m_n(gen)$ by open sets determined by Hilbert stairs and find representatives of the orbits in σ-standard form, that is replacing every i-colored stone in σ by a 1 at the same spot in A_i and fill the remaining spots in the same column of A_i by zeroes



As this fixes $(n-1)n$ entries of the $mn^2 + n$ entries of V, one recovers the following result of M. Van den Bergh [31]

Theorem 7.26 *The generic Brauer-Severi variety $BS^m_n(gen)$ of degree n in m generators is a smooth variety which can be covered by affine open subsets each isomorphic to $\mathbb{C}^{(m-1)n^2+n}$.*

For an arbitrary affine $\mathbb{C}$-algebra A, one defines the Brauer stable points to be the open subset of $\underline{rep}_n\ A \times \mathbb{C}^n$

$$Brauer^s_n(A) = \{(\phi, v) \in \underline{rep}_n\ A \times \mathbb{C}^n \ \mid \ \phi(A)v = \mathbb{C}^n\}$$

As Brauer stable points have trivial stabilizer in GL_n all orbits are closed and we can define the Brauer-Severi variety of A of degree n to be the orbit space

$$BS_n(A) = Brauer^s_n(A)/GL_n$$

We claim that Quillen-smooth algebras have smooth Brauer-Severi varieties. Indeed, as the quotient morphism

$$Brauer^s_n(A) \twoheadrightarrow BS_n(A)$$

is a principal GL_n-fibration, the base is smooth whenever the total space is smooth. The total space is an open subvariety of $\underline{rep}_n A \times \mathbb{C}^n$ which is smooth whenever A is Quillen-smooth.

Proposition 7.27 *If A is Quillen-smooth, then for every n we have that the Brauer-Severi variety of A at degree n is smooth.*

Next, we bring in the approximation at level n. Observe that for every affine $\mathbb{C}$-algebra A we have a GL_n-equivariant isomorphism

$$\underline{rep}_n\ A \simeq \underline{rep}_n^{tr}\ A @_n$$

More generally, we can define for every Cayley-Hamilton algebra A of degree n the trace preserving Brauer-Severi variety to be the orbit space of the Brauer stable points in $\underline{rep}_n^{tr}\ A \times \mathbb{C}^n$. We denote this variety with $BS_n^{tr}(A)$. Again, the same argument applies

Proposition 7.28 *If A is Cayley-smooth of degree n , then the trace preserving Brauer-Severi variety $BS_n^{tr}(A)$ is smooth.*

We have seen that the moduli spaces are projective fiber bundles over the variety determined by the invariants,

$$M_\alpha^{ss}(Q,\theta) \twoheadrightarrow iss_\alpha\ Q$$

Similarly, the (trace preserving) Brauer-Severi variety is a projective fiber bundle over the quotient variety of $\underline{rep}_n\ A$, that is, there is a proper map

$$BS_n(A) \xrightarrow{\pi} \underline{iss}_n\ A$$

and we would like to study the fibers of this map. Recall that when A is an order in a central simple algebra of degree n, then the general fiber will be isomorphic to the projective space $\mathbb{P}^{n-1}$ embedded in a higher dimensional $\mathbb{P}^N$. Over non-Azumaya points we expect this $\mathbb{P}^{n-1}$ to degenerate to more complex projective varieties which we would like to describe. To perform this study we need to control the étale local structure of the fiber bundle π in a neighborhood of $\xi \in \underline{iss}_n\ A$. Again, it is helpful to consider first the generic case, that is when $A = \mathbb{C}\langle x_1, \ldots, x_m\rangle$ or $\mathbb{T}_n^m$. In this case, we have seen that the following two fiber bundles are isomorphic

$$BS_n^m(gen) \twoheadrightarrow iss_n\ \mathbb{T}_n^m \quad \text{and} \quad M_\alpha^{ss}(Q,\theta) \twoheadrightarrow iss_\alpha\ Q$$

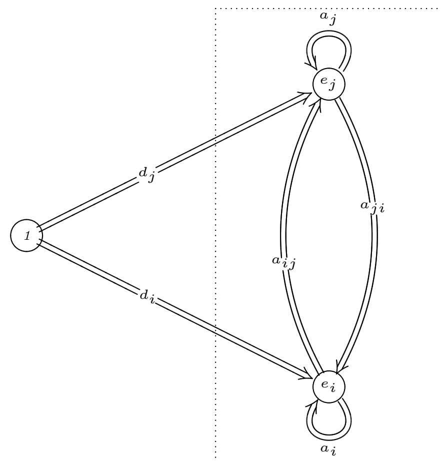
where $\alpha = (1,n)$, $\theta = (-n,1)$ and the quiver

$$\begin{tikzcd} \textcircled{1} \arrow[rr] & & \textcircled{n} \arrow[loop above, "m"] \end{tikzcd} \quad \text{has Euler form} \quad \chi_Q = \begin{bmatrix} 1 & -1 \\ 0 & 1-m \end{bmatrix}$$

A semi-simple α-dimensional representation V_ζ of Q has representation type

$$(1,0) \oplus (0,d_1)^{\oplus e_1} \oplus \ldots \oplus (0,d_k)^{\oplus e_k} \quad \text{with} \quad \sum_i d_i e_i = n$$

and hence corresponds uniquely to a point $\xi \in iss_n\ \mathbb{T}_n^m$ of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$. The étale local structure of $rep_\alpha\ Q$ and of $iss_\alpha\ Q$ near ζ is determined by the local quiver Q_ζ on $k+1$-vertices, say $\{v_0, v_1, \ldots, v_k\}$ with dimension vector $\alpha_\zeta = (1, e_1, \ldots, e_k)$ and where Q_ζ has the following local form for *every triple* (v_0, v_i, v_j) *as can be verified from the Euler-form*



where $a_{ij} = (m-1)d_id_j = a_{ji}$ *and* $a_i = (m-1)d_i^2+1$, $a_j = (m-1)d_j^2+1$. *The dashed part of* Q_ζ *is the same as the local quiver* Q_ξ *describing the étale local structure of* $iss_n\ \mathbb{T}_n^m$ *near* ξ. *Hence, we see that the fibration* $BS_n^m(gen) \twoheadrightarrow iss_n\ \mathbb{T}_n^m$ *is étale isomorphic in a neighborhood of* ξ *to the fibration of the moduli space*

$$M_{\alpha_\zeta}^{ss}(Q_\zeta, \theta_\zeta) \twoheadrightarrow iss_{\alpha_\zeta}\ Q_\zeta \simeq iss_{\alpha_\xi}\ Q_\xi$$

in a neighborhood of the trivial representation and where $\theta_\zeta = (-n, d_1, \ldots, d_k)$. *Another application of the Luna slice results gives the following*

Theorem 7.29 *Let* A *be a Cayley-smooth algebra of degree* n. *Let* $\xi \in iss_n^{tr}\ A$ *correspond to the trace preserving* n*-dimensional semi-simple representation*

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

where the S_i *are distinct simple representations of dimension* d_i *and occurring with multiplicity* e_i. *Then, the projective fibration*

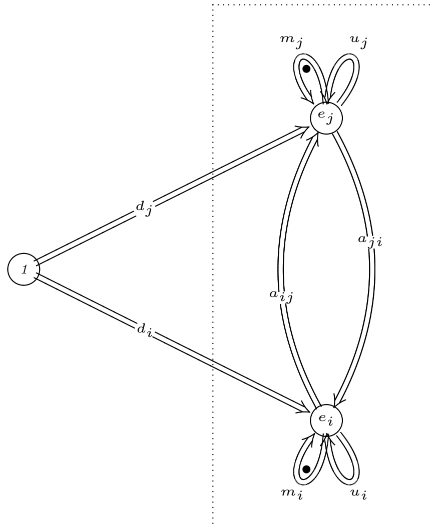
$$BS_n^{tr}(A) \overset{\pi}{\twoheadrightarrow} iss_n^{tr}\ A$$

is étale isomorphic in a neighborhood of ξ *to the fibration of the moduli space*

$$M_{\alpha_\zeta}^{ss}(Q_\zeta^\bullet, \theta_\zeta) \twoheadrightarrow iss_{\alpha_\zeta}\ Q_\zeta^\bullet \simeq iss_{\alpha_\xi}\ Q_\xi^\bullet$$

in a neighborhood of the trivial representation. Here, $Q_\xi^\bullet$ *is the local marked quiver describing the étale local structure of* $rep_n^{tr}\ A$ *near* ξ, *where* $Q_\zeta^\bullet$ *is the extended marked quiver situation, which locally for every triple* (v_0, v_i, v_j) *has the following shape where the dashed region is the local marked quiver* $Q_\xi^\bullet$ *describing*

$Ext^{tr}_A(M_\xi, M_\xi)$ and where $\alpha_\zeta = (1, e_1, \ldots, e_k)$ and $\theta_\zeta = (-n, d_1, \ldots, d_k)$.



In the next chapter we will use this local description to describe the fibers of the Brauer-Severi fibration.

Chapter 8

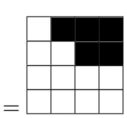
Nullcones.

When A is a Quillen-smooth algebra we have found a local description of the variety $rep_n\ A$ of its n-dimensional representations, of the variety $iss_n\ A$ os isomorphism classes of semi-simple n-dimensional representations and of the Brauer-Severi variety $BS_n\ A$. In this chapter we will develop tools to study the fibers of the structural morphisms

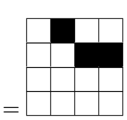
$$\begin{tikzcd} & BS_n\ A \arrow[d, two heads, "\psi"'] \\ rep_n\ A \arrow[r, two heads, "\pi"] & iss_n\ A \end{tikzcd}$$

In particular, we will be able to compute the number and dimensions of their irreducible components allowing to determine the flat locus of these morphisms. The basic observation is that these fibers are nullcones of certain quiver settings, that is, quiver representations on which all polynomial invariants evaluate to zero. What we will do is to give a representation theoretic interpretation of a stratification by vectorbundles over flag varieties of these nullcones, due to W. Hesselink [10].

The strategy is easy to explain in the generic case, that is, of m-tuples of $n \times n$ matrices. Here, $(A_1, \ldots, A_m)$ lies in the nullcone if and only if by applying permutation Jordan-moves simultaneously to the components A_i, they all become strictly upper triangular matrices. Sometimes, we can do better and bring all the non-zero entries of the A_i together in a smaller upper right-hand side corner, such as

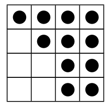
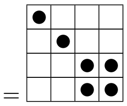
$C =$ 

for 4×4 matrices. For a given m-tuple it is easy to determine the smallest corner which can be obtained by only applying permutation moves. Remains the problem whether we can simultaneously conjugate the m-tuple to produce another tuple $(A'_1 \ldots, A'_m)$ in the orbit which can be permuted to a strictly smaller corner. If this is not possible, we will say that the corner type C is optimal for $(A_1, \ldots, A_m)$. To verify this it is clear that the border region of the corner, such as

$B =$ 

will be relevant. We will assign a new quiver setting to the border region. Observe that there is a parabolic subgroup P of GL_n preserving the corner C and its action on the border region is coming from its Levi subgroup L which is a product of GL_l's.

In the example,

$P =$  *and* $L =$ 

and the corresponding quiver setting is easily seen to be

$$\begin{tikzcd} 1 & 1 \arrow[l, Rightarrow, "m"'] & 2 \arrow[l, Rightarrow, "m"'] \end{tikzcd}$$

The crucial observation is now, to assign a character to $GL(\alpha) = L$ such that an m-tuple $(A_1, \ldots, A_m)$ has optimal corner type C if and only if the representation in $rep_n\ Q = B$ it determines by looking only at the border entries is θ-semistable. By Schofield's criterium we have a combinatorial way to verify whether there are such θ-semistable representations. The corresponding stratum in Hesselinks's stratification is $S = GL_n.U$ where U is the collection of all such m-tuples. We have the following size-reduction of the problem : there is a natural one-to-one correspondence between

- *GL_n-orbits in S, and*
- *P-orbits in U*

Moreover, as U is determined by θ-semistables, there is a moduli space of quiver representation $M^{ss}_\alpha(Q, \theta)$ ar the heart of the stratum.

8.1 Cornering matrices.

In this section we will outline the basic idea of the Hesselink stratification of the nullcone [10] in the generic case, that is, the action of GL_n by simultaneous conjugation on m-tuples of matrices $M^m_n = M_n \oplus \ldots \oplus M_n$. With $Null^m_n$ we denote the nullcone of this action

$$Null^m_n = \{x = (A_1, \ldots, A_m) \in M^m_n \ \mid\ \underline{0} = (0, \ldots, 0) \in \overline{\mathcal{O}(x)}\}$$

By the Hilbert criterium we know that $x = (A_1, \ldots, A_m)$ belongs to the nullcone if and only if there is a one-parameter subgroup $\mathbb{C}^* \xrightarrow{\lambda} GL_n$ such that

$$\lim_{t \to 0} \lambda(t).(A_1, \ldots, A_m) = (0, \ldots, 0).$$

We recall from chapter 4 that any one-parameter subgroup of GL_n is conjugated to one determined by an integral n-tuple $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ by

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}$$

Moreover, permuting the basis if necessary, we can conjugate this λ to one where the n-tuple if dominant, *that is, $r_1 \geq r_2 \geq \ldots \geq r_n$. By applying permutation Jordan-moves, that is, by simultaneously interchanging certain rows and columns in all A_i, we may therefore assume that the limit-formula holds for a dominant one-parameter subgroup λ of the maximal torus*

$$T_n \simeq \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{n} = \{ \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} \mid c_i \in \mathbb{C}^* \} \hookrightarrow GL_n$$

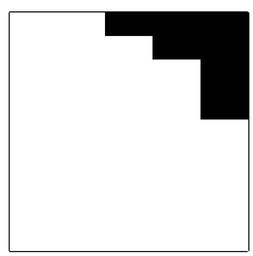
of GL_n. Computing its action on a $n \times n$ matrix A we obtain

$$\begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} t^{-r_1} & & 0 \\ & \ddots & \\ 0 & & r^{-r_n} \end{bmatrix} = \begin{bmatrix} t^{r_1-r_1}a_{11} & \dots & t^{r_1-r_n}a_{1n} \\ \vdots & & \vdots \\ t^{r_n-r_1}a_{n1} & \dots & t^{r_n-r_n}a_{nn} \end{bmatrix}$$

But then, using dominance $r_i \leq r_j$ for $i \geq j$, we see that the limit is only defined if $a_{ij} = 0$ for $i \geq j$, that is, when A is a strictly upper triangular matrix. We have proved our first 'cornering' result.

Lemma 8.1 *Any m-tuple $x = (A_1, \dots, A_m) \in Null_n^m$ has a point in its orbit $\mathcal{O}(x)$ under simultaneous conjugation $(A'_1, \dots, A'_m)$ with all A'_i strictly upper triangular matrices. In fact permutation Jordan-moves suffice.*

For specific m-tuples $x = (A_1, \dots, A_m)$ it might be possible to improve on this result. That is, we want to determine the smallest 'corner' C in the upper right hand corner of the matrix, such that all the component matrices A_i can be conjugated simultaneously to matrices A'_i having only non-zero entries in the corner C

$C \ =$ 

and no strictly smaller corner C' can be found with this property. Our first task will be to compile a list of the relevant corners and to define an order relation on this set. Consider the weight space decomposition of M_n^m for the action by simultaneous conjugation of the maximal torus T_n,

$$M_n^m = \oplus_{1 \leq i,j \leq n} M_n^m(\pi_i - \pi_j) = \oplus_{1 \leq i,j \leq n} \mathbb{C}^{\oplus m}_{\pi_i - \pi_j}$$

where $c = diag(c_1, \dots, c_n) \in T_m$ acts on any element of $M_n^m(\pi_i - \pi_j)$ by multiplication with $c_i c_j^{-1}$, that is, the eigenspace $M_n^m(\pi_i - \pi_j)$ is the space of the (i,j)-entries of the m-matrices. We call

$$\mathcal{W} = \{\pi_i - \pi_j \ \mid \ 1 \leq i,j \leq n\}$$

the set of T_n-weights of M_n^m. Let $x = (A_1, \dots, A_m) \in Null_n^m$ and consider the subset $E_x \subset \mathcal{W}$ consisting of the elements $\pi_i - \pi_j$ such that for at least one of the matrix components A_k the (i,j)-entry is non-zero. Repeating the argument above, we see that if λ is a one-parameter subgroup of T_n determined by the integral n-tuple $(r_1, \dots, r_n) \in \mathbb{Z}^n$ such that $\lim \lambda(t).x = \underline{0}$ we have

$$\forall \ \pi_i - \pi_j \in E_x \quad \text{we have} \quad r_i - r_j \geq 1$$

Conversely, let $E \subset \mathcal{W}$ be a subset of weights, we want to determine the subset

$$\{s = (s_1, \dots, s_n) \in \mathbb{R}^n \ \mid \ s_i - s_j \geq 1 \ \forall \ \pi_i - \pi_j \in E \ \}$$

and determine a point in this set, minimal with respect to the usual norm

$$\parallel s \parallel = \sqrt{s_1^2 + \dots + s_n^2}$$

Let $s = (s_1, \ldots, s_n)$ attain such a minimum. We can partition the entries of s in a disjoint union of strings

$$\{p_i, p_i + 1, \ldots, p_i + k_i\}$$

with $k_i \in \mathbb{N}$ and subject to the condition that all the numbers $p_{ij} \overset{def}{=} p_i + j$ with $0 \leq j \leq k_i$ occur as components of s, possibly with a multiplicity that we denote by a_{ij}. We call a string $string_i = \{p_i, p_i + 1, \ldots, p_i + k_i\}$ of s balanced *if and only if*

$$\sum_{s_k \in string_i} s_j = \sum_{j=0}^{k_i} a_{ij}(p_i + j) = 0$$

In particular, all balanced strings consists entirely of rational numbers. We have

Lemma 8.2 *Let $E \subset \mathcal{W}$, then the subset of $\mathbb{R}^n$ determined by*

$$\mathbb{R}^n_E = \{ \ (r_1, \ldots, r_n) \ \mid \ r_i - r_j \geq 1 \ \forall \ \pi_i - \pi_j \in E\}$$

has a unique point $s_E = (s_1, \ldots, s_n)$ of minimal norm $\parallel s_E \parallel$. This point is determined by the characteristic feature that all its strings are balanced. In particular, $s_E \in \mathbb{Q}^n$.

Proof. *Let s be a minimal point for the norm in $\mathbb{R}^n_E$ and consider a string of s and denote with S the indices $k \in \{1, \ldots, n\}$ such that $s_k \in string$. Let $\pi_i - \pi_j \in E$, then if only one of i or j belongs to S we have a strictly positive number a_{ij}*

$$s_i - s_j = 1 + r_{ij} \quad with \quad r_{ij} > 0$$

Take $\epsilon_0 > 0$ smaller than all r_{ij} and consider the n-tuple

$$s_\epsilon = s + \epsilon(\delta_{1S}, \ldots, \delta_{nS}) \quad with \quad \delta_{kS} = 1 \ if \ k \in S \ and \ 0 \ otherwise$$

with $\mid \epsilon \mid \leq \epsilon_0$. Then, $s_\epsilon \in \mathbb{R}^n_E$ for if $\pi_i - \pi_j \in E$ and i and j both belong to S or both do not belong to S then $(s_\epsilon)_i - (s_\epsilon)_j = s_i - s_j \geq 1$ and if one of i or j belong to S, then

$$(s_\epsilon)_i - (s_\epsilon)_j = 1 + r_{ij} \pm \epsilon \geq 1$$

by the choice of ϵ_0. However, the norm of s_ϵ is

$$\parallel s_\epsilon \parallel = \sqrt{\parallel s \parallel + 2\epsilon \sum_{k \in S} s_k + \epsilon^2 \# S}$$

Hence, if the string would not be balanced, $\sum_{k \in S} s_k \neq 0$ and we can choose ϵ small enough such that $\parallel s_\epsilon \parallel < \parallel s \parallel$, contradicting minimality of s. $\square$

For given n we can compile a list $\mathcal{S}_n$ of all dominant n-tuples $(s_1, \ldots, s_n)$ (that is, $s_i \leq s_j$ whenever $i \geq j$) having all its strings balanced, as follows.

- *List all Young-diagrams $\mathcal{Y}_n = \{Y_1, \ldots\}$ having $\leq n$ boxes.*
- *For every diagram Y_l fill the boxes with strictly positive integers subject to the rules*

  1. *the total sum is equal to n*
  2. *no two rows are filled identically*
  3. *at most one row has length 1*

This gives a list $\mathcal{T}_n = \{T_1, \ldots\}$ of tableaux.

- *For every tableau $T_l \in \mathcal{T}_n$, for each of its rows $(a_1, a_2, \ldots, a_k)$ find a solution p to the linear equation*

$$a_1 x + a_2(x+1) + \ldots + a_k(x+k) = 0$$

and define the $\sum a_i$-tuple of rational numbers

$$(\underbrace{p, \ldots, p}_{a_1}, \underbrace{p+1, \ldots, p+1}_{a_2}, \ldots \underbrace{p+k, \ldots, p+k}_{a_k})$$

Repeating this process for every row of T_l we obtain an n-tuple, which we then order.

For example, for $n = 5$ and the tableaux $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$ *the linear equations are*

$$\begin{cases} 2x + x + 1 & = 0 \quad \textit{giving } p_1 = -\frac{1}{3} \\ x + x + 1 & = 0 \quad \textit{giving } p_2 = -\frac{1}{2} \end{cases}$$

The corresponding 5-tuple is therefore $s = (\frac{2}{3}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2})$. The list $\mathcal{S}_n$ will be the combinatorial object underlying the relevant corners and the stratification of the nullcone.

Example 8.3 ($\mathcal{S}_n$ for small n)
For $n = 2$, we have $\boxed{1}\boxed{1}$ giving $(\frac{1}{2}, -\frac{1}{2})$ and $\boxed{2}$ giving $(0, 0)$. For $n = 3$ we have five types

$\mathcal{S}_3 =$

tableau	s_1	s_2	s_3	$\parallel s \parallel^2$
$\boxed{1}\boxed{1}\boxed{1}$	1	0	-1	2
$\boxed{1}\boxed{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
$\boxed{2}\boxed{1}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
$\boxed{1}\boxed{1}$ $\boxed{1}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
$\boxed{3}$	0	0	0	0

$\mathcal{S}_4$ has eleven types

$\mathcal{S}_4 =$

tableau	s_1	s_2	s_3	s_4	$\parallel s \parallel^2$
$\boxed{1}\boxed{1}\boxed{1}\boxed{1}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	5
$\boxed{2}\boxed{1}\boxed{1}$	$\frac{5}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{11}{4}$
$\boxed{1}\boxed{1}\boxed{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$\frac{11}{4}$
$\boxed{1}\boxed{2}\boxed{1}$	1	0	0	-1	2
$\boxed{2}\boxed{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
$\boxed{3}\boxed{1}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$
$\boxed{1}\boxed{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$
$\boxed{1}\boxed{2}$ $\boxed{1}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$
$\boxed{2}\boxed{1}$ $\boxed{1}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
$\boxed{1}\boxed{1}$ $\boxed{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
$\boxed{4}$	0	0	0	0	0

Observe that we ordered the elements in $\mathcal{S}_n$ according to $\parallel s \parallel$. The reader is invited to verify that $\mathcal{S}_5$ has 28 different types.

To every $s = (s_1, \ldots, s_n) \in \mathcal{S}_n$ we associate the following data

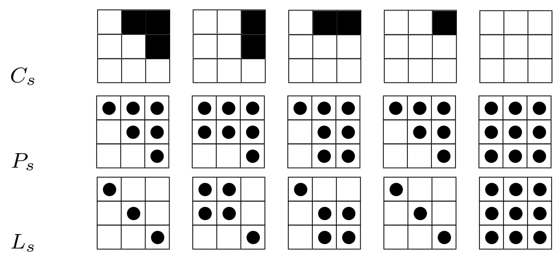
- *the* corner C_s *is the subspace of* M_n^m *consisting of those* m *tuples of* $n \times n$ *matrices with zero entries except perhaps at position* (i,j) *where* $s_i - s_j \geq 1$. *A partial ordering is defined on these corners by the rule*

$$C_{s'} < C_s \;\Leftrightarrow\; \parallel s' \parallel \;<\; \parallel s \parallel$$

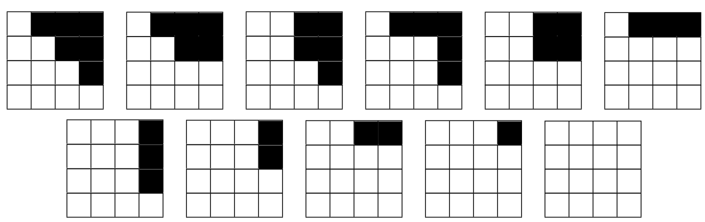
- *the* parabolic subgroup P_s *which is the subgroup of* GL_n *consisting of matrices with zero entries except perhaps at entry* (i,j) *when* $s_i - s_j \geq 0$.

- *the* Levi subgroup L_s *which is the subgroup of* GL_n *consisting of matrices with zero entries except perhaps at entry* (i,j) *when* $s_i - s_j = 0$. *Observe that* $L_s = \prod GL_{a_{ij}}$ *where the* a_{ij} *are the multiplicities of* $p_i + j$.

Example 8.4 Using the sequence of types in the previous example, we have that the relevant corners and subgroup for 3×3 matrices are



For 4×4 matrices the relevant corners are



Returning to the corner-type of an m*-tuple* $x = (A_1, \ldots, A_m) \in Null_n^m$, *we have seen that* $E_x \subset \mathcal{W}$ *determines a unique* $s_{E_x} \in \mathbb{Q}^n$ *which up to permuting the entries an element* s *of* $\mathcal{S}_n$. *As permuting the entries of* s *translates into permuting rows and columns in* $M_n(\mathbb{C})$ *we have*

Theorem 8.5 *Every* $x = (A_1, \ldots, A_m) \in Null_n^m$ *can be brought by permutation Jordan-moves to an* m*-tuple* $x' = (A'_1, \ldots, A'_m) \in C_s$. *Here,* s *is the dominant reordering of* s_{E_x} *with* $E_x \subset \mathcal{W}$ *the subset* $\pi_i - \pi_j$ *determined by the non-zero entries at place* (i,j) *of one of the components* A_k. *The permutation of rows and columns is determined by the dominant reordering.*

The m*-tuple* s *(or* s_{E_x}*) determines a one-parameter subgroup* λ_s *of* T_n *where* λ *corresponds to the unique* n*-tuple of integers*

$$(r_1, \ldots, r_n) \in \mathbb{N}_+ s \cap \mathbb{Z}^n \quad with \quad gcd(r_i) = 1$$

For any one-parameter subgroup μ *of* T_n *determined by an integral* n*-tuple* $\mu = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ *and any* $x = (A_1, \ldots, A_n) \in Null_n^m$ *we define the integer*

$$m(x,\mu) = min\ \{a_i - a_j \mid x \text{ contains a non-zero entry in } M_n^m(\pi_i - \pi_j)\ \}$$

From the definition of $\mathbb{R}_E^n$ *it follows that the minimal value* s_E *and* λ_{s_E} *is*

$$s_{E_x} = \frac{\lambda_{s_{E_x}}}{m(x, \lambda_{s_{E_x}})} \quad and \quad s = \frac{\lambda_s}{m(x,\lambda_s)}$$

We can now state to what extend λ_s *is an optimal one-parameter subgroup of* T_n.

Theorem 8.6 *Let $x = (A_1, \ldots, A_m) \in Null_n^m$ and let μ be a one-parameter subgroup contained in T_n such that $\lim\limits_{t \to 0} \lambda(t).x = \underline{0}$, then*

$$\frac{\| \lambda_{s_{E_x}} \|}{m(x, \lambda_{s_{E_x}})} \leq \frac{\| \mu \|}{m(x, \mu)}$$

The proof follows immediately from the observation that $\frac{\mu}{m(x,\mu)} \in \mathbb{R}^n_{E_x}$ and the minimality of s_{E_x}. Phrased differently, there is no simultaneous reordering of rows and columns that admit an m-tuple $x" = (A"_1, \ldots, A"_m) \in C_{s'}$ for a corner $C_{s'} < C_s$.

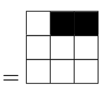
8.2 Optimal corners.

In the foregoing section we have transformed an m-tuple $x = (A_1, \ldots, A_m) \in Null_n^m$ by interchanging rows and columns to an m-tuple in corner-form C_s. However, it is still possible that another point in the orbit $\mathcal{O}(x)$ say $y = g.x = (B_1, \ldots, B_m)$ can be transformed by interchanging rows and columns in a smaller corner.

Example 8.7 Consider one 3×3 nilpotent matrix of the form

$$x = \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad ab \neq 0$$

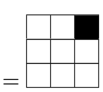
Then, $E_x = \{\pi_1 - \pi_2, \pi_1 - \pi_3\}$ and the corresponding $s = s_{E_x} = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ so x is clearly of corner type

$C_s =$ 

However, x is a nilpotent matrix of rank 1 and by the Jordan-normalform we can conjugate it in standard form, that is, there is some $g \in GL_3$ such that

$$y = g.x = gxg^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For this y we have $E_y = \{\pi_1 - \pi_2\}$ and the corresponding $s_{E_y} = (\frac{1}{2}, -\frac{1}{2}, 0)$, which can be brought into standard dominant form $s' = (\frac{1}{2}, 0, -\frac{1}{2})$ by interchanging the two last entries. Hence, by interchanging the last two rows and columns, y is indeed of corner type

$C_{s'} =$ 

and we have that $C_{s'} < C_s$.

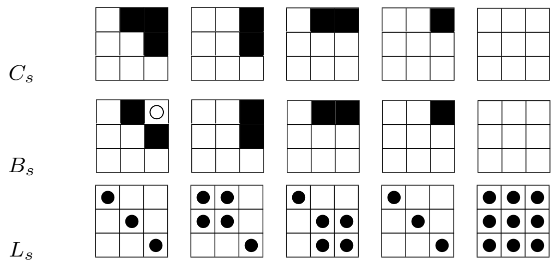
Trivial as this example seems, we needed the Jordan-normalform to produce it. As there are no known canonical forms for m tuples of $n \times n$ matrices, it is a difficult problem to determine the optimal corner type in general.

Definition 8.8 We say that $x = (A_1, \ldots, A_m) \in Null_n^m$ is of *optimal corner type* C_s if after reordering rows and columns, x is of corner type C_s and there is no point $y = g.x$ in the orbit which is of corner type $C_{s'}$ with $C_{s'} < C_s$.

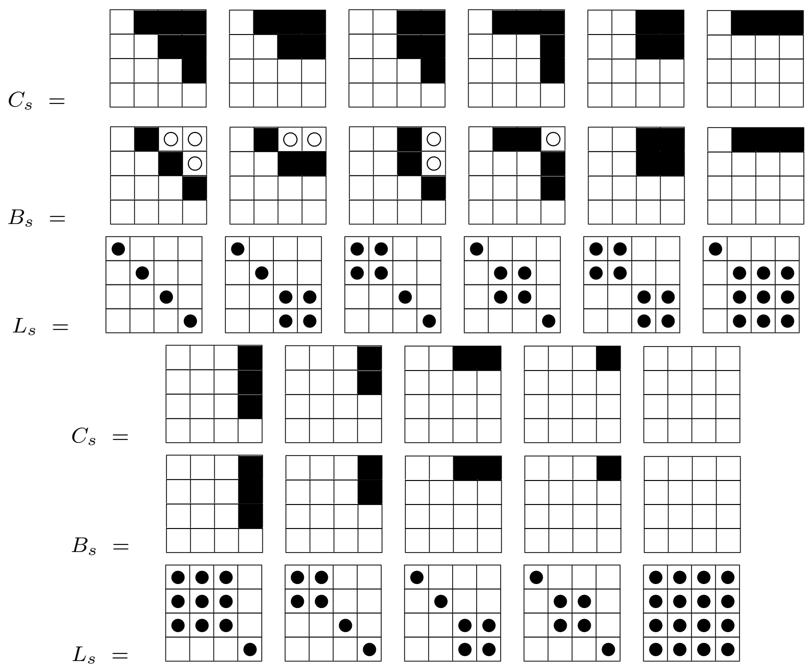
Using the results of the foregoing chapter we can give an elegant solution to the problem of determining the optimal corner type of an m-tuple in $Null_n^m$. We assume that $x = (A_1, \ldots, A_m)$ is brought into corner type C_s with $s = (s_1, \ldots, s_n) \in \mathcal{S}_n$. We will associate a quiver-representation to x. As we are interested in checking whether we can transform x to a smaller corner-type, it is intuitively clear that the border region of C_s will be important.

- *the* border B_s *is the subspace of* C_s *consisting of those* m*-tuples of* $n \times n$ *matrices with zero entries except perhaps at entries* (i,j) *where* $s_i - s_j = 1$.

Example 8.9 For 3×3 matrices we have the following corner-types C_s having border-regions B_s and associated Levi-subgroups L_s



For 4×4 matrices the relevant data are as follows



From these examples, it is clear that the action of the Levi-subgroup L_s *on the border* B_s *is a quiver-setting. In general, let* $s \in \mathcal{S}_n$ *be determined by the tableau* T_s*, the associated quiver-setting* (Q_s, α_s) *is*

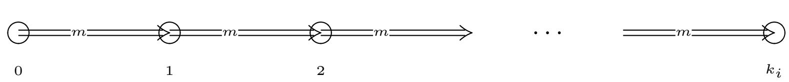
- Q_s *is the quiver having as many connected components as there are rows in the tableau* T_s*. If the i-th row in* T_s *is*

$$(a_{i0}, a_{i1}, \ldots, a_{ik_i})$$

then the corresponding string of entries in s *is of the form*

$$\{\underbrace{p_i, \ldots, p_i}_{a_{i0}}, \underbrace{p_i+1, \ldots, p_i+1}_{a_{i1}}, \ldots, \underbrace{p_i+k_i, \ldots, p_i+k_i}_{a_{ik_i}}\}$$

and the i-th component of Q_s *is defined to be the quiver* Q_i *on* $k_i + 1$ *vertices having* m *arrows between the consecutive vertices, that is* Q_i *is*



- *the dimension vector* α_i *for the* i*-th component quiver* Q_i *is equal to the* i*-th row of the tableau* T_s*, that is*

$$\alpha_i = (a_{i0}, a_{i1}, \ldots, a_{ik_i})$$

and the total dimension vector α_s *is the collection of these component dimension vectors.*

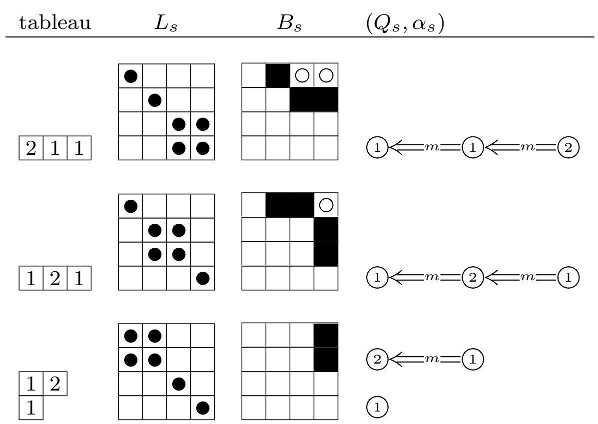
With these notations we have

Proposition 8.10 *The action of the Levi-subgroup* $L_s = \prod_{i,j} GL_{a_{ij}}$ *on the border* B_s *coincides with the base-change action of* $GL(\alpha_s)$ *on the representation space* $rep_{\alpha_s}\, Q_s$*. The isomorphism*

$$B_s \longrightarrow rep_{\alpha_s}\, Q_s$$

is given by sending an m*-tuple of border* B_s*-matrices* $(A_1, \ldots, A_m)$ *to the representation in* $rep_{\alpha_s}\, Q_s$ *where the* j*-th arrow between the vertices* v_a *and* v_{a+1} *of the* i*-th component quiver* Q_i *is given by the relevant block in the matrix* A_j*.*

Example 8.11 Let us give a couple of examples for 4×4 matrices



Finally, we associate to $s \in \mathcal{S}_n$ *a character* χ_s *of the Levi-subgroup* $L_s = GL(\alpha_s)$

- *the character* $GL(\alpha_s) \xrightarrow{\chi_s} \mathbb{C}^*$ *is determined by the integral* n*-tuple* $\theta_s = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ *where if entry* k *corresponds to the* j*-th vertex of the* i*-th component of* Q_s *we have*

$$t_k = n_{ij} \overset{def}{=} d.(p_i + j)$$

where d *is the least common multiple of the numerators of the* p_i*'s for all* i*. Equivalently, the* n_{ij} *are the integers appearing in the description of the one-parameter subgroup* $\lambda_s = (r_1, \ldots, r_n)$ *grouped together according to the ordering of vertices in the quiver* Q_s*. Recall that the character* χ_s *is then defined to be*

$$\chi_s(g_1 \ldots, g_n) = \prod_{i=1}^{n} det(g_i)^{t_i}$$

or in terms of $GL(\alpha_s)$ *it sends an element* $g_{ij} \in GL(\alpha_s)$ *to* $\prod_{i,j} det(g_{ij})^{n_{ij}}$*.*

Example 8.12 For the $n = 4$ examples above we obtain the following characters (indicated as top-labels of the vertices)

tableau	t_1	t_2	t_3	t_4	$(Q_s, \alpha_s, \theta_s)$
$\boxed{2}\boxed{1}\boxed{1}$	5	1	-3	-3	$\overset{5}{1} \xLeftarrow{m} \overset{1}{1} \xLeftarrow{m} \overset{-3}{2}$
$\boxed{1}\boxed{2}\boxed{1}$	1	0	0	-1	$\overset{1}{1} \xLeftarrow{m} \overset{0}{2} \xLeftarrow{m} \overset{-1}{1}$
$\begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{1}\end{array}$	1	1	0	-2	$\overset{1}{2} \xLeftarrow{m} \overset{-2}{1}$ $\overset{0}{1}$

Observe that $\theta_s(\alpha_s) = 0$.

Using these conventions we can now state the main result of this section, giving a solution to the problem of optimal corners.

Theorem 8.13 *Let $x = (A_1, \ldots, A_m) \in Null_n^m$ be of corner type C_s. Then, x is of optimal corner type C_s if and only if under the natural maps*

$$C_s \twoheadrightarrow B_s \xrightarrow{\simeq} rep_{\alpha_s}\ Q_s$$

(the first map forgets the non-border entries) x is mapped to a θ_s-semistable representation in $rep_{\alpha_s}\ Q_s$.

8.3 The Hesselink stratification.

We have seen that every orbit in $Null_n^m$ has a representative $x = (A_1, \ldots, A_m)$ with all A_i strictly upper triangular matrices. That is, if $N \subset M_n$ is the subspace of strictly upper triangular matrices, then the action map determines a surjection

$$GL_n \times N^m \overset{ac}{\twoheadrightarrow} Null_n^m$$

Recall that the standard Borel subgroup B is the subgroup of GL_n consisting of all upper triangular matrices and consider the action of B on $GL_n \times M_n^m$ determined by

$$b.(g, x) = (gb^{-1}, b.x)$$

Then, B-orbits in $GL_n \times N^m$ are mapped under the action map ac to the same point in the nullcone $Null_n^m$. Consider the morphisms

$$GL_n \times M_n^m \overset{\pi}{\twoheadrightarrow} GL_n/B\ \times M_n^m$$

which sends a point (g, x) to $(gB, g.x)$. The quotient GL_n/B is called a flag variety *and is a projective manifold. Its points are easily seen to correspond to complete* flags

$$\mathcal{F}\ :\ 0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n \quad with \quad dim_{\mathbb{C}}\ F_i = i$$

of subspaces of $\mathbb{C}^n$. For example, if $n = 2$ then $GL_2/B \simeq \mathbb{P}^1$. Consider the fiber π^{-1} of a point $(\overline{g}, (B_1, \ldots, B_m)) \in GL_n/B \times M_n^m$. These are the points

$$(h, (A_1, \ldots, A_m)) \quad such\ that \quad \begin{cases} g^{-1}h & = b \in B \\ bA_ib^{-1} & = g^{-1}B_ig \quad for\ all\ 1 \leq i \leq m. \end{cases}$$

Therefore, the fibers of π are precisely the B-orbits in $GL_n \times M_n^m$. That is, there exists a quotient variety for the B-action on $GL_n \times M_n^m$ which is the trivial vector-bundle of rank mn^2

$$\mathcal{T} = GL_n/B \times M_n^m \overset{p}{\twoheadrightarrow} GL_n/B$$

over the flag variety GL_n/B. We will denote with $GL_n \times^B N^m$ the image of the subvariety $GL_n \times N^m$ of $GL_n \times M_n^m$ under this quotient map. That is, we have a commuting diagram

$$\begin{tikzcd} GL_n \times N^m \arrow[r, hook] \arrow[d] & GL_n \times M_n^m \arrow[d] \\ GL_n \times^B N^m \arrow[r, hook] & GL_n/B \times M_n^m \end{tikzcd}$$

Hence, $\mathcal{V} = GL_n \times^B N^m$ is a sub-bundle of rank $m.\frac{n(n-1)}{2}$ of the trivial bundle $\mathcal{T}$ over the flag variety. Note however that $\mathcal{V}$ itself is not trivial as the action of GL_n does not map N^m to itself.

Theorem 8.14 *Let U be the open subvariety of m-tuples of strictly upper triangular matrices N^m consisting of those tuples such that one of the component matrices has rank $n-1$. the action map ac induces a commuting diagram*

$$\begin{tikzcd} GL_n \times^B U \arrow[r, "\simeq"] \arrow[d, hook] & GL_n.U \arrow[d, hook] \\ GL_n \times^B N^m \arrow[r, "ac"] & Null_n^m \end{tikzcd}$$

where the upper map is an isomorphism of GL_n-varieties if we define the action on fiber bundles to be given by left multiplication in the first component. Therefore, there is a natural one-to-one correspondence between GL_n-orbits in $GL_n.U$ and B-orbits in U. Further, ac is a desingularization of the nullcone and $Null_n^m$ is irreducible of dimension

$$(m+1)\frac{n(n-1)}{2}.$$

Proof. *Let $A \in N$ be a strictly upper triangular matrix of rank $n-1$ and $g \in GL_n$ such that $gAg^{-1} \in N$, then $g \in B$ as one verifies by first bringing A into Jordan-normal form $J_n(0)$. This implies that over a point $x = (A_1, \ldots, A_m) \in U$ the fiber of the action map*

$$GL_n \times N^m \overset{ac}{\twoheadrightarrow} Null_n^m$$

has dimension $\frac{n(n-1)}{2} = dim\ B$. Over all other points the fiber has at least dimension $\frac{n(n-1)}{2}$.But then, by the dimension formula we have

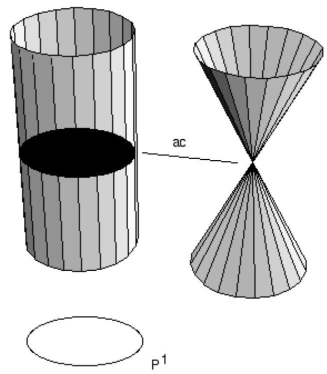
$$dim\ Null_n^m = dim\ GL_n + dim\ N^m - dim\ B = (m+1)\frac{n(n-1)}{2}$$

Over $GL_n.U$ this map is an isomorphism of GL_n-varieties. Irreducibility of $Null_n^m$ follows from surjectivity of ac as $\mathbb{C}[Null_n^m] \hookrightarrow \mathbb{C}[GL_n] \otimes \mathbb{C}[N^m]$ and the latter is a domain. These facts imply that the induced action map

$$GL_n \times^B N^m \xrightarrow{ac} Null_n^m$$

is birational and as the former is a smooth variety (being a vectorbundle over the flag manifold), this is a desingularization. □

Example 8.15 Let $n = 2$ and $m = 1$. We have seen in chapter 3 that $Null_2^1$ is a cone in 3-space with the singular top the orbit of the zero-matrix and the open complement the orbit of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.



In this case the flag variety is $\mathbb{P}^1$ and the fiber bundle $GL_2 \times^B N$ has rank one. The action map can be depicted as above and is a GL_2-isomorphism over the complement of the fiber of the top.

The foregoing theorem gives us a reduction in the complexity, both in the dimension of the acting group as in the dimension of the space acted upon, of the study of

- *GL_n-orbits in the nullcone $Null_n^m$, to*
- *B-orbits in N^m.*

at least on the stratum $GL_n.U$ described before. The aim of the Hesselink stratification of the nullcone is to extend this reduction also to the complement.

Let $s \in \mathcal{S}_n$ and let C_s be the vectorspace of all m-tuples in M_n^m which are of corner-type C_s. We have seen that there is a Zariski open subset (but, possibly empty) U_s of C_s consisting of m-tuples of optimal corner type C_s. Observe that the action of conjugation of GL_n on M_n^m induces an action of the associated parabolic subgroup P_s on C_s.

- *The Hesselink stratum S_s associated to s is the subvariety $GL_n.U_s$ where U_s is the open subset of C_s consisting of the optimal C_s-type tuples*

The results of the foregoing section allow us to prove, similar to the foregoing result, the following reduction of complexity result from

- *GL_n-orbits in the Hesselink stratum S_s to*
- *P_s-orbits in optimal corner tuples U_s.*

Theorem 8.16 *With notations as before we have a commuting diagram*

$$\begin{tikzcd} GL_n \times^{P_s} U_s \arrow[r, "\simeq"] \arrow[d, hook] & S_s \arrow[d, hook] \\ GL_n \times^{P_s} C_s \arrow[r, "ac"] & \overline{S_s} \end{tikzcd}$$

where ac is the action map, $\overline{S_s}$ is the Zariski closure of S_s in $Null_n^m$ and the upper map is an isomorphism of GL_n-varieties. Here, GL_n/P_s is the flag variety associated to the parabolic subgroup P_s and is a projective manifold. The variety

$GL_n \times^{P_s} C_s$ is a vectorbundle over the flag variety GL_n/P_s and is a subbundle of the trivial bundle $GL_n \times^{P_s} M_n^m$. Therefore, the Hesselink stratum S_s is an irreducible smooth variety of dimension

$$\begin{aligned} dim\ S_s &= dim\ GL_n/P_s + rk\ GL_n \times^{P_s} C_s \\ &= n^2 - dim\ P_s + dim_{\mathbb{C}}\ C_s \end{aligned}$$

and there is a natural one-to-one correspondence between the GL_n-orbits in S_s and the P_s-orbits in U_s. Moreover, the vectorbundle $GL_n \times^{P_s} C_s$ is a desingularization of $\overline{S_s}$ hence 'feels' the gluing of S_s to the remaining strata. Finally, the ordering of corners has the geometric interpretation

$$\overline{S_s} \subset \bigcup_{\|s'\| \leq \|s\|} S_{s'}$$

In the previous section we have seen that $U_s = p^{-1}\ rep^{ss}_{\alpha_s}(Q_s, \theta_s)$ where $C_s \overset{p}{\twoheadrightarrow} B_s$ is the canonical projection forgetting the non-border entries. As the action of the parabolic subgroup P_s restricts to the action of its Levi-part L_s on $B_s = rep_{\alpha_s}\ Q$ we have a canonical projection

$$U_s/P_s \overset{p}{\twoheadrightarrow} M^{ss}_{\alpha_s}(Q_s, \theta_s)$$

to the moduli space of θ_s-semistable representations in $rep_{\alpha_s}\ Q_s$. As none of the components of Q_s admits cycles, these moduli spaces are projective varieties. For small values of m and n these moduli spaces give good approximations to the study of the orbits in the nullcone.

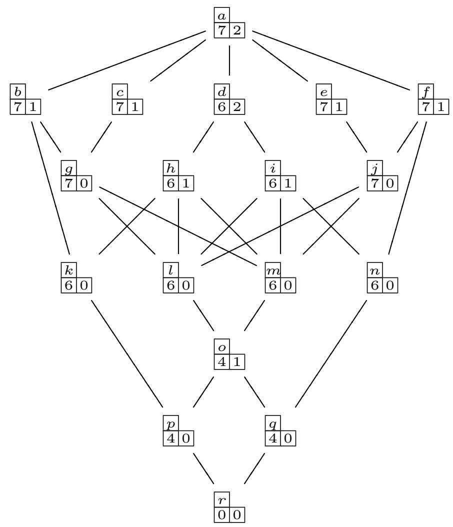
Example 8.17 (Nullcone of m-tuples of 2×2 matrices)

In chapter 3 we have proved by brute force that the orbits in $Null_2^2$ correspond to points on $\mathbb{P}^1$ together with one extra orbit, the zero representation. For arbitrary m, the relevant strata-information for $Null_2^m$ is contained in the following table

tableau	s	$B_s = C_s$	P_s	$(Q_s, \alpha_s, \theta_s)$
$\boxed{1}\boxed{1}$	$(\frac{1}{2}, -\frac{1}{2})$	$\begin{bmatrix} & \blacksquare \\ & \end{bmatrix}$	$\begin{bmatrix} \bullet & \bullet \\ & \bullet \end{bmatrix}$	$\overset{1}{(1)} \xLeftarrow{m} \overset{-1}{(1)}$
$\boxed{2}$	$(0,0)$	$\begin{bmatrix} & \\ & \end{bmatrix}$	$\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}$	$\overset{0}{(2)}$

Because $B_s = C_s$ we have that the orbit space $U_s/P_s \simeq M^{ss}_{\alpha_s}(Q_s, \theta_s)$. For the first stratum, every representation in $rep_{\alpha_s}\ Q_s$ is θ_s-semistable except the zero-representation (as it contains a subrepresentation of dimension $\beta = (1,0)$ and $\theta_s(\beta) = -1 < 0$. The action of $L_s = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^m - \underline{0}$ has as orbit space $\mathbb{P}^{m-1}$, classifying the orbits in the maximal stratum. The second stratum consists of one point, the zero representation.

Example 8.18 A more interesting application, illustrating all of the general phenomena, is the description of orbits in the nullcone of two 3×3 matrices. As we mentioned in chapter 4, H. Kraft described them in [14, p. 202] by brute force. The orbit space decomposes as a disjoint union of tori and can be represented by the picture

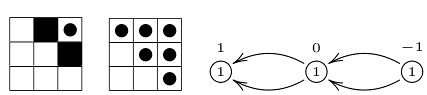
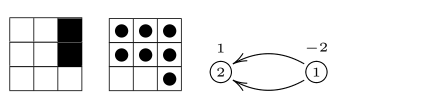
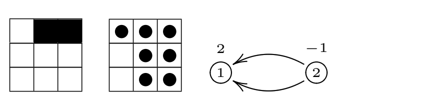
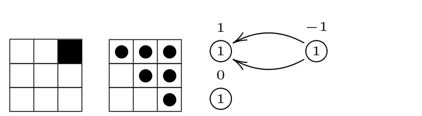
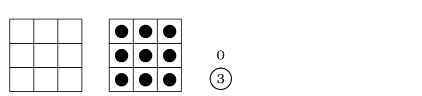


Here, each node corresponds to a torus of dimension the right-hand side number in the bottom row. A point in this torus represents an orbit with dimension the left-hand side number. The top letter is included for classification purposes. That is, every orbit has a unique representant in the following list of couples of 3×3 matrices (A, B). The top letter gives the torus, the first 2 rows give the first two rows of A and the last two rows give the first two rows of B, $x, y \in \mathbb{C}^*$

a	b	c	d	e	f	g	h	i
0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 0 0	0 1 0	0 1 0	0 0 x
0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 0	0 0 1	0 0 0
0 x 0	0 0 0	0 x 0	0 x y	0 x 0	0 1 0	0 0 0	0 0 x	0 1 0
0 0 y	0 0 x	0 0 0	0 0 x	0 0 0	0 0 x	0 0 1	0 0 0	0 0 1

j	k	l	m	n	o	p	q	r
0 0 0	0 0 1	0 0 0	0 0 1	0 0 0	0 1 0	0 1 0	0 0 0	0 0 0
0 0 1	0 0 0	0 0 1	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0
0 1 0	0 1 0	0 0 1	0 1 0	0 1 0	0 x 0	0 0 0	0 1 0	0 0 0
0 0 0	0 0 0	0 0 0	0 0 0	0 0 1	0 0 0	0 0 0	0 0 0	0 0 0

We will now derive this result from the above description of the Hesselink stratification. To begin, the relevant data concerning $\mathcal{S}_3$ is summarized in the following table

tableau	s	B_s, C_s P_s $(Q_s, \alpha_s, \theta_s)$
1 1 1	$(1, 0, -1)$	
1 2	$(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$	
2 1	$(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$	
1 1 / 1	$(\frac{1}{2}, 0, -\frac{1}{2})$	
3	$(0, 0, 0,)$	

For the last four corner types, $B_s = C_s$ whence the orbit space U_s/P_s is isomorphic to the moduli space $M^{ss}_{\alpha_s}(Q_s, \theta_s)$. Consider the quiver-setting

$$\begin{tikzcd} \overset{1}{\textcircled{2}} & \overset{-2}{\textcircled{1}} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

If the two arrows are not linearly independent, then the representation contains a proper subrepresentation of dimension-vector $\beta = (1,1)$ or $(1,0)$ and in both cases $\theta_s(\beta) < 0$ whence the representation is not θ_s-semistable. If the two arrows are linearly independent, we can use the GL_2-component to bring them in the form $(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$, whence $M^{ss}_{\alpha_s}(Q_s, \alpha_s)$ is reduced to one point, corresponding to the matrix-couple of type l

$$(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

A similar argument, replacing linear independence by common zero-vector shows that also the quiver-setting corresponding to the tableau $\boxed{2}\boxed{1}$ has one point as its moduli space, the matrix-tuple of type k. Incidentally, this shows that the corners corresponding to the tableaux $\boxed{2}\boxed{1}$ or $\boxed{1}\boxed{2}$ cannot be optimal when $m = 1$ as then the row or column vector always has a kernel or cokernel whence cannot be θ_s-semistable. This of course corresponds to the fact that the only orbits in $Null^1_3$ are those corresponding to the Jordan-matrixes

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which are respectively of corner type $\boxed{1}\boxed{1}\boxed{1}$, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ and $\boxed{3}$, whence the two other types do not occur. Next, consider the quiver setting

$$\begin{tikzcd} \overset{1}{\textcircled{1}} & \overset{-1}{\textcircled{1}} \arrow[l, bend right] \arrow[l, bend left] \\ \overset{0}{\textcircled{1}} & \end{tikzcd}$$

A representation in $rep_{\alpha_s}\ Q_s$ is θ_s-semistable if and only if the two maps are not both zero (otherwise, there is a subrepresentation of dimension $\beta = (1,0)$ with $\theta_s(\beta) < 0$). The action of $GL(\alpha_s) = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2 - \underline{0}$ has a s orbit space $\mathbb{P}^1$ and they are represented by matrix-couples

$$(\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$$

with $[a : b] \in \mathbb{P}^1$ giving the types o,p and q. Clearly, the stratum $\boxed{3}$ consists just of the zero-matrix, which is type r. Remains to investigate the quiver-setting

$$\begin{tikzcd} \overset{1}{\textcircled{1}} & \overset{0}{\textcircled{1}} \arrow[l, bend right, "c"'] \arrow[l, bend left, "d"] & \overset{-1}{\textcircled{1}} \arrow[l, bend right, "a"'] \arrow[l, bend left, "b"] \end{tikzcd}$$

Again, one easily verifies that a representation in $rep_{\alpha_s}\ Q_s$ is θ_s-semistable if and only if $(a,b) \neq (0,0) \neq (c,d)$ (for otherwise one would have subrepresentations of dimensions $(1,1,0)$ or $(1,0,0)$). The corresponding $GL(\alpha_s)$-orbits are classified by

$$M^{ss}_{\alpha_s}(Q_s.\theta_s) \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

corresponding to the matrix-couples of types a, b, c, e, f, g, j, k and n

$$(\begin{bmatrix} 0 & c & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix})$$

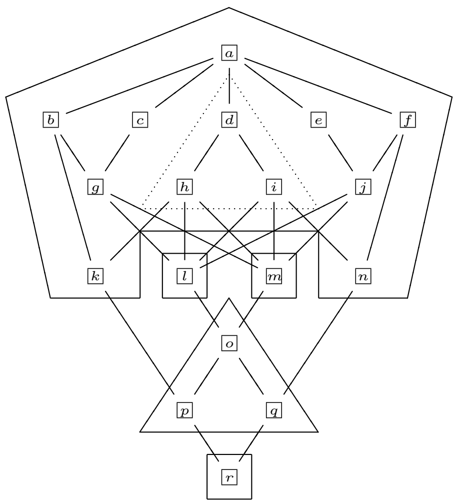
where $[a : b]$ and $[c : d]$ are points in $\mathbb{P}^1$. In this case, however, $C_s \neq B_s$ and we need to investigate the fibers of the projection

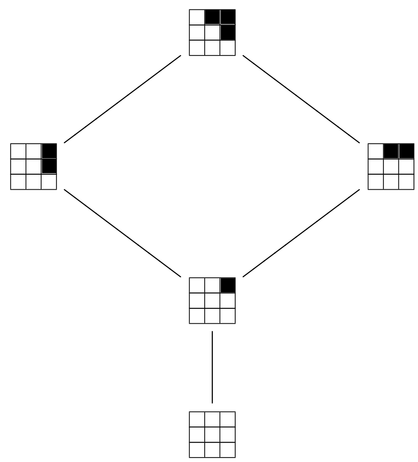
$$U_s/P_s \overset{p}{\twoheadrightarrow} M^{ss}_{\alpha_s}(Q_s, \alpha_s)$$

Now, P_s is the Borel subgroup of upper triangular matrices and one verifies that the following two couples

$$\left(\begin{bmatrix} 0 & c & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} 0 & c & x \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & d & y \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \right)$$

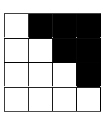
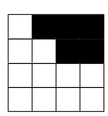
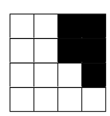
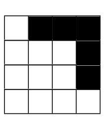
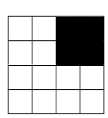
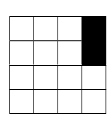
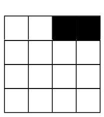
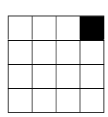
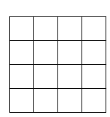
lie in the same B-orbit if and only if $det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \neq 0$, that is, if and only if $[a : b] \neq [c : d]$ in $\mathbb{P}^1$. Hence, away from the diagonal p is an isomorphism. On the diagonal one can again verify by direct computation that the fibers of p are isomorphic to $\mathbb{C}$, giving rise to the cases d, h and i in the classification.
The connection between this approach and Kraft's result is depicted by the following two pictures



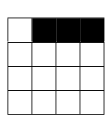


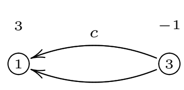
The picture on the left is Kraft's toric degeneration picture where we enclosed all orbits belonging to the same Hesselink strata, that is, having the same optimal corner type. The dashed region enclosed the orbits which do not come from the moduli spaces $M^{ss}_{\alpha_s}(Q_s, \theta_s)$, that is, those coming from the projection $U_s/P_s \twoheadrightarrow M^{ss}_{\alpha_s}(Q_s, \theta_s))$. The picture on the right gives the ordering of the relevant corners.

Example 8.19 We see that we get most orbits in the nullcone from the moduli spaces $M^{ss}_{\alpha_s}(Q_s, \theta_s)$. The reader is invited to work out the orbits in $Null^2_4$. We list here the moduli spaces of the relevant corners

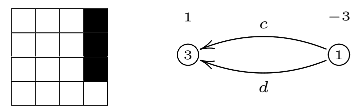
corner	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$	corner	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$	corner	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$
	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$		$\mathbb{P}^1$		$\mathbb{P}^1$
	$\mathbb{P}^3 \sqcup \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1$		$\mathbb{P}^1 \sqcup S^2(\mathbb{P}^1)$		$\mathbb{P}^0$
	$\mathbb{P}^1$		$\mathbb{P}^1$		$\mathbb{P}^0$

Observe that two potential corners are missing in this list. This is because we have the following quiver setting for the corner





and there are no θ_s-semistable representations as the two maps have a common kernel, whence a subrepresentation of dimension $\beta = (1,0)$ and $\theta_s(\beta) < 0$. A similar argument holds for the other missing corner and quiver setting



For general n*, a similar argument proves that the corners associated to the tableaux* $\boxed{1}\boxed{n}$ *and* $\boxed{n}\boxed{1}$ *are not optimal for tuples in* $Null^m_{n+1}$ *unless* $m \geq n$*. It is also easy to see that with* $m \geq n$ *all relevant corners appear in* $Null^m_{n+1}$*, that is all potential Hesselink strata are non-empty.*

8.4 Cornering quiver representations.

One again, generalizing the results from m*-tuples of* $n \times n$ *matrices to arbitrary quiver representations presents more a notational than an intellectual challenge. Let* Q *be a (marked) quiver on* k *vertices* $\{v_1, \ldots, v_k\}$ *and fix a dimension vector* $\alpha = (a_1, \ldots, a_k)$ *and denote the total dimension* $\sum_{i=1}^k a_i$ *by* a*. A representation* $V \in \underline{rep_\alpha}\ Q$ *is said to belong to the nullcone* $Null_\alpha\ Q$ *if the trivial representation* $\underline{0} \in \overline{\mathcal{O}(V)}$*. Equivalently, all polynomial invariants are zero when evaluated in* V*, that is, the traces of all oriented cycles in* Q *are zero in* V*. By the Hilbert criterium for* $GL(\alpha)$*,* $V \in Null_\alpha\ Q$ *if and only if there is a one-parameter subgroup*

$$\mathbb{C}^* \xrightarrow{\lambda} GL(\alpha) = \begin{bmatrix} GL_{a_1} & & \\ & \ddots & \\ & & GL_{a_k} \end{bmatrix} \hookrightarrow GL_a$$

such that $\lim_{\rightarrow} \lambda(t).V = \underline{0}$*. Up to conjugation in* $GL(\alpha)$*, or equivalently, replacing* V *by another point in the orbit* $\mathcal{O}(V)$*, we may assume that* λ *lies in the maximal torus* T_a *of* $GL(\alpha)$ *(and of* GL_a*) and can be represented by an integral* a*-tuple* $(r_1, \ldots, r_a) \in \mathbb{Z}^a$ *such that*

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & \\ & \ddots & \\ & & t^{r_a} \end{bmatrix}$$

We have to take the vertices into account, so we decompose the integer interval $[1, 2, \ldots, a]$ *into* vertex intervals I_{v_i} *such that*

$$[1, 2, \ldots, a] = \sqcup_{i=1}^k I_{v_i} \quad \textit{with} \quad I_{v_i} = [\sum_{j=1}^{i-1} a_j + 1, \ldots, \sum_{j=1}^{i} a_j]$$

If we recall that the weights of T_a *are isomorphic to* $\mathbb{Z}^a$ *having canonical generators* π_p *for* $1 \leq p \leq a$ *we can decompose the representation space into weight spaces*

$$rep_\alpha\ Q = \bigoplus_{\pi_{pq} = \pi_q - \pi_p} rep_\alpha\ Q(\pi_{pq})$$

where the eigenspace of π_{pq} *is non-zero if and only if for* $p \in I_{v_i}$ *and* $q \in I_{v_j}$*, there is an arrow*

$$\textcircled{j} \longleftarrow \textcircled{i}$$

in the quiver Q. Call $\pi_\alpha\ Q$ the set of weights π_{pq} which have non-zero eigenspace in $rep_\alpha\ Q$. Using this weight space decomposition we can write every representation as $V = \sum_{p,q} V_{pq}$ where V_{pq} is a vector of the (p,q)-entries of the maps $V(a)$ for all arrows a in Q from v_i to v_j. Using the fact that the action of T_a on $rep_\alpha\ Q$ is induced by conjugation, we deduce as before that for λ determined by $(r_1, \dots, r_a)$

$$\lim_{t \to 0} \lambda(t).V = \underline{0} \ \Leftrightarrow\ r_q - r_p \geq 1 \text{ whenever } V_{pq} \neq 0$$

Again, we can define the corner type C of the representation V by defining the subset of real a-tuples

$$E_V = \{(x_1, \dots, x_a) \in \mathbb{R}^a \ \mid\ x_q - x_p \geq 1\ \forall\ V_{pq} \neq 0\}$$

and determine a minimal element s_V in it, minimal with respect to the usual norm on $\mathbb{R}^a$. Similar to the case of matrices considered before, it follows that s_V is a uniquely determined point in $\mathbb{Q}^a$, having the characteristic property that its entries can be partitioned into strings

$$\{\underbrace{p_l, \dots, p_l}_{a_{l0}}, \underbrace{p_l+1, \dots, p_l+1}_{a_{l1}}, \dots, \underbrace{p_l+k_l, \dots, p_l+k_l}_{a_{lk_l}}\} \qquad \text{with all } a_{lm} \geq 1$$

which are balanced, that is $\sum_{m=0}^{k_l} a_{lm}(p_l + m) = 0$. Note however that this time we are not allowed to bring s_V into dominant form, as we can only permute base-vectors of the vertex-spaces. That is, we can only use the action of the vertex-symmetric groups

$$S_{a_1} \times \dots \times S_{a_k} \hookrightarrow S_a$$

to bring s_V into vertex dominant form, *that is if $s_V = (s_1, \dots, s_a)$ then*

$$s_q \leq s_p \quad \text{whenever} \quad p, q \in I_{v_i} \text{ for some } i \text{ and } p < q$$

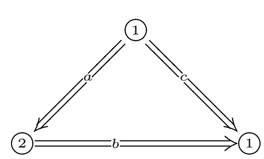
We can compile a list $\mathcal{S}_\alpha$ of such rational a-tuples as follows

- *Start with the list $\mathcal{S}_a$ of matrix corner types.*
- *For every $s \in \mathcal{S}_a$ consider all permutations $\sigma \in S_a/(S_{a_1} \times \dots \times S_{a_k})$ such that $\sigma.s = (s_{\sigma(1)}, \dots, s_{\sigma(a)})$ is vertex dominant.*
- *Take $\mathcal{H}_\alpha$ to be the list of the distinct a-tuples $\sigma.s$ which are vertex dominant.*
- *Remove $s \in \mathcal{H}_\alpha$ whenever there is an $s' \in \mathcal{H}_\alpha$ such that*

  $$\pi_s\ Q = \{\pi_{pq} \in \pi_\alpha\ Q \mid s_q - s_p \geq 1\} \subset \pi_{s'}\ Q = \{\pi_{pq} \in \pi_\alpha\ Q \ \mid\ s'_q - s'_p \geq 1\}$$

  and $\parallel s \parallel > \parallel s' \parallel$.
- *The list $\mathcal{S}_\alpha$ are the remaining entries s from $\mathcal{H}_\alpha$.*

Example 8.20 Let us give an example illustrating the removing condition. Consider the quiver setting



with $a, b, c \geq 1$. As $\alpha = (1,2,1)$ we have that the set of occurring weights in $rep_\alpha\ Q$ is

$$\pi_\alpha\ Q = \{\pi_{12}, \pi_{13}, \pi_{14}, \pi_{24}, \pi_{34}\}$$

The total dimension $a = 4$ and we have compiled the list $\mathcal{S}_4$ before. Consider the vertex-dominant reordering of $(1, 0, 0, -1)$

$$s = (0, 1, -1, 0) \quad \text{then} \quad \pi_s\ Q = \{\pi_{12}, \pi_{34}\} \quad \text{and} \quad \| s \| = 2$$

However, we have a vertex-dominant reordering of $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

$$s' = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \quad \text{with} \quad \pi_{s'}\ Q = \{\pi_{12}, \pi_{34}, \pi_{14}\} \quad \text{and} \quad \| s' \| = 1$$

and we need to remove s from the possible corner types. Indeed, s cannot be a minimum for the set E_V where $V_{12} \neq 0 \neq V_{34}$. In fact, the list $\mathcal{S}_\alpha$ for this quiver-setting consists of the following types

s	$\pi_s\ Q$	$\| s \|$
$(-1, 0, 0, 1)$	$\pi_\alpha\ Q$	2
$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$\{\pi_{12}, \pi_{14}, \pi_{34}\}$	1
$(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$\{\pi_{12}, \pi_{13}, \pi_{14}\}$	$\frac{3}{4}$
$(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4})$	$\{\pi_{14}, \pi_{24}, \pi_{34}\}$	$\frac{3}{4}$
$(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$\{\pi_{12}, \pi_{13}\}$	$\frac{2}{3}$
$(-\frac{2}{3}, \frac{1}{3}, 0, \frac{1}{3})$	$\{\pi_{12}, \pi_{14}\}$	$\frac{2}{3}$
$(-\frac{1}{3}, 0, -\frac{1}{3}, \frac{2}{3})$	$\{\pi_{14}, \pi_{34}\}$	$\frac{2}{3}$
$(0, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$	$\{\pi_{24}, \pi_{34}\}$	$\frac{2}{3}$
$(-\frac{1}{2}, \frac{1}{2}, 0, 0)$	$\{\pi_{12}\}$	$\frac{1}{2}$
$(-\frac{1}{2}, 0, 0, \frac{1}{2})$	$\{\pi_{14}\}$	$\frac{1}{2}$
$(0, 0, -\frac{1}{2}, \frac{1}{2})$	$\{\pi_{34}\}$	$\frac{1}{2}$
$(0, 0, 0, 0)$	$\emptyset$	0

and one verifies that all other vertex-dominant reorderings of elements from $\mathcal{S}_4$ have to be removed. Observe that we do not have to worry about this additional restriction if each vertex has a loop and any two vertices of Q are connected by arrows in both ways, that is, when $\pi_\alpha\ Q$ is the set of all weights π_{ij} with $1 \leq i, j \leq a$.

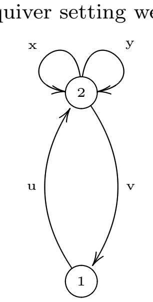
For $s \in \mathcal{S}_\alpha$, we can then define similar associated data as in the case of matrices

- *The corner C_s is the subspace of $rep_\alpha\ Q$ such that all arrow matrices V_b, when viewed as $a \times a$ matrices using the partitioning in vertex-entries, have only non-zero entries at spot (p, q) when $s_q - s_p \geq 1$.*
- *The border B_s is the subspace of $rep_\alpha\ Q$ such that all arrow matrices V_b, when viewed as $a \times a$ matrices using the partitioning in vertex-entries, have only non-zero entries at spot (p, q) when $s_q - s_p = 1$.*
- *The parabolic subgroup $P_s(\alpha)$ is the intersection of $P_s \subset GL_a$ with $GL(\alpha)$ embedded along the diagonal. $P_s(\alpha)$ is a parabolic subgroup of $GL(\alpha)$, that is, contains the product of the Borels $B(\alpha) = B_{a_1} \times \ldots \times B_{a_k}$.*
- *The Levi-subgroup $L_s(\alpha)$ is the intersection of $L_s \subset GL_a$ with $GL(\alpha)$ embedded along the diagonal.*

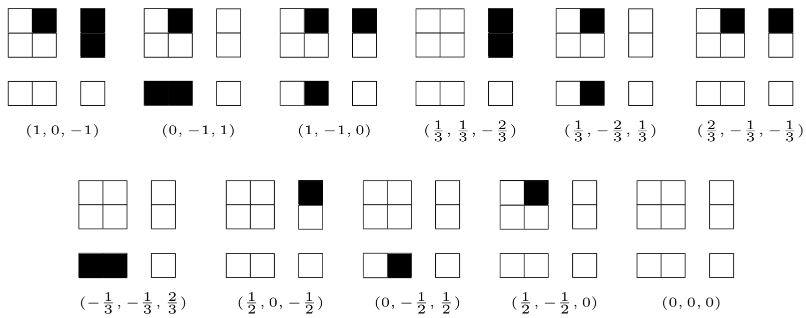
We say that a representation $V \in rep_\alpha\ Q$ is of corner type C_s whenever $V \in C_s$.

Theorem 8.21 *By permuting the vertex-bases, every representation $V \in rep_\alpha\ Q$ can be brought to a corner type C_s for a uniquely determined s which is a vertex-dominant reordering of s_V.*

Example 8.22 Let us consider the quiver setting we encountered in chapter 1



Then, the relevant corners have the following block decomposition



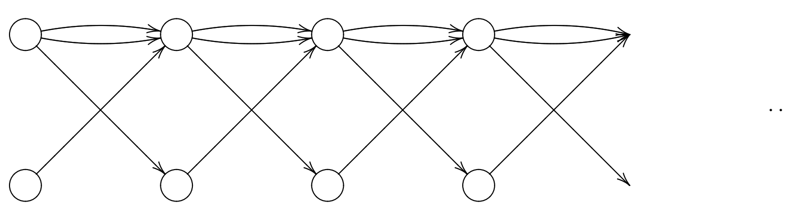
Again we will solve the problem of the optimal corner representations by introducing a new quiver setting. Fix a type $s \in \mathcal{S}_\alpha\ Q$ and let $J_1, \ldots, J_u$ be the distinct strings partitioning the entries of s, say with

$$J_l = \{\underbrace{p_l, \ldots, p_l}_{\sum_{i=1}^k b_{i,l0}}, \underbrace{p_l+1, \ldots, p_l+1}_{\sum_{i=1}^k b_{i,l1}}, \ldots, \underbrace{p_l+k_l, \ldots, p_l+k_l}_{\sum_{i=1}^k b_{i,lk_l}}\}$$

where $b_{i,lm}$ is the number of entries $p \in I_{v_i}$ such that $s_p = p_l + m$. To every string l we will associate a quiver $Q_{s,l}$ and dimension vector $\alpha_{s,l}$ as follows

- *$Q_{s,l}$ has $k.(k_l+1)$ vertices labeled (v_i, m) with $1 \leq i \leq k$ and $0 \leq m \leq k_l$.*
- *In $Q_{s,l}$ there are as many arrows from vertex (v_i, m) to vertex $(v_j, m+1)$ as there are arrows in Q from vertex v_i to vertex v_j. There are no arrows between (v_i, m) and (v_j, m') if $m' - m \neq 1$.*
- *The dimension-component of $\alpha_{s,l}$ in vertex (v_i, m) is equal to $b_{i,lm}$.*

Example 8.23 For the above quiver, all component quivers $Q_{s,l}$ are pieces of the quiver below



Clearly, we only need to consider that part of the quiver $Q_{s,l}$ where the dimensions of the vertex spaces are non-zero.

The quiver-setting (Q_s, α_s) associated to a type $s \in \mathcal{S}_\alpha\ Q$ will be the disjoint union of the string quiver-settings $(Q_{s,l}, \alpha_{s,l})$ for $1 \leq l \leq u$. The purpose of all these definitions is

Theorem 8.24 *With notations as before, for $s \in \mathcal{S}_\alpha\ Q$ we have isomorphisms*

$$\begin{cases} B_s & \simeq rep_{\alpha_s}\ Q_s \\ L_s(\alpha) & \simeq GL(\alpha_s) \end{cases}$$

Moreover, the base-change action of $GL(\alpha_s)$ on $rep_{\alpha_s}\ Q_s$ coincides under the isomorphisms with the action of the Levi-subgroup $L_s(\alpha)$ on the border B_s.

In order to determine the representations in $rep_{\alpha_s}\ Q_s$ which have optimal corner type C_s we define the following character on the Levi-subgroup

$$L_s(\alpha) = \prod_{l=1}^{u} \times_{i=1}^{k} \times_{m=0}^{k_l} GL_{b_{i,lm}} \xrightarrow{\chi\theta_s} \mathbb{C}^*$$

determined by sending a tuple $(g_{i,lm})_{ilm} \longrightarrow \prod_{ilm} det\ g_{i,lm}^{m_{i,lm}}$ where the exponents are determined by

$$\theta_s = (m_{i,lm})_{ilm} \quad where \quad m_{i,lm} = d(p_l + m)$$

with d the least common multiple of the numerators of the rational numbers p_l for all $1 \leq l \leq u$. As in the case of m-tuples of $n \times n$ matrices we can prove

Theorem 8.25 *Consider a representation $V \in Null_\alpha\ Q$ of corner type C_s. Then, V is of optimal corner type C_s if and only if under the natural maps*

$$C_s \overset{\pi}{\twoheadrightarrow} B_s \xrightarrow{\simeq} rep_{\alpha_s}\ Q_s$$

V is mapped to a θ_s-semistable representation in $rep_{\alpha_s}\ Q_s$. If U_s is the open subvariety of C_s consisting of all representations of optimal corner type C_s, then

$$U_s = \pi^{-1}\ rep_{\alpha_s}^{ss}(Q_s, \theta_s)$$

For the corresponding Hesselink stratum $S_s = GL(\alpha).U_s$ we have the commuting diagram

$$\begin{array}{ccc} GL(\alpha) \times^{P_s(\alpha)} U_s & \xrightarrow{\simeq} & S_s \\ \big\downarrow & & \big\downarrow \\ GL(\alpha) \times^{P_s(\alpha)} C_s & \xrightarrow{ac} & \overline{S_s} \end{array}$$

where ac is the action map, $\overline{S_s}$ is the Zariski closure of S_s in $Null_\alpha\ Q$ and the upper map is an isomorphism as $GL(\alpha)$-varieties. Here, $GL(\alpha)/P_s(\alpha)$ is the flag variety associated to the parabolic subgroup $P_s(\alpha)$ and is a projective manifold. The variety $GL(\alpha) \times^{P_s(\alpha)} C_s$ is a vectorbundle over the flag variety $GL(\alpha)/P_s(\alpha)$ and is a subbundle of the trivial bundle $GL(\alpha) \times^{P_s(\alpha)} rep_\alpha\ Q$. Hence, the Hesselink stratum S_s is an irreducible smooth variety of dimension

$$\begin{aligned} dim\ S_s &= dim\ GL(\alpha)/P_s(\alpha) + rk\ GL(\alpha) \times^{P_s(\alpha)} C_s \\ &= \sum_{i=1}^{k} a_i^2 - dim\ P_s(\alpha) + dim_{\mathbb{C}}\ C_s \end{aligned}$$

and there is a natural one-to-one correspondence between the $GL(\alpha)$-orbits in S_s and the $P_s(\alpha)$-orbits in U_s. Moreover, the vectorbundle $GL(\alpha) \times^{P_s(\alpha)} C_s$ is a desingularization of $\overline{S_s}$ hence 'feels' the gluing of S_s to the remaining strata. Finally, the ordering of corners has the geometric interpretation

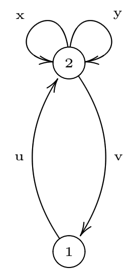
$$\overline{S_s} \subset \bigcup_{\|s'\| \leq \|s\|} S_{s'}$$

Finally, because $P_s(\alpha)$ acts on B_s by the restriction to its subgroup $L_s(\alpha) = GL(\alpha_s)$ we have a projection from the orbit space

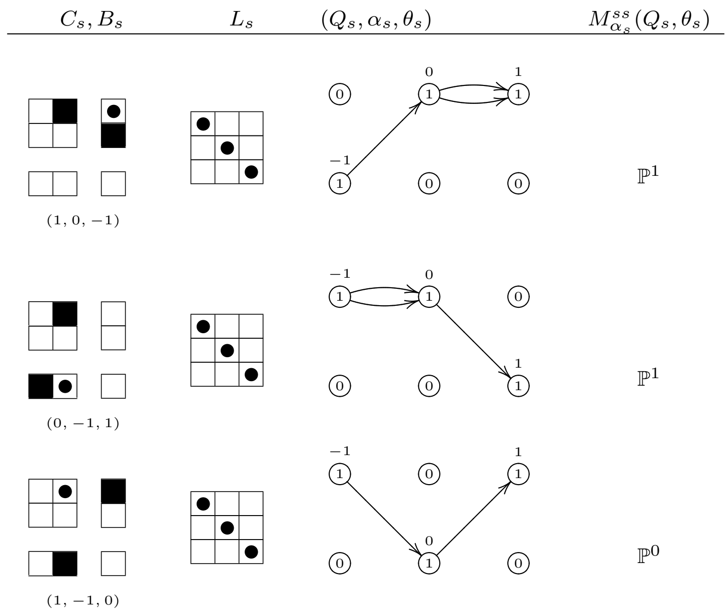
$$U_s/P_s \overset{p}{\twoheadrightarrow} M_{\alpha_s}^{ss}(Q_s, \theta_s)$$

to the moduli space of θ_s-semistable quiver representations.

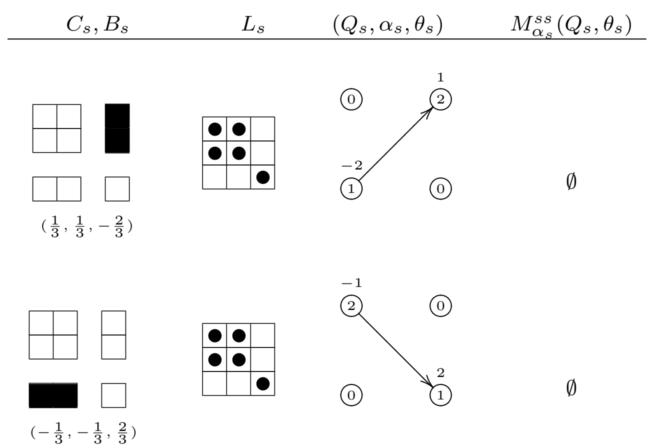
Example 8.26 Above we have listed the relevant corner-types for the nullcone of the quiver-setting



In the table below we list the data of the three irreducible components of $Null_\alpha\ Q/GL(\alpha)$ corresponding to the three maximal Hesselink strata :



There are 6 other Hesselink strata consisting of precisely one orbit. Finally, two possible corner-types do not appear as there are no θ_s-semistable representations for the corresponding quiver setting



8.5 Etale fibers.

Having obtained some control on the nullcone of arbitrary quiver settings, we want to apply these results to obtain information on the representations of smooth algebras. Let us recall the setting : A will be an affine $\mathbb{C}$-algebra and M_ξ is a semi-

simple n-dimensional module such that the (trace preserving) representation variety $\underline{rep}_n\ A@_n$ is smooth in M_ξ. If M_ξ is of representation type $\tau = (e_1, d_1; \ldots ; e_k, d_k)$, that is, if M_ξ decomposes as

$$M_\xi = S_1^{\oplus e_1} \oplus \ldots \oplus S_k^{\oplus e_k}$$

with distinct simple components S_i of dimension d_i and occurring in M_ξ with multiplicity e_i, then the $GL(\alpha) = Stab\ M_\xi$-structure on the normal space N_ξ to the orbit $\mathcal{O}(M_\xi)$ is isomorphic to that of the representation space

$$rep_\alpha\ Q^\bullet$$

of a certain marked quiver on k vertices (the number of distinct simple components) and the dimension vector $\alpha = (e_1, \ldots, e_k)$ is given by the multiplicities. Moreover, we have seen in chapter 6 that the arrows in $Q^\bullet$ are determined by the (trace preserving) self-extensions of M_ξ. The Luna slice theorem asserts the existence of a slice $S_\xi \xrightarrow{\phi} N_\xi$ such that there is a local commuting diagram

$$\begin{tikzcd}
& GL_n \times^{GL(\alpha)} S_\xi \arrow[dl, "GL_n \times^{GL(\alpha)} \phi"'] \arrow[dd] \arrow[dr, "\psi"] & \\
GL_n \times^{GL(\alpha)} N_\xi \arrow[dd, "\pi_2"] & & \underline{rep}_n\ A@_n \arrow[dd, "\pi_1"] \\
& S_\xi/GL(\alpha) \arrow[dl, "\phi/GL(\alpha)"'] \arrow[dr, "\psi/GL_n"] & \\
N_\xi/GL(\alpha) & & \underline{iss}_n\ A@_n
\end{tikzcd}$$

in a neighborhood of $\xi \in \underline{iss}_n\ A@_n$ on the right and a neighborhood of the image $\underline{0}$ of the trivial representation in $N_\xi/GL(\alpha)$ on the left. In this diagram, the vertical maps are the quotient maps, all diagonal maps are étale and the upper diagonal maps are GL_n-equivariant. In particular, there is a GL_n-isomorphism between the fibers

$$\pi_2^{-1}(\underline{0}) \simeq \pi_1^{-1}(\xi)$$

and as $\pi_2^{-1}(\underline{0}) \simeq GL_n \times^{GL(\alpha)} \pi^{-1}(\underline{0})$ where π is the quotient morphism for the marked quiver representations $N_\xi = rep_\alpha\ Q^\bullet \xrightarrow{\pi} iss_\alpha\ Q^\bullet = N_x/GL(\alpha)$ we have a GL_n-isomorphism

$$\pi_1^{-1}(\xi) \simeq GL_n \times^{GL(\alpha)} \pi^{-1}(\underline{0})$$

That is, there is a natural one-to-one correspondence between

- *GL_n-orbits in the fiber $\pi_1^{-1}(\zeta)$, that is, isomorphism classes of n-dimensional (trace preserving) representations of A with Jordan-Hölder decomposition M_ξ, and*
- *$GL(\alpha)$-orbits in $\pi^{-1}(\underline{0})$, that is, the nullcone of the marked quiver $Null_\alpha\ Q^\bullet$.*

Summarizing we have the following

Theorem 8.27 *Let A be an affine Quillen-smooth $\mathbb{C}$-algebra and M_ξ a semi-simple n-dimensional representation of A. Then, the isomorphism classes of n-dimensional representations of A with Jordan-Hölder sum isomorphic to M_ξ are given by the $GL(\alpha)$-orbits in the nullcone $Null_\alpha\ Q^\bullet$ of the local marked quiver setting.*

8.6 Simultaneous conjugacy classes.

We have come a long way from our bare hands description of the simultaneous conjugacy classes of couples of 2×2 matrices in chapter 3. In this section we will summarize what we have learned so far to approach this hopeless problem. The problem of classifying simultaneous conjugacy classes of m-tuples of $n \times n$ matrices, is the same as studying the GL_n-orbits in

$$M_n^m \simeq rep_n\ \mathbb{C}\langle x_1, \ldots, x_m \rangle$$

The best continuous approximation to the non-existent Hausdorff orbit-space is given by the algebraic quotient map

$$rep_n \mathbb{C}\langle x_1, \ldots, x_m \rangle \xrightarrow{\pi} iss_n\ \mathbb{C}\langle x_1, \ldots, x_m \rangle = iss_n^m$$

where the points ξ in iss_n^m classify the isomorphism classes of n-dimensional semi-simple modules M_ξ. If M_ξ has a simple decomposition

$$M_\xi \simeq S_1^{\oplus e_1} \oplus \ldots S_k^{\oplus e_k}$$

with the S_i distinct simples of dimension d_i (so that $n = \sum_i d_i e_i$) we say that M_ξ is of representation type

$$\tau(M_\xi = (e_1, d_1; \ldots; e_k, d_k)$$

We have calculated the coordinate ring $\mathbb{C}[iss_n^m] = \mathbb{N}_n^m$ which is the necklace algebra, that is, is generated by traces of monomials in the generic $n \times n$ matrices $X_1, \ldots, X_m$ of length bounded by $n^2 + 1$. Moreover, we know that if we collect all M_ξ with fixed representation type together in the subset $iss_n^m(\tau)$, then

$$iss_n = \bigsqcup_\tau iss_n^m(\tau)$$

is a finite stratification of iss_n^m into locally closed smooth algebraic subvarieties. Moreover, we know that a stratum $iss_n^m(\tau')$ is contained in the Zariski closure $\overline{iss_n^m(\tau)}$ of another stratum if and only if $\tau' < \tau$. Here, the order relation is induced by the direct ordering

$$\tau' = (e'_1, d'_1; \ldots; e'_{k'}, d'_{k'}) <^{dir} \tau = (e_1, d_1; \ldots; e_k, d_k)$$

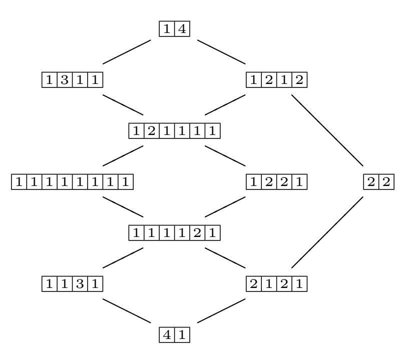
if there exist a permutation σ on $[1, 2, \ldots, k']$ such that there exist numbers

$$1 = j_0 < j_1 < j_2 \ldots < j_k = k'$$

such that for every $1 \leq i \leq k$ we have the following relations

$$\begin{cases} e_i d_i & = \sum_{j=j_{i-1}+1}^{j_i} e'_{\sigma(j)} d'_{\sigma(j)} \\ e_i & \leq e'_{\sigma(j)} \text{ for all } j_{i-1} < j \leq j_i \end{cases}$$

For example,the order relation on the representation types of dimension $n = 4$ has the following Hasse diagram.



Because iss_n^m is irreducible, there is an open stratum corresponding to the simple representations, that is type $(1, n)$. The sub-generic strata are all of the form

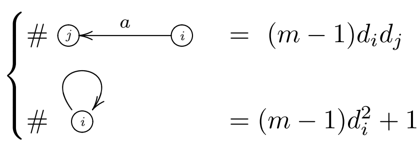
$$\tau = (1, m_1; 1, m_2) \quad \textit{with} \quad m_1 + m_2 = n.$$

The (in)equalities describing the locally closed subvarieties $iss_n^m(\tau)$ can (in principle) be deduced from the theory of trace identities. Remains to study the local structure of the quotient variety iss_n^m near ξ and the description of the fibers $\pi^{-1}(\xi)$.

Both problems can be tackled by studying the local quiver setting (Q_ξ, α_ξ) corresponding to ξ which describes the $GL(\alpha_\xi) = Stab(M_\xi)$-module structure of the normal space to the orbit of M_ξ. If ξ is of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ then the local quiver Q_ξ has k-vertices $\{v_1, \ldots, v_k\}$ corresponding the the k distinct simple components $S_1, \ldots, S_k$ of M_ξ and the number of arrows (resp. loops) from v_i to v_j (resp. in v_i) are given by the dimensions

$$dim_{\mathbb{C}} Ext^1(S_i, S_j) \quad \textit{resp.} \quad dim_{\mathbb{C}} Ext^1(S_i, S_i)$$

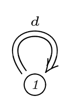
and these numbers can be computed from the dimensions of the simple components,



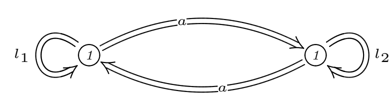
Further, the local dimension vector α_ξ is given by the multiplicities $(e_1, \ldots, e_k)$. The étale local structure of iss_n^m in a neighborhood of ξ is the same as that of the quotient variety $iss_{\alpha_\xi}\ Q_\xi$ in a neighborhood of $\underline{0}$. The local algebra of the latter is generated by traces along oriented cycles in the quiver Q_ξ. A direct application is

Proposition 8.28 *For $m \geq 2$, ξ is a smooth point of iss_n^m if and only if M_ξ is a simple representation, unless $(m, n) = (2, 2)$ i which case $iss_2^2 \simeq \mathbb{C}^5$ is a smooth variety.*

Proof. *If ξ is of representation type $(1, n)$, the local quiver setting (Q_ξ, α_ξ) is*



where $d = (m-1)n^2 + 1$, whence the local algebra is the formal power series ring in d variables and so iss_n^m is smooth in ξ. As the singularities form a Zariski closed subvariety of iss_n^m, the result follows if we prove that all points ξ lying in sub-generic strata, say of type $(1, m_1; 1, m_2)$ are singular. In this case the local quiver setting is equal to



where $a = (m-1)m_1m_2$ and $l_i = (m-1)m_i^2 + 1$. Let us denote the arrows from v_1 to v_2 by $x_1, \ldots, x_a$ and those from v_2 to v_1 by $y_1, \ldots, y_a$. If $(m, n) \neq (2, 2)$ then $a \geq 2$, but then we have traces along cycles

$$\{x_iy_j \mid 1 \leq i, j \leq a\}$$

that is, the polynomial ring of invariants is the polynomial algebra in $l_1 + l_2$ variables (the traces of the loops) over the homogeneous coordinate ring of the Segre embedding

$$\mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \hookrightarrow \mathbb{P}^{a^2-1}$$

which has a singularity at the top (for example we have equations of the form $(x_1y_2)(x_2y_1) - (x_1y_1)(x_2y_2)$). Thus, the local algebra of iss_n^m cannot be a formal power series ring in ξ whence iss_n^m is singular in ξ. We have proved in chapter 3 that in the exceptional case $iss_2^2 \simeq \mathbb{C}^5$. □

To determine the fibers of the quotient map $M_n^m \xrightarrow{\pi} iss_n^m$ we have to study the nullcone of this local quiver setting, $Null_{\alpha_\xi}\ Q_\xi$. Observe that the quiver Q_ξ has loops in every vertex and arrows connecting each ordered pair of vertices, whence we do not have to worry about potential corner-type removals. Denote $\sum e_i = z \leq n$ and let $\mathcal{C}_z$ be the set of all $s = (s_1, \ldots, s_z) \in \mathbb{Q}^z$ which are disjoint unions of strings of the form

$$\{p_i, p_i + 1, \ldots, p_i + k_i\}$$

where $l_i \in \mathbb{N}$, all intermediate numbers $p_i + j$ with $j \leq k_i$ do occur as components in s with multiplicity $a_{ij} \geq 1$ and p_i satisfies the balance-condition

$$\sum_{j=0}^{k_i} a_{ij}(p_i + j) = 0$$

for every string in s. For fixed $s \in \mathcal{C}_z$ we can distribute the components s_i over the vertices of Q_ξ (e_j of them to vertex v_j) in all possible ways modulo the action of the small Weyl group $S_{e_1} \times \ldots S_{e_k} \hookrightarrow S_z$. That is, we can rearrange the s_i's belonging to a fixed vertex such that they are in decreasing order. This gives us the list $\mathcal{S}_{\alpha_\xi}$ or $\mathcal{S}_\tau$ of all corner-types in $Null_{\alpha_\xi}\ Q_\xi$. For each $s \in \mathcal{S}_{\alpha_\xi}$ we then construct the corner-quiver setting

$$(Q_{\xi\ s}, \alpha_{\xi\ s}, \theta_{\xi\ s})$$

and study the Hesselink strata S_s which actually do appear, which is equivalent to verifying whether there are $\theta_{\xi\ s}$-semistable representations in $rep_{\alpha_{\xi\ s}}\ Q_{\xi\ s}$. Using Schofield's criterium proved in chapter 7 we have a purely combinatorial way to settle this (in general quite hard) problem of optimal corner-types.

That is, we can determine which Hesselink strata S_s actually occur in $\pi^{-1}(\xi) \simeq Null_{\alpha_{xi}}\ Q_\xi$. The $GL(\alpha_{\xi\ s})$-orbits in the stratum S_s are in natural one-to-one correspondence with the orbits under the associated parabolic subgroup P_s acting on the semistable representations

$$U_s = \pi^{-1}\ rep^{ss}_{\alpha_{\xi\ s}}(Q_{\xi\ s}, \theta_{\xi\ s})$$

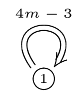
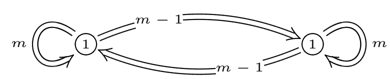
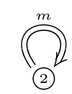
and there is a natural projection morphism from the corresponding orbit-space

$$U_s/P_s \xrightarrow{p_s} M^{ss}_{\alpha_{\xi\ s}}(Q_{\xi\ s}, \theta_{\xi\ s})$$

to the moduli space of $\theta_{\xi\ s}$-semistable representations which we can study locally because we know how to construct all semi-invariants of quivers. The 'only' (usually hard) remaining problem in the classification of m-tuples of $n \times n$ matrices under simultaneous conjugation is the description of the fibers of this projection map p_s.

Example 8.29 (m-tuples of 2×2 matrices)
There are three different representation types τ of 2-dimensional representations of $\mathbb{C}\langle x_1, \ldots, x_m\rangle$

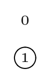
with corresponding local quiver settings (Q_τ, α_τ)

type	τ	(Q_τ, α_τ)
2_a	$(1,2)$	
2_b	$(1,1;1,1)$	
2_c	$(2,1)$	

The defining (in)equalities of the strata $iss_2^m(\tau)$ are given by $k \times k$ minors (with $k \leq 4$ of the symmetric $m \times m$ matrix

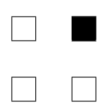
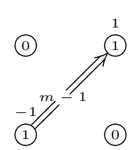
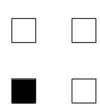
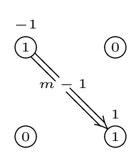
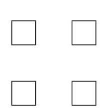
$$\begin{bmatrix} tr(x_1^0 x_1^0) & \dots & tr(x_1^0 x_m^0) \\ \vdots & & \vdots \\ tr(x_m^0 x_1^0) & \dots & tr(x_m^0 x_m^0) \end{bmatrix}$$

where $x_i^0 = x_i - \frac{1}{2} tr(x_i)$ is the generic trace zero matrix. These facts follow from the description of the trace algebras $\mathbb{T}_2^m$ as polynomial algebras over the generic Clifford algebras of rank ≤ 4 (determined by the above symmetric matrix) and the classical matrix decomposition of Clifford algebras over $\mathbb{C}$. Full details can be found my Habilitation thesis **??**. To study the fibers $M_2^m \longrightarrow\!\!\!\!\!\rightarrow iss_2^m$ we need to investigate the different Hesselink strata in the nullcones of these local quiver settings. Type 2_a has just one potential corner type corresponding to $s = (0) \in \mathcal{S}_1$ and with corresponding corner-quiver setting



which obviously has $\mathbb{P}^0$ (one point) as corresponding moduli (and orbit) space. This corresponds to the fact that for $\xi \in iss_2^m(1,2)$, M_ξ is simple and hence the fiber $\pi^{-1}(\xi)$ consists of the closed orbit $\mathcal{O}(M_\xi)$.

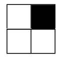
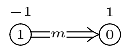
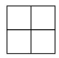
For type 2_b the following list gives the potential corner-types C_s together with their associated corner-quiver settings and moduli spaces (note that as $B_s = C_s$ in all cases, these moduli spaces describe the full fiber)

s	B_s, C_s	$(Q_s, \alpha_s, \theta_s)$	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$
$(\frac{1}{2}, -\frac{1}{2})$			$\mathbb{P}^{m-2}$
$(-\frac{1}{2}, \frac{1}{2})$			$\mathbb{P}^{m-2}$
$(0,0)$			$\mathbb{P}^0$

That is, for $\xi \in iss_2^m(1,1;1,1)$, the fiber $\pi^{-1}(\xi)$ consists of the unique closed orbit $\mathcal{O}(M_\xi)$ (corresponding to the $\mathbb{P}^0$) and two families $\mathbb{P}^{m-2}$ of non-closed orbits. Observe that in the special case $m = 2$ we recover the two non-closed orbits found in chapter 3.

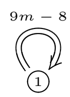
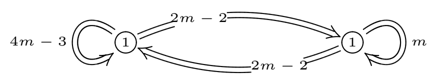
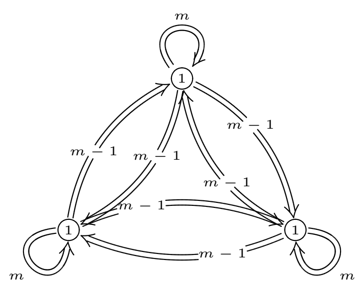
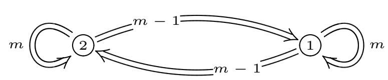
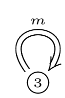
Finally, for type 2_c, the fibers are isomorphic to the nullcones of m-tuples of 2×2 matrices. We have the following list of corner-types, corner-quiver settings and moduli spaces. Again, as

$B_s = C_s$ in all cases, these moduli spaces describe the full fiber.

s	B_s, C_s	$(Q_s, \alpha_s, \theta_s)$	$M^{ss}_{\alpha_s}(Q_s, \theta_s)$
$(\frac{1}{2}, -\frac{1}{2})$			$\mathbb{P}^{m-1}$
$(0,0)$			$\mathbb{P}^0$

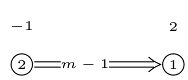
whence the fiber $\pi^{-1}(\xi)$ consists of the closed orbit, together wit a $\mathbb{P}^{m-1}$-family of non-closed orbits. Again, in the special case $m = 2$, we recover the $\mathbb{P}^1$-family found in chapter 3.

Example 8.30 (m-tuples of 3×3 matrices)
There are 5 different representation-types for 3-dimensional representations. Their associated local quiver settings are depicted in the following table

type	τ	(Q_τ, α_τ)
3_a	$(1,3)$	
3_b	$(1,2;1,1)$	
3_c	$(1,1;1,1;1,1)$	
3_d	$(2,1;1,1)$	
3_e	$(3,1)$	

For each of these types we can perform an analysis of the nullcones as before. We leave the details to the interested reader and mention only the end-result

- For type 3_a the fiber is one closed orbit.
- For type 3_b the fiber consists of the closed orbit together with two $\mathbb{P}^{2m-3}$-families of non-closed orbits.
- For type 3_c the fiber consists of the closed orbit together with twelve $\mathbb{P}^{m-2} \times \mathbb{P}^{m-2}$-families and one $\mathbb{P}^{m-2}$-family of non-closed orbits.
- For type 3_d the fiber consists of the closed orbit together with four $\mathbb{P}^{m-1} \times \mathbb{P}^{m-2}$-families, one $\mathbb{P}^{m-2} \times \mathbb{P}^{m-2}$-family, two $\mathbb{P}^{m-2}$-families, one $\mathbb{P}^{m-1}$-family and two M-families of non-closed orbits determined by moduli spaces of quivers, where M is the moduli space of the following quiver setting

  together with some additional orbits coming from the projection maps p_s.
- For type 3_e we have to study the nullcone of m-tuples of 3×3 matrices, which can be done as in the case of couples but for $m \geq 3$ the two extra strata do occur.

We see that in this case the only representation-types where the fiber is not fully determined by moduli spaces of quivers are 3_d and 3_e.

8.7 Representation fibers.

Let A be a Cayley-Hamilton algebra of degree n and consider the algebraic quotient map

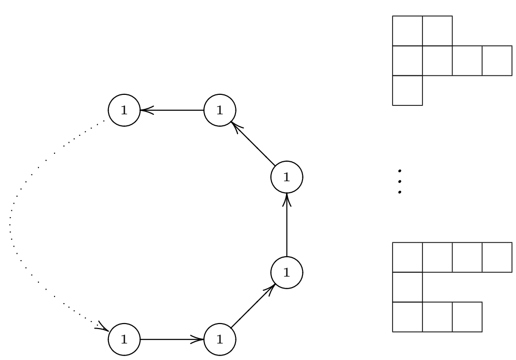
$$\underline{rep}_n^{tr}\ A \xrightarrow{\pi} \underline{iss}_n^{tr}\ A$$

from the variety of n-dimensional trace preserving representations to the variety classifying isomorphism classes of trace preserving n-dimensional semi-simple representations. Assume $\xi \in Sm_n\ A \hookrightarrow \underline{iss}_n^{tr}\ A$. That is, the representation variety $\underline{rep}_n^{tr}\ A$ is smooth along the GL_n-orbit of M_ξ where M_ξ is the semi-simple representation determined by $\xi \in \underline{iss}_n^{tr}\ A$. In chapter 5 we have seen that the local structure of A and $\underline{rep}_n^{tr}\ A$ near ξ is fully determined by a local marked quiver setting $(Q_\xi^\bullet, \alpha_\xi)$. That is, we have a GL_n-isomorphism between the fiber of the quotient map, that is, the n-dimensional trace preserving representation degenerating to M_ξ

$$\pi^{-1}(\xi) \simeq GL_n \times^{GL(\alpha_\xi)} Null_{\alpha_\xi}\ Q_\xi$$

and the nullcone of the marked quiver-setting. In this section we will apply the results on nullcones to the study of these representation fibers $\pi^{-1}(\xi)$. Observe that all the facts on nullcones of quivers extend verbatim to marked quivers $Q^\bullet$ using the underlying quiver Q with the proviso that we drop all loops in vertices with vertex-dimension 1 which get a marking in $Q^\bullet$. This is clear as nilpotent quiver representations obviously have zero trace along each oriented cycle, in particular in each loop. The examples given before illustrate that a complete description of the nullcone is rather cumbersome. For this reason we restrict here to the determination of the number of irreducible components and their dimensions in the representation fibers. Modulo the GL_n-isomorphism above this study amounts to describing the irreducible components of $Null_{\alpha_\xi}\ Q_\xi$ which are determined by the maximal corner-types C_s, that is such that the set of weights in C_s is maximal among subsets of $\pi_{\alpha_{xi}}\ Q_\xi$ (and hence $\parallel s \parallel$ is maximal among $\mathcal{S}_{\alpha_\xi}\ Q_\xi$.

To illustrate our strategy, consider the case of curve orders. In chapter 6 we proved that if A is a Cayley-Hamilton order of degree n over an affine curve $X = \underline{iss}_n^t\ A$ and if $\xi \in Sm_n\ A$, then the local quiver setting (Q, α) is determined by an oriented cycle Q on k vertices with $k \leq n$ being the number of distinct simple components of M_ξ, the dimension vector $\alpha = (1, \dots, 1)$



and an unordered partition $p = (d_1, \dots, d_k)$ having precisely k parts such that $\sum_i d_i = n$, determining the dimensions of the simple components of M_ξ. Fixing a cyclic ordering of the k-vertices $\{v_1, \dots, v_k\}$ we have that the set of weights of the maximal torus $T_k = \mathbb{C}^* \times \dots \times \mathbb{C}^* = GL(\alpha)$ occurring in $rep_\alpha\ Q$ is the set

$$\pi_\alpha\ Q = \{\pi_{k1}, \pi_{12}, \pi_{23}, \dots, \pi_{k-1k}\}$$

Denote $K = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}$ *and consider the one string vector*

$$s = (\ \ldots, k-2-\frac{K}{k}, k-1-\frac{K}{k}, \underbrace{-\frac{K}{k}}_{i}, 1-\frac{K}{k}, 2-\frac{K}{k}, \ldots \ )$$

then s *is balanced and vertex-dominant,* $s \in \mathcal{S}_\alpha\ Q$ *and* $\pi_s\ Q = \Pi$*. To check whether the corresponding Hesselink strata in* $Null_\alpha\ Q$ *is nonempty we have to consider the associated quiver-setting* $(Q_s, \alpha_s, \theta_s)$ *which is*

$$\begin{tikzcd} \underset{v_i}{\overset{-K}{\textcircled{1}}} \arrow[r] & \underset{v_{i+1}}{\overset{-K+k}{\textcircled{1}}} \arrow[r] & \underset{v_{i+2}}{\overset{-K+2k}{\textcircled{1}}} \arrow[r] & \cdots \quad \cdots \arrow[r] & \underset{v_{i-2}}{\overset{-K+k^2-2k}{\textcircled{1}}} \arrow[r] & \underset{v_{i-1}}{\overset{-K+k^2-k}{\textcircled{1}}} \end{tikzcd}$$

It is well known and easy to verify that $rep_{\alpha_s}\ Q_s$ *has an open orbit with representative all arrows equal to* 1*. For this representation all proper subrepresentations have dimension vector* $\beta = (0,\ldots,0,1,\ldots,1)$ *and hence* $\theta_s(\beta) > 0$*. That is, the representation is* θ_s*-stable and hence the corresponding Hesselink stratum* $S_s \neq \emptyset$*. Finally, because the dimension of* $rep_{\alpha_s}\ Q_s$ *is* $k-1$ *we have that the dimension of this component in the representation fiber* $\pi^{-1}(x)$ *is equal to*

$$dim\ GL_n - dim\ GL(\alpha) + dim\ rep_{\alpha_s}\ Q_s = n^2 - k + k - 1 = n^2 - 1$$

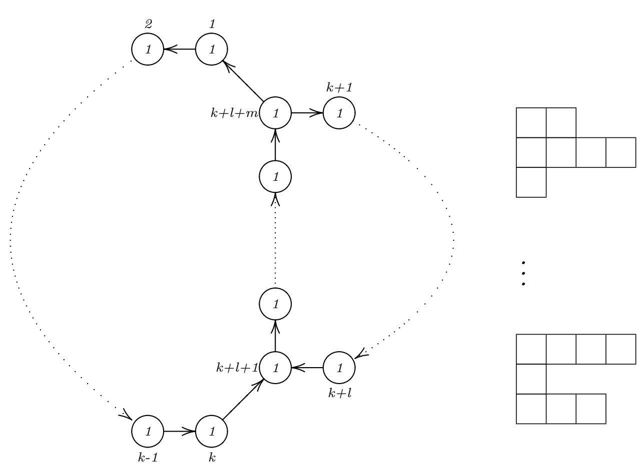
which completes the proof of the following

Theorem 8.31 *Let* A *be a Cayley-Hamilton order of degree* n *over an affine curve* X *such that* A *is smooth in* $\xi \in X$*. Then, the representation fiber* $\pi^{-1}(\xi)$ *has exactly* k *irreducible components of dimension* n^2-1*, each the closure of one orbit. In particular, if* A *is Cayley-smooth over* X*, then the quotient map*

$$\underline{rep}_n^t\ A \xrightarrow{\pi} \underline{iss}_n^t\ A = X$$

is flat, that is, all fibers have the same dimension n^2-1*.*

For Cayley-Hamilton orders over surfaces, the situation is slightly more complicated. From chapter 6 we recall that if A *is a Cayley-Hamilton order of degree* n *over an affine surface* $S = \underline{iss}_n^t\ A$ *and if* A *is smooth in* $\xi \in X$*, then the local structure of* A *is determined by a quiver setting* (Q,α) *where* $\alpha = (1,\ldots,1)$ *and* Q *is a two-circuit quiver on* $k+l+m \leq n$ *vertices, corresponding to the distinct simple components of* M_ξ



and an unordered partition $p = (d_1, \ldots, d_{k+l+m})$ *of* n *with* $k+l+m$ *non-zero parts determined by the dimensions of the simple components of* M_ξ. *With the indicated ordering of the vertices we have that*

$$\pi_\alpha\ Q = \{\pi_{i\ i+1} \mid \begin{cases} 1 & \leq i \leq k-1 \\ k+1 & \leq i \leq k+l-1 \\ k+l+1 & \leq i \leq k+l+m-1 \end{cases} \} \cup \{\pi_{k\ k+l+1}, \pi_{k+l\ k+l+1}, \pi_{k+l+m\ 1}, \pi_{k+l+m\ k+1}\}$$

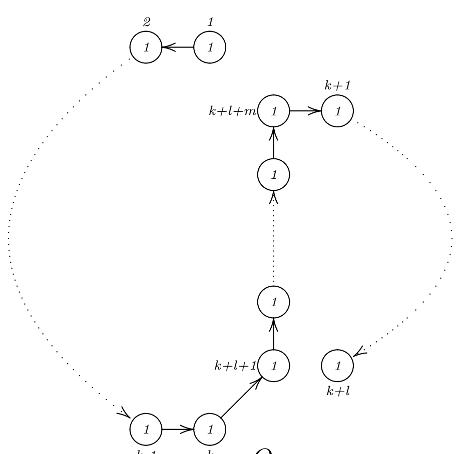
As the weights of a corner cannot contain all weights of an oriented cycle in Q *we have to consider the following two types of potential corner-weights* Π *of maximal cardinality*

- *(outer type) :* $\Pi = \pi_\alpha\ Q - \{\pi_a, \pi_b\}$ *where* a *is an edge in the interval* $[v_1, \ldots, v_k]$ *and* b *is an edge in the interval* $[v_{k+1}, \ldots, v_{k+l}]$.
- *(inner type) :* $\Pi = \pi_\alpha\ Q - \{\pi_c\}$ *where* c *is an edge in the interval* $[v_{k+l+1}, v_{k+l+m}]$.

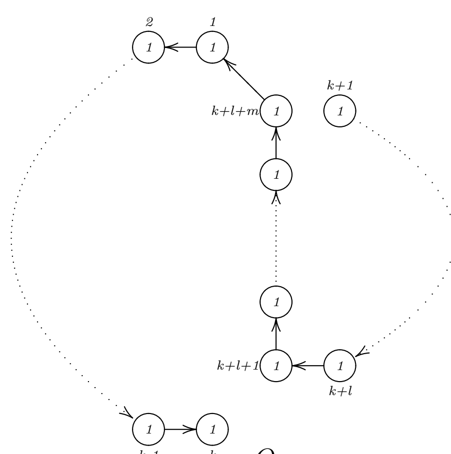
There are $2 + (k-1)(l-1)$ *different subsets* Π *of outer type, each occurring as the set of weights of a corner* C_s, *that is* $\Pi = \pi_s\ Q$ *for some* $s \in \mathcal{S}_\alpha\ Q$. *The two exceptional cases correspond to*

$$\begin{cases} \Pi_1 & = \pi_\alpha\ Q - \{\pi_{k+l+m\ 1}, \pi_{k+l\ k+l+1}\} \\ \Pi_2 & = \pi_\alpha\ Q - \{\pi_{k+l+m\ k+1}, \pi_{k\ k+l+1}\} \end{cases}$$

which are of the form $\pi_{s_i}\ Q$ *with associated border quiver-setting* $(Q_{s_i}, \alpha_{s_i}, \theta_{s_i})$ *where* $\alpha_{s_i} = (1, \ldots, 1)$, Q_{s_i} *are the following full line subquivers of* Q



Q_{s_1}



Q_{s_1}

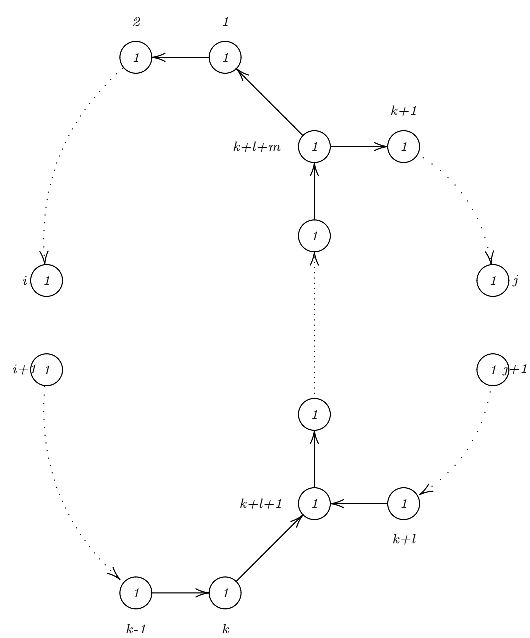
with starting point v_1 *(resp.* v_{k+1}*). The corresponding* $s_i \in \mathcal{S}_\alpha\ Q$ *is a single string with minimal entry*

$$-\frac{\sum_{i=0}^{k+l+m-1} i}{k+l+m} = -\frac{k+l+m-1}{2} \qquad \textit{at place} \quad \begin{cases} 1 \\ k+1 \end{cases}$$

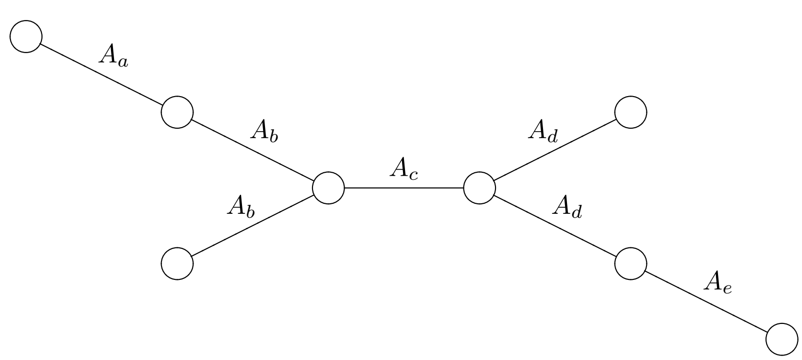
and going with increments equal to one along the unique path. Again, one verifies that $rep_{\alpha_s}\ Q_s$ *has a unique open and* θ_s*-stable orbit, whence these Hesselink strata do occur and the border* B_s *is the full corner* C_s. *The corresponding irreducible* *component in* $\pi^{-1}(\xi)$ *has therefore dimension equal to* $n^2 - 1$ *and is the closure of a unique orbit. The remaining* $(k-1)(l-1)$ *subsets* Π *of outer type are of the form*

$$\Pi_{ij} = \pi_\alpha\ Q - \{\pi_{i\ i+1}, \pi_{j\ j+1}\}$$

with $1 \leq i \leq k-1$ *and* $k+1 \leq j \leq k+l-1$. *We will see in a moment that they are again of type* $\pi_s\ Q$ *for some* $s \in \mathcal{S}_\alpha\ Q$ *with associated border quiver-setting* $(Q_s, \alpha_s, \theta_s)$ *where* $\alpha_s = (1, \ldots, 1)$ *and* Q_s *is the full subquiver of* Q



If we denote with A_l *the directed line quiver on* $l+1$ *vertices, then* Q_s *can be decomposes into full line subquivers*



but then we consider the one string $s \in \mathcal{S}_\alpha\ Q$ *with minimal entry equal to* $-\frac{x}{k+l+m}$ *where with notations as above*

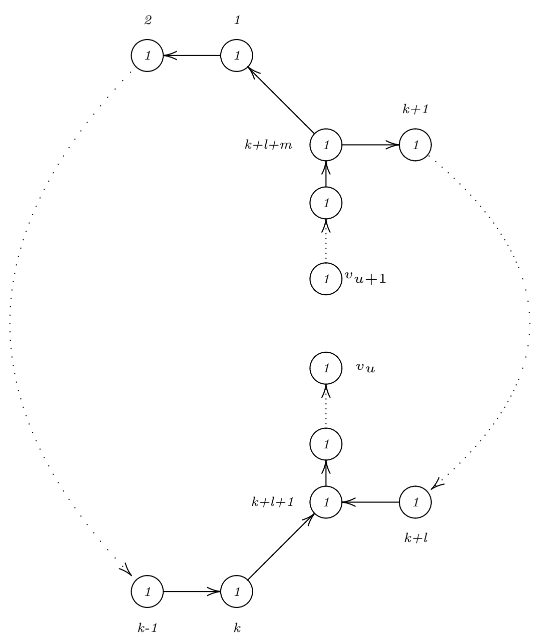
$$x = \sum_{i=1}^{a} i + 2\sum_{i=1}^{b}(a+i) + \sum_{i=1}^{c}(a+b+i) + 2\sum_{i=1}^{d}(a+b+c+i) + \sum_{i=1}^{e}(a+b+c+d+i)$$

where the components of s *are given to the relevant vertex-indices. Again, there is a unique open orbit in* $rep_{\alpha_s}\ Q_s$ *which is a* θ_s*-stable representation and the border* B_s

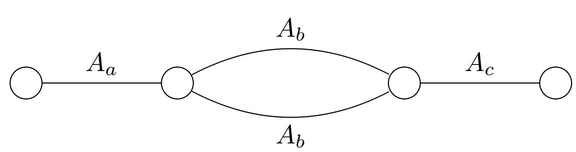
coincides with the corner C_s. That is, the corresponding Hesselink stratum occurs and the irreducible component of $\pi^{-1}(\xi)$ it determines had dimension equal to

$$\begin{aligned} dim\ GL_n - dim\ GL(\alpha) + dim\ rep_{\alpha_s}\ Q_s &= n^2 - (k+l+m) + (k+l+m-1) \\ &= n^2 - 1 \end{aligned}$$

There are $m-1$ different subsets Π_u of inner type, where for $k+l+1 \leq u < k+l+m$ we define $\Pi_u = \pi_\alpha\ Q - \{\pi_{u\ u+1}\}$, that is dropping an edge in the middle



First assume that $k = l$. In this case we can walk through the quiver (with notations as before)



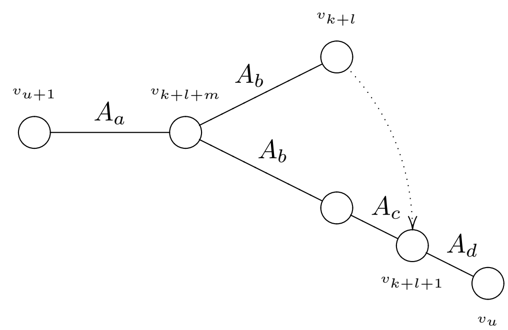
and hence the full subquiver of Q is part of a corner quiver-setting $(Q_s, \alpha_s, \theta_s)$ where $\alpha = (1, \ldots, 1)$ and where s has as its minimal entry $-\frac{x}{k+l+m}$ where

$$x = \sum_{i=1}^{a} i + 2\sum_{i=1}^{b}(a+i) + \sum_{i=1}^{c}(a+b+i)$$

In this case we see that $rep_{\alpha_s}\ Q_s$ has θ_s-stable representations, in fact, there is a $\mathbb{P}^1$-family of such orbits. The corresponding Hesselink stratum is nonempty and the irreducible component of $\pi^{-1}(\xi)$ determined by it has dimension

$$dim\ GL_n - dim\ GL(\alpha) + dim\ rep_{\alpha_s}\ Q_s = n^2 - (k+l+m) + (k+l+m) = n^2$$

If $l < k$, then $\Pi_u = \pi_s\ Q$ for some $s \in \mathcal{S}_\alpha\ Q$ but this time the border quiver-setting $(Q_s, \alpha_s, \theta_s)$ is determined by $\alpha_s = (1, \ldots, 1)$ and Q_s the full subquiver of Q by also *dropping the arrow corresponding to* $\pi_{k+l+1\ k+l}$, *that is*



If Q_s *is this quiver (without the dashed arrow) then* $B_s = rep_{\alpha_s}\ Q_s$ *and it contains an open orbit of a* θ_s*-stable representation. Observe that* s *is determines as the one string vector with minimal entry* $-\frac{x}{k+l+m}$ *where*

$$x = \sum_{i=1}^{a} i + 2\sum_{i=1}^{b}(a+i) + \sum_{i=1}^{c}(a+b+i) + \sum_{i=1}^{d}(a+b+c+i)$$

However, in this case $B_s \neq C_s$ *and we can identify* C_s *with* $rep_{\alpha_s}\ Q'_s$ *where* Q'_s *is* Q_s *together with the dashed arrow. There is an* $\mathbb{A}^1$*-family of orbits in* C_s *mapping to the* θ_s*-stable representation. In particular, the Hesselink stratum exists and the corresponding irreducible component in* $\pi^{-1}(\xi)$ *has dimension equal to*

$$dim\ GL_n - dim\ GL(\alpha) + dim\ C_s = n^2 - (k+l+m) + (k+l+m) = n^2.$$

This concludes the proof of the description of the representation fibers of smooth orders over surfaces, summarized in the following result.

Theorem 8.32 *Let* A *be a Cayley-Hamilton order of degree* n *over an affine surface* $X = \underline{iss}_n^t\ A$ *and assume that* A *is smooth in* $\xi \in X$ *of local type* (A_{klm}, α)*. Then, the representation fiber* $\pi^{-1}(\xi)$ *has exactly* $2 + (k-1)(l-1) + (m-1)$ *irreducible components of which* $2 + (k-1)(l-1)$ *are of dimension* $n^2 - 1$ *and are closure of one orbit and the remaining* $m - 1$ *have dimension* n^2 *and are closures of a one-dimensional family of orbits. In particular, if* A *is Cayley-smooth, then the algebraic quotient map*

$$\underline{rep}_n^t\ A \xrightarrow{\pi} \underline{iss}_n^t\ A = X$$

is flat if and only if all local quiver settings of A *have quiver* A_{klm} *with* $m = 1$*.*

8.8 Brauer-Severi fibers.

In the foregoing chapter we have given a description of the generic Brauer-Severi variety $BS_n^m(gen)$ *as a moduli space of quiver representation. Moreover, we have given a local description of the fibration*

$$BS_n^m(gen) \xrightarrow{\psi} iss_n^m$$

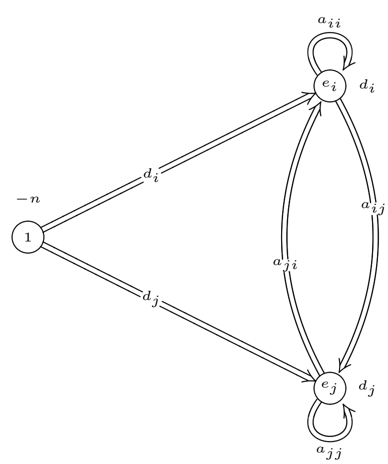
in an étale neighborhood of a point $\xi \in iss_n^m$ *of representation type* $\tau = (e_1, d_1; \ldots; e_k, d_k)$*. We proved that it is étale locally isomorphic to the fibration*

$$M_{\alpha_\zeta}^{ss}(Q_\zeta, \theta_\zeta) \longrightarrow iss_{\alpha_\zeta}\ Q_\zeta$$

in a neighborhood of the trivial representation. That is, we can obtain the generic Brauer-Severi fiber $\psi^{-1}(\xi)$ from the description of the nullcone $Null_{\alpha_\zeta}\ Q_\zeta$ provided we can keep track of θ_ζ-semistable representations. Let us briefly recall the description of the quiver-setting $(Q_\zeta, \alpha_\zeta, \theta_\zeta)$.

- *The quiver Q_ζ has $k+1$ vertices $\{v_0, v_1, \ldots, v_k\}$ such that there are d_i arrows from v_0 to v_i for $1 \leq i \leq k$. For $1 \leq i, j \leq k$ there are $a_{ij} = (m-1)d_i d_j + \delta_{ij}$ directed arrows from v_i to v_j.*
- *The dimension vector $\alpha_\zeta = (1, e_1, \ldots, e_k)$.*
- *The character θ_ζ is determined by the integral $k+1$-tuple $(-n, d_1, \ldots, d_k)$.*

That is, for any triple (v_0, v_i, v_j) of vertices, the full subquiver of Q_ζ on these three vertices has the following form



Let $E = \sum_{i=1}^k e_i$ and T the usual (diagonal) maximal torus of dimension $1+E$ in $GL(\alpha_\zeta) \hookrightarrow GL_E$ and let $\{\pi_0, \pi_1, \ldots, \pi_E\}$ be the obvious basis for the weights of T.. As there are loops in every v_i for $i \geq 1$ and there are arrows from v_i to v_j for all $i, j \geq 1$ we see that the set of weights of $rep_{\alpha_\zeta}\ Q_\zeta$ is

$$\pi_{\alpha_\zeta}\ Q_\zeta = \{\pi_{ij} = \pi_j - \pi_i \ \mid\ 0 \leq i \leq E, 1 \leq j \leq E\}$$

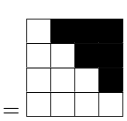
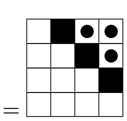
The maximal sets $\pi_s\ Q_\zeta$ for $s \in \mathcal{S}_{\alpha_\zeta}\ Q_\zeta$ are of the form

$$\pi_s\ Q_\zeta \overset{dfn}{=} \pi_\sigma = \{\pi_{ij} \ \mid\ i = 0 \ or\ \sigma(i) < \sigma(j)\}$$

for some fixed permutation $\sigma \in S_E$ of the last E entries. To begin, there can be no larger subset as this would imply that for some $1 \leq i, j \leq E$ both π_{ij} and π_{ji} would belong to it which cannot be the case for a subset $\pi_{s'}\ Q_\zeta$. Next, $\pi_\sigma = \pi_s\ Q_\zeta$ where

$$s = (p, p+\sigma(1), p+\sigma(2), \ldots, p+\sigma(E)) \quad where \quad p = -\tfrac{E}{2}$$

If we now make s vertex-dominant, or equivalently if we only take a σ in the factor $S_E/(S_{e_1} \times S_{e_2} \times \ldots \times S_{e_k})$, then s belongs to $\mathcal{S}_{\alpha_\zeta}\ Q_\zeta$. For example, if $E = 3$ and $\sigma = id \in S_3$, then the corresponding border and corner regions for π_s are

$C_s =$  *and* $B_s =$ 

We have to show that the corresponding Hesselink stratum is non-empty in $Null_{\alpha_\zeta}\ Q_\zeta$ and that it contains θ_ζ-semistable representations. For s corresponding to a fixed $\sigma \in S_E$ the border quiver-setting $(Q_s, \alpha_s, \theta_s)$ is equal to

$$\begin{tikzcd} \overset{-E}{1} \arrow[r, Rightarrow, "z_0"] & \overset{-E+2}{1} \arrow[r, Rightarrow, "z_1"] & \overset{-E+4}{1} \arrow[r, Rightarrow, "z_2"] & \cdots & \cdots \arrow[r, Rightarrow, "z_{E-1}"] & \overset{E-2}{1} \arrow[r, Rightarrow, "z_E"] & \overset{E}{1} \end{tikzcd}$$

where the number of arrows z_i are determined by

$$\begin{cases} z_0 & = p_u \ \text{if } \sigma(1) \in I_{v_u} \\ z_i = a_{uv} \ \text{if } \sigma(i) \in I_{v_u} \text{ and } \sigma(i+1) \in I_{v_v} \end{cases}$$

where we recall that I_{v_i} is the interval of entries in $[1, \ldots, E]$ belonging to vertex v_i. As all the $z_i \geq 1$ it follows that $rep_{\alpha_s}\ Q_s$ contains θ_s-stable representations, so the stratum in $Null_{\alpha_\zeta}\ Q_\zeta$ determined by the corner-type C_s is non-empty. We can depict the $L_s = T$-action on the corner as a representation space of the extended quiver-setting

$$\begin{tikzcd} 1 \arrow[r, Rightarrow, "z_0"] \arrow[rr, bend left, Rightarrow, "v_{02}"] \arrow[rrrrr, bend left=60, Rightarrow, "v_{0E}"] & 1 \arrow[r, Rightarrow, "z_1"] \arrow[rrrr, bend left=45, Rightarrow, "v_{1E}"] & 1 \arrow[r, Rightarrow, "z_2"] & {} & \cdots \quad \cdots \arrow[r, Rightarrow, "z_E"] & 1 \end{tikzcd}$$

Translating representations of this extended quiver back to the original quiver-setting (Q_ζ, α_ζ) we see that the corner C_s indeed contains θ_ζ-semistable representations and hence that this stratum in the nullcone determines an irreducible component in the Brauer-Severi fiber $\psi(\xi)$ of the generic Brauer-Severi variety.

Theorem 8.33 *Let $\xi \in iss_n^m$ be of representation type $\tau = (e_1, d_1; \ldots; e_k, d_k)$ and let $E = \sum_{i=1}^k e_i$. Then, the fiber $\pi^{-1}(\xi)$ of the Brauer-Severi fibration*

$$\begin{tikzcd} Brauer^s(gen) \arrow[d, two heads] \arrow[dr, "\psi"] & \\ BS_n^m(gen) \arrow[r, two heads, "\pi"'] & iss_n^m \end{tikzcd}$$

has exactly $\frac{E!}{e_1!e_2!\ldots e_k!}$ irreducible components, all of dimension

$$n + (m-1)\sum_{i<j} e_i e_j d_i d_j + (m-1)\sum_i \frac{e_i(e_i-1)}{2} - \sum_i e_i$$

Proof. *In view of the foregoing remarks we only have to compute the dimension of the irreducible components. For a corner type C_s as above we have that the corresponding irreducible component in $Null_{\alpha_\zeta}\ Q_\zeta$ has dimension*

$$dim\ GL(\alpha_\zeta) - dim\ P_s + dim\ C_s$$

and from the foregoing description of C_s as a quiver-representation space we see that

- $dim\ P_s = 1 + \frac{e_i(e_i+1)}{2}$.

- $dim\ C_s = n + \sum_i \frac{e_i(e_i-1)}{2}((m-1)d_i^2+1) + \sum_{i<j}(m-1)e_ie_jd_id_j$.

as we can identify $P_s \simeq \mathbb{C}^* \times B_{e_1} \times \ldots \times B_{e_k}$ *where* B_e *is the Borel subgroup of* GL_e. *Moreover, as* $\psi^{-1}(\xi)$ *is a Zariski open subset of*

$$(\mathbb{C}^* \times GL_n) \times^{GL(\alpha_\zeta)} Null_{\alpha_\zeta}\ Q_\zeta$$

we see that the corresponding irreducible component of $\psi^{-1}(\xi)$ *has dimension*

$$1 + dim\ GL_n - dim\ P_s + dim\ C_s$$

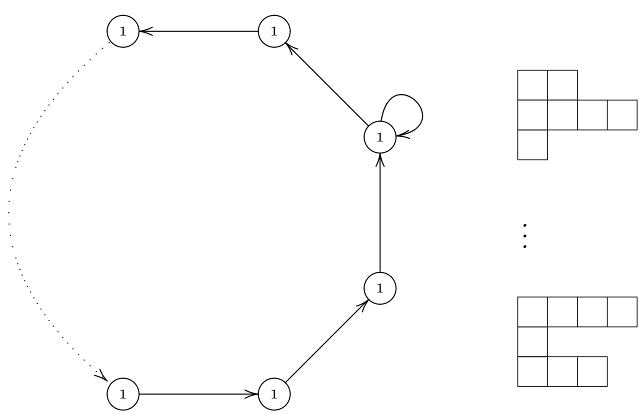
As the quotient morphism $\psi^{-1}(\xi) \twoheadrightarrow \pi^{-1}(\xi)$ *is surjective, we have that the Brauer-Severi fiber* $\pi^{-1}(\xi)$ *has the same number of irreducible components of* $\psi^{-1}(\xi)$. *As the quotient*

$$\psi^{-1}(\xi) \twoheadrightarrow \pi^{-1}(\xi)$$

is by Brauer-stability of all point a principal $PGL(1,n)$*-fibration, substituting the obtained dimensions finishes the proof.* □

In particular, we deduce that the Brauer-Severi fibration $BS_n^m(gen) \xrightarrow{\pi} iss_n^m$ *is a flat morphism if and only if* $(m,n) = (2,2)$ *in which case all Brauer-Severi fibers have dimension one.*

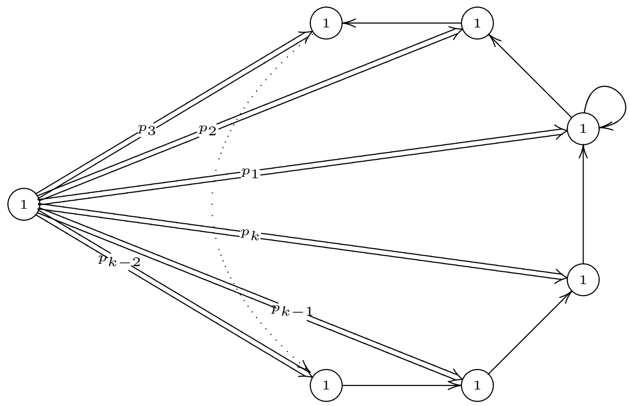
As a final application, let us compute the Brauer-Severi fibers in a point $\xi \in X = \underline{iss}_n^t\ A$ *of the smooth locus* $Sm_n\ A$ *of a Cayley-Hamilton order of degree n which is of local quiver type* (Q,α) *where* $\alpha = (1,\ldots,1)$ *and Q is the quiver*



where the cycle has k vertices and $p = (p_1,\ldots,p_k)$ *is an unordered partition of n having exactly k parts. That is, A is a local Cayley-smooth order over a surface of type* A_{k-101}. *These are the only types that can occur for smooth surface orders which are maximal orders and have a non-singular ramification divisor. Observe also that in the description of nullcones, the extra loop will play no role, so the discussion below also gives the Brauer-Severi fibers of smooth curve orders. The Brauer-Severi fibration is étale locally isomorphic to the fibration*

$$M_{\alpha'}^{ss}(Q',\theta') \xrightarrow{\pi} iss_\alpha\ Q = iss_{\alpha'}\ Q'$$

in a neighborhood of the trivial representation. Here, Q' is the extended quiver by one vertex v_0



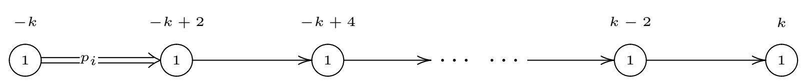
the extended dimension vector is $\alpha' = (1, 1, \ldots, 1)$ and the character is determined by the integral $k+1$-tuple $(-n, p_1, p_2, \ldots, p_k)$. The weights of the maximal torus $T = GL(\alpha')$ of dimension $k+1$ that occur in representations in the nullcone are

$$\pi_{\alpha'}\ Q' = \{\pi_{0\ i}, \pi_{i\ i+1}, 1 \leq i \leq k\}$$

Therefore, maximal corners C_s are associated to $s \in \mathcal{S}_{\alpha'}\ Q'$ where

$$\pi_s\ Q' = \{\pi_{0\ j}, 1 \leq j \leq k\} \cup \{\pi_{i\ i+1}, \pi_{i+1\ i+2}, \ldots, \pi_{i-2\ i-1}\}$$

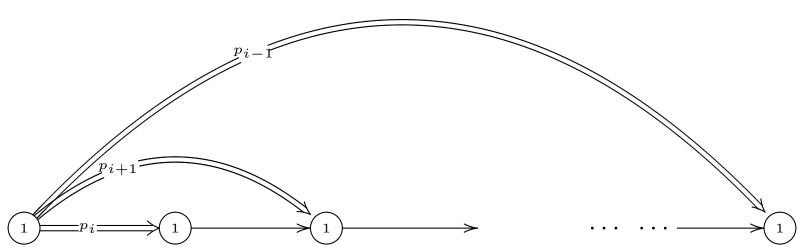
for some fixed i. For such a subset the corresponding s is a one string $k+1$-tuple having as minimal value $-\frac{k}{2}$ at entry 0, $-\frac{k}{2}+1$ at entry i, $-\frac{k}{2}+2$ at entry $i+1$ and so on. To verify that this corner-type occurs in $Null_{\alpha'}\ Q'$ we have to consider the corresponding border quiver-setting $(Q'_s, \alpha'_s, \theta'_s)$ which is



which clearly has θ'_s-semistable representations, in fact, the corresponding moduli space $M^{ss}_{\alpha'_s}(Q'_s, \theta'_s) \simeq \mathbb{P}^{p_1-1}$. In this case we have that $L_s = P_s = GL(\alpha'_s)$ and therefore we can also interpret the corner as an open subset of the representation space

$$C_s \hookrightarrow rep_{\alpha'_s}\ Q"_s$$

where the embedding is $P_s = GL(\alpha'_s)$-equivariant and the extended quiver $Q"_s$ is



Translating corner representations back to $rep_{\alpha'}\ Q'$ we see that C_s contains θ'-semistable representations, so will determine an irreducible component in the Brauer-Severi fiber $\pi^{-1}(\xi)$. Let us calculate its dimension. The irreducible component N_s of $Null_{\alpha'}\ Q'$ determined by the corner C_s has dimension

$$\begin{aligned} dim\ GL(\alpha') - dim\ P_s + dim\ C_s &= (k+1) - (k+1) + \sum_i p_i + (k-1) \\ &= n + k - 1 \end{aligned}$$

But then, the corresponding component in the Brauer-stable is an open subvariety of $(\mathbb{C}^* \times GL_n) \times^{GL(\alpha')} N_s$ *and therefore has dimension*

$$\begin{aligned} dim\ \mathbb{C}^* \times GL_n - dim\ GL(\alpha') + dim\ N_s &= 1 + n^2 - (k+1) + n + k - 1 \\ &= n^2 + n - 1 \end{aligned}$$

But then, as the stabilizer subgroup of all Brauer-stable points is one dimensional in $\mathbb{C}^* \times GL_n$ *the corresponding irreducible component in the Brauer-Severi fiber* $\pi^{-1}(\xi)$ *has dimension* $n-1$*. This completes the proof of the*

Theorem 8.34 *Let* A *be a Cayley-Hamilton order of degree* n *over a surface* $X = \underline{iss}^t_n\ A$ *and let* A *be Cayley-smooth in* $\xi \in X$ *of type* A_{k-101} *and* p *as before. Then,the fiber of the Brauer-Severi fibration*

$$BS^t_n(A) \twoheadrightarrow X$$

in ξ *has exactly* k *irreducible components, each of dimension* $n-1$*. In particular, if* A *is a Cayley-smooth order over the surface* X *such that all local types are* $(A_{k-101}.p)$ *for some* $k \geq 1$ *and partition* p *of* n *in having* k*-parts, then the Brauer-Severi fibration is a flat morphism.*

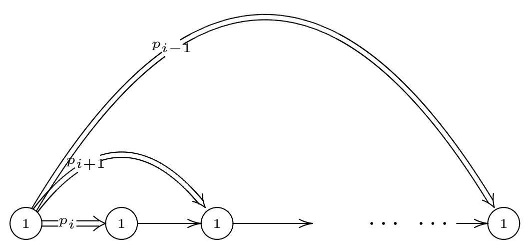
In fact, one can give a nice geometric interpretation to the different components. Consider the component corresponding to the corner C_s *with notations as before. Consider the sequence of* $k-1$ *rational maps*

$$\mathbb{P}^{n-1} \twoheadrightarrow \mathbb{P}^{n-1-p_{i-1}} \twoheadrightarrow \mathbb{P}^{n-1-p_{i-1}-p_{i-2}} \twoheadrightarrow \ldots \twoheadrightarrow \mathbb{P}^{p_i-1}$$

defined by killing the right hand coordinates

$$[x_1 : \ldots : x_n] \mapsto [x_1 : \ldots : x_{n-p_{i-1}} : \underbrace{0 : \ldots : 0}_{p_{i-1}}] \mapsto \ldots \mapsto [x_1 : \ldots : x_{p_i} : \underbrace{0 : \ldots : 0}_{n-p_i}]$$

that is in the extended corner-quiver setting



we subsequently set all entries of the arrows from v_0 *to* v_{i-j} *zero for* $j \geq 1$*, the extreme projection* $\mathbb{P}^{n-1} \twoheadrightarrow \mathbb{P}^{p_i-1}$ *corresponds to the projection* $C_s/P_s \twoheadrightarrow B_s/L_s = M^{ss}_{\alpha'_s}(Q'_s, \theta'_s)$*. Let* V_i *be the subvariety in* $\times_{j=1}^k \mathbb{P}^{n-1}$ *be the closure of the graph of this sequence of rational maps. If we label the coordinates in the* $k-j$*-th component* $\mathbb{P}^{n-1}$ *as* $x(j) = [x_1(j) : \ldots : x_n(j)]$*, then the multi-homogeneous equations defining* V_i *are*

$$\begin{cases} x_a(j) & = 0 \text{ if } a > p_i + p_{i+1} + \ldots + p_{i+j} \\ x_a(j)x_b(j-1) & = x_b(j)x_a(j-1) \text{ if } 1 \leq a < b \leq p_i + \ldots + p_{i+l-1} \end{cases}$$

One verifies that V_i *is a smooth variety of dimension* $n-1$*. If we would have the patience to work out the whole nullcone (restricting to the* θ'*-semistable representations) rather than just the irreducible components, we would see that the Brauer-Severi fiber* $\pi^{-1}(\xi)$ *consists of the varieties* $V_1, \ldots, V_k$ *intersecting transversally. The reader is invited to compare our description of the Brauer-Severi fibers with that of M. Artin [2] in the case of Cayley-smooth maximal curve orders.*

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